## IMPROPER INTEGRAL OF THE SECOND KIND

➢ **Definition**

Let $f$ be defined on the half open interval $(a,b]$ and assume that $f \in R(\alpha; x, b)$ for every $x \in (a,b]$. Define a function $I$ on $(a,b]$ as follows:

$$I(x) = \int_x^b f\, d\alpha \quad \text{if} \quad x \in (a,b] \;\ldots\ldots\ldots.. (i)$$

The function $I$ so defined is called an improper integral of the second kind and is denoted by the symbol $\int_{a+}^{b} f(t)\, d\alpha(t)$ or $\int_{a+}^{b} f\, d\alpha$.

The integral $\int_{a+}^{b} f\, d\alpha$ is said to converge if the limit

$$\lim_{x \to a+} I(x) \;\ldots\ldots..(ii) \quad \text{exists (finite).}$$

Otherwise, $\int_{a+}^{b} f\, d\alpha$ is said to diverge. If the limit in (*ii*) exists and equals $A$, the number $A$ is called the value of the integral and we write $\int_{a+}^{b} f\, d\alpha = A$.

Similarly, if $f$ is defined on $[a,b)$ and $f \in R(\alpha; a, x)$ $\forall$ $x \in [a,b)$ then $I(x) = \int_a^x f\, d\alpha$ if $x \in [a,b)$ is also an improper integral of the second kind and is denoted as $\int_a^{b-} f\, d\alpha$ and is convergent if $\lim_{x \to b-} I(x)$ exists (finite).

➢ **Example**

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f \in R(x,b)$ for every $x \in (0,b]$.

$$I(x) = \int_x^b x^{-p}\, dx \qquad \text{if} \quad x \in (0,b]$$

$$= \int_{0+}^{b} x^{-p}\, dx = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^{b} x^{-p}\, dx$$

$$= \lim_{\varepsilon \to 0} \left| \frac{x^{1-p}}{1-p} \right|_{\varepsilon}^{b} = \lim_{\varepsilon \to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \qquad , \qquad (p \neq 1)$$

$$= \left[ \begin{array}{ll} \textit{finite} & , \; p < 1 \\ \textit{infinite} & , \; p > 1 \end{array} \right.$$

When $p = 1$, we get $\int_{\varepsilon}^{b} \frac{1}{x}\, dx = \log b - \log \varepsilon \to \infty$ as $\varepsilon \to 0$.

$$\Rightarrow \int_{0+}^{b} x^{-1}\,dx \text{ also diverges.}$$

Hence the integral converges when $p<1$ and diverges when $p\ge 1$.

## ➢ *Note*

If the two integrals $\int_{a+}^{c} f\,d\alpha$ and $\int_{c}^{b-} f\,d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\,d\alpha = \int_{a+}^{c} f\,d\alpha + \int_{c}^{b-} f\,d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^{b} f\,d\alpha + \int_{b}^{\infty} f\,d\alpha \quad \text{which can be written as} \quad \int_{a+}^{\infty} f\,d\alpha.$$

## ➢ *Example*

Consider $\int_{0+}^{\infty} e^{-x}x^{p-1}\,dx \quad , \quad (p>0)$

This integral must be interpreted as a sum as

$$\int_{0+}^{\infty} e^{-x}x^{p-1}\,dx = \int_{0+}^{1} e^{-x}x^{p-1}\,dx + \int_{1}^{\infty} e^{-x}x^{p-1}\,dx$$
$$= I_1 + I_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

$I_2$, the second integral, converges for every real $p$ as proved earlier.

To test $I_1$, put $t=\dfrac{1}{x} \quad \Rightarrow \quad dx = -\dfrac{1}{t^2}\,dt$

$$\Rightarrow \quad I_1 = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} e^{-x}x^{p-1}\,dx \;=\lim_{\varepsilon\to 0}\int_{1/\varepsilon}^{1} e^{-\frac{1}{t}}\,t^{1-p}\left(-\frac{1}{t^2}dt\right) \;=\lim_{\varepsilon\to 0}\int_{1}^{1/\varepsilon} e^{-\frac{1}{t}}\,t^{-p-1}\,dt$$

Take $f(t)=e^{-\frac{1}{t}}\,t^{-p-1}$ and $g(t)=t^{-p-1}$

Then $\lim\limits_{t\to\infty}\dfrac{f(t)}{g(t)} = \lim\limits_{t\to\infty}\dfrac{e^{-\frac{1}{t}}\cdot t^{-p-1}}{t^{-p-1}} = 1$ and since $\int_{1}^{\infty} t^{-p-1}\,dt$ converges when $p>0$

$$\therefore \int_{1}^{\infty} e^{-\frac{1}{t}}\,t^{-p-1}\,dt \text{ converges when } p>0$$

Thus $\int_{0+}^{\infty} e^{-x}x^{p-1}\,dx$ converges when $p>0$.

When $p>0$, the value of the sum in $(i)$ is denoted by $\Gamma(p)$. The function so defined is called the Gamma function.

### ➢ *Note*

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

### ➢ *A Useful Comparison Integral*

$$\int_a^b \frac{dx}{(x-a)^n}$$

We have, if $n \neq 1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \left|\frac{1}{(1-n)(x-a)^{n-1}}\right|_{a+\varepsilon}^{b}$$

$$= \frac{1}{(1-n)}\left(\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right)$$

Which tends to $\dfrac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n<1$ or $n>1$, as $\varepsilon \to 0$.

Again, if $n=1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a) - \log\varepsilon \to +\infty \quad \text{as} \quad \varepsilon \to 0.$$

Hence the improper integral $\displaystyle\int_a^b \frac{dx}{(x-a)^n}$ converges iff $n<1$.

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### ➢ *Question*

Examine the convergence of

$(i)$ $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ $(ii)$ $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$ $(iii)$ $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

### *Solution*

**(*i*)** $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

Here '0' is the only point of infinite discontinuity of the integrand.
We have

$$f(x) = \frac{1}{x^{1/3}(1+x^2)}$$

Take $g(x) = \dfrac{1}{x^{1/3}}$

Then $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 0}\dfrac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

**(*ii*)** $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2}$$

Take $g(x)=\dfrac{1}{x^2}$

Then $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to 0}\dfrac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

**(*iii*)** $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\displaystyle\int_0^{1/2} f(x)\,dx$ and $\displaystyle\int_{1/2}^1 f(x)\,dx$.

To examine the convergence of $\displaystyle\int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\dfrac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1-x)^{1/3}}=1$$

$\because \int_0^{1/2} \frac{1}{x^{1/2}} dx$ converges $\quad \therefore \int_0^{1/2} \frac{1}{x^{1/2}(1-x)^{1/3}} dx$ is convergent.

To examine the convergence of $\int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{1/3}} dx$, we take $g(x) = \frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{1}{x^{1/2}} = 1$$

$\because \int_{1/2}^{1} \frac{1}{(1-x)^{1/3}} dx$ converges $\quad \because \int_{1/2}^{1} \frac{1}{x^{1/2}(1-x)^{1/3}} dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

## ➢ *Question*

Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ exists iff $m, n$ are both positive.

### *Solution*

The integral is proper if $m \geq 1$ and $n \geq 1$.

The number '0' is a point of infinite discontinuity if $m < 1$ and the number '1' is a point of infinite discontinuity if $n < 1$.

Let $m < 1$ and $n < 1$.

We take any number, say $1/2$, between 0 & 1 and examine the convergence of the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1} dx$ and $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1} dx$ at '0' and '1' respectively.

**Convergence at 0:**

We write

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take} \quad g(x) = \frac{1}{x^{1-m}}$$

Then $\frac{f(x)}{g(x)} \to 1$ as $x \to 0$

As $\int_0^{1/2} \frac{1}{x^{1-m}} dx$ is convergent at 0 iff $1-m < 1$ i.e. $m > 0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1} dx$ is convergent at 0, iff $m$ is +ive.

**Convergence at 1:**

We write $f(x)=x^{m-1}(1-x)^{n-1}=\dfrac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x)=\dfrac{1}{(1-x)^{1-n}}$

Then $\dfrac{f(x)}{g(x)}\to 1$ as $x\to 1$

As $\displaystyle\int_{1/2}^{1}\frac{1}{(1-x)^{1-n}}dx$ is convergent, iff $1-n<1$ i.e. $n>0$.

We deduce that the integral $\displaystyle\int_{1/2}^{1}x^{m-1}(1-x)^{n-1}dx$ converges iff $n>0$.

Thus $\displaystyle\int_{0}^{1}x^{m-1}(1-x)^{n-1}dx$ exists for positive values of $m,n$ only.

It is a function which depends upon $m$ & $n$ and is defined for all positive values of $m$ & $n$. It is called Beta function.

## ➢ *Question*

Show that the following improper integrals are convergent.

(*i*) $\displaystyle\int_{1}^{\infty}\sin^2\frac{1}{x}\,dx$ (*ii*) $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ (*iii*) $\displaystyle\int_{0}^{1}\frac{x\log x}{(1+x)^2}\,dx$ (*iv*) $\displaystyle\int_{0}^{1}\log x\cdot\log(1+x)\,dx$

## *Solution*

**(*i*)** Let $f(x)=\sin^2\dfrac{1}{x}$ and $g(x)=\dfrac{1}{x^2}$

then $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2=1$

$\Rightarrow \displaystyle\int_{1}^{\infty}f(x)\,dx$ and $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ behave alike.

$\because \displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ is convergent $\therefore \displaystyle\int_{1}^{\infty}\sin^2\frac{1}{x}\,dx$ is also convergent.

**(*ii*)** $\displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$\sin^2 x\le 1 \quad\Rightarrow\quad \dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2} \quad \forall\ x\in(1,\infty)$

and $\displaystyle\int_{1}^{\infty}\frac{1}{x^2}\,dx$ converges $\therefore \displaystyle\int_{1}^{\infty}\frac{\sin^2 x}{x^2}\,dx$ converges.

## ➢ *Note*

$\int_0^1 \frac{\sin^2 x}{x^2}\, dx$ is a proper integral because $\lim_{x\to 0}\frac{\sin^2 x}{x^2} = 1$ so that '0' is not a point of infinite discontinuity. Therefore $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx$ is convergent.

**(*iii*)** $\int_0^1 \frac{x \log x}{(1+x)^2}\, dx$

$\because\ \log x < x\ , \quad x \in (0,1)$

$\therefore\ x \log x < x^2$

$\Rightarrow\ \frac{x \log x}{(1+x)^2} < \frac{x^2}{(1+x)^2}$

Now $\int_0^1 \frac{x^2}{(1+x)^2}\, dx$ is a proper integral.

$\therefore\ \int_0^1 \frac{x \log x}{(1+x)^2}\, dx$ is convergent.

**(*iv*)** $\int_0^1 \log x \cdot \log(1+x)\, dx$

$\because\ \log x < x \qquad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x \cdot \log(1+x) < x(x+1)$

$\because\ \int_0^1 x(x+1)\, dx$ is a proper integral $\quad \therefore\ \int_0^1 \log x \cdot \log(1+x)\, dx$ is convergent.

## ➢ *Note*

(*i*) $\int_0^a \frac{1}{x^p} dx$ diverges when $p \geq 1$ and converges when $p < 1$.

(*ii*) $\int_a^\infty \frac{1}{x^p}\, dx$ converges iff $p > 1$.