Improper Integrals

Course Title: Real Analysis II — Course Code: MTH322
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We discussed (in MTH321: Real Analysis I) Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. To extend the concept, we shall relax these restrictions on f and α.

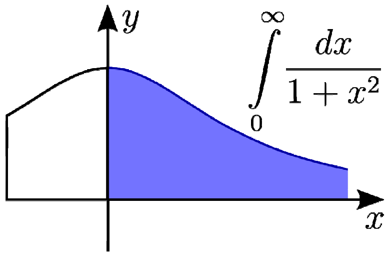
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

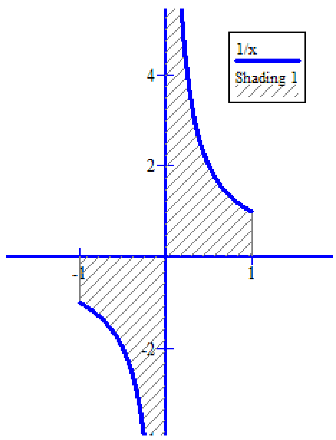
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a\le x\le b$. Such points are called singularities of $f(x)$.

➢ *Examples*



- $\int_0^\infty \frac{1}{1+x^2}dx$, $\int_{-\infty}^1 \frac{1}{x-2}dx$ and $\int_{-\infty}^\infty (x^2+1)dx$ are examples of improper integrals of first kind.



- $\int_{-1}^1 \frac{1}{x}dx$ and $\int_0^1 \frac{1}{2x-1}dx$ are examples of improper integrals of second kind.

➢ *Notations*

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

IMPROPER INTEGRAL OF THE FIRST KIND

➢ Definition

Assume that $f \in R(\alpha;a,b)$ for every $b \geq a$. Keep a,α and f fixed and define a function I on $[a,\infty)$ as follows:

$$I(b)=\int_a^b f(x)\,d\alpha(x) \quad \text{if} \quad b \geq a \quad \ldots\ldots\ldots\ldots (i)$$

The function I so defined is called an infinite (or an improper) integral of first kind and is denoted by the symbol $\int_a^\infty f(x)\,d\alpha(x)$ or by $\int_a^\infty f\,d\alpha$.

The integral $\int_a^\infty f\,d\alpha$ is said to converge if the limit

$$\lim_{b\to\infty} I(b) \quad \ldots\ldots\ldots\ldots (ii)$$

exists (finite). Otherwise, $\int_a^\infty f\,d\alpha$ is said to diverge.

If the limit in (ii) exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,d\alpha = A$

➢ Example

Consider and integral $\int_1^\infty x^{-p}\,dx$, where p is any real number.

Now $I(b)=\int_1^b x^{-p}\,dx = \left.\dfrac{x^{1-p}}{1-p}\right|_1^b = \dfrac{1-b^{1-p}}{p-1}$ if $p \neq 1$.

As we know

$$\lim_{b\to\infty} I(b) = \lim_{b\to\infty}\frac{1-b^{1-p}}{p-1} = \begin{cases} \infty & \text{if} \quad p<1, \\ \dfrac{1}{p-1} & \text{if} \quad p>1. \end{cases}$$

Thus integral $\int_1^\infty x^{-p}\,dx$ diverges if $p<1$ and converges if $p>1$ and has the value $\dfrac{1}{p-1}$.

If $p=1$, we get $\int_1^b x^{-1}\,dx = \log b \to \infty$ as $b \to \infty$. $\Rightarrow \int_1^\infty x^{-1}\,dx$ diverges.

Hence we concluded: $\displaystyle\int_1^\infty x^{-p}\,dx = \begin{cases} \text{diverges} & \text{if} \quad p \leq 1, \\ \dfrac{1}{p-1} & \text{if} \quad p>1. \end{cases}$

➢ *Example*

Consider $\int_0^{\infty} \sin 2\pi x\, dx$

Since $\int_0^{b} \sin 2\pi x\, dx = \dfrac{1-\cos 2\pi b}{2\pi} \to l$ as $b \to \infty$, where l has values between 0 and $\dfrac{1}{\pi}$, that is, limit is not unique.

Therefore the integral $\int_0^{\infty} \sin 2\pi x\, dx$ diverges.

➢ *Note*

If $\int_{-\infty}^{a} f\, d\alpha$ and $\int_{a}^{\infty} f\, d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\, d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\, d\alpha = \int_{-\infty}^{a} f\, d\alpha + \int_{a}^{\infty} f\, d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} f\, d\alpha$ converges, its value is equal to the limit: $\lim_{b\to+\infty} \int_{-b}^{b} f\, d\alpha$.

➢ *Theorem*

Assume that α is monotonically increasing on $[a,+\infty)$ and suppose that $f \in R(\alpha;a,b)$ for every $b \geq a$. Assume that $f(x) \geq 0$ for each $x \geq a$. Then $\int_a^{\infty} f\, d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^{b} f\, d\alpha \leq M \quad \text{for every } b \geq a.$$

Proof

Let $I(b) = \int_a^{b} f\, d\alpha$ and suppose that $\int_a^{\infty} f\, d\alpha$ is convergent, then $\lim_{b\to+\infty} I(b)$ exists, that is, $I(b)$ is bounded.

So there exists a constant $M > 0$ such that

$$|I(b)| < M \quad \text{for every } b \geq a.$$

As $f(x) \geq 0$ for each $x \geq a$, therefore $\int_a^{b} f\, d\alpha \geq 0$.

This gives $I(b)=\int_a^b f\,d\alpha \le M$ for every $b\ge a$.

Conversely, suppose that there exists a constant $M>0$ such that $\int_a^b f\,d\alpha \le M$ for every $b\ge a$. This give $|I(b)|\le M$ for every $b\ge a$.

That is, I is bounded on $[a,+\infty)$.

Now for $b_2\ge b_1>a$, we have

$$I(b_2)=\int_a^{b_2} f(x)\,d\alpha(x)=\int_a^{b_1} f(x)\,d\alpha(x)+\int_{b_1}^{b_2} f(x)\,d\alpha(x)$$

$$\ge\int_a^{b_1} f(x)\,d\alpha(x)=I(b_1) \qquad \because \int_{b_1}^{b_2} f(x)\,d\alpha(x)\ge 0 \text{ as } f(x)\ge 0.$$

This gives I is monotonically increasing on $[a,+\infty)$.

As I is monotonically increasing and bounded on $[a,+\infty)$, therefore it is convergent, that is $\int_a^\infty f\,d\alpha$ converges.

➢ *Theorem: (Comparison Test)*

Assume that α is monotonically increasing on $[a,+\infty)$. If $f\in R(\alpha;a,b)$ for every $b\ge a$, if $0\le f(x)\le g(x)$ for every $x\ge a$, and if $\int_a^\infty g\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have

$$\int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$$

Proof

Let $I_1(b)=\int_a^b f\,d\alpha$ and $I_2(b)=\int_a^b g\,d\alpha$, $b\ge a$

$\because\ 0\le f(x)\le g(x)$ for every $x\ge a$

$\therefore\ I_1(b)\le I_2(b)$ …………………. (i)

$\because \int_a^\infty g\,d\alpha$ converges $\therefore\ \exists$ a constant $M>0$ such that

$$\int_a^b g\,d\alpha\le M \quad , \quad b\ge a \ \ldots\ldots\ldots\ldots (ii)$$

From (i) and (ii) we have $I_1(b)\le M$ for every $b\ge a$.

$\Rightarrow \lim_{b\to\infty} I_1(b)$ exists and is finite.

$\Rightarrow \int_a^\infty f\,d\alpha$ converges.

Also $\lim_{b\to\infty} I_1(b) \le \lim_{b\to\infty} I_2(b) \le M$

$\Rightarrow \int_a^\infty f\,d\alpha \le \int_a^\infty g\,d\alpha$.

> ***Theorem (Limit Comparison Test)***
>
> Assume that α is monotonically increasing on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ if $x \ge a$. If
>
> $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$$
>
> then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

For all $b \ge a$, we can find some $N > 0$ such that

$$\left|\frac{f(x)}{g(x)} - 1\right| < \varepsilon \qquad \forall\ x \ge N \quad \text{for every} \quad \varepsilon > 0.$$

$$\Rightarrow\ 1-\varepsilon < \frac{f(x)}{g(x)} < 1+\varepsilon$$

Let $\varepsilon = \frac{1}{2}$. Then we have

$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2}.$$

$\Rightarrow\ g(x) < 2f(x)$ ……….(i) and $2f(x) < 3g(x)$ ………..(ii)

From (i) $\int_a^\infty g\,d\alpha < 2\int_a^\infty f\,d\alpha$,

$\Rightarrow \int_a^\infty g\,d\alpha$ converges if $\int_a^\infty f\,d\alpha$ converges and $\int_a^\infty f\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

From (ii) $2\int_a^\infty f\,d\alpha < 3\int_a^\infty g\,d\alpha$,

$\Rightarrow \int_a^\infty f\,d\alpha$ converges if $\int_a^\infty g\,d\alpha$ converges and $\int_a^\infty g\,d\alpha$ diverges if $\int_a^\infty f\,d\alpha$ diverges.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

➢ *Note*

The above theorem also holds if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=c$, provided that $c\neq 0$. If $c=0$, we can only conclude that convergence of $\int_a^\infty g\,d\alpha$ implies convergence of $\int_a^\infty f\,d\alpha$.

➢ *Example*

For every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2}dx$.

Let $f(x)=e^{-x}x^p$ and $g(x)=\frac{1}{x^2}$.

Now $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{e^{-x}x^p}{1/x^2}$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}e^{-x}x^{p+2}=\lim_{x\to\infty}\frac{x^{p+2}}{e^x}=0.$$

Since $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore the given integral $\int_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ *Remark*

It is easy to show that if $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ are convergent, then

- $\int_a^\infty (f\pm g)\,d\alpha$ is convergent.
- $\int_a^\infty cf\,d\alpha$, where c is some constant, is convergent.

➢ *Theorem*

Assume $\alpha\uparrow$ on $[a,+\infty)$. If $f\in R(\alpha;a,b)$ for every $b\geq a$ and if $\int_a^\infty |f|\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

If $x \ge a$, $\pm f(x) \le |f(x)|$

$\Rightarrow |f(x)| - f(x) \ge 0$

$\Rightarrow 0 \le |f(x)| - f(x) \le 2|f(x)|$

$\Rightarrow \int_a^\infty (|f| - f)\,d\alpha$ converges.

Now difference of $\int_a^\infty |f|\,d\alpha$ and $\int_a^\infty (|f| - f)\,d\alpha$ is convergent,

that is, $\int_a^\infty f\,d\alpha$ is convergent.

➢ *Note*

$\int_a^\infty f\,d\alpha$ is said to converge absolutely if $\int_a^\infty |f|\,d\alpha$ converges. It is said to be convergent conditionally if $\int_a^\infty f\,d\alpha$ converges but $\int_a^\infty |f|\,d\alpha$ diverges.

➢ *Remark*

Every absolutely convergent integral is convergent.

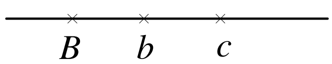
➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f \in R(\alpha; a, b)$ for every $b \ge a$. Then the integral $\int_a^\infty f\,d\alpha$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies

$$\left| \int_b^c f(x)\,d\alpha(x) \right| < \varepsilon$$

Proof

Let $\int_a^\infty f\,d\alpha$ be convergent. Then $\exists\ B > 0$ such that



$$\left| \int_a^b f\,d\alpha - \int_a^\infty f\,d\alpha \right| < \frac{\varepsilon}{2} \quad \text{for every } b \ge B \quad \ldots\ldots\ldots\ldots(i)$$

Also for $c > b > B$,

$$\left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| < \frac{\varepsilon}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (ii)$$

Consider

$$\left| \int_b^c f\,d\alpha \right| = \left| \int_a^c f\,d\alpha - \int_a^b f\,d\alpha \right|$$

$$= \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha + \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right|$$

$$\le \left| \int_a^c f\,d\alpha - \int_a^\infty f\,d\alpha \right| + \left| \int_a^\infty f\,d\alpha - \int_a^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \left| \int_b^c f\,d\alpha \right| < \varepsilon \quad \text{when} \quad c > b > B.$$

Conversely, assume that the Cauchy condition holds.

Define $a_n = \int_a^{a+n} f\,d\alpha$ if $n = 1, 2, \ldots\ldots$

Consider n, m such that $a+n, a+m > b > B$, then

$$|a_n - a_m| = \left| \int_a^{a+n} f\,d\alpha - \int_a^{a+m} f\,d\alpha \right| = \left| \int_a^b f\,d\alpha + \int_b^{a+n} f\,d\alpha - \int_a^b f\,d\alpha - \int_b^{a+m} f\,d\alpha \right|$$

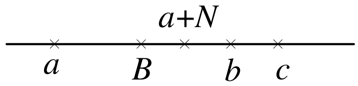
$$= \left| \int_b^{a+n} f\,d\alpha - \int_b^{a+m} f\,d\alpha \right| \le \left| \int_b^{a+n} f\,d\alpha \right| + \left| \int_b^{a+m} f\,d\alpha \right| < \varepsilon + \varepsilon = 2\varepsilon$$

This gives, the sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it converges.

Let $\lim_{n\to\infty} a_n = A$

Given $\varepsilon > 0$, choose B so that $\left| \int_b^c f\,d\alpha \right| < \frac{\varepsilon}{2}$ if $c > b > B$.

and also that $|a_n - A| < \frac{\varepsilon}{2}$ whenever $a + n \ge B$.



Choose an integer N such that $a + N > B$ i.e. $N > B - a$.

Then, if $b > a + N$, we have

$$\left| \int_a^b f\,d\alpha - A \right| = \left| \int_a^{a+N} f\,d\alpha - A + \int_{a+N}^b f\,d\alpha \right|$$

$$\le |a_N - A| + \left| \int_{a+N}^b f\,d\alpha \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_a^\infty f\,d\alpha = A$$

This completes the proof.

➢ ***Remarks***

It follows from the above theorem that convergence of $\int_a^\infty f\,d\alpha$ implies $\lim_{b\to\infty} \int_b^{b+\varepsilon} f\,d\alpha = 0$ for every fixed $\varepsilon > 0$.

However, this does not imply that $f(x) \to 0$ as $x \to \infty$.

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