MTH103: Exploring Quantitative Skills

https://www.mathcity.org/atiq/fa23-mth103

Objectives

The main aim of the lecture is as follows:

- Introduction To Number Systems,
- Square Roots,
- Cube Roots,
- Highest Common Factors,
- Lowest Common Multiples,
- Visualizing Fractions,
- Decimals,
- Systems Of Measurements

References:

1. R. N. Aufmann, J. S. Lockwood, R. D. Natio and D. K. Clegg, Mathematical Thinking and Quantitative Reasoning, Houghton Mifflin Company, New York, 2008.
2. J. Bennett and W. Briggs, Using and Understanding Mathematics (6th Edition). Pearson Education Limited, New York, 2015.
3. R. Blitzer, Precalculus. (5th Edition). Pearson Education Limited, 2014.

Introduction to Number Systems

Mathematics is just logic with numbers attached. --- Marilyn vos Savant, American author

Without mathematics, there's nothing you can do. Everything around you is mathematics. Everything around you is numbers. --- Shakuntala Devi

- Set of digits: $\{0,1,2,3,4,5,6,7,8,9\}$
- Set of natural numbers: $\mathbb{N}=\{1,2,3,\ldots\}$.
- Set of whole numbers: $W=\{0,1,2,\ldots\}$.
- Set of integers: $\mathbb{Z}=\{0,\pm1,\pm2,\ldots\}$.
- Set of rational numbers: $\mathbb{Q}=\left\{\frac{p}{q}: p,q\in\mathbb{Z},q\neq 0\right\}$.
- Set of real number is denoted by $\mathbb{R}$.

It is very important to know the field of working:

For example, divide PKR1000/- into three persons equally.

- First thing first --- > What is the share of each?
- How will you pay in currency?
- What is the easiest way?

➤➤➤ First, we talked about the set of integers.

- $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Positive integers: $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
- Negative integers: $\mathbb{Z}^- = \{-1, -2, -3, \ldots\}$

Remark: An integer '0' is neither positive nor negative.

Highest Common Factor or Greatest Common Divisor

First there is need to know about the factors or divisors.

For any integers *a,b* and *c*.

If a=bc, then b is called divisor of a if b is non-zero.

e.g.

10 = 5x2=2x5.	5 and 2 are divisor of 10.
0 = 2x0,	2 is divisor of zero. (Every integer is divisor of zero.)
5 = 1x5 = 5x1,	1 and 5 are divisors of 5 (1 and integer itself are divisors of itself.)
5 = (-1)(-5),	-1 and -5 are divisors of 5.

- All divisors of 10: ±1, ±2, ±5, ±10
- All positive divisors of 60: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

Remark: 60 is very interesting number, it is the smallest number with exactly 12 divisors.

An even number is any integer that is divisible by two.

An odd number is any integer that is not even.

e.g.

-4 and 200 are even numbers.

-31 and 201 are odd numbers.

Write a set of all odd numbers.

Write a set of all even numbers.

Is zero an even or odd number?

A positive integer is called prime if it has exactly two positive divisors.

A positive integer which has more than two divisors is called composite number.

e.g.

2 and 11 are prime numbers.

4 and 526 are composite numbers.

Remark:

The prime numbers from 1 to 100 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

1 is neither prime nor composite.

An integer a is called factor of b if a is divisor of b.

e.g.

1 and 5 are factors of 50. (Is there any other factor?)

7 and 37 are factors of 259.

What is the difference between divisor and factor?

Properties of Factors

The factors of a number have the following properties:

- The number of factors of a number is finite. This means that a number has a fixed number of factors.
- The factor of a number is always less than or equal to the given number.
- Every number except 0 and 1 has at least two factors, 1 and the number itself.
- Division and multiplication are the operations that are used in finding the factors of a number.

Application

The product of the ages, in years, of three teenagers is 4590. None of the teens are the same age. What are the ages of the teenagers? [1, page23]

Understand the Problem

We need to determine three distinct whole numbers, from the list 13, 14, 15, 16, 17, 18, and 19, that have a product of 4590.

Devise a Plan

If we represent the ages by x, y, and z, then

$$xyz = 4590 .$$

We are unable to solve this equation, but we notice that 4590 ends in a zero.

Hence, 4590 has a factor of 2 and a factor of 5, which means that at least one of the numbers we seek must be an even number and at least one number must have 5 as a factor. The only number in our list that has 5 as a factor is 15. Thus 15 is one of the numbers and at least one of the other numbers must be an even number. At this point we try to solve by guessing and checking.

Carry Out the Plan

- $15 \cdot 16 \cdot 18 = 4320$ No. This product is too small.
- $15 \cdot 16 \cdot 19 = 4560$ No. This product is too small.
- $15 \cdot 17 \cdot 18 = 4590$ Yes. This is the correct product.

The ages of the teenagers are 15, 17, and 18.

Review the Solution

Because $15 \cdot 17 \cdot 18 = 4590$ and each of the ages represents the age of a teenager, we know our solution is correct. None of the numbers 13, 14, 16, and 19 is a factor (divisor) of 4590, so there are no other solutions.

Remark: The above technique is called Polya's Strategy.

Question: Bank pays his client to an amount of PKR46435/-. Using the largest currency notes, break the amount into the number of currency notes paid by bank. Verify your answer.

Prime Factorization

Consider a composite integer, e.g., 60. We can express 60 as a product of prime factors, i.e., factors which are prime number.

$$60 = 2 \times 2 \times 3 \times 5$$
$$= 2^2 \times 3 \times 5.$$

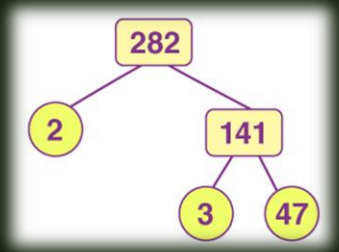
The integers 2, 3 and 5 are prime factor of the 60.

The process of expressing the integer as a product of its prime factors is called the prime factorization.

Examples:

$147 = 3 \times 7^2$, $90 = 2 \times 3^2 \times 5$, $282 = 2 \times 3 \times 47$.

Finding prime factorization is not an easy task, especially for large numbers. What about prime factorization of 527?



Why find Prime Factors?

- A prime number can only be divided by 1 or itself, so it cannot be factored any further!
- Every other whole number can be broken down into prime number factors.
- It is like the Prime Numbers are the basic building blocks of all numbers.
- This idea can be very useful when working with big numbers, such as in Cryptography. (For more see at https://youtu.be/JD72Ry60eP4?si=Y1IJ9QIfGXUuVT6P)

Another Example

Suppose there are items weighing 100 Kg that need to be shipped somewhere. You can factorize 100 in the following ways-

$100 = 2 \times 50$ or, $100 = 4 \times 25$

So, either you can take two packing boxes capable of carrying 50 Kg each, or you can choose four packing boxes of 25 Kg capability each. Either way, factorization comes in handy.

There are other ways also as $100 = 2^2 \times 5^2$.

For more examples, see: https://numberdyslexia.com/prime-factorization-real-world-examples/

Highest Common Factor (HCF)

The common factors of 12 and 30:

Factors of 12 are 1, 2, 3, 4, 6 and 12 Factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30

Then the common factors are those that are found in both lists:

Notice that 1, 2, 3 and 6 appear in both lists?

So, the common factors of 12 and 30 are: 1, 2, 3 and 6

The common factors of 15, 30 and 105

Factors of 15 are 1, 3, 5, and 15 Factors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30

Factors of 105 are 1, 3, 5, 7, 15, 21, 35 and 105

The factors that are common to all three numbers are 1, 3, 5 and 15

What about Highest Common Factor (HCF)?

It is simply the highest of largest of the common factors.

Other Names

The "Highest Common Factor" is often abbreviated to "HCF", and is also known as:
the "Greatest Common Divisor (GCD)", or
the "Largest Common Factor (LCF)".

Lowest Common Multiple (LCM)

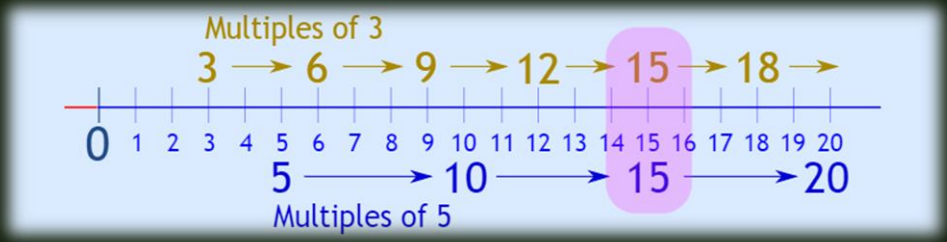
We start with an example:

Least Common Multiple of 3 and 5:

List the Multiples of each number,

The multiples of 3 are 3, 6, 9, 12, 15, 18, ... etc

The multiples of 5 are 5, 10, 15, 20, 25, ... etc



Find the first Common (same) value:

LCM of 3 and 5 is 15 (15 is a multiple of both 3 and 5, and is the smallest number like that.)

What is the "Least Common Multiple" ?

It is simply the smallest of the common multiples.

Visualizing Fraction

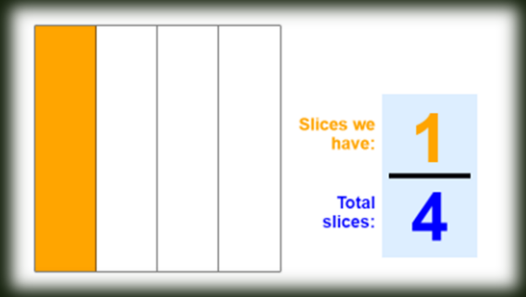
Slice a pizza, and we get fractions:

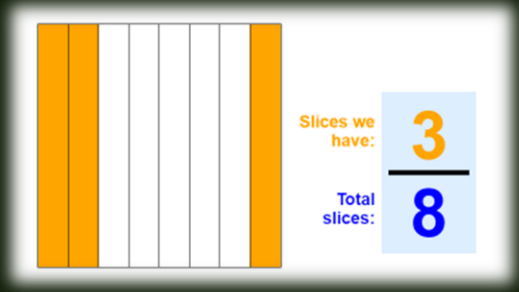
The top number says how many slices we have.

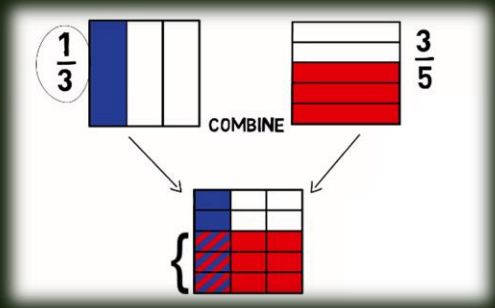
The bottom number says how many equal slices the whole pizza was cut into.

Please note that fractions are the part of set of rational numbers.

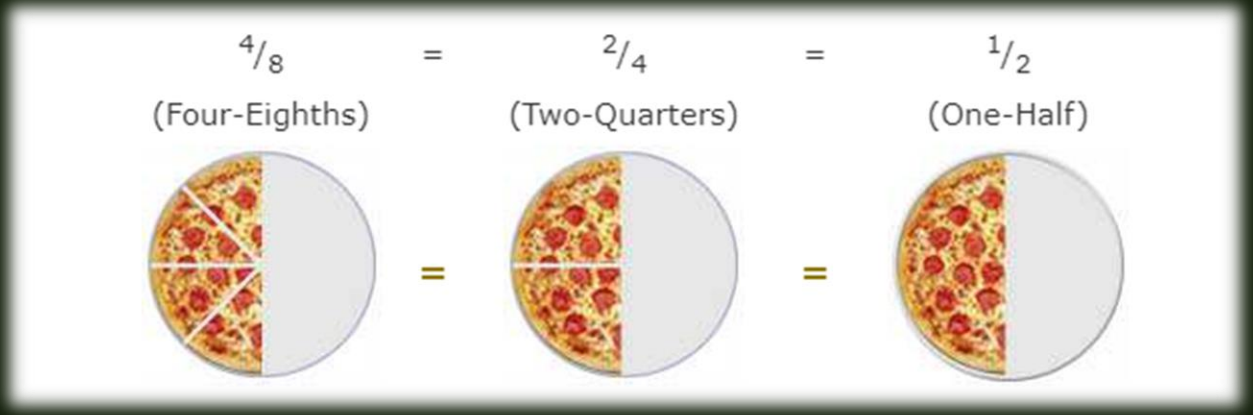
We may use square to visualize the fraction:







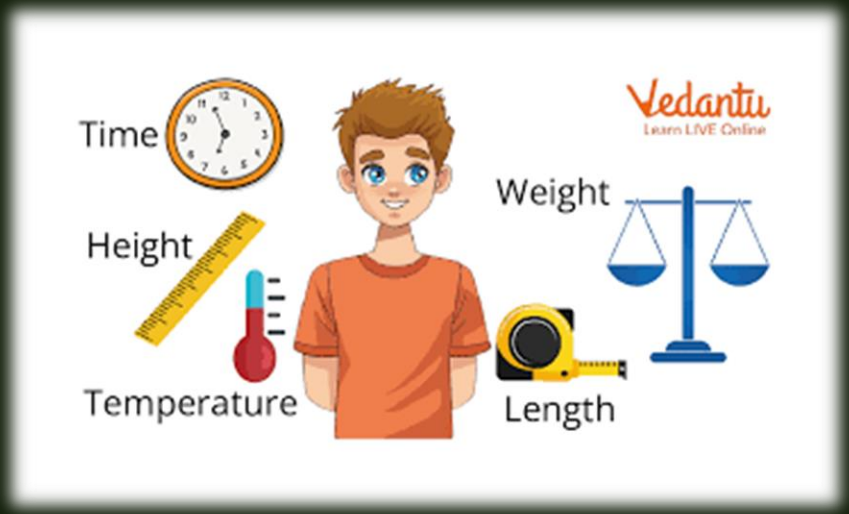
Some fractions may look different, but are really the same, for example:



But there might be hidden meaning behind these fractions.

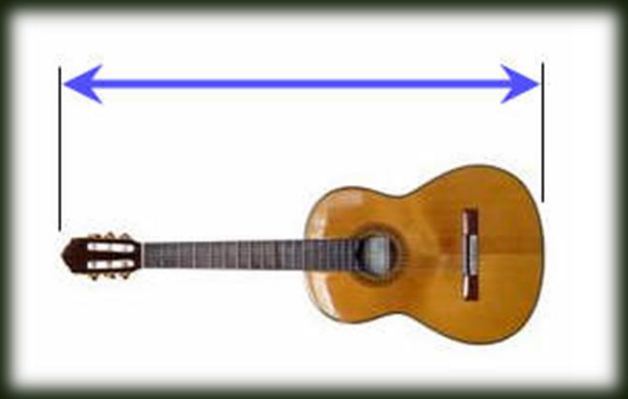
System of measurement:

Measurement is finding a number that shows the size or amount of something.



Length

Length is how far from end to end. Also called Distance.



The length of this guitar is about 1 meter (slightly more than 1 yard).

Area

Area is the size of a surface: how much is inside the boundary of a flat (2-dimensional) object such as a triangle or circle.

Here Ariel the Dog is waiting patiently inside 1 square meter.

Volume

Volume is the amount of 3-dimensional space an object occupies. Also called Capacity.

This jug has exactly 1 liter of water in it.

Mass (Weight)

Mass is how much matter something contains.

This bar of Gold has a Mass of 1 kilogram (slightly more than 2 pounds)

(Weight is "Heaviness" ... the downward force caused by gravity on an object.)

Temperature

Temperature is measured using a thermometer, usually in the Celsius or Fahrenheit scale.

This block of ice would measure 0° Celsius, or 32° Fahrenheit.

Time

Time is the ongoing sequence of events taking place.

We measure time using seconds, minutes, hours, days, weeks, months and years.

There are lots of other things we can measure, but those are the most common.

History of Measurement

Years ago, people came up with standard length measures, but they didn't all agree on one system.

- The Imperial System (which uses yards, feet, inches, etc to measure length) was developed over hundreds of years in the UK.
- Then the French came up with the Metric System or System International (SI) (meters, kilograms, liters, etc.) (origins in 1670, but developed in the 1790s), which soon spread through Europe, and then most of the world, even to England itself in 1965.
- The USA developed their own version of the Imperial system (US Standard Units) (feet, pounds, pints, etc.), but the Metric System is also used in the USA, particularly in science.

Metric System or System International (SI)

There are seven SI base units.

From these base units, all other SI units of measurement can be derived.

Derived units are used for measurements such as volume, density, and pressure.

Quantity	SI base unit	Symbol
Length	meter	m
Mass	kilogram	kg
Temperature	kelvin	K
Time	second	s
Amount of substance	mole	mol
Luminous intensity	candela	cd
Electric current	ampere	A

Unit analysis as a problem-solving tool

Unit analysis means using the rules of multiplying and reducing fractions to solve problems involving different units.

Example:

A desk is $4\frac{1}{2}$ feet long. Find the length of the desk in inches.

Solution:

There are 12 inches in 1 foot.

$$4\frac{1}{2}ft \times \frac{12in}{1ft} = \frac{9ft}{2} \times \frac{12in}{1ft}$$

$$= 54\ inches.$$

Example:

If Faheem runs one lap around the track at his campus, he has gone a total distance of 400 yards. Yesterday, he ran once around the track in 3 minutes. What was Faheem's running speed in kilometer per hour?

Solution:

$$\frac{400\,yards}{3\,minutes} = \frac{400\,yards}{3\,minutes} \times \frac{60\,minutes}{1\,hour} \times \frac{1\,km}{1093.61\,yards}$$

$$= \frac{24000\,km}{3280.83\,hours}$$

$$= 7.315\,km/hour\,.$$

1 hours = 60 minutes
1 minutes = 60 seconds
1 km = 1093.61 yards
1 km = 1000 meters

Questions:

1. Find the lowest number which is exactly divisible by 18 and 24.
2. A florist wants to arrange 24 boquets of flowers in different rows. Find out in how many ways he can arrange the bouquets with same number in each row.
3. A shopkeeper sells candles in packets of 12 and candle stands in packet of 8. What is the least number of candles and candle stands Areesha should buy so that there will be one candle for each candle stand.
4. Find the least number, which, when divided by 35, 56 and 91, leaves the same remainder of 7, respectively.
5. Two alarm clocks ring their alarms at regular intervals of 72 seconds and 50 seconds respectively. If they beep together at noon, at what time will they beep again together?