

Chapter 2 – Sequences

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Sequences form an important component of Mathematical Analysis and arise in many situations. The first rigorous treatment of sequences was made by A. Cauchy (1789-1857) and George Cantor (1845-1918). A sequence (of real numbers, of sets, of functions, of anything) is simply a list. There is a first element in the list, a second element, a third element, and so on continuing in an order forever. In mathematics a finite list is not called a sequence (some authors considered it finite sequence); a sequence must continue without interruption. Formally it is defined as follows:

Sequence

A function whose domain is the set of natural numbers and range is a subset of real numbers is called *real sequence*.

Since in this chapter, we shall be concerned with *real sequences* only, we shall refer to them as just *sequences*.

Notation:

A sequence is usually denoted as

$$\{s_n\}_{n=1}^{\infty} \text{ or } \{s_n : n \in \mathbb{N}\} \text{ or } \{s_1, s_2, s_3, \dots\} \text{ or simply as } \{s_n\} \text{ or by } (s_n).$$

But it is not limited to above notations only.

The values s_n are called the *terms* or the *elements* of the sequence $\{s_n\}$.

e.g. i) $\{n\} = \{1, 2, 3, \dots\}$.

$$\text{ii) } \left\{ \frac{1}{n} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\}.$$

$$\text{iii) } \{(-1)^{n+1}\} = \{1, -1, 1, -1, \dots\}.$$

iv) $\{2, 3, 5, 7, 11, \dots\}$, a sequence of positive prime numbers.

v) $\{s_n\}$ such that $s_1 = 1$, $s_2 = 1$ and $s_{n+2} = s_{n+1} + s_n$.

Range of a sequence

The set of all distinct terms of a sequence is called its range.

Remark:

In a sequence $\{s_n\}$, since $n \in \mathbb{N}$ and \mathbb{N} is an infinite set, the number of the terms of a sequence is always infinite. However, the range of the sequence may be finite.

Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{s_n\}$ is usually written as $\{s_{n_k}\}$.

Examples:

1. $\{2, 4, 6, \dots\}$ is subsequence of $\{1, 2, 3, \dots\}$
2. $\left\{\frac{1}{2n}\right\}$ and $\left\{\frac{1}{n+1}\right\}$ is subsequence of $\left\{\frac{1}{n}\right\}$.

Increasing sequence

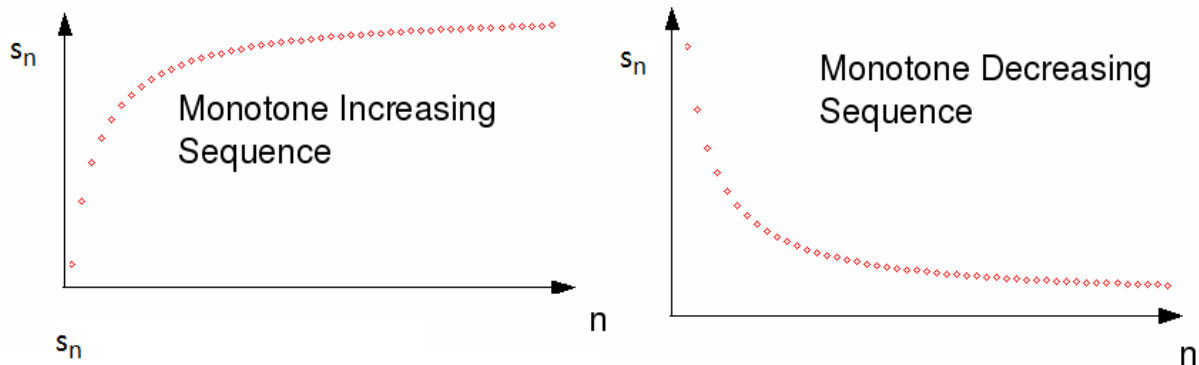
A sequence $\{s_n\}$ is said to be an increasing sequence if $s_{n+1} \geq s_n \quad \forall n \geq 1$.

Decreasing sequence

A sequence $\{s_n\}$ is said to be a decreasing sequence if $s_{n+1} \leq s_n \quad \forall n \geq 1$.

Monotonic sequence

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.

**Remarks:**

- A sequence $\{s_n\}$ is monotonically increasing if $s_{n+1} - s_n \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically increasing if $\frac{s_{n+1}}{s_n} \geq 1, \quad \forall n \geq 1$.
- A sequence $\{s_n\}$ is monotonically decreasing if $s_n - s_{n+1} \geq 0$.
- A positive term sequence $\{s_n\}$ is monotonically decreasing if $\frac{s_n}{s_{n+1}} \geq 1, \quad \forall n \geq 1$.

Strictly Increasing or Decreasing

A sequence $\{s_n\}$ is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \quad \text{or} \quad s_{n+1} < s_n \quad \forall n \geq 1.$$

Examples:

- $\{n\} = \{1, 2, 3, \dots\}$ is an increasing sequence (also it is strictly increasing).
- $\left\{\frac{1}{n}\right\}$ is a decreasing sequence. (also it is strictly decreasing).
- $\{1, 1, 2, 2, 3, 3, \dots\}$ is increasing sequence but it is not strictly increasing.
- $\{\cos n\pi\} = \{-1, 1, -1, 1, \dots\}$ is neither increasing nor decreasing.

Questions:

- 1) Prove that $\left\{1 + \frac{1}{n}\right\}$ is a decreasing sequence.
- 2) Is $\left\{\frac{n+1}{n+2}\right\}$ is increasing or decreasing sequence?

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there is a positive number λ such that

$$|s_n| \leq \lambda \quad \forall n \in \mathbb{N}.$$

For such a sequence, every term belongs to the interval $[-\lambda, \lambda]$. Also inequality in the above definition can be replaced with strict inequality. Alternatively, a sequence is bounded if its range is a bounded set.

It can be noted that if the sequence is bounded then its supremum and infimum exist.

If S and s are the supremum and infimum of the bounded sequence $\{s_n\}$, then we write

$$S = \sup s_n \quad \text{and} \quad s = \inf s_n.$$

Remarks:

It is easy to conclude that if $\{s_n\}$ is bounded sequence and n_0 is positive integer then there exists $\lambda > 0$ such that

$$|s_n| \leq \lambda \quad \text{whenever } n \geq n_0.$$

Examples:

- (i) $\{u_n\} = \left\{\frac{(-1)^n}{n}\right\}$ is a bounded sequence
- (ii) $\{v_n\} = \{\sin n\}$ is also bounded sequence. Its supremum is 1 and infimum is -1 .
- (iii) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by a .
- (iv) $\{\exp(n)\}$ is an unbounded sequence.

Convergence of the sequence

The sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots$$

is getting closer and closer to the number 0. We say that this sequence converges to 0 or that the limit of the sequence is the number 0. How should this idea be properly defined?

The study of convergent sequences was undertaken and developed in the eighteenth century without any precise definition. The closest one might find to a definition in the early literature would have been something like

A sequence $\{s_n\}$ converges to a number L if the terms of the sequence get closer and closer to L .

However, this is too vague and too weak to serve as definition but a rough guide for the intuition, this is misleading in other respects. What about the sequence

0.1, 0.01, 0.02, 0.001, 0.002, 0.0001, 0.0002, 0.00001, 0.00002, ...?

Surely this should converge to 0 but the terms do not get steadily “closer and closer” but back off a bit at each second step.

The definition that captured the idea in the best way was given by Augustin Cauchy in the 1820s. He found a formulation that expressed the idea of “arbitrarily close” using inequalities.

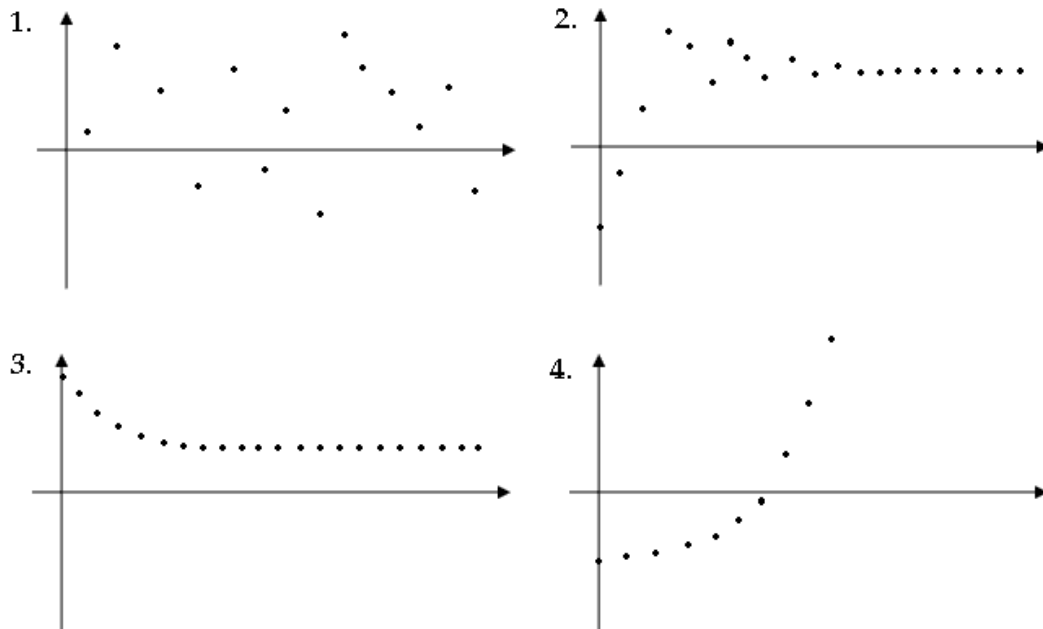
Definition

A sequence $\{s_n\}$ of real numbers is said to be convergent to limit ‘ s ’ as $n \rightarrow \infty$, if for every real number $\varepsilon > 0$, there exists a positive integer n_0 , depending on ε , so that

$$|s_n - s| < \varepsilon \quad \text{whenever } n > n_0.$$

A sequence that converges is said to be *convergent*. A sequence that fails to converge is said to be *divergent* (it will be discussed later).

We will try to understand it by graph of some sequence. Graphs of any four sequences is drawn in the picture below.



Examples

a) Prove that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ (or $\left\{\frac{1}{n}\right\}$ converges to 0).

Solution: Let $\varepsilon > 0$ be given. By the Archimedean Property, there is a positive integer $n_0 = n_0(\varepsilon)$ such that $n_0 \cdot \varepsilon > 1$, that is, $\frac{1}{n_0} < \varepsilon$. Then, if $n > n_0$, we have

$$\frac{1}{n} < \frac{1}{n_0} < \varepsilon.$$

Thus we proved that for all $\varepsilon > 0$, there exists n_0 , depending upon ε , such that

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} < \varepsilon \quad \text{whenever } n > n_0.$$

Hence $\left\{ \frac{1}{n} \right\}$ converges to point '0'.

b) Prove that $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1} = 0$ (by definition).

Solution: Let $\varepsilon > 0$ be given. Now consider

$$\left| \frac{1}{n^2 + 1} - 0 \right| = \frac{1}{n^2 + 1} < \frac{1}{n^2} < \frac{1}{n}. \quad (\text{Since } n^2 + 1 > n^2 > 0)$$

Now if we choose n_0 such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives

us

$$\left| \frac{1}{n^2 + 1} - 0 \right| < \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon \quad \text{whenever } n \geq n_0 > \frac{1}{\varepsilon}.$$

Hence, we conclude that, $\lim_{n \rightarrow \infty} \frac{1}{n^2 + 1} = 0$.

c) Prove that $\lim_{n \rightarrow \infty} \frac{3n + 2}{n + 1} = 3$ (by definition).

Solution: Let $\varepsilon > 0$ be given. Now consider

$$\begin{aligned} \left| \frac{3n + 2}{n + 1} - 3 \right| &= \left| \frac{3n + 2 - 3n - 3}{n + 1} \right| \\ &= \left| \frac{-1}{n + 1} \right| = \frac{1}{n + 1} < \frac{1}{n} \quad (\because n + 1 > n > 0) \end{aligned}$$

Now if we take n_0 such that $\frac{1}{n_0} < \varepsilon$ (or $n_0 > \frac{1}{\varepsilon}$), then the above expression gives us

$$\left| \frac{3n + 2}{n + 1} - 3 \right| < \varepsilon \quad \text{whenever } n \geq n_0.$$

Hence, we conclude that $\lim_{n \rightarrow \infty} \frac{3n + 2}{n + 1} = 3$.

Questions:

Use definition of the limits to prove the followings:

$$\text{a) } \lim_{n \rightarrow \infty} \frac{2n}{n + 1} = 2. \quad \text{b) } \lim_{n \rightarrow \infty} \frac{n^2 - 1}{2n^2 + 3} = \frac{1}{2} \quad \text{c) } \lim_{n \rightarrow \infty} \frac{1}{3^n} = 0$$

Definitions

- i. A bounded sequence which does not converge is said to *oscillate finitely*.
- ii. A sequence $\{s_n\}$ is said to be *divergent to ∞* , if to each given positive number Δ , there correspond an integer m such that

$$s_n > \Delta \text{ for all } n \geq m.$$
- iii. A sequence $\{s_n\}$ is said to be *divergent to $-\infty$* , if to each given positive number Δ , there correspond an integer m such that

$$s_n < -\Delta \text{ for all } n \geq m.$$
- iv. A sequence $\{s_n\}$ is said to *oscillate infinitely*, if it is unbounded and is divergent neither to ∞ nor to $-\infty$.

Examples

- a. $\{1 + (-1)^n\}$ oscillates finitely.
- b. $\{(-1)^n n\}$ oscillates infinitely.
- c. $\{2^n\}$ diverges to ∞ .
- d. $\{-2n\}$ diverges to $-\infty$.

Question

Prove that $\{-e^n\}$ diverges to $-\infty$ (by definition)

Solution.

Suppose $\Delta > 0$ be given and $s_n = -e^n$.

Take $s_n < -\Delta$, i.e. $-e^n < -\Delta \Rightarrow e^n > \Delta \Rightarrow n > \log \Delta$.

Now if m is positive integer such that $m > \log \Delta$, then

$s_n < -\Delta$ for all $n > m$.

This implies $\{-e^n\}$ is diverges to $-\infty$.

Question

Prove that $\{5^n\}$ diverges to ∞ (by definition).

Prove that $\{n^2\}$ diverges to ∞ (by definition).

Review

- Triangular inequality: If $a, b \in \mathbb{R}$, then $||a| - |b|| \leq |a \pm b| \leq |a| + |b|$.
- If $0 \leq a < \varepsilon$ for all $\varepsilon > 0$, then $a = 0$.

Theorem

A convergent sequence of real number has one and only one limit (i.e. limit of the sequence is unique.)

Proof:

Suppose $\{s_n\}$ converges to two limits s and t , where $s \neq t$.

Then for all $\varepsilon > 0$, there exists two positive integers n_1 and n_2 such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall n > n_1 \dots\dots\dots (1)$$

$$\text{and } |s_n - t| < \frac{\varepsilon}{2} \quad \forall n > n_2 \dots\dots\dots (2)$$

As (1) and (2) hold simultaneously for all $n > \max\{n_1, n_2\}$.

Thus, for all $n > \max\{n_1, n_2\}$ we have

$$\begin{aligned} 0 \leq |s - t| &= |s - s_n + s_n - t| \\ &\leq |s_n - s| + |s_n - t| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As ε is arbitrary, we get $|s - t| = 0$, this gives $s = t$, that is, the limit of the sequence is unique. \square

Theorem

If the sequence $\{s_n\}$ converges to s , where $s \neq 0$, then there exists a positive integer n_1 such that $|s_n| > \frac{1}{2}|s|$ for all $n > n_1$.

Proof:

Since $\{s_n\}$ converges to s , therefore for all real $\varepsilon > 0$, there exists positive integer n_1 such that

$$|s_n - s| < \varepsilon \quad \text{for } n > n_1.$$

We fix $\varepsilon = \frac{1}{2}|s| > 0$ to get

$$|s_n - s| < \frac{1}{2}|s| \quad \text{for } n > n_1,$$

that is,

$$-\frac{1}{2}|s| > -|s_n - s| \quad \text{for } n > n_1 \dots\dots\dots (1)$$

Now

$$\begin{aligned} \frac{1}{2}|s| &= |s| - \frac{1}{2}|s| \\ &< |s| - |s_n - s| \quad \text{for } n > n_1 \quad (\text{by using (1)}) \\ &\leq |s + (s_n - s)| \quad \text{for } n > n_1 \end{aligned}$$

This ultimately gives us

$$|s_n| > \frac{1}{2}|s| \quad \text{for all } n > n_1. \quad \square$$

Theorem

Let a and b be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, then

- (i) $\{as_n + bt_n\}$ converges to $as + bt$.
(ii) $\{s_n t_n\}$ converges to st .
(iii) $\left\{\frac{s_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n \neq 0$ for all n and $t \neq 0$.

Proof:

Since $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, therefore

$$|s_n - s| < \varepsilon \quad \forall n > n_1 \in \mathbb{N}$$

$$|t_n - t| < \varepsilon \quad \forall n > n_2 \in \mathbb{N}$$

Also $\exists \lambda > 0$ such that $|s_n| < \lambda \quad \forall n > 1 \quad (\because \{s_n\} \text{ is bounded})$

(i) We have

$$\begin{aligned} |(as_n + bt_n) - (as + bt)| &= |a(s_n - s) + b(t_n - t)| \\ &\leq |a(s_n - s)| + |b(t_n - t)| \\ &< |a|\varepsilon + |b|\varepsilon \quad \forall n > \max\{n_1, n_2\} \\ &= \varepsilon_1, \end{aligned}$$

where $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$ a certain number.

This implies $\{as_n + bt_n\}$ converges to $as + bt$.

$$\begin{aligned} (ii) \quad |s_n t_n - st| &= |s_n t_n - s_n t + s_n t - st| \\ &= |s_n(t_n - t) + t(s_n - s)| \\ &\leq |s_n| \cdot |(t_n - t)| + |t| \cdot |(s_n - s)| \\ &< \lambda \varepsilon + |t|\varepsilon \quad \forall n > \max\{n_1, n_2\} \\ &= \varepsilon_2, \quad \text{where } \varepsilon_2 = \lambda \varepsilon + |t|\varepsilon \text{ a certain number.} \end{aligned}$$

This implies $\{s_n t_n\}$ converges to st .

$$\begin{aligned} (iii) \quad \left| \frac{1}{t_n} - \frac{1}{t} \right| &= \left| \frac{t - t_n}{t_n t} \right| \\ &= \frac{|t_n - t|}{|t_n||t|} < \frac{\varepsilon}{\frac{1}{2}|t||t|} \quad \forall n > \max\{n_1, n_2\} \\ &\because |t_n| > \frac{1}{2}|t| \\ &= \frac{\varepsilon}{\frac{1}{2}|t|^2} = \varepsilon_3, \quad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}|t|^2} \text{ a certain number.} \end{aligned}$$

This implies $\left\{\frac{1}{t_n}\right\}$ converges to $\frac{1}{t}$.

Hence $\left\{ \frac{s_n}{t_n} \right\} = \left\{ s_n \cdot \frac{1}{t_n} \right\}$ converges to $s \cdot \frac{1}{t} = \frac{s}{t}$. (from (ii)) \square

Question

Prove that if $\lim_{n \rightarrow \infty} s_n = t$, then $\lim_{n \rightarrow \infty} |s_n| = |t|$ but converse is not true in general.

Question

Prove that every convergent sequence is bounded.

Solution:

Consider a sequence $\{s_n\}$ converges to limit l , that is, for all $\varepsilon > 0$, there exists positive integer n_0 such that

$$|s_n - l| < \varepsilon \quad \text{for all } n > n_0.$$

For $\varepsilon = 1$, we have

$$|s_n - l| < 1 \quad \text{for all } n > n_0 \dots\dots\dots (i)$$

$$\text{Now} \quad |s_n| < |s_n - l + l| \leq |s_n - l| + |l|$$

Using (i), in above expression, we get

$$|s_n| < 1 + |l| \quad \text{for all } n > n_0.$$

Now take $\lambda = \max\{|s_1|, |s_2|, \dots, |s_{n_0}|, 1 + |l|\}$, then we have

$$|s_n| \leq \lambda \quad \text{for all } n \in \mathbb{N}.$$

This implies $\{s_n\}$ is bounded. \square

Review:

- For all $a, b, c \in \mathbb{R}$, $|a - b| < c \Leftrightarrow b - c < a < b + c$ or $a - c < b < a + c$.

Theorem (Sandwich Theorem or Squeeze Theorem)

Suppose that $\{s_n\}$ and $\{t_n\}$ be two convergent sequences such that $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} t_n = s$.

If $s_n < u_n < t_n$ for all $n \geq n_0$, then the sequence $\{u_n\}$ also converges to s .

Proof:

Since the sequence $\{s_n\}$ and $\{t_n\}$ converge to the same limit s (say), therefore for given $\varepsilon > 0$ there exists two positive integers n_1 and n_2 such that

$$|s_n - s| < \varepsilon \quad \forall n > n_1,$$

$$|t_n - s| < \varepsilon \quad \forall n > n_2.$$

$$\text{i.e.} \quad s - \varepsilon < s_n < s + \varepsilon \quad \forall n > n_1,$$

$$s - \varepsilon < t_n < s + \varepsilon \quad \forall n > n_2.$$

Also, we have given

$$s_n < u_n < t_n \quad \forall n > n_0.$$

Consider $n_3 = \max\{n_0, n_1, n_2\}$, then we have

$$s - \varepsilon < s_n < u_n < t_n < s + \varepsilon \quad \forall \quad n > n_3$$

$$\Rightarrow s - \varepsilon < u_n < s + \varepsilon \quad \forall \quad n > n_3$$

$$\text{i.e. } |u_n - s| < \varepsilon \quad \forall \quad n > n_3$$

$$\text{i.e. } \lim_{n \rightarrow \infty} u_n = s.$$

□

Example

$$\text{Show that } \lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2} \right) = 0.$$

Solution.

Consider

$$s_n = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \dots + \frac{1}{(2n)^2}$$

$$\text{As } \underbrace{\frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \dots + \frac{1}{(2n)^2}}_{n \text{ times}} \leq s_n < \underbrace{\frac{1}{n^2} + \frac{1}{n^2} + \dots + \frac{1}{n^2}}_{n \text{ times}},$$

that is,

$$n \cdot \frac{1}{(2n)^2} \leq s_n < n \cdot \frac{1}{n^2} \quad \Rightarrow \quad \frac{1}{4n} \leq s_n < \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{4n} \leq \lim_{n \rightarrow \infty} s_n < \lim_{n \rightarrow \infty} \frac{1}{n} \quad \Rightarrow \quad 0 \leq \lim_{n \rightarrow \infty} s_n < 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} s_n = 0.$$

□

Theorem

For each irrational number x , there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim_{n \rightarrow \infty} r_n = x$.

Proof:

Since x and $x + 1$ are two different real numbers, so there exist a rational number r_1 such that

$$x < r_1 < x + 1$$

Similarly there exists a rational number $r_2 \neq r_1$ such that

$$x < r_2 < \min \left\{ r_1, x + \frac{1}{2} \right\} < x + 1$$

Continuing in this manner we have

$$x < r_3 < \min \left\{ r_2, x + \frac{1}{3} \right\} < x + 1$$

$$x < r_4 < \min \left\{ r_3, x + \frac{1}{4} \right\} < x + 1$$

.....

$$\dots\dots\dots$$

$$x < r_n < \min \left\{ r_{n-1}, x + \frac{1}{n} \right\} < x + 1$$

This implies that there is a sequence $\{r_n\}$ of the distinct rational number such that

$$x < r_n < x + \frac{1}{n}.$$

Since $\lim_{n \rightarrow \infty} (x) = \lim_{n \rightarrow \infty} \left(x + \frac{1}{n} \right) = x.$

Therefore

$$\lim_{n \rightarrow \infty} r_n = x.$$

□

Theorem

Let a sequence $\{s_n\}$ be a bounded sequence.

- (i) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.
- (ii) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

Proof

(i) Let $S = \sup s_n$ and take $\varepsilon > 0$.

Since there exists s_{n_0} such that $S - \varepsilon < s_{n_0}$

Since $\{s_n\}$ is monotonically increasing,
therefore

$$\begin{aligned} S - \varepsilon < s_{n_0} < s_n < S < S + \varepsilon & \text{for } n > n_0 \\ \Rightarrow S - \varepsilon < s_n < S + \varepsilon & \text{for } n > n_0 \\ \Rightarrow |s_n - S| < \varepsilon & \text{for } n > n_0 \\ \Rightarrow \lim_{n \rightarrow \infty} s_n = S \end{aligned}$$

(ii) Let $s = \inf s_n$ and take $\varepsilon > 0$.

Since there exists s_{n_1} such that $s_{n_1} < s + \varepsilon$

Since $\{s_n\}$ is monotonically decreasing,
therefore

$$\begin{aligned} s - \varepsilon < s < s_n < s_{n_1} < s + \varepsilon & \text{for } n > n_1 \\ \Rightarrow s - \varepsilon < s_n < s + \varepsilon & \text{for } n > n_1 \\ \Rightarrow |s_n - s| < \varepsilon & \text{for } n > n_1 \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} s_n = s$

□

Questions:

1. Let $\{s_n\}$ be a sequence and $\lim_{n \rightarrow \infty} s_n = s$. Then prove that $\lim_{n \rightarrow \infty} s_{n+1} = s$.
2. Prove that a bounded increasing sequence converges to its supremum.

3. Prove that a bounded decreasing sequence converges to its infimum.
4. Prove that if a sequence $\{s_n\}$ converges to l , then every subsequence of $\{s_n\}$ converges to l .
5. If the subsequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$ of sequence $\{s_n\}$ converges to the same limit l then $\{s_n\}$ converges to l .

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Example:

Let $t_1 > 1$ and let $\{t_n\}$ be defined by $t_{n+1} = 2 - \frac{1}{t_n}$ for $n \geq 1$.

- (i) Show that $\{t_n\}$ is decreasing sequence.
- (ii) It is bounded below.
- (iii) Find the limit of the sequence.

Since $t_1 > 1$ and $\{t_n\}$ is defined by $t_{n+1} = 2 - \frac{1}{t_n}$; $n \geq 1$

$$\Rightarrow t_n > 0 \quad \forall n \geq 1$$

$$\begin{aligned} \text{Also } t_n - t_{n+1} &= t_n - 2 + \frac{1}{t_n} \\ &= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n - 1)^2}{t_n} > 0. \end{aligned}$$

$$\Rightarrow t_n > t_{n+1} \quad \forall n \geq 1.$$

This implies that t_n is monotonically decreasing.

Since $t_n > 1 \quad \forall n \geq 1$,

$\Rightarrow t_n$ is bounded below.

Since t_n is decreasing and bounded below therefore t_n is convergent.

Let us suppose $\lim_{n \rightarrow \infty} t_n = t$.

$$\text{Then } \lim_{n \rightarrow \infty} t_{n+1} = \lim_{n \rightarrow \infty} t_n \quad \Rightarrow \quad \lim_{n \rightarrow \infty} \left(2 - \frac{1}{t_n} \right) = \lim_{n \rightarrow \infty} t_n$$

$$\Rightarrow 2 - \frac{1}{t} = t \quad \Rightarrow \quad \frac{2t-1}{t} = t \quad \Rightarrow \quad 2t-1 = t^2 \quad \Rightarrow \quad t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow \quad t = 1.$$



Question:

- Let $\{t_n\}$ be a positive term sequence. Find the limit of the sequence if $4t_{n+1} = \frac{2}{5} - 3t_n$ for all $n \geq 1$.
- Let $\{u_n\}$ be a sequence of positive numbers. Then find the limit of the sequence if $u_{n+1} = \frac{1}{u_n} + \frac{1}{4}u_{n-1}$ for $n \geq 1$.
- The Fibonacci numbers are: $F_1 = F_2 = 1$, and for every $n \geq 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$. Find the $\lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}}$ (this limit is known as golden number)

Cauchy Sequence

A sequence $\{s_n\}$ of real number is said to be a *Cauchy sequence* if for given number $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that

$$|s_n - s_m| < \varepsilon \quad \forall \quad m, n > n_0$$

Example

The sequence $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

Suppose $s_n = \frac{1}{n}$ and $\varepsilon > 0$ be given. We choose a positive integer $n_0 = n_0(\varepsilon)$ such that $n_0 > \frac{2}{\varepsilon}$.

Then if $m, n > n_0$, we have $\frac{1}{n} < \frac{1}{n_0} < \frac{\varepsilon}{2}$ and similarly $\frac{1}{m} < \frac{1}{n_0} < \frac{\varepsilon}{2}$. Therefore, it follows that if $m, n > n_0$, then

$$|s_n - s_m| = \left| \frac{1}{n} - \frac{1}{m} \right| \leq \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\left\{\frac{1}{n}\right\}$ is Cauchy sequence.

Theorem

A Cauchy sequence of real numbers is bounded.

Proof:

Let $\{s_n\}$ be a Cauchy sequence. Then for given number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$|s_n - s_m| < \varepsilon \quad \forall \quad m, n > n_0.$$

Take $\varepsilon = 1$, then we have

$$|s_n - s_m| < 1 \quad \forall m, n > n_0.$$

Fix $m = n_0 + 1$ then

$$\begin{aligned} |s_n| &= |s_n - s_{n_0+1} + s_{n_0+1}| \\ &\leq |s_n - s_{n_0+1}| + |s_{n_0+1}| \\ &< 1 + |s_{n_0+1}| \quad \forall n > n_0. \end{aligned}$$

Now take $\lambda = \max\{|s_1|, |s_2|, \dots, |s_{n_0}|, 1 + |s_{n_0+1}|\}$, then we have

$$|s_n| \leq \lambda \quad \text{for all } n \in \mathbb{N}.$$

Hence we conclude that $\{s_n\}$ is a Cauchy sequence, which is bounded one. \square

Remarks:

The converse of the above theorem does not hold, that is, every bounded sequence is not Cauchy.

Consider the sequence $\{s_n\}$, where $s_n = (-1)^n$, $n \geq 1$. It is bounded sequence because

$$|(-1)^n| = 1 < 2 \quad \forall n \geq 1.$$

But it is not a Cauchy sequence if it is then for $\varepsilon = 1$ we should be able to find a positive integer n_0 such that $|s_n - s_m| < 1$ for all $m, n > n_0$.

But with $m = 2k + 1$, $n = 2k + 2$ when $2k + 1 > n_0$, we arrive at

$$\begin{aligned} |s_n - s_m| &= |(-1)^{2n+2} - (-1)^{2k+1}| \\ &= |1 + 1| = 2 < 1 \quad \text{is absurd.} \end{aligned}$$

Hence $\{s_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence.

Questions:

- Prove that every Cauchy sequence of real number is bounded but converse is not true.
- Prove that every convergent sequence is bounded but converse is not true.

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Proof:

Suppose $\{s_n\}$ is a Cauchy sequence, therefore it is bounded.

First, we assume that $\{s_n : n \in \mathbb{N}\}$ has maximum value, then set

$$\begin{aligned} s_{n_1} &= \max\{s_n : n \geq 1\} \\ s_{n_2} &= \max\{s_n : n > n_1\} \\ s_{n_3} &= \max\{s_n : n > n_2\} \text{ and so on} \end{aligned}$$

Then clearly $\{s_{n_k}\}$ is subsequence of $\{s_n\}$ and it is decreasing and bounded.

Hence it is convergent.

On the other hand, if $\{s_n : n \in \mathbb{N}\}$ has no maximum value, then there exist some positive integer N such that $\{s_n : n > N\}$ has no maximum value.

Now for $m > N$, we can find some s_m such that $s_m > s_N$, otherwise one of the $s_{N+1}, s_{N+2}, \dots, s_m$ will be the maximum value of $\{s_n : n > N\}$.

So assume $s_{n_1} = s_{N+1}$.

Now s_{n_2} can be the first term after s_{n_1} such that $s_{n_2} > s_{n_1}$.

Then s_{n_3} can be the first term after s_{n_2} such that $s_{n_3} > s_{n_2}$.

Continuing in this way, we get $\{s_{n_k}\}$ be a subsequence of $\{s_n\}$ such that it is increasing and bounded. Thus it is convergent. \square

Question:

Prove that every bounded sequence has convergent subsequence.

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof:

Let $\{s_n\}$ be a convergent sequence, which converges to s .

Then for given $\varepsilon > 0$ \exists a positive integer n_0 , such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall \quad n > n_0$$

Now for $n > m > n_0$

$$\begin{aligned} |s_n - s_m| &= |s_n - s + s - s_m| \\ &\leq |s_n - s| + |s - s_m| = |s_n - s| + |s_m - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{s_n\}$ is a Cauchy sequence.

Conversely, suppose that $\{s_n\}$ is a Cauchy sequence then for $\varepsilon > 0$, there exists a positive integer m_1 such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \quad \forall \quad n, m > m_1 \dots\dots\dots (i)$$

Since $\{s_n\}$ is a Cauchy sequence,

therefore it has a subsequence $\{s_{n_k}\}$ converging to s (say).

This implies there exists a positive integer m_2 such that

$$|s_{n_k} - s| < \frac{\varepsilon}{2} \quad \forall \quad n_k > m_2 \dots\dots\dots (ii)$$

Now

$$|s_n - s| = |s_n - s_{n_k} + s_{n_k} - s|$$

$$\begin{aligned} &\leq |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n > \max\{m_1, m_2\}, \end{aligned}$$

this shows that $\{s_n\}$ is a convergent sequence. \square

Example

Prove that $\left\{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right\}$ is divergent sequence.

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

For $m, n \in \mathbb{N}$, $n > m$ we have

$$\begin{aligned} |t_n - t_m| &= \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \\ &> \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} \quad (n-m \text{ times}) \\ &= (n-m) \frac{1}{n} = 1 - \frac{m}{n}. \end{aligned}$$

In particular if $n = 2m$ then

$$|t_n - t_m| > \frac{1}{2}.$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent. \square

Theorem (Nested intervals)

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n \quad \forall n \geq 1$, and $(b_n - a_n) \rightarrow 0$ as $n \rightarrow \infty$ then $\bigcap I_n$ contains one and only one point.

Proof:

Since $I_{n+1} \subset I_n$, therefore

$$a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n < b_n < b_{n-1} < \dots < b_3 < b_2 < b_1.$$

Note that $\{a_n\}$ is increasing sequence, bounded above by b_1 and bounded below by a_1 .

Also note that $\{b_n\}$ is decreasing sequence bounded below by a_1 and bounded above by b_1 .

This implies both $\{a_n\}$ and $\{b_n\}$ are monotone and bounded sequences and hence convergent.

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b .

$$\begin{aligned} \text{But } |a - b| &= |a - a_n + a_n - b_n + b_n - b| \\ &\leq |a_n - a| + |a_n - b_n| + |b_n - b| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

$$\Rightarrow a = b$$

and $a_n < a < b_n \quad \forall n \geq 1.$

This given $\cap I_n = \{a\}$, that is, $\cap I_n$ contains only one point. □

Limit inferior of the sequence

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\liminf_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} u_n, \quad \text{where } u_n = \inf \{s_k : k \geq n\}.$$

If s_n is not bounded below then we define

$$\liminf_{n \rightarrow \infty} s_n = -\infty.$$

Limit superior of the sequence

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} v_n, \quad \text{where } v_n = \sup \{s_k : k \geq n\}$$

If s_n is not bounded above then we define

$$\limsup_{n \rightarrow \infty} s_n = +\infty.$$

Remarks:

- i. *Limit inferior* is also known as *lower limit* and *limit superior* is also known as *upper limit* of the sequence in the literature with the notations $\underline{\lim}$ and $\overline{\lim}$ respectively.
- ii. A bounded sequence has unique limit inferior and superior.
- iii. It is easy to prove that limit inferior is less than or equal to limit superior.

Examples

(i) Let $s_n = (-1)^n$, then limit superior of $\{s_n\}$ is 1 and limit inferior of $\{s_n\}$ is -1.

(ii) Let $s_n = (-1)^n \left(1 + \frac{1}{n}\right)$

then limit superior of $\{s_n\}$ is 1 and limit inferior of $\{s_n\}$ is -1.

(iii) Let $s_n = \left(1 + \frac{1}{n}\right) \cos n\pi$.

Then $u_n = \inf \{s_k : k \geq n\}$

$$\begin{aligned} &= \inf \left\{ \left(1 + \frac{1}{n}\right) \cos n\pi, \left(1 + \frac{1}{n+1}\right) \cos(n+1)\pi, \left(1 + \frac{1}{n+2}\right) \cos(n+2)\pi, \dots \right\} \\ &= \begin{cases} \left(1 + \frac{1}{n}\right) \cos n\pi & \text{if } n \text{ is odd} \\ \left(1 + \frac{1}{n+1}\right) \cos(n+1)\pi & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

$$\Rightarrow \liminf_{n \rightarrow \infty} s_n s_n = \lim_{n \rightarrow \infty} u_n = -1.$$

$$\text{Also } v_n = \sup \{s_k : k \geq n\}$$

$$= \begin{cases} \left(1 + \frac{1}{n+1}\right) \cos(n+1)\pi & \text{if } n \text{ is odd} \\ \left(1 + \frac{1}{n}\right) \cos n\pi & \text{if } n \text{ is even} \end{cases}$$

$$\Rightarrow \limsup_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} v_n = 1. \quad \square$$

Theorem

If $\{s_n\}$ is a convergent sequence, then

$$\lim_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n$$

Proof:

Let $\lim_{n \rightarrow \infty} s_n = s$ then for a real number $\varepsilon > 0$, there exists a positive integer n_0 such that

$$|s_n - s| < \varepsilon \text{ whenever } n \geq n_0.$$

$$\text{i.e. } s - \varepsilon < s_n < s + \varepsilon \text{ whenever } n \geq n_0. \dots\dots\dots (i)$$

If we take $u_n = \inf \{s_k : k \geq n\}$ and $v_n = \sup \{s_k : k \geq n\}$, then (i) gives us

$$s - \varepsilon < u_n \leq v_n < s + \varepsilon \text{ whenever } n \geq n_0.$$

This gives $\lim_{n \rightarrow \infty} u_n = s$ and $\lim_{n \rightarrow \infty} v_n = s$

that is, $\liminf_{n \rightarrow \infty} s_n = \limsup_{n \rightarrow \infty} s_n = s.$ □

$$\geq \dots\dots\dots \leq$$

References:

1. W. Rudin, *Principle of Mathematical Analysis*, 3rd Edition, McGraw-Hill, Inc., 1976.
2. R.G. Bartle and D.R. Sherbert, *Introduction to Real Analysis*, 4th Edition, John Wiley & Sons, Inc., 2011.
3. S. Narayan and M.D. Raisinghania, *Elements of Real Analysis*, 17th Edition, S. Chand & Company, New Delhi, 2018.
4. B.S. Thomson, J.B. Bruckner and A.M. Bruckner, *Elementary Real Analysis*, Prentice Hall (Pearson), 2001. URL: <http://www.classicalrealanalysis.com>

