Chapter 1: Real Number System

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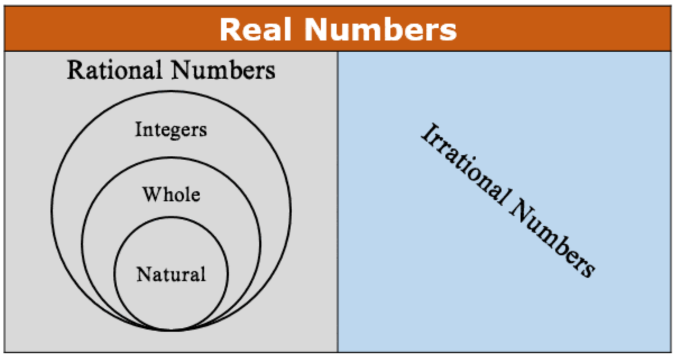
You don't have to be a mathematician to have a feel for numbers.
John Forbes Nash, Jr.

Historical Note: Numbers are like blood cells in the body of mathematics. Just as the understanding of anatomy and physiology of an organic system depends much on the knowledge of blood cells, so does the understanding of mathematics depend on the knowledge of numbers. In fact, a major part of mathematics bases its development on numbers and their multifarious properties.



It is very difficult, if not impossible, to spell out as to when did the concept of numbers came to human civilization. History, however, reveals that a formal study of numbers started almost five thousand years ago and that too by the Hindus who studied numbers purely as abstract symbols and were very proficient not only in discovering very large and very small numbers but also in using them effectively. Evidence are there that the Greek studied numbers purely on geometric conceptualization as they were very proficient in geometry and as a result had a relatively retarded progress. The greatest contribution of the Hindus is the discovery of zero, negative numbers and the decimal scale of representing numbers. In fact, they showed commendable mastery over rational numbers as early as the 5th century after Christ. The formal rigorous study of numbers, however, began even much later when mathematics faced several foundational crises. All these started in the 17th century but reached a climax after George Cantor (1845-1925) in 18th and 19th century. The contribution of 20th century in this regard is, on the one hand, stunning remarkable but on the other hand, devastating from the foundation point of view. The work and criticism by Russell (1872-1970), Lowenheim (1887-1940), Skolem (1887-1963) and Church (1903-1995) have been instrumental in bringing about a drastic change in our attitude and approach towards mathematics in general. In our modern approach, we start directly from real numbers defined axiomatically and then pass on to the related concept. (for more details see [4]). Many authors have different approach to define set of real numbers. Here we use the idea of Rudin introduced in [1].

❖ Preliminaries

In this section, we give some basic definitions and facts. These will help to learn and understand our main topic.

Definition: The set $\{1,2,3,\ldots\}$, which is usually denoted by $\mathbb{N}$ is called set of natural numbers.

Definition: The set $\{\ldots,-2,-1,0,1,2,\ldots\}$, which is usually denoted by $\mathbb{Z}$ is called set of integers.

❖ **Remarks:**

a. A set $\mathbb{Z}=\{\ldots,-2,-1,0,1,2,\ldots\}$ can also be written as $\mathbb{Z}=\{0,\pm 1,\pm 2,\pm 3,\ldots\}$.

b. A set of positive integers is denoted by $\mathbb{Z}^{+}=\{1,2,3,\ldots\}$ and set of negative integers is denoted by $\mathbb{Z}^{-}=\{-1,-2,-3,\ldots\}$

c. $\mathbb{Z}=\mathbb{Z}^{-}\cup\{0\}\cup\mathbb{Z}^{+}$, that is, a number 0 is neither positive nor negative.

Definition: Given two integers $a,b\in\mathbb{Z}$, $a\neq 0$, we say a divides b if there exists some integer q such that $b=a\cdot q$.

Notation: If a divides b, then we write $a\mid b$ and if a doesn't divides b, then we write $a\nmid b$.

Examples: (i) 2 divides 6, i.e. $2\mid 6$ because if $a=2$ and $b=6$, then $q=3$.

(ii) -2 divides 6, i.e. $-2\mid 6$ because if $a=-2$ and $b=6$, then $q=-3$.

(iii) -1, 1, -*a* and *a* divide every integer *a*.

(iv) Every non-zero integer divides 0.

Definition: An integer is called even if it is divisible by 2, otherwise it is called odd.

Note: A set $E:=\{0,\pm 2,\pm 4,\ldots\}$ represents set of all even integers and a set of odd integers is represented as $O:=\{\pm 1,\pm 3,\pm 5,\ldots\}$.

Definition: A positive integer *p* is called prime if it has exactly four divisors (or two positive divisors).

Examples: 2, 3, 11, 29 are prime numbers.

Definition: A set $\left\{\frac{p}{q}\mid p,q\in\mathbb{Z}\wedge q\neq 0\right\}$ is called set of rational numbers and it is usually denoted by $\mathbb{Q}$.

❖ **Remarks:**

a. All the integers are rational number but there are numbers which are rational but not integer.

b. One rational number can be written as infinitely many ways e.g. $\dfrac{1}{3}$ can be written as $0.333\ldots$ or $\dfrac{2}{6}$ or $\dfrac{-4}{-12}$.

c. Between any two rational numbers there exist a rational number, that is, there are infinity many rational between any two rational numbers.

d. There are operations of addition $(+)$ and multiplication $(\cdot)$ on $\mathbb{N},\mathbb{Z}$ and $\mathbb{Q}$, which has nice properties.

e. The set of integers is exclusively the point of interest in Number Theory.

❖ Preparation to Define Set of Real Numbers

It is not easy to define set of real numbers as we define $\mathbb{N}, \mathbb{Z}$ or $\mathbb{Q}$. The real number system can be described as a "complete ordered field". Therefore, let's discusses and understand these notions first.

❖ Order or Ordered Set

Definition: Let *S* be a non-empty set. An *order* on a set *S* is a relation denoted by "$<$" with the following two properties

(*i*) If $x, y \in S$, then one and only one of the statements

$x < y$, $x = y$, $y < x$ is true.

(*ii*) If $x, y, z \in S$ and if $x < y$, $y < z$ then $x < z$.

Examples: Consider the following sets:

- $A = \{1, 2, 3, ..., 50\}$
- $B = \{a, e, i, o, u\}$
- $C = \{x : x \in \mathbb{Z} \wedge x^2 \leq 19\}$

There is an order on *A* and *C* but there is no order on *B* (we can define order on *B*).

Definition: A non-empty set *S* is said to be *ordered set* if an order is defined on *S*.

Examples: (i) The set $\{2, 4, 6, 7, 8, 9\}$, $\mathbb{Z}$ and $\mathbb{Q}$ are examples of ordered set with standard order relation.

(ii) The set $\{a, b, c, d\}$ and $\{\alpha, \beta, \chi, \vartheta\}$ are examples of set with no order.

❖ Bounded & Unbounded Set

Definition: Let *S* be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, then we say that *E* is *bounded above*. The number β is known as *upper bound* of *E*.

Definition: Let *S* be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \geq \beta$ for all $x \in E$, then we say that *E* is *bounded below*. The number β is known as *lower bound* of *E*.

Definition: Let *S* be an ordered set and $E \subset S$. A set *E* is said to be bounded if it has both upper and lower bounds. Otherwise it is said to be an unbounded.

Examples: (i) Consider $S = \{1, 2, 3, ..., 50\}$ and $E = \{5, 10, 15, 20\}$.

Set of all lower bounds of $E = \{1, 2, 3, 4, 5\}$.

Set of all upper bounds of $E = \{20, 21, 22, ..., 50\}$.

(ii) Consider $S = \mathbb{N}$, $E = \{1, 2, 3, ..., 100\}$ and $F = \{10, 20, 30, ...\}$.

Set of lower bounds of $E = \{1\}$.

Set of lower bounds of $F = \{1,2,3,...,10\}$.

Set of upper bounds of $E = \{100,101,102,...\}$.

Set of upper bounds of $F = \varphi$.

❖ Least Upper Bound (Supremum) and Greatest Lower Bound (Infimum)

Definition: Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exists an $\alpha \in S$ such that

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ for $\gamma \in S$, then γ is not an upper bound of E.

Then α is called *least upper bound* of E or *supremum* of E and written as $\sup E = \alpha$.

Example: Consider $S = \{1,2,3,...,50\}$ and $E = \{5,10,15,20\}$.

(i) It is clear that 20 is upper bound of E.

(ii) For $\gamma \in S$ if $\gamma < 20$ then clearly γ is not an upper bound of E. Hence $\sup E = 20$.

Definition: Suppose S is an ordered set, $E \subset S$ and E is bounded below. Suppose there exists a $\beta \in S$ such that

(i) β is a lower bound of E.

(ii) If $\beta < \gamma$ for $\gamma \in S$, then γ is not a lower bound of E.

Then β is called *greatest lower bound* or *infimum* of E and written as $\inf E = \beta$.

Example: Consider $S = \{1,2,3,...,50\}$ and $E = \{5,10,15,20\}$.

(i) It is clear that 5 is lower bound of E.

(ii) For $\gamma \in S$ if $5 < \gamma$, then clearly γ is not lower bound of E. Hence $\inf E = 5$.

❖ Remarks

- A set is unbounded if either its set of upper bounds or set of lower bounds is empty.
- Supremum is the least member of the set of upper bound of the given set.
- Infimum is the greatest member of the set of lower bound of the given set.
- If α is supremum or infimum of E, then α may or may not belong to E.
  - Let $E_1 = \{r : r \in \mathbb{Q} \wedge r < 0\}$ and $E_2 = \{r : r \in \mathbb{Q} \wedge r \geq 0\}$. Then $\sup E_1 = \inf E_2 = 0$ but $0 \notin E_1$ and $0 \in E_2$.
  - Let $E \subset \mathbb{Q}$ be the set of all numbers of the form $1/n$, where n is the natural numbers, that is,

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}.$$

Then $\sup E = 1$ which is in E, but $\inf E = 0$ which is not in E.

❖ Least Upper Bound Property and Greatest Lower Bound Property

Definition: A set S is said to have the *least upper bound property* if the followings is true

(*i*) S is non-empty and ordered.

(*ii*) If $E \subset S$ and E is non-empty and bounded above then sup$E$ exists in S.

Definition: A set S is said to have the *greatest lower bound property* if the followings is true

(*i*) S is non-empty and ordered.

(*ii*) If $E \subset S$ and E is non-empty and bounded below then inf$E$ exists in S.

Examples: (i) The sets $\mathbb{N}$ and $\mathbb{Z}$ satisfies least upper bound property.

(ii) The set of rational numbers $\mathbb{Q}$ doesn't satisfy completeness axiom.

Consider a set $E = \left\{x : x \in \mathbb{Q} \wedge x^2 \leq 2\right\}$. One can prove that supremum of E doesn't exist in $\mathbb{Q}$ even E is a bounded set.

If U and L denotes the set of upper and lower bounds of E respectively, then $U = \left\{x : x \in \mathbb{Q} \wedge x^2 \geq 2 \wedge x > 0\right\}$ and $L = \left\{x : x \in \mathbb{Q} \wedge x^2 \geq 2 \wedge x < 0\right\}$.

If r is the supremum of E, then clearly $r^2 = 2$.

Here, we prove there is no rational p such that $p^2 = 2$.

Let us suppose that there exists a rational p such that $p^2 = 2$.

This implies we can write

$p = \dfrac{m}{n}$ where $m, n \in \mathbb{Z}$, $n \neq 0$ & m, n have no common factor.

Then $p^2 = 2 \quad \Rightarrow \quad \dfrac{m^2}{n^2} = 2 \quad \Rightarrow \quad m^2 = 2n^2$

$\Rightarrow$ m^2 is even $\quad \Rightarrow$ m is even

$\Rightarrow$ m is divisible by 2 and so m^2 is divisible by 4.

$\Rightarrow$ $2n^2$ is divisible by 4 and so n^2 is divisible by 2. $\quad \because m^2 = 2n^2$.

i.e. n^2 is even $\Rightarrow$ n is an even

$\Rightarrow$ m and n both have common factor 2.

which is contradiction because m and n have no common factor.

Hence $p^2 = 2$ is impossible for rational p.

Finally, we conclude that the set E, which is bounded in $\mathbb{Q}$ doesn't have supremun and infimum in $\mathbb{Q}$, hence set of rational $\mathbb{Q}$ doesn't satisfy the least upper bound property.

❖ Remark

The above property is known as *completeness axiom* or *LUB axiom* or *continuity axiom* or *order completeness axiom*.

❖ Theorem

Suppose S is an ordered set with least upper bound property, $B \subset S$, B is non-empty and is bounded below. Let L be set of all lower bound of B. Then

$$\alpha := \sup L$$

exists in S and $\alpha = \inf B$.

Proof

Since B is bounded below therefore L is non-empty.

Since L consists of exactly those $y \in S$ which satisfy the inequality.

$$y \le x \qquad \forall\ x \in B.$$

We see that every $x \in B$ is an upper bound of L.

This implies L is bounded above.

Since S is ordered and non-empty with least upper bound property therefore L has a supremum in S, that is, $\alpha := \sup L$ exists in S.

If $\gamma < \alpha$, then (by definition of supremum) γ is not upper bound of L.

$\Rightarrow \quad \gamma \notin B$.

It follows that $\alpha \le x \ \ \forall\ x \in B$.

Thus α is lower bound of B.

Now if $\alpha < \beta$, then $\beta \notin L$ because $\alpha = \sup L$, that is, β is not lower bound of B.

this means (by definition of infumum) $\alpha = \inf B$.

❖ Remark

Above theorem can be stated as follows:

An ordered set which has the least upper bound property has also the greatest lower bound property.

❖ Field

A set F with two operations called addition and multiplication satisfying the following axioms is known to be field.

Axioms for Addition:

(i) If $x, y \in F$ then $x + y \in F$. *Closure Law*

(ii) $x + y = y + x, \quad \forall\ x, y \in F$. *Commutative Law*

(iii) $x + (y + z) = (x + y) + z \quad \forall\ x, y, z \in F$. *Associative Law*

(iv) For any $x \in F$, $\exists\ 0 \in F$ such that $x + 0 = 0 + x = x$ *Additive Identity*

(v) For any $x \in F$, $\exists\ -x \in F$ such that $x + (-x) = (-x) + x = 0$ *+tive Inverse*

Axioms for Multiplication:

(i) If $x, y \in F$ then $x\,y \in F$. *Closure Law*

(ii) $x\,y = y\,x, \quad \forall\ x, y \in F$ *Commutative Law*

(iii) $x(y\,z) = (x\,y)z \quad \forall\ x, y, z \in F$

(iv) For any $x \in F$, $\exists\ 1 \in F$ such that $x \cdot 1 = 1 \cdot x = x$ *Multiplicative Identity*

(v) For any $x \in F$, $x \neq 0$, $\exists\ \dfrac{1}{x} \in F$, such that $x\left(\dfrac{1}{x}\right) = \left(\dfrac{1}{x}\right)x = 1$ $\times$*tive Inverse*.

Distributive Law

For any $x, y, z \in F$, (i) $x(y+z) = xy + xz$

(ii) $(x+y)z = xz + yz$

❖ Existence of Real Field

It is worth mentioning that $\mathbb{N}$ and $\mathbb{Z}$ are completely order sets but not a field. While $\mathbb{Q}$ is ordered field but not satisfy completeness axiom. What about a set which satisfy all three properties, that is, *i*. ordered *ii*. field and *iii*. satisfy completeness axiom. Amazingly, $\mathbb{R}$ (set of real numbers) is the only set which satisfy all these properties.

❖ Theorem:

There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover $\mathbb{R}$ contains $\mathbb{Q}$ (set of rational numbers) as a subfield.

Proof

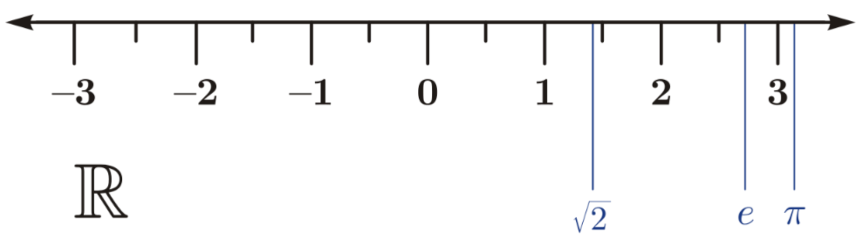
The proof of the theorem is rather long and a bit tedious. So, we are skipping the proof, one can see it at [1, Page 17]. □

Definition: The members of $\mathbb{R}$ are called *real numbers*.

Definition: Real numbers which are not rational are called *irrational* numbers.

❖ Explanation about $\mathbb{R}$

The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as $\sqrt{2}$ (1.41421356…, the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number). Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one. Or a real number is a value that represents any quantity along a number line. Because they lie on a number line, their size can be compared. You can say one is greater or less than another and do arithmetic with them.



By using the fact that $\mathbb{R}$, the set of real numbers, is a completely order field, one can prove the following theorem.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then axioms for addition imply the following.

(***a***) If $x+y=x+z$ then $y=z$

(***b***) If $x+y=x$ then $y=0$

(***c***) If $x+y=0$ then $y=-x$.

(***d***) $-(-x)=x$

Proof

(Note: *We have given the proofs here just to show that the things which looks simple must have valid analytical proofs under some consistence theory of mathematics*)

(a) Suppose $x+y=x+z$.

$$\begin{aligned}
\text{Since}\quad y &= 0+y \\
&= (-x+x)+y && \text{since } -x+x=0. \\
&= -x+(x+y) && \text{by associative law.} \\
&= -x+(x+z) && \text{by supposition.} \\
&= (-x+x)+z && \text{by associative law.} \\
&= 0+z && \text{since } -x+x=0. \\
&= z
\end{aligned}$$

(b) Take $z=0$ in (a)

$$x+y=x+0 \qquad \Rightarrow \quad y=0$$

(c) Take $z=-x$ in (a)

$$x+y=x+(-x) \qquad \Rightarrow \quad y=-x$$

(d) Since $(-x)+x=0$,

then (c) gives $x=-(-x)$. □

We are skipping the proofs of following three theorems as these may be the part of the mathematics of FSc.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then axioms of multiplication imply the following.

(***a***) If $x \neq 0$ and $xy=xz$ then $y=z$.

(***b***) If $x \neq 0$ and $xy=x$ then $y=1$.

(***c***) If $x \neq 0$ and $xy=1$ then $y=\frac{1}{x}$.

(***d***) If $x \neq 0$, then $\frac{1}{1/x}=x$.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then field axioms imply the following.

(***i***) $0 \cdot x = 0$. (***ii***) if $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.

(***iii***) $(-x)y = -(xy) = x(-y)$. (***iv***) $(-x)(-y) = xy$.

❖ **Theorem**

Let $x, y, z \in \mathbb{R}$. Then the following statements are true:

i) If $x > 0$ then $-x < 0$ and vice versa.

ii) If $x > 0$ and $y < z$ then $xy < xz$.

iii) If $x < 0$ and $y < z$ then $xy > xz$.

iv) If $x \neq 0$ then $x^2 > 0$ in particular $1 > 0$.

v) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

❖ **Theorem (Archimedean Property)**

If $x, y \in \mathbb{R}$ and $x > 0$ then there exists a positive integer *n* such that $nx > y$.

Proof

Let $A = \left\{ nx : n \in \mathbb{Z}^+ \wedge x > 0, x \in \mathbb{R} \right\}$

Suppose the given statement is false i.e. $nx \leq y$.

This implies *y* is an upper bound of *A*, that is, *A* is bounded above.

Since we are dealing with a set of real and it satisfies the least upper bound property,

Therefore supremum of *A* exists in $\mathbb{R}$.

Assume that $\alpha = \sup A$.

As $x > 0$ so we have $\alpha - x < \alpha$.

This gives $\alpha - x$ is not an upper bound of *A*.

Hence $\alpha - x < mx$, where $mx \in A$ for some positive integer *m*.

So, we have $\alpha < (m+1)x$, where $m + 1$ is integer.

This implies $(m+1)x \in A$.

This is impossible because α is least upper bound of A i.e. $\alpha = \sup A$.

Hence, we conclude that our supposition is wrong and the given statement is true. □

❖ **Theorem**

The set $\mathbb{N}$ of natural numbers is not bounded above.

Proof.

By Archimedean property in real number, for each positive real numbers x, there exist $n \in \mathbb{N}$ such that $n \cdot 1 > x$, that is, $n > x$.

This implies, there is no positive real number x such that $n \leq x$ for all $n \in \mathbb{N}$.

This implies no real number is an upper bound of $\mathbb{N}$.

Hence $\mathbb{N}$ is not bounded above. □

❖ The Density Theorem

If $x, y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

i.e., between any two real numbers there is a rational number *or* $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

Let us assume that $x, y \in \mathbb{R}$ with $x < y$. Then $y - x > 0$.

By Archimedean property, for $y - x, 1 \in \mathbb{R}$, $y - x > 0$, there exists positive integer n such that

$$n(y - x) > 1,$$

$$\Rightarrow \; 1 + nx < ny \, . \; \ldots\ldots\ldots\ldots\ldots \; (i)$$

Again, we use Archimedean property, for $1, nx \in \mathbb{R}$ and $1, -nx \in \mathbb{R}$, $1 > 0$, to obtain two positive integers m_1 and m_2 such that

$$m_1 \cdot 1 > nx \quad \text{and} \quad m_2 \cdot 1 > -nx,$$

that is,

$$nx < m_1 \quad \text{and} \quad -m_2 < nx,$$

$$\Rightarrow \; -m_2 < nx < m_1 .$$

Then there is an integer m(with $-m_2 \le m \le m_1$) such that

$$m - 1 \le nx < m,$$

$$\Rightarrow \; nx < m \quad \text{and} \quad m \le 1 + nx,$$

$$\Rightarrow \; nx < m < 1 + nx .$$

Using (i) in the above inequality, we get

$$nx < m < ny$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y$$

$$\Rightarrow \; x < p < y, \quad \text{where} \quad p = \frac{m}{n} \text{ is a rational.}$$

This completes the proof. □

❖ Relatively Prime

Definition: For $a, b \in \mathbb{Z}$, the numbers a and b are said to be *relatively prime* or *co-prime* if a and b don't have common factor other than 1. If a and b are relatively prime, then we write $(a, b) = 1$.

❖ Theorem

(i) If r is rational and x is irrational, then $r + x$ is irrational.

(ii) If r is non-zero rational and x is irrational, then $r\,x$ are irrational.

Proof

(i) Suppose the contrary that $r + x$ is rational. Then

$$r+x=\frac{a}{b},\qquad \text{where } a,b\in\mathbb{Z}\ ,\ b\neq 0 \text{ such that } (a,b)=1,$$

$$\Rightarrow\ x=\frac{a}{b}-r.\quad \ldots\ldots\ (1)$$

Since r is rational, there exists $c,d\in\mathbb{Z},\ d\neq 0$ and $(c,d)=1$ such that

$$r=\frac{c}{d}.$$

Using it in (1) to get

$$x=\frac{a}{b}-\frac{c}{d}\quad\Rightarrow\ x=\frac{ad-bc}{bd},\ \text{where } bd\neq 0.$$

As $ad-bc, bd\in\mathbb{Z}$, we get x is rational.

This cannot happen because x is given to be irrational, hence we conclude that $r+x$ is irrational. □

(ii) Let us suppose the contrary that $r\,x$ is rational. Then

$$r\,x=\frac{a}{b}\qquad \text{for some } a,b\in\mathbb{Z},\ b\neq 0 \text{ such that } (a,b)=1.$$

$$\Rightarrow\ x=\frac{a}{b}\cdot\frac{1}{r}\ \ldots\ldots\ (2)$$

Since r is rational, there exists $c,d\in\mathbb{Z},\ d\neq 0$ and $(c,d)=1$ such that

$$r=\frac{c}{d}.$$

Using it in (2) to get

$$x=\frac{a}{b}\cdot\frac{1}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc},\ \text{where } bc\neq 0.$$

This shows that x is rational, which is again contradiction; hence we conclude that $r\,x$ is irrational. □

❖ Theorem

Given two real numbers x and y, $x<y$ there is an irrational number u such that $x<u<y$.

Proof

We have given $x<y$, therefore $\dfrac{x}{\sqrt{2}}<\dfrac{y}{\sqrt{2}}$.

By density theorem, for real numbers $\dfrac{x}{\sqrt{2}}$ and $\dfrac{y}{\sqrt{2}}$, we can obtain a rational number $r\neq 0$ such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$
$$\Rightarrow \; x < r\sqrt{2} < y$$
$$\Rightarrow \; x < u < y,$$

where $u = r\sqrt{2}$ is an irrational as product of non-zero rational and irrational is irrational. □

❖ Theorem

For every real number x there is a set E of rational number such that $x = \sup E$.

Proof

Take $E = \{q \in \mathbb{Q} : q < x\}$, where x is a real.

Then E is bounded above. Since $E \subset \mathbb{R}$ therefore supremum of E exists in $\mathbb{R}$.
Suppose $\sup E = \lambda$.

It is clear that $\lambda \leq x$.
If $\lambda = x$ then there is nothing to prove.
If $\lambda < x$ then $\exists \; q \in \mathbb{Q}$ such that $\lambda < q < x$,

which cannot happen as λ is the upper bound of E.
Hence, we conclude that real x is sup$E$. □

❖ Question

Let E be a non-empty subset of an ordered set, suppose α is a lower bound of E and β is an upper bound then prove that $\alpha \leq \beta$.

Proof

Since E is a subset of an ordered set S i.e. $E \subseteq S$.
Also α is a lower bound of E therefore by definition of lower bound

$$\alpha \leq x \quad \forall \quad x \in E \;\; \ldots\ldots\ldots\ldots\ldots \; (i)$$

Since β is an upper bound of E therefore by the definition of upper bound

$$x \leq \beta \quad \forall \quad x \in E \;\; \ldots\ldots\ldots\ldots\ldots \; (ii)$$

Combining (i) and (ii)

$$\alpha \leq x \leq \beta \quad \Rightarrow \quad \alpha \leq \beta \quad \text{as required.}$$ □

❖ Question

Show that for any two real numbers a and b.

(i) $\max\{a,b\} = \frac{1}{2}(a+b+|a-b|)$ (ii) $\min\{a,b\} = \frac{1}{2}(a+b-|a-b|)$.

Note: Above question is proposed to know the difference between supremum & maximum.

❖ The Extended Real Numbers

Definition: The extended real number system consists of real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty \quad \forall\ x \in \mathbb{R}.$$

❖ Remarks

It is clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. If, for example, E is a nonempty set of real numbers which is not bounded above in $\mathbb{R}$, then sup$E$ = $+\infty$ in the extended real number system.

The same observations apply to lower bounds.

❖ Extension of Operation in Extended Real Numbers

The extended real number system does not form a field. But it is customary to make the following conventions:

a) If x is real, then

$$x+\infty = +\infty, \qquad x-\infty = -\infty, \qquad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

b) If $x > 0$ then $x \cdot (+\infty) = +\infty,\ \ x \cdot (-\infty) = -\infty$.

c) If $x < 0$ then $x \cdot (+\infty) = -\infty,\ x \cdot (-\infty) = +\infty$.

Note: (*i*) Mostly we write $+\infty = \infty$.

(*ii*) The above operations hold in extended real number system not in $\mathbb{R}$.

❖ Euclidean Space

Definitions: For each positive integer k, let $\mathbb{R}^k$ be the set of all ordered k-tuples

$$\underline{x} = (x_1, x_2, ..., x_k)$$

where $x_1, x_2, ..., x_k$ are real numbers, called the *coordinates* of $\underline{x}$.

The elements of $\mathbb{R}^k$ are called *points* or *vectors*, especially when $k > 1$.

If $\underline{y} = (y_1, y_2, ..., y_n)$ and α is a real number, we define

$$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, ..., x_k + y_k)$$

and

$$\alpha\underline{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_k).$$

Observation: It is clear that $\underline{x} + \underline{y} \in \mathbb{R}^k$ and $\alpha\underline{x} \in \mathbb{R}^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws and make $\mathbb{R}^k$ into a vector space over the real field. The *zero element* of $\mathbb{R}^k$ (sometimes called the *origin* or the *null vector*) is the point $\underline{0}$ (or we simply write 0), all of whose coordinates are 0. These operations make $\mathbb{R}^k$ into a *vector space over the real field*.

Definitions: The *inner product* or *scalar product* of $\underline{x}$ and $\underline{y}$ from $\mathbb{R}^k$ is defined as

$$\underline{x}.\underline{y} = \sum_{i=1}^{k} x_i y_i = x_1 y_1 + x_2 y_2 + \ldots + x_k y_k$$

and the norm of $\underline{x}$ is defined by

$$\| \underline{x} \| = (x \cdot x)^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}.$$

Definition: The vector space $\mathbb{R}^k$ with the above inner product and norm is called *Euclidean k-space* or *Euclidean space*.

❖ **Theorem**

Let $\underline{x}, \underline{y} \in \mathbb{R}^k$ then

i) $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$,

ii) $\left| \underline{x} \cdot \underline{y} \right| \leq \| \underline{x} \| \, \| \underline{y} \|$. (*Cauchy-Schwarz's inequality*)

Proof

i) Since $\| \underline{x} \| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}}$ therefore $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$

ii) If $\underline{x} = 0$ or $\underline{y} = 0$, then Cauchy-Schwarz's inequality holds with equality.

If $\underline{x} \neq 0$ and $\underline{y} \neq 0$, then for $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} 0 \leq \left\| \underline{x} - \lambda \underline{y} \right\|^2 &= \left(\underline{x} - \lambda \underline{y} \right) \cdot \left(\underline{x} - \lambda \underline{y} \right) \\ &= \underline{x} \cdot \left(\underline{x} - \lambda \underline{y} \right) + \left(-\lambda \underline{y} \right) \cdot \left(\underline{x} - \lambda \underline{y} \right) \\ &= \underline{x} \cdot \underline{x} + \underline{x} \cdot (-\lambda \underline{y}) + (-\lambda \underline{y}) \cdot \underline{x} + (-\lambda \underline{y}) \cdot (-\lambda \underline{y}) \\ &= \| \underline{x} \|^2 - 2\lambda (\underline{x} \cdot \underline{y}) + \lambda^2 \| \underline{y} \|^2 \end{aligned}$$

Now put $\lambda = \dfrac{\underline{x} \cdot \underline{y}}{\| \underline{y} \|^2}$ (*certain real number*)

$$\Rightarrow \ 0 \leq \| \underline{x} \|^2 - 2 \frac{\left(\underline{x} \cdot \underline{y} \right)\left(\underline{x} \cdot \underline{y} \right)}{\| \underline{y} \|^2} + \frac{\left(\underline{x} \cdot \underline{y} \right)^2}{\| \underline{y} \|^4} \| \underline{y} \|^2 \quad \Rightarrow \ 0 \leq \| \underline{x} \|^2 - \frac{\left(\underline{x} \cdot \underline{y} \right)^2}{\| \underline{y} \|^2}$$

$$\Rightarrow \ 0 \leq \| \underline{x} \|^2 \| \underline{y} \|^2 - \left| \underline{x} \cdot \underline{y} \right|^2 \qquad \because \ a^2 = |a|^2 \ \forall \ a \in \mathbb{R},$$

$$\Rightarrow \ 0 \leq \left(\| \underline{x} \| \, \| \underline{y} \| + \left| \underline{x} \cdot \underline{y} \right| \right) \left(\| \underline{x} \| \, \| \underline{y} \| - \left| \underline{x} \cdot \underline{y} \right| \right).$$

This holds if and only if

$$0 \leq \| \underline{x} \| \, \| \underline{y} \| - \left| \underline{x} \cdot \underline{y} \right|$$

i.e., $\left| \underline{x} \cdot \underline{y} \right| \leq \| \underline{x} \| \, \| \underline{y} \|$. □

❖ Question

Suppose $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^k$, then prove that

$a)\ \|\underline{x}+\underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|.$

$b)\ \|\underline{x}-\underline{z}\| \le \|\underline{x}-\underline{y}\| + \|\underline{y}-\underline{z}\|.$

Solution

$a)$ Consider
$$\begin{aligned}
\|\underline{x}+\underline{y}\|^2 &= (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y}) \\
&= \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y} \\
&= \|\underline{x}\|^2 + 2(\underline{x}\cdot\underline{y}) + \|\underline{y}\|^2 \\
&\le \|\underline{x}\|^2 + 2|\underline{x}\cdot\underline{y}| + \|\underline{y}\|^2 && \because\ |a| \ge a\ \forall\ a \in \mathbb{R}. \\
&\le \|\underline{x}\|^2 + 2\|\underline{x}\|\,\|\underline{y}\| + \|\underline{y}\|^2 && \because\ \|\underline{x}\|\,\|\underline{y}\| \ge |\underline{x}\cdot\underline{y}| \\
&= (\|\underline{x}\| + \|\underline{y}\|)^2
\end{aligned}$$
$$\Rightarrow\ \|\underline{x}+\underline{y}\| \le \|\underline{x}\| + \|\underline{y}\| \quad \ldots\ldots\ldots\ldots (i)$$

$b)$ We have
$$\begin{aligned}
\|\underline{x}-\underline{z}\| &= \|\underline{x}-\underline{y}+\underline{y}-\underline{z}\| \\
&\le \|\underline{x}-\underline{y}\| + \|\underline{y}-\underline{z}\| && \text{from } (i) \qquad \square
\end{aligned}$$

References:

[1] Principles of Mathematical Analysis by Walter Rudin (McGraw-Hill, Inc.)
[2] Introduction to Real Analysis by R.G.Bartle, and D.R. Sherbert (John Wiley & Sons, Inc.)
[3] Mathematical Analysis by Tom M. Apostol, (Pearson; 2nd edition.)
[4] Real Analysis by Dipak Chatterjee (PHI Learning, 2nd edition.)

A password protected "zip" archive of above three resources can be downloaded from the following URL:
http://bit.ly/2BViMnB