Ch 01: Improper Integrals of 1st and 2nd Kinds

Course Title: Real Analysis II | Course Code: MTH322
Course instructor: Dr. Atiq ur Rehman | Class: BS-VI
Course URL: *www.mathcity.org/atiq/fa21-mth322*

The objective of this chapter is to

- *learn about different kinds of improper integrals.*
- *learn the meaning of convergence and divergence of improper integrals.*
- *learn the theory to develop different tests and techniques to find convergence or divergence of improper integrals.*

We discussed Riemann's integrals of the form $\int_a^b f(x)\,dx$ under the restrictions that both f is defined and bounded on a finite interval $[a,b]$. To extend the concept, we shall relax some condition on definite integral like f on finite interval or boundedness of f on finite interval.

First of all we recall few things about symbol $+\infty$ (or ∞) and $-\infty$. These symbols don't behave like usual real numbers. Historically, with these concepts, mathematicians were never very comfortable, and these were some sources of much confusion and debate. Indeed, the concepts of infinite sets and infinity took nearly a century for a definite implication. To get the idea of infinity one can read the articles at following URLs:

- https://www.mathsisfun.com/numbers/infinity.html
- https://en.wikipedia.org/wiki/Infinity

An integral is said to be improper integral if either the function f is unbounded on $[a,b]$ or the interval of integration is unbounded. Now we are going to give formal definitions of improper integrals.

➢ ***Definition***

The integral $\int_a^b f(x)\,dx$ or $\int_a^b f\,dx$ is called an improper integral of first kind if $a=-\infty$ or $b=\infty$ or *both* i.e. one or both integration limits are infinite.

➢ ***Definition***

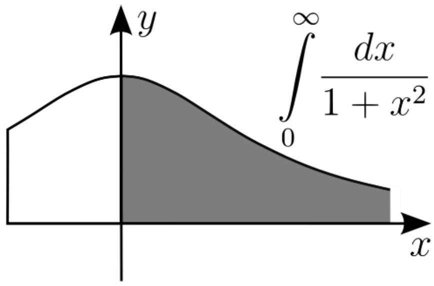
The integral $\int_a^b f\,dx$ is called an improper integral of second kind if $f(x)$ is unbounded with infinite discontinuity at one or more points of $a \le x \le b$.

➢ ***Remark:***

Some time we deal with an improper integral which involves both kinds of integral at once. It is known as improper integral of mixed kind. It can be break into the sum of improper integrals of first and second kinds.

➢ *Examples*

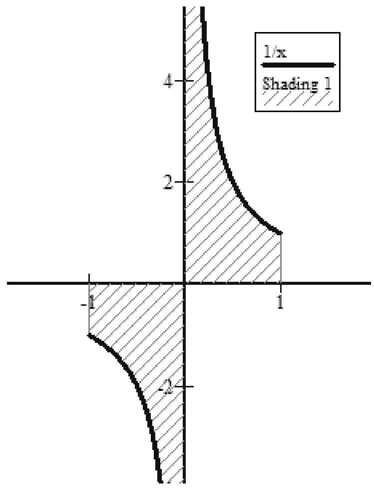
- $\int_0^{\infty}\frac{1}{1+x^2}dx$, $\int_{-\infty}^{1}\frac{1}{x-2}dx$ and $\int_{-\infty}^{\infty}(x^2+1)dx$ are examples of improper integrals of first kind.



- $\int_{-1}^{1}\frac{1}{x}dx$ and $\int_0^{1}\frac{1}{2x-1}dx$ are examples of improper integrals of second kind.

- $\int_0^{\infty}\frac{1}{x}dx$ is an example of improper integral of mixed kind and it can be written as follow:

$$\int_0^{\infty}\frac{1}{x}dx=\int_0^{1}\frac{1}{x}dx+\int_1^{\infty}\frac{1}{x}dx$$



➢ *MCQs*

(i) Which of the following is an improper integral of 1^{st} kind.

(I) $\int_1^{2}\frac{1}{x}dx$ (II) $\int_1^{\infty}\frac{1}{x^2}dx$ (III) $\int_{-\infty}^{\infty}(2t+1)dt$

A. I and III only B. III only
C. II only D. II and III only

(ii) Which of the following is an improper integral of 2^{nd} kind.

(I) $\int_{-1}^{0}\frac{1}{x}dx$ (II) $\int_2^{3}\frac{1}{x^2-1}dx$ (III) $\int_0^{1}\tan\frac{\pi t}{2}dt$

A. I and III only B. III only
C. I only D. II and III only

(iii) The integral $\int_0^1\frac{\sin\theta}{\theta}d\theta$ is ………………..

A. improper integral of 1^{st} kind. B. improper integral of 2^{nd} kind.
C. improper integral of mixed kind. D. none of these.

(iv) The integral $\int_0^{\infty}\frac{1}{x}dx$ is ………………..

A. improper integral of 1^{st} kind. B. improper integral of 2^{nd} kind.
C. improper integral of mixed kind. D. none of these.

IMPROPER INTEGRAL OF THE FIRST KIND

➤ ***Definition***

Assume that $f \in \mathcal{R}[a,b]$ for every $b \geq a$. Define a function I on $[a,\infty)$ as follows:

$$I(b) = \int_a^b f(x)\,dx \quad \text{for} \quad b \geq a.$$

The integral $\int_a^\infty f(x)\,dx$ is said to converge if the $\lim_{b\to\infty} I(b)$ exists (finite). Otherwise, $\int_a^\infty f\,dx$ is said to diverge.

If the $\lim_{b\to\infty} I(b)$ exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\,dx = A$.

For the integral $\int_{-\infty}^a f(x)\,dx$, we define $I(b) = \int_{-b}^a f(x)\,dx$ and for the integral $\int_{-\infty}^\infty f(x)\,dx$, we define $I(b) = \int_{-b}^b f(x)\,dx$.

➤ ***Example***

Consider and integral $\int_1^\infty \frac{1}{x^2}\,dx$. Discuss its convergence or divergence.

Solution

Let $I(b) = \int_1^b \frac{1}{x^2}\,dx$, where $b \geq 1$.

Then $I(b) = \int_1^b x^{-2}\,dx = -x^{-1}\Big|_1^b = -\frac{1}{x}\Big|_1^b = 1 - \frac{1}{b}$.

Now $\lim_{b\to\infty} I(b) = \lim_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1.$

Hence $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent.

➤ ***Example***

Consider and integral $\int_1^\infty x^{-p}\,dx$, where p is any real number. Discuss its convergence or divergence.

Solution

Let $I(b) = \int_1^b x^{-p}\,dx$, where $b \geq 1$.

Then $I(b)=\int_1^b x^{-p}\,dx=\left.\dfrac{x^{1-p}}{1-p}\right|_1^b=\dfrac{1-b^{1-p}}{p-1}$ if $p\neq 1$.

If $b\to\infty$, then

$$b^{1-p}\to\begin{cases}\infty & if \quad 1>p,\\ 0 & if \quad 1<p.\end{cases}$$

Thus, we have

$$\lim_{b\to\infty} I(b)=\lim_{b\to\infty}\frac{1-b^{1-p}}{p-1}=\begin{cases}\infty & if \quad p<1,\\ \dfrac{1}{p-1} & if \quad p>1.\end{cases}$$

Now if $p=1$, we get $\int_1^b x^{-1}\,dx=\log b\to\infty$ as $b\to\infty$.

Hence, we concluded: $\int_1^\infty x^{-p}\,dx=\begin{cases} diverges & if \quad p\le 1,\\ \dfrac{1}{p-1} & if \quad p>1.\end{cases}$

➢ ***Review***

If $\lim_{x\to a} f(x)=L$, then for every sequence $\{x_n\}$ such that $x_n\to a$ when $n\to\infty$, one has $\lim_{n\to\infty} f(x_n)=L$.

➢ ***Example***

Is the integral $\int_0^\infty \sin 2\pi x\,dx$ converges or diverges?

Solution:

Consider $I(b)=\int_0^b \sin 2\pi x\,dx$, where $b\ge 0$.

We have $\int_0^b \sin 2\pi x\,dx=\left.\dfrac{-\cos 2\pi x}{2\pi}\right|_0^b=\dfrac{1-\cos 2\pi b}{2\pi}$.

Consider $b_n=n$ and $c_n=n+\dfrac{1}{2}$. Clearly $b_n\to\infty$ and $c_n\to\infty$ as $n\to\infty$.

But $\cos 2\pi b_n\to 1$ and $\cos 2\pi c_n\to -1$ as $n\to\infty$.

Thus $\lim_{b\to\infty}\cos 2\pi b$ doesn't exist and hence given integral is divergent.

➢ ***Remark***

- If $\int_a^\infty f\,dx$ is convergent(divergent), then $\int_c^\infty f\,dx$ is convergent(divergent) for $c>a$.

- If $\int_c^\infty f\,dx$ is convergent (divergent), then $\int_a^\infty f\,dx$ is convergent (divergent) for $a<c$ if f in bounded in $[a,c]$.

➢ *Exercises*

- Show that $\int_a^\infty \frac{1}{x^p}dx$ converges if $p>1$.
- Evaluate: (i) $\int_{-\infty}^0 \sin x\,dx$ (ii) $\int_{-\infty}^0 e^x\,dx$

➢ *Note*

If $\int_{-\infty}^a f\,dx$ and $\int_a^\infty f\,dx$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^\infty f\,dx$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\,dx = \int_{-\infty}^{a} f\,dx + \int_{a}^{\infty} f\,dx .$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^\infty f dx$ converges, its value is equal to the limit: $\lim_{b\to+\infty}\int_{-b}^b f\,dx$.

For improper integral of first kind we will discuss the results for integral of the type $\int_a^\infty f\,dx$. The results for other cases can be derived in a similar manner.

➢ *Exercises*

Evaluate the improper integral $\int_{-\infty}^\infty e^x\,dx$.

➢ *MCQ*

(i) For what value of m the integral $\int_1^\infty \frac{dx}{x^{m+1}}$ is convergent.

A. $m>1$ B. $m\le 1$ C. $m>0$ D. $m\ge 0$

(ii) Which of the following integrals is divergent.

A. $\int_2^\infty \frac{dx}{x^2}$ B. $\int_1^\infty \frac{dt}{t^{\alpha+1}},\ \alpha>0$ C. $\int_1^\infty z^{-\frac{3}{2}}dz$ D. $\int_1^\infty x^{\frac{3}{2}}dx$

(iii) If $\int_2^\infty fdx$ is convergent then ……………… is convergent.

A. $\int_0^\infty fdx$ B. $\int_1^\infty fdx$ C. $\int_3^\infty fdx$ D. $\int_{-2}^\infty fdx$

➢ *Review:*

- A function f is said to be increasing, if for all $x_1, x_2 \in D_f$ (domain of f) and $x_1 \le x_2$ implies $f(x_1)\le f(x_2)$.

- A function f is said to be bounded if there exist some positive number μ such that $|f(t)| \le \mu$ for all $t \in D_f$.
- If f is define on $[a,+\infty)$ and $\lim_{x\to\infty} f(x)$ exists then f is bounded on $[a,+\infty)$.
- If $f \in \mathcal{R}[a,b]$ and $c \in [a,b]$, then $\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$.
- If $f \in \mathcal{R}[a,b]$ and $f(x) \ge 0$ for all $x \in [a,b]$, then $\int_a^b f\,dx \ge 0$.
- If f is monotonically increasing and bounded on $[a,+\infty)$, then $\lim_{x\to\infty} f(x) = \sup_{x\in[a,\infty)} f(x)$.
- If $f, g \in \mathcal{R}[a,b]$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f\,dx \le \int_a^b g\,dx$.

➤ ***Theorem***

Suppose that $f \in \mathcal{R}[a,b]$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^\infty f(x)\,dx$ converges if, and only if, there exists a constant $M > 0$ such that

$$\int_a^b f(x)\,dx \le M \quad \text{for every } b \ge a.$$

Proof

Let $I(b) = \int_a^b f\,dx$ for $b \ge a$.

First suppose that $\int_a^\infty f(x)\,dx$ is convergent, then $\lim_{b\to+\infty} I(b)$ exists, that is, $I(b)$ is bounded on $[a,+\infty)$.

So there exists a constant $M > 0$ such that

$$|I(b)| < M \text{ for every } b \ge a.$$

As $f(x) \ge 0$ for each $x \ge a$, therefore $\int_a^b f(x)\,dx \ge 0$.

This gives $I(b) = \int_a^b f(x)dx \le M$ for every $b \ge a$.

Conversely, suppose that there exists a constant $M > 0$ such that

$$\int_a^b f(x)dx \le M \text{ for every } b \ge a.$$

This give $|I(b)| \le M$ for every $b \ge a$, that is, I is bounded on $[a,+\infty)$.

Now for $b_2 \ge b_1 > a$, we have

$$I(b_2)=\int_a^{b_2} f(x)\,dx=\int_a^{b_1} f(x)dx+\int_{b_1}^{b_2} f(x)\,dx$$

$$\geq\int_a^{b_1} f(x)dx=I(b_1), \qquad \because \int_{b_1}^{b_2} f(x)\,dx\geq 0 \text{ as } f(x)\geq 0 \text{ for all } x\geq a$$

.

This gives I is monotonically increasing on $[a,+\infty)$.
As I is monotonically increasing and bounded on $[a,+\infty)$, therefore $\lim\limits_{b\to\infty} I(b)$ exists, that is, $\int_a^\infty f(x)\,dx$ converges.

➢ ***Theorem: (Comparison Test)***
Assume $f\in\mathcal{R}[a,b]$ for every $b\geq a$. If $0\leq f(x)\leq g(x)$ for every $x\geq a$ and $\int_a^\infty g\,dx$ converges, then $\int_a^\infty f\,dx$ converges and we have $\int_a^\infty f\,dx \leq \int_a^\infty g\,dx$.

Proof

Let $I_1(b)=\int_a^b f\,dx$ and $I_2(b)=\int_a^b g\,dx$, $b\geq a$.

Since $0\leq f(x)\leq g(x)$ for every $x\geq a$, therefore

$$\int_a^b f\,dx \leq \int_a^b g\,dx,$$

that is, $I_1(b)\leq I_2(b)$, $b\geq a$ …………………. (i)

Since $\int_a^\infty g\,dx$ converges, there exists a constant $M>0$ such that

$$\int_a^b g\,dx\leq M, \quad b\geq a.$$

That is, $I_2(b)\leq M$, $b\geq a$ …………………(ii)

From (i) and (ii), we have $I_1(b)\leq M$ for every $b\geq a$.

This implies $\int_a^\infty f\,dx$ converges, that is, $\lim\limits_{b\to\infty} I_1(b)$ exists and is finite.

So we have $\lim\limits_{b\to\infty} I_1(b)\leq \lim\limits_{b\to\infty} I_2(b)\leq M$,

this gives $\int_a^\infty f\,dx\leq\int_a^\infty g\,dx$.

➢ ***Remark***

In comparison test, if $\int_a^\infty f\,dx$ is divergent, then $\int_a^\infty g\,dx$ is divergent.

➢ ***Example***

Is the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ convergent or divergent?

Solution:

Since $\sin^2 x \le 1$ for all $x \in [1,+\infty)$,

therefore $\frac{\sin^2 x}{x^2} \le \frac{1}{x^2}$ for all $x \in [1,+\infty)$.

This gives $\int_1^\infty \frac{\sin^2 x}{x^2}dx \le \int_1^\infty \frac{1}{x^2}dx$.

Now $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore $\int_1^\infty \frac{\sin^2 x}{x^2}dx$ is convergent.

➢ *MCQs*

(i) A function f is said to be bounded if there exist a positive number α such that ………….. for all $t \in D_f$ (domain of f)

A. $f(t) \le \alpha$ B. $|f(t)| \le \alpha$ C. $|f(t)| > \alpha$ D. $f(t) > \alpha$

(ii) If $f:[a,b] \to (0,\infty)$ is a bounded function then

A. $\int_a^b f(t)dt \ge 0$ B. $\int_a^\infty f(t)dt \ge 0$ C. $f(t) \ge \mu$ for $\mu \in \mathbb{R}$ D. None of these

➢ *Review:*

- For all $a,b,c \in \mathbb{R}$, $|a-b| < c \iff b-c < a < b+c$ or $a-c < b < a+c$.
- If $\lim_{x\to\infty} f(x) = m$, then for all real $\varepsilon > 0$, there exists $N > 0$ such that

$$|f(x)-m| < \varepsilon \text{ whenever } x > N.$$

- If $\int_a^\infty f\,dx$ converges(diverges), then $\int_N^\infty f\,dx$ converges(diverges) if $N > a$.
- If $\int_N^\infty f\,dx$ is convergent (divergent), then $\int_a^\infty f\,dx$ is convergent (divergent) for $a < N$ if f is bounded in $[a,N]$.

➢ *Theorem (Limit Comparison Test)*

Suppose that $f,g \in \mathcal{R}[a,b]$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ for $x \ge a$. If

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1,$$

then $\int_a^\infty f\,dx$ and $\int_a^\infty g\,dx$ both converge, or both diverge.

Proof

Suppose $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$, then for all real $\varepsilon>0$, we can find some $N>0$, such that

$$\left|\frac{f(x)}{g(x)}-1\right|<\varepsilon \qquad \forall\ \ x>N\geq a.$$

$$\Rightarrow\ 1-\varepsilon<\frac{f(x)}{g(x)}<1+\varepsilon \qquad \forall\ \ x>N\geq a.$$

If we choose $\varepsilon=\frac{1}{2}$, then we have

$$\frac{1}{2}<\frac{f(x)}{g(x)}<\frac{3}{2} \qquad \forall\ \ x>N\geq a.$$

This implies $g(x)<2f(x)$ ……….(*i*) and $2f(x)<3g(x)$ ……....(*ii*)

From (*i*) $\int_N^\infty g\,dx\ <\ 2\int_N^\infty f\,dx$,

so if $\int_a^\infty f\,dx$ converges, then $\int_N^\infty f\,dx$ converges and hence by comparison test we get $\int_N^\infty g\,dx$ is convergent, which implies $\int_a^\infty g\,dx$ is convergent.

Now if $\int_a^\infty g\,dx$ diverges, then $\int_N^\infty g\,dx$ diverges and hence by comparison test we get $\int_N^\infty f\,dx$ is divergent, which implies $\int_a^\infty f\,dx$ is divergent.

From (*ii*), we have $2\int_N^\infty f\,dx\ <\ 3\int_N^\infty g\,dx$,

so if $\int_a^\infty g\,dx$ converges, then $\int_N^\infty g\,dx$ converges and hence by comparison test we get $\int_N^\infty f\,dx$ is convergent, which implies $\int_a^\infty f\,dx$ is convergent.

Now if $\int_a^\infty f\,dx$ diverges, then $\int_N^\infty f\,dx$ diverges and hence by comparison test we get $\int_N^\infty g\,dx$ is divergent, which implies $\int_a^\infty g\,dx$ is divergent.

$\Rightarrow$ The integrals $\int_a^\infty f\,dx$ and $\int_a^\infty g\,dx$ converge or diverge together.

➢ *Note*

The above theorem also holds if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=c$, provided that $c>0$. If $c=0$, we can only conclude that convergence of $\int_a^\infty g dx$ implies convergence of $\int_a^\infty f dx$.

➢ *Questions*

(i) Suppose $f(x)$ and $g(x)$ are positive integrable functions for $x>a$. If $\lim_{x\to\infty}\frac{f(x)}{g(x)}=c$, where $c>0$, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or both diverge.

(ii) Suppose $f(x)$ and $g(x)$ are positive integrable functions for $x>a$. If $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$, then convergence of $\int_a^\infty g(x)dx$ implies convergence of $\int_a^\infty f(x)dx$.

(iii) Suppose $f(x)$ and $g(x)$ are positive integrable functions for $x>a$.. If $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$, then convergence of $\int_a^\infty f(x)dx$ implies convergence of $\int_a^\infty g(x)dx$.

➢ *Example*

Prove that, for every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

Solution:

Let $f(x)=e^{-x}x^p$ and $g(x)=\frac{1}{x^2}$.

Now $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{e^{-x}x^p}{1/x^2}$

$$\Rightarrow \lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}e^{-x}x^{p+2}=\lim_{x\to\infty}\frac{x^{p+2}}{e^x}=0.$$ (find this limit yourself)

Since $\int_1^\infty \frac{1}{x^2}dx$ is convergent, therefore the given integral $\int_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ *Remark*

It is easy to show that if $\int_a^\infty f\,dx$ and $\int_a^\infty g\,dx$ are convergent, then

- $\int_a^\infty (f \pm g)\,dx$ is convergent.
- $\int_a^\infty cf\,dx$, where c is some constant, is convergent.

➢ *Review*

- If $\lim_{x\to\infty} f(x) = m$, then for all real $\varepsilon > 0$, there exists real $N > 0$ such that
$$|f(x) - m| < \varepsilon \text{ whenever } x > N.$$
- A sequence $\{a_n\}$ is said to be convergent if there exist a number l such that for all $\varepsilon > 0$, there exists a positive integer n_0 (depending on ε) such that
$$|a_n - l| < \varepsilon \text{ whenever } n > n_0.$$
The number l is called limit of the sequence and we write $\lim_{n\to\infty} a_n = l$.
- A sequence $\{a_n\}$ is said to be Cauchy if for all $\varepsilon > 0$, there exists a positive integer n_0 such that
$$|a_n - a_m| < \varepsilon \text{ whenever } n, m > n_0.$$
- A sequence of real numbers is Cauchy if and only if it is convergent.

➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f \in \mathcal{R}[a,b]$ for every $b \geq a$. Then the integral $\int_a^\infty f\,dx$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $c > b > B$ implies
$$\left|\int_b^c f\,dx\right| < \varepsilon.$$

Proof

Let $\int_a^\infty f\,dx$ be convergent, that is, $\lim_{b\to\infty}\int_a^b f\,dx$ exists and assume it to be A.

Then for all $\varepsilon > 0$, there exists $B > a$ such that
$$\left|\int_a^b f\,dx - A\right| < \frac{\varepsilon}{2} \quad \text{for every } b > B. \quad \ldots\ldots\ldots\ldots(i)$$
Also for $c > b > B$,

$$\left|\int_a^c f\,dx - A\right| < \frac{\varepsilon}{2} \quad \text{................ } (ii)$$

As we know $\int_a^c f\,dx = \int_a^b f\,dx + \int_b^c f\,dx$, this gives

$$\left|\int_b^c f\,dx\right| = \left|\int_a^c f\,dx - \int_a^b f\,dx\right| = \left|\int_a^c f\,dx - A + A - \int_a^b f\,dx\right|$$

$$\leq \left|\int_a^c f\,dx - A\right| + \left|A - \int_a^b f\,dx\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

$$\Rightarrow \left|\int_b^c f\,dx\right| < \varepsilon \quad \text{when } c > b > B.$$

Conversely, assume that the Cauchy condition holds.
For a positive integer $n \geq a$, define

$$a_n = \int_a^n f\,dx.$$

Consider n, m such that $n > m > B$, then

$$|a_n - a_m| = \left|\int_a^n f\,dx - \int_a^m f\,dx\right| = \left|\int_a^m f\,dx + \int_m^n f\,dx - \int_a^m f\,dx\right|$$

$$= \left|\int_m^n f\,dx\right| < \varepsilon \quad \text{(by Cauchy condition).}$$

This gives us that the sequence $\{a_n\}$ is a Cauchy sequence.
This implies $\{a_n\}$ is convergent and consider that $\lim_{n\to\infty} a_n = A$.
Then for given $\varepsilon > 0$, choose B so that

$$|a_n - A| < \frac{\varepsilon}{2} \quad \text{whenever } n \geq B.$$

Also, for $\varepsilon > 0$, we can have (by Cauchy condition)

$$\left|\int_b^c f\,dx\right| < \frac{\varepsilon}{2} \quad \text{if } c > b > B.$$

Now if $n, a, b > B$ such that $b \geq B + 1$, then we have

$$\left|\int_a^b f\,dx - A\right| = \left|\int_a^n f\,dx - A + \int_n^b f\,dx\right|$$

$$\leq |a_n - A| + \left|\int_n^b f\,dx\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This gives us $\lim_{b\to\infty}\int_a^b f\,dx = A$, that is, $\int_a^\infty f\,dx$ is convergent.

This completes the proof.

➢ ***Question:***

Suppose $f \in \mathcal{R}[a,b]$ for every $b \ge a$ and for every $\varepsilon > 0$ there exists a $B > 0$ such that $\left|\int_b^c f\ dx\right| < \varepsilon$ for $b,c > B$, then $\int_a^\infty f\,dx$ is convergent.

➢ ***Example:***

Use Cauchy criterion to prove that $\int_1^\infty \frac{\sin x}{x}dx$ is convergent.

Proof.

Consider

$$\int_b^c \frac{\sin x}{x}dx = \left.\frac{-\cos x}{x}\right|_b^c - \int_b^c \frac{\cos x}{x^2}dx\ .$$

This gives us

$$\left|\int_b^c \frac{\sin x}{x}dx\right| = \left|\frac{\cos b}{b} - \frac{\cos c}{c} - \int_b^c \frac{\cos x}{x^2}dx\right|$$

$$\le \left|\frac{\cos b}{b}\right| + \left|\frac{\cos c}{c}\right| + \left|\int_b^c \frac{\cos x}{x^2}dx\right| \quad \text{by triangular inequality}$$

$$\le \left|\frac{\cos b}{b}\right| + \left|\frac{\cos c}{c}\right| + \int_b^c \left|\frac{\cos x}{x^2}\right|dx \qquad \because \left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx\ .$$

$$\le \frac{1}{b} + \frac{1}{c} + \int_b^c \frac{1}{x^2}dx \qquad \because |\cos x| \le 1\ \forall\ x \in \mathbb{R}\ .$$

$$= \frac{1}{b} + \frac{1}{c} - \left.\frac{1}{x}\right|_b^c \ = \frac{1}{b} + \frac{1}{c} + \frac{1}{b} - \frac{1}{c} \ = \frac{2}{b}$$

Now let $\varepsilon > 0$ be an arbitrary and take $B = \frac{2}{\varepsilon}$ such that $c > b > B = \frac{2}{\varepsilon}$. Then

$$\left|\int_b^c \frac{\sin x}{x}dx\right| = \frac{2}{b} < \ 2\cdot\frac{\varepsilon}{2} \ = \varepsilon \qquad \because\ \frac{1}{b} < \frac{\varepsilon}{2}\ .$$

Hence by Cauchy criterion $\int_1^\infty \frac{\sin x}{x}dx$ is convergent.

➢ *Absolutely convergent*

An improper integral $\int_a^{\infty} f\,dx$ is said to converge absolutely if $\int_a^{\infty} |f|\,dx$ converges.

➢ *Conditionally convergent*

An improper integral $\int_a^{\infty} f\,dx$ is said to be converge conditionally if $\int_a^{\infty} f\,dx$ converges but $\int_a^{\infty} |f|\,dx$ is divergent.

➢ ***Note:*** The definition of absolutely and conditionally convergence is the same as above for other type of improper integrals.

➢ *Theorem*

If $f \in \mathcal{R}[a,b]$ for every $b \geq a$ and if $\int_a^{\infty} f\,dx$ is absolutely converges, then it is convergent.

Proof

Let $\int_a^{\infty} fdx$ be absolutely convergent, i.e., $\int_a^{\infty} |f|dx$ is convergent.

It is easy to see that

$$0 \leq |f(x)| - f(x) \leq 2|f(x)| \text{ for all } x \geq a.$$

Since we have given that $\int_a^{\infty} |f|dx$ is convergent, therefore by comparison test, we have $\int_a^{\infty} (|f| - f)\,dx$ converges.

Now difference of $\int_a^\infty |f|dx$ and $\int_a^\infty (|f|-f)dx$ is convergent,

that is, $\int_a^\infty |f|dx - \int_a^\infty (|f|-f)dx = \int_a^\infty f\,dx$ is convergent.

➢ ***Note:*** The converse of the above theorem doesn't hold in general.

For example: The integral $\int_1^\infty \frac{\sin x}{x}dx$ is convergent (prove yourself) but $\int_1^\infty \left|\frac{\sin x}{x}\right|dx$ is divergent (it is hard to prove).

➢ ***Remarks:***

- The above theorem can be stated as "an absolutely convergent integral is convergent".

➢ ***Questions***

- Use Cauchy criterion to prove that if an improper integral $\int_a^\infty f(x)dx$ is absolutely convergent then it is convergent.
- Show that $\int_1^\infty \exp(-x^2)dx$ is convergent.
- Show that $\int_1^\infty \frac{1+e^{-x}}{x}dx$ is divergent.

➢ ***Review***

- A function $f(x)$ is bounded for $x \ge a$ if there exist some positive number K such that $|f(x)| \le K$ for all $x \ge a$.
- An integral $\int_a^\infty f(x)dx$ is said to be absolutely convergent if $\int_a^\infty |f(x)|dx$ is convergent.
- Let $f \in R(a,b)$ for each $b \ge a$. An integral $\int_a^\infty f\,dx$ converges if, and only if, there exists a constant $M > 0$ such that $\int_a^b f dx \le M$ for every $b \ge a$.

The following theorem provide useful test of convergence for product of function.

➢ ***Theorem***

If $f(x)$ is bounded for all $x \geq a$, integrable on every closed subinterval of $[a,\infty)$ (i.e. $f \in \mathcal{R}[a,b]$ for each $b \geq a$) and $\int_a^\infty g(x)dx$ is absolutely convergent, then $\int_a^\infty f(x)g(x)dx$ is absolutely convergent.

Proof

Since $f(x)$ is bounded for all $x \geq a$, there exists $K > 0$ such that

$$|f(x)| \leq K \text{ for all } x \geq a. \quad \ldots\ldots \text{(i)}$$

Since $\int_a^\infty g(x)dx$ is absolutely convergent, that is, $\int_a^\infty |g(x)|dx$ is convergent, there exists $M > 0$ such that

$$\int_a^b |g(x)|dx \leq M \text{ for all } b \geq a. \quad \ldots\ldots\text{. (ii)}$$

Now

$$\int_a^b |f(x)g(x)|\, dx = \int_a^b |f(x)||g(x)|\, dx$$

$$\leq K\int_a^b |g(x)|\, dx \qquad \text{from (i)}$$

$$\leq KM \quad \text{for all } b \geq a \text{ by using (ii).}$$

Hence $\int_a^\infty |f(x)g(x)|\, dx$ is convergent, this implies $\int_a^\infty f(x)g(x)\, dx$ is absolutely convergent.

➢ ***Review***

- ***Second Mean Value Theorem (Bonnet's theorem):*** If $f, g \in \mathcal{R}[a,b]$ and f is monotonic on $[a,b]$, then there exist point $c \in [a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

- A function $f(x)$ is bounded for $x \geq a$ if there exist some positive number K such that $|f(x)| \leq K$ for all $x \geq a$.
- An integral $\int_a^b f(x)dx$ converges if, and only if, for every $\varepsilon > 0$ there exists a $B > 0$ such that $\left|\int_b^c f(x)dx\right| < \varepsilon$ when $c > b > B$.
- If $\lim_{x\to\infty} f(x) = m$, then for all real $\varepsilon > 0$, there exists real $N > 0$ such that

$$|f(x) - m| < \varepsilon \text{ whenever } x > N.$$

➢ ***Theorem (Abel)***

If $f(x)$ is bounded and monotone for all $x \ge a$ and $\int_a^\infty g(x)dx$ is convergent, then $\int_a^\infty f(x)g(x)dx$ is convergent.

Proof

As f is bounded and monotone on $[a,\infty)$, so it is integrable on $[a,b]$, $b > a$.
Also g is integrable on $[a,b]$ for $b > a$.
By using second mean value theorem, we have

$$\int_b^c f(x)g(x)dx = f(b)\int_b^{c_0} g(x)dx + f(c)\int_{c_0}^c g(x)dx, \quad \text{….. (i)}$$

where $a < b < c_0 < c.$

Since f is given to be bounded on $[a,\infty)$, there exists positive number K such that

$$|f(x)| \le K \text{ for } x \ge a.$$

In particular:

$$|f(b)| \le K \text{ and } |f(c)| \le K. \quad \text{…….. (ii)}$$

Also $\int_a^\infty g(x)dx$ is convergent, by Cauchy criterion, for all $\varepsilon > 0$, there exist positive number B such that

$$\left|\int_b^c g(x)dx\right| < \frac{\varepsilon}{2K} \text{ for } b,c > B.$$

In particular:

$$\left|\int_b^{c_0} g(x)dx\right| < \frac{\varepsilon}{2K} \text{ and } \left|\int_{c_0}^c g(x)dx\right| < \frac{\varepsilon}{2K} \quad \text{……… (iii)}$$

From (i), (ii) and (iii), if $b,c > B$, we have

$$\left|\int_b^c f(x)g(x)dx\right| = \left|f(b)\int_b^{c_0} g(x)dx + f(c)\int_{c_0}^c g(x)dx\right|$$

$$\le |f(b)|\left|\int_b^{c_0} g(x)dx\right| + |f(c)|\left|\int_{c_0}^c g(x)dx\right|$$

$$< K\cdot\frac{\varepsilon}{2K} + K\cdot\frac{\varepsilon}{2K} = \varepsilon.$$

Hence by Cauchy criterion, we have that $\int_a^\infty f(x)g(x)dx$ is convergent.

➢ ***Theorem (Dirichlet)***

If $f(x)$ is bounded, monotone for all $x \ge a$ and $\lim_{x\to\infty} f(x) = 0$. Also $\int_a^X g(x)dx$ is is bounded for all $X \ge a$, then $\int_a^\infty f(x)g(x)dx$ is convergent.

Proof

As f is bounded and monotone on $[a,\infty)$, so it is integrable on $[a,b]$, $b > a$.
Also g is integrable on $[a,b]$ for $b > a$.
By using second mean value theorem, we have

$$\int_b^c f(x)g(x)dx = f(b)\int_b^{c_0} g(x)dx + f(c)\int_{c_0}^c g(x)dx, \quad \text{..... (i)}$$

where $a < b < c_0 < c$.

Since $\int_a^X g(x)dx$ is bounded for all $X \ge a$, there exists positive number K such that

$$\left|\int_a^X g(x)dx\right| \le K \text{ for } X \ge a. \quad \text{...... (ii)}$$

Now for $a < b < c_0$, we have

$$\int_a^{c_0} g(x)dx = \int_a^b g(x)dx + \int_b^{c_0} g(x)dx.$$

This gives

$$\left|\int_b^{c_0} g(x)dx\right| = \left|\int_a^{c_0} g(x)dx - \int_a^b g(x)dx\right|$$

$$\le \left|\int_a^{c_0} g(x)dx\right| + \left|\int_a^b g(x)dx\right|$$

$$\le K + K \quad \text{by using (ii) as } b, c_0 > a.$$

That is, $\left|\int_b^{c_0} g(x)dx\right| \le 2K$. .... (iii)

Similarly, $\left|\int_{c_0}^c g(x)dx\right| \le 2K$. .... (iv)

Also we have $\lim_{x\to\infty} f(x) = 0$, so for all $\varepsilon > 0$, there exist $B > 0$ such that

$$|f(x) - 0| < \frac{\varepsilon}{4K} \text{ for } x > B.$$

In particular, for $b, c > B$, we have

$$|f(b)| < \frac{\varepsilon}{4K} \text{ and } |f(c)| < \frac{\varepsilon}{4K}. \quad \text{...... (v)}$$

From (i), (iii), (iv) and (v), if $b, c > B$, we have

$$\left|\int_b^c f(x)g(x)dx\right| = \left|f(b)\int_b^{c_0} g(x)dx + f(c)\int_{c_0}^{c} g(x)dx\right|$$

$$\leq |f(b)|\left|\int_b^{c_0} g(x)dx\right| + |f(c)|\left|\int_{c_0}^{c} g(x)dx\right|$$

$$< \frac{\varepsilon}{4K}\cdot 2K + \frac{\varepsilon}{4K}\cdot 2K = \varepsilon.$$

Hence by Cauchy criterion, we have that $\int_a^\infty f(x)g(x)dx$ is convergent.

> **Example**

Prove that $\int_0^\infty \frac{\sin x}{x}dx$ is convergent;

Solution:

Since $\frac{\sin x}{x} \to 1$ as $x \to 0$, therefore 0 is not a point of infinite discontinuity.

We write $\int_0^\infty \frac{\sin x}{x}dx = \int_0^1 \frac{\sin x}{x}dx + \int_1^\infty \frac{\sin x}{x}dx$ and note that $\int_0^1 \frac{\sin x}{x}dx$ is a proper integral. Therefore, it is enough to test the convergence of $\int_1^\infty \frac{\sin x}{x}dx$.

Denote $f(x) = \frac{1}{x}$ and $g(x) = \sin x$, where $x \geq 1$.

Clearly $|f(x)| = \frac{1}{x} \leq 1$ for $x \geq 1$ implies $f(x)$ is bounded.

Now for $x_1 \geq x_2 \geq 1$, we have $\frac{1}{x_1} \leq \frac{1}{x_2}$, that is, $f(x_1) \leq f(x_2)$. This gives us $f(x)$ is decreasing for all for all $x \geq 1$.

Also $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\frac{1}{x} = 0$.

Now $\left|\int_1^X g(x)\,dx\right| = \left|\int_1^X \sin x\,dx\right|$

$$= \left|-\cos X + \cos(1)\right| \le \left|\cos X\right| + \left|\cos(1)\right| < 2$$

This gives $\int_1^X g(x)\,dx$ is bounded for every $X \ge 1$.

Hence by Dirichlet theorem $\int_1^\infty f(x)g(x)\,dx = \int_1^\infty \frac{\sin x}{x}\,dx$ is convergent.

➢ ***Example***

Discuss the convergence of $\int_1^\infty \sin x^2\,dx$.

Solution: We write $\sin x^2 = \frac{1}{2x}\cdot 2x\cdot \sin x^2$, i.e.

$$\int_1^\infty \sin x^2\,dx = \int_1^\infty \frac{1}{2x}\cdot 2x\cdot \sin x^2\,dx$$

Take $f(x) = \frac{1}{2x}$ and $g(x) = 2x\sin x^2$, where $x \ge 1$.

Note that $|f(x)| \le \frac{1}{2}$ and $f(x)$ is decreasing for all for all $x \ge 1$, it gives $f(x)$ is bounded and monotone for all $x \ge 1$. Also $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\frac{1}{x} = 0$.

Now $\left|\int_1^X g(x)\,dx\right| = \left|\int_1^X 2x\sin x^2\,dx\right| = \left|-\cos X^2 + \cos(1)\right| < 2$.

This gives $\int_1^X g(x)\,dx$ is bounded for every $X \ge 1$.

Hence by Dirichlet's theorem $\int_1^\infty f(x)g(x)\,dx = \int_1^\infty \frac{1}{2x}\cdot 2x\sin x^2\,dx$

i.e. $\int_1^\infty \sin x^2\,dx$ is convergent.

➢ ***Example***

Discus the convergence of $\int_0^\infty e^{-x}\frac{\sin x}{x}\,dx$.

Solution:

Let $f(x)=e^{-x}$ and $g(x)=\dfrac{\sin x}{x}$, where $x\geq 0$.

As $x\geq 0$, then we have $e^{x}\geq 1$ (as exponential function is increasing).

This gives $\dfrac{1}{e^{x}}\leq 1$, that is, $|f(x)|\leq 1$ for all $x\geq 0$.

Also $f'(x)=-e^{-x}<0$ for all $x\geq 0$. Hence, we conclude that $f(x)$ is bounded and monotonically decreasing for all $x\geq 0$.

Also $\int\limits_{0}^{\infty} g(x)\,dx=\int\limits_{0}^{\infty}\dfrac{\sin x}{x}dx$ is convergent (by previous example).

Hence by Abel's theorem $\int\limits_{0}^{\infty} f(x)g(x)dx=\int\limits_{0}^{\infty} e^{-x}\dfrac{\sin x}{x}dx$ is convergent.

➢ ***Question***

Show that $\int\limits_{0}^{\infty}\dfrac{\sin x}{(1+x)^{\alpha}}dx$ converges for $\alpha>0$.

➢ ***Question***

Show that $\int\limits_{0}^{\infty} e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because$ $\left|e^{-x}\cos x\right|<e^{-x}$ and $\int\limits_{0}^{\infty} e^{-x}\,dx=1$

$\therefore$ the given integral is absolutely convergent. (comparison test).

➢ ***MCQs***

1. An improper integral $\int_{a}^{\infty} f(x)dx$ is conditionally convergent if it is ………………. but not ………………

(A) convergent; divergent
(B) convergent; absolutely convergent
(C) divergent, convergent
(D) divergent; absolutely convergent

2. If f is continuous and decreasing for $x \geq a$ and $\lim_{x\to\infty} f(x) = 0$, then the integral $\int_a^\infty f(x)dx$ ……………………

(A) is convergent. (B) is divergent.
(C) may convergent. (D) absolutely convergent.

3. The integral $\int_1^\infty \frac{f(x)}{x^2} dx$ is ……………….. if f is bounded on $[1,\infty)$.

(A) convergent (B) divergent
(C) conditionally convergent (D) None of these

IMPROPER INTEGRAL OF THE SECOND KIND

➢ ***Definition***

Let f be defined on the half open interval $(a,b]$ (having point of infinite discontinuity at a) and assume that $f \in \mathcal{R}[t,b]$ for every $t \in (a,b]$. Define a function I on $(a,b]$ as follows:

$$I(t) = \int_t^b f\,dx \quad \text{if } \; t \in (a,b]$$

If $\lim_{t\to a+} I(t)$ exists then the integral $\int_{a+}^b f\,dx$ is said to be convergent. Otherwise, $\int_{a+}^b f\,dx$ is said to be divergent.

If $\lim_{x\to a+} I(t) = A$, the number A is called the value of the integral and we write $\int_{a+}^b f\,dx = A.$

Similarly, if f is defined on $[a,b)$ (having point of infinite discontinuity at b) and $f \in \mathcal{R}[a,t]$ for all $x \in [a,b)$ then define $I(t) = \int_a^t f\,dx$ if $t \in [a,b)$. If $\lim_{t \to b-} I(t)$ exists (finite) then we say $\int_a^{b-} f\,dx$ is convergent.

➤ ***Note***

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➤ ***Example***

Discuss the convergence or divergence of $\int_0^b x^{-p}\,dx$ for real p.

Solution:

$f(x) = x^{-p}$ is defined on $(0,b]$ and $f, g \in \mathcal{R}[x,b]$ for every $x \in (0,b]$.

$$I(x) = \int_x^b u^{-p}\,du \quad \text{if} \quad x \in (0,b]$$

$$\int_{0+}^b u^{-p}\,du = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^b u^{-p}\,du = \lim_{\varepsilon \to 0} \left| \frac{u^{1-p}}{1-p} \right|_\varepsilon^b = \lim_{\varepsilon \to 0} \frac{b^{1-p} - \varepsilon^{1-p}}{1-p} \quad , \quad (p \neq 1)$$

$$= \begin{bmatrix} \text{finite} \ , & p < 1 \\ \text{infinite} \ , & p > 1 \end{bmatrix}$$

When $p = 1$, we get $\int_\varepsilon^b \frac{1}{x}\,dx = \log b - \log \varepsilon \to \infty$ as $\varepsilon \to 0$.

$$\Rightarrow \int_{0+}^b x^{-1}\,dx \text{ also diverges.}$$

Hence the integral converges when $p < 1$ and diverges when $p \geq 1$.

➤ ***Note***

If the two integrals $\int_{a+}^c f\,dx$ and $\int_c^{b-} f\,dx$ both converge, we write

$$\int_{a+}^{b-} f\,dx = \int_{a+}^c f\,dx + \int_c^{b-} f\,dx$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^b f\,dx + \int_b^\infty f\,dx \quad \text{which can be written as} \quad \int_{a+}^\infty f\,dx.$$

➢ ***Question:***

Prove that $\int_a^b \frac{dx}{(x-a)^n}$ and $\int_a^b \frac{dx}{(b-x)^n}$ converges if $n<1$. (see [4, page 490])

➢ ***Question***

Examine the convergence of

(*i*) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ (*ii*) $\int_0^1 \frac{dx}{x^2(1+x)^2}$ (*iii*) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Solution: (*i*) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$

Here '0' is the only point of infinite discontinuity of the integrand.

Let $f(x)=\frac{1}{x^{1/3}(1+x^2)}$ and take $g(x)=\frac{1}{x^{1/3}}$.

Then $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ also converges.

(*ii*) $\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2} \text{ and take } g(x)=\frac{1}{x^2}.$$

Then $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge.
Hence the given integral also does not converge.

(*iii*) $\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Here '0' and '1' are the two points of infinite discontinuity of the integrand.

We have

$$f(x)=\frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^{1} f(x)\,dx$.

To examine the convergence of $\int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{x^{1/2}}$

Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1-x)^{1/3}}=1$$

$\because \int_0^{1/2}\frac{1}{x^{1/2}}dx$ converges $\therefore \int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

To examine the convergence of $\int_{1/2}^{1}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to 1}\frac{f(x)}{g(x)}=\lim_{x\to 1}\frac{1}{x^{1/2}}=1$$

$\because \int_{1/2}^{1}\frac{1}{(1-x)^{1/3}}dx$ converges $\because \int_{1/2}^{1}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ ***Question***

Show that the following improper integrals are convergent.

(*i*) $\int_1^{\infty}\sin^2\frac{1}{x}\,dx$ (*ii*) $\int_1^{\infty}\frac{\sin^2 x}{x^2}\,dx$

(*iii*) $\int_0^1\frac{x\log x}{(1+x)^2}\,dx$ (*iv*) $\int_0^1 \log x\cdot\log(1+x)\,dx$

Solution: (*i*) Let $f(x)=\sin^2\frac{1}{x}$ and $g(x)=\frac{1}{x^2}$.

Then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim_{y\to 0}\left(\frac{\sin y}{y}\right)^2=1$

$$\Rightarrow \int_1^\infty f(x)\,dx \text{ and } \int_1^\infty \frac{1}{x^2}\,dx \text{ behave alike.}$$

$$\because \int_1^\infty \frac{1}{x^2}\,dx \text{ is convergent } \therefore \int_1^\infty \sin^2\frac{1}{x}\,dx \text{ is also convergent.}$$

(*ii*) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$$\sin^2 x \le 1 \quad \Rightarrow \quad \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \forall \ x \in (1,\infty)$$

and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges $\therefore \displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

➢ ***Note***

$\displaystyle\int_0^1 \frac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\displaystyle\lim_{x\to 0}\frac{\sin^2 x}{x^2}=1$ so that '0' is not a point of infinite discontinuity. Therefore $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

(*iii*) $\displaystyle\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$

$$\because \log x < x, \quad x\in(0,1) \quad \therefore x\log x < x^2$$

$$\Rightarrow \frac{x\log x}{(1+x)^2} < \frac{x^2}{(1+x)^2}$$

Now $\displaystyle\int_0^1 \frac{x^2}{(1+x)^2}\,dx$ is a proper integral, therefore $\displaystyle\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$ is convergent.

(*iv*) $\displaystyle\int_0^1 \log x\cdot\log(1+x)\,dx$

$$\because \log x < x \qquad \therefore \log(x+1) < x+1$$

$$\Rightarrow \log x\cdot\log(1+x) < x(x+1)$$

$\because \int_0^1 x(x+1)\,dx$ is a proper integral

$\therefore \int_0^1 \log x \cdot \log(1+x)\,dx$ is convergent.

➢ *Note*

(i) $\int_0^a \frac{1}{x^p}dx$ diverges when $p \geq 1$ and converges when $p<1$.

(ii) $\int_a^\infty \frac{1}{x^p}\,dx$ converges iff $p>1$.

➢ *Questions*

Examine the convergence of

(i) $\int_1^\infty \frac{x}{(1+x)^3}dx$ (ii) $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ (iii) $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Solution: (***i***) Let $f(x)=\frac{x}{(1+x)^3}$ and take $g(x)=\frac{1}{x^2}$.

As $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x^3}{(1+x)^3}=1$

Therefore the two integrals $\int_1^\infty \frac{x}{(1+x)^3}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ have identical behaviour for convergence at ∞.

$\because \int_1^\infty \frac{1}{x^2}dx$ is convergent $\therefore \int_1^\infty \frac{x}{(1+x)^3}dx$ is convergent.

(***ii***) Let $f(x)=\frac{1}{(1+x)\sqrt{x}}$ and take $g(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{3/2}}$

We have $\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{x}{1+x}=1$

and $\int_1^\infty \frac{1}{x^{3/2}}dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ is convergent.

(***iii***) Let $f(x)=\frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x)=\frac{1}{x^{1/3}\cdot x^{1/2}}=\frac{1}{x^{5/6}}$

We have $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1$ and $\displaystyle\int_1^\infty \frac{1}{x^{5/6}}dx$ is divergent $\therefore \displaystyle\int_1^\infty f(x)\,dx$ is divergent.

➢ *Question*

Show that $\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx$ is convergent.

Solution: We have

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{a\to\infty}\left[\int_{-a}^{0}\frac{1}{1+x^2}dx+\int_{0}^{a}\frac{1}{1+x^2}dx\right]$$

$$=\lim_{a\to\infty}\left[\int_{0}^{a}\frac{1}{1+x^2}dx+\int_{0}^{a}\frac{1}{1+x^2}dx\right]=2\lim_{a\to\infty}\left[\int_{0}^{a}\frac{1}{1+x^2}dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ *Question*

Show that $\displaystyle\int_{0}^{\infty}\frac{\tan^{-1}x}{1+x^2}dx$ is convergent.

Solution: $\because\ (1+x^2)\cdot\dfrac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\dfrac{\pi}{2}$ as $x\to\infty$

Here $f(x)=\dfrac{\tan^{-1}x}{1+x^2}$ and $g(x)=\dfrac{1}{1+x^2}$

$\displaystyle\int_{0}^{\infty}\frac{\tan^{-1}x}{1+x^2}dx$ & $\displaystyle\int_{0}^{\infty}\frac{1}{1+x^2}dx$ behave alike.

$\because \displaystyle\int_{0}^{\infty}\frac{1}{1+x^2}dx$ is convergent $\therefore$ A given integral is convergent.

➢ *Question*

Show that $\displaystyle\int_{0}^{1}\frac{e^{-x}}{\sqrt{1-x^4}}dx$ is convergent.

Solution: $\because\ e^{-x}<1$ and $1+x^2>1$ for all $x\in(0,1)$.

$$\therefore\ \frac{e^{-x}}{\sqrt{1-x^4}}<\frac{1}{\sqrt{(1-x^2)(1+x^2)}}<\frac{1}{\sqrt{1-x^2}}$$

Also $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \lim_{\varepsilon\to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent. (by comparison test)

References:

1. Tom M. Apostol, Mathematical Analysis, 2nd Edition, MA: Addison-Wesley, 1974.
2. S.C. Malik and Savita Arora, Mathematical Analysis, 2nd Edition, New Age International, 1992.
3. S. Narayan and M.D. Raisinghania, Elements of Real Analysis, S. Chand & Company, New Delhi, 2007.
4. D. Chatterjee, Real Analysis, 2nd Edition, PHI Learning Private Limited, Delhi, 2012.

For online resource related to the course please visit: www.mathcity.org/atiq/fa20-mth322