Chapter 6 – Riemann Integrals

Course Title: Real Analysis II Course Code: MTH322
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Course URL: *www.mathcity.org/atiq/fa19-mth322*

We assume that the reader is familiar at least informally with the integral from a calculus course (FSc or BSc). In addition, they know about integrating a function on an interval $[a,b]$ and know few of its interpretation as the "area under the graph", or its many applications to physics, engineering, economics, etc. Here our aim is to focus on the purely mathematical aspects of the integral. However, we first recall some basic terms that will be frequently used (see [1]).

Partition

Let $[a,b]$ be a given interval. By a partition P of $[a,b]$, we mean a finite set of points $x_0, x_1, ..., x_n$, where

$$a = x_0 < x_1 < ... < x_{n-1} < x_n = b.$$

The points of P are used to divide $[a,b]$ into n non-overlapping subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], ..., [x_{n-1}, x_n].$$

Each sub-interval is called a *component* of the partition.
Obviously, corresponding to different choices of the points x_i we shall have different partition.
The maximum of the length of the components is defined as the *norm* of the partition and it is denoted by $\|P\|$, that is,

$$\|P\| = \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}.$$

Examples

Consider an interval $[1,10]$ and following partitions of this interval.

$P_1 = \{1, 2, 3, 10\}$,

$P_2 = \{1, 2, 3, 6, 9, 10\}$,

$$P_3 = \left\{1, 1 + \frac{9}{100}, 1 + 2\left(\frac{9}{100}\right), 1 + 3\left(\frac{9}{100}\right), ..., 1 + 99\left(\frac{9}{100}\right), 10\right\}$$

and more generally for any positive integer *n*, we can write

$$P_4 = \left\{1, 1 + \frac{9}{n}, 1 + 2\left(\frac{9}{n}\right), 1 + 3\left(\frac{9}{n}\right), ..., 1 + (n-1)\left(\frac{9}{n}\right), 1 + n\left(\frac{9}{n}\right) = 10\right\}.$$

Also note that $\|P_1\| = 7$, $\|P_2\| = 3$, $\|P_3\| = \dfrac{9}{100}$, $\|P_4\| = \dfrac{9}{n}$.

Refinement of a Partition

Let P and P^* be two partitions of an interval $[a,b]$ such that $P \subset P^*$ i.e. P^* contains all the points of P and possibly some other points as well. Then P^* is said to be a *refinement* of P.

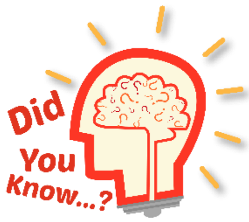
Example

Note that P_2 is refinement of P_1.

Remark

Note that if $P_1 \subseteq P_2$ implies $\|P_1\| \geq \|P_2\|$, that is, refinement of a partition decreases its norm but the convers does not necessarily hold.

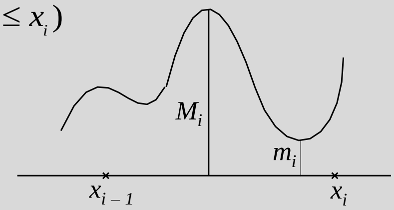
- How many partition can be made for any closed interval $[a,b]$?
- Can you write two different partitions of $[1,3]$ with same norm?
- Can you write two partitions P_1 and P_2 of $[0,5]$ such that $\|P_1\| < \|P_2\|$ but $P_1 \not\supset P_2$.



Riemann Integral

Let f be a real-valued function defined and bounded on $[a,b]$.
Corresponding to each partition P of $[a,b]$, we put

$$M_i = \sup f(x) \qquad (x_{i-1} \leq x \leq x_i)$$
$$m_i = \inf f(x) \qquad (x_{i-1} \leq x \leq x_i)$$



We define upper and lower sums as

$$U(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i$$

and
$$L(P,f) = \sum_{i=1}^{n} m_i \, \Delta x_i \, ,$$

where $\Delta x_i = x_i - x_{i-1} \qquad (i = 1, 2, \ldots, n)$.

Now we define
$$\overline{\int_a^b} f \, dx = \inf U(P,f) \, , \ldots\ldots\ldots\ldots\ldots (i)$$

$$\underline{\int_a^b} f \, dx = \sup L(P,f) \, , \ldots\ldots\ldots\ldots\ldots (ii)$$

where the infimum and the supremum are taken over all partitions P of $[a,b]$. Then $\overline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} f(x)dx$ are called the upper and lower Riemann integrals of f over $[a,b]$ respectively.

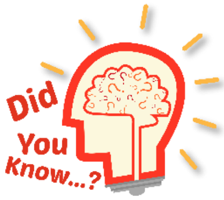
In case the upper and lower integrals are equal, we say that f is Riemann integrable on $[a,b]$ and we write $f \in \mathcal{R}[a,b]$, where $\mathcal{R}[a,b]$ denotes the set of Riemann integrable functions over $[a,b]$.

The common value of (i) and (ii) is denoted by $\int_a^b f \, dx$ or by $\int_a^b f(x) \, dx$.

Which is known as the Riemann integral of f over $[a,b]$.

Exercises

1. Let $P_1 = \{1,2,3,4,5\}$ be partition of $[1,5]$ and $f:[1,5]\to\mathbb{R}$ be function defined by $f(x)=x^2$. Find $U(P_1, f)$ and $L(P_1, f)$.
2. Let $P_2 = \{0, \frac{\pi}{3}, \frac{\pi}{6}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi\}$ be partition of $[0,\pi]$ and $f:[0,\pi]\to\mathbb{R}$ be function defined by $f(x)=\sin x$. Find $U(P_1, f)$ and $L(P_1, f)$.



- If a function f is increasing on $[a,b]$, then $\max_{x\in[a,b]} f(x) = f(b)$ and $\min_{x\in[a,b]} f(x) = f(a)$.
- If a function f is decreasing on $[a,b]$, then what about its maximum and minimum value over interval $[a,b]$.
- Let f be bounded on interval $[a,b]$. Can you guess its maximum and minimum value over interval $[a,b]$.

Theorem

The upper and lower integrals are defined for every bounded function f over interval $[a,b]$.

Proof

Since f is bounded on $[a,b]$, so its supremum and infimum values exist over $[a,b]$.

Take M and m to be the maximum and minimum value of f in $[a,b]$ respectively, that is,

$$m \le f(x) \le M \qquad (a\le x\le b)$$

Let M_i and m_i denote the supremum and infimum of f in $[x_{i-1}, x_i]$ for certain partition P of $[a,b]$ respectively. Then

$$M_i \le M \quad \text{and} \quad m_i \ge m \quad (i=1,2,.....,n).$$

This gives

$$L(P,f) = \sum_{i=1}^{n} m_i\,\Delta x_i \ge \sum_{i=1}^{n} m\,\Delta x_i \qquad (\Delta x_i = x_i - x_{i-1})$$

$$\Rightarrow L(P,f) \ge m\sum_{i=1}^{n}\Delta x_i$$

But
$$\sum_{i=1}^{n}\Delta x_i = (x_1-x_0)+(x_2-x_1)+(x_3-x_2)+....+(x_n-x_{n-1}),$$
$$= x_n - x_0 = b-a.$$

This gives

$$L(P,f) \ge m(b-a). \quad .... \text{(i)}$$

Similarity one can have

$$U(P,f) \le M(b-a). \quad .... \text{(ii)}$$

Also we have $L(P,f) \le U(P,f)$ .... (iii)

Combining (i), (ii) and (iii), we have

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

This shows that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set over all the partitions P of $[a,b]$.
This gives the upper and lower integrals are defined for every function f over interval.

Remark: In mathematics, different author approached to Riemann integral with the same ideas but slightly different than above e.g. see [2] and [3].

Theorem

If P^* is a refinement of P, then following holds:

(i) $L(P,f) \leq L(P^*,f)$,

(ii) $U(P,f) \geq U(P^*,f)$.

Theorem

Let f be a real and bounded function defined on $[a,b]$. Then

$$\sup L(P,f) \leq \inf U(P,f) \qquad \text{i.e.} \int_{\underline{a}}^{b} f\,dx \leq \int_{a}^{\bar{b}} f\,dx .$$

Theorem (Condition of Integrability or Cauchy's Criterion for Integrability.)

A function $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$. there exists a partition P such that $U(P,f) - L(P,f) < \varepsilon$.

Theorem

If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$ and

$$\left| \int_a^b f\,dx \right| \leq \int_a^b |f|\,dx .$$

Theorem (Fundamental Theorem of Calculus)

If $f \in \mathcal{R}[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) .$$

Theorem

Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$. Then $f \in \mathcal{R}[a,b]$.

References:

1. Walter Rudin, Principles of mathematical analysis. Vol. 3. New York: McGraw-hill, 1964.
2. Bartle, Robert Gardner, and Donald R. Sherbert. Introduction to real analysis. Vol. 2. New York: Wiley, 2000.
3. Tom M. Apostol, Mathematical Analysis, 2nd Edition, MA: Addison-Wesley, 1974.