Ch 01: Improper Integrals of 1st and 2nd Kinds	
Course Title: Real Analysis II	Course Code: MTH322
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"The objective of this chapter is to learn about different kind of improper integral. To learn the meaning of convergence and divergence of improper integrals. Learn the theory to develop different tests and techniques to find convergence or divergence of improper integrals"

We discussed Riemann-Stieltjes's integrals of the form $\int_a^b f\, d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. To extend the concept, we shall relax some condition on definite integral like f on finite interval or boundedness of f on finite interval.

First of all we recall few things about symbol $+\infty$ (or ∞) and $-\infty$. These symbols don't behave like usual real numbers. Historically, with these concepts, mathematicians were never very comfortable and these were some sources of much confusion and debate. Indeed, the concepts of infinite sets and infinity took nearly a century for a definite implication. To get the idea of infinity one can read the articles at following URLs:

- https://www.mathsisfun.com/numbers/infinity.html
- https://en.wikipedia.org/wiki/Infinity

An integral is said to be improper integral if either the function f is unbounded on $[a,b]$ or the interval of integration is unbounded. Now we are going to give formal definitions of improper integrals.

- ***Definition***

The integral $\int_a^b f\, d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=\infty$ or *both* i.e. one or both integration limits are infinite.

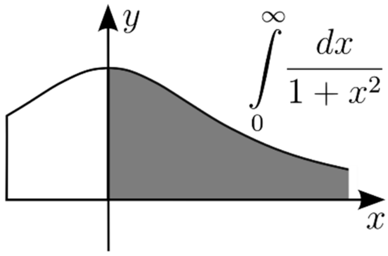
- ***Definition***

The integral $\int_a^b f\, d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded with infinite discontinuity at one or more points of $a \le x \le b$.

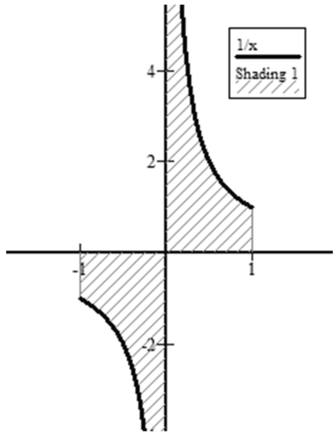
- ***Remark:***

Some time we deal with an improper integral which involves both kinds of integral at once. It is known as improper integral of mixed kind. It can be break in to the sum of improper integrals of first and second kinds.

➢ *Examples*



- $\int_0^{\infty}\frac{1}{1+x^2}dx$, $\int_{-\infty}^{1}\frac{1}{x-2}dx$ and $\int_{-\infty}^{\infty}(x^2+1)dx$ are examples of improper integrals of first kind.



- $\int_{-1}^{1}\frac{1}{x}dx$ and $\int_0^{1}\frac{1}{2x-1}dx$ are examples of improper integrals of second kind.

- $\int_0^{\infty}\frac{1}{x}dx$ is an example of improper integral of mixed kind and it can be written as follow:
$$\int_0^{\infty}\frac{1}{x}dx=\int_0^{1}\frac{1}{x}dx+\int_1^{\infty}\frac{1}{x}dx$$

➢ *Notations*

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

➢ *MCQs*

(i) Which of the following is an improper integral of 1st kind.

(I) $\int_1^{2}\frac{1}{x}dx$ (II) $\int_1^{\infty}\frac{1}{x^2}dx$ (III) $\int_{-\infty}^{\infty}(2t+1)dt$

A. I and III only B. III only
C. II only D. II and III only

(ii) Which of the following is an improper integral of 2nd kind.

(I) $\int_{-1}^{0}\frac{1}{x}dx$ (II) $\int_2^{3}\frac{1}{x^2-1}dx$ (III) $\int_0^{1}\tan\frac{\pi t}{2}dt$

A. I and III only B. III only
C. I only D. II and III only

(ii) The integral $\int_0^1\frac{\sin\theta}{\theta}d\theta$ is ………………..

A. Improper integral of 1st kind. B. Improper integral of 2nd kind.
C. Improper integral of mixed kind. D. None of these.

IMPROPER INTEGRAL OF THE FIRST KIND

➢ ***Definition***

Assume that $f \in R(\alpha; a, b)$ for every $b \geq a$. Keep a, α and f fixed and define a function I on $[a, \infty)$ as follows:

$$I(b) = \int_a^b f(x)\, d\alpha(x) \quad \text{if} \quad b \geq a.$$

The integral $\int_a^\infty f(x)\, d\alpha(x)$ is said to converge if the $\lim\limits_{b\to\infty} I(b)$ exists (finite). Otherwise, $\int_a^\infty f\, d\alpha$ is said to diverge.

If the $\lim\limits_{b\to\infty} I(b)$ exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\, d\alpha = A$.

➢ ***Remark***

- If $\int\limits_a^\infty f\, d\alpha$ is convergent(divergent), then $\int\limits_c^\infty f\, d\alpha$ is convergent(divergent) for $c > a$.
- If $\int\limits_c^\infty f\, d\alpha$ is convergent (divergent), then $\int\limits_a^\infty f\, d\alpha$ is convergent (divergent) for $a < c$ if f in bounded in $[a, c]$.

➢ ***Example***

Consider and integral $\int_1^\infty \frac{1}{x^2} dx$. Discuss its convergence or divergence.

Solution

Let $I(b) = \int\limits_1^b \frac{1}{x^2} dx$, where $b \geq 1$.

Then $I(b) = \int\limits_1^b x^{-2}\, dx = -x^{-1}\Big|_1^b = -\frac{1}{x}\Big|_1^b = 1 - \frac{1}{b}$.

Now $\lim\limits_{b\to\infty} I(b) = \lim\limits_{b\to\infty}\left(1 - \frac{1}{b}\right) = 1$.

Hence $\int_1^\infty \frac{1}{x^2} dx$ is convergent.

➢ ***Example***

Consider and integral $\int_1^\infty x^{-p}\,dx$, where p is any real number. Discuss its convergence or divergence.

Solution

Let $I(b)=\int\limits_1^b x^{-p}\,dx$ where $b\ge 1$.

Then $I(b)=\int\limits_1^b x^{-p}\,dx=\left.\dfrac{x^{1-p}}{1-p}\right|_1^b=\dfrac{1-b^{1-p}}{p-1}$ if $p\neq 1$.

If $b\to\infty$, then $b^{1-p}\to 0$ for $p>1$ and $b^{1-p}\to\infty$ for $p<1$.

Therefore we have

$$\lim_{b\to\infty} I(b)=\lim_{b\to\infty}\frac{1-b^{1-p}}{p-1}=\begin{cases}\infty & \textit{if} \quad p<1,\\ \dfrac{1}{p-1} & \textit{if} \quad p>1.\end{cases}$$

Now if $p=1$, we get $\int_1^b x^{-1}\,dx=\log b\to\infty$ as $b\to\infty$.

Hence we concluded: $\int\limits_1^\infty x^{-p}\,dx=\begin{cases}\textit{diverges} & \textit{if} \quad p\le 1,\\ \dfrac{1}{p-1} & \textit{if} \quad p>1.\end{cases}$

➢ ***Example***

Is the integral $\int\limits_0^\infty \sin 2\pi x\,dx$ converges or diverges?

Solution:

Consider $I(b)=\int\limits_0^b \sin 2\pi x\,dx$, where $b\ge 0$.

We have $\int\limits_0^b \sin 2\pi x\,dx=\left.\dfrac{-\cos 2\pi x}{2\pi}\right|_0^b=\dfrac{1-\cos 2\pi b}{2\pi}$.

Also $\cos 2\pi b\to l$ as $b\to\infty$, where l has values between -1 and 1, that is, limit is not unique.

Therefore the integral $\int\limits_0^\infty \sin 2\pi x\,dx$ diverges.

➢ ***Exercises***

- Show that $\int_a^\infty \dfrac{1}{x^p}\,dx$ converges if $p>1$.
- Evaluate: (i) $\int_{-\infty}^0 \sin x\,dx$ (ii) $\int_{-\infty}^0 e^x\,dx$

➢ *Note*

If $\int_{-\infty}^{a} f\,d\alpha$ and $\int_{a}^{\infty} f\,d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^{\infty} f\,d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^{\infty} f\,d\alpha = \int_{-\infty}^{a} f\,d\alpha + \int_{a}^{\infty} f\,d\alpha .$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^{\infty} fd\alpha$ converges, its value is equal to the limit: $\lim\limits_{b\to+\infty}\int_{-b}^{b} fd\alpha$.

For improper integral of first kind we will discuss the results for integral of the type $\int_{a}^{\infty} f\,d\alpha$. The results for other cases can be derived in a similar manner.

➢ *Exercises*

Evaluate the improper integral $\int_{-\infty}^{\infty} e^{x}\,dx$.

➢ *MCQ*

(i) For what value of m the integral $\int_{1}^{\infty}\frac{dx}{x^{m+1}}$ is convergent.

A. $m>1$ B. $m\le 1$ C. $m>0$ D. $m\ge 0$

(ii) Which of the following integrals is divergent.

A. $\int_{2}^{\infty}\frac{dx}{x^2}$ B. $\int_{1}^{\infty}\frac{dt}{t^{\alpha+1}},\ \alpha>0$ C. $\int_{1}^{\infty} z^{-\frac{3}{2}}dz$ D. $\int_{1}^{\infty} x^{\frac{3}{2}}dx$

(iii) If $\int_{2}^{\infty} fdx$ is convergent then ……………… is convergent.

A. $\int_{0}^{\infty} fdx$ B. $\int_{1}^{\infty} fdx$ C. $\int_{3}^{\infty} fdx$ D. $\int_{-2}^{\infty} fdx$

➢ *Review:*

- A function f is said to be increasing, if for all $x_1,x_2\in D_f$ (domain of f) and $x_1\le x_2$ implies $f(x_1)\le f(x_2)$.
- A function f is said to be bounded if there exist some positive number μ such that $|f(t)|\le\mu$ for all $t\in D_f$.
- If f is define on $[a,+\infty)$ and $\lim\limits_{x\to\infty} f(x)$ exists then f is bounded on $[a,+\infty)$.
- If $f\in R(\alpha;a,b)$ and $c\in[a,b]$, then $\int_{a}^{b} fd\alpha=\int_{a}^{c} fd\alpha+\int_{c}^{b} fd\alpha$.
- If $f\in R(\alpha;a,b)$ and $f(x)\ge 0$ for all $x\in[a,b]$, then $\int_{a}^{b} fd\alpha\ge 0$.
- If f is monotonically increasing and bounded on $[a,+\infty)$, then $\lim\limits_{x\to\infty} f(x)=\sup\limits_{x\in[a,\infty)} f(x)$.

- If $f, g \in R(\alpha; a, b)$ and $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f d\alpha \le \int_a^b g d\alpha$.

> ### ➢ *Theorem*
> Assume that α is monotonically increasing on $[a, +\infty)$ and suppose that $f \in R(\alpha; a, b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^\infty f\, d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that
> $$\int_a^b f\, d\alpha \le M \quad \text{for every } b \ge a.$$

Proof

Let $I(b) = \int_a^b f\, d\alpha$ for $b \ge a$.

First suppose that $\int_a^\infty f\, d\alpha$ is convergent, then $\lim\limits_{b \to +\infty} I(b)$ exists, that is, $I(b)$ is bounded on $[a, +\infty)$.

So there exists a constant $M > 0$ such that
$$|I(b)| < M \text{ for every } b \ge a.$$

As $f(x) \ge 0$ for each $x \ge a$, therefore $\int_a^b f\, d\alpha \ge 0$.

This gives $I(b) = \int_a^b f\, d\alpha \le M$ for every $b \ge a$.

Conversely, suppose that there exists a constant $M > 0$ such that $\int_a^b f\, d\alpha \le M$ for every $b \ge a$. This give $|I(b)| \le M$ for every $b \ge a$.

That is, I is bounded on $[a, +\infty)$.

Now for $b_2 \ge b_1 > a$, we have
$$I(b_2) = \int_a^{b_2} f\, d\alpha = \int_a^{b_1} f\, d\alpha + \int_{b_1}^{b_2} f\, d\alpha$$
$$\ge \int_a^{b_1} f\, d\alpha = I(b_1), \qquad \because \int_{b_1}^{b_2} f\, d\alpha \ge 0 \text{ as } f(x) \ge 0 \text{ for all } x \ge a.$$

This gives I is monotonically increasing on $[a, +\infty)$.

As I is monotonically increasing and bounded on $[a, +\infty)$, therefore $\lim\limits_{b \to \infty} I(b)$ exists, that is, $\int_a^\infty f\, d\alpha$ converges.

➢ ***Theorem: (Comparison Test)***

Assume that α is monotonically increasing on $[a,+\infty)$ and $f \in R(\alpha;a,b)$ for every $b \geq a$. If $0 \leq f(x) \leq g(x)$ for every $x \geq a$ and $\int_a^\infty g d\alpha$ converges, then $\int_a^\infty f\, d\alpha$ converges and we have $\int_a^\infty f\, d\alpha \leq \int_a^\infty g\, d\alpha$.

Proof

Let $I_1(b) = \int_a^b f\, d\alpha$ and $I_2(b) = \int_a^b g\, d\alpha$, $b \geq a$.

Since $0 \leq f(x) \leq g(x)$ for every $x \geq a$, therefore

$$\int_a^b f\, d\alpha \leq \int_a^b g\, d\alpha,$$

that is, $I_1(b) \leq I_2(b)$ .... (i)

Since $\int_a^\infty g\, d\alpha$ converges, there exists a constant $M > 0$ such that

$$\int_a^b g\, d\alpha \leq M \quad , \quad b \geq a \text{ ....}(ii)$$

From (i) and (ii) we have $I_1(b) \leq M$ for every $b \geq a$.

This implies $\int_a^\infty f d\alpha$ converges, that is, $\lim\limits_{b\to\infty} I_1(b)$ exists and is finite.

Also $\lim\limits_{b\to\infty} I_1(b) \leq \lim\limits_{b\to\infty} I_2(b) \leq M$,

this gives $\int_a^\infty f\, d\alpha \leq \int_a^\infty g\, d\alpha$.

➢ ***Example***

Is the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ convergent or divergent?

Solution:

Since $\sin^2 x \leq 1$ for all $x \in [1,+\infty)$, therefore $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for all $x \in [1,+\infty)$.

This gives $\int_1^\infty \frac{\sin^2 x}{x^2} dx \leq \int_1^\infty \frac{1}{x^2} dx$.

Now $\int_1^\infty \frac{1}{x^2} dx$ is convergent, therefore $\int_1^\infty \frac{\sin^2 x}{x^2} dx$ is convergent.

➢ *MCQs*

(i) A function f is said to be bounded if there exist a positive number α such that ………….. for all $t \in D_f$ (domain of f)

A. $f(t) \le \alpha$ B. $|f(t)| \le \alpha$ C. $|f(t)| > \alpha$ D. $f(t) > \alpha$

(i) If $f : [a,b] \to (0,\infty)$ is a bounded function then

A. $\int_a^b f(t)dt \ge 0$ B. $\int_a^\infty f(t)dt \ge 0$ C. $f(t) \ge \mu$ for $\mu \in \mathbb{R}$ D. None of these

➢ *Review:*

- For all $a,b,c \in \mathbb{R}$, $|a-b| < c \iff c-b < a < c+b$ or $c-a < b < c+a$.
- If $\lim\limits_{x\to\infty} f(x) = m$, then for all real $\varepsilon > 0$, there exists $N > 0$ such that
$$|f(x)-m| < \varepsilon \text{ whenever } |x| > N.$$
- If $\int_a^\infty f\,d\alpha$ converges(diverges), then $\int_N^\infty f\,d\alpha$ converges(diverges) if $N > a$.
- If $\int\limits_N^\infty f\,d\alpha$ is convergent (divergent), then $\int\limits_a^\infty f\,d\alpha$ is convergent (divergent) for $a < N$ if f is bounded in $[a,N]$.

➢ *Theorem (Limit Comparison Test)*

Assume that α is monotonically increasing on $[a,+\infty)$. Suppose that $f \in R(\alpha;a,b)$ and that $g \in R(\alpha;a,b)$ for every $b \ge a$, where $f(x) \ge 0$ and $g(x) \ge 0$ for $x \ge a$. If
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$
then $\int\limits_a^\infty f\,d\alpha$ and $\int\limits_a^\infty g\,d\alpha$ both converge, or both diverge.

Proof

Suppose $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = 1$, then for all real $\varepsilon > 0$, we can find some $N > 0$, such that
$$\left|\frac{f(x)}{g(x)} - 1\right| < \varepsilon \qquad \forall\ x > N \ge a.$$
$$\Rightarrow\ 1-\varepsilon < \frac{f(x)}{g(x)} < 1+\varepsilon \qquad \forall\ x > N \ge a.$$
If we choose $\varepsilon = \dfrac{1}{2}$, then we have

$$\frac{1}{2} < \frac{f(x)}{g(x)} < \frac{3}{2} \qquad \forall \;\; x > N \geq a.$$

This implies $g(x) < 2f(x)$ ……..(i) and $2f(x) < 3g(x)$ ……...(ii)

From (i) $\int_N^\infty g\,d\alpha < 2\int_N^\infty f\,d\alpha$,

so if $\int_a^\infty f\,d\alpha$ converges, then $\int_N^\infty f\,d\alpha$ converges and hence by comparison test we get $\int_N^\infty g\,d\alpha$ is convergent, which implies $\int_a^\infty g\,d\alpha$ is convergent.

Now if $\int_a^\infty g\,d\alpha$ diverges, then $\int_N^\infty g\,d\alpha$ diverges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is divergent, which implies $\int_a^\infty f\,d\alpha$ is divergent.

From (ii), we have $2\int_N^\infty f\,d\alpha < 3\int_N^\infty g\,d\alpha$,

so if $\int_a^\infty g\,d\alpha$ converges, then $\int_N^\infty g\,d\alpha$ converges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is convergent, which implies $\int_a^\infty f\,d\alpha$ is convergent.

Now if $\int_a^\infty f\,d\alpha$ diverges, then $\int_N^\infty f\,d\alpha$ diverges and hence by comparison test we get $\int_N^\infty g\,d\alpha$ is divergent, which implies $\int_a^\infty g\,d\alpha$ is divergent.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

➢ ***Note***

The above theorem also holds if $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = c$, provided that $c > 0$. If $c = 0$, we can only conclude that convergence of $\int_a^\infty g d\alpha$ implies convergence of $\int_a^\infty f d\alpha$.

➢ ***Questions***

(i) Suppose $f(x)$ and $g(x)$ are positive integrable functions for $x > a$. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c$, where $c > 0$, then $\int\limits_a^\infty f(x)\,dx$ and $\int\limits_a^\infty g(x)dx$ both converge or both diverge.

(ii) Suppose $f(x)$ and $g(x)$ are positive integrable functions for $x > a$. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$, then convergence of $\int_a^\infty g(x)dx$ implies convergence of $\int_a^\infty f(x)dx$.

(iii) Suppose $f(x)$ and $g(x)$ are positive integrable functions for $x > a$.. If $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, then convergence of $\int_a^\infty f(x)dx$ implies convergence of $\int_a^\infty g(x)dx$.

➢ ***Example***

For every real p, the integral $\int\limits_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int\limits_1^\infty \frac{1}{x^2}dx$.

Let $f(x) = e^{-x}x^p$ and $g(x) = \frac{1}{x^2}$.

Now $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \frac{e^{-x}x^p}{1/x^2}$

$$\Rightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} e^{-x}x^{p+2} = \lim_{x\to\infty} \frac{x^{p+2}}{e^x} = 0. \quad \text{(find this limit yourself)}$$

Since $\int\limits_1^\infty \frac{1}{x^2}dx$ is convergent, therefore the given integral $\int\limits_1^\infty e^{-x}x^p\,dx$ is also convergent.

➢ ***Remark***

It is easy to show that if $\int\limits_a^\infty f\,d\alpha$ and $\int\limits_a^\infty g\,d\alpha$ are convergent, then

- $\int_a^\infty (f \pm g)d\alpha$ is convergent.
- $\int_a^\infty cf\,d\alpha$, where c is some constant, is convergent.