Chapter 1: Real Number System

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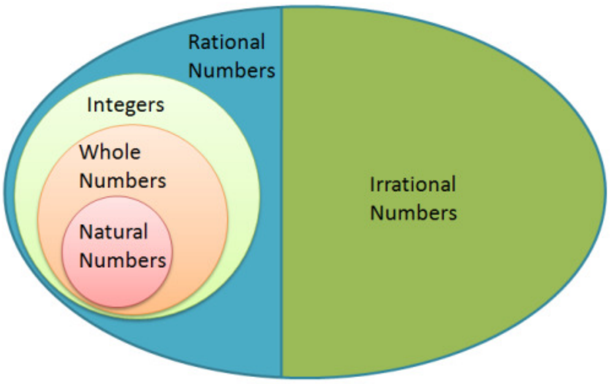
You don't have to be a mathematician to have a feel for numbers.
John Forbes Nash, Jr.

Historical Note: Numbers are like blood cells in the body of mathematics. Just as the understanding of anatomy and physiology of an organic system depends much on the knowledge of blood cells, so does the understanding of mathematics depend on the knowledge of numbers. In fact, a major part of mathematics bases its development on numbers and their multifarious properties.



It is very difficult, if not impossible, to spell out as to when did the concept of numbers came to human civilization. History, however, reveals that a formal study of numbers started almost five thousand years ago and that too by the Hindus who studied numbers purely as abstract symbols and were very proficient not only in discovering very large and very small numbers but also in using them effectively. Evidences are there that the Greek studied numbers purely on geometric conceptualization as they were very proficient in geometry and as a result had a relatively retarded progress. The greatest contribution of the Hindus is the discovery of zero, negative numbers and the decimal scale of representing numbers. In fact, they showed commendable mastery over rational numbers as early as the 5th century after Christ. The formal rigorous study of numbers, however began even much later when mathematics faced several foundational crises. All these started in the 17th century but reached a climax after George Cantor (1845-1925) in 18th and 19th century. The contribution of 20th century in this regard is, on the one hand, stunning remarkable but on the other hand, devastating from the foundation point of view. The work and criticism by Russell (1872-1970), Lowenheim (1887-1940), Skolem (1887-1963) and Church (1903-1995) have been instrumental in bringing about a drastic change in our attitude and approach towards mathematics in general. In our modern approach, we start directly from real numbers defined axiomatically and then pass on to the related concept. (for more details see [4]). Many authors have different approach to define set of real numbers. Here we use the idea of Rudin introduced in [1].

For the understanding of the topic we consider that we know about $\mathbb{N}=\{1,2,3,...\}$, $\mathbb{Z}=\{0,\pm1,\pm2,...\}$ and $\mathbb{Q}=\left\{\frac{p}{q} \mid p,q\in\mathbb{Z}\wedge q\neq 0\right\}$. Some authors define these sets after defining the set of real numbers.

The real number system can be described as a **"complete ordered field".** Therefore, let's discusses and understand these notions first.

❖ Order

Let *S* be a non-empty set. An *order* on a set *S* is a relation denoted by "<" with the following two properties

(i) If $x, y \in S$,

then one and only one of the statements $x < y$, $x = y$, $y < x$ is true.

(ii) If $x, y, z \in S$ and if $x < y$, $y < z$ then $x < z$.

❖ Examples:

Consider the following sets:

- $A = \{1, 2, 3, ..., 50\}$
- $B = \{a, e, i, o, u\}$
- $C = \{x : x \in \mathbb{Z} \wedge x^2 \leq 19\}$

There is an order on *A* and *C* but there is no order on *B* (we can define order on *B*).

❖ Ordered Set

A set is said to be *ordered set* if an order is defined on *S*.

❖ Examples

The set $\{2, 4, 6, 7, 8, 9\}$, $\mathbb{Z}$ and $\mathbb{Q}$ are examples of ordered set with standard order relation.

The set $\{a, b, c, d\}$ and $\{\alpha, \beta, \chi, \vartheta\}$ are examples of set with no order. Also set of complex numbers have no order.

❖ Bound

Upper Bound

Let *S* be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for all $x \in E$, then we say that *E* is bounded above. The number β is known as upper bound of *E*.

Lower Bound

Let *S* be an ordered set and $E \subset S$. If there exists a $\beta \in S$ such that $x \geq \beta$ for all $x \in E$, then we say that *E* is bounded below. And β is known as lower bound of *E*.

❖ Example

(i) Consider $S = \{1, 2, 3, ..., 50\}$ and $E = \{5, 10, 15, 20\}$.

Set of all lower bounds of $E=\{1,2,3,4,5\}$.

Set of all upper bounds of $E=\{20,21,22,...,50\}$.

(ii) Consider $S=\mathbb{N}$, $E=\{1,2,3,...,100\}$ and $F=\{10,20,30,...\}$.

Set of lower bounds of E = $\{1\}$.

Set of lower bounds of F = $\{1,2,3,...,10\}$.

Set of upper bounds of E = $\{100,101,102,...\}$.

Set of upper bounds of F = φ.

❖ **Least Upper Bound (Supremum)**

Suppose S is an ordered set, $E \subset S$ and E is bounded above. Suppose there exists an $\alpha \in S$ such that

(i) α is an upper bound of E.

(ii) If $\gamma < \alpha$ for $\gamma \in S$, then γ is not an upper bound of E.

Then α is called *least upper bound* of E or *supremum* of E and written as $\sup E = \alpha$.
In other words, α is the least member of the set of upper bound of E.

❖ **Example**

Consider $S=\{1,2,3,...,50\}$ and $E=\{5,10,15,20\}$.

(i) It is clear that 20 is upper bound of E.

(ii) For $\gamma \in S$ if $\gamma < 20$ then clearly γ is not an upper bound of E. Hence $\sup E = 20$.

❖ **Greatest Lower Bound (Infimum)**

Suppose S is an ordered set, $E \subset S$ and E is bounded below. Suppose there exists a $\beta \in S$ such that

(i) β is a lower bound of E.

(ii) If $\beta < \gamma$ for $\gamma \in S$, then γ is not a lower bound of E.

Then β is called *greatest lower bound* of E or *infimum* of E and written as $\inf E = \beta$.
In other words, β is the greatest member of the set of lower bound of E.

❖ **Example**

Consider $S=\{1,2,3,...,50\}$ and $E=\{5,10,15,20\}$.

(i) It is clear that 5 is lower bound of E.

(ii) For $\gamma \in S$ if $5 < \gamma$, then clearly γ is not lower bound of E. Hence $\inf E = 5$.

❖ **Remark**

If α is supremum or infimum of *E*, then α may or may not belong to *E*.
Let $E_1 = \{r : r \in \mathbb{Q} \wedge r < 0\}$ and $E_2 = \{r : r \in \mathbb{Q} \wedge r \geq 0\}$.
Then $\sup E_1 = \inf E_2 = 0$ but $0 \notin E_1$ and $0 \in E_2$.

❖ Example

Let E be the set of all numbers of the form $1/n$, where *n* is the natural numbers, that is,

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}.$$

Then $\sup E = 1$ which is in *E*, but $\inf E = 0$ which is not in *E*.

❖ Least Upper Bound Property

A set *S* is said to have the *least upper bound property* if the followings is true
(i) S is non-empty and ordered.
(ii) If $E \subset S$ and *E* is non-empty and bounded above then sup*E* exists in *S*.
Greatest lower bound property can be defined in a similar manner.

❖ Remark

The above property is known as completeness axiom or LUB axiom or continuity axiom or order completeness axiom.

The set of rational numbers $\mathbb{Q}$ doesn't satisfy completeness axiom. Consider a set $E = \{x : x \in \mathbb{Q} \wedge x^2 \leq 2\}$. One can prove that supremum of E doesn't exist in $\mathbb{Q}$.

To prove it, consider r is the supremum of E, then clearly $r^2 = 2$.

We have left for the reader to prove that there doesn't exist any rational number r, which satisfy the above expression (or alternatively $\sqrt{2}$ is not a rational number).

❖ Theorem

Suppose *S* is an ordered set with least upper bound property, $B \subset S$, *B* is non-empty and is bounded below. Let *L* be set of all lower bound of *B*. Then

$$\alpha = \sup L$$

exists in *S* and $\alpha = \inf B$.

Proof

Since *B* is bounded below therefore *L* is non-empty.

Since *L* consists of exactly those $y \in S$ which satisfy the inequality.

$$y \leq x \qquad \forall\ x \in B$$

We see that every $x \in B$ is an upper bound of *L*.

$\Rightarrow$ L is bounded above.

Since S is ordered and non-empty with least upper bound property therefore L has a supremum in S. Let us call it α.

If $\gamma < \alpha$, then (by definition of supremum) γ is not upper bound of L.

$\Rightarrow$ $\gamma \notin B$.

It follows that $\alpha \le x \ \forall \ x \in B$.

Thus α is lower bound of B.

Now if $\alpha < \beta$, then $\beta \notin L$ because $\alpha = \sup L$, that is, β is not lower bound of B.

this means (by definition of infumum) $\alpha = \inf B$.

❖ Remark

Above theorem can be stated as follows:

An ordered set which has the least upper bound property has also the greatest lower bound property.

❖ Field

A set F with two operations called addition and multiplication satisfying the following axioms is known to be field.

Axioms for Addition:

(i) If $x,\ y \in F$ then $x + y \in F$. *Closure Law*

(ii) $x + y = y + x, \quad \forall \ x, y \in F$. *Commutative Law*

(iii) $x + (y + z) = (x + y) + z \quad \forall \ x, y, z \in F$. *Associative Law*

(iv) For any $x \in F$, $\exists \ 0 \in F$ such that $x + 0 = 0 + x = x$ *Additive Identity*

(v) For any $x \in F$, $\exists \ -x \in F$ such that $x + (-x) = (-x) + x = 0$ *+tive Inverse*

Axioms for Multiplication:

(i) If $x,\ y \in F$ then $x\,y \in F$. *Closure Law*

(ii) $x\,y = y\,x, \quad \forall \ x, y \in F$ *Commutative Law*

(iii) $x(y\,z) = (x\,y)\,z \quad \forall \ x, y, z \in F$

(iv) For any $x \in F$, $\exists \ 1 \in F$ such that $x \cdot 1 = 1 \cdot x = x$ *Multiplicative Identity*

(v) For any $x \in F$, $x \ne 0$, $\exists \ \dfrac{1}{x} \in F$, such that $x\left(\dfrac{1}{x}\right) = \left(\dfrac{1}{x}\right)x = 1$ $\times$*tive Inverse.*

Distributive Law

For any $x, y, z \in F$, (i) $x(y + z) = xy + xz$

(ii) $(x + y)z = xz + yz$

❖ Ordered Field

An ordered field is a field F which is also an ordered set such that *i*) $x+y<x+z$ if $x,y,z\in F$ and $y<z$.

ii) $xy>0$ if $x,y\in F$, $x>0$ and $y>0$.

For example, the set $\mathbb{Q}$ of rational number is an ordered field.

❖ Existence of Real Field

There exists an ordered field $\mathbb{R}$ (set of real numbers) which has the least upper bound property.

Moreover $\mathbb{R}$ contains $\mathbb{Q}$ (set of rational numbers) as a subfield.

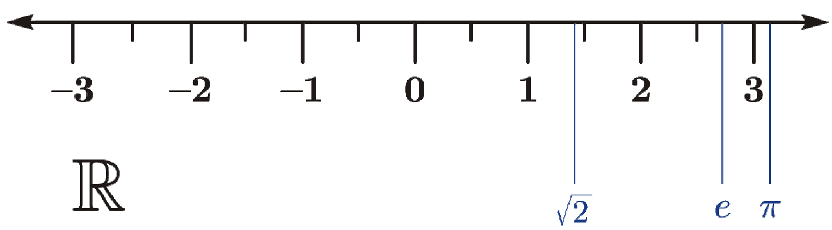
The members of $\mathbb{R}$ are called real numbers. The real numbers which are not rational are called irrational numbers.

(To see complete proof of the existence of real field from set $\mathbb{Q}$, see [1, Page 17])

> *There are many other ways to construct a set of real numbers. We are not interested to do so therefore we leave it to the reader if they are interested then following page is useful:*
> *http://en.wikipedia.org/wiki/Construction_of_the_real_numbers*

❖ Remarks

The real numbers include all the rational numbers, such as the integer −5 and the fraction 4/3, and all the irrational numbers such as $\sqrt{2}$ (1.41421356…, the square root of two, an irrational algebraic number) and π (3.14159265…, a transcendental number). Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation such as that of 8.632, where each consecutive digit is measured in units one tenth the size of the previous one. Or in other words, any real number can be though as length of line in such a way that



❖ Theorem

Let $x,y,z\in\mathbb{R}$. Then axioms for addition imply the following.

(***a***) If $x+y=x+z$ then $y=z$

(***b***) If $x+y=x$ then $y=0$

(***c***) If $x+y=0$ then $y=-x$.

(***d***) $-(-x)=x$

Proof

(Note: *We have given the proofs here just to show that the things which looks simple must have valid analytical proofs under some consistence theory of mathematics*)

(a) Suppose $x+y=x+z$.

Since
$$\begin{aligned} y &= 0+y \\ &= (-x+x)+y && \because\ -x+x=0 \\ &= -x+(x+y) && \text{by Associative law} \\ &= -x+(x+z) && \text{by supposition} \\ &= (-x+x)+z && \text{by Associative law} \\ &= (0)+z && \because\ -x+x=0 \\ &= z \end{aligned}$$

(b) Take $z=0$ in (a)

$$x+y=x+0$$
$$\Rightarrow\ y=0$$

(c) Take $z=-x$ in (a)

$$x+y=x+(-x)$$
$$\Rightarrow\ y=-x$$

(d) Since $(-x)+x=0$

then (c) gives $x=-(-x)$

❖ Theorem

Let $x, y, z \in \mathbb{R}$. Then axioms of multiplication imply the following.

(***a***) If $x\neq 0$ and $xy=xz$ then $y=z$.

(***b***) If $x\neq 0$ and $xy=x$ then $y=1$.

(***c***) If $x\neq 0$ and $xy=1$ then $y=\dfrac{1}{x}$.

(***d***) If $x\neq 0$, then $\dfrac{1}{1/x}=x$.

Proof

(Note: *We have given the proofs here just to show that the things which looks simple must have valid analytical proofs under some consistence theory of mathematics*)

(a) Suppose $xy=xz$

Since
$$\begin{aligned} y &= 1\cdot y \\ &= \left(\frac{1}{x}\cdot x\right)y && \because\ \frac{1}{x}\cdot x=1 \end{aligned}$$

$$=\frac{1}{x}(x\,y) \qquad \text{by associative law}$$

$$=\frac{1}{x}(x\,z) \qquad \because x\,y=x\,z$$

$$=\left(\frac{1}{x}\cdot x\right)z \qquad \text{by associative law}$$

$$=1\cdot z=z$$

(b) Take $z=1$ in (a)

$$x\,y=x\cdot 1 \quad \Rightarrow \quad y=1$$

(c) Take $z=\frac{1}{x}$ in (a)

$$x\,y=x\cdot\frac{1}{x} \quad \text{i.e. } x\,y=1$$

$$\Rightarrow \quad y=\frac{1}{x}$$

(d) Since $\frac{1}{x}\cdot x=1$

then (c) give

$$x=\frac{1}{1/x}$$

❖ **Theorem**

Let $x,y,z\in\mathbb{R}$. Then field axioms imply the following.

$(\boldsymbol{i})$ $0\cdot x=x$ $\qquad$ $(\boldsymbol{ii})$ if $x\neq 0$, $y\neq 0$ then $xy\neq 0$.

$(\boldsymbol{iii})$ $(-x)y=-(xy)=x(-y)$ $\qquad$ $(\boldsymbol{iv})$ $(-x)(-y)=xy$

Proof

$(\boldsymbol{i})$ Since $0x+0x=(0+0)x$

$$\Rightarrow \quad 0x+0x=0x$$

$$\Rightarrow \quad 0x=0 \qquad \because x+y=x \Rightarrow y=0$$

$(\boldsymbol{ii})$ Suppose $x\neq 0$, $y\neq 0$ but $x\,y=0$

Since $1=\frac{1}{x\,y}\cdot x\,y$

$$\Rightarrow 1=\frac{1}{x\,y}(0) \qquad \because xy=0$$

$$\Rightarrow \quad 1=0 \qquad \text{from } (i) \quad \because x0=0$$

a contradiction, thus (*ii*) is true.

(***iii***) Since $(-x)y + xy = (-x + x)y = 0y = 0$ …….. (1)

Also $x(-y) + xy = x(-y + y) = x0 = 0$ ……… (2)

Also $-(xy) + xy = 0$ …………. (3)

Combining (1) and (2)

$$(-x)y + xy = x(-y) + xy$$

$\Rightarrow$ $(-x)y = x(-y)$ ……….. (4)

Combining (2) and (3)

$$x(-y) + xy = -(xy) + xy$$

$\Rightarrow$ $x(-y) = -xy$ …………. (5)

From (4) and (5)

$$(-x)y = x(-y) = -xy$$

(***iv***) $(-x)(-y) = -[x(-y)] = -[-xy] = xy$ using (*iii*)

❖ Theorem

Let $x, y, z \in \mathbb{R}$. Then the following statements are true in every ordered field.

i) If $x > 0$ then $-x < 0$ and vice versa.

ii) If $x > 0$ and $y < z$ then $xy < xz$.

iii) If $x < 0$ and $y < z$ then $xy > xz$.

iv) If $x \neq 0$ then $x^2 > 0$ in particular $1 > 0$.

v) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof

i) If $x > 0$ then $0 = -x + x > -x + 0$ so that $-x < 0$.

If $x < 0$ then $0 = -x + x < -x + 0$ so that $-x > 0$.

ii) Since $z > y$ we have $z - y > y - y = 0$

which means that $z - y > 0$ also $x > 0$

$\therefore$ $x(z - y) > 0$

$\Rightarrow$ $xz - xy > 0$

$\Rightarrow$ $xz - xy + xy > 0 + xy$

$\Rightarrow$ $xz + 0 > 0 + xy$

$\Rightarrow$ $xz > xy$

iii) Since $y < z$ $\Rightarrow$ $-y + y < -y + z$

$\Rightarrow$ $z - y > 0$

Also $x < 0$ $\Rightarrow -x > 0$

Therefore $-x(z-y)>0$

$\Rightarrow -xz+xy>0 \qquad \Rightarrow -xz+xy+xz>0+xz$

$\Rightarrow xy>xz$

iv) If $x>0$ then $x\cdot x>0 \quad \Rightarrow x^2>0$

If $x<0$ then $-x>0 \Rightarrow (-x)(-x)>0 \Rightarrow (-x)^2>0 \quad \Rightarrow x^2>0$

i.e. if $x>0$ then $x^2>0$, since $1^2=1$ then $1>0$.

v) If $y>0$ and $v\le 0$ then $yv\le 0$, But $y\left(\frac{1}{y}\right)=1>0 \quad \Rightarrow \frac{1}{y}>0$

Likewise, $\frac{1}{x}>0$ as $x>0$

If we multiply both sides of the inequality $x<y$ by the positive quantity,

$\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)$ we obtain $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x<\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$

i.e. $\frac{1}{y}<\frac{1}{x}$

finally, $0<\frac{1}{y}<\frac{1}{x}$.

❖ **Theorem (Archimedean Property)**

If $x\in\mathbb{R}$, $y\in\mathbb{R}$ and $x>0$ then there exists a positive integer *n* such that $nx>y$.

Proof

Let $A=\{nx:\ n\in\mathbb{Z}^+\wedge x>0, x\in\mathbb{R}\}$

Suppose the given statement is false i.e. $nx\le y$.

$\Rightarrow$ y is an upper bound of *A*.

Since we are dealing with a set of real therefore it has the least upper bound property.

Let $\alpha=\sup A$

$\Rightarrow$ $\alpha-x$ is not an upper bound of A.

$\Rightarrow$ $\alpha-x<mx$ where $mx\in A$ for some positive integer *m*.

$\Rightarrow$ $\alpha<(m+1)x$ where $m+1$ is integer, therefore $(m+1)x\in A$.

This is impossible because α is least upper bound of *A* i.e. $\alpha=\sup A$.

Hence we conclude that the given statement is true i.e. $nx>y$.

❖ The Density Theorem

If $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$ then there exists $p \in \mathbb{Q}$ such that $x < p < y$.

i.e. between any two real numbers there is a rational number *or* $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof

Since $x < y$, therefore $y - x > 0$

$\Rightarrow$ there exists a positive integer n such that

$$n(y-x) > 1 \qquad (\textit{by Archimedean Property})$$

$$\Rightarrow \; ny > 1 + nx \; \ldots\ldots\ldots\ldots\ldots (i)$$

Again we use Archimedean property to obtain two positive integers m_1 and m_2 such that $m_1 \cdot 1 > nx$ and $m_2 \cdot 1 > -nx$

$$\Rightarrow \; -m_2 < nx < m_1,$$

then there exists and integers $m(-m_2 \le m \le m_1)$ such that

$$m - 1 \le nx < m$$

$$\Rightarrow \; nx < m \quad \text{and} \quad m \le 1 + nx$$

$$\Rightarrow \; nx < m < 1 + nx$$

$$\Rightarrow \; nx < m < ny \qquad \text{from } (i)$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y$$

$$\Rightarrow \; x < p < y \quad \text{where} \quad p = \frac{m}{n} \text{ is a rational.}$$

❖ Theorem

Given two real numbers x and y, $x < y$ there is an irrational number u such that $x < u < y$.

Proof

If we apply density theorem to real numbers $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, we obtain a rational number $r \neq 0$ such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$$

$$\Rightarrow \; x < r\sqrt{2} < y$$

$$\Rightarrow \; x < u < y,$$

where $u = r\sqrt{2}$ is an irrational as product of rational and irrational is irrational.

❖ Theorem

For every real number x there is a set E of rational number such that $x = \sup E$.

Proof

Take $E = \{q \in \mathbb{Q} : q < x\}$ where x is a real.

Then E is bounded above. Since $E \subset \mathbb{R}$ therefore supremum of E exists in $\mathbb{R}$.

Suppose $\sup E = \lambda$.

It is clear that $\lambda \leq x$.

If $\lambda = x$ then there is nothing to prove.

If $\lambda < x$ then $\exists\ q \in \mathbb{Q}$ such that $\lambda < q < x$,

which can not happened hence we conclude that real x is sup$E$.

❖ Question

Let E be a non-empty subset of an ordered set, suppose α is a lower bound of E and β is an upper bound then prove that $\alpha \leq \beta$.

Proof

Since E is a subset of an ordered set S i.e. $E \subseteq S$.

Also α is a lower bound of E therefore by definition of lower bound

$$\alpha \leq x \quad \forall \ x \in E \ \ldots\ldots\ldots\ldots\ldots \ (i)$$

Since β is an upper bound of E therefore by the definition of upper bound

$$x \leq \beta \quad \forall \ x \in E \ \ldots\ldots\ldots\ldots\ldots \ (ii)$$

Combining (i) and (ii)

$$\alpha \leq x \leq \beta$$

$\Rightarrow \ \alpha \leq \beta$ as required.

❖ The Extended Real Numbers

The extended real number system consists of real field $\mathbb{R}$ and two symbols $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$ and define

$$-\infty < x < +\infty \quad \forall \ x \in \mathbb{R}.$$

The extended real number system does not forms a field. Mostly we write $+\infty = \infty$.

We make following conventions:

i) If x is real the $x + \infty = \infty$, $x - \infty = -\infty$, $\dfrac{x}{\infty} = \dfrac{x}{-\infty} = 0$.

ii) If $x > 0$ then $x(\infty) = \infty$, $x(-\infty) = -\infty$.

iii) If $x < 0$ then $x(\infty) = -\infty$, $x(-\infty) = \infty$.

❖ Euclidean Space

For each positive integer *k*, let $\mathbb{R}^k$ be the set of all ordered *k*-tuples

$$\underline{x} = (x_1, x_2, ..., x_k)$$

where $x_1, x_2, ..., x_k$ are real numbers, called the *coordinates* of $\underline{x}$. The elements of $\mathbb{R}^k$ are called points, or vectors, especially when $k > 1$.

If $\underline{y} = (y_1, y_2, ..., y_n)$ and α is a real number, put

$$\underline{x} + \underline{y} = (x_1 + y_1, x_2 + y_2, ..., x_k + y_k)$$

and $$\alpha \underline{x} = (\alpha x_1, \alpha x_2, ..., \alpha x_k)$$

So that $\underline{x} + \underline{y} \in \mathbb{R}^k$ and $\alpha \underline{x} \in \mathbb{R}^k$. These operations make $\mathbb{R}^k$ into a vector space over the real field.

The inner product or scalar product of $\underline{x}$ and $\underline{y}$ is defined as

$$\underline{x} . \underline{y} = \sum_{i=1}^{k} x_i y_i = (x_1 y_1 + x_2 y_2 + ... + x_k y_k)$$

And the norm of $\underline{x}$ is defined by

$$\| \underline{x} \| = (x \cdot x)^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}$$

The vector space $\mathbb{R}^k$ with the above inner product and norm is called *Euclidean k-space*.

❖ Theorem

Let $\underline{x}, \underline{y} \in \mathbb{R}^n$ then

i) $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$

ii) $\left| \underline{x} \cdot \underline{y} \right| \leq \| \underline{x} \| \, \| \underline{y} \|$ (*Cauchy-Schwarz's inequality*)

Proof

i) Since $\| \underline{x} \| = (\underline{x} \cdot \underline{x})^{\frac{1}{2}}$ therefore $\| \underline{x} \|^2 = \underline{x} \cdot \underline{x}$

ii) If $\underline{x} = 0$ or $\underline{y} = 0$, then Cauchy-Schwarz's inequality holds with equality.

If $\underline{x} \neq 0$ and $\underline{y} \neq 0$, then for $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} 0 \leq \left\| \underline{x} - \lambda \underline{y} \right\|^2 &= \left(\underline{x} - \lambda \underline{y} \right) \cdot \left(\underline{x} - \lambda \underline{y} \right) \\ &= \underline{x} \cdot \left(\underline{x} - \lambda \underline{y} \right) + \left(-\lambda \underline{y} \right) \cdot \left(\underline{x} - \lambda \underline{y} \right) \\ &= \underline{x} \cdot \underline{x} + \underline{x} \cdot (-\lambda \underline{y}) + (-\lambda \underline{y}) \cdot \underline{x} + (-\lambda \underline{y}) \cdot (-\lambda \underline{y}) \\ &= \| \underline{x} \|^2 - 2\lambda (\underline{x} \cdot \underline{y}) + \lambda^2 \| \underline{y} \|^2 \end{aligned}$$

Now put $\lambda = \dfrac{\underline{x}\cdot\underline{y}}{\|\underline{y}\|^2}$ (*certain real number*)

$$\Rightarrow\ 0 \le \|\underline{x}\|^2 - 2\frac{(\underline{x}\cdot\underline{y})(\underline{x}\cdot\underline{y})}{\|\underline{y}\|^2} + \frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{y}\|^4}\|\underline{y}\|^2 \quad \Rightarrow\ 0 \le \|\underline{x}\|^2 - \frac{(\underline{x}\cdot\underline{y})^2}{\|\underline{y}\|^2}$$

$$\Rightarrow\ 0 \le \|\underline{x}\|^2\|\underline{y}\|^2 - |\underline{x}\cdot\underline{y}|^2 \qquad \because\ a^2 = |a|^2\ \forall\ a\in\mathbb{R},$$

$$\Rightarrow\ 0 \le \left(\|\underline{x}\|\,\|\underline{y}\| + |\underline{x}\cdot\underline{y}|\right)\left(\|\underline{x}\|\,\|\underline{y}\| - |\underline{x}\cdot\underline{y}|\right).$$

Which hold if and only if

$$0 \le \|\underline{x}\|\,\|\underline{y}\| - |\underline{x}\cdot\underline{y}|$$

i.e. $|\underline{x}\cdot\underline{y}| \le \|\underline{x}\|\,\|\underline{y}\|$.

❖ Question

Suppose $\underline{x},\underline{y},\underline{z}\in\mathbb{R}^n$ the prove that

a) $\|\underline{x}+\underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|$

b) $\|\underline{x}-\underline{z}\| \le \|\underline{x}-\underline{y}\| + \|\underline{y}-\underline{z}\|$

Proof

a) Consider

$$\begin{aligned}\|\underline{x}+\underline{y}\|^2 &= (\underline{x}+\underline{y})\cdot(\underline{x}+\underline{y})\\ &= \underline{x}\cdot\underline{x} + \underline{x}\cdot\underline{y} + \underline{y}\cdot\underline{x} + \underline{y}\cdot\underline{y}\\ &= \|\underline{x}\|^2 + 2(\underline{x}\cdot\underline{y}) + \|\underline{y}\|^2\\ &\le \|\underline{x}\|^2 + 2|\underline{x}\cdot\underline{y}| + \|\underline{y}\|^2 && \because\ |a| \ge a\ \forall\ a\in\mathbb{R}.\\ &\le \|\underline{x}\|^2 + 2\|\underline{x}\|\,\|\underline{y}\| + \|\underline{y}\|^2 && \because\ \|\underline{x}\|\,\|\underline{y}\| \ge |\underline{x}\cdot\underline{y}|\\ &= \left(\|\underline{x}\| + \|\underline{y}\|\right)^2\end{aligned}$$

$$\Rightarrow\ \|\underline{x}+\underline{y}\| \le \|\underline{x}\| + \|\underline{y}\| \ \ \ldots\ldots\ldots\ldots\ (i)$$

b) We have

$$\begin{aligned}\|\underline{x}-\underline{z}\| &= \|\underline{x}-\underline{y}+\underline{y}-\underline{z}\|\\ &\le \|\underline{x}-\underline{y}\| + \|\underline{y}-\underline{z}\| && \text{from } (i)\end{aligned}$$

❖ Relatively Prime

Let $a,b\in\mathbb{Z}$. Then a and b are said to be relatively prime or co-prime if a and b don't have common factor other than 1. If a and b are relatively prime then we write $(a,b)=1$.

❖ **Question**

If r is non-zero rational and x is irrational then prove that $r+x$ and rx are irrational.

Proof

Let $r+x$ be rational.

$\Rightarrow\ r+x=\dfrac{a}{b}$ where $a,b\in\mathbb{Z}$, $b\neq 0$ such that $(a,b)=1$,

$$\Rightarrow\ x=\frac{a}{b}-r$$

Since r is rational therefore $r=\dfrac{c}{d}$ where $c,d\in\mathbb{Z}$, $d\neq 0$ such that $(c,d)=1$,

$$\Rightarrow\ x=\frac{a}{b}-\frac{c}{d}\quad\Rightarrow\ x=\frac{ad-bc}{bd}.$$

Which is rational, which can not happened because x is given to be irrational.

Similarly let us suppose that rx is rational then

$rx=\dfrac{a}{b}$ for some $a,b\in\mathbb{Z}$, $b\neq 0$ such that $(a,b)=1$

$$\Rightarrow\ x=\frac{a}{b}\cdot\frac{1}{r}$$

Since r is rational therefore $r=\dfrac{c}{d}$ where $c,d\in\mathbb{Z}$, $d\neq 0$ such that $(c,d)=1$

$$\Rightarrow\ x=\frac{a}{b}\cdot\frac{1}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Which shows that x is rational, which is again contradiction; hence we conclude that $r+x$ and rx are irrational.

References:

[1] Principles of Mathematical Analysis by Walter Rudin (McGraw-Hill, Inc.)
[2] Introduction to Real Analysis by R.G.Bartle, and D.R. Sherbert (John Wiley & Sons, Inc.)
[3] Mathematical Analysis by Tom M. Apostol, (Pearson; 2nd edition.)
[4] Real Analysis by Dipak Chatterjee (PHI Learning, 2nd edition.)

A password protected "zip" archive of above three resources can be downloaded from the following URL: http://bit.ly/2BViMnB