# *Summary: Riemann-Stieltjes Integral*

Course Title: Real Analysis II  Course Code: MTH322
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## ➢ *Partition*

Let $[a,b]$ be a given interval. A finite set $P=\{a=x_0,x_1,x_2,\ldots,x_k,\ldots,x_n=b\}$ is said to be a partition of $[a,b]$ which divides it into $n$ such intervals

$$[x_0,x_1],[x_1,x_2],[x_2,x_3],\ldots,[x_{n-1},x_n].$$

Each sub-interval is called a *component* of the partition.

Obviously, corresponding to different choices of the points $x_i$ we shall have different partition.

The maximum of the length of the components is defined as the *norm* of the partition and it is denoted by $\|P\|$.

## ➢ *Refinement of a Partition*

Let $P$ and $P^*$ be two partitions of an interval $[a,b]$ such that $P\subset P^*$ i.e. $P^*$ contains all the points of $P$ and possibly some other points as well. Then $P^*$ is said to be a *refinement* of $P$.

## ➢ *Common Refinement*

Let $P_1$ and $P_2$ be two partitions of $[a,b]$. Then a partition $P^*$ is said to be their *common refinement* if $P^*=P_1\cup P_2$.

## ➢ *Examples*

Consider an interval $[1,10]$ and following partitions of this interval.

$P_1=\{1,2,3,10\}$,

$P_2=\{1,2,3,6,9,10\}$,

$$P_3=\left\{1,1+\frac{9}{100},1+2\left(\frac{9}{100}\right),1+3\left(\frac{9}{100}\right),\ldots,1+99\left(\frac{9}{100}\right),10\right\}$$

and more generally for any positive integer $n$, we can write

$$P_4=\left\{1,1+\frac{9}{n},1+2\left(\frac{9}{n}\right),1+3\left(\frac{9}{n}\right),\ldots,1+(n-1)\left(\frac{9}{n}\right),1+n\left(\frac{9}{n}\right)=10\right\}.$$

One can note that $P_2$ is refinement of $P_1$.

Also note that $\|P_1\|=7$, $\|P_2\|=3$, $\|P_3\|=\dfrac{9}{100}$, $\|P_4\|=\dfrac{9}{n}$.

## ➢ *Remark*

Note that if $P\subseteq P'$ implies $\|P'\|\le\|P\|$. That is, refinement of a partition decreases its norm but the convers does not necessarily hold.

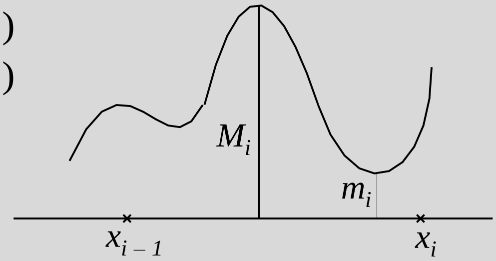
Also note that $P_1\subseteq P_2$ and $\|P_2\|\le\|P_1\|$.

## ➢ *Riemann Integral*

Let $f$ be a real-valued function defined and bounded on $[a,b]$. Corresponding to each partition $P$ of $[a,b]$, we put

$$M_i = \sup f(x) \qquad (x_{i-1} \le x \le x_i)$$

$$m_i = \inf\, f(x) \qquad (x_{i-1} \le x \le x_i)$$




We define upper and lower sums as

$$U(P,f) = \sum_{i=1}^{n} M_i\, \Delta x_i$$

and $$L(P,f) = \sum_{i=1}^{n} m_i\, \Delta x_i\,.$$

where $\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \ldots, n)$

and finally $$\overline{\int_a^b} f\, dx = \inf U(P,f) \ \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\underline{\int_a^b} f\, dx = \sup L(P,f) \ \ldots\ldots\ldots\ldots\ldots (ii)$$

Where the infimum and the supremum are taken over all partitions $P$ of $[a,b]$. Then $\overline{\int_a^b} f\, dx$ and $\underline{\int_a^b} f\, dx$ are called the upper and lower Riemann Integrals of $f$ over $[a,b]$ respectively.

In case the upper and lower integrals are equal, we say that $f$ is Riemann-Integrable on $[a,b]$ and we write $f \in R$, where $R$ denotes the set of Riemann integrable functions.

The common value of $(i)$ and $(ii)$ is denoted by $\int_a^b f\, dx$ or by $\int_a^b f(x)\, dx$.

Which is known as the Riemann integral of $f$ over $[a,b]$.

## ➢ *Theorem*

The upper and lower integrals are defined for every bounded function $f$.

### *Proof*

Take $M$ and $m$ to be the upper and lower bounds of $f(x)$ in $[a,b]$.

$$\Rightarrow \quad m \ \le\ f(x) \ \le\ M \qquad (a \le x \le b)$$

Then $\quad M_i \le M \quad$ and $\quad m_i \ge m \qquad (i = 1, 2, \ldots\ldots, n)$

Where $M_i$ and $m_i$ denote the supremum and infimum of $f(x)$ in $(x_{i-1}, x_i)$ for certain partition $P$ of $[a,b]$.

$$\Rightarrow\ L(P,f) = \sum_{i=1}^{n} m_i\, \Delta x_i \ge \sum_{i=1}^{n} m\, \Delta x_i \qquad (\Delta x_i = x_{i-1} - x_i)$$

$$\Rightarrow\ L(P,f) \ge m \sum_{i=1}^{n} \Delta x_i$$

But $\displaystyle\sum_{i=1}^{n}\Delta x_i = (x_1 - x_0)+(x_2 - x_1)+(x_3 - x_2)+....+(x_n - x_{n-1})$

$$= x_n - x_0 = b - a$$

$\Rightarrow\ L(P,f) \geq m(b-a)$

Similarity $U(P,f) \leq M(b-a)$

$\Rightarrow\ m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

Which shows that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set.

$\Rightarrow$ The upper and lower integrals are defined for every bounded function $f$. ⊙

## ➢ *Riemann-Stieltjes Integral*

It is a generalization of the Riemann Integral. Let $\alpha(x)$ be a monotonically increasing function on $[a,b]$. $\alpha(a)$ and $\alpha(b)$ being finite, it follows that $\alpha(x)$ is bounded on $[a,b]$. Corresponding to each partition $P$ of $[a,b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

( Difference of values of $\alpha$ at $x_i$ & $x_{i-1}$ )

$\because\ \alpha(x)$ is monotonically increasing.

$\therefore\ \Delta\alpha_i \geq 0$

Let $f$ be a real function which is bounded on $[a,b]$.

Put $\displaystyle U(P,f,\alpha) = \sum_{i=1}^{n} M_i\,\Delta\alpha_i$

$$L(P,f,\alpha) = \sum_{i=1}^{n} m_i\,\Delta\alpha_i\ ,$$

where $M_i$ and $m_i$ have their usual meanings.

Define

$$\overline{\int_a^b} f\,d\alpha = \inf U(P,f,\alpha)\ \ldots\ldots\ldots\ldots\ldots\ (i)$$

$$\underline{\int_a^b} f\,d\alpha = \sup L(P,f,\alpha)\ \ldots\ldots\ldots\ldots\ldots\ (ii)$$

Where the infimum and supremum are taken over all partitions of $[a,b]$.

If $\displaystyle\overline{\int_a^b} f\,d\alpha = \underline{\int_a^b} f\,d\alpha$, we denote their common value by $\displaystyle\int_a^b f\,d\alpha$ or $\displaystyle\int_a^b f(x)\,d\alpha(x)$.

This is the Riemann-Stieltjes integral or simply the Stieltjes Integral of $f$ w.r.t. $\alpha$ over $[a,b]$.

If $\int_a^b f\,d\alpha$ exists, we say that $f$ is integrable w.r.t. $\alpha$, in the Riemann sense, and write $f \in \mathcal{R}(\alpha)$.

## ➢ *Note*

The Riemann-integral is a special case of the Riemann-Stieltjes integral when we take $\alpha(x)=x$.

$\because$ The integral depends upon $f,\alpha,a$ and $b$ but not on the variable of integration.

$\therefore$ We can omit the variable and prefer to write $\int_a^b f\,d\alpha$ instead of $\int_a^b f(x)\,d\alpha(x)$.

In the following discussion $f$ will be assume to be real and bounded, and $\alpha$ monotonically increasing on $[a,b]$.

## ➢ *Theorem*

If $P^*$ is a refinement of $P$, then following holds:

(i) $L(P,f,\alpha) \le L(P^*,f,\alpha)$,

(ii) $U(P,f,\alpha) \ge U(P^*,f,\alpha)$.

## ➢ *Theorem*

Let $f$ be a real valued function defined on $[a,b]$ and $\alpha$ be a monotonically increasing function on $[a,b]$. Then

$$\sup L(P,f,\alpha) \le \inf U(P,f,\alpha)$$

$$\text{i.e. } \int_{\underline{a}}^{b} f\,d\alpha \le \int_{a}^{\bar{b}} f\,d\alpha .$$

## ➢ *Theorem* *(Condition of Integrability or Cauchy's Criterion for Integrability.)*

A function $f\in R(\alpha)$ on $[a,b]$ if and only if for every $\varepsilon>0$ there exists a partition $P$ such that $U(P,f,\alpha)-L(P,f,\alpha)<\varepsilon$ .

## ➢ *Theorem*

If $f\in R(\alpha)$ on $[a,b]$, then $|f|\in R(\alpha)$ on $[a,b]$ and $\left|\int_a^b f\,d\alpha\right| \le \int_a^b |f|\,d\alpha$ .

## ➢ *Theorem (Fundamental Theorem of Calculus)*

If $f\in R$ on $[a,b]$ and if there is a differentiable function $F$ on $[a,b]$ such that $F'=f$, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

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