Ch 01: Improper Integrals of 1st and 2nd Kinds

Course Title: Real Analysis II — Course Code: MTH322
Course instructor: Dr. Atiq ur Rehman — Class: MSc-III
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We discussed Riemann-Stieltjes's integrals of the form $\int_a^b f\,d\alpha$ under the restrictions that both f and α are defined and bounded on a finite interval $[a,b]$. The integral of the form $\int_a^b f\,d\alpha$ are called definite integrals. To extend the concept, we shall relax some condition on definite integral like f on finite interval or boundedness of f on finite interval.

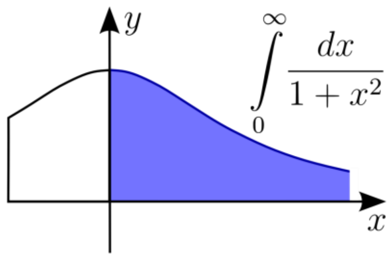
➢ *Definition*

The integral $\int_a^b f\,d\alpha$ is called an improper integral of first kind if $a=-\infty$ or $b=+\infty$ or *both* i.e. one or both integration limits is infinite.

➢ *Definition*

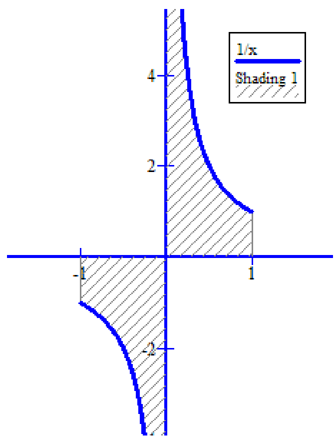
The integral $\int_a^b f\,d\alpha$ is called an improper integral of second kind if $f(x)$ is unbounded at one or more points of $a\le x\le b$. Such points are called singularities of $f(x)$.

➢ *Examples*



- $\int_0^\infty \frac{1}{1+x^2}dx$, $\int_{-\infty}^1 \frac{1}{x-2}dx$ and $\int_{-\infty}^\infty (x^2+1)dx$ are examples of improper integrals of first kind.

- $\int_{-1}^1 \frac{1}{x}dx$ and $\int_0^1 \frac{1}{2x-1}dx$ are examples of improper integrals of second kind.



➢ *Notations*

We shall denote the set of all functions f such that $f\in R(\alpha)$ on $[a,b]$ by $R(\alpha;a,b)$. When $\alpha(x)=x$, we shall simply write $R(a,b)$ for this set. The notation $\alpha\uparrow$ on $[a,\infty)$ will mean that α is monotonically increasing on $[a,\infty)$.

IMPROPER INTEGRAL OF THE FIRST KIND

➢ Definition

Assume that $f \in R(\alpha; a, b)$ for every $b \geq a$. Keep a, α and f fixed and define a function I on $[a, \infty)$ as follows:

$$I(b) = \int_a^b f(x)\, d\alpha(x) \quad \text{if} \quad b \geq a.$$

The integral $\int_a^\infty f(x)\, d\alpha(x)$ is said to converge if the $\lim_{b\to\infty} I(b)$ exists (finite). Otherwise, $\int_a^\infty f\, d\alpha$ is said to diverge.

If the $\lim_{b\to\infty} I(b)$ exists and equals A, the number A is called the value of the integral and we write $\int_a^\infty f\, d\alpha = A$.

➢ Remark

If $\int_a^\infty f\, d\alpha$ is convergent(divergent), then $\int_c^\infty f\, d\alpha$ is convergent(divergent) for $c > a$.

If $\int_c^\infty f\, d\alpha$ is convergent (divergent), then $\int_a^\infty f\, d\alpha$ is convergent (divergent) for $a < c$ if f in bounded in $[a, c]$.

➢ Example

Consider and integral $\int_1^\infty x^{-p}\, dx$, where p is any real number. Discuss its convergence or divergence.

Solution

Let $I(b) = \int_1^b x^{-p}\, dx$ where $b \geq 1$.

Then $I(b) = \int_1^b x^{-p}\, dx = \left.\dfrac{x^{1-p}}{1-p}\right|_1^b = \dfrac{1-b^{1-p}}{p-1}$ if $p \neq 1$.

If $b \to \infty$, then $b^{1-p} \to 0$ for $p > 1$ and $b^{1-p} \to \infty$ for $p < 1$.

Therefore we have

$$\lim_{b\to\infty} I(b) = \lim_{b\to\infty} \frac{1-b^{1-p}}{p-1} = \begin{cases} \infty & \text{if} \quad p < 1, \\ \dfrac{1}{p-1} & \text{if} \quad p > 1. \end{cases}$$

Now if $p = 1$, we get $\int_1^b x^{-1}\, dx = \log b \to \infty$ as $b \to \infty$.

Hence we concluded: $\int_1^\infty x^{-p}\,dx = \begin{cases} diverges & if \quad p \le 1, \\ \dfrac{1}{p-1} & if \quad p > 1. \end{cases}$

➢ *Example*

Is the integral $\int_0^\infty \sin 2\pi x\,dx$ converges or diverges?

Solution:

Consider $I(b) = \int_0^b \sin 2\pi x\,dx$, where $b \ge 0$.

We have $\int_0^b \sin 2\pi x\,dx = \left.\dfrac{-\cos 2\pi x}{2\pi}\right|_0^b = \dfrac{1-\cos 2\pi b}{2\pi}$.

Also $\cos 2\pi b \to l$ as $b \to \infty$, where l has values between -1 and 1, that is, limit is not unique.

Therefore the integral $\int_0^\infty \sin 2\pi x\,dx$ diverges.

➢ *Note*

If $\int_{-\infty}^a f\,d\alpha$ and $\int_a^\infty f\,d\alpha$ are both convergent for some value of a, we say that the integral $\int_{-\infty}^\infty f\,d\alpha$ is convergent and its value is defined to be the sum

$$\int_{-\infty}^\infty f\,d\alpha = \int_{-\infty}^a f\,d\alpha + \int_a^\infty f\,d\alpha$$

The choice of the point a is clearly immaterial.

If the integral $\int_{-\infty}^\infty f\,d\alpha$ converges, its value is equal to the limit: $\lim\limits_{b\to+\infty} \int_{-b}^b f\,d\alpha$.

➢ *Review:*

- A function f is said to be increasing, if for all $x_1, x_2 \in D_f$ (domain of f) and $x_1 \le x_2$ implies $f(x_1) \le f(x_2)$.
- A function f is said to be bounded if there exist some positive number μ such that $|f(t)| \le \mu$ for all $t \in D_f$.
- If f is define on $[a,+\infty)$ and $\lim\limits_{x\to\infty} f(x)$ exists then f is bounded on $[a,+\infty)$.
- If $f \in R(\alpha; a,b)$ and $c \in [a,b]$, then $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.
- If $f \in R(\alpha; a,b)$ and $f(x) \ge 0$ for all $x \in [a,b]$, then $\int_a^b f d\alpha \ge 0$.

- If f is monotonically increasing and bounded on $[a,+\infty)$, then $\lim_{x\to\infty} f(x) = \sup_{x\in[a,\infty)} f(x)$.

> **Theorem**
>
> Assume that α is monotonically increasing on $[a,+\infty)$ and suppose that $f \in R(\alpha;a,b)$ for every $b \ge a$. Assume that $f(x) \ge 0$ for each $x \ge a$. Then $\int_a^\infty f\,d\alpha$ converges if, and only if, there exists a constant $M > 0$ such that
> $$\int_a^b f\,d\alpha \le M \quad \text{for every } b \ge a.$$

Proof

Let $I(b) = \int_a^b f\,d\alpha$ for $b \ge a$.

First suppose that $\int_a^\infty f\,d\alpha$ is convergent, then $\lim_{b\to+\infty} I(b)$ exists, that is, $I(b)$ is bounded on $[a,+\infty)$.

So there exists a constant $M > 0$ such that
$$|I(b)| < M \text{ for every } b \ge a.$$

As $f(x) \ge 0$ for each $x \ge a$, therefore $\int_a^b f\,d\alpha \ge 0$.

This gives $I(b) = \int_a^b f\,d\alpha \le M$ for every $b \ge a$.

Conversely, suppose that there exists a constant $M > 0$ such that $\int_a^b f\,d\alpha \le M$ for every $b \ge a$. This give $|I(b)| \le M$ for every $b \ge a$.

That is, I is bounded on $[a,+\infty)$.

Now for $b_2 \ge b_1 > a$, we have
$$I(b_2) = \int_a^{b_2} f\,d\alpha = \int_a^{b_1} f\,d\alpha + \int_{b_1}^{b_2} f\,d\alpha$$
$$\ge \int_a^{b_1} f\,d\alpha = I(b_1), \qquad \because \int_{b_1}^{b_2} f\,d\alpha \ge 0 \text{ as } f(x) \ge 0 \text{ for all } x \ge a.$$

This gives I is monotonically increasing on $[a,+\infty)$.

As I is monotonically increasing and bounded on $[a,+\infty)$, therefore $\lim_{b\to\infty} I(b)$ exists, that is, $\int_a^\infty f\,d\alpha$ converges.

➢ *Theorem: (Comparison Test)*

Assume that α is monotonically increasing on $[a,+\infty)$ and $f \in R(\alpha;a,b)$ for every $b \geq a$. If $0 \leq f(x) \leq g(x)$ for every $x \geq a$ and $\int_a^\infty g\,d\alpha$ converges, then $\int_a^\infty f\,d\alpha$ converges and we have

$$\int_a^\infty f\,d\alpha \leq \int_a^\infty g\,d\alpha.$$

Proof

Let $I_1(b) = \int_a^b f\,d\alpha$ and $I_2(b) = \int_a^b g\,d\alpha$, $b \geq a$.

Since $0 \leq f(x) \leq g(x)$ for every $x \geq a$,
therefore

$$I_1(b) \leq I_2(b) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\;(i)$$

Since $\int_a^\infty g\,d\alpha$ converges, there exists a constant $M > 0$ such that

$$\int_a^b g\,d\alpha \leq M \quad , \quad b \geq a \;\ldots\ldots\ldots\ldots\ldots\ldots\;(ii)$$

From (i) and (ii) we have $I_1(b) \leq M$ for every $b \geq a$.

This implies $\lim_{b\to\infty} I_1(b)$ exists and is finite, that is, $\int_a^\infty f\,d\alpha$ converges.

Also $\lim_{b\to\infty} I_1(b) \leq \lim_{b\to\infty} I_2(b) \leq M$,

this gives $\int_a^\infty f\,d\alpha \leq \int_a^\infty g\,d\alpha$.

➢ *Example*

Is the improper integral $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ convergent or divergent?

Solution:

Since $\sin^2 x \leq 1$ for all $x \in [1,+\infty)$, therefore $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ for all $x \in [1,+\infty)$.

This gives $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx \leq \int_1^\infty \frac{1}{x^2}\,dx$.

Now $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent, therefore $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

➢ Review:

- For all $a,b,c \in \mathbb{R}$, $|a-b|<c \Leftrightarrow c-b<a<c+b$ or $c-a<b<c+a$.
- If $\lim_{x\to\infty} f(x)=m$, then for all real $\varepsilon>0$, there exists $N>0$ such that
$$|f(x)-m|<\varepsilon \text{ whenever } |x|>N.$$
- If $\int_a^\infty f\,d\alpha$ converges(diverges), then $\int_N^\infty f\,d\alpha$ converges(diverges) if $N>a$.
- If $\int_N^\infty f\,d\alpha$ is convergent (divergent), then $\int_a^\infty f\,d\alpha$ is convergent (divergent) for $a<N$ if f is bounded in $[a,N]$.

➢ Theorem (Limit Comparison Test)

Assume that α is monotonically increasing on $[a,+\infty)$. Suppose that $f\in R(\alpha;a,b)$ and that $g\in R(\alpha;a,b)$ for every $b\geq a$, where $f(x)\geq 0$ and $g(x)\geq 0$ for $x\geq a$. If
$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=1,$$
then $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ both converge or both diverge.

Proof

Suppose $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$, then for all real $\varepsilon>0$, we can find some $N>0$, such that
$$\left|\frac{f(x)}{g(x)}-1\right|<\varepsilon \qquad \forall\ x>N\geq a.$$
$$\Rightarrow\ 1-\varepsilon<\frac{f(x)}{g(x)}<1+\varepsilon \qquad \forall\ x>N\geq a.$$
If we choose $\varepsilon=\frac{1}{2}$, then we have
$$\frac{1}{2}<\frac{f(x)}{g(x)}<\frac{3}{2} \qquad \forall\ x>N\geq a.$$
This implies $g(x)<2f(x)$ ……….(i) and $2f(x)<3g(x)$ ……....(ii)

From (i) $\int_N^\infty g\,d\alpha < 2\int_N^\infty f\,d\alpha$,

so if $\int_a^\infty f\,d\alpha$ converges, then $\int_N^\infty f\,d\alpha$ converges and hence by comparison test we get $\int_N^\infty g\,d\alpha$ is convergent, which implies $\int_a^\infty g\,d\alpha$ is convergent.

Now if $\int_a^\infty g\,d\alpha$ diverges, then $\int_N^\infty g\,d\alpha$ diverges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is divergent, which implies $\int_a^\infty f\,d\alpha$ is divergent.

From (*ii*), we have $2\int_N^\infty f\,d\alpha < 3\int_N^\infty g\,d\alpha$,

so if $\int_a^\infty g\,d\alpha$ converges, then $\int_N^\infty g\,d\alpha$ converges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is convergent, which implies $\int_a^\infty f\,d\alpha$ is convergent.

Now if $\int_a^\infty f\,d\alpha$ diverges, then $\int_N^\infty f\,d\alpha$ diverges and hence by comparison test we get $\int_N^\infty f\,d\alpha$ is divergent, which implies $\int_a^\infty f\,d\alpha$ is divergent.

$\Rightarrow$ The integrals $\int_a^\infty f\,d\alpha$ and $\int_a^\infty g\,d\alpha$ converge or diverge together.

➢ *Note*

The above theorem also holds if $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = c$, provided that $c \neq 0$. If $c = 0$, we can only conclude that convergence of $\int_a^\infty g\,d\alpha$ implies convergence of $\int_a^\infty f\,d\alpha$.

➢ *Example*

For every real p, the integral $\int_1^\infty e^{-x}x^p\,dx$ converges.

This can be seen by comparison of this integral with $\int_1^\infty \frac{1}{x^2}\,dx$.

Let $f(x) = e^{-x}x^p$ and $g(x) = \dfrac{1}{x^2}$.

Now $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty}\dfrac{e^{-x}x^p}{1/x^2}$

$\Rightarrow \lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} e^{-x}x^{p+2} = \lim\limits_{x\to\infty}\dfrac{x^{p+2}}{e^x} = 0$. (find this limit yourself)

Since $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent, therefore the given integral $\int_1^{\infty} e^{-x} x^p\,dx$ is also convergent.

➢ *Remark*

It is easy to show that if $\int_a^{\infty} f\,d\alpha$ and $\int_a^{\infty} g\,d\alpha$ are convergent, then

- $\int_a^{\infty} (f \pm g)\,d\alpha$ is convergent.
- $\int_a^{\infty} cf\,d\alpha$, where c is some constant, is convergent.

➢ *Note*

An improper integral $\int_a^{\infty} f\,d\alpha$ is said to converge absolutely if $\int_a^{\infty} |f|\,d\alpha$ converges.

It is said to be convergent conditionally if $\int_a^{\infty} f\,d\alpha$ converges but $\int_a^{\infty} |f|\,d\alpha$ diverges.

➢ *Theorem*

Assume α is monotonically increasing on $[a,+\infty)$. If $f \in R(\alpha;a,b)$ for every $b \ge a$ and if $\int_a^{\infty} |f|\,d\alpha$ converges, then $\int_a^{\infty} f\,d\alpha$ also converges.

Or: An absolutely convergent integral is convergent.

Proof

If $x \ge a$, $\pm f(x) \le |f(x)|$

$\Rightarrow |f(x)| - f(x) \ge 0 \qquad \Rightarrow 0 \le |f(x)| - f(x) \le 2|f(x)|$

$\Rightarrow \int_a^{\infty} \left(|f| - f\right) d\alpha$ converges.

Now difference of $\int_a^{\infty} |f|\,d\alpha$ and $\int_a^{\infty} \left(|f| - f\right) d\alpha$ is convergent,

that is, $\int_a^{\infty} f\,d\alpha$ is convergent.

➢ *Remark*

Every absolutely convergent integral is convergent.

➢ *Review*

- If $\lim\limits_{x\to\infty} f(x)=m$, then for all real $\varepsilon>0$, there exists real $N>0$ such that
$$|f(x)-m|<\varepsilon \text{ whenever } |x|>N.$$
- A sequence $\{a_n\}$ is said to be convergent if there exist a number l such that for all $\varepsilon>0$, there exists an integer n_0 (depending on ε) such that
$$|a_n-l|<\varepsilon \text{ whenever } n>n_0.$$
A number l is called limit of the sequence and we write $\lim\limits_{n\to\infty} a_n=l$.

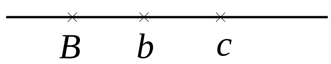
  A sequence $\{a_n\}$ is said to be Cauchy if there
- A sequence of real numbers is Cauchy if and only if it is convergent.

➢ *Theorem (Cauchy condition for infinite integrals)*

Assume that $f\in R(\alpha;a,b)$ for every $b\ge a$. Then the integral $\int\limits_a^\infty f\,d\alpha$ converges if, and only if, for every $\varepsilon>0$ there exists a $B>0$ such that $c>b>B$ implies
$$\left|\int\limits_b^c f\,d\alpha\right|<\varepsilon$$

Proof

Let $\int\limits_a^\infty f\,d\alpha$ be convergent, that is $\lim\limits_{b\to\infty}\int\limits_a^b f\,d\alpha=\int\limits_a^\infty f\,d\alpha$.



Then $\exists\ B>0$ such that
$$\left|\int\limits_a^b f\,d\alpha-\int\limits_a^\infty f\,d\alpha\right|<\frac{\varepsilon}{2} \quad \text{for every } b>B. \ \ldots\ldots\ldots(i)$$
Also for $c>b>B$,
$$\left|\int\limits_a^c f\,d\alpha-\int\limits_a^\infty f\,d\alpha\right|<\frac{\varepsilon}{2} \ \ldots\ldots\ldots\ldots\ldots (ii)$$

As we know $\int\limits_a^c f\,d\alpha=\int\limits_a^b f\,d\alpha+\int\limits_b^c f\,d\alpha$, this gives
$$\left|\int\limits_b^c f\,d\alpha\right|=\left|\int\limits_a^c f\,d\alpha-\int\limits_a^b f\,d\alpha\right|$$
$$=\left|\int\limits_a^c f\,d\alpha-\int\limits_a^\infty f\,d\alpha+\int\limits_a^\infty f\,d\alpha-\int\limits_a^b f\,d\alpha\right|$$
$$\le\left|\int\limits_a^c f\,d\alpha-\int\limits_a^\infty f\,d\alpha\right|+\left|\int\limits_a^\infty f\,d\alpha-\int\limits_a^b f\,d\alpha\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$
$$\Rightarrow \left|\int\limits_b^c f\,d\alpha\right|<\varepsilon \quad \text{when } c>b>B.$$

Conversely, assume that the Cauchy condition holds.

Define $a_n = \int_a^{a+n} f\,d\alpha$ if $n = 1, 2, \ldots\ldots$

Consider n, m such that $a+n, a+m > b > B$, then

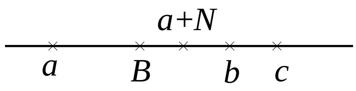
$$|a_n - a_m| = \left|\int_a^{a+n} f\,d\alpha - \int_a^{a+m} f\,d\alpha\right| = \left|\int_a^{b} f\,d\alpha + \int_b^{a+n} f\,d\alpha - \int_a^{b} f\,d\alpha - \int_b^{a+m} f\,d\alpha\right|$$

$$= \left|\int_b^{a+n} f\,d\alpha - \int_b^{a+m} f\,d\alpha\right| \le \left|\int_b^{a+n} f\,d\alpha\right| + \left|\int_b^{a+m} f\,d\alpha\right| < \varepsilon + \varepsilon = 2\varepsilon$$

This gives, the sequence $\{a_n\}$ is a Cauchy sequence $\Rightarrow$ it is convergent.

Let $\lim\limits_{n\to\infty} a_n = A$. Then for given $\varepsilon > 0$, choose B so that

$$|a_n - A| < \frac{\varepsilon}{2} \quad \text{whenever} \quad a+n \ge B.$$



Also for $\varepsilon > 0$, we can have (by Cauchy condition)

$$\left|\int_b^{c} f\,d\alpha\right| < \frac{\varepsilon}{2} \quad \text{if} \quad c > b > B.$$

Choose an integer N such that $a + N > B$.
Then, if $b > a + N$, we have

$$\left|\int_a^{b} f\,d\alpha - A\right| = \left|\int_a^{a+N} f\,d\alpha - A + \int_{a+N}^{b} f\,d\alpha\right|$$

$$\le |a_N - A| + \left|\int_{a+N}^{b} f\,d\alpha\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\Rightarrow \int_a^{\infty} f\,d\alpha = A$$

This completes the proof.

➢ *Remarks*

It follows from the above theorem that convergence of $\int_a^{\infty} f\,d\alpha$ implies $\lim\limits_{b\to\infty}\int_b^{b+\varepsilon} f\,d\alpha = 0$ for every fixed $\varepsilon > 0$.

However, this does not imply that $f(x) \to 0$ as $x \to \infty$.

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IMPROPER INTEGRAL OF THE SECOND KIND

➢ Definition

Let f be defined on the half open interval $(a,b]$ (having point of infinite discontinuity at a) and assume that $f\in R(\alpha;x,b)$ for every $x\in(a,b]$. Define a function I on $(a,b]$ as follows:

$$I(x)=\int_x^b f\,d\alpha \quad \text{if} \quad x\in(a,b]$$

If $\lim_{x\to a+} I(x)$ exists then the integral $\int_{a+}^b f\,d\alpha$ is said to be convergent. Otherwise, $\int_{a+}^b f\,d\alpha$ is said to be divergent. If $\lim_{x\to a+} I(x)=A$, the number A is called the value of the integral and we write $\int_{a+}^b f\,d\alpha=A.$

Similarly, if f is defined on $[a,b)$ (having point of infinite discontinuity at b) and $f\in R(\alpha;a,x)\ \ \forall\ x\in[a,b)$ then define $I(x)=\int_a^x f\,d\alpha$ if $x\in[a,b)$. If $\lim_{x\to b-} I(x)$ exists (finite) then we say $\int_a^{b-} f\,d\alpha$ is convergent.

➢ Note

The tests developed to check the behaviour of the improper integrals of Ist kind are applicable to improper integrals of IInd kind after making necessary modifications.

➢ Example

$f(x)=x^{-p}$ is defined on $(0,b]$ and $f\in R(x,b)$ for every $x\in(0,b]$.

$$I(x)=\int_x^b x^{-p}\,dx \qquad \text{if} \quad x\in(0,b]$$

$$=\int_{0+}^b x^{-p}\,dx=\lim_{\varepsilon\to0}\int_{0+\varepsilon}^b x^{-p}\,dx=\lim_{\varepsilon\to0}\left|\frac{x^{1-p}}{1-p}\right|_\varepsilon^b=\lim_{\varepsilon\to0}\frac{b^{1-p}-\varepsilon^{1-p}}{1-p}\quad , \quad (p\neq1)$$

$$=\begin{cases} \text{finite}\ , & p<1\\ \text{infinite}\ , & p>1\end{cases}$$

When $p=1$, we get $\int_\varepsilon^b \frac{1}{x}\,dx=\log b-\log\varepsilon\to\infty$ as $\varepsilon\to0$.

$\Rightarrow \int_{0+}^{b} x^{-1}\,dx$ also diverges.

Hence the integral converges when $p<1$ and diverges when $p\geq 1$.

➢ *Note*

If the two integrals $\int_{a+}^{c} f\,d\alpha$ and $\int_{c}^{b-} f\,d\alpha$ both converge, we write

$$\int_{a+}^{b-} f\,d\alpha = \int_{a+}^{c} f\,d\alpha + \int_{c}^{b-} f\,d\alpha$$

The definition can be extended to cover the case of any finite number of sums. We can also consider mixed combinations such as

$$\int_{a+}^{b} f\,d\alpha + \int_{b}^{\infty} f\,d\alpha \quad \text{which can be written as} \quad \int_{a+}^{\infty} f\,d\alpha\,.$$

➢ *Assignment*

Consider $\Gamma(p)=\int_{0}^{\infty} e^{-x}\,x^{p-1}\,dx$, $(p>0)$. Evaluate the convergence of this improper integral.

➢ *A Useful Comparison Integral*

$$\int_{a}^{b} \frac{dx}{(x-a)^n}$$

We have, if $n\neq 1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{(x-a)^n} = \left| \frac{1}{(1-n)(x-a)^{n-1}} \right|_{a+\varepsilon}^{b}$$

$$= \frac{1}{(1-n)}\left(\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}} \right)$$

Which tends to $\dfrac{1}{(1-n)(b-a)^{n-1}}$ or $+\infty$ according as $n<1$ or $n>1$, as $\varepsilon\to 0$.

Again, if $n=1$,

$$\int_{a+\varepsilon}^{b} \frac{dx}{x-a} = \log(b-a)-\log\varepsilon \to +\infty \quad \text{as} \quad \varepsilon\to 0\,.$$

Hence the improper integral $\int_{a}^{b} \dfrac{dx}{(x-a)^n}$ converges iff $n<1$.

➢ *Question*

Examine the convergence of

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ (*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$ (*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Solution

(*i*) $\displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$

Here '0' is the only point of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/3}\left(1+x^2\right)}$$

Take $g(x)=\dfrac{1}{x^{1/3}}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{1+x^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ have identical behaviours.

$\because \displaystyle\int_0^1 \frac{dx}{x^{1/3}}$ converges $\therefore \displaystyle\int_0^1 \frac{dx}{x^{1/3}\left(1+x^2\right)}$ also converges.

(*ii*) $\displaystyle\int_0^1 \frac{dx}{x^2(1+x)^2}$

Here '0' is the only point of infinite discontinuity of the given integrand.
We have

$$f(x)=\frac{1}{x^2(1+x)^2}$$

Take $g(x)=\dfrac{1}{x^2}$

Then $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1}{(1+x)^2}=1$

$\Rightarrow \int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ behave alike.

But $n=2$ being greater than 1, the integral $\int_0^1 g(x)\,dx$ does not converge. Hence the given integral also does not converge.

(*iii*) $\displaystyle\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/3}}$

Here ‘0’ and ‘1’ are the two points of infinite discontinuity of the integrand.
We have

$$f(x)=\frac{1}{x^{1/2}(1-x)^{1/3}}$$

We take any number between 0 and 1, say $1/2$, and examine the convergence of the improper integrals $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^{1} f(x)\,dx$.

To examine the convergence of $\int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{x^{1/2}}$

Then

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{1}{(1-x)^{1/3}}=1$$

$\because \int_0^{1/2}\frac{1}{x^{1/2}}dx$ converges $\therefore \int_0^{1/2}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

To examine the convergence of $\int_{1/2}^{1}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$, we take $g(x)=\frac{1}{(1-x)^{1/3}}$

Then

$$\lim_{x\to1}\frac{f(x)}{g(x)}=\lim_{x\to1}\frac{1}{x^{1/2}}=1$$

$\because \int_{1/2}^{1}\frac{1}{(1-x)^{1/3}}dx$ converges $\because \int_{1/2}^{1}\frac{1}{x^{1/2}(1-x)^{1/3}}dx$ is convergent.

Hence $\int_0^1 f(x)\,dx$ converges.

➢ *Question*

Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists iff m, n are both positive.

Solution

The integral is proper if $m\geq1$ and $n\geq1$.
The number ‘0’ is a point of infinite discontinuity if $m<1$ and the number ‘1’ is a point of infinite discontinuity if $n<1$.
Let $m<1$ and $n<1$.

We take any number, say $\frac{1}{2}$, between 0 & 1 and examine the convergence of the improper integrals $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ and $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ at '0' and '1' respectively.

Convergence at 0:

We write

$$f(x) = x^{m-1}(1-x)^{n-1} = \frac{(1-x)^{n-1}}{x^{1-m}} \quad \text{and take} \quad g(x) = \frac{1}{x^{1-m}}$$

Then $\frac{f(x)}{g(x)} \to 1$ as $x \to 0$

As $\int_0^{1/2} \frac{1}{x^{1-m}}\,dx$ is convergent at 0 iff $1-m<1$ i.e. $m>0$

We deduce that the integral $\int_0^{1/2} x^{m-1}(1-x)^{n-1}dx$ is convergent at 0, iff m is +ive.

Convergence at 1:

We write $f(x) = x^{m-1}(1-x)^{n-1} = \frac{x^{m-1}}{(1-x)^{1-n}}$ and take $g(x) = \frac{1}{(1-x)^{1-n}}$

Then $\frac{f(x)}{g(x)} \to 1$ as $x \to 1$

As $\int_{1/2}^{1} \frac{1}{(1-x)^{1-n}}dx$ is convergent, iff $1-n<1$ i.e. $n>0$.

We deduce that the integral $\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}dx$ converges iff $n>0$.

Thus $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ exists for positive values of m, n only.

It is a function which depends upon m & n and is defined for all positive values of m & n. It is called Beta function.

➢ *Question*

Show that the following improper integrals are convergent.

(*i*) $\int_1^{\infty} \sin^2 \frac{1}{x}\, dx$ (*ii*) $\int_1^{\infty} \frac{\sin^2 x}{x^2}\, dx$ (*iii*) $\int_0^1 \frac{x \log x}{(1+x)^2}\, dx$ (*iv*) $\int_0^1 \log x \cdot \log(1+x)\, dx$

Solution

(*i*) Let $f(x) = \sin^2 \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$

then $\lim_{x\to\infty}\dfrac{f(x)}{g(x)}=\lim_{x\to\infty}\dfrac{\sin^2\frac{1}{x}}{\frac{1}{x^2}}=\lim_{y\to 0}\left(\dfrac{\sin y}{y}\right)^2=1$

$\Rightarrow \int_1^\infty f(x)\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$ behave alike.

$\because \int_1^\infty \frac{1}{x^2}\,dx$ is convergent $\therefore \int_1^\infty \sin^2\frac{1}{x}\,dx$ is also convergent.

(*ii*) $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$

Take $f(x)=\dfrac{\sin^2 x}{x^2}$ and $g(x)=\dfrac{1}{x^2}$

$\sin^2 x\le 1 \quad\Rightarrow \dfrac{\sin^2 x}{x^2}\le\dfrac{1}{x^2} \quad \forall\ x\in(1,\infty)$

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges $\therefore \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges.

➢ *Note*

$\int_0^1 \frac{\sin^2 x}{x^2}\,dx$ is a proper integral because $\lim_{x\to 0}\frac{\sin^2 x}{x^2}=1$ so that '0' is not a point of infinite discontinuity. Therefore $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

(*iii*) $\int_0^1 \frac{x\log x}{(1+x)^2}\,dx$

$\because \log x < x, \quad x\in(0,1)$

$\therefore x\log x < x^2$

$\Rightarrow \dfrac{x\log x}{(1+x)^2} < \dfrac{x^2}{(1+x)^2}$

Now $\int_0^1 \frac{x^2}{(1+x)^2}\,dx$ is a proper integral.

$\therefore \int_0^1 \frac{x\log x}{(1+x)^2}\,dx$ is convergent.

(*iv*) $\int_0^1 \log x\cdot\log(1+x)\,dx$

$\because\ \log x < x \qquad \therefore\ \log(x+1) < x+1$

$\Rightarrow\ \log x\cdot\log(1+x) < x(x+1)$

$\because\ \int_0^1 x(x+1)\,dx$ is a proper integral $\quad\therefore\ \int_0^1 \log x\cdot\log(1+x)\,dx$ is convergent.

➢ *Note*

(i) $\int_0^a \frac{1}{x^p}\,dx$ diverges when $p\ge 1$ and converges when $p<1$.

(ii) $\int_a^\infty \frac{1}{x^p}\,dx$ converges iff $p>1$.

➢ *Questions*

Examine the convergence of

(i) $\int_1^\infty \frac{x}{(1+x)^3}\,dx \qquad$ (ii) $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx \qquad$ (iii) $\int_1^\infty \frac{dx}{x^{1/3}(1+x)^{1/2}}$

Solution

(***i***) Let $f(x)=\frac{x}{(1+x)^3}$ and take $g(x)=\frac{1}{x^2}$.

As $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{x^3}{(1+x)^3}=1$

Therefore the two integrals $\int_1^\infty \frac{x}{(1+x)^3}\,dx$ and $\int_1^\infty \frac{1}{x^2}\,dx$ have identical behaviour for convergence at ∞.

$\because\ \int_1^\infty \frac{1}{x^2}\,dx$ is convergent $\quad\therefore\ \int_1^\infty \frac{x}{(1+x)^3}\,dx$ is convergent.

(***ii***) Let $f(x)=\frac{1}{(1+x)\sqrt{x}}$ and take $g(x)=\frac{1}{x\sqrt{x}}=\frac{1}{x^{3/2}}$

We have $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=\lim\limits_{x\to\infty}\frac{x}{1+x}=1$

and $\int_1^\infty \frac{1}{x^{3/2}}\,dx$ is convergent. Thus $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$ is convergent.

(***iii***) Let $f(x)=\frac{1}{x^{1/3}(1+x)^{1/2}}$

we take $g(x)=\frac{1}{x^{1/3}\cdot x^{1/2}}=\frac{1}{x^{5/6}}$

We have $\lim\limits_{x\to\infty}\dfrac{f(x)}{g(x)}=1$ and $\int\limits_1^\infty \dfrac{1}{x^{5/6}}dx$ is divergent $\therefore$ $\int\limits_1^\infty f(x)\,dx$ is divergent.

➢ *Question*

Show that $\int\limits_{-\infty}^{\infty}\dfrac{1}{1+x^2}dx$ is convergent.

Solution

We have

$$\int\limits_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\lim_{a\to\infty}\left[\int\limits_{-a}^{0}\frac{1}{1+x^2}dx+\int\limits_0^a\frac{1}{1+x^2}dx\right]$$

$$=\lim_{a\to\infty}\left[\int\limits_0^a\frac{1}{1+x^2}dx+\int\limits_0^a\frac{1}{1+x^2}dx\right]=2\lim_{a\to\infty}\left[\int\limits_0^a\frac{1}{1+x^2}dx\right]$$

$$=2\lim_{a\to\infty}\left|\tan^{-1}x\right|_0^a=2\left(\frac{\pi}{2}\right)=\pi$$

therefore the integral is convergent.

➢ *Question*

Show that $\int\limits_0^\infty\dfrac{\tan^{-1}x}{1+x^2}dx$ is convergent.

Solution

$\because\ (1+x^2)\cdot\dfrac{\tan^{-1}x}{(1+x^2)}=\tan^{-1}x\to\dfrac{\pi}{2}$ as $x\to\infty$

$\int\limits_0^\infty\dfrac{\tan^{-1}x}{1+x^2}dx$ & $\int\limits_0^\infty\dfrac{1}{1+x^2}dx$ behave alike.

Here $f(x)=\dfrac{\tan^{-1}x}{1+x^2}$ and $g(x)=\dfrac{1}{1+x^2}$

$\because\ \int\limits_0^\infty\dfrac{1}{1+x^2}dx$ is convergent $\therefore$ A given integral is convergent.

➢ *Question*

Show that $\int\limits_0^\infty e^{-x}\cos x\,dx$ is absolutely convergent.

Solution

$\because\ \left|e^{-x}\cos x\right|<e^{-x}$ and $\int\limits_0^\infty e^{-x}\,dx=1$

$\therefore$ the given integral is absolutely convergent. (Comparison test)

➢ *Question*

Show that $\int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent.

Solution

$\because$ $e^{-x} < 1$ and $1 + x^2 > 1$ for all $x \in (0,1)$.

$$\therefore \quad \frac{e^{-x}}{\sqrt{1-x^4}} < \frac{1}{\sqrt{(1-x^2)(1+x^2)}} < \frac{1}{\sqrt{1-x^2}}$$

Also $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\varepsilon\to 0} \int_0^{1-\varepsilon} \frac{1}{\sqrt{1-x^2}}\,dx$

$$= \lim_{\varepsilon\to 0} \sin^{-1}(1-\varepsilon) = \frac{\pi}{2}$$

$\Rightarrow \int_0^1 \frac{e^{-x}}{\sqrt{1-x^4}}\,dx$ is convergent. (by comparison test)

References:

1. Tom M. Apostol, Mathematical Analysis, 2nd Edition, 1974.
2. S.C. Malik and Savita Arora, Mathematical Analysis, 2nd Edition, 1992.