Chapter 2 – Sequences

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Course instructor: Dr. Atiq ur Rehman Class: MSc-II
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Sequence

A sequence is a function whose domain of definition is the set of natural numbers.

Or it can also be defined as an ordered set.

Notation:

An infinite sequence is denoted as

$\{s_n\}_{n=1}^{\infty}$ or $\{s_n : n \in \mathbb{N}\}$ or $\{s_1, s_2, s_3, ...\}$ or simply as $\{s_n\}$,

e.g. i) $\{n\} = \{1, 2, 3, \ldots\}$.

ii) $\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$.

iii) $\left\{(-1)^{n+1}\right\} = \{1, -1, 1, -1, \ldots\}$.

Subsequence

It is a sequence whose terms are contained in given sequence.

A subsequence of $\{s_n\}_{n=1}^{\infty}$ is usually written as $\{s_{n_k}\}^{\infty}$.

Increasing Sequence

A sequence $\{s_n\}$ is said to be an increasing sequence if $s_{n+1} \geq s_n \quad \forall\ n \geq 1$.

Decreasing Sequence

A sequence $\{s_n\}$ is said to be an decreasing sequence if $s_{n+1} \leq s_n \quad \forall\ n \geq 1$.

Monotonic Sequence

A sequence $\{s_n\}$ is said to be monotonic sequence if it is either increasing or decreasing.

$\{s_n\}$ is monotonically increasing if $s_{n+1} - s_n \geq 0$ or $\frac{s_{n+1}}{s_n} \geq 1, \ \forall\ n \geq 1$.

$\{s_n\}$ is monotonically decreasing if $s_n - s_{n+1} \geq 0$ or $\frac{s_n}{s_{n+1}} \geq 1, \ \forall\ n \geq 1$.

Strictly Increasing or Decreasing

$\{s_n\}$ is called strictly increasing or decreasing according as

$$s_{n+1} > s_n \quad \text{or} \quad s_{n+1} < s_n \qquad \forall\ n \geq 1.$$

Bernoulli's Inequality

Let $p \in \mathbb{R}$, $p \geq -1$ and $p \neq 0$ then for $n \geq 2$ we have

$$(1+p)^n > 1+np.$$

Proof:

We shall use mathematical induction to prove this inequality.

If $n = 2$

$L.H.S = (1+p)^2 = 1+2p+p^2$,

$R.H.S = 1+2p$,

$\Rightarrow L.H.S > R.H.S$,

i.e. condition I of mathematical induction is satisfied.

Suppose $(1+p)^k > 1+kp$ .... (i) where $k \geq 2$

$$\begin{aligned} \text{Now} \quad (1+p)^{k+1} &= (1+p)(1+p)^k \\ &> (1+p)(1+kp) \qquad \text{using } (i) \\ &= 1+kp+p+kp^2 \\ &= 1+(k+1)p+kp^2 \\ &\geq 1+(k+1)p \qquad \text{ignoring } kp^2 \geq 0, \end{aligned}$$

$\Rightarrow (1+p)^{k+1} > 1+(k+1)p$.

Since the truth for $n = k$ implies the truth for $n = k+1$ therefore condition II of mathematical induction is satisfied. Hence we conclude that $(1+p)^n > 1+np$. ❑

Example:

Prove that $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is an increasing sequence.

Let $s_n = \left(1+\frac{1}{n}\right)^n$ where $n \geq 1$.

To prove that this sequence is an increasing sequence, we use $p = \frac{-1}{n^2}$, $n \geq 2$ in Bernoulli's inequality to have

$$\left(1-\frac{1}{n^2}\right)^n > 1-\frac{n}{n^2}$$

$$\Rightarrow \left(\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right)^n > 1-\frac{1}{n}$$

$$\Rightarrow \left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1} = \left(1+\frac{1}{n-1}\right)^{n-1}$$

$$\Rightarrow s_n > s_{n-1} \qquad \forall\ n \geq 1.$$

This shows that $\{s_n\}$ is increasing sequence. ❑

Example:

Prove that a sequence $\left\{\left(1+\dfrac{1}{n}\right)^{n+1}\right\}$ is a decreasing sequence.

Let $t_n = \left(1+\dfrac{1}{n}\right)^{n+1}$; $n \ge 1$.

We use $p = \dfrac{1}{n^2-1}$ in Bernoulli's inequality.

$$\left(1+\frac{1}{n^2-1}\right)^n > 1+\frac{n}{n^2-1} \quad \ldots\ldots\ldots.. (i)$$

where

$$1+\frac{1}{n^2-1} = \frac{n^2}{n^2-1} = \left(\frac{n}{n-1}\right)\left(\frac{n}{n+1}\right)$$

$$\Rightarrow \left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right) = \left(\frac{n}{n-1}\right) \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Now $t_{n-1} = \left(1+\dfrac{1}{n-1}\right)^n = \left(\dfrac{n}{n-1}\right)^n$

$$= \left(\left(1+\frac{1}{n^2-1}\right)\left(\frac{n+1}{n}\right)\right)^n \qquad \text{from } (ii)$$

$$= \left(1+\frac{1}{n^2-1}\right)^n \left(\frac{n+1}{n}\right)^n$$

$$> \left(1+\frac{n}{n^2-1}\right)\left(\frac{n+1}{n}\right)^n \qquad \text{from } (i)$$

$$> \left(1+\frac{1}{n}\right)\left(\frac{n+1}{n}\right)^n \qquad \because \frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n}$$

$$= \left(\frac{n+1}{n}\right)^{n+1} = t_n,$$

i.e. $t_{n-1} > t_n$.

Hence the given sequence is decreasing sequence. ❑

Bounded Sequence

A sequence $\{s_n\}$ is said to be bounded if there exists a positive real number λ such that $|s_n| < \lambda \quad \forall \; n \in \mathbb{N}$.

If S and s are the supremum and infimum of elements forming the bounded sequence $\{s_n\}$ we write $S = \sup s_n$ and $s = \inf s_n$.

All the elements of the sequence s_n such that $|s_n| < \lambda \quad \forall \; n \in \mathbb{N}$ lie with in the strip $\{y : -\lambda < y < \lambda\}$. But the elements of the unbounded sequence can not be contained in any strip of a finite width.

Examples

(i) $\{u_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$ is a bounded sequence

(ii) $\{v_n\} = \{\sin nx\}$ is also bounded sequence. Its supremum is 1 and infimum is -1.

(iii) The geometric sequence $\{ar^{n-1}\}$, $r > 1$ is an unbounded above sequence. It is bounded below by a.

(iv) $\left\{\tan \dfrac{n\pi}{2}\right\}$ is an unbounded sequence.

Convergence of the Sequence

A sequence $\{s_n\}$ of real numbers is said to convergent to limit 's' as $n \to \infty$, if for every positive real number $\varepsilon > 0$, there exists a positive integer n_0, depending upon ε, such that $|s_n - s| < \varepsilon \quad \forall \; n > n_0$.

Theorem

A convergent sequence of real number has one and only one limit (i.e. Limit of the sequence is unique.)

Proof:

Suppose $\{s_n\}$ converges to two limits s and t, where $s \neq t$.

Put $\varepsilon = \dfrac{|s-t|}{2}$ then there exits two positive integers n_1 and n_2 such that

$$|s_n - s| < \varepsilon \qquad \forall \; n > n_1$$

and $$|s_n - t| < \varepsilon \qquad \forall \; n > n_2.$$

$\Rightarrow$ $|s_n - s| < \varepsilon$ and $|s_n - t| < \varepsilon$ hold simultaneously $\forall \; n > \max(n_1, n_2)$.

Thus for all $n > \max(n_1, n_2)$ we have

$$\begin{aligned} |s-t| &= |s - s_n + s_n - t| \\ &\leq |s_n - s| + |s_n - t| \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

$$\Rightarrow \left| s-t \right| < 2\left(\frac{\left| s-t \right|}{2} \right)$$

$$\Rightarrow \left| s-t \right| < \left| s-t \right|$$

Which is impossible, therefore the limit of the sequence is unique. ❑

Note: If $\{s_n\}$ converges to s then all of its infinite subsequence converge to s.

Cauchy Sequence

A sequence $\{x_n\}$ of real number is said to be a *Cauchy sequence* if for given positive real number ε, $\exists$ a positive integer $n_0(\varepsilon)$ such that

$$\left| x_n - x_m \right| < \varepsilon \qquad \forall \; m,n > n_0$$

Theorem

A Cauchy sequence of real numbers is bounded.

Proof:

Let $\{s_n\}$ be a Cauchy sequence.

Take $\varepsilon = 1$, then there exits a positive integers n_0 such that

$$\left| s_n - s_m \right| < 1 \qquad \forall \; m,n > n_0 .$$

Fix $m = n_0 + 1$ then

$$\begin{aligned} \left| s_n \right| &= \left| s_n - s_{n_0+1} + s_{n_0+1} \right| \\ &\le \left| s_n - s_{n_0+1} \right| + \left| s_{n_0+1} \right| \\ &< 1 + \left| s_{n_0+1} \right| \qquad \forall \; n > n_0 \\ &< \lambda \qquad \forall \; n > 1 \text{ , and } \lambda = 1 + \left| s_{n_0+1} \right| \quad (n_0 \text{ changes as } \varepsilon \text{ changes}) \end{aligned}$$

Hence we conclude that $\{S_n\}$ is a Cauchy sequence, which is bounded one. ❑

Note:

(i) Convergent sequence is bounded.

(ii) The converse of the above theorem does not hold.

i.e. every bounded sequence is not Cauchy.

Consider the sequence $\{s_n\}$ where $s_n = (-1)^n$, $n \ge 1$. It is bounded sequence because

$$\left| (-1)^n \right| = 1 < 2 \qquad \forall \; n \ge 1.$$

But it is not a Cauchy sequence if it is then for $\varepsilon = 1$ we should be able to find a positive integer n_0 such that $\left| s_n - s_m \right| < 1$ for all $m,n > n_0$.

But with $m = 2k+1$, $n = 2k+2$ when $2k+1 > n_0$, we arrive at

$$\begin{aligned} \left| s_n - s_m \right| &= \left| (-1)^{2n+2} - (-1)^{2k+1} \right| \\ &= \left| 1+1 \right| = 2 < 1 \quad \text{is absurd.} \end{aligned}$$

Hence $\{s_n\}$ is not a Cauchy sequence. Also this sequence is not a convergent sequence. (it is an oscillatory sequence).

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Divergent Sequence

A $\{s_n\}$ is said to be divergent if it is not convergent or it is unbounded.

e.g. $\{n^2\}$ is divergent, it is unbounded.

(ii) $\{(-1)^n\}$ tends to 1 or -1 according as *n* is even or odd. It oscillates finitely.

(iii) $\{(-1)^n n\}$ is a divergent sequence. It oscillates infinitely.

Note: If two subsequence of a sequence converges to two different limits then the sequence itself is a divergent.

Theorem

If $s_n < u_n < t_n \quad \forall\ n \geq n_0$ and if both the $\{s_n\}$ and $\{t_n\}$ converge to same limits as *s*, then the sequence $\{u_n\}$ also converges to *s*.

Proof:

Since the sequence $\{s_n\}$ and $\{t_n\}$ converge to the same limit *s*, therefore, for given $\varepsilon > 0$ there exists two positive integers $n_1, n_2 > n_0$ such that

$$|s_n - s| < \varepsilon \qquad \forall\ n > n_1$$

$$|t_n - s| < \varepsilon \qquad \forall\ n > n_2$$

i.e.

$$s - \varepsilon < s_n < s + \varepsilon \qquad \forall\ n > n_1$$

$$s - \varepsilon < t_n < s + \varepsilon \qquad \forall\ n > n_2$$

Since we have given

$$s_n < u_n < t_n \qquad \forall\ n > n_0$$

$$\therefore\ s - \varepsilon < s_n < u_n < t_n < s + \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$$

$$\Rightarrow\ s - \varepsilon < u_n < s + \varepsilon \qquad \forall\ n > \max(n_0, n_1, n_2)$$

i.e. $|u_n - s| < \varepsilon \quad \forall\ n > \max(n_0, n_1, n_2)$

i.e. $\lim\limits_{n\to\infty} u_n = s$. ❑

Example

Show that $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

Solution

Using Bernoulli's Inequality

$$\left(1 + \frac{1}{\sqrt{n}}\right)^n \geq 1 + \frac{n}{\sqrt{n}} \geq \sqrt{n} \geq 1 \qquad \forall\ n.$$

Also

$$\left(1 + \frac{1}{\sqrt{n}}\right)^2 = \left[\left(1 + \frac{1}{\sqrt{n}}\right)^n\right]^{\frac{2}{n}} > \left(\sqrt{n}\right)^{\frac{2}{n}} > n^{\frac{1}{n}} \geq 1,$$

$$\Rightarrow\ 1 \le n^{\frac{1}{n}} < \left(1 + \frac{1}{\sqrt{n}}\right)^2,$$

$$\Rightarrow\ \lim_{n\to\infty} 1 \le \lim_{n\to\infty} n^{\frac{1}{n}} < \lim_{n\to\infty}\left(1 + \frac{1}{\sqrt{n}}\right)^2,$$

$$\Rightarrow\ 1 \le \lim_{n\to\infty} n^{\frac{1}{n}} < 1.$$

i.e. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1.$ ❑

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Example

Show that $\lim_{n\to\infty}\left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \text{..........} + \frac{1}{(2n)^2}\right) = 0$

Solution

Consider

$$s_n = \left(\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \ldots + \frac{1}{(2n)^2}\right)$$

and

$$\frac{n}{(2n)^2} < s_n < \frac{n}{n^2}$$

$$\Rightarrow\ \frac{1}{4n} < s_n < \frac{1}{n}$$

$$\Rightarrow\ \lim_{n\to\infty}\frac{1}{4n} < \lim_{n\to\infty} s_n < \lim_{n\to\infty}\frac{1}{n}$$

$$\Rightarrow\ 0 < \lim_{n\to\infty} s_n < 0$$

$$\Rightarrow\ \lim_{n\to\infty} S_n = 0$$ ❑

Theorem

If the sequence $\{s_n\}$ converges to s then $\exists$ a positive integer n such that $|s_n| > \frac{1}{2}s$.

Proof:

We fix $\varepsilon = \frac{1}{2}|s| > 0$

$\Rightarrow\ \exists$ a positive integer n_1 such that

$$|s_n - s| < \varepsilon \qquad \text{for } n > n_1$$

$$\Rightarrow\ |s_n - s| < \frac{1}{2}|s|$$

Now

$$\frac{1}{2}|s| = |s| - \frac{1}{2}|s|$$

$$< |s| - |s_n - s| \quad \le |s + (s_n - s)|$$

$$\Rightarrow \frac{1}{2}|s| < |s_n|.$$ ❑

Theorem

Let a and b be fixed real numbers if $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively, then

(i) $\{as_n + bt_n\}$ converges to $as + bt$.

(ii) $\{s_n t_n\}$ converges to st.

(iii) $\left\{\frac{s_n}{t_n}\right\}$ converges to $\frac{s}{t}$, provided $t_n \ne 0 \;\; \forall \; n$ and $t \ne 0$.

Proof:

Since $\{s_n\}$ and $\{t_n\}$ converge to s and t respectively,

$$\therefore \; |s_n - s| < \varepsilon \qquad \forall \; n > n_1 \in \mathbb{N}$$

$$|t_n - t| < \varepsilon \qquad \forall \; n > n_2 \in \mathbb{N}$$

Also $\exists \; \lambda > 0$ such that $|s_n| < \lambda \quad \forall \; n > 1$ $\qquad$ ($\because$ $\{s_n\}$ is bounded)

(i) We have

$$|(as_n + bt_n) - (as + bt)| = |a(s_n - s) + b(t_n - t)|$$

$$\le |a(s_n - s)| + |b(t_n - t)|$$

$$< |a|\varepsilon + |b|\varepsilon \qquad \forall \; n > \max(n_1, n_2)$$

$$= \varepsilon_1,$$

where $\varepsilon_1 = |a|\varepsilon + |b|\varepsilon$ a certain number.

This implies $\{as_n + bt_n\}$ converges to $as + bt$.

(ii) $$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st|$$

$$= |s_n(t_n - t) + t(s_n - s)| \quad \le |s_n| \cdot |(t_n - t)| + |t| \cdot |(s_n - s)|$$

$$< \lambda\varepsilon + |t|\varepsilon \qquad \forall \; n > \max(n_1, n_2)$$

$= \varepsilon_2,$ $\qquad$ where $\varepsilon_2 = \lambda\varepsilon + |t|\varepsilon$ a certain number.

This implies $\{s_n t_n\}$ converges to st.

(iii) $$\left|\frac{1}{t_n} - \frac{1}{t}\right| = \left|\frac{t - t_n}{t_n t}\right|$$

$$= \frac{|t_n - t|}{|t_n||t|} < \frac{\varepsilon}{\frac{1}{2}|t||t|} \qquad \forall \; n > \max(n_1, n_2) \qquad \because \; |t_n| > \frac{1}{2}t$$

$$= \frac{\varepsilon}{\frac{1}{2}|t|^2} = \varepsilon_3, \qquad \text{where } \varepsilon_3 = \frac{\varepsilon}{\frac{1}{2}|t|^2} \text{ a certain number.}$$

This implies $\left\{\frac{1}{t_n}\right\}$ converges to $\frac{1}{t}$.

Hence $\left\{\frac{s_n}{t_n}\right\} = \left\{s_n \cdot \frac{1}{t_n}\right\}$ converges to $s \cdot \frac{1}{t} = \frac{s}{t}$. (from (ii)) ❑

Theorem

For each irrational number *x*, there exists a sequence $\{r_n\}$ of distinct rational numbers such that $\lim_{n\to\infty} r_n = x$.

Proof:

Since *x* and *x* + 1 are two different real numbers

∵ ∃ a rational number r_1 such that

$$x < r_1 < x + 1$$

Similarly ∃ a rational number $r_2 \neq r_1$ such that

$$x < r_2 < \min\left(r_1, x + \frac{1}{2}\right) < x + 1$$

Continuing in this manner we have

$$x < r_3 < \min\left(r_2, x + \frac{1}{3}\right) < x + 1$$

$$x < r_4 < \min\left(r_3, x + \frac{1}{4}\right) < x + 1$$

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$$x < r_n < \min\left(r_{n-1}, x + \frac{1}{n}\right) < x + 1$$

This implies that ∃ a sequence $\{r_n\}$ of the distinct rational number such that

$$x - \frac{1}{n} < x < r_n < x + \frac{1}{n}.$$

Since

$$\lim_{n\to\infty}\left(x - \frac{1}{n}\right) = \lim_{n\to\infty}\left(x + \frac{1}{n}\right) = x.$$

Therefore

$$\lim_{n\to\infty} r_n = x.$$ ❑

Theorem

Let a sequence $\{s_n\}$ be a bounded sequence.

(i) If $\{s_n\}$ is monotonically increasing then it converges to its supremum.

(ii) If $\{s_n\}$ is monotonically decreasing then it converges to its infimum.

Proof

Let $S = \sup s_n$ and $s = \inf s_n$

Take $\varepsilon > 0$

(i) Since $S = \sup s_n$

$\therefore\ \exists\ s_{n_0}$ such that $S - \varepsilon < s_{n_0}$

Since $\{s_n\}$ is $\uparrow$ ($\uparrow$ stands for monotonically increasing)

$\therefore\ S - \varepsilon < s_{n_0} < s_n < S < S + \varepsilon$ for $n > n_0$

$\Rightarrow\ S - \varepsilon < s_n < S + \varepsilon$ for $n > n_0$

$\Rightarrow\ |s_n - S| < \varepsilon$ for $n > n_0$

$\Rightarrow\ \lim\limits_{n\to\infty} s_n = S$

(ii) Since $s = \inf S_n$

$\therefore\ \exists\ s_{n_1}$ such that $s_{n_1} < s + \varepsilon$

Since $\{s_n\}$ is $\downarrow$. ($\downarrow$ stands for monotonically decreasing)

$\therefore\ s - \varepsilon < s < s_n < s_{n_1} < s + \varepsilon$ for $n > n_1$

$\Rightarrow\ s - \varepsilon < s_n < s + \varepsilon$ for $n > n_1$

$\Rightarrow\ |s_n - s| < \varepsilon$ for $n > n_1$

Thus $\lim\limits_{n\to\infty} s_n = s$ ❑

Note

A monotonic sequence can not oscillate infinitely.

Example:

Show that $\left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ is bounded sequence.

Consider $\{s_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$

As shown earlier it is an increasing sequence

Take $s_{2n} = \left(1+\dfrac{1}{2n}\right)^{2n}$, then $\sqrt{s_{2n}} = \left(1+\dfrac{1}{2n}\right)^n$,

$$\Rightarrow\ \frac{1}{\sqrt{s_{2n}}} = \left(\frac{2n}{2n+1}\right)^n \qquad \Rightarrow\ \frac{1}{\sqrt{s_{2n}}} = \left(1-\frac{1}{2n+1}\right)^n$$

Using Bernoulli's Inequality we have

$$\Rightarrow \frac{1}{\sqrt{s_{2n}}} \geq 1-\frac{n}{2n+1} > 1-\frac{n}{2n} = \frac{1}{2} \qquad \because \left(1-\frac{1}{2n+1}\right)^n \geq 1-\frac{n}{2n+1}$$

$$\Rightarrow \sqrt{s_{2n}} < 2 \qquad \forall\ n=1,2,3,\ldots$$

$$\Rightarrow s_{2n} < 4 \qquad \forall\ n=1,2,3,\ldots$$

$$\Rightarrow s_n < s_{2n} < 4 \qquad \forall\ n=1,2,3,\ldots$$

Which show that the sequence $\{s_n\}$ is bounded one.

Hence $\{s_n\}$ is a convergent sequence the number to which it converges is its supremum, which is denoted by 'e' and $2<e<3$. ❑

Recurrence Relation

A sequence is said to be defined *recursively* or *by recurrence relation* if the general term is given as a relation of its preceding and succeeding terms in the sequence together with some initial condition.

Example:

Let $t_1 > 0$ and let $\{t_n\}$ be defined by $t_{n+1} > 2-\frac{1}{t_n}$ for $n \geq 1$.

(i) Show that $\{t_n\}$ is decreasing sequence.

(ii) It is bounded below.

(iii) Find the limit of the sequence.

Let $t_1 > 0$ and let $\{t_n\}$ be defined by $t_{n+1} > 2-\frac{1}{t_n}$; $n \geq 1$

$$\Rightarrow t_n > 0 \qquad \forall\ n \geq 1$$

Also $\quad t_n - t_{n+1} = t_n - 2 + \frac{1}{t_n}$

$$= \frac{t_n^2 - 2t_n + 1}{t_n} = \frac{(t_n-1)^2}{t_n} > 0.$$

$$\Rightarrow t_n > t_{n+1} \qquad \forall\ n \geq 1.$$

This implies that t_n is monotonically decreasing.

Since $t_n > 1 \quad \forall\ n \geq 1$,

$\Rightarrow t_n$ is bounded below $\quad \Rightarrow t_n$ is convergent.

Let us suppose $\lim\limits_{n\to\infty} t_n = t$.

Then $\quad \lim\limits_{n\to\infty} t_{n+1} = \lim\limits_{n\to\infty} t_n \quad \Rightarrow \lim\limits_{n\to\infty}\left(2-\frac{1}{t_n}\right) = \lim\limits_{n\to\infty} t_n$

$$\Rightarrow 2-\frac{1}{t} = t \quad \Rightarrow \frac{2t-1}{t} = t \quad \Rightarrow 2t-1 = t^2 \quad \Rightarrow t^2 - 2t + 1 = 0$$

$$\Rightarrow (t-1)^2 = 0 \quad \Rightarrow t = 1.$$

❑

Theorem

Every Cauchy sequence of real numbers has a convergent subsequence.

Proof:

Suppose $\{s_n\}$ is a Cauchy sequence.

Let $\varepsilon > 0$ then $\exists$ a positive integer $n_0 \geq 1$ such that

$$\left|s_{n_k} - s_{n_{k-1}}\right| < \frac{\varepsilon}{2^k} \qquad \forall\ n_k, n_{k-1},\ k = 1,2,3,\ldots\ldots\ldots$$

Put $\quad b_k = \left(s_{n_1} - s_{n_0}\right) + \left(s_{n_2} - s_{n_1}\right) + \ldots + \left(s_{n_k} - s_{n_{k-1}}\right)$

$$\Rightarrow |b_k| = \left|\left(s_{n_1} - s_{n_0}\right) + \left(s_{n_2} - s_{n_1}\right) + \ldots + \left(s_{n_k} - s_{n_{k-1}}\right)\right|$$

$$\leq \left|\left(s_{n_1} - s_{n_0}\right)\right| + \left|\left(s_{n_2} - s_{n_1}\right)\right| + \ldots + \left|\left(s_{n_k} - s_{n_{k-1}}\right)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \ldots + \frac{\varepsilon}{2^k}$$

$$= \varepsilon\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k}\right)$$

$$= \varepsilon\left(\frac{\frac{1}{2}\left(1 - \frac{1}{2^k}\right)}{1 - \frac{1}{2}}\right) = \varepsilon\left(1 - \frac{1}{2^k}\right)$$

$$\Rightarrow |b_k| < \varepsilon \qquad \forall\ k \geq 1$$

$\Rightarrow \{b_k\}$ is convergent

$\because\ b_k = s_{n_k} - s_{n_0} \qquad \therefore\ s_{n_k} = b_k + s_{n_0}$,

where s_{n_0} is a certain fix number therefore $\{s_{n_k}\}$ which is a subsequence of $\{s_n\}$ is convergent. ❑

Theorem (Cauchy's General Principle for Convergence)

A sequence of real number is convergent if and only if it is a Cauchy sequence.

Proof:

<u>Necessary Condition</u>

Let $\{s_n\}$ be a convergent sequence, which converges to s.

Then for given $\varepsilon > 0$ $\exists$ a positive integer n_0, such that

$$|s_n - s| < \frac{\varepsilon}{2} \quad \forall\ n > n_0$$

Now for $n > m > n_0$

$$|s_n - s_m| = |s_n - s + s_m - s|$$

$$\leq |s_n - s| + |s_m - s|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Which shows that $\{s_n\}$ is a Cauchy sequence.

Sufficient Condition

Let us suppose that $\{s_n\}$ is a Cauchy sequence then for $\varepsilon > 0$, $\exists$ a positive integer m_1 such that

$$|s_n - s_m| < \frac{\varepsilon}{2} \;\; \forall \;\; n, m > m_1 \;\ldots\ldots\ldots\; (i)$$

Since $\{s_n\}$ is a Cauchy sequence

therefore it has a subsequence $\{s_{n_k}\}$ converging to s (say).

$\Rightarrow$ $\exists$ a positive integer m_2 such that

$$|s_{n_k} - s| < \frac{\varepsilon}{2} \qquad \forall \;\; n > m_2 \;\ldots\ldots\ldots\; (ii)$$

Now

$$\begin{aligned}|s_n - s| &= |s_n - s_{n_k} + s_{n_k} - s| \\ &\le |s_n - s_{n_k}| + |s_{n_k} - s| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall \;\; n > \max(m_1, m_2),\end{aligned}$$

this shows that $\{s_n\}$ is a convergent sequence. ❑

Example

Prove that $\left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots + \frac{1}{n}\right\}$ is divergent sequence.

Let $\{t_n\}$ be defined by

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots + \frac{1}{n}.$$

For $m, n \in \mathbb{N}$, $n > m$ we have

$$\begin{aligned}|t_n - t_m| &= \frac{1}{m+1} + \frac{1}{m+2} + \ldots\ldots\ldots\ldots + \frac{1}{n} \\ &> (n-m)\frac{1}{n} \;=\; 1 - \frac{m}{n}.\end{aligned}$$

In particular if $n = 2m$ then

$$|t_n - t_m| > \frac{1}{2}.$$

This implies that $\{t_n\}$ is not a Cauchy sequence therefore it is divergent. ❑

Theorem (nested intervals)

Suppose that $\{I_n\}$ is a sequence of the closed interval such that $I_n = [a_n, b_n]$, $I_{n+1} \subset I_n \ \forall \ n \geq 1$, and $(b_n - a_n) \to 0$ as $n \to \infty$ then $\bigcap I_n$ contains one and only one point.

Proof:

Since $I_{n+1} \subset I_n$

$$\therefore \ a_1 < a_2 < a_3 < \ldots < a_{n-1} < a_n < b_n < b_{n-1} < \ldots < b_3 < b_2 < b_1$$

$\{a_n\}$ is increasing sequence, bounded above by b_1 and bounded below by a_1.

And $\{b_n\}$ is decreasing sequence bounded below by a_1 and bounded above by b_1.

$\Rightarrow$ $\{a_n\}$ and $\{b_n\}$ both are convergent.

Suppose $\{a_n\}$ converges to a and $\{b_n\}$ converges to b.

But $|a-b| = |a - a_n + a_n - b_n + b_n - b|$

$$\leq |a_n - a| + |a_n - b_n| + |b_n - b| \to 0 \quad \text{as} \quad n \to \infty.$$

$$\Rightarrow a = b$$

and $\quad a_n < a < b_n \quad \forall \ n \geq 1.$ ❑

Theorem (Bolzano-Weierstrass theorem)

Every bounded sequence has a convergent subsequence.

Proof:

Let $\{s_n\}$ be a bounded sequence.

Take $a_1 = \inf s_n$ and $b_1 = \sup s_n$

Then $a_1 < s_n < b_1 \quad \forall \ n \geq 1$.

Now bisect interval $[a_1, b_1]$ such that at least one of the two sub-intervals contains infinite numbers of terms of the sequence.

Denote this sub-interval by $[a_2, b_2]$.

If both the sub-intervals contain infinite number of terms of the sequence then choose the one on the right hand.

Then clearly $a_1 \leq a_2 < b_2 \leq b_1$.

Suppose there exist a subinterval $[a_k, b_k]$ such that

$$a_1 \leq a_2 \leq \ldots \leq a_k < b_k \leq \ldots \leq b_2 \leq b_1$$

$$\Rightarrow (b_k - a_k) = \frac{1}{2^k}(b_1 - a_1)$$

Bisect the interval $[a_k, b_k]$ in the same manner and choose $[a_{k+1}, b_{k+1}]$ to have

$$a_1 \leq a_2 \leq \ldots \leq a_k \leq a_{k+1} < b_{k+1} \leq b_k \leq \ldots \leq b_2 \leq b_1$$

and

$$b_{k+1} - a_{k+1} = \frac{1}{2^{k+1}}(b_1 - a_1)$$

This implies that we obtain a sequence of interval $[a_n, b_n]$ such that

$$b_n - a_n = \frac{1}{2^n}(b_1 - a_1) \to 0 \text{ as } n \to \infty.$$

$\Rightarrow$ we have a unique point s such that

$$s = \bigcap [a_n, b_n]$$

there are infinitely many terms of the sequence whose length is $\varepsilon > 0$ that contain s. For $\varepsilon = 1$ there are infinitely many values of n such that

$$|s_n - s| < 1$$

Let n_1 be one of such value then

$$|s_{n_1} - s| < 1$$

Again choose $n_2 > n_1$ such that

$$|s_{n_2} - s| < \frac{1}{2}$$

Continuing in this manner we find a sequence $\{s_{n_k}\}$ for each positive integer k such that $n_k < n_{k+1}$ and

$$|s_{n_k} - s| < \frac{1}{k} \qquad \forall\ k = 1,2,3,\ldots\ldots\ldots\ldots$$

Hence there is a subsequence $\{s_{n_k}\}$ which converges to s. ❑

Limit Inferior of the sequence

Suppose $\{s_n\}$ is bounded below then we define limit inferior of $\{s_n\}$ as follow

$$\lim_{n\to\infty}(\inf s_n) = \lim_{n\to\infty} u_k, \quad \text{where} \quad u_k = \inf\{s_n : n \geq k\}$$

If s_n is not bounded below then

$$\lim_{n\to\infty}(\inf s_n) = -\infty.$$

Limit Superior of the sequence

Suppose $\{s_n\}$ is bounded above then we define limit superior of $\{s_n\}$ as follow

$$\lim_{n\to\infty}(\sup s_n) = \lim_{n\to\infty} v_k, \quad \text{where} \quad v_k = \inf\{s_n : n \geq k\}$$

If s_n is not bounded above then we have

$$\lim_{n\to\infty}(\sup s_n) = +\infty.$$

Note:

(i) A bounded sequence has unique limit inferior and superior

(ii) Let $\{s_n\}$ contains all the rational numbers, then every real number is a subsequencial limit then limit superior of s_n is $+\infty$ and limit inferior of s_n is $-\infty$

(iii) Let $\{s_n\} = (-1)^n\left(1 + \frac{1}{n}\right)$

then limit superior of s_n is 1 and limit inferior of s_n is -1.

(*iv*) Let $s_n = \left(1+\frac{1}{n}\right)\cos n\pi$.

Then $u_k = \inf\{s_n : n \geq k\}$

$$= \inf\left\{\left(1+\frac{1}{k}\right)\cos k\pi, \left(1+\frac{1}{k+1}\right)\cos(k+1)\pi, \left(1+\frac{1}{k+2}\right)\cos(k+2)\pi, \ldots\ldots\ldots\ldots\right\}$$

$$= \begin{cases} \left(1+\frac{1}{k}\right)\cos k\pi & \text{if } k \text{ is odd} \\ \left(1+\frac{1}{k+1}\right)\cos(k+1)\pi & \text{if } k \text{ is even} \end{cases}$$

$\Rightarrow \lim\limits_{n\to\infty}(\inf s_n) = \lim\limits_{n\to\infty} u_k = -1$

Also $v_k = \sup\{s_n : n \geq k\}$

$$= \begin{cases} \left(1+\frac{1}{k+1}\right)\cos(k+1)\pi & \text{if } k \text{ is odd} \\ \left(1+\frac{1}{k}\right)\cos k\pi & \text{if } k \text{ is even} \end{cases}$$

$\Rightarrow \lim\limits_{n\to\infty}(\inf s_n) = \lim\limits_{n\to\infty} v_k = 1$ ❑

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Theorem

If $\{s_n\}$ is a convergent sequence then

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}(\inf s_n) = \lim_{n\to\infty}(\sup s_n)$$

Proof:

Let $\lim\limits_{n\to\infty} s_n = s$ then for a real number $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$|s_n - s| < \varepsilon \qquad \forall\ n \geq n_0 \ \ldots\ldots\ldots\ (i)$$

i.e. $\quad s-\varepsilon < s_n < s+\varepsilon \qquad \forall\ n \geq n_0$

If $\quad v_k = \sup\{s_n : n \geq k\}$

Then $\quad s-\varepsilon < v_n < s+\varepsilon \qquad \forall\ k \geq n_0$

$\Rightarrow s-\varepsilon < \lim\limits_{k\to\infty} v_n < s+\varepsilon \qquad \forall\ k \geq n_0 \ \ldots\ldots\ldots\ldots\ (ii)$

from (*i*) and (*ii*) we have

$$s = \lim_{k\to\infty} \sup\{s_n\}$$

We can have the same result for limit inferior of $\{s_n\}$ by taking

$$u_k = \inf\{s_n : n \geq k\}.$$ ❑

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