

# **Continuous Time Markov Chains (CTMC)**

**(Updated)**

**Start with real numbers (15 minutes)**

## Motivation

Imagine a toddler who has only two moods (states):

- ☀️ Happy (State 0)
- ☁️ Crying (State 1)

This is a CTMC because the toddler stays in one mood for a random amount of time before switching to the other.

### The States

- State 0: Happy
- State 1: Crying

### The Transition Rates

In continuous time, we don't use fixed probabilities (like "50% chance to switch"). Instead, we use **rates** (how fast something tends to happen).



**Rate  $\lambda$ :** The rate at which the toddler goes from **Happy**  $\rightarrow$  **Crying**.

Let's say  $\lambda = 2$  times per hour. (This means, on average, they get upset twice an hour).

**Rate  $\mu$ :** The rate at which the toddler goes from **Crying**  $\rightarrow$  **Happy**.

Let's say  $\mu = 4$  times per hour. (They recover quickly, usually within 15 minutes).

### **How it works in Continuous Time**

- **The Waiting Game:** If the toddler is currently **Happy**, they will stay happy for a random duration. In a CTMC, this duration follows an **Exponential Distribution**.
- **Memoryless:** It doesn't matter how long the toddler has already been happy. The probability of them crying in the next minute is always the same. They don't "get tired" of being happy; the switch happens randomly based on the rate  $\lambda$ .

## A Slightly More Complex Example: The Coffee Shop Queue (M/M/1 Queue)

This is the most famous example, often called the Birth-Death Process.

- States: The number of people in line (0, 1, 2, 3, ...).
- Transitions:
  - Arrival ( $\lambda$ ): A new customer arrives, and the state goes up (0  $\rightarrow$  1, or 1  $\rightarrow$  2). This happens randomly (e.g., 10 customers per hour).
  - Service ( $\mu$ ): The barista finishes serving a customer, and the state goes down (2  $\rightarrow$  1). This also happens randomly (e.g., the barista serves 12 customers per hour).

### Why is this "Continuous"?

In a discrete model (like a turn-based game), a customer would arrive exactly every minute. In this Continuous model, a customer might arrive at 10:01, the next at 10:02:30, and the next at 10:15. The system evolves smoothly in real-time.

## Continuous Time Markov Chain (CTMC)

A **Continuous-Time Markov Chain (CTMC)** is a stochastic process that evolves over a continuous time set (usually  $[0, \infty)$ ) and satisfies the Markov property.

### Formal Definition

A continuous-time Markov chain (CTMC) is a stochastic process

$$\{X(t), t \geq 0\}$$

with a countable state space  $S$ , satisfying the Markov property:

$$P(X(t+h) = j \mid X(t) = i, \text{ past history}) = P(X(t+h) = j \mid X(t) = i), \forall i, j \in S.$$

and whose transition behavior is characterized by a generator matrix  $Q = [q_{ij}]$ .

## Steps to Construct the Generator Matrix $Q$

### 1. List the states:

Identify all possible states of the system and label them (e.g., 0,1,2, ...).

### 2. Determine transition rates:

For each pair of *different* states  $i \neq j$ , find the rate  $q_{ij}$ .

### 3. Fill off-diagonal entries:

Place the transition rates in the matrix:

$q_{ij}$  = rate of transition from  $i$  to  $j$ ,  $i \neq j$ .

### 4. Set impossible transitions to zero:

If a transition from  $i$  to  $j$  is not allowed, then set  $q_{ij} = 0$ .

### 5. Compute diagonal entries

For each state  $i$ , set the diagonal entry as:

$$q_{ii} = - \sum_{j \neq i} q_{ij}$$

(negative of the total rate of leaving state  $i$ ).

### 6. Check row sums

Verify that each row of the  $Q$ -matrix sums to zero:

$$\sum_j q_{ij} = 0.$$

Finding the **stationary distribution (or stationary probabilities)** is *one important way* to study the long-term behavior of a Continuous-Time Markov Chain (CTMC)

Let's see the example of the Toddler, which was discussed above:

### **Example 1:**

A toddler has two moods modelled by a continuous-time Markov chain (CTMC):

State 0: Happy,            State 1: Crying

The toddler switches from Happy to Crying at rate 2 per hour, and from Crying to Happy at rate 4 per hour.

- (a) Write the  $Q$ -matrix of the CTMC.
- (b) Find the stationary distribution.

### **Step 1: Define the States and Rates**

- **State 0:** Happy
- **State 1:** Crying

**Step 2: Determine the transition rates**

- **Rate 0 → 1:**  $q_{01} = 2$  per hour (gets upset)
- **Rate 1 → 0:**  $q_{10} = 4$  per hour (recovers)

**Step 3: Construct the Generator Matrix Q**

From state 0 (Happy):

- Can only go to state 1 at rate 1, that is

$$q_{01} = 2 \text{ and } q_{00} = -2 \text{ (negative total exit rate)}$$

From state 1 (Crying):

- Can only go to state 0 at rate 4, that is

$$q_{10} = 4 \text{ and } q_{11} = -4 \text{ (negative total exit rate)}$$

**Generator Matrix:**

$$Q = \begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}$$

Check: Row sums = 0.

#### Step 4: Set Up Stationary Equations

Let  $\pi = [\pi_0 \ \pi_1]$ , that is

- $\pi_0$  = long-run probability of being in state 0 (Happy)
- $\pi_1$  = long-run probability of being in state 1 (Crying)

We solve  $\pi Q = 0$  with  $\pi_0 + \pi_1 = 1$ .

**From**  $\pi Q = 0$ :

$$\pi_0 \cdot (-2) + \pi_1 \cdot 4 = 0$$

$$\pi_0 \cdot 2 + \pi_1 \cdot (-4) = 0$$

These are the same equation. Using the first:

$$-2\pi_0 + 4\pi_1 = 0$$

$$2\pi_0 = 4\pi_1$$

$$\pi_0 = 2\pi_1$$

**Step 4: Add Normalization Condition**

$$\pi_0 + \pi_1 = 1$$

Substitute  $\pi_0 = 2\pi_1$ :

$$2\pi_1 + \pi_1 = 1$$

$$3\pi_1 = 1$$

$$\pi_1 = \frac{1}{3} \approx 0.333$$

$$\pi_0 = 2 \cdot \frac{1}{3} = \frac{2}{3} \approx 0.667$$

### Step 5: Stationary Distribution

$$\pi = [\pi_0 \quad \pi_1] = \left[ \frac{2}{3} \quad \frac{1}{3} \right] \approx [0.667 \quad 0.333]$$

#### Interpretation:

- **66.7%** of time: Toddler is happy (state 0)
- **33.3%** of time: Toddler is crying (state 1)

## Example 2:

A machine in a factory can be in one of three conditions:

- **State 0:** The machine is working normally
- **State 1:** The machine has a minor failure
- **State 2:** The machine has a major failure

The machine's condition changes over time according to the following rules:

- When the machine is working normally (State 0), it develops a **minor failure** at a rate of **2 per hour**.
- When the machine has a minor failure (State 1), it worsens into a **major failure** at a rate of **1 per hour**.
- When the machine has a major failure (State 2), it is repaired and returns to the **normal working state** at a rate of **3 per hour**.

Assume the machine behaviour can be modelled as a **continuous-time Markov chain**.

**States:**

A machine has 3 states:

- 0 = Working normally                      1 = Minor failure                      2 = Major failure

**Rates:**

- $0 \rightarrow 1$  at rate 2, i.e.  $q_{01} = 2, q_{02} = 0$ .
- $1 \rightarrow 2$  at rate 1, i.e.  $q_{12} = 1, q_{10} = 0$ .
- $2 \rightarrow 0$  after repair at rate 3, i.e.  $q_{20} = 3, q_{21} = 0$ .

**Q-matrix:**

$$Q = \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ 3 & 0 & -3 \end{bmatrix}$$

This is a simple CTMC with 3 states and realistic interpretation.

## 2) Balance equations $\pi Q = 0$

Take  $\pi = [\pi_0 \quad \pi_1 \quad \pi_2]$ , then

$$\pi Q = 0 \Rightarrow [\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} -2 & 2 & 0 \\ 0 & -1 & 1 \\ 3 & 0 & -3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2\pi_0 + 3\pi_2 \\ 2\pi_0 - \pi_1 \\ \pi_1 - 3\pi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus

$$-2\pi_0 + 3\pi_2 = 0 \Rightarrow 3\pi_2 = 2\pi_0 \Rightarrow \pi_2 = \frac{2}{3}\pi_0.$$

and

$$2\pi_0 - \pi_1 = 0 \Rightarrow \pi_1 = 2\pi_0.$$

### 3) Normalize $\sum \pi_i = 1$

$$\pi_0 + \pi_1 + \pi_2 = \pi_0 + 2\pi_0 + \frac{2}{3}\pi_0 = \frac{11}{3}\pi_0 = 1$$

so  $\pi_0 = 3/11$ .

Therefore

$$\pi = (\pi_0, \pi_1, \pi_2) = \left( \frac{3}{11}, \frac{6}{11}, \frac{2}{11} \right)$$

### Interpretation

- Long-run fraction of time machine is **working**:  $\pi_0 = 3/11 \approx 0.2727$ .
- In **minor failure**:  $6/11 \approx 0.5455$ .
- In **major failure**:  $2/11 \approx 0.1818$ .

### Example 3

During a study session in the university library, a student's activity changes randomly over time and can be modelled as a continuous-time Markov chain (CTMC) with three states:

Studying, Distracted and Taking a Break

The transition rates (per five minutes) are given as follows:

Studying  $\rightarrow$  Distracted at rate 3      Studying  $\rightarrow$  Taking a Break at rate 2

Distracted  $\rightarrow$  Studying at rate 2      Distracted  $\rightarrow$  Taking a Break at rate 1

Taking a Break  $\rightarrow$  Studying at rate 1      Taking a Break  $\rightarrow$  Distracted at rate 2

- (a) Write the generator matrix (Q-matrix) of the CTMC.
- (b) Find the stationary probabilities of the CTMC.

**Solution:**

**State Space:** Let the states be ordered as

0 = Studying,    1 = Distracted,    2 = Taking a Break.

**Rates:**

- $0 \rightarrow 1$  at rate 3, i.e.  $q_{01} = 3$  and  $0 \rightarrow 2$  at rate 2, i.e.  $q_{02} = 2$ .
- $1 \rightarrow 0$  at rate 2, i.e.  $q_{10} = 2$  and  $1 \rightarrow 2$  at rate 1, i.e.  $q_{12} = 1$ .
- $2 \rightarrow 0$  at rate 1, i.e.  $q_{20} = 1$  and  $2 \rightarrow 1$  at rate 2, i.e.  $q_{21} = 2$ .
- $q_{00} = -(q_{01} + q_{02}) = -5$ . Also  $q_{11} = -3$ , and  $q_{22} = -3$ .

**Generator matrix (Q-matrix):**

Hence the generator matrix is

$$Q = \begin{bmatrix} -5 & 3 & 2 \\ 2 & -3 & 1 \\ 1 & 2 & -3 \end{bmatrix}.$$

**Balance equations  $\pi Q = 0$** 

Take  $\pi = [\pi_0 \quad \pi_1 \quad \pi_2]$ , then

$$\pi Q = 0 \Rightarrow [\pi_0 \quad \pi_1 \quad \pi_2] \begin{bmatrix} -5 & 3 & 2 \\ 2 & -3 & 1 \\ 1 & 2 & -3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -5\pi_0 + 2\pi_1 + \pi_2 \\ 3\pi_0 - 3\pi_1 + 2\pi_2 \\ 2\pi_1 + \pi_1 - 3\pi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$-5\pi_0 + 2\pi_1 + \pi_2 = 0 \quad \text{--- (i)}$$

$$3\pi_0 - 3\pi_1 + 2\pi_2 = 0 \quad \text{--- (ii)}$$

$$2\pi_1 + \pi_1 - 3\pi_2 = 0 \quad \text{--- (iii)}$$

From (i):

$$\pi_2 = 5\pi_0 - 2\pi_1 \quad \text{--- (iv)}$$

Substitute into (ii):

$$\begin{aligned} 3\pi_0 - 3\pi_1 + 2(5\pi_0 - 2\pi_1) &= 0 \\ \Rightarrow 13\pi_0 - 7\pi_1 &= 0 \quad \Rightarrow \pi_1 = \frac{13}{7}\pi_0. \end{aligned}$$

Using in (iv)

$$\pi_2 = 5\pi_0 - 2 \cdot \frac{13}{7}\pi_0 = \frac{9}{7}\pi_0.$$

**Normalize  $\sum \pi_i = 1$**

$$\begin{aligned} \pi_0 + \pi_1 + \pi_2 &= 1 \\ \Rightarrow \pi_0 + \frac{13}{7}\pi_0 + \frac{9}{7}\pi_0 &= 1 \\ \Rightarrow \frac{29}{7}\pi_0 &= 1 \quad \Rightarrow \pi_0 = \frac{7}{29} \end{aligned}$$

Therefore,

$$\pi_1 = \frac{13}{7} \cdot \frac{7}{29} = \frac{13}{29},$$

$$\pi_2 = \frac{9}{7} \cdot \frac{7}{29} = \frac{9}{29}.$$

Thus

$$\pi = [\pi_0 \quad \pi_1 \quad \pi_2] = \left[ \frac{7}{29} \quad \frac{13}{29} \quad \frac{9}{29} \right].$$

**Conclusion:**

**Q-matrix** =  $Q = \begin{bmatrix} -5 & 3 & 2 \\ 2 & -3 & 1 \\ 1 & 2 & -3 \end{bmatrix}$  and **Stationary distribution** =  $\pi = \left[ \frac{7}{29} \quad \frac{13}{29} \quad \frac{9}{29} \right].$

*-: Thank you so much :-*