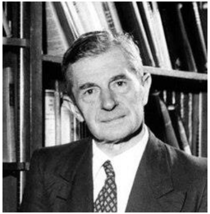
Chapman–Kolmogorov Equation



Sydney Chapman (1888–1970)
British Mathematician

Andrey Kolmogorov (1903-1987)
Russian Mathematician

Review:

◆ **Conditional law of total probability**

For any events A *and a partition of events* $\{B_k\}$*, and for any condition* C *with* $P(C) > 0$*:*

$$P(A \mid C) = \sum_k P(A, B_k \mid C).$$

◆ ◆ **Example:**

Suppose:

- C: “Today is Monday.”
- The day is divided into two possibilities:
  - B_1: “It is sunny.”
  - B_2: “It is cloudy.”
- A: “The teacher comes to school.”

Assume the following probabilities conditional on Monday:

$$P(B_1 \mid C) = 0.6\,, \qquad P(B_2 \mid C) = 0.4.$$

And:

$$P(A \mid B_1, C) = 0.8, \quad P(A \mid B_2, C) = 0.3$$

Then by the conditional law of total probability:

$$P(A \mid C) = P(A, B_1 \mid C) + P(A, B_2 \mid C)$$

Compute each term:

$$P(A, B_1 \mid C) = P(A \mid B_1, C)P(B_1 \mid C) = 0.8 \times 0.6 = 0.48$$
$$P(A, B_2 \mid C) = P(A \mid B_2, C)P(B_2 \mid C) = 0.3 \times 0.4 = 0.12$$

Add then:

$$P(A \mid C) = 0.48 + 0.12 = 0.60.$$

- **Matrix multiplication**

If $A = [a_{ik}]$ and $B = [b_{kj}]$, then

$$C = AB = \left[\sum_k a_{ik} b_{kj}\right].$$

Chapman-Kolmogorov Equation

Let $\{X_n : n \geq 0\}$ be a Markov chain with a discrete state space S. Let

$$P_{ij}^{(n)} = P(X_{m+n} = j \,|X_m = i)$$

denote the n-step transition probability from state i to state j.

The Chapman-Kolmogorov Equation states that for all $i, j \in S$ and for any non-negative integers $m, n \geq 0$, the following holds:

$$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)} .$$

Proof

The (i, j) th entry of the matrix $P^{(m+n)}$ is given by

$$\begin{aligned}
P_{ij}^{(m+n)} &= P(X_{m+n} = j \mid X_0 = i) \\
&= \sum_k P(X_{m+n} = j, X_m = k \mid X_0 = i) \qquad \textit{by total law of prob.} \\
&= \sum_k \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_0 = i)} \qquad \because P(A|B) = \frac{P(A \cap B)}{P(B)} \\
&= \sum_k \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_m = k, X_0 = i)} \cdot \frac{P(X_m = k, X_0 = i)}{P(X_0 = i)} \\
&= \sum_k P(X_{m+n} = j \mid X_m = k, X_0 = i) \cdot P(X_m = k \mid X_0 = i) \\
&= \sum_k P(X_{m+n} = j \mid X_m = k) \cdot P(X_m = k \mid X_0 = i).
\end{aligned}$$

Hence, we proved

$$P_{ij}^{(m+n)} = \sum_{k} P_{ik}^{(m)} P_{kj}^{(n)} .$$

Chapman-Kolmogorov Differential Equation

Take $P_{ij}^{(m+n)} = P_{ij}(m+n)$, then differentiating w.r.t. n, we get

$$P'_{ij}\,(m+n) = \frac{\partial}{\partial n}\,P_{ij}\,(m+n) \ldots\ldots (1)$$

By Chapman-Kolmogorov equation, we have

$$P^{(m+n)} = P^{(m)}P^{(n)} \;\text{ or }\; P(m+n) = P(m)P(n)$$

Using it in (1), we have

$$P'_{ij}(m+n) = \sum_{k} P_{ik}(m)\,\frac{d}{dn}P_{kj}(n)$$

At $n = 0$, we have

$$P'_{ij}(m) = \sum_k P_{ik}(m) a_{kj},$$

where $a_{kj} = \frac{d}{dn} p_{kj}(n)$ at $n = 0$.

This gives Chapman-Kolmogorov equation as differential equation.

If P represents the transition probability, that is square matrix, so in matrix notation,

$$\frac{d}{dn} P = PA.$$

Chapman-Kolmogorov Theorem

Let $\{X_n, n \geq 0\}$ be Markov chain with TPM (transition probability matrix) $P = \left[P_{ij}\right]$ and n-step TPM $P^{(n)} = \left[P_{ij}^{(n)}\right]$, where

$$P_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \text{ and } P_{ij}^{(1)} = P_{ij}.$$

Then the following properties hold:

1. $P^{(n+m)} = P^{(n)}P^{(m)}$
2. $P^{(n)} = P^n$

Proof: (1) The (i, j) th entry of the matrix $P^{(m+n)}$ is given by

$$\begin{aligned}
P_{ij}^{(m+n)} &= P(X_{m+n} = j \mid X_0 = i) \\
&= \sum_k P(\,X_{m+n} = j, X_m = k \mid X_0 = i\,) \qquad \textit{by total law of prob.} \\
&= \sum_k \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_0 = i)} \qquad \because P(A|B) = \frac{P(A \cap B)}{P(B)} \\
&= \sum_k \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_m = k, X_0 = i)} \cdot \frac{P(X_m = k, X_0 = i)}{P(X_0 = i)} \\
&= \sum_k P(X_{m+n} = j \mid X_m = k, X_0 = i) \cdot P(X_m = k \mid X_0 = i) \\
&= \sum_k \; P(X_{m+n} = j \mid X_m = k) \cdot P(X_m = k \mid X_0 = i).
\end{aligned}$$

Hence, we proved

$$P_{ij}^{(m+n)} = \sum_{k} P_{ik}^{(m)} P_{kj}^{(n)} .$$

This ultimately gives us

$$\begin{aligned}\left[P_{ij}^{(m+n)}\right] &= \left[\sum_{k} P_{ik}^{(m)} P_{kj}^{(n)}\right] \\ &= \left[P_{ik}^{(m)}\right]\left[P_{kj}^{(n)}\right]\end{aligned}$$

So, we have

$$P^{(m+n)} = P^{(m)} P^{(n)} \ldots\ldots (2)$$

(2) We prove this result by induction.
By definition, we have $P = P^{(1)}$.

Thus, the assertion $P^{(n)} = P^n$ is true for $n = 1$
Assume $P^{(m)} = P^m$ for some positive integer m.
Then by using (2), we have $P^{(m+n)} = P^{(m)}P^{(n)}$, that is,

$$P^{(m+1)} = P^{(m)}P^{(1)} = P^mP = P^{m+1}.$$

Thus, the assertion $P^{(n)} = P^n$ is also true for $m + 1$.
Hence, by the principle of mathematical induction, $P^{(n)} = P^n$ is true for any positive integer n.

Thank you so much