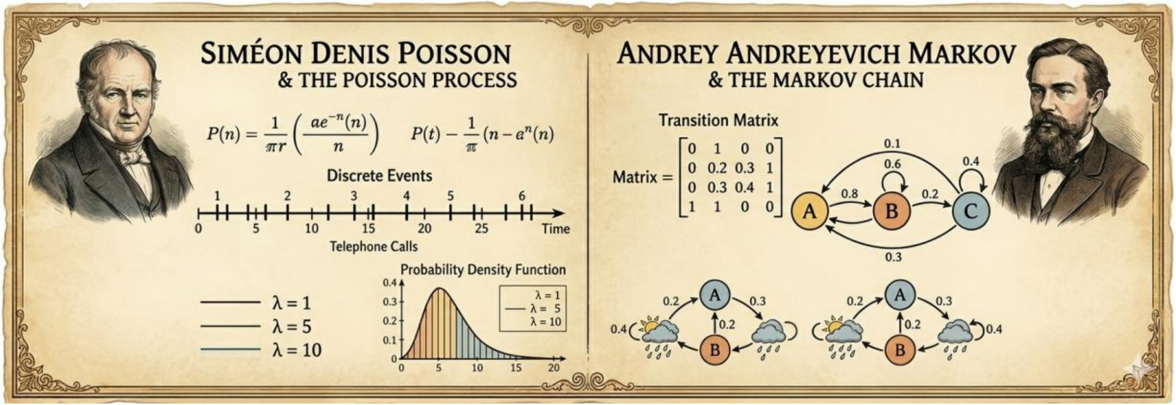
Poisson Process vs Markov Chain



What they are

Poisson Process is a model for counting random events over time (or space).

It answers:

"How many events happen in a given interval?"

Markov Chain is a model for a system moving between states.

It answers:

"What state will the system be in next?"

The Markov Property

Both share the **memoryless property**, the future depends only on the present, not the past. This is actually why a Poisson process is a Markov chain (a special case).

Key formulas

Poisson Process*: Probability of k events in time t with rate λ:*

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Markov Chain: *Probability of being in state j after n steps from state i:*

$$P(X_n = j \mid X_0 = i) = P_{ij}^{(n)}$$

Relationship

A Poisson process is a continuous-time Markov chain where:

- States are counts (0, 1, 2, …)
- Transitions only go up by 1
- Holding time in each state is ***exponential*** with rate λ

More generally, **Continuous-Time Markov Chains (CTMCs)** generalize Poisson processes by allowing transitions between any states, each with their own rate.

Question:

A hospital emergency room receives patients at an average rate of 4 per hour. Assuming arrivals follow a Poisson process, what is the probability that exactly 6 patients arrive in a 2-hour window?

Solution:

Here

$$\lambda = 4/hr, \;\; t \; = \; 2\,hr \;\Rightarrow\; \lambda t \; = \; 8.$$

We have

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

$$\Longrightarrow \;\; P(N(2) = 6) = \frac{8^6 \cdot e^{-6}}{6!} \approx \; 0.1221$$

Hence there is 0.1221 or 12.21% probability that exactly 6 patients arrive in a 2-hour window.

Question:

A coffee shop receives customers at an average rate of 5 per hour. Assuming arrivals follow a Poisson process, find the probability that *at most 2* customers arrive in a 30-minute period.

Solution:

Step 1: Identify the rate for the given problem:

$$\lambda = 5/hr, \qquad t = 0.5\,hr \implies \lambda t = 5 \times 0.5 = 2.5$$

Step 2: Write out "at most 2" as a sum.

$$P(N(0.5) \le 2) = P(N(0.5) = 0) + P(N(0.5) = 1) + P(N(0.5) = 2).$$

Step 3: Apply $P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ for each term.

$$P(N(0.5) = 0) = \frac{(2.5)^0 \cdot e^{-2.5}}{0!} \approx 0.0821$$

$$P(N(0.5) = 1) = \frac{(2.5)^1 \cdot e^{-2.5}}{1!} \approx 0.2052$$

$$P(N(0.5) = 2) = \frac{(2.5)^2 \cdot e^{-2.5}}{2!} \approx 0.2565$$

Step 4: Sum the three terms.

$$P(N(0.5) \leq 2) = 0.0821 + 0.2052 + 0.2565 = 0.5438$$

Hence $P(N(0.5) \leq 2) \approx 0.5438$ $(\approx 54.38\%)$, that is,

there is 0.5438 (54.38%) probability that *at most 2* customers arrive in a 30-minute period.

Question: (Time in a different way)

Defects appear on a production line as a Poisson process with rate 3 defects per metre. A quality inspector examines a 2-metre strip. What is the probability of finding *fewer than 3* defects?

Solution:

Step 1: Identify the rate for the given problem:

$$\lambda = \frac{3}{m}, \qquad length = t = 2\,m \Longrightarrow \lambda t = 3 \times 2 = 6$$

Step 2: "Fewer than 3" means $k = 0, 1, or\ 2$.

$$P(N < 3) = P(N = 0) + P(N = 1) + P(N = 2)$$

Now solve it yourself

Answer: $P(N < 3) \approx 0.0620\ (\approx 6.2\%)$.

-: Thank you so much :-