Review

Stochastic Process:

At the moment, we formally define three main things

1. Stochastic process.
2. Index Set
3. State Space

Definitions:

1. A stochastic process $\{X_t, t \in T\}$ is a collection of random variables.
2. The index set T represents the set of coordinates (usually time or space) over which the random variables are collected.
3. The state space S is the set of all possible values that the random variables X_t can take.

Types of Process & Index Set

The nature of T determines the "type" of process:

- **Discrete-time process:** If $T = \{0, 1, 2, \dots\}$ or $T = \mathbb{Z}$ or any subset of $\mathbb{Z}$.
- **Continuous-time process:** If $T = [0, \infty)$ or $T = \mathbb{R}$ or any closed interval $[a, b]$.

Questions:

1. Define stochastic processes and give its example.
2. Give an example of the stochastic process. Write index set and state space of the process.

Sample Space and Events

Suppose that we are about to perform an experiment whose outcome is not predictable in advance. However, while the outcome of the experiment will not be known in advance, ***let us suppose that the set of all possible outcomes is known (or maybe we can calculate).*** This set of all possible outcomes of an experiment is known as the ***sample space*** of the experiment and is denoted by S.

Any subset E *(or denoted by any other letter)* of the sample space S is known as an event.

Probability of Events

Consider an experiment whose sample space is S. For each event E of the sample space S, we define

$$P(E) = \frac{|E|}{|S|} \; or = \frac{n(E)}{n(S)}$$

Here $|E|$ or $n(E)$ means number of elements in E.

It is worth mentioning that $P(E)$ satisfies the following three conditions:

(i) $0 \leqslant P(E) \leqslant 1$.

(ii) $P(S) = 1$.

(iii) For any sequence of events $E_1, E_2, \ldots$ that are mutually exclusive, that is, events for which $E_n E_m = \emptyset$ when $n \neq m$, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

We refer to $P(E)$ as the probability of the event E.

Conditional Probability

Conditional probability is the likelihood of an event occurring given that another event has already occurred. It is represented as $P(E|F)$, which is read as "the probability of E given F". The formula is

$$P(E|F) = \frac{P(EF)}{P(F)},$$

where $P(F)$ is the probability of the condition event occurring which must be positive.

Markov Chain: Formal Definition

Let $\{X_n : n = 0,1,2, \ldots\}$ be a stochastic process taking values in a finite or countably infinite state space $\mathcal{S}$. If $X_n = i$, the process is said to be in state i at time n.

The process is called a **Markov chain** if it satisfies the **Markov property**:

For every $n \geq 0$ and for all states $i_0, i_1, \ldots, i_{n-1}, i, j \in \mathcal{S}$,

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i).$$

That is, the conditional probability of the future state X_{n+1} depends only on the present state X_n and not on the past states $X_0, \ldots, X_{n-1}$.

Assume

$$P_{ij} = P\{X_{n+1} = j \mid X_n = i\} \text{ for all } n \geq 0,$$

The quantities P_{ij} are called **transition probabilities**, and they satisfy

$$P_{ij} \geq 0 \text{ for all } i, j \in \mathcal{S}, \text{and} \sum_{j \in \mathcal{S}} P_{ij} = 1 \text{ for each } i \in \mathcal{S}.$$

Let $\mathbf{P}$ denote the matrix of one-step transition probabilities P_{ij}, so that

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

The matrix P is called **Transition Probability Matrix .**

Chapman-Kolmogorov Theorem

Let $\{X_n, n \geq 0\}$ be Markov chain with Transition Probability Matrix (TPM) $P = [P_{ij}]$ and n-step TPM $P^{(n)} = \left[P_{ij}^{(n)}\right]$, where

$$P_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \text{ and } P_{ij}^{(1)} = P_{ij}.$$

Then the following properties hold:

1. $P^{(n+m)} = P^{(n)} P^{(m)}$
2. $P^{(n)} = P^n$

The Initial State Vector Method

Let the system have n states, and define the initial state vector

$$\pi^{(0)} = (P_1^{(0)}, P_2^{(0)}, \dots, P_n^{(0)}),$$

where $P_i^{(0)} = \mathrm{P}(X_0 = i)$ and $\sum_{i=1}^{n} P_i^{(0)} = 1$.

Let $P = [P_{ij}]$ be the transition probability matrix, where $P_{ij} = P(X_{k+1} = j \mid X_k = i)$.

Then the state probability vector at time k is obtained by

$$\pi^{(k)} = \pi^{(0)} P^k,$$

where P^k is the k-step transition matrix. Equivalently, the recursive form is

$$\pi^{(k+1)} = \pi^{(k)} P.$$

Stationary Probabilities

Let $\{X_n, n \geq 0\}$ be a Markov chain with transition probability matrix P. A probability vector π is stationary if:

$$\pi P = \pi$$

and

$$\sum_i \pi_i = 1\,, \pi_i \geq 0.$$

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Note: A discrete-time Markov chain has a stationary distribution if and only if it is irreducible and positive recurrent.

- A Markov chain is **irreducible** if for every pair of states i, j, $i \to j$, that is, $\exists\, n \geq 1$ such that $P^n(i, j) > 0$.
- A state i is **positive recurrent** if its expected return time $\mathbb{E}_i[T_i]$ is **finite**, where $T_i = \inf\{n \geq 1 : X_n = i\}$.
- A chain is **positive recurrent** if *all* of its states are positive recurrent.

Second-Order Markov Property

The process $\{X_n\}$ is said to be a **second-order Markov chain** if, for all $n \geq 1$and for all states $i_0, i_1, \ldots, i_{n+1} \in S$,

$$P(X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$$
$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1}).$$

A second-order Markov chain can be **converted into a first-order chain** by defining an *augmented state variable*:

$$Y_n = (X_n, X_{n-1}),$$

whose state space is $S \times S = \{(i,j): i, j \in S\}$.

Then

$$P_{ij} = P\Big(Y_{n+1} = j \coloneqq (i_{n+1}, i_n) \;\Big|\; Y_n = i \coloneqq (i_n, i_{n-1}) \Big)$$
$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n, X_{n-1} = i_{n-1})$$

Thus $\{Y_n\}$forms a **Markov chain** on $S \times S$ (it is also known as first order Markov chain).