Stationary Distribution in Markov Chain & Initial State Vector Method

Motivation

Consider two state Markov model $\{X_n, n \geq 0\}$ as follows:

A student moves between two places on campus every hour:

- State 0: Football Ground
- State 1: Canteen

with

$$P(X_{n+1} = 1|X_n = 0) = 0.3,$$
$$P(X_{n+1} = 0|X_n = 1) = 0.4.$$

Then the transition probability matrix is as follows:

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

One can note

$$P^8 \approx \begin{bmatrix} 0.5714 & 0.4286 \\ 0.5714 & 0.4286 \end{bmatrix}$$

Now what P^9, P^{10}, ...

Let $\pi = (\pi_0, \pi_1)$.
Solve:

$$\pi P = \pi, \qquad \pi_0 + \pi_1 = 1$$

Equations:

$$\pi_0 = 0.7\pi_0 + 0.4\pi_1 \dots (1)$$
$$\pi_1 = 0.3\pi_0 + 0.6\pi_1 \dots (2)$$

Substitute $\pi_1 = 1 - \pi_0$ in (1):

$$\pi_0 = 0.7\pi_0 + 0.4(1 - \pi_0)$$
$$\pi_0 = 0.7\pi_0 + 0.4 - 0.4\pi_0$$
$$\pi_0 = 0.3\pi_0 + 0.4$$
$$0.7\pi_0 = 0.4$$
$$\pi_0 = \frac{4}{7} \approx 0.5714$$

Similarly, one can get $\pi_1 = \frac{3}{7} \approx 0.4286$.

Interpretation

In the long run:

- The student spends 57.14% of the time in the football ground
- The student spends 42.86% of the time in the canteen

Even if the student starts in one place, the proportions eventually settle to these stationary probabilities.

Stationary Probabilities

Let $\{X_n, n \geq 0\}$ be a Markov chain with transition probability matrix P. A probability vector π is stationary if:

$$\pi P = \pi$$

and

$$\sum_i \pi_i = 1\,, \pi_i \geq 0.$$

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Note: A discrete-time Markov chain has a stationary distribution if and only if it is irreducible and positive recurrent.

- A Markov chain is **irreducible** if for every pair of states i, j, $i \to j$, that is, $\exists\ n \geq 1$ such that $P^n(i,j) > 0$.
- A state i is **positive recurrent** if its expected return time $\mathbb{E}_i[T_i]$ is **finite**, where $T_i = \inf\{n \geq 1 : X_n = i\}$.
- A chain is **positive recurrent** if *all* of its states are positive recurrent.

The Initial State Vector Method

Let the system have n states, and define the initial state vector

$$\pi^{(0)} = (P_1^{(0)}, P_2^{(0)}, \dots, P_n^{(0)}),$$

where $P_i^{(0)} = \mathrm{P}(X_0 = i)$ and $\sum_{i=1}^{n} P_i^{(0)} = 1$.

Let $P = [P_{ij}]$ be the transition probability matrix, where $P_{ij} = P(X_{k+1} = j \mid X_k = i)$.

Then the state probability vector at time k is obtained by

$$\pi^{(k)} = \pi^{(0)} P^k,$$

where P^k is the k-step transition matrix. Equivalently, the recursive form is

$$\pi^{(k+1)} = \pi^{(k)} P.$$

We have the transition probability matrix:

$$P = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

What is the probability of change in state from 1 to 2 after five steps (initial state is 0)?

Since we start from state 1:

$$\pi^{(0)} = [0,1,0]$$

This means: $P(X_0 = 0) = 0$, $P(X_0 = 1) = 1$ and $P(X_0 = 0) = 0$.

After 1 Step

$$\pi^{(1)} = \pi^{(0)} \times P = [0,1,0] \times \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$

$$= [0.3,0.4,0.3]$$

After 2 Step

$$\pi^{(2)} = \pi^{(1)} \times P = [0.3,0.4,0.3] \times \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix}$$
$$= [0.33,0.37,0.30]$$

After 3 Step

$$\pi^{(3)} = \pi^{(2)} \times P = [0.33,0.37,0.30] \times P$$
$$= [0.336,0.370,0.294]$$

After 4 Step

$$\pi^{(4)} = \pi^{(3)} \times P = [0.336,0.370,0.294] \times P$$
$$= [0.3378,0.3706,0.2916]$$

After 5 Step

$$\pi^{(5)} = \pi^{(4)} \times P = [0.3378,0.3706,0.2916] \times P$$
$$= [0.3384,0.3708,0.2908]$$

Finally, the probability of going from state 1 to state 2 in exactly 5 steps is **0.2908.**

Problem:

A mobile phone market has three companies: **A**, **B**, and **C**. Each year, customers may switch between companies according to the following patterns:

- **Company A** retains 80% of its customers, loses 10% to B, and 10% to C
- **Company B** retains 70% of its customers, loses 20% to A, and 10% to C
- **Company C** retains 60% of its customers, loses 30% to A, and 10% to B

Current market share: A = 40%, B = 35%, C = 25%

Question: What will be the market share of each company after 3 years?

Solution:

Let's order the states as [A, B, C]. Then the transition probability matrix is as follows:

$$P = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$$

The current market share gives us our initial state vector:

$$\pi^{(0)} = [0.40, 0.35, 0.25]$$

Market Share After 1 Year:

$$\pi^{(1)} = \pi^{(0)} \times P = [0.40, 0.35, 0.25] \times \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix}$$
$$= [0.465, 0.310, 0.225]$$

Market Share After 2 Years:

$$\pi^{(2)} = \pi^{(1)} \times P = [0.465, 0.310, 0.225] \times P$$
$$= [0.5015, 0.2860, 0.2125]$$

Market Share After 3 Years:

$$\pi^{(3)} = \pi^{(2)} \times P = [0.5015, 0.2860, 0.2125] \times P$$
$$= [0.522, 0.272, 0.206]$$

Final Answer:

After 3 years, the projected market shares are:

- Company A: 52.2%
- Company B: 27.2%
- Company C: 20.6%

Thank you so much