Stochastic Processes
&
Probability

Our objective is to understand Stochastic Processes and review the notion of Probability.

Introduction to Stochastic Processes

What is a Stochastic Process?

Our aim is to understand this process by some activities.

Stochastic Process:

At the moment, we formally define three main things

1. Stochastic process.
2. Index Set
3. State Space

Definitions:

1. A stochastic process $\{X_t, t \in T\}$ is a collection of random variables.
2. The index set T represents the set of coordinates (usually time or space) over which the random variables are collected.
3. The state space S is the set of all possible values that the random variables X_t can take.

Types of Process & Index Set

The nature of T determines the "type" of process:

- **Discrete-time process:** If $T = \{0, 1, 2, \ldots\}$ or $T = \mathbb{Z}$ or any subset of $\mathbb{Z}$.
- **Continuous-time process:** If $T = [0, \infty)$ or $T = \mathbb{R}$ or any closed interval $[a, b]$.

Questions:

1. Define stochastic processes and give its example.
2. Give an example of the stochastic process. Write index set and state space of the process.

Sample Space and Events

Suppose that we are about to perform an experiment whose outcome is not predictable in advance. However, while the outcome of the experiment will not be known in advance, ***let us suppose that the set of all possible outcomes is known (or maybe we can calculate).***

This set of all possible outcomes of an experiment is known as the ***sample space*** of the experiment and is denoted by S.

Any subset E *(or denoted by any other letter)* of the sample space S is known as an event.

Examples:

1. If the experiment consists of the flipping of a coin, then

$$S = \{H, T\}$$

where H means that the outcome of the toss is a head and T that it is a tail.

Let E be an event that tail appears. Then $E = \{T\} \subseteq S.$

2. If the experiment consists of flipping three coins, then the sample space consists of the following elements:

$$S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

Suppose E_1 be an event that at least two head appears and E_2 be an event that consecutive two head appear then

$$E_1 = \{(H,H,T),(H,T,H),(T,H,H),(H,H,H)\}$$

$$E_2 = \{(H,H,T),(T,H,H),(H,H,H)\}$$

3. When a team deploys a new version of a service to a production environment, the final state of that deployment can be considered a random process (due to unpredictable network conditions, underlying infrastructure, race conditions, etc.).

The sample space S could be the set of all possible final states of the deployment:

$$S = \{ Success, Failure\_Rollback\_Complete, Failure\_Rollback\_Incomplete, Build\_Failed, Deployment\_Timeout \}$$

Assume the following events

$$E = \{ Failure\_Rollback\_Incomplete, Deployment\_Timeout \}$$

$$F = \{ Success, Failure\_Rollback\_Complete \}$$

It seems that for event E, the deployment requires manual intervention.

For and event F, the system is in a stable state after the process completes.

More about events

For any two events E and F of a sample space S we define the new event $\boldsymbol{E \cup F}$ to consist of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if either E or F occurs.

Also we define the new event ***EF*, *sometimes written* $\boldsymbol{E \cap F}$**, and referred to as the intersection of E and F, as follows. EF consists of all outcomes which are both in E and in F. That is, the event EF will occur only if both E and F occur.

If $EF = \emptyset$, then E and F are said to be ***mutually exclusive***.

We define the new event $\boldsymbol{E^c}$ **or** $\boldsymbol{E'}$**,** referred to as the ***complement of E***, to consist of all outcomes in the sample space S that are not in E.

Example:

If the experiment consists of flipping three coins, then the sample space consists of the following elements:

$$S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

Suppose E be an event that all head or all tail appears, then

$$E = \{(H,H,H),(T,T,T)\}.$$

Suppose F be an event that head appear at first then

$$F = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T)\}.$$

Now

$$E \cup F = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\}.$$

This means either *all head or all tail appears* **or** *head appears at first.*

$$EF = E \cap F = \{(H,H,H)\}.$$

This means either *all head or all tail appears* **and** *head appears at first.*

Events E and F are not mutually exclusive.

Probability of Events

Consider an experiment whose sample space is S. For each event E of the sample space S, we define

$$P(E) = \frac{|E|}{|S|} \; or = \frac{n(E)}{n(S)}$$

Here $|E|$ or $n(E)$ means number of elements in E.

It is worth mentioning that $P(E)$ satisfies the following three conditions:

(i) $0 \leqslant P(E) \leqslant 1$.

(ii) $P(S) = 1$.

(iii) For any sequence of events $E_1, E_2, \ldots$ that are mutually exclusive, that is, events for which $E_n E_m = \emptyset$ when $n \neq m$, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

We refer to $P(E)$ as the probability of the event E.

Example:

$$S = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,H,H),(T,H,T),(T,T,H),(T,T,T)\}$$

$$E = \{(H,H,H),(T,T,T)\},\ \ P(E) = \frac{2}{8} = \frac{1}{4}$$

$$F = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T)\},\ \ P(F) = \frac{4}{8} = \frac{1}{2}.$$

$$E \cup F = \{(H,H,H),(H,H,T),(H,T,H),(H,T,T),(T,T,T)\},\ \ P(E \cup F) = \frac{5}{8}$$

$$EF = E \cap F = \{(H,H,H)\},\ \ P(EF) = P(E \cap F) = \frac{1}{8}$$

Note that

$$P(E \cap F) = P(E) \cdot P(F)$$
$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$
$$P(E^c) = 1 - P(E)$$

Conditional Probability

Conditional probability is the likelihood of an event occurring given that another event has already occurred. It is represented as $P(E|F)$, which is read as "the probability of E given F". The formula is

$$P(E|F) = \frac{P(EF)}{P(F)},$$

where $P(F)$ is the probability of the condition event occurring which must be positive.

Example:

Suppose cards numbered one through ten are placed in a hat, mixed up, and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

Let E denote the event that the number of the drawn card is ten and let F be the event that it is at least five. Then

$$n(E) = 1,\ \ n(F) = 6,\ \ n(EF) = (1),$$

So we have

$$P(E) = \frac{1}{10},\quad P(F) = \frac{6}{10},\quad P(EF) = \frac{1}{10}$$

Now

$$P(E|F) = \frac{1/10}{6/10} = \frac{1}{6}.$$

Independent Events

Two events are said to be independent if the occurrence (or non-occurrence) of one event does ***not affect*** the probability of the other event occurring. Formally two events E and F are said to be independent if

$$P(EF) = P(E) \cdot P(F)$$

or we have

$$P(E|F) = P(E).$$

Example

Suppose we toss two fair dice. Let E_1 denote the event that the sum of the dice is six and F denote the event that the first die equals four. Then

$$P(E_1F) = P(\{(4,2)\}) = \frac{1}{36}$$

while

$$P(E_1) = P(\{(1,5),(2,4),(3,3),(4,2),(1,5)\}) = \frac{5}{36}$$

and

$$P(F) = (\{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}) = \frac{6}{36} = \frac{1}{6}$$

Thus

$$P(E_1)P(F) = \frac{5}{36} \cdot \frac{1}{6} = \frac{5}{216} \neq P(E_1F).$$

Hence E_1 and F are not independent.

Let E_2 be the event that the sum of the dice equals seven. Then

$$P(E_2F) = P(\{(4,3)\}) = \frac{1}{36}.$$

Now

$$P(E_2) = P(\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}) = \frac{6}{36} = \frac{1}{6}$$

Also we have

$$P(F) = \frac{1}{6}$$

Thus

$$P(E_2)P(F) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36} = P(E_1F).$$

Hence E_2 and F are independent.

Ref: Introduction to probability models, Ross, S. M., Academic press, 2014.