Question # 1

(i) $\quad \underline{u} = 3\hat{\underline{i}} + \hat{\underline{j}} - \hat{\underline{k}} \quad , \quad \underline{v} = 2\hat{\underline{i}} - \hat{\underline{j}} + \hat{\underline{k}}$

$$|\underline{u}| = \sqrt{(3)^2 + (1)^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$$

$$|\underline{v}| = \sqrt{(2)^2 + (-1)^2 + (1)^2} = \sqrt{4+1+1} = \sqrt{6}$$

$$\underline{u} \cdot \underline{v} = \left(3\hat{\underline{i}} + \hat{\underline{j}} - \hat{\underline{k}}\right) \cdot \left(2\hat{\underline{i}} - \hat{\underline{j}} + \hat{\underline{k}}\right)$$
$$= (3)(2) + (1)(-1) + (-1)(1) = 6-1-1 = 4$$

Now $\quad \underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\theta$

$$\Rightarrow \cos\theta = \frac{\underline{u} \cdot \underline{v}}{|\underline{u}||\underline{v}|} = \frac{4}{\sqrt{11} \times \sqrt{6}} \qquad \Rightarrow \boxed{\cos\theta = \frac{4}{\sqrt{66}}}$$

(ii) $\qquad$ *Do yourself as above*

(iii) $\qquad \underline{u} = [-3, 5] = -3\hat{\underline{i}} + 5\hat{\underline{j}} \quad , \quad \underline{v} = [6, -2] = 6\hat{\underline{i}} - 2\hat{\underline{j}}$

$\qquad$ *Now do yourself as above*

(iv) $\qquad \underline{u} = [2, -3, 1] = 2\hat{\underline{i}} - 3\hat{\underline{j}} + \hat{\underline{k}} \quad , \quad \underline{v} = 2\hat{\underline{i}} + 4\hat{\underline{j}} + \hat{\underline{k}} \quad$ *Now do yourself as (i)*

Question # 2

(i) $\qquad \underline{a} = \hat{\underline{i}} - \hat{\underline{k}} \quad , \quad \underline{b} = \hat{\underline{j}} + \hat{\underline{k}}$

$$|\underline{a}| = \sqrt{(1)^2 + (0)^2 + (-1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$|\underline{b}| = \sqrt{(0)^2 + (1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\underline{a} \cdot \underline{b} = \left(\hat{\underline{i}} - \hat{\underline{k}}\right) \cdot \left(\hat{\underline{j}} + \hat{\underline{k}}\right) = (1)(0) + (0)(1) + (-1)(1) = 0+0-1 = -1$$

Since $\quad \underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$

So projection of $\underline{a}$ along $\underline{b} = |\underline{a}| \cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{b}|} = \dfrac{-1}{\sqrt{2}}$

Also projection of $\underline{b}$ along $\underline{a} = |\underline{b}| \cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}|} = \dfrac{-1}{\sqrt{2}}$

(ii) $\qquad$ *Do yourself as above*

Question # 3

(i) $\qquad$ *Do yourself as (ii) below*

(ii) $\quad \underline{u} = \alpha\hat{\underline{i}} + 2\alpha\hat{\underline{j}} - \hat{\underline{k}} \quad , \quad \underline{v} = \hat{\underline{i}} + \alpha\hat{\underline{j}} + 3\hat{\underline{k}}$

$\qquad$ Since $\underline{u}$ and $\underline{v}$ are perpendicular therefore $\underline{u}.\underline{v} = 0$

$$\Rightarrow \left(\alpha\hat{\underline{i}} + 2\alpha\hat{\underline{j}} - \hat{\underline{k}}\right) \cdot \left(\hat{\underline{i}} + \alpha\hat{\underline{j}} + 3\hat{\underline{k}}\right) = 0$$

$$\Rightarrow (\alpha)(1) + (2\alpha)(\alpha) + (-1)(3) = 0$$

$$\Rightarrow \alpha + 2\alpha^2 - 3 = 0 \quad \Rightarrow 2\alpha^2 + \alpha - 3 = 0$$

$$\Rightarrow 2\alpha^2 + 3\alpha - 2\alpha - 3 = 0 \quad \Rightarrow \alpha(2\alpha + 3) - 1(2\alpha + 3) = 0$$

$$\Rightarrow (2\alpha + 3)(\alpha - 1) = 0$$

$$\Rightarrow 2\alpha + 3 = 0 \quad \text{or} \quad \alpha - 1 = 0$$

$$\Rightarrow \alpha = -\frac{3}{2} \quad \text{or} \quad \alpha = 1$$

Question # 4

Given vertices: $A(1, -1, 0)$, $B(-2, 2, 1)$ and $C(0, 2, z)$

$$\overrightarrow{CA} = (1 - 0)\hat{\underline{i}} + (-1 - 2)\hat{\underline{j}} + (0 - z)\hat{\underline{k}} \quad = \hat{\underline{i}} - 3\hat{\underline{j}} - z\hat{\underline{k}}$$

$$\overrightarrow{CB} = (-2 - 0)\hat{\underline{i}} + (2 - 2)\hat{\underline{j}} + (1 - z)\hat{\underline{k}} \quad = -2\hat{\underline{i}} + (1 - z)\hat{\underline{k}}$$

Now $\overrightarrow{CA}$ is $\perp$ to $\overrightarrow{CB}$ therefore $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0$

$$\Rightarrow \left(\hat{\underline{i}} - 3\hat{\underline{j}} - z\hat{\underline{k}}\right) \cdot \left(-2\hat{\underline{i}} + (1 - z)\hat{\underline{k}}\right) = 0$$

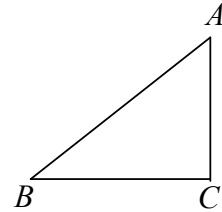$$\Rightarrow (1)(-2) + (-3)(0) + (-z)(1 - z) = 0$$

$$\Rightarrow -2 + 0 - z + z^2 = 0 \quad \Rightarrow z^2 - z - 2 = 0$$

$$\Rightarrow z^2 - 2z + z - 2 = 0 \quad \Rightarrow z(z - 2) + 1(z - 2) = 0$$

$$\Rightarrow (z - 2)(z + 1) = 0$$

$$\Rightarrow z - 2 = 0 \quad \text{or} \quad z + 1 = 0$$

$$\Rightarrow z = 2 \quad \text{or} \quad z = -1$$

Question # 5

Suppose $\underline{v} = a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}$

Since $\underline{v} \cdot \hat{\underline{i}} = 0 \quad \Rightarrow \left(a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}\right) \cdot \hat{\underline{i}} = 0$

$$\Rightarrow a_1\hat{\underline{i}} \cdot \hat{\underline{i}} + a_2\hat{\underline{j}} \cdot \hat{\underline{i}} + a_3\hat{\underline{k}} \cdot \hat{\underline{i}} = 0$$

$$\Rightarrow a_1(1) + a_2(0) + a_3(0) = 0 \quad \Rightarrow a_1 = 0$$

Also $\underline{v} \cdot \hat{\underline{j}} = 0 \quad \Rightarrow \left(a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}\right) \cdot \hat{\underline{j}} = 0$

$$\Rightarrow a_1\hat{\underline{i}} \cdot \hat{\underline{j}} + a_2\hat{\underline{j}} \cdot \hat{\underline{j}} + a_3\hat{\underline{k}} \cdot \hat{\underline{j}} = 0$$

$$\Rightarrow a_1(0) + a_2(1) + a_3(0) = 0 \quad \Rightarrow a_2 = 0$$

Also $\underline{v} \cdot \hat{\underline{k}} = 0 \quad \Rightarrow \left(a_1\hat{\underline{i}} + a_2\hat{\underline{j}} + a_3\hat{\underline{k}}\right) \cdot \hat{\underline{k}} = 0$

$$\Rightarrow a_1\hat{\underline{i}} \cdot \hat{\underline{k}} + a_2\hat{\underline{j}} \cdot \hat{\underline{k}} + a_3\hat{\underline{k}} \cdot \hat{\underline{k}} = 0$$

$$\Rightarrow a_1(0) + a_2(0) + a_3(1) = 0 \quad \Rightarrow a_3 = 0$$

Hence

$$\underline{v} = (0)\hat{\underline{i}} + (0)\hat{\underline{j}} + (0)\hat{\underline{k}} = 0$$

Question # 6 (i)

Let $\underline{a} = 3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}$, $\underline{b} = \hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}$ and $\underline{c} = 2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}$

Now $\qquad \underline{b} + \underline{c} = \hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}} + 2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}$

$\qquad\qquad\qquad = 3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}} = \underline{a}$

Hence $\underline{a}$, $\underline{b}$ and $\underline{c}$ form a triangle.

Now $\underline{a} \cdot \underline{b} = \left(3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}\right) \cdot \left(\hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}\right)$

$\qquad\qquad = (3)(1) + (-2)(-3) + (1)(5) \quad = 3 + 6 + 5 \ = 14$

$\underline{b} \cdot \underline{c} = \left(\hat{\underline{i}} - 3\hat{\underline{j}} + 5\hat{\underline{k}}\right) \cdot \left(2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}\right)$

$\qquad\qquad = (1)(2) + (-3)(1) + (5)(-4) = 2 - 3 - 20 = -21$

$\underline{c} \cdot \underline{a} = \left(2\hat{\underline{i}} + \hat{\underline{j}} - 4\hat{\underline{k}}\right) \cdot \left(3\hat{\underline{i}} - 2\hat{\underline{j}} + \hat{\underline{k}}\right)$

$\qquad\qquad = (2)(3) + (1)(-2) + (-4)(1) \ = 6 - 2 - 4 \ = 0$

Since $\underline{c} \cdot \underline{a} = 0$ therefore $\underline{c} \perp \underline{a}$

Hence $\underline{a}$, $\underline{b}$ and $\underline{c}$ represents sides of right triangle.

Given: $P(1,3,2)$, $Q(4,1,4)$ and $R(6,5,5)$

$\qquad\qquad \overrightarrow{PQ} = (4-1)\hat{\underline{i}} + (1-3)\hat{\underline{j}} + (4-2)\hat{\underline{k}} \ = 3\hat{\underline{i}} - 2\hat{\underline{j}} + 2\hat{\underline{k}}$

$\qquad\qquad \overrightarrow{QR} = (6-4)\hat{\underline{i}} + (5-1)\hat{\underline{j}} + (5-4)\hat{\underline{k}} \ = 2\hat{\underline{i}} + 4\hat{\underline{j}} + \hat{\underline{k}}$

$\qquad\qquad \overrightarrow{RP} = (1-6)\hat{\underline{i}} + (3-5)\hat{\underline{j}} + (2-5)\hat{\underline{k}} \ = -5\hat{\underline{i}} - 2\hat{\underline{j}} - 3\hat{\underline{k}}$

Now

$\qquad \overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$

$\qquad = 3\hat{\underline{i}} - 2\hat{\underline{j}} + 2\hat{\underline{k}} + 2\hat{\underline{i}} + 4\hat{\underline{j}} + \hat{\underline{k}} - 5\hat{\underline{i}} - 2\hat{\underline{j}} - 3\hat{\underline{k}} \ = 0$

Hence P, Q and R are vertices of triangle.

Now

$\qquad \overrightarrow{PQ} \cdot \overrightarrow{QR} = \left(3\hat{\underline{i}} - 2\hat{\underline{j}} + 2\hat{\underline{k}}\right) \cdot \left(2\hat{\underline{i}} + 4\hat{\underline{j}} + \hat{\underline{k}}\right)$

$\qquad\qquad\qquad = (3)(2) + (-2)(4) + (2)(1) \ = 6 - 8 + 2 \ = 0$

$\Rightarrow \overrightarrow{PQ} \perp \overrightarrow{QR}$

Hence P, Q and R are vertices of right triangle.

Question # 7

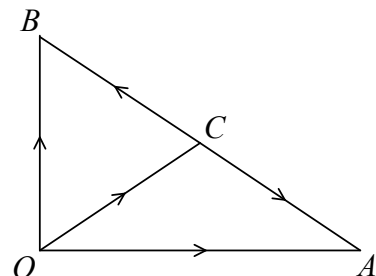Suppose a right triangle OAB. Let C be a midpoint of hypotenuse AB, then

$\qquad\qquad \overrightarrow{CA} = -\overrightarrow{CB} \quad \Rightarrow \left|\overrightarrow{CA}\right| = \left|\overrightarrow{CB}\right|$ .... (i)

Now $\quad \overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{CA}$

$\qquad\qquad \overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

Since $\overrightarrow{OA} \perp \overrightarrow{OB}$ therefore $\overrightarrow{OA} \cdot \overrightarrow{OB} = 0$

$\qquad \Rightarrow \left(\overrightarrow{OC} + \overrightarrow{CA}\right) \cdot \left(\overrightarrow{OC} + \overrightarrow{CB}\right) = 0$

$\qquad \Rightarrow \left(\overrightarrow{OC} - \overrightarrow{CB}\right) \cdot \left(\overrightarrow{OC} + \overrightarrow{CB}\right) = 0 \quad \because \ \overrightarrow{CA} = -\overrightarrow{CB}$

$$\Rightarrow \overrightarrow{OC} \cdot \left(\overrightarrow{OC} + \overrightarrow{CB} \right) - \overrightarrow{CB} \cdot \left(\overrightarrow{OC} + \overrightarrow{CB} \right) = 0$$

$$\Rightarrow \overrightarrow{OC} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{CB} - \overrightarrow{CB} \cdot \overrightarrow{OC} - \overrightarrow{CB} \cdot \overrightarrow{CB} = 0$$

$$\Rightarrow \left| \overrightarrow{OC} \right|^2 + \overrightarrow{OC} \cdot \overrightarrow{CB} - \overrightarrow{OC} \cdot \overrightarrow{CB} - \left| \overrightarrow{CB} \right|^2 = 0 \qquad \because \ \overrightarrow{OC} \cdot \overrightarrow{CB} = \overrightarrow{CB} \cdot \overrightarrow{OC}$$

$$\Rightarrow \left| \overrightarrow{OC} \right|^2 - \left| \overrightarrow{CB} \right|^2 = 0$$

$$\Rightarrow \left| \overrightarrow{OC} \right|^2 = \left| \overrightarrow{CB} \right|^2 \qquad \Rightarrow \left| \overrightarrow{OC} \right| = \left| \overrightarrow{CB} \right| \ \ldots\ldots\ldots\ldots\ (ii)$$

Combining (i) and (ii), we have

$$\left| \overrightarrow{OC} \right| = \left| \overrightarrow{CA} \right| = \left| \overrightarrow{CB} \right|$$

Hence midpoint of hypotenuse of right triangle is equidistant from its vertices.
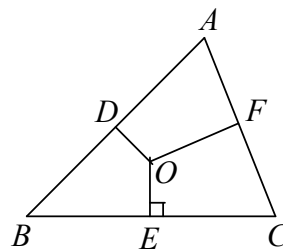
Question # 8

Let A, B and C be a vertices of a triangle having position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively.

Also consider D, E and F are midpoints of sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$, then

p.v of $D = \overrightarrow{OD} = \dfrac{\underline{a} + \underline{b}}{2}$

p.v of $E = \overrightarrow{OE} = \dfrac{\underline{b} + \underline{c}}{2}$

p.v of $F = \overrightarrow{OF} = \dfrac{\underline{c} + \underline{a}}{2}$

Let right bisector on $\overline{AB}$ and $\overline{BC}$ intersect at point O, which is an origin.

Since $\overrightarrow{OD}$ is $\perp$ to $\overline{AB}$

Therefore $\overrightarrow{OD} \cdot \overline{AB} = 0$

$$\Rightarrow \left(\dfrac{\underline{a} + \underline{b}}{2} \right) \cdot \left(\underline{b} - \underline{a} \right) = 0 \quad \Rightarrow \dfrac{1}{2} \left(\underline{b} + \underline{a} \right) \cdot \left(\underline{b} - \underline{a} \right) = 0$$

$$\Rightarrow \left(\underline{b} + \underline{a} \right) \cdot \left(\underline{b} - \underline{a} \right) = 0 \qquad \Rightarrow \underline{a} \cdot \left(\underline{b} - \underline{a} \right) + \underline{b} \cdot \left(\underline{b} - \underline{a} \right) = 0$$

$$\Rightarrow \underline{a} \cdot \underline{b} - \underline{a} \cdot \underline{a} + \underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{a} = 0$$

$$\Rightarrow \underline{a} \cdot \underline{b} - \left| \underline{a} \right|^2 + \left| \underline{b} \right|^2 - \underline{a} \cdot \underline{b} = 0 \qquad \because \ \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

$$\Rightarrow \left| \underline{b} \right|^2 - \left| \underline{a} \right|^2 = 0 \ \ldots\ldots\ldots\ldots\ (i)$$

Also $\overrightarrow{OE}$ is $\perp$ to $\overline{BC}$

Therefore $\overrightarrow{OE} \cdot \overline{BC} = 0 \quad \Rightarrow \left(\dfrac{\underline{b} + \underline{c}}{2} \right) \cdot \left(\underline{c} - \underline{b} \right) = 0$

Similarly solving as above, we get

$$\left| \underline{c} \right|^2 - \left| \underline{b} \right|^2 = 0 \ \ldots\ldots\ldots\ldots\ (ii)$$

Adding (i) and (ii), we have

$$\left| \underline{b} \right|^2 - \left| \underline{a} \right|^2 + \left| \underline{c} \right|^2 - \left| \underline{b} \right|^2 = 0 + 0$$

$$\Rightarrow \left|\underline{c}\right|^2 - \left|\underline{a}\right|^2 = 0$$
$$\Rightarrow (\underline{c} + \underline{a}) \cdot (\underline{c} - \underline{a}) = 0$$
$$\Rightarrow \left(\frac{\underline{c} + \underline{a}}{2}\right) \cdot (\underline{c} - \underline{a}) = 0$$
$$\Rightarrow \overrightarrow{OF} \cdot \overrightarrow{AC} = 0 \quad \Rightarrow \overrightarrow{OF} \text{ is } \perp \text{ to } \overrightarrow{AC}$$

i.e. $\overrightarrow{OF}$ is also right bisector of $\overrightarrow{AC}$.

Hence perpendicular bisector of the sides of the triangle are concurrent.

Question # 9

Consider A, B and C are vertices of triangle having position vectors $\underline{a}, \underline{b}$ and $\underline{c}$ respectively. Let altitude on $\overrightarrow{AB}$ and $\overrightarrow{BC}$ intersect at origin $O(0,0)$.

Since $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$

$$\Rightarrow \overrightarrow{OC} \cdot \overrightarrow{AB} = 0$$
$$\Rightarrow \underline{c} \cdot (\underline{b} - \underline{a}) = 0$$
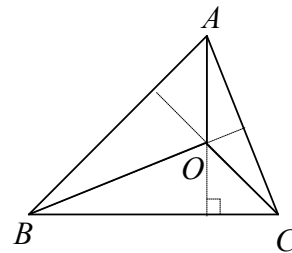$$\Rightarrow \underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} = 0 \quad \text{...............(i)}$$

Also $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{BC}$

$$\Rightarrow \overrightarrow{OA} \cdot \overrightarrow{BC} = 0$$
$$\Rightarrow \underline{a} \cdot (\underline{c} - \underline{b}) = 0$$
$$\Rightarrow \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} = 0 \quad \text{...............(ii)}$$

Adding (i) and (ii)

$$\underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} + \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{b} = 0 + 0$$
$$\Rightarrow \underline{c} \cdot \underline{b} - \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{a} - \underline{a} \cdot \underline{b} = 0 \qquad \because \underline{a} \cdot \underline{c} = \underline{c} \cdot \underline{a}$$
$$\Rightarrow \underline{c} \cdot \underline{b} - \underline{a} \cdot \underline{b} = 0$$
$$\Rightarrow (\underline{c} - \underline{a}) \cdot \underline{b} = 0$$
$$\Rightarrow \overrightarrow{AC} \cdot \overrightarrow{OB} = 0 \qquad \because \overrightarrow{AC} = \underline{c} - \underline{a}$$
$$\Rightarrow \overrightarrow{AC} \text{ is perpendicular to } \overrightarrow{OB}.$$

Hence altitude of the triangle are concurrent.

Question # 10

Consider a semicircle having centre at origin $O(0,0)$ and A, B are end points of diameter having position vectors $\underline{a}$, $-\underline{a}$ respectively. Let C be any point on a circle having position vector $\underline{c}$.
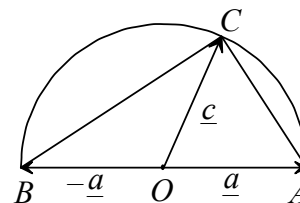
Clearly radius of semicircle $= \left|\underline{a}\right| = \left|-\underline{a}\right| = \left|\underline{c}\right|$

Now $\overrightarrow{AC} = \underline{c} - \underline{a}$

$\overrightarrow{BC} = \underline{c} - (-\underline{a}) = \underline{c} + \underline{a}$

Consider

$$\overrightarrow{AC} \cdot \overrightarrow{BC} = (\underline{c} - \underline{a}) \cdot (\underline{c} + \underline{a})$$
$$= \underline{c} \cdot (\underline{c} + \underline{a}) - \underline{a} \cdot (\underline{c} + \underline{a})$$
$$= \underline{c} \cdot \underline{c} + \underline{c} \cdot \underline{a} - \underline{a} \cdot \underline{c} - \underline{a} \cdot \underline{a}$$

$$= \left|\underline{c}\right|^2 + \underline{a}\cdot\underline{c} - \underline{a}\cdot\underline{c} + \left|\underline{a}\right|^2 \qquad \because \ \underline{a}\cdot\underline{c} = \underline{c}\cdot\underline{a}$$

$$= \left|\underline{c}\right|^2 - \left|\underline{a}\right|^2$$

$$= \left|\underline{c}\right|^2 - \left|\underline{c}\right|^2 = 0 \qquad \because \ \left|\underline{a}\right| = \left|\underline{c}\right|$$

This show $\overrightarrow{AC}$ is $\perp$ to $\overrightarrow{BC}$ i.e. $\angle ACB = 90°$

Hence angle in a semi circle is a right angle.

Question # 11

Consider two unit vectors $\hat{\underline{a}}$ and $\hat{\underline{b}}$ making angle α and $-\beta$ with + ive $x-$axis.

Then $\quad \hat{\underline{a}} = OA = \cos\alpha\,\hat{\underline{i}} + \sin\alpha\,\hat{\underline{j}}$

and $\quad \hat{\underline{b}} = OB = \cos(-\beta)\hat{\underline{i}} + \sin(-\beta)\hat{\underline{j}}$

$$= \cos\beta\,\hat{\underline{i}} - \sin\beta\,\hat{\underline{j}}$$

Now

$$\hat{\underline{a}}\cdot\hat{\underline{b}} = \left(\cos\alpha\,\hat{\underline{i}} + \sin\alpha\,\hat{\underline{j}}\right)\cdot\left(\cos\beta\,\hat{\underline{i}} - \sin\beta\,\hat{\underline{j}}\right)$$

$$\Rightarrow \hat{\underline{a}}\cdot\hat{\underline{b}} = \cos\alpha\cos\beta - \sin\alpha\sin\beta \ \ldots\ldots\ldots\ldots(i)$$

But we have $\angle AOB = \alpha + \beta$

$$\Rightarrow \hat{\underline{a}}\cdot\hat{\underline{b}} = \left|\hat{\underline{a}}\right|\left|\hat{\underline{b}}\right|\cos(\alpha + \beta)$$

$$= (1)(1)\cos(\alpha + \beta) \qquad \because \ \left|\hat{\underline{a}}\right| = \left|\hat{\underline{b}}\right| = 1$$

$$\Rightarrow \hat{\underline{a}}\cdot\hat{\underline{b}} = \cos(\alpha + \beta) \ \ldots\ldots\ldots\ldots(ii)$$

Comparing (i) and (ii), we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
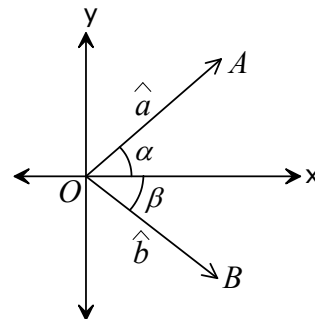
Question # 12

Consider $\underline{a}, \underline{b}$ and $\underline{c}$ are vectors along the sides of triangle BC, CA and AB,

also let $\quad \left|\underline{a}\right| = a$, $\left|\underline{b}\right| = b$ and $\left|\underline{c}\right| = c$

then form triangle,

$$\underline{a} + \underline{b} + \underline{c} = 0 \ \ldots\ldots\ldots\ldots(i)$$

(i) $\quad \Rightarrow \underline{b} = -\underline{a} - \underline{c}$

Taking dot product of above with $\underline{b}$, we have

$$\underline{b}\cdot\underline{b} = (-\underline{a} - \underline{c})\cdot\underline{b}$$

$$\left|\underline{b}\right|^2 = -\underline{a}\cdot\underline{b} - \underline{c}\cdot\underline{b}$$

$$= -\left|\underline{a}\right|\left|\underline{b}\right|\cos(\pi - C) - \left|\underline{c}\right|\left|\underline{b}\right|\cos(\pi - A)$$

$$= \left|\underline{a}\right|\left|\underline{b}\right|\cos C + \left|\underline{c}\right|\left|\underline{b}\right|\cos A \qquad \because \ \cos(\pi - B) = -\cos B$$
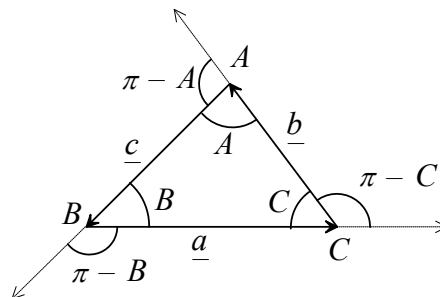
$$\Rightarrow b^2 = ab\cos C + cb\cos A$$

$$\Rightarrow b = a\cos C + c\cos A \qquad \qquad \div\text{ing by } b$$

(ii) From equation (i)

$$\underline{c} = -\underline{a} - \underline{b}$$

Taking dot product of above equation with $\underline{c}$.

$$\underline{c} \cdot \underline{c} = \left(-\underline{a} - \underline{b}\right) \cdot \underline{c}$$

Now do yourself as above.

(iii) From equation (i)

$$\underline{b} = -\underline{a} - \underline{c}$$

Taking dot product of above equation with $\underline{b}$

$$\underline{b} \cdot \underline{b} = (-\underline{a} - \underline{c}) \cdot \underline{b}$$
$$= (-\underline{a} - \underline{c}) \cdot (-\underline{a} - \underline{c}) \qquad \because \underline{b} = -\underline{a} - \underline{c}$$
$$\left|\underline{b}\right|^2 = -\underline{a} \cdot (-\underline{a} - \underline{c}) - \underline{c} \cdot (-\underline{a} - \underline{c})$$
$$= \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{a} + \underline{c} \cdot \underline{c}$$
$$= \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{c} + \underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{c} \qquad \because \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$
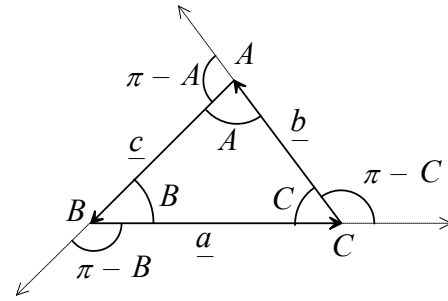$$= \underline{a} \cdot \underline{a} + 2\underline{a} \cdot \underline{c} + \underline{c} \cdot \underline{c}$$
$$= \left|\underline{a}\right|^2 + 2\left|\underline{a}\right|\left|\underline{c}\right|\cos\left(\pi - B\right) + \left|\underline{c}\right|^2$$
$$\Rightarrow b^2 = a^2 + ac\left(-\cos B\right) + c^2 \qquad \because \cos\left(\pi - B\right) = -\cos B$$

Hence $$b^2 = c^2 + a^2 - 2ca \cos B$$

(iv) From equation (i)

$$\underline{c} = -\underline{a} - \underline{b}$$

Taking dot product of above equation with $\underline{c}$

$$\underline{c} \cdot \underline{c} = \left(-\underline{a} - \underline{b}\right) \cdot \underline{c}$$
$$= (-\underline{a} - \underline{b}) \cdot (-\underline{a} - \underline{b}) \qquad \because \underline{c} = -\underline{a} - \underline{b}$$

Now do yourself as above *(iii)*

Error Analyst		
Saqib Aleem (2015)	-	Punjab College of Sciences
Uzair Amin (2016)		