

Question # 1

Given $A(2,5)$, $B(-1,1)$ and $C(2,-6)$

(i) $\overrightarrow{AB} = (-1-2)\hat{i} + (1-5)\hat{j} = -3\hat{i} - 4\hat{j}$

(ii) From above $\overrightarrow{AB} = -3\hat{i} - 4\hat{j}$

Also $\overrightarrow{CB} = (2+1)\hat{i} + (-6-1)\hat{j} = 3\hat{i} - 7\hat{j}$

Now

$$\begin{aligned} 2\overrightarrow{AB} - \overrightarrow{CB} &= 2(-3\hat{i} - 4\hat{j}) - (3\hat{i} - 7\hat{j}) \\ &= -6\hat{i} - 8\hat{j} - 3\hat{i} + 7\hat{j} \\ &= -9\hat{i} - \hat{j} \end{aligned}$$

(iii) Do yourself as above

Question # 2

(i) $\underline{u} = \hat{i} + 2\hat{j} - \hat{k}$

$\underline{v} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

$\underline{w} = 5\hat{i} - \hat{j} + 3\hat{k}$

$$\begin{aligned} \underline{u} + 2\underline{v} + \underline{w} &= \hat{i} + 2\hat{j} - \hat{k} + 2(3\hat{i} - 2\hat{j} + 2\hat{k}) \\ &\quad + (5\hat{i} - \hat{j} + 3\hat{k}) \\ &= \hat{i} + 2\hat{j} - \hat{k} + 6\hat{i} - 4\hat{j} + 4\hat{k} + 5\hat{i} - \hat{j} + 3\hat{k} \\ &= 12\hat{i} - 3\hat{j} - 6\hat{k} \end{aligned}$$

(ii) Do yourself

(iii)

$$\begin{aligned} 3\underline{v} + \underline{w} &= 3(3\hat{i} - 2\hat{j} + 2\hat{k}) + 5\hat{i} - \hat{j} + 3\hat{k} \\ &= 9\hat{i} - 6\hat{j} + 6\hat{k} + 5\hat{i} - \hat{j} + 3\hat{k} \\ &= 14\hat{i} - 7\hat{j} + 9\hat{k} \end{aligned}$$

Now $|3\underline{v} + \underline{w}| = \sqrt{(14)^2 + (-7)^2 + (9)^2} = \sqrt{196 + 49 + 81} = \sqrt{326}$

Question # 3

(i) $\underline{v} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$$\Rightarrow |\underline{v}| = \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$\begin{aligned} \text{Unit vector of } \underline{v} &= \hat{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\hat{i} + 3\hat{j} + 4\hat{k}}{\sqrt{29}} \\ &= \frac{2}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k} \end{aligned}$$

Hence direction cosines of \underline{v} are

$$\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}$$

(ii) Do yourself as above.

(iii) Do yourself as (i)

Question # 4

$$\begin{aligned} \text{Since } |\alpha\hat{i} + (\alpha+1)\hat{j} + 2\hat{k}| &= 3 \\ \Rightarrow \sqrt{\alpha^2 + (\alpha+1)^2 + (2)^2} &= 3 \\ \Rightarrow \sqrt{\alpha^2 + \alpha^2 + 2\alpha + 1 + 4} &= 3 \end{aligned}$$

On squaring both sides

$$\begin{aligned} 2\alpha^2 + 2\alpha + 5 &= 9 \\ \Rightarrow 2\alpha^2 + 2\alpha + 5 - 9 &= 0 \\ \Rightarrow 2\alpha^2 + 2\alpha - 4 &= 0 \\ \Rightarrow \alpha^2 + \alpha - 2 &= 0 \\ \Rightarrow \alpha^2 + 2\alpha - \alpha - 2 &= 0 \\ \Rightarrow \alpha(\alpha+2) - 1(\alpha+2) &= 0 \\ \Rightarrow (\alpha+2)(\alpha-1) &= 0 \\ \Rightarrow \alpha+2=0 & \text{ or } \alpha-1=0 \\ \Rightarrow \alpha=-2 & \text{ or } \alpha=1 \end{aligned}$$

Question # 5

Given $\underline{v} = \hat{i} + 2\hat{j} - \hat{k}$

$$\begin{aligned} |\underline{v}| &= \sqrt{(1)^2 + (2)^2 + (-1)^2} \\ &= \sqrt{1+4+1} = \sqrt{6} \end{aligned}$$

Now

$$\begin{aligned} \hat{v} &= \frac{\underline{v}}{|\underline{v}|} = \frac{\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{6}} \\ &= \frac{1}{\sqrt{6}}\hat{i} + \frac{2}{\sqrt{6}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k} \end{aligned}$$

Question # 6

Given $\underline{a} = 3\underline{i} - \underline{j} - 4\underline{k}$
 $\underline{b} = -2\underline{i} - 4\underline{j} - 3\underline{k}$
 $\underline{c} = \underline{i} + 2\underline{j} - \underline{k}$

Suppose that

$$\begin{aligned}\underline{d} &= 3\underline{a} - 2\underline{b} + 4\underline{c} \\ \Rightarrow \underline{d} &= 3(3\underline{i} - \underline{j} - 4\underline{k}) \\ &\quad - 2(-2\underline{i} - 4\underline{j} - 3\underline{k}) \\ &\quad + 4(\underline{i} + 2\underline{j} - \underline{k}) \\ &= 9\underline{i} - 3\underline{j} - 12\underline{k} + 4\underline{i} + 8\underline{j} + 6\underline{k} + 4\underline{i} + 8\underline{j} - 4\underline{k} \\ &= 17\underline{i} + 13\underline{j} - 10\underline{k}\end{aligned}$$

Now

$$\begin{aligned}|\underline{d}| &= \sqrt{(17)^2 + (-13)^2 + (-10)^2} \\ &= \sqrt{289 + 169 + 100} = \sqrt{558} = 3\sqrt{62}\end{aligned}$$

Now

$$\begin{aligned}\underline{\hat{d}} &= \frac{\underline{d}}{|\underline{d}|} = \frac{17\underline{i} + 13\underline{j} - 10\underline{k}}{3\sqrt{62}} \\ &= \frac{17}{3\sqrt{62}}\underline{i} + \frac{13}{3\sqrt{62}}\underline{j} - \frac{10}{3\sqrt{62}}\underline{k}.\end{aligned}$$

Question # 7

Consider $\underline{a} = 2\underline{i} - 3\underline{j} + 6\underline{k}$
 $|\underline{a}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

Now

$$\begin{aligned}\underline{\hat{a}} &= \frac{\underline{a}}{|\underline{a}|} = \frac{2\underline{i} - 3\underline{j} + 6\underline{k}}{7} \\ &= \frac{2}{7}\underline{i} - \frac{3}{7}\underline{j} + \frac{6}{7}\underline{k}\end{aligned}$$

Let \underline{b} be a vector having magnitude 4

i.e. $|\underline{b}| = 4$

Since \underline{b} is parallel to \underline{a}

therefore $\underline{b} = \underline{\hat{a}} = \frac{2}{7}\underline{i} - \frac{3}{7}\underline{j} + \frac{6}{7}\underline{k}$

Now $\underline{b} = |\underline{b}| \underline{\hat{b}} = 4 \left(\frac{2}{7}\underline{i} - \frac{3}{7}\underline{j} + \frac{6}{7}\underline{k} \right)$

$$= \frac{8}{7}\underline{i} - \frac{12}{7}\underline{j} + \frac{24}{7}\underline{k}$$

(ii)

Do yourself.

Question # 8

Given $\underline{u} = 2\underline{i} + 3\underline{j} + 4\underline{k}$
 $\underline{v} = -\underline{i} + 3\underline{j} - \underline{k}$
 $\underline{w} = \underline{i} + 6\underline{j} + z\underline{k}$

Since \underline{u} , \underline{v} and \underline{w} are sides of triangle therefore

$$\underline{u} + \underline{v} = \underline{w}$$

$$\Rightarrow 2\underline{i} + 3\underline{j} + 4\underline{k} - \underline{i} + 3\underline{j} - \underline{k} = \underline{i} + 6\underline{j} + z\underline{k}$$

$$\Rightarrow \underline{i} + 6\underline{j} + 3\underline{k} = \underline{i} + 6\underline{j} + z\underline{k}$$

Equating coefficient of \underline{k} only, we have

$$3 = z \text{ i.e. } \boxed{z = 3}$$

Question # 9

Position vector (p.v) of point $A = 2\underline{i} - \underline{j} + \underline{k}$

p.v of point $B = 3\underline{i} + \underline{j}$

p.v. of point $C = 2\underline{i} + 4\underline{j} - 2\underline{k}$

p.v. of point $D = -\underline{i} - 2\underline{j} + \underline{k}$

$$\overrightarrow{AB} = \text{p.v. of } B - \text{p.v. of } A$$

$$= 3\underline{i} + \underline{j} - 2\underline{i} + \underline{j} - \underline{k} = \underline{i} + 2\underline{j} - \underline{k}$$

$$\overrightarrow{CD} = \text{p.v. of } D - \text{p.v. of } C$$

$$= -\underline{i} - 2\underline{j} + \underline{k} - 2\underline{i} - 4\underline{j} + 2\underline{k}$$

$$= -3\underline{i} - 6\underline{j} + 3\underline{k}$$

$$= -3(\underline{i} + 2\underline{j} - \underline{k}) = -3\overrightarrow{AB}$$

$$\text{i.e. } \overrightarrow{CD} = \lambda \overrightarrow{AB} \text{ where } \lambda = -3$$

Hence \overrightarrow{AB} and \overrightarrow{CD} are parallel.

Question # 10 (i)

$$\underline{v} = 2\underline{i} - 4\underline{j} + 4\underline{k}$$

$$\begin{aligned}|\underline{v}| &= \sqrt{(2)^2 + (-4)^2 + (4)^2} \\ &= \sqrt{4 + 16 + 16} = \sqrt{36} = 6\end{aligned}$$

Now $\underline{\hat{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - 4\underline{j} + 4\underline{k}}{6}$

$$= \frac{1}{3}\underline{i} - \frac{2}{3}\underline{j} + \frac{2}{3}\underline{k}$$

The two vectors of length 2 and parallel to \underline{v} are $2\underline{\hat{v}}$ and $-2\underline{\hat{v}}$.

$$2\hat{v} = 2\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}\hat{i} - \frac{4}{3}\hat{j} + \frac{4}{3}\hat{k}$$

$$-2\hat{v} = -2\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = -\frac{2}{3}\hat{i} + \frac{4}{3}\hat{j} - \frac{4}{3}\hat{k}$$

Question # 10 (ii)

Given $\underline{v} = \hat{i} - 3\hat{j} + 4\hat{k}$, $\underline{w} = a\hat{i} + 9\hat{j} - 12\hat{k}$

Since \underline{v} and \underline{w} are parallel therefore there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}\underline{v} &= \lambda \underline{w} \\ \Rightarrow \hat{i} - 3\hat{j} + 4\hat{k} &= \lambda(a\hat{i} + 9\hat{j} - 12\hat{k}) \\ \Rightarrow \hat{i} - 3\hat{j} + 4\hat{k} &= a\lambda\hat{i} + 9\lambda\hat{j} - 12\lambda\hat{k}\end{aligned}$$

Comparing coefficients of \hat{i} , \hat{j} and \hat{k}

$$\begin{aligned}1 &= a\lambda \quad \dots \dots \dots (i) \\ -3 &= 9\lambda \quad \dots \dots \dots (ii) \\ 4 &= -12\lambda \quad \dots \dots \dots (iii)\end{aligned}$$

$$\text{From (ii)} \quad \lambda = -\frac{3}{9} \Rightarrow \lambda = -\frac{1}{3}$$

Putting in equation (i)

$$1 = a\left(-\frac{1}{3}\right) \Rightarrow -3 = a \quad \text{i.e. } \boxed{a = -3}$$

Question # 10 (c)

Consider $\underline{v} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\begin{aligned}|\underline{v}| &= \sqrt{(1)^2 + (-2)^2 + (3)^2} \\ &= \sqrt{1+4+9} = \sqrt{14}\end{aligned}$$

Now

$$\begin{aligned}\hat{v} &= \frac{\underline{v}}{|\underline{v}|} = \frac{\hat{i} - 2\hat{j} + 3\hat{k}}{\sqrt{14}} \\ &= \frac{1}{\sqrt{14}}\hat{i} - \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}\end{aligned}$$

Let \underline{a} be a vector having magnitude 5 i.e.

$$|\underline{a}| = 5$$

Since \underline{a} is parallel to \underline{v} but opposite in direction,
therefore

$$\hat{a} = -\hat{v} = -\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} - \frac{3}{\sqrt{14}}\hat{k}$$

Now

$$\underline{a} = |\underline{a}| \hat{a} = 5\left(-\frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} - \frac{3}{\sqrt{14}}\hat{k}\right)$$

$$= -\frac{5}{\sqrt{14}}\hat{i} + \frac{10}{\sqrt{14}}\hat{j} - \frac{15}{\sqrt{14}}\hat{k}.$$

Question # 10 (d)

Suppose that $\underline{v} = 3\hat{i} - \hat{j} + 4\hat{k}$ and

$$\underline{w} = a\hat{i} + b\hat{j} - 2\hat{k}$$

$\because \underline{v}$ and \underline{w} are parallel

\therefore there exists $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}\underline{v} &= \lambda \underline{w} \\ \Rightarrow 3\hat{i} - \hat{j} + 4\hat{k} &= \lambda(a\hat{i} + b\hat{j} - 2\hat{k}) \\ \Rightarrow 3\hat{i} - \hat{j} + 4\hat{k} &= a\lambda\hat{i} + b\lambda\hat{j} - 2\lambda\hat{k}\end{aligned}$$

Comparing coefficients of \hat{i} , \hat{j} and \hat{k}

$$3 = a\lambda \quad \dots \dots \dots (i)$$

$$-1 = b\lambda \quad \dots \dots \dots (ii)$$

$$4 = -2\lambda \quad \dots \dots \dots (iii)$$

From equation (iii)

$$-\frac{4}{2} = \lambda \Rightarrow \lambda = -2$$

Putting value of λ in equation (i)

$$3 = a(-2) \Rightarrow \boxed{a = -\frac{3}{2}}$$

Putting value of λ in equation (ii)

$$-1 = b(-2) \Rightarrow \boxed{b = \frac{1}{2}}$$

Question # 11 (i)

$$\begin{aligned}\underline{v} &= 3\hat{i} - \hat{j} + 2\hat{k} \\ |\underline{v}| &= \sqrt{(3)^2 + (-1)^2 + (2)^2} \\ &= \sqrt{9+1+4} = \sqrt{14}\end{aligned}$$

Let \hat{v} be unit vector along \underline{v} . Then

$$\begin{aligned}\hat{v} &= \frac{\underline{v}}{|\underline{v}|} = \frac{3\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{14}} \\ &= \frac{3}{\sqrt{14}}\hat{i} - \frac{1}{\sqrt{14}}\hat{j} + \frac{2}{\sqrt{14}}\hat{k} \\ \hat{v} &= \left[\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} \right]\end{aligned}$$

Hence the direction cosines of \underline{v} are

$$\frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}.$$

Question # 11 (ii)

$$\underline{v} = 6\underline{i} - 2\underline{j} + \underline{k}$$

$$|\underline{v}| = \sqrt{(6)^2 + (-2)^2 + (1)^2} \\ = \sqrt{36 + 4 + 1} = \sqrt{41}$$

Let \hat{v} be unit vector along \underline{v} . Then

$$\begin{aligned}\hat{v} &= \frac{\underline{v}}{|\underline{v}|} = \frac{6\underline{i} - 2\underline{j} + \underline{k}}{\sqrt{41}} \\ &= \frac{6}{\sqrt{41}}\underline{i} - \frac{2}{\sqrt{41}}\underline{j} + \frac{1}{\sqrt{41}}\underline{k} \\ \hat{v} &= \left[\frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}} \right]\end{aligned}$$

Hence the direction cosines of \underline{v} are

$$\frac{6}{\sqrt{41}}, \frac{-2}{\sqrt{41}}, \frac{1}{\sqrt{41}}.$$

Question # 11 (iii)

$$P = (2, 1, 5), Q = (1, 3, 1)$$

$$\begin{aligned}\overrightarrow{PQ} &= (1-2)\underline{i} + (3-1)\underline{j} + (1-5)\underline{k} \\ &= -\underline{i} + 2\underline{j} - 4\underline{k} \\ |\overrightarrow{PQ}| &= \sqrt{(-1)^2 + (2)^2 + (-4)^2} \\ &= \sqrt{1+4+16} = \sqrt{21}\end{aligned}$$

Let \hat{v} be unit vector along \overrightarrow{PQ} . Then

$$\begin{aligned}\hat{v} &= \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{-\underline{i} + 2\underline{j} - 4\underline{k}}{\sqrt{21}} \\ &= \frac{-1}{\sqrt{21}}\underline{i} + \frac{2}{\sqrt{21}}\underline{j} - \frac{4}{\sqrt{21}}\underline{k} \\ \hat{v} &= \left[\frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}} \right]\end{aligned}$$

Hence the direction cosines of \overrightarrow{PQ} are

$$\frac{-1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}.$$

Question # 12(i)

$45^\circ, 45^\circ, 60^\circ$ will be direction angles of the vectors if

$$\cos^2 45^\circ + \cos^2 45^\circ + \cos^2 60^\circ = 1$$

$$\text{L.H.S} = \cos^2 45^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$= \left(\frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{2} \right)^2$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{5}{4} \neq \text{R.H.S}$$

Therefore given angles are not direction angles.

Question # 12(ii)

$30^\circ, 45^\circ, 60^\circ$ will be direction angles of the vectors if

$$\cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ = 1$$

$$\text{L.H.S} = \cos^2 30^\circ + \cos^2 45^\circ + \cos^2 60^\circ$$

$$\begin{aligned}&= \left(\frac{\sqrt{3}}{2} \right)^2 + \left(\frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{2} \right)^2 \\ &= \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2} \neq \text{R.H.S}\end{aligned}$$

Therefore given angles are not direction angles.

Question # 12 (iii)

$30^\circ, 60^\circ, 60^\circ$ will be direction angles of the vectors if

$$\cos^2 45^\circ + \cos^2 60^\circ + \cos^2 60^\circ = 1$$

$$\text{L.H.S} = \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 60^\circ$$

$$\begin{aligned}&= \left(\frac{1}{\sqrt{2}} \right)^2 + \left(\frac{1}{2} \right)^2 + \left(\frac{1}{2} \right)^2 \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 = \text{R.H.S}\end{aligned}$$

Therefore given angles are direction angles.

Error Analyst**Saqib Aleem**

Punjab College of Sciences

Muhammad Tayyab Riaz (2009-10)

Pakistan International School Al-Khobar, Saudi Arabia.

Awais (2009-10)

Punjab College, Lahore.

Salman Ali (2009-2010)

Superior College Multan.

Become an Error analyst, submit errors at
<http://www.mathcity.org/errors>

Book: Exercise 7.2

Calculus and Analytic Geometry

Mathematic 12

Punjab Textbook Board, Lahore.

Edition: August 2003.

Made by: Atiq ur Rehman (atiq@mathcity.org)

Available online at <http://www.MathCity.org> in PDF Format (Picture format to view online).

Page Setup used: A4

Updated: December 06, 2015.