Ð *Binomial Theorem*

If a and x are two real number and n is a positive integer then

$$(a+x)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}x^1 + \binom{n}{2}a^{n-2}x^2 + \ldots\ldots\ldots + \binom{n}{n-1}a\,x^{n-1} + \binom{n}{n}x^n$$

PROOF

We will use mathematical induction to prove this so let $S(n)$ be the given statement.
Put $n = 1$

$$S(1): (a+x)^1 = \binom{1}{0}a^1 + \binom{1}{1}a^{1-1}x^1 = (1)a + (1)(1)\,x \quad \Rightarrow \quad a+x = a+x$$

$S(1)$ is true so condition I is satisfied.

Now suppose that $S(n)$ is true for $n = k$.

$$S(k): (a+x)^k = \binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^1 + \binom{k}{2}a^{k-2}x^2 + \ldots\ldots\ldots + \binom{k}{k-1}a\,x^{k-1} + \binom{k}{k}x^k \quad \ldots..(i)$$

The statement for $n = k+1$

$$S(k+1): \ (a+x)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^{k+1-1}x^1 + \binom{k+1}{2}a^{k+1-2}x^2 + \ldots\ldots\ldots$$
$$+ \binom{k+1}{k+1-1}a\,x^{k+1-1} + \binom{k+1}{k+1}x^{k+1}$$

$$\Rightarrow (a+x)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k x^1 + \binom{k+1}{2}a^{k-1}x^2 + \ldots\ldots\ldots$$
$$+ \binom{k+1}{k}a\,x^k + \binom{k+1}{k+1}x^{k+1}$$

Multiplying both sides of equation (i) by $(a+x)$

$$(a+x)^k(a+x) = \left(\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^1 + \binom{k}{2}a^{k-2}x^2 + \ldots\ldots\ldots + \binom{k}{k-1}a\,x^{k-1} + \binom{k}{k}x^k\right)(a+x)$$

$$\Rightarrow (a+x)^{k+1} = \left(\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^1 + \binom{k}{2}a^{k-2}x^2 + \ldots\ldots\ldots + \binom{k}{k-1}a\,x^{k-1} + \binom{k}{k}x^k\right)(a)$$
$$+ \left(\binom{k}{0}a^k + \binom{k}{1}a^{k-1}x^1 + \binom{k}{2}a^{k-2}x^2 + \ldots\ldots\ldots + \binom{k}{k-1}a\,x^{k-1} + \binom{k}{k}x^k\right)(x)$$

$$\Rightarrow (a+x)^{k+1} = \binom{k}{0}a^{k+1} + \binom{k}{1}a^k x^1 + \binom{k}{2}a^{k-1}x^2 + \ldots\ldots\ldots + \binom{k}{k-1}a^2\,x^{k-1} + \binom{k}{k}a\,x^k$$
$$+ \binom{k}{0}a^k x + \binom{k}{1}a^{k-1}x^2 + \binom{k}{2}a^{k-2}x^3 + \ldots\ldots\ldots + \binom{k}{k-1}a\,x^k + \binom{k}{k}x^{k+1}$$

$$\Rightarrow (a+x)^{k+1} = \binom{k}{0}a^{k+1} + \left(\binom{k}{1}+\binom{k}{0}\right)a^k x^1 + \left(\binom{k}{2}+\binom{k}{1}\right)a^{k-1}x^2 + \ldots\ldots\ldots$$
$$+ \left(\binom{k}{k}+\binom{k}{k-1}\right)a\,x^k + \binom{k}{k}x^{k+1}$$

Since $\binom{n}{0} = \binom{n+1}{0}$, $\binom{n}{r}+\binom{n}{r-1} = \binom{n+1}{r}$ and $\binom{n}{n} = \binom{n+1}{n+1}$

$$\Rightarrow (a+x)^{k+1} = \binom{k+1}{0}a^{k+1} + \binom{k+1}{1}a^k x^1 + \binom{k+1}{2}a^{k-1}x^2 + \ldots\ldots\ldots$$
$$+ \binom{k+1}{k}a\,x^k + \binom{k+1}{k+1}x^{k+1}$$

Thus $S(k+1)$ is true when $S(k)$ is true so condition II is satisfied and $S(n)$ is true for all positive integral value of n.

Đ *Question # 1 (i)*

$$(a+2b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^{5-1}(2b)^1 + \binom{5}{2}a^{5-2}(2b)^2 + \binom{5}{3}a^{5-3}(2b)^3 + \binom{5}{4}a^{5-4}(2b)^4 + \binom{5}{5}a^{5-5}(2b)^5$$

$$= (1)a^5 + (5)a^4(2b) + (10)a^3(4b^2) + (10)a^2(8b^3) + (5)a^1(16b^4) + (1)a^0(32b^5)$$

$$= a^5 + 10a^4b + 40a^3b^2 + 80a^2b^3 + 80ab^4 + 32b^5 \qquad \because a^0 = 1$$

Đ *Question # 1 (ii)*

$$\left(\frac{x}{2} - \frac{2}{x^2}\right)^6 = \binom{6}{0}\left(\frac{x}{2}\right)^6 + \binom{6}{1}\left(\frac{x}{2}\right)^{6-1}\left(-\frac{2}{x^2}\right)^1 + \binom{6}{2}\left(\frac{x}{2}\right)^{6-2}\left(-\frac{2}{x^2}\right)^2 + \binom{6}{3}\left(\frac{x}{2}\right)^{6-3}\left(-\frac{2}{x^2}\right)^3$$

$$+ \binom{6}{4}\left(\frac{x}{2}\right)^{6-4}\left(-\frac{2}{x^2}\right)^4 + \binom{6}{5}\left(\frac{x}{2}\right)^{6-5}\left(-\frac{2}{x^2}\right)^5 + \binom{6}{6}\left(\frac{x}{2}\right)^{6-6}\left(-\frac{2}{x^2}\right)^6$$

$$= (1)\left(\frac{x}{2}\right)^6 - (6)\left(\frac{x}{2}\right)^5\left(\frac{2}{x^2}\right) + (15)\left(\frac{x}{2}\right)^4\left(\frac{2}{x^2}\right)^2 - (20)\left(\frac{x}{2}\right)^3\left(\frac{2}{x^2}\right)^3$$

$$+ (15)\left(\frac{x}{2}\right)^2\left(\frac{2}{x^2}\right)^4 - (6)\left(\frac{x}{2}\right)^1\left(\frac{2}{x^2}\right)^5 + (1)(1)\left(\frac{2}{x^2}\right)^6$$

$$= \left(\frac{x^6}{64}\right) - 6\left(\frac{x^5}{32}\right)\left(\frac{2}{x^2}\right) + 15\left(\frac{x^4}{16}\right)\left(\frac{4}{x^4}\right) - 20\left(\frac{x^3}{8}\right)\left(\frac{8}{x^6}\right)$$

$$+ 15\left(\frac{x^2}{4}\right)\left(\frac{16}{x^8}\right) - 6\left(\frac{x}{2}\right)\left(\frac{32}{x^{10}}\right) + \left(\frac{64}{x^{12}}\right)$$

$$= \frac{x^6}{64} - \frac{3x^3}{8} + \frac{15}{4} - \frac{20}{x^3} + \frac{60}{x^6} - \frac{96}{x^9} + \frac{64}{x^{12}}$$

Question # 1 (iii), (iv) and (v)

Do yourself

Đ *Question # 1 (vi)*

$$\left(\sqrt{\frac{a}{x}} - \sqrt{\frac{x}{a}}\right)^6 = \binom{6}{0}\left(\sqrt{\frac{a}{x}}\right)^6 + \binom{6}{1}\left(\sqrt{\frac{a}{x}}\right)^{6-1}\left(-\sqrt{\frac{x}{a}}\right)^1 + \binom{6}{2}\left(\sqrt{\frac{a}{x}}\right)^{6-2}\left(-\sqrt{\frac{x}{a}}\right)^2$$

$$+ \binom{6}{3}\left(\sqrt{\frac{a}{x}}\right)^{6-3}\left(-\sqrt{\frac{x}{a}}\right)^3 + \binom{6}{4}\left(\sqrt{\frac{a}{x}}\right)^{6-4}\left(-\sqrt{\frac{x}{a}}\right)^4 + \binom{6}{5}\left(\sqrt{\frac{a}{x}}\right)^{6-5}\left(-\sqrt{\frac{x}{a}}\right)^5 + \binom{6}{6}\left(\sqrt{\frac{a}{x}}\right)^{6-6}\left(-\sqrt{\frac{x}{a}}\right)^6$$

$$= (1)\left(\sqrt{\frac{a}{x}}\right)^6 - (6)\left(\sqrt{\frac{a}{x}}\right)^5\left(\sqrt{\frac{x}{a}}\right)^1 + (15)\left(\sqrt{\frac{a}{x}}\right)^4\left(\sqrt{\frac{x}{a}}\right)^2$$

$$- (20)\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{x}{a}}\right)^3 + (15)\left(\sqrt{\frac{a}{x}}\right)^2\left(\sqrt{\frac{x}{a}}\right)^4 - (6)\left(\sqrt{\frac{a}{x}}\right)^1\left(\sqrt{\frac{x}{a}}\right)^5 + (1)\left(\sqrt{\frac{a}{x}}\right)^0\left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^5\left(\sqrt{\frac{a}{x}}\right)^{-1} + 15\left(\sqrt{\frac{a}{x}}\right)^4\left(\sqrt{\frac{a}{x}}\right)^{-2}$$

$$- 20\left(\sqrt{\frac{a}{x}}\right)^3\left(\sqrt{\frac{a}{x}}\right)^{-3} + 15\left(\sqrt{\frac{x}{a}}\right)^{-2}\left(\sqrt{\frac{x}{a}}\right)^4 - 6\left(\sqrt{\frac{x}{a}}\right)^{-1}\left(\sqrt{\frac{x}{a}}\right)^5 + 1(1)\left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^{5-1} + 15\left(\sqrt{\frac{a}{x}}\right)^{4-2} - 20\left(\sqrt{\frac{a}{x}}\right)^{3-3} + 15\left(\sqrt{\frac{x}{a}}\right)^{-2+4} - 6\left(\sqrt{\frac{x}{a}}\right)^{-1+5} + 1\left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\sqrt{\frac{a}{x}}\right)^6 - 6\left(\sqrt{\frac{a}{x}}\right)^4 + 15\left(\sqrt{\frac{a}{x}}\right)^2 - 20\left(\sqrt{\frac{a}{x}}\right)^0 + 15\left(\sqrt{\frac{x}{a}}\right)^2 - 6\left(\sqrt{\frac{x}{a}}\right)^4 + \left(\sqrt{\frac{x}{a}}\right)^6$$

$$= \left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^6 - 6\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^4 + 15\left(\left(\frac{a}{x}\right)^{\frac{1}{2}}\right)^2 - 20(1) + 15\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^2 - 6\left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^4 + \left(\left(\frac{x}{a}\right)^{\frac{1}{2}}\right)^6$$

$$= \left(\frac{a}{x}\right)^3 - 6\left(\frac{a}{x}\right)^2 + 15\left(\frac{a}{x}\right) - 20 + 15\left(\frac{x}{a}\right) - 6\left(\frac{x}{a}\right)^2 + \left(\frac{x}{a}\right)^3$$

$$= \frac{a^3}{x^3} - 6\frac{a^2}{x^2} + 15\frac{a}{x} - 20 + 15\frac{x}{a} - 6\frac{x^2}{a^2} + \frac{x^3}{a^3}$$

Đ *Question # 2 (i)*

$$(0.97)^3 = (1 - 0.03)^3$$

$$= \binom{3}{0}(1)^3 + \binom{3}{1}(1)^2(-0.03) + +\binom{3}{2}(1)^1(-0.03)^2 + +\binom{3}{3}(-0.03)^3$$

$$= (1)(1) + 3(1)(-0.03) + 3(1)(0.0009) + +(1)(-0.000024)$$

$$= 1 - 0.09 + 0.0027 - 0.000027 = 0.912673$$

Đ *Question # 2 (ii)*

$$(2.02)^4 = (2 + 0.02)^4 \qquad \textit{Now do yourself.}$$

Đ *Question # 2 (iii)*

$$(9.98)^4 = (10 - 0.02)^4$$

$$= \binom{4}{0}(10)^4 + \binom{4}{1}(10)^3(-0.02) + \binom{4}{2}(10)^2(-0.02)^2 + \binom{4}{3}(10)^1(-0.02)^3 + \binom{4}{4}(10)^0(-0.02)^4$$

$$= (1)(10000) + 4(1000)(-0.02) + 6(100)(0.0004) + 4(10)(-0.000008) + (1)(1)(0.00000016)$$

$$= 10000 - 80 + 0.24 - 0.00032 + 0.00000016 = 9920.23968$$

Đ *Question # 2 (iv)*

$$(2.1)^5 = (2 + 0.1)^5$$

Đ *Question # 3(i)*

$$\left(a + \sqrt{2}x\right)^4 + \left(a - \sqrt{2}x\right)^4$$

We take

$$\left(a + \sqrt{2}x\right)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3\left(\sqrt{2}x\right)^1 + \binom{4}{2}a^2\left(\sqrt{2}x\right)^2 + \binom{4}{3}a^1\left(\sqrt{2}x\right)^3 + \binom{4}{4}a^0\left(\sqrt{2}x\right)^4$$

$$= (1)a^4 + (4)a^3\left(\sqrt{2}x\right) + (6)a^2\left(2x^2\right) + (4)a\left(2\sqrt{2}x^3\right) + (1)(1)\left(4x^4\right)$$

$$\Rightarrow \left(a + \sqrt{2}x\right)^4 = a^4 + 4\sqrt{2}a^3x + 12a^2x^2 + 8\sqrt{2}\,ax^3 + 4x^4 \quad \text{............. } (i)$$

Replacing $\sqrt{2}$ by $-\sqrt{2}$ in eq. (i)

$$\left(a - \sqrt{2}x\right)^4 = a^4 + 4\left(-\sqrt{2}\right)a^3x + 12a^2x^2 + 8\left(-\sqrt{2}\right)ax^3 + 4x^4$$

$$= a^4 - 4\sqrt{2}a^3x + 12a^2x^2 - 8\sqrt{2}\,ax^3 + 4x^4 \quad \text{............... } (ii)$$

Adding (i) & (ii)

$$\left(a + \sqrt{2}x\right)^4 + \left(a - \sqrt{2}x\right)^4 = 2a^4 + 24a^2x^2 + 8x^4$$

Đ *Question # 3 (ii)*

Do yourself.

Đ *Question # 3 (iii)*

Since

$$(2+i)^5 = \binom{5}{0}2^5 + \binom{5}{1}2^{5-1}i + \binom{5}{2}2^{5-2}i^2 + \binom{5}{3}2^{5-3}i^3 + \binom{5}{4}2^{5-4}i^4 + \binom{5}{5}2^{5-5}i^5$$

$$= (1)2^5 + (5)2^4 i + (10)2^3 i^2 + (10)2^2 i^3 + (5)2^1 i^4 + (1)2^0 i^5$$

$$= 32 + 80i + 80i^2 + 40i^3 + 10i^4 + i^5 \quad \ldots\ldots\ldots (i)$$

Replacing i by $-i$ in eq. (i)

$$(2+i)^5 = 32 + 80(-i) + 80(-i)^2 + 40(-i)^3 + 10(-i)^4 + (-i)^5$$

$$= 32 - 80i + 80i^2 - 40i^3 + 10i^4 - i^5$$

Subtracting (i) & (ii)

$$(2+i)^5 - (2-i)^5 = 160i + 80i^3 + 2i^5$$

$$= 160i + 80(-1)\cdot i + 2(-1)^2 \cdot i$$

$$= 160i - 80i + 2i \quad = 82i$$

Đ *Question # 3 (iv)*

$$\left(x + \sqrt{x^2-1}\right)^3 + \left(x + \sqrt{x^2-1}\right)^3$$

Suppose $t = \sqrt{x^2-1}$ then

$$\left(x + \sqrt{x^2-1}\right)^3 + \left(x + \sqrt{x^2-1}\right)^3 = (x+t)^3 + (x+t)^3$$

$$= \left((x)^3 + 3(x)^2(t) + 3(x)(t)^2 + (t)^3\right) + \left((x)^3 + 3(x)^2(-t) + 3(x)(-t)^2 + (-t)^3\right)$$

$$= x^3 + 3x^2 t + 3xt^2 + t^3 + x^3 - 3x^2 t + 3xt^2 - t^3$$

$$= 2x^3 + 6xt^2$$

$$= 2x^3 + 6x\left(\sqrt{x^2-1}\right)^2 \qquad \because t = \sqrt{x^2-1}$$

$$= 2x^3 + 6x(x^2-1) = 2x^3 + 6x^3 - 6x = 8x^3 - 6x$$

Đ *Question # 4 (i)*

$$(2 + x - x^2)^4$$

Put $t = 2 + x$ then

$$(2 + x - x^2)^4 = (t - x^2)^4$$

$$= \binom{4}{0}(t)^4 + \binom{4}{1}(t)^3(-x^2) + \binom{4}{2}(t)^2(-x^2)^2 + \binom{4}{3}(t)^1(-x^2)^3 + \binom{4}{4}(t)^0(-x^2)^4$$

$$= (1)(t)^4 - (4)(t)^3(x^2) + (6)(t)^2(x^4) - (4)(t)(x^6) + (1)(1)(x^8)$$

$$= t^4 - 4t^3 x^2 + 6t^2 x^4 - 4t x^6 + x^8 \quad \ldots\ldots\ldots\ldots\ldots (i)$$

Now

$$t^4 = (2+x)^4 = \binom{4}{0}(2)^4 + \binom{4}{1}(2)^3(x) + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4$$

$$= (1)(16) + (4)(8)(x) + (6)(4)(x^2) + (4)(2)(x^3) + (1)(1)(x^4)$$

$$= 16 + 32x + 24x^2 + 8x^3 + x^4$$

Also

$$t^3 = (2+x)^3 = (2)^3 + (3)(2)^2(x) + (3)(2)^1(x)^2 + (x)^3$$

$$= 8 + 12x + 6x^2 + x^3$$

$$t^2 = (2+x)^2 = 4 + 4x + x^2$$

Putting values of t^4, t^3, t^2 and t in equation (i)

$$(2 + x - x^2)^4 = (16 + 32x + 24x^2 + 8x^3 + x^4) - 4(8 + 12x + 6x^2 + x^3)x^2$$

$$+ 6(4 + 4x + x^2)x^4 - 4(2 + x)x^6 + x^8$$

$$= 16 + 32x + 24x^2 + 8x^3 + x^4 - 32x^2 - 48x^3 - 24x^4 - 4x^5$$
$$+ 24x^4 + 24x^5 + 6x^6 - 8x^6 + 4x^7 + x^8$$
$$= 16 + 32 - 8x^2 - 40x^3 + x^4 + 20x^5 - 2x^6 - 4x^7 - x^8$$

Ð *Question # 4 (ii)*

Suppose $t = 1 - x$ *Do yourself*

Ð *Question # 4 (iii)*

Suppose $t = 1 - x$ *Do yourself*

Ð *Question # 5 (i)*

Suppose $t = x - 1$ *Do yourself*

Ð *Question # 5 (ii)*

$$\left(x - 1 - \frac{1}{x} \right)^3$$

Suppose $t = x - 1$ then

$$\left(t - \frac{1}{x} \right)^3 = (t)^3 + 3(t)^2 \left(-\frac{1}{x} \right) + 3(t) \left(-\frac{1}{x} \right)^2 + \left(-\frac{1}{x} \right)^3$$

$$= t^3 - 3t^2 \cdot \frac{1}{x} + 3t \cdot \frac{1}{x^2} - \frac{1}{x^3} \quad \cdots\cdots\cdots \ (i)$$

Now

$$t^3 = (x-1)^3 = (x)^3 + 3(x)^2(-1) + 3(x)(-1)^2 + (-1)^3$$
$$= x^3 - 3x^2 + 3x - 1$$
$$t^2 = (x-1)^2 = x^2 - 2x + 1$$

Putting values of t^3, t^2 and t in equation (i)

$$\left(x - 1 - \frac{1}{x} \right)^3 = (x^3 - 3x^2 + 3x - 1) - 3(x^2 - 2x + 1) \cdot \frac{1}{x} + 3(x-1) \cdot \frac{1}{x^2} - \frac{1}{x^3}$$

$$= x^3 - 3x^2 + 3x - 1 - 3x + 6 - 3\frac{1}{x} + 3\frac{1}{x} - 3\frac{1}{x^2} - \frac{1}{x^3}$$

$$= x^3 - 3x^2 + 5 - \frac{3}{x^2} - \frac{1}{x^3}$$

Ð *Question # 6 (i)*

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

Here $a = 3$, $x = -2x$, $n = 7$ so

$$T_{r+1} = \binom{7}{r} (3)^{7-r} (-2x)^r = \binom{7}{r} (3)^{7-r} (-2)^r (x)^r$$

For term involving x^4 we must have

$$x^r = x^4 \ \Rightarrow \ r = 4$$

So

$$T_{4+1} = \binom{7}{4} (3)^{7-4} (-2)^4 (x)^4$$

$$\Rightarrow \ T_5 = (35)(3)^3(-2)^4(x)^4 = (35)(27)(16)(x)^4$$

$$= 15120x^4$$

Đ *Question # 6 (ii)*

Since $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

Here $\quad a = x, \quad x = -\dfrac{2}{x^2}, \quad n = 13$ so

$$T_{r+1} = \binom{13}{r}(x)^{13-r}\left(-\dfrac{2}{x^2}\right)^r = \binom{13}{r}(x)^{13-r}(-2)^r (x)^{-2r}$$

$$= \binom{13}{r}(x)^{13-r-2r}(-2)^r = \binom{13}{r}(x)^{13-3r}(-2)^r$$

For term involving x^{-2} we must have

$$x^{13-3r} = x^{-2} \implies 13 - 3r = -2 \implies -3r = -2 - 13$$

$$\implies -3r = -15 \implies r = 5$$

So

$$T_{5+1} = \binom{13}{5}(x)^{13-3(5)}(-2)^5$$

$$\implies T_6 = (1287)(x)^{13-15}(-32) = -41184 x^{-2}$$

Đ *Question # 6 (iii)*

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

Here $\quad a = \dfrac{2}{x}, \quad x = -a, \quad n = 9 \quad$ so

$$T_{r+1} = \binom{9}{r}\left(\dfrac{2}{x}\right)^{9-r}(-a)^r = \binom{9}{r}\left(\dfrac{2}{x}\right)^{9-r}(-1)^r (a)^r$$

For term involving a^4 we must have

$$a^r = a^4 \implies r = 4$$

So $\quad T_{4+1} = \binom{9}{4}\left(\dfrac{2}{x}\right)^{9-4}(-1)^4 (a)^4$

$$\implies T_5 = (126)\left(\dfrac{2}{x}\right)^5 (1) a^4 = (126)\left(\dfrac{32}{x^5}\right) a^4 = 4032 \dfrac{a^4}{x^5}$$

Đ *Question # 6 (iii)*

Here $\quad a = x, \quad x = -\sqrt{y}, \quad n = 11 \quad$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{11}{r}(x)^{11-r}\left(-\sqrt{y}\right)^r = \binom{11}{r}(x)^{11-r}\left(-y^{\frac{1}{2}}\right)^r$$

$$= \binom{11}{r}(x)^{11-r}(-1)^r\left(y^{\frac{r}{2}}\right)$$

For term involving y^3 we must have

$$y^{\frac{r}{2}} = y^3 \implies \dfrac{r}{2} = 3 \implies r = 6$$

So $\quad T_{6+1} = \binom{11}{6}(x)^{11-6}(-1)^6\left(y^{\frac{6}{2}}\right)$

$$\implies T_7 = (462)(x)^5 (1)\left(y^3\right) = 462 x^5 y^3$$

Ð *Question # 7 (i)*

Here $a = x^2$, $x = -\dfrac{3}{2x}$, $n = 10$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{10}{r} \left(x^2\right)^{10-r} \left(-\frac{3}{2x}\right)^r = \binom{10}{r} (x)^{2(10-r)} (-1)^r \frac{(3)^r}{(2)^r (x)^r}$$

$$= \binom{10}{r} (x)^{20-2r} (-1)^r (3)^r (2)^{-r} (x)^{-r} = \binom{10}{r} (x)^{20-2r-r} (-1)^r (3)^r (2)^{-r}$$

$$= \binom{10}{r} (x)^{20-3r} (-1)^r (3)^r (2)^{-r}$$

For term involving x^5 we must have

$$x^{20-3r} = x^5 \quad \Rightarrow \quad 20 - 3r = 5 \quad \Rightarrow \quad -3r = 5 - 20$$

$$\Rightarrow \quad -3r = -15 \quad \Rightarrow \quad r = 5$$

So $$T_{5+1} = \binom{10}{5} (x)^{20-3(5)} (-1)^5 (3)^5 (2)^{-5}$$

$$\Rightarrow T_6 = 252 (x)^{20-15} (-1)^5 (3)^5 \frac{1}{2^5} = -252 (x)^5 (243) \frac{1}{32}$$

$$= -\frac{61236}{32} x^5 = -\frac{15309}{8} x^5$$

Hence coefficient of $x^5 = -\dfrac{15309}{8}$

Ð *Question # 7 (ii)*

Here $a = x^2$, $x = -\dfrac{1}{x}$, $n = 2n$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{2n}{r} \left(x^2\right)^{2n-r} \left(-\frac{1}{x}\right)^r = \binom{2n}{r} (x)^{2(2n-r)} (-1)^r \frac{1}{x^r}$$

$$= \binom{2n}{r} (x)^{4n-2r} (-1)^r x^{-r} = \binom{2n}{r} (x)^{4n-2r-r} (-1)^r$$

$$= \binom{2n}{r} (x)^{4n-3r} (-1)^r$$

For term involving x^n we must have

$$x^{4n-3r} = x^n \quad \Rightarrow \quad 4n - 3r = n \quad \Rightarrow \quad -3r = n - 4n$$

$$\Rightarrow \quad -3r = -3n \quad \Rightarrow \quad r = n$$

So $$T_{n+1} = \binom{2n}{n} (x)^{4n-3n} (-1)^n$$

$$= \frac{(2n)!}{(2n-n)! \cdot n!} (x)^n (-1)^n = \frac{(2n)!}{n! \cdot n!} (x)^n (-1)^n$$

$$= (-1)^n \frac{(2n)!}{(n!)^2} x^n$$

Hence coefficient of $x^n = (-1)^n \dfrac{(2n)!}{(n!)^2}$

Đ *Question # 8*

Here $a = x^2$, $x = -\dfrac{3}{2x}$, $n = 10$ and $r + 1 = 6 \Rightarrow r = 5$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{5+1} = \binom{10}{5} (x^2)^{10-5} \left(-\frac{3}{2x}\right)^5$$

$$\Rightarrow T_6 = 252(x^2)^5 \left(-\frac{3^5}{(2x)^5}\right) = 252 x^{10} \left(-\frac{243}{32x^5}\right)$$

$$= -\frac{61236}{32} x^{10-5} = -\frac{15309}{8} x^5$$

Đ *Question # 9 (i)*

Do yourself as Q # 9 (ii)

Đ *Question # 9 (ii)*

Here $a = \sqrt{x}$, $x = \dfrac{1}{2x^2}$, $n = 10$ so

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$T_{r+1} = \binom{10}{r} (\sqrt{x})^{10-r} \left(\frac{1}{2x^2}\right)^r = \binom{10}{r} \left(x^{\frac{1}{2}}\right)^{10-r} \left(\frac{1}{2^r x^{2r}}\right)$$

$$= \binom{10}{r} (x)^{\frac{1}{2}(10-r)} \frac{1}{2^r} x^{-2r} = \binom{10}{r} (x)^{5-\frac{r}{2}} \frac{1}{2^r} x^{-2r}$$

$$= \binom{10}{r} (x)^{5-\frac{r}{2}-2r} \frac{1}{2^r} = \binom{10}{r} (x)^{5-\frac{5r}{2}} \frac{1}{2^r}$$

For term independent of x we must have

$$x^{5-\frac{5r}{2}} = x^0 \quad \Rightarrow \quad 5 - \frac{5r}{2} = 0 \quad \Rightarrow \quad -\frac{5r}{2} = -5$$

$$\Rightarrow \quad r = (-5)\left(-\frac{2}{5}\right) \quad \Rightarrow \quad r = 2$$

So $\quad T_{2+1} = \binom{10}{2} (x)^{5-\frac{5(2)}{2}} \frac{1}{2^2}$

$$\Rightarrow T_3 = 45 (x)^{5-5} \frac{1}{4} = 45 x^0 \frac{1}{4}$$

$$= 45 (1) \frac{1}{4} = \frac{45}{4}$$

Đ *Question # 9 (iii)*

$$(1+x^2)^3 \left(1 + \frac{1}{x^2}\right)^4 = (1+x^2)^3 \left(\frac{x^2+1}{x^2}\right)^4$$

$$= (1+x^2)^3 \frac{(x^2+1)^4}{(x^2)^4} = (1+x^2)^3 \frac{(1+x^2)^4}{x^8}$$

$$= x^{-8}(1+x^2)^{3+4} = x^{-8}(1+x^2)^7$$

Now $\quad T_{r+1} = x^{-8} \binom{n}{r} a^{n-r} x^r$

Where $n=7$, $a=1$, $x=x^2$

$$T_{r+1} = x^{-8}\binom{7}{r}(1)^{7-r}\left(x^2\right)^r = x^{-8}\binom{7}{r}(1)\ x^{2r}$$

$$= \binom{7}{r}x^{2r-8}$$

For term independent of x we must have

$$x^{2r-8} = x^0 \implies 2r-8=0 \implies 2r=8 \implies r=4$$

So

$$T_{4+1} = \binom{7}{4}x^{2(4)-8}$$

$$\implies T_5 = 35\ x^{8-8} = 35\ x^0 = 35$$

Đ *Question # 10*

$$\left(\frac{1}{x}-\frac{x^2}{2}\right)^{12}$$

Since $n=12$ is an even so middle terms is $\dfrac{n+2}{2}=\dfrac{12+2}{2}=7$

Therefore $r+1=7 \implies r=7-1=6$

And $a=\dfrac{1}{x}$, $x=-\dfrac{x^2}{2}$ and $n=12$

Now

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r$$

$$\implies T_{6+1} = \binom{12}{6}\left(\frac{1}{x}\right)^{12-6}\left(-\frac{x^2}{2}\right)^6$$

$$\implies T_7 = 924\ \frac{1}{x^6}\frac{x^{12}}{64} = \frac{924}{64}x^{12-6}$$

$$= \frac{231}{16}x^6$$

Thus the middle terms of the given expansion is $\dfrac{231}{16}x^6$.

Đ *Question # 10 (ii)*

Since $n=11$ is odd so the middle terms are $\dfrac{n+1}{2}=\dfrac{11+1}{2}=6$ and

$$\frac{n+3}{2}=\frac{11+3}{2}=7$$

So for first middle term

$$a=\frac{3}{2}x, \quad x=-\frac{1}{3x}, \quad n=11 \text{ and } r+1=6 \implies r=5$$

Now

$$T_{r+1} = \binom{n}{r}a^{n-r}x^r \qquad \implies T_{5+1} = \binom{11}{5}\left(\frac{3}{2}x\right)^{11-5}\left(-\frac{1}{3x}\right)^5$$

Now simplify yourself.

Now for second middle term

$$r+1=7 \implies r=6$$

so $\qquad T_{6+1} = \binom{11}{6}\left(\frac{3}{2}x\right)^{11-6}\left(-\frac{1}{3x}\right)^6$ *Now simplify yourself.*

Ð *Question # 10 (iii)*

Since $n = 2m + 1$ is odd so there are two middle terms

First middle term $= \dfrac{n+1}{2} = \dfrac{2m+1+1}{2} = \dfrac{2m+2}{2} = m+1$

Second middle terms $= \dfrac{n+3}{2} = \dfrac{2m+1+3}{2} = \dfrac{2m+4}{2} = m+2$

Here $a = 2x$, $x = -\dfrac{1}{2x}$ and $n = 2m+1$

For first middle term $r+1 = m+1 \Rightarrow r = m$.

Since

$$T_{r+1} = \binom{n}{r} a^{n-r} x^r$$

$$\Rightarrow T_{m+1} = \binom{2m+1}{m}(2x)^{2m+1-m}\left(-\frac{1}{2x}\right)^m = \frac{(2m+1)!}{(2m+1-m)!\cdot m!}(2x)^{m+1}\left(-\frac{1}{2x}\right)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1}(x)^{m+1}(-1)^m\left(\frac{1}{2}\right)^m\left(\frac{1}{x}\right)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1}(x)^{m+1}(-1)^m(2)^{-m}(x)^{-m}$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^{m+1-m}(x)^{m+1-m}(-1)^m = \frac{(2m+1)!}{(m+1)!\cdot m!}(2)^1(x)^1(-1)^m$$

$$= \frac{(2m+1)!}{(m+1)!\cdot m!}2x(-1)^m$$

For second middle term

$$r+1 = m+2 \quad \Rightarrow r = m+2-1 \quad \Rightarrow r = m+1$$

As $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

$$\Rightarrow T_{m+1+1} = \binom{2m+1}{m+1}(2x)^{(2m+1)-(m+1)}\left(-\frac{1}{2x}\right)^{m+1}$$

Now simplify yourself

Ð *Question # 11* (Submitted by Waqas Ahmad - FAZMIC Sargodha – 2004-06)

Here $a = x$, $x = -\dfrac{1}{2x}$,

Number of term from the end $= 2n + 1$

To make it from beginning we take $a = -\dfrac{1}{2x}$, $x = x$ and $r+1 = 2n+1 \Rightarrow r = 2n$

As $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

$$\Rightarrow T_{2n+1} = \binom{3n}{2n}\left(-\frac{1}{2x}\right)^{3n-2n}(x)^{2n} = \frac{(3n)!}{(3n-2n)!\cdot (2n)!}\left(-\frac{1}{2x}\right)^n x^{2n}$$

$$= \frac{(3n)!}{(n)!\cdot (2n)!}(-1)^n\frac{1}{2^n \cdot x^n} x^{2n} = \frac{(3n)!}{n!\cdot (2n)!}(-1)^n\frac{1}{2^n} x^{2n-n}$$

$$= \frac{(-1)^n}{2^n}\frac{(3n)!}{n!\cdot (2n)!} x^n \quad \text{Answer}$$

Note: If there are p term in some expansion and qth term is from the end then the term from the beginning will be $= p-q+1$.

So in above you can use term from the end $= (3n+1)-(2n+1)+1 = n+1$

Đ *Question # 12*

Since $2n$ is even so the middle term is $\dfrac{2n+2}{2} = n+1$ and

$$a=1, \quad x=x, \quad n=2n, \quad r+1=n+1 \implies r=n$$

Now $\quad T_{r+1} = \binom{n}{r} a^{n-r} x^r$

$$\implies T_{n+1} = \binom{2n}{n}(1)^{2n-n} x^n$$

$$\implies T_{n+1} = \frac{(2n)!}{(2n-n)! \cdot n!}(1)^n x^n = \frac{(2n)!}{n! \cdot n!} x^n$$

$$= \frac{2n(2n-1)(2n-2)(2n-3)(2n-4)\ldots\ldots\ldots\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{n! \cdot n!} x^n$$

$$= \frac{\left[2n\,(2n-2)(2n-4)\ldots\ldots\ldots\cdot 4\cdot 2\right]\left[(2n-1)(2n-3)\ldots\ldots\ldots\cdot 5\cdot 3\cdot 1\right]}{n! \cdot n!} x^n$$

$$= \frac{2^n\left[n\,(n-1)(n-2)\ldots\ldots\ldots\cdot 2\cdot 1\right]\left[(2n-1)(2n-3)\ldots\ldots\ldots\cdot 5\cdot 3\cdot 1\right]}{n! \cdot n!} x^n$$

$$= \frac{2^n\, n!\left[(2n-1)(2n-3)\ldots\ldots\ldots\cdot 5\cdot 3\cdot 1\right]}{n! \cdot n!} x^n$$

$$= \frac{2^n\left[1\cdot 3\cdot 5\cdot\ldots\ldots\ldots\cdot (2n-1)\right]}{n!} x^n$$

$$= \frac{1\cdot 3\cdot 5\cdot\ldots\ldots\ldots\cdot (2n-1)}{n!}\, 2^n x^n$$

Đ *Question # 13*

Consider

$$(1+x)^n = \binom{n}{0}+\binom{n}{1}x+\binom{n}{2}x^2+\binom{n}{3}x^3+\binom{n}{4}x^4+\binom{n}{5}x^5+\ldots\ldots\ldots\ldots+\binom{n}{n-1}x^{n-1}+\binom{n}{n}x^n$$

$$\ldots\ldots\ldots (i)$$

Put $x=1$

$$(1+1)^n = \binom{n}{0}+\binom{n}{1}(1)+\binom{n}{2}(1)^2+\binom{n}{3}(1)^3+\binom{n}{4}(1)^4+\binom{n}{5}(1)^5+\ldots\ldots\ldots+\binom{n}{n-1}(1)^{n-1}+\binom{n}{n}(1)^n$$

$$\implies 2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\binom{n}{4}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}+\binom{n}{n}$$

$$\implies 2^n = \left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots\ldots\ldots+\binom{n}{n}\right]+\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]$$

$$\ldots\ldots\ldots (ii)$$

Now put $x=-1$ in equation (i)

$$(1-1)^n = \binom{n}{0}+\binom{n}{1}(-1)+\binom{n}{2}(-1)^2+\binom{n}{3}(-1)^3+\binom{n}{4}(-1)^4+\binom{n}{5}(-1)^5+\ldots\ldots$$

$$\ldots\ldots+\binom{n}{n-1}(-1)^{n-1}+\binom{n}{n}(-1)^n$$

If we consider n is even then

$$\implies (0)^n = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\binom{n}{3}+\binom{n}{4}-\binom{n}{5}+\ldots\ldots\ldots-\binom{n}{n-1}+\binom{n}{n}$$

$$\implies 0 = \left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots\ldots\ldots+\binom{n}{n}\right]-\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]$$

$$\implies \left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]=\left[\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\ldots\ldots\ldots+\binom{n}{n}\right]$$

Using it in equation (*ii*)

$$2^n = \left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]+\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow \quad 2^n = 2\left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow \quad \frac{2^n}{2} = \left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow \quad 2^{n-1} = \left[\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1}\right]$$

$$\Rightarrow \quad \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\ldots\ldots\ldots+\binom{n}{n-1} = 2^{n-1}$$

Ð *Question # 14*

$$\text{L.H.S} = \binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\frac{1}{4}\binom{n}{3}+\ldots\ldots\ldots+\frac{1}{n+1}\binom{n}{n}$$

$$= \left[\binom{n}{0}+\frac{1}{2}\left(\frac{n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{n!}{(n-3)!\cdot 3!}\right)+\ldots\ldots\ldots+\frac{1}{n+1}\binom{n}{n}\right]$$

$$= \frac{n+1}{n+1}\left[1+\frac{1}{2}\left(\frac{n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{n!}{(n-3)!\cdot 3!}\right)+\ldots\ldots\ldots+\frac{1}{n+1}\cdot 1\right]$$

$$= \frac{1}{n+1}\left[(n+1)+\frac{1}{2}\left(\frac{(n+1)n!}{(n-1)!\cdot 1!}\right)+\frac{1}{3}\left(\frac{(n+1)n!}{(n-2)!\cdot 2!}\right)+\frac{1}{4}\left(\frac{(n+1)n!}{(n-3)!\cdot 3!}\right)+\ldots\ldots\ldots+\frac{n+1}{n+1}\cdot 1\right]$$

$$= \frac{1}{n+1}\left[(n+1)+\left(\frac{(n+1)!}{(n-1)!\cdot 2\cdot 1!}\right)+\left(\frac{(n+1)!}{(n-2)!\cdot 3\cdot 2!}\right)+\left(\frac{(n+1)!}{(n-3)!\cdot 4\cdot 3!}\right)+\ldots\ldots\ldots+1\right]$$

$$= \frac{1}{n+1}\left[(n+1)+\left(\frac{(n+1)!}{(n+1-2)!\cdot 2!}\right)+\left(\frac{(n+1)!}{(n+1-3)!\cdot 3!}\right)+\left(\frac{(n+1)!}{(n+1-4)!\cdot 4!}\right)+\ldots\ldots\ldots+1\right]$$

$$= \frac{1}{n+1}\left[\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots\ldots\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+1+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots\ldots\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+\binom{n+1}{0}+\binom{n+1}{1}+\binom{n+1}{2}+\binom{n+1}{3}+\binom{n+1}{4}+\ldots\ldots\ldots+\binom{n+1}{n+1}\right]$$

$$= \frac{1}{n+1}\left[-1+2^{n+1}\right]$$

$$= \frac{2^{n+1}-1}{n+1} = \text{R.H.S}$$

Ð *Remember*

$$\binom{n+1}{0}=1 \ , \quad \binom{n+1}{1}=n+1 \quad \text{and} \quad \binom{n+1}{n+1}=1$$

ᴊ The End ᴊ