Principle of Mathematical Induction

A given statement $S(n)$ is true for each positive integer n if two below conditions hold

Condition I: $S(1)$ is true i.e. $S(n)$ is true for $n = 1$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers

Question # 1

Suppose $S(n):\ 1+5+9+.............+(4n-3)=n(2n-1)$

Put $n = 1$

$$S(1):\ 1=1\big(2(1)-1\big) \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+5+9+.............+(4k-3)=k(2k-1) \ ………… (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+5+9+.............+(4(k+1)-3)=(k+1)(2(k+1)-1)$$

$$\Rightarrow\ 1+5+9+.............+(4k+1)=(k+1)(2k+2-1)$$

$$=(k+1)(2k+1)$$

$$=2k^2+2k+k+1$$

$$=2k^2+3k+1$$

Adding $4k+1$ on both sides of equation (i)

$$1+5+9+.............+(4k-3)+(4k+1)=k(2k-1)+4k+1$$

$$\Rightarrow\ 1+5+9+.............+(4k+1)=2k^2-k+4k+1$$

$$=2k^2+3k+1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 2

Suppose $S(n):\ 1+3+5+.............+(2n-1)=n^2$

Put $n = 1$

$$S(1):\ 1=(1)^2 \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+3+5+.............+(2k-1)=k^2 \ ………… (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+3+5+.............+(2(k+1)-1)=(k+1)^2$$

$$\Rightarrow\ 1+3+5+.............+(2k+1)=(k+1)^2$$

Adding $2k+1$ on both sides of equation (i)

$$1+3+5+.............+(2k-1)+(2k+1)=k^2+2k+1$$

$$\Rightarrow\ 1+3+5+.............+(2k+1)=(k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 3

Suppose $S(n):\ 1+4+7+.............+(3n-2)=\dfrac{n(3n-1)}{2}$

Put $n = 1$

$$S(1):\ 1=\frac{1\big(3(1)-1\big)}{2} \ \Rightarrow\ 1=\frac{2}{2} \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+4+7+.............+(3k-2)=\frac{k(3k-1)}{2} \quad ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+4+7+.............+(3(k+1)-2)=\frac{(k+1)(3(k+1)-1)}{2}$$

$$\Rightarrow\ 1+4+7+.............+(3k+1)=\frac{(k+1)(3k+3-1)}{2}$$

$$=\frac{(k+1)(3k+2)}{2}$$

Adding $3k+1$ on both sides of equation (i)

$$1+4+7+.............+(3k-2)+(3k+1)=\frac{k(3k-1)}{2}+3k+1$$

$$\Rightarrow\ 1+4+7+.............+(3k+1)=\frac{k(3k-1)+2(3k+1)}{2}$$

$$=\frac{3k^2-k+6k+2}{2}$$

$$=\frac{3k^2+5k+2}{2}$$

$$=\frac{3k^2+3k+2k+2}{2}$$

$$=\frac{3k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)(3k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 4

Suppose $S(n):\ 1+2+4+.............+2^{n-1}=2^n-1$

Put n = 1

$$S(1):\ 1=2^1-1 \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+2+4+.............+2^{k-1}=2^k-1 \quad ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+2+4+.............+2^{k+1-1}=2^{k+1}-1$$

$$\Rightarrow\ 1+2+4+.............+2^k=2^{k+1}-1$$

Adding 2^k on both sides of equation (i)

$$1+2+4+.............+2^{k-1}+2^k=2^k-1+2^k$$

$$\Rightarrow\ 1+2+4+.............+2^k=2(2^k)-1 \qquad \because\ 2^k+2^k=2(2^k)$$

$$=2^{k+1}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 5

Suppose $S(n):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{n-1}}=2\left(1-\frac{1}{2^n}\right)$

Put $n = 1$

$$S(1):\ 1=2\left(1-\frac{1}{2^1}\right)\ \Rightarrow\ 1=2\left(\frac{1}{2}\right)\ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{k-1}}=2\left(1-\frac{1}{2^k}\right)\ \ldots\ldots\ldots\ldots\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{k+1-1}}=2\left(1-\frac{1}{2^{k+1}}\right)$$

$$\Rightarrow\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^k}=2-\frac{2}{2^{k+1}}$$

$$=2-\frac{2}{2^k\cdot 2}$$

$$=2-\frac{1}{2^k}$$

Adding $\frac{1}{2^k}$ on both sides of equation (i)

$$1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{k-1}}+\frac{1}{2^{k+1-1}}=2\left(1-\frac{1}{2^k}\right)+\frac{1}{2^k}$$

$$\Rightarrow\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{2^{k+1-1}}=2-\frac{2}{2^k}+\frac{1}{2^k}$$

$$=2-\frac{1}{2^k}(2-1)$$

$$=2-\frac{1}{2^k}(1)\ =2-\frac{1}{2^k}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 6 *Do yourself as Question # 1*

Question # 7

Suppose $S(n):\ 2+6+18+.............+2\times3^{n-1}=3^n-1$

Put $n = 1$

$$S(1):\ 2=3^1-1\ \Rightarrow\ 2=2$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 2+6+18+.............+2\times3^{k-1}=3^k-1\ \ldots\ldots\ldots\ldots\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 2+6+18+.............+2\times3^{k+1-1}=3^{k+1}-1$$

Adding 2×3^k on both sides of equation (i)

$$2+6+18+.............+2\times3^{k-1}+2\times3^k=3^k-1+2\times3^k$$

$$\Rightarrow\ 2+6+18+.............+2\times3^{k+1-1}=3(3^k)-1 \qquad \because\ 3^k+2\times3^k=3(3^k)$$

$$=3^{k+1}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 8

Suppose $S(n):\ 1\times3+2\times5+3\times7+\ldots\ldots\ldots\ldots..+n\times(2n+1)=\dfrac{n(n+1)(4n+5)}{6}$

Put $n = 1$

$$S(1):\ 1\times3=\frac{1(1+1)(4(1)+5)}{6} \quad\Rightarrow\ 3=\frac{(2)(9)}{6} \quad\Rightarrow\ 3=3$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1\times3+2\times5+3\times7+\ldots\ldots\ldots\ldots..+k\times(2k+1)=\frac{k(k+1)(4k+5)}{6}\ \ldots\ldots.\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):1\times3+2\times5+3\times7+\ldots\ldots\ldots\ldots..+(k+1)\times(2(k+1)+1)=\frac{(k+1)(k+1+1)(4(k+1)+5)}{6}$$

$$\Rightarrow\ 1\times3+2\times5+3\times7+\ldots\ldots\ldots\ldots..+(k+1)\times(2k+3)=\frac{(k+1)(k+2)(4k+9)}{6}$$

Adding $(k+1)\times(2k+3)$ on both sides of equation (i)

$$1\times3+2\times5+3\times7+\ldots\ldots\ldots\ldots..+k\times(2k+1)+(k+1)\times(2k+3)=\frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+3)$$

$$\Rightarrow\ 1\times3+2\times5+3\times7+\ldots\ldots\ldots\ldots..+(k+1)\times(2k+3)=(k+1)\left(\frac{k(4k+5)}{6}+(2k+3)\right)$$

$$=(k+1)\left(\frac{k(4k+5)+6(2k+3)}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+5k+12k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+17k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+17k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+8k+9k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k(k+2)+9(k+2)}{6}\right)$$

$$=(k+1)\left(\frac{(k+2)(4k+9)}{6}\right)$$

$$=\frac{(k+1)(k+2)(4k+9)}{6}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 9

Do yourself as Question # 8

Question # 10

Do yourself as Question # 8

Question # 11

Suppose $S(n):\ \dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+\dfrac{1}{3\times 4}+.............+\dfrac{1}{n(n+1)}=1-\dfrac{1}{n+1}$

Put n = 1

$$S(1):\ \frac{1}{1\times 2}=1-\frac{1}{1+1}\quad\Rightarrow\ \frac{1}{2}=1-\frac{1}{2}\quad\Rightarrow\ \frac{1}{2}=\frac{1}{2}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.............+\frac{1}{k(k+1)}=1-\frac{1}{k+1}\ \ ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{(k+1)(k+1+1)}=1-\frac{1}{k+1+1}$$

$$\Rightarrow\ 1+\frac{1}{2}+\frac{1}{4}+.............+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}$$

Adding $\dfrac{1}{(k+1)(k+2)}$ on both sides of equation (i)

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.............+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\Rightarrow\ \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+.............+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}\left(1-\frac{1}{(k+2)}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+1}{k+2}\right)$$

$$=1-\frac{1}{k+2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 12

Suppose $S(n):\ \dfrac{1}{1\times 3}+\dfrac{1}{3\times 5}+\dfrac{1}{5\times 7}+.............+\dfrac{1}{(2n-1)(2n+1)}=\dfrac{n}{2n+1}$

Put n = 1

$$S(1):\ \frac{1}{1\times 3}=\frac{1}{2(1)+1}\quad\Rightarrow\ \frac{1}{3}=\frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+.............+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}\ ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+.............+\frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)}=\frac{k+1}{2(k+1)+1}$$

$$\Rightarrow\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+.............+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Adding $\dfrac{1}{(2k+1)(2k+3)}$ on both sides of equation (i)

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+.............+\frac{1}{(2k-1)(2k+1)}+\frac{1}{\left(2k+1\right)\left(2k+3\right)}=\frac{k}{2k+1}+\frac{1}{\left(2k+1\right)\left(2k+3\right)}$$

$$\Rightarrow\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+.............+\frac{1}{\left(2k+1\right)\left(2k+3\right)}=\frac{1}{2k+1}\left(k+\frac{1}{\left(2k+3\right)}\right)$$

$$=\frac{1}{2k+1}\left(\frac{k(2k+3)+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{2k^2+3k+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{2k^2+2k+k+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{2k(k+1)+1(k+1)}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{(2k+1)(k+1)}{2k+3}\right)$$

$$=\left(\frac{k+1}{2k+3}\right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 13 *Do yourself as Question # 12*

Question # 14

Suppose $S(n):\ r+r^2+r^3+.............+r^n=\frac{r(1-r^n)}{1-r}$

Put $n = 1$

$$S(1):\ r=\frac{r\left(1-r^1\right)}{1-r}\ \Rightarrow\ r=r$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ r+r^2+r^3+.............+r^k=\frac{r(1-r^k)}{1-r}\ ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ r+r^2+r^3+.............+r^{k+1}=\frac{r(1-r^{k+1})}{1-r}$$

Adding r^{k+1} on both sides of equation (i)

$$r+r^2+r^3+.............+r^k+r^{k+1}=\frac{r(1-r^k)}{1-r}+r^{k+1}$$

$$\Rightarrow\ r+r^2+r^3+.............+r^{k+1}=\frac{r(1-r^k)+r^{k+1}(1-r)}{1-r}$$

$$=\frac{r-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}$$

$$=\frac{r-r^{k+2}}{1-r}$$

$$=\frac{r(1-r^{k+1})}{1-r}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 15

Suppose $S(n):\ a+(a+d)+(a+2d)+...............+\left[a+(n-1)d\right]=\dfrac{n}{2}\left[2a+(n-1)d\right]$

Put $n = 1$

$S(1):\ a=\dfrac{1}{2}\left[2a+(1-1)d\right]\ \Rightarrow\ a=\dfrac{1}{2}\left[2a+(0)d\right]\ \Rightarrow\ a=\dfrac{1}{2}\left[2a\right]=a$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ a+(a+d)+(a+2d)+...............+\left[a+(k-1)d\right]=\dfrac{k}{2}\left[2a+(k-1)d\right]$ ……… (i)

The statement for $n=k+1$ becomes

$S(k+1):\ a+(a+d)+(a+2d)+...............+\left[a+(k+1-1)d\right]=\dfrac{k+1}{2}\left[2a+(k+1-1)d\right]$

$\Rightarrow\ a+(a+d)+(a+2d)+...............+\left[a+kd\right]=\dfrac{k+1}{2}\left[2a+kd\right]$

Adding $a+kd$ on both sides of equation (i)

$a+(a+d)+(a+2d)+...............+\left[a+(k-1)d\right]+\left[a+kd\right]=\dfrac{k}{2}\left[2a+(k-1)d\right]+\left[a+kd\right]$

$\Rightarrow\ a+(a+d)+(a+2d)+...............+\left[a+kd\right]=\dfrac{k}{2}\left[2a+kd-d\right]+\left[a+kd\right]$

$=\dfrac{k\left[2a+kd-d\right]+2\left[a+kd\right]}{2}$

$=\dfrac{2ak+k^2d-kd+2a+2kd}{2}$

$=\dfrac{2ak+k^2d+kd+2a}{2}$

$=\dfrac{2ak+2a+k^2d+kd}{2}$

$=\dfrac{2a(k+1)+kd(k+1)}{2}$

$=\dfrac{(k+1)(2a+kd)}{2}$

$=\dfrac{k+1}{2}\left[2a+kd\right]$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 16

Suppose $S(n):\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+..............+n\cdot\underline{|n}=\underline{|n+1}-1$

Put $n = 1$

$S(1):\ 1\cdot\underline{|1}=\underline{|1+1}-1\ \Rightarrow\ 1=\underline{|2}-1\ \Rightarrow\ 1=2-1\ \Rightarrow\ 1=1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+..............+k\cdot\underline{|k}=\underline{|k+1}-1$ ………… (i)

The statement for $n=k+1$ becomes

$S(k+1):\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+..............+(k+1)\cdot\underline{|k+1}=\underline{|k+1+1}-1$

$\Rightarrow\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+..............+(k+1)\cdot\underline{|k+1}=\underline{|k+2}-1$

Adding $(k+1)\cdot\underline{|k+1}$ on both sides of equation (i)

$1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+..............+k\cdot\underline{|k}+(k+1)\cdot\underline{|k+1}=\underline{|k+1}-1+(k+1)\cdot\underline{|k+1}$

$$\Rightarrow\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+(k+1)\cdot\underline{|k+1}=\underline{|k+1}+\underline{|k+1}(k+1)-1$$
$$=\underline{|k+1}\left(1+k+1\right)-1$$
$$=\underline{|k+1}\left(k+2\right)-1$$
$$=\left(k+2\right)\underline{|k+1}-1$$
$$=\underline{|k+2}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 17

Suppose $S(n):\ a_n = a_1 + (n-1)d$

Put $n=1$

$S(1):\ a_1 = a_1 + (1-1)d \ \Rightarrow\ a_1 = a_1 + 0d = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ a_k = a_1 + (k-1)d$ ………… (i)

The statement for $n=k+1$ becomes

$S(k+1):\ a_{k+1} = a_1 + (k+1-1)d$
$= a_1 + (k)d$

Adding d on both sides of equation (i)

$a_k + d = a_1 + (k-1)d + d$
$\Rightarrow\ a_{k+1} = a_1 + (k-1+1)d$
$\Rightarrow\ a_{k+1} = a_1 + (k)d$

$\because\ a_2 = a_1 + d$
$a_3 = a_2 + d$
$\therefore\ a_{k+1} = a_k + d$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 18

Suppose $S(n):\ a_n = a_1 r^{n-1}$

Put $n=1$

$S(1):\ a_1 = a_1 r^{1-1} \ \Rightarrow\ a_1 = a_1 r^0 = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ a_k = a_1 r^{k-1}$ ………… (i)

The statement for $n=k+1$ becomes

$S(k+1):\ a_{k+1} = a_1 r^{k+1-1}$
$= a_1 r^k$

Multiplying r on both sides of equation (i)

$a_k \cdot r = a_1 r^{k-1} \cdot r^1$
$\Rightarrow\ a_{k+1} = a_1 r^{k-1+1}$
$\Rightarrow\ a_{k+1} = a_1 r^k$

$\because\ a_2 = a_1 r$
$a_3 = a_2 r$
$\therefore\ a_{k+1} = a_k r$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 19

Suppose $S(n):\ 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots + (2n-1)^2 = \dfrac{n(4n^2-1)}{3}$

Put $n = 1$

$S(1):\ 1^2 = \dfrac{1(4(1)^2-1)}{3} \ \Rightarrow\ 1 = \dfrac{1(4-1)}{3} \ \Rightarrow\ 1 = \dfrac{3}{3} = 1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1^2+3^2+5^2+...............+(2k-1)^2=\frac{k(4k^2-1)}{3}\ ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1^2+3^2+5^2+...............+(2(k+1)-1)^2=\frac{(k+1)(4(k+1)^2-1)}{3}$$

$$\Rightarrow\ 1^2+3^2+5^2+...............+(2k+1)^2=\frac{(k+1)(4(k^2+2k+1)-1)}{3}$$

$$=\frac{(k+1)(4k^2+8k+4-1)}{3}$$

$$=\frac{(k+1)(4k^2+8k+3)}{3}$$

$$=\frac{4k^3+8k^2+3k+4k^2+8k+3}{3}$$

$$=\frac{4k^3+12k^2+11k+3}{3}$$

Adding $\left(2k+1\right)^2$ on both sides of equation (i)

$$1^2+3^2+5^2+...............+(2k-1)^2+\left(2k+1\right)^2=\frac{k(4k^2-1)}{3}+\left(2k+1\right)^2$$

$$\Rightarrow\ 1^2+3^2+5^2+...............+\left(2k+1\right)^2=\frac{k(4k^2-1)+3(2k+1)^2}{3}$$

$$=\frac{k(4k^2-1)+3(4k^2+4k+1)}{3}$$

$$=\frac{4k^3-k+12k^2+12k+3}{3}$$

$$=\frac{4k^3+12k^2+11k+3}{3}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 20

Suppose $S(n):\ \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+...............+\binom{n+2}{3}=\binom{n+3}{4}$

Put n = 1

L.H.S = $\binom{3}{3}=1$

R.H.S = $\binom{1+3}{4}=\binom{4}{4}=1$

L.H.S = R.H.S

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+...............+\binom{k+2}{3}=\binom{k+3}{4}\ ....\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+...............+\binom{k+1+2}{3}=\binom{k+1+3}{4}$$

$$\Rightarrow \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+3}{3}=\binom{k+4}{4}$$

Adding $\binom{k+3}{3}$ on both sides of equation (i)

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+3}{4}+\binom{k+3}{3}$$

$$\Rightarrow \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+3}{3}=\binom{k+3+1}{4} \qquad \because \binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

$$\Rightarrow \binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+3}{3}=\binom{k+4}{4}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 21(i)

Suppose $S(n):\ n^2+n$

Put $n=1$

$S(1):\ 1^2+1=2$

$S(1)$ is clearly divisible by 2, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ k^2+k$

Then there exists quotient Q such that

$$k^2+k=2Q$$

$$\begin{array}{r|l} & Q \\ 2 & k^2+k \\ & \underline{-k^2-k} \\ & 0 \end{array}$$

The statement for $n=k+1$

$$S(k+1):\ (k+1)^2+k+1$$
$$=k^2+2k+1+k+1$$
$$=k^2+k+2k+2$$
$$=2Q+2k+2 \qquad \because k^2+k=2Q$$
$$=2(Q+k+1)$$

Clearly $S(k+1)$ is divisible by 2.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 21 (ii)

Suppose $S(n):\ 5^n-2^n$

Put $n=1$

$S(1):\ 5^1-2^1=3$

$S(1)$ is clearly divisible by 3, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ 5^k-2^k$

Then there exists quotient Q such that

$$5^k-2^k=3Q$$

The statement for $n=k+1$

$$S(k+1):\ 5^{k+1}-2^{k+1}$$
$$=\ 5\cdot5^k-2\cdot2^k$$
$$=\ 5\cdot5^k-5\cdot2^k+5\cdot2^k-2\cdot2^k$$
$$=\ 5\left(5^k-2^k\right)+2^k(5-2)$$

$= 5(3Q)+2^k\cdot 3 \qquad \because 5^k-2^k=3Q$

$=3(5Q+2^k)$

Clearly $S(k+1)$ is divisible by 3.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 21 (iii)

Same as Question # 21 (ii)

Hint: $S(k+1):\ 5^{k+1}-1$

$= 5\cdot 5^k-1 \quad =5\cdot 5^k-5+5-1$

$= 5\left(5^k-1\right)+4 \quad = 5(4Q)-4 \qquad \because 5^k-1=4Q$

Question # 21 (iv)

Suppose $S(n):\ 8\times 10^n-2$

Put $n=1$

$S(1):\ 8\times 10^1-2=80-2=78=6\times 13$

$S(1)$ is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ 8\times 10^k-2$

Then there exists quotient Q such that

$8\times 10^k-2\ =\ 6Q$

The statement for $n=k+1$

$S(k+1):\ 8\times 10^{k+1}-2$

$=8\times 10\cdot 10^k-2$

$= 8\times 10\cdot 10^k-2\cdot 10+2\cdot 10-2 \qquad$ –ing & +ing $2\cdot 10$

$= 10\left(8\times 10^k-2\right)+20-2$

$= 10\left(6Q\right)+18 \qquad \because 8\times 10^k-2=6Q$

$= 6\left(10Q+3\right)$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 21 (v)

Suppose $S(n):\ n^3-n$

Put $n=1$

$S(1):\ 1^3-1=0$

$S(1)$ i.e. 0 is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ k^3-k$

Then there exists quotient Q such that

$k^3-k=6Q$

The statement for $n=k+1$

$S(k+1):\ (k+1)^3-(k+1)$

$=k^3+3k^2+3k+1-k-1$

$=k^3+3k^2+3k-k$

$=(k^3-k)+3(k^2+k)$

$=6Q+3(2Q')$

Since n^2+n is divisible by 2

Therefore $n^2+n=2Q'$

Or $k^2+k=2Q'$

$$= 6Q + 6Q'$$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n \in \mathbb{Z}^+$.

Question # 22

Suppose $S(n):\ \frac{1}{3}+\frac{1}{3^2}+.............+\frac{1}{3^n}=\frac{1}{2}\left(1-\frac{1}{3^n}\right)$

Put n = 1

$$S(1):\ \frac{1}{3}=\frac{1}{2}\left(1-\frac{1}{3^1}\right) \Rightarrow \frac{1}{3}=\frac{1}{2}\left(\frac{2}{3}\right) \Rightarrow \frac{1}{3}=\frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{3}+\frac{1}{3^2}+.............+\frac{1}{3^k}=\frac{1}{2}\left(1-\frac{1}{3^k}\right) \ldots\ldots\ldots\ldots (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ \frac{1}{3}+\frac{1}{3^2}+.............+\frac{1}{3^{k+1}}=\frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$$

Adding $\frac{1}{3^k}$ on both sides of equation (i)

$$\frac{1}{3}+\frac{1}{3^2}+.............+\frac{1}{3^k}+\frac{1}{3^{k+1}}=\frac{1}{2}\left(1-\frac{1}{3^k}\right)+\frac{1}{3^{k+1}}$$

$$\Rightarrow \frac{1}{3}+\frac{1}{3^2}+.............+\frac{1}{3^{k+1}}=\frac{1}{2}-\frac{1}{2\cdot 3^k}+\frac{1}{3\cdot 3^k}$$

$$=\frac{1}{2}-\frac{1}{3^k}\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\frac{1}{2}-\frac{1}{3^k}\left(\frac{3-2}{6}\right) = \frac{1}{2}-\frac{1}{3^k}\left(\frac{1}{6}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{3^k}\left(\frac{1}{3}\right)\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 23

Suppose $S(n):\ 1^2-2^2+3^2-4^2+.............+(-1)^{n-1}\cdot n^2=\frac{(-1)^{n-1}\cdot n(n+1)}{2}$

Put n = 1

$$S(1):\ 1^2=\frac{(-1)^{1-1}\cdot 1(1+1)}{2} \Rightarrow 1=\frac{(-1)^0\cdot 2}{2} \Rightarrow 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1^2-2^2+3^2-4^2+.............+(-1)^{k-1}\cdot k^2=\frac{(-1)^{k-1}\cdot k(k+1)}{2} \ldots\ldots\ldots\ldots (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1^2-2^2+3^2-4^2+.............+(-1)^{k+1-1}\cdot(k+1)^2=\frac{(-1)^{k+1-1}\cdot(k+1)(k+1+1)}{2}$$

$$\Rightarrow\ 1^2-2^2+3^2-4^2+\dots\dots\dots\dots+(-1)^k\cdot(k+1)^2=\frac{(-1)^k\cdot(k+1)(k+2)}{2}$$

Adding $(-1)^k\cdot(k+1)^2$ on both sides of equation (i)

$$1^2-2^2+3^2-4^2+\dots\dots\dots\dots+(-1)^{k-1}\cdot k^2+(-1)^k\cdot(k+1)^2=\frac{(-1)^{k-1}\cdot k(k+1)}{2}+(-1)^k\cdot(k+1)^2$$

$$\Rightarrow\ 1^2-2^2+3^2-4^2+\dots\dots\dots\dots+(-1)^k\cdot(k+1)^2=\frac{(-1)^{k-1}\cdot k(k+1)+2(-1)^k\cdot(k+1)^2}{2}$$

$$=\frac{(-1)^k(k+1)\left[(-1)^{-1}k+2(k+1)\right]}{2}$$

$$=\frac{(-1)^k(k+1)\left[-k+2k+2\right]}{2}$$

$$=\frac{(-1)^k(k+1)\left(k+2\right)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 24

Suppose $S(n):\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2n-1)^3=n^2(2n^2-1)$

Put n = 1

$$S(1):\ 1^3=1^2\left(2(1)^2-1\right)\quad\Rightarrow\ 1=1(2-1)\quad\Rightarrow 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2k-1)^3=k^2(2k^2-1)\ \dots\dots\dots\dots\ (i)$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2(k+1)-1)^3=(k+1)^2(2(k+1)^2-1)$$

$$\Rightarrow\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2k+1)^3=(k^2+2k+1)(2(k^2+2k+1)-1)$$

$$=(k^2+2k+1)(2k^2+4k+2-1)$$

$$=(k^2+2k+1)(2k^2+4k+1)$$

$$=2k^4+4k^3+2k^2+4k^3+8k^2+4k+k^2+2k+1$$

$$=2k^4+8k^3+11k^2+6k+1$$

Adding $(2k+1)^3$ on both sides of equation (i)

$$S(k):\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2k-1)^3+(2k+1)^3=k^2(2k^2-1)+(2k+1)^3$$

$$\Rightarrow\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2k+1)^3=k^2(2k^2-1)+(2k)^3+3(2k)^2(1)+3(2k)(1)^2+(1)^3$$

$$\Rightarrow\ 1^3+3^3+5^3+\dots\dots\dots\dots+(2k+1)^3=2k^4-k^2+8k^3+12k^2+6k+1$$

$$=2k^4+8k^3+11k^2+6k+1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

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Question # 25

Suppose $S(n):\ x^{2n}-1$

Put $n=1$

$S(1):\ x^{2(1)}-1=x^2-1=(x-1)(x+1)$

$x+1$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ x^{2k}-1$

Then there exists quotient Q such that

$$x^{2k}-1=(x+1)Q$$

The statement for $n=k+1$

$S(k+1):\ x^{2(k+1)}-1$

$$\begin{aligned}
&=x^{2k+2}-1\\
&=x^{2k+2}-x^{2k}+x^{2k}-1 && \text{+ing and –ing } x^{2k}\\
&=x^{2k}(x^2-1)+(x^{2k}-1)\\
&=x^{2k}(x-1)(x+1)+(x+1)Q && \because\ x^{2k}-1=(x+1)Q\\
&=(x+1)\left(x^{2k}(x-1)+Q\right)
\end{aligned}$$

Clearly $x+1$ is a factor of $S(k+1)$.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 26

Suppose $S(n):\ x^n-y^n$

Put $n=1$

$S(1):\ x^1-y^1=x-y$

$x-y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ x^k-y^k$

Then there exists quotient Q such that

$$x^k-y^k=(x-y)Q$$

The statement for $n=k+1$

$S(k+1):\ x^{k+1}-y^{k+1}$

$$\begin{aligned}
&=\ x\cdot x^k-y\cdot y^k\\
&=\ x\cdot x^k-x\cdot y^k+x\cdot y^k-y\cdot y^k && \text{–ing \& +ing } xy^k\\
&=x(x^k-y^k)+y^k(x-y)\\
&=\ x(x-y)Q+y^k(x-y) && \because\ x^k-y^k=(x-y)Q
\end{aligned}$$

Clearly $x-y$ is a factor of $S(k+1)$.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 27

Suppose $S(n):\ x^{2n-1}+y^{2n-1}$

Put $n=1$

$S(1):\ x^{2(1)-1}+y^{2(1)-1}=x^1+y^1=x+y$

$x+y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ x^{2k-1}+y^{2k-1}$

Then there exists quotient Q such that

$x^{2k-1}+y^{2k-1}=(x+y)Q$

The statement for $n=k+1$

$S(k+1):\ x^{2(k+1)-1}+y^{2(k+1)-1}$

$=x^{2k+2-1}+y^{2k+2-1}$

$=x^{2k+2-1}-x^{2k-1}y^2+x^{2k-1}y^2+y^{2k+2-1}$ +ing and –ing $x^{2k-1}y^2$

$=x^{2k-1}(x^2-y^2)+y^2(x^{2k-1}+y^{2k-1})$

$=x^{2k-1}(x-y)(x+y)+y^2(x+y)Q$ $\because\ x^{2k-1}+y^{2k-1}=(x+y)Q$

$=(x+y)\left(x^{2k-1}(x-y)+y^2\ Q\right)$

Clearly $x+y$ is a factor of $S(k+1)$.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Principle of Extended Mathematical Induction

A given statement $S(n)$ is true for $n\geq i$ if the following two conditions hold

Condition I: $S(i)$ is true i.e. $S(n)$ is true for $n = i$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers

Question # 28

Suppose $S(n):\ 1+2+2^2+.............+2^n=2^{n+1}-1$

Put $n = 0$

$S(1):\ 1=2^{0+1}-1=\ \Rightarrow\ 1=2-1\ \Rightarrow\ 1=1$

> ***Note***: Non- negative number are $0,1,2,3,...........$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ 1+2+2^2+.............+2^k=2^{k+1}-1$ …………. (i)

The statement for $n=k+1$ becomes

$S(k+1):\ 1+2+2^2+.............+2^{k+1}=2^{k+1+1}-1$

$=2^{k+2}-1$

Adding 2^{k+1} on both sides of equation (i)

$1+2+2^2+.............+2^k+2^{k+1}=2^{k+1}-1+2^{k+1}$

$\Rightarrow\ 1+2+4+.............+2^{k+1}=2(2^{k+1})-1$ $\because\ 2^{k+1}+2^{k+1}=2(2^{k+1})$

$=2^{k+1+1}-1$

$=2^{k+1+1}-1$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all non-negative integers n.

Question # 29

Suppose $S(n):\ AB^n=B^nA$

Put $n=1$

$S(1):\ \mathrm{AB}^1=B^1A\ \Rightarrow\ \mathrm{AB}=BA$

$S(1)$ is true as we have given $\mathrm{AB}=BA$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ AB^k=B^kA$ ……………. (i)

The statement for $n=k+1$

$S(k+1):\ AB^{k+1}=B^{k+1}A$

Post-multiplying equation (i) by B.

$(AB^k)B=(B^kA)B$

$\Rightarrow\ A(B^kB)=B^k(AB)$ by associative law

$\Rightarrow \quad AB^{k+1} = B^k(BA) \qquad \because \ \mathrm{AB} = BA \quad (given)$

$= (B^k B)A$

$= B^{k+1}A$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integers *n*.

Question # 30

Suppose $S(n):\ n^2 - 1$

Put $n = 1$

$S(1):\ (1)^2 - 1 = 0$

$S(1)$ is clearly divisible by 8, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$ where k is odd.

$S(k):\ k^2 - 1$

Then there exists quotient Q such that

$k^2 - 1 = 8\,Q$

As $k+2$ is the next odd integer after k The statement for $n = k+1$

$S(k+2):\ (k+2)^2 - 1$

$= k^2 + 4k + 4 - 1$

$= k^2 - 1 + 4k + 4$

$= 8Q + 4(k+1) \qquad \because\ k^2 + k = 2Q$

Since k is odd therefore $k+1$ is even so their exists integer t such that $k+1 = 2t$

$\Rightarrow\ S(k+2) := 8Q + 4(2t)$

$= 8Q + 8t$

Clearly $S(k+2)$ is divisible by 8 so condition II is satisfied.

Therefore the given statement is true for odd positive integers.

Question # 31

Suppose $S(n):\ \ln x^n = n \ln x$

Put $n = 1$

$S(1):\ \ln x^1 = (1)\ln x \qquad \Rightarrow\ \ln x = \ln x$

$S(1)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k):\ \ln x^k = k \ln x$ .... (i)

The statement for $n = k+1$

$S(k+1):\ \ln x^{k+1} = (k+1)\ln x$

Now adding $\ln x$ on both sides of equation (i)

$\ln x^k + \ln x = k \ln x + \ln x$

$\Rightarrow\ \ln x^k \cdot x = (k+1)\ln x \qquad \because\ \ln x + \ln y = \ln x y$

$\Rightarrow\ \ln x^{k+1} = (k+1)\ln x$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all $n \in \mathbb{Z}^+$.

Question # 32

Suppose $S(n):\ n! > 2^n - 1 \qquad ;\ n \geq 4$

Put $n = 4$

$S(4):\ 4! > 2^4 - 1 \quad \Rightarrow\ 24 > 16 - 1 \quad \Rightarrow\ 24 > 15$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k):\ k! > 2^k - 1$................ (i)

The statement for $n = k+1$

$S(k+1):\ (k+1)! > 2^{k+1} - 1$

Multiplying both sides of equation (i) by $k+1$

$$(k+1)k! > (k+1)(2^k - 1)$$

$$\Rightarrow\ (k+1)! > (k+1+2-2)(2^k - 1) \qquad \because\ (k+1)k! = (k+1)!$$

$$\Rightarrow\ (k+1)! > (k-1+2)(2^k - 1)$$

$$\Rightarrow\ (k+1)! > k\cdot 2^k - k - 2^k + 1 + 2\cdot 2^k - 2$$

$$\Rightarrow\ (k+1)! > (k\cdot 2^k - 2^k - k) + 2^{k+1} - 1$$

$$\Rightarrow\ (k+1)! > 2^{k+1} - 1 \qquad \because\ k\cdot 2^k - 2^k - k \geq 0\ \ \forall\ k \geq 4$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 33

Suppose $S(n):\ n^2 > n+3 \qquad ;\ n \geq 3$

Put $n = 3$

$S(3):\ 3^2 > 3+3 \quad \Rightarrow\ 9 > 6$

$S(3)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k):\ k^2 > k+3$ ……………. (i)

The statement for $n = k+1$

$S(k+1):\ (k+1)^2 > k+1+3 \quad \Rightarrow\ (k+1)^2 > k+4$

Adding $2k+1$ on both sides of equation (i)

$$k^2 + 2k + 1 > k + 3 + 2k + 1$$

$$\Rightarrow\ (k+1)^2 > k + 4 + 2k$$

$$\Rightarrow\ (k+1)^2 > k + 4 \qquad \text{ignoring } 2k \text{ as } 2k > 0$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 34

Suppose $S(n):\ 4^n > 3^n + 2^{n-1} \qquad ;\ n \geq 2$

Put $n = 2$

$S(2):\ 4^2 > 3^2 + 2^{2-1} \quad \Rightarrow\ 16 > 9 + 2 \quad \Rightarrow\ 16 > 11$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k):\ 4^k > 3^k + 2^{k-1}$ ……………. (i)

The statement for $n = k+1$

$S(k+1):\ 4^{k+1} > 3^{k+1} + 2^{k+1-1}$

$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k$

Multiplying both sides of equation (i) by 4.

$$4(4^k) > 4(3^k + 2^{k-1})$$

$$\Rightarrow\ 4^{k+1} > 4\cdot 3^k + 4\cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > (3+1)\cdot 3^k + (2+2)\cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > 3\cdot 3^k + 3^k + 2\cdot 2^{k-1} + 2\cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k + (3^k + 2^k)$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k \qquad \text{ignoring } 3^k + 2^k \text{ as } 3^k + 2^k > 0$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 35

Suppose $S(n):\ 3^n < n!\qquad ;\ n>6$

Put $n=7$

$S(7):\ 3^7 < 7! \quad \Rightarrow\ 2187 < 5040$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$S(k):\ 3^k < k! \ldots\ldots\ldots\ldots\ldots\ (i)$

The statement for $n=k+1$

$S(k+1):\ 3^{k+1} < (k+1)!$

Multiplying both sides of equation (i) by $k+1$.

$$(k+1)3^k < (k+1)k!$$
$$\Rightarrow\ \big((k-2)+3\big)3^k < (k+1)!$$
$$\Rightarrow\ (k-2)3^k + 3^{k+1} < (k+1)!$$
$$\Rightarrow\ 3^{k+1} < (k+1)! \qquad \because\ (k-2)3^k > 0\ \ \forall\ k>6$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n>6$.

Question # 36

Suppose $S(n):\ n! > n^2\qquad ;\ n\geq 4$

Put $n=4$

$S(4):\ 4! > 4^2 \quad \Rightarrow\ 24 > 16$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$S(k):\ k! > k^2 \ldots\ldots\ldots\ldots\ldots\ (i)$

The statement for $n=k+1$

$S(k+1):\ (k+1)! > (k+1)^2$

Multiplying both sides of equation (i) by $k+1$.

$$(k+1)k! > (k+1)k^2$$
$$\Rightarrow\ (k+1)! > (k+1)(k+1) \qquad \because\ k+1 < k^2\ \ \forall\ k\geq 4$$
$$\Rightarrow\ (k+1)! > (k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n\geq 4$.

Question # 37

Suppose $S(n):\ 3+5+7+\ldots\ldots\ldots+(2n+5)=(n+2)(n+4)\qquad ;\ n\geq -1$

Put $n=-1$

$S(-1):\ 3=(-1+2)(-1+4) \quad \Rightarrow\ 3=(1)(3) \quad \Rightarrow\ 3=3$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ 3+5+7+\ldots\ldots\ldots+(2k+5)=(k+2)(k+4)\ \ldots\ldots\ldots\ldots\ (i)$

The statement for $n=k+1$ becomes

$$S(k+1):\ 3+5+7+\ldots\ldots\ldots+(2(k+1)+5)=\big((k+1)+2\big)\big((k+1)+4\big)$$
$$\Rightarrow\ 3+5+7+\ldots\ldots\ldots+(2k+7)=\big(k+3\big)\big(k+5\big)$$

Adding $(2k+7)$ on both sides of equation (i)

$$S(k):\ 3+5+7+\ldots\ldots\ldots+(2k+5)+(2k+7)=(k+2)(k+4)+(2k+7)$$
$$\Rightarrow\ 3+5+7+\ldots\ldots\ldots+(2k+7)=k^2+2k+4k+8+2k+7$$
$$=k^2+8k+15$$
$$=k^2+5k+3k+15$$

$= k(k+5)+3(k+5)$
$=(k+5)(k+3)$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq -1$.

Question # 38

Suppose $S(n):\ 1+nx \leq (1+x)^n$; $n \geq 2$

Put $n=2$

$S(2):\ 1+2x \leq (1+x)^2 \quad \Rightarrow \ 1+2x \leq 1+2x+x^2$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$S(k):\ 1+kx \leq (1+x)^k$ ……………. (i)

The statement for $n=k+1$

$S(k+1):\ 1+(k+1)x \leq (1+x)^{k+1}$

Multiplying both sides of equation (i) by $1+x$.

$(1+kx)(1+x) \leq (1+x)^k(1+x)$

$\Rightarrow\ 1+kx+x+kx^2 \leq (1+x)^{k+1}$

$\Rightarrow\ 1+kx+x \leq (1+x)^{k+1} \qquad \because\ kx^2 > 0$

$\Rightarrow\ 1+(k+1)x \leq (1+x)^{k+1}$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 2$

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