1·4–1

Logarithmic Function:-

Let $z \in$ where $z \neq 0$ & $\omega \in$ such that $e^{\omega} = z \Rightarrow \omega = \ln z$, '$\omega$' is Natural logarithm of z.

A particular value of 'ω' satisfying the eq. $e^{\omega} = z$ is given by $\ln|z| + i\arg z$

$e^{\ln|z| + i\arg z} = e^{\ln|z|} e^{i\arg z} = |z| e^{i\theta} = r(\cos\theta + i\sin\theta) = z$

This particular value of ω is called Principal logarithm of z denoted by $\text{Log } z$.

$\boxed{\text{Log } z = \ln|z| + i\arg z} = \boxed{\ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x} \quad \text{if } z = x+iy}$

The general value of ω satisfying the eq $e^{\omega} = z$ is given by $\ln|z| + i\arg z + 2n\pi i$

$e^{\ln|z| + i\arg z + 2n\pi i} = e^{\ln|z|} \cdot e^{i\arg z} \cdot e^{2n\pi i} = |z| e^{i\theta} e^{2\pi n i}$

$= r e^{i\theta} (\cos 2\pi n + i\sin 2\pi n)$

$= r e^{i\theta}(1+0)$

$= r(\cos\theta + i\sin\theta)$

$= z$

Ex 1.4

$\text{Log } z = \ln|z| + i\arg z$

when $z = x+iy$: $\text{Log}(x+iy) = \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$

Q1 (i) Prove that $\text{Log } i = \frac{\pi i}{2}$

$\text{Log } i = \text{Log}(0+1i)$

$= \ln\sqrt{0^2+1^2} + i\tan^{-1}\left(\frac{1}{0}\right)$

$= \ln 1 + i\frac{\pi}{2}$

$= 0 + \frac{i\pi}{2}$

$= \frac{i\pi}{2}$ Ans

$x+ive, y+ive$
So Ist Quad
$\therefore \theta$
$\theta = \tan^{-1}(\infty) = \frac{\pi}{2}$

(ii) $\text{Log}(-5) = \ln 5 + i\pi$ To Prove

$\text{Log}(-5) = \ln\sqrt{(-5)^2+0^2} + i\tan^{-1}\left(\frac{0}{-5}\right)$

$= \ln\sqrt{25} + i\tan^{-1}(0)$

$= \ln 5 + i\pi$

x–ive y+ive
So IInd Q
$\therefore \pi - \theta$
$\theta = \tan^{-1}(0) = 0$
Principal $\arg z = \pi - \theta = \pi - 0$
$= \pi$

$\boxed{1\cdot 4-2}$

Q1 (iii) $Log(-1+i) = \frac{1}{2}\ln 2 + \frac{3}{4}\pi i$ To Prove

$$Log(-1+i) = \ln\sqrt{1+1} + i\tan^{-1}\left(\frac{1}{-1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1}(-1)$$
$$= \ln 2^{\frac{1}{2}} + i\tan^{-1}(-1)$$
$$= \frac{1}{2}\ln 2 + i\,\frac{3\pi}{4}$$

x –ive y +ive
So IInd Quad.
$\therefore \pi - \theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
Principal arg z $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

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iv) $Log(1+i) = \frac{1}{2}\ln 2 + \frac{\pi}{4} i$ To Prove

$$Log(1+i) = \ln\sqrt{1^2+1^2} + i\tan^{-1}\left(\frac{1}{1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1} 1$$
$$= \frac{1}{2}\ln 2 + i\,\frac{\pi}{4}$$

x +ive, y +ive
Ist Quad $\therefore \theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

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v) $Log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = -\frac{2}{3}\pi i$ To Prove

$$Log\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = \ln\sqrt{\frac{1}{4}+\frac{3}{4}} + i\tan^{-1}\left(\frac{-\sqrt{3}/2}{-1/2}\right)$$
$$= \ln\sqrt{\frac{4}{4}} + i\tan^{-1}(\sqrt{3})$$
$$= \ln 1 + i\left(-\frac{2\pi}{3}\right)$$
$$= 0 - \frac{2}{3}\pi i$$

x –ive, y –ive
3rd Quad.
$\therefore \theta - \pi$
$\theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$
$\therefore \theta - \pi = \frac{\pi}{3} - \pi$
$= -\frac{2\pi}{3}$

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vi) $Log(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4} i$ To Prove

$$Log(1-i) = \ln\sqrt{1^2+(-1)^2} + i\tan^{-1}\left(\frac{-1}{1}\right)$$
$$= \ln\sqrt{2} + i\tan^{-1}(-1)$$
$$= \frac{1}{2}\ln 2 + i\left(-\frac{\pi}{4}\right)$$
$$= \frac{1}{2}\ln 2 - \frac{\pi}{4} i$$

x +ive, y –ive
So 4th Quad
$\therefore -\theta$
$\theta = \tan^{-1}(1) = \frac{\pi}{4}$
Principal arg z $= -\theta = -\frac{\pi}{4}$

Q2 i) To Prove $\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$

Let $\coth^{-1} z = w \Rightarrow z = \coth w$

$$\frac{z+1}{z-1} = \frac{\coth w + 1}{\coth w - 1}$$

$$= \frac{\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}} + 1}{\frac{e^{w}+e^{-w}}{e^{w}-e^{-w}} - 1}$$

$$= \frac{e^{w}\cancel{+e^{-w}}+e^{w}\cancel{-e^{-w}}}{e^{w}+e^{-w}-e^{w}+e^{-w}} \quad \text{LCM}$$

$$= \frac{\cancel{2}e^{w}}{\cancel{2}e^{-w}}$$

$$= e^{w}e^{w}$$

$$= e^{2w}$$

$$\log\left(\frac{z+1}{z-1}\right) = 2w$$

$$\frac{1}{2}\log\left(\frac{z+1}{z-1}\right) = w = \coth^{-1} z$$

To Prove $\operatorname{Sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\operatorname{Sech}^{-1} z = w \Rightarrow z = \operatorname{Sech} w$

So, $$\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\operatorname{Sech}^2 w}}{\operatorname{Sech} w}$$

$(\because \operatorname{Sech}^2 z + \tanh^2 z = 1)$

$$= \frac{1+\tanh w}{\operatorname{Sech} w}$$

$$= \frac{1}{\operatorname{Sech} w} + \cancel{\left(\frac{\sinh w}{\cosh w}\right)\cdot \cosh w}$$

$$= \cosh w + \sinh w$$

$$= \frac{e^{w}+e^{-w}}{2} + \frac{e^{w}-e^{-w}}{2}$$

$$= \frac{\cancel{2}e^{w}}{\cancel{2}} \quad \text{LCM}$$

$$\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = w$$

$$\boxed{\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = \operatorname{Sech}^{-1} z} \text{ proved}$$

(6) To Prove $\operatorname{Cosech}^{-1} z = \log\left(\frac{1+\sqrt{1+z^2}}{z}\right)$

Let $\operatorname{Cosech}^{-1} z = w \Rightarrow z = \operatorname{Cosech} w$

So $$\frac{1+\sqrt{1+z^2}}{z} = \frac{1+\sqrt{1+\operatorname{Cosech}^2 w}}{\operatorname{Cosech} w}$$

$$= \frac{1+\sqrt{\coth^2 w}}{\operatorname{Cosech} w}$$

$\because \coth^2 z - \operatorname{Cosech}^2 z = 1$

$$= \frac{1+\coth w}{\operatorname{Cosech} w}$$

$$= \frac{1}{\operatorname{Cosech} w} + \frac{\coth w}{\operatorname{Cosech} w}$$

$$= \sinh w + \frac{\cosh w}{\cancel{\sinh w}}\cdot\frac{\cancel{\sinh w}}{1}$$

$$= \frac{e^{w}-e^{-w}}{2} + \frac{e^{w}+e^{-w}}{2}$$

$$= \frac{\cancel{2}e^{w}}{\cancel{2}}$$

$$\log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = w$$

$$\boxed{\log\left(\frac{1+\sqrt{1+z^2}}{z}\right) = \operatorname{Cosech}^{-1} z} \text{ proved}$$

Hyperbolic Identities

$$\begin{cases} \cosh^2 z - \sinh^2 z = 1 \\ \operatorname{Sech}^2 z + \tanh^2 z = 1 \\ \coth^2 z - \operatorname{Cosech}^2 z = 1 \end{cases}$$

Trig Identities

$$\begin{cases} \cos^2\theta + \sin^2\theta = 1 \\ \sec^2\theta - \tan^2\theta = 1 \\ \operatorname{Cosec}^2\theta - \cot^2\theta = 1 \end{cases}$$

Inverse Hyperbolic Functions :-

To Prove $\sinh^{-1} z = \log(z+\sqrt{1+z^2})$

Let $\sinh^{-1} z = \omega \Rightarrow z = \sinh\omega$

So $z+\sqrt{1+z^2} = \sinh\omega + \sqrt{1+\sinh^2\omega}$

$= \sinh\omega + \sqrt{\cosh^2\omega}$

$= \sinh\omega + \cosh\omega$

$= \frac{e^{\omega}-e^{-\omega}}{2} + \frac{e^{\omega}+e^{-\omega}}{2}$

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$ LCM

$\log(z+\sqrt{z^2+1}) = \omega$

$\boxed{\log(z+\sqrt{z^2+1}) = \sinh^{-1} z}$ proved

To Prove $\cosh^{-1} z = \log(z+\sqrt{z^2-1})$

Let $\cosh^{-1} z = \omega \Rightarrow z = \cosh\omega$

So $z+\sqrt{z^2-1} = \cosh\omega + \sqrt{\cosh^2\omega - 1}$

$= \cosh\omega + \sqrt{\sinh^2\omega}$

$= \cosh\omega + \sinh\omega$

$= \frac{e^{\omega}+e^{-\omega}}{2} + \frac{e^{\omega}-e^{-\omega}}{2}$

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}}$ LCM

$\log(z+\sqrt{z^2-1}) = \omega$

$\boxed{\log(z+\sqrt{z^2-1}) = \cosh^{-1} z}$

IHF

1) $\sinh^{-1} z = \log(z+\sqrt{1+z^2})$
2) $\cosh^{-1} z = \log(z+\sqrt{z^2-1})$
3) $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$
4) $\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$
5) $\operatorname{sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$
6) $\operatorname{cosech}^{-1} z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right)$

3) **To Prove** $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

Let $\tanh^{-1} z = \omega \Rightarrow z = \tanh\omega$

So $\frac{1+z}{1-z} = \frac{1+\tanh\omega}{1-\tanh\omega}$

$= \frac{1+\left(\frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}\right)}{1-\left(\frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}}\right)}$

$= \frac{e^{\omega}+\cancel{e^{-\omega}}+e^{\omega}-\cancel{e^{-\omega}}}{\cancel{e^{\omega}}+e^{-\omega}-\cancel{e^{\omega}}+e^{-\omega}}$ LCM

$= \frac{\cancel{2}e^{\omega}}{\cancel{2}e^{-\omega}}$

$= e^{\omega}\cdot e^{\omega}$

$\frac{1+z}{1-z} = e^{2\omega}$

$\log\left(\frac{1+z}{1-z}\right) = 2\omega$

$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \omega = \tanh^{-1} z$

Inverse Trigonometric Functions :-

1) To Prove $\sin^{-1}z = \frac{1}{i}\log(iz+\sqrt{1-z^2})$

Let $\sin^{-1}z = w \Rightarrow z = \sin w$

So $iz+\sqrt{1-z^2} = i\sin w + \sqrt{1-\sin^2 w}$

$= i\sin w + \sqrt{\cos^2 w}$

$= i\sin w + \cos w$

$= e^{iw}$

$\log(iz+\sqrt{1-z^2}) = iw$

$\frac{1}{i}\log(iz+\sqrt{1-z^2}) = w$

$\boxed{\frac{1}{i}\log(iz+\sqrt{1-z^2}) = \sin^{-1}z}$

Q 3(i) 2) To Prove $\cos^{-1}z = \frac{1}{i}\log(z+\sqrt{z^2-1})$

Let $\cos^{-1}z = w \Rightarrow z = \cos w$

So $z+\sqrt{z^2-1} = \cos w + \sqrt{\cos^2 w - 1}$

$= \cos w + \sqrt{-(1-\cos^2 w)}$

$= \cos w + \sqrt{-\sin^2 w}$

$= \cos w + i\sin w$

$z+\sqrt{z^2-1} = e^{iw}$

$\log(z+\sqrt{z^2-1}) = iw$

$\frac{1}{i}\log(z+\sqrt{z^2-1}) = w$

$\boxed{\frac{1}{i}\log(z+\sqrt{z^2-1}) = \cos^{-1}z}$

$$\sin^{-1}z = \frac{1}{i}\log(iz+\sqrt{1-z^2})$$
$$\cos^{-1}z = \frac{1}{i}\log(z+\sqrt{z^2-1})$$
$$\tan^{-1}z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$$
$$\sec^{-1}z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$
$$\csc^{-1}z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$$
$$\cot^{-1}z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$$

3) To Prove $\tan^{-1}z = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$

Let $\tan^{-1}z = w \Rightarrow z = \tan w$

So $\frac{1+iz}{1-iz} = \frac{1+i\tan w}{1-i\tan w}$

$= \dfrac{1+i\frac{\sin w}{\cos w}}{1-i\frac{\sin w}{\cos w}}$

LCM $= \frac{\cos w + i\sin w}{\cos w - i\sin w}$

$= \frac{e^{iw}}{e^{-iw}}$

$= e^{iw}e^{iw}$

$\frac{1+iz}{1-iz} = e^{2iw}$

$\log\left(\frac{1+iz}{1-iz}\right) = 2iw$

$\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = w$

$\boxed{\frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right) = \tan^{-1}z}$

Q3

(iii) To Prove $\cot^{-1}z = \frac{1}{2i}\log\left(\frac{z+i}{z-i}\right)$

Let $\cot^{-1}z = \omega \Rightarrow z = \cot\omega$

So $\frac{z+i}{z-i} = \frac{\cot\omega + i}{\cot\omega - i}$

$= \frac{\frac{\cos\omega}{\sin\omega} + i}{\frac{\cos\omega}{\sin\omega} - i}$

$= \frac{\cos\omega + i\sin\omega}{\cos\omega - i\sin\omega}$

$= \frac{e^{i\omega}}{e^{-i\omega}} = e^{i\omega}e^{i\omega}$

$= e^{2i\omega}$

$\log\left(\frac{z+i}{z-i}\right) = 2i\omega$

$\frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) = \omega$

$\boxed{\frac{1}{2i}\log\left(\frac{z+i}{z-i}\right) = \cot^{-1}z}$ Proved

(v) To Prove $\operatorname{Cosec}^{-1}z = \frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

Let $\operatorname{Cosec}^{-1}z = \omega \Rightarrow z = \operatorname{Cosec}\omega$

So $\frac{i+\sqrt{z^2-1}}{z} = \frac{i+\sqrt{\operatorname{Cosec}^2\omega - 1}}{\operatorname{Cosec}\omega}$

$= \frac{i+\sqrt{\cot^2\omega}}{\operatorname{Cosec}\omega}$

$= \frac{i+\cot\omega}{\operatorname{Cosec}\omega}$

$= \frac{i+\frac{\cos\omega}{\sin\omega}}{\frac{1}{\sin\omega}}$ Lcm

$= \frac{i\sin\omega + \cos\omega}{1}$

$= e^{i\omega}$

$\log\frac{i+\sqrt{z^2-1}}{z} = i\omega$

$\frac{1}{i}\log\left[\frac{i+\sqrt{z^2-1}}{z}\right] = \omega$

$\boxed{\frac{1}{i}\log\left(\frac{i+\sqrt{z^2-1}}{z}\right) = \operatorname{Cosec}^{-1}z}$ Proved

Q3 (iv) To Prove $\operatorname{Sec}^{-1}z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Let $\operatorname{Sec}^{-1}z = \omega \Rightarrow z = \operatorname{Sec}\omega$

So $\frac{1+\sqrt{1-z^2}}{z} = \frac{1+\sqrt{1-\operatorname{Sec}^2\omega}}{\operatorname{Sec}\omega}$

$= \frac{1+\sqrt{-\tan^2\omega}}{\operatorname{Sec}\omega}$ $\left(\because \sec^2\omega - \tan^2\omega = 1 \quad -\tan^2\omega = 1-\sec^2\omega\right)$

$= \frac{1+i\tan\omega}{\operatorname{Sec}\omega}$

$= \left(1 + i\frac{\sin\omega}{\cos\omega}\right)\cdot\frac{\cos\omega}{1}$

$= \frac{\cos\omega + i\sin\omega}{\cancel{\cos\omega}}\cdot\frac{\cancel{\cos\omega}}{1}$

$\frac{1+\sqrt{1-z^2}}{z} = e^{i\omega}$

$\log\left(\frac{1+\sqrt{1-z^2}}{z}\right) = i\omega \Rightarrow \omega = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)^2$

$\operatorname{Sec}^{-1}z = \frac{1}{i}\log\left(\frac{1+\sqrt{1-z^2}}{z}\right)^2$ proved

Complex Power If z and w are any two complex numbers s.t $z \neq 0$

$$\boxed{z^{w} = e^{w\,\mathrm{Log}\, z}}$$

then we define

Q i Prove that $i^{i} = e^{-\pi/2}$

LHS $i^{i} = e^{i\,\mathrm{Log}\, i}$

$= e^{i\{\ln\sqrt{0^2+1^2} + i\tan^{-1}(\frac{1}{0})\}}$ $\because Q_1$

$= e^{i(0 + i\frac{\pi}{2})}$

$i^{i} = e^{-\frac{\pi}{2}}$ proved

ii) Prove that $(-1)^{i} = e^{-\pi}$

LHS $(-1)^{i} = e^{i\,\mathrm{Log}(-1)}$

$= e^{i\{\ln\sqrt{(-1)^2+0^2} + i\tan^{-1}(0)\}}$

$= e^{i\{0 + i\pi\}}$ $\because Q\text{II}$

$= e^{-\pi}$

$(-1)^{i} = e^{-\pi}$ proved

iii) Prove that $(-i)^{-i} = e^{-\frac{\pi}{2}}$

LHS $(-i)^{-i} = e^{-i\,\mathrm{Log}(-i)}$

$= e^{-i\{\ln\sqrt{0^2+(-1)^2} + i\tan^{-1}(\frac{-1}{0})\}}$

$= e^{-i\{0 + i\tan^{-1}(-\infty)\}}$ $\because Q\text{II}$

$= e^{-i\{0 + i(-\frac{\pi}{2})\}}$

$= e^{-\frac{\pi}{2}}$ proved

iv) Prove that $a^{i} = \cos(\ln a) + i\sin(\ln a)$

LHS $a^{i} = e^{i\,\mathrm{Log}\, a}$

$= e^{i\{\ln\sqrt{a^2+0^2} + i\tan^{-1}(\frac{0}{a})\}}$

$= e^{i\{\ln a + i(0)\}}$ $\because \text{IV}^{th}\, Q$

$= e^{i\ln a}$

$a^{i} = \cos\ln a + i\sin\ln a$ proved

Q5 Prove that $\tanh^{-1} z = \sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right)$

Sol Let $\tanh^{-1} z = w \Rightarrow z = \tanh w$

So $\dfrac{z}{\sqrt{1-z^2}} = \dfrac{\tanh w}{\sqrt{1-\tanh^2 w}}$

$= \dfrac{\tanh w}{\sqrt{\operatorname{sech}^2 w}}$

$= \dfrac{\tanh w}{\operatorname{sech} w}$

$= \dfrac{\sinh w}{\cosh w} \cdot \dfrac{\cosh w}{1}$

$= \sinh w$

$\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) = w$

$\sinh^{-1}\left(\frac{z}{\sqrt{1-z^2}}\right) = \tanh^{-1} z$

Q6 Show that if $z = x + iy$ then $\mathrm{Log}\left(\frac{z}{\bar{z}}\right) = 2i\tan^{-1}\left(\frac{y}{x}\right)$

Sol $z = x + iy$

$\bar{z} = x - iy$

$\mathrm{Log}\left(\frac{z}{\bar{z}}\right) = \mathrm{Log}\, z - \mathrm{Log}\, \bar{z}$

$= \mathrm{Log}(x+iy) - \mathrm{Log}(x-iy)$

$= \left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right)\right] - \left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{-y}{x}\right)\right]$

$= \ln\sqrt{x^2+y^2} + i\tan^{-1}\left(\frac{y}{x}\right) - \ln\sqrt{x^2+y^2} - \left(-i\tan^{-1}\frac{y}{x}\right)$

$\mathrm{Log}\frac{z}{\bar{z}} = 2i\tan^{-1}\frac{y}{x}$

proved

1·4–7

Q. If $a^{\alpha+i\beta} = (x+iy)^{p+iq}$ then Prove that

i) $\alpha = \frac{1}{2}\log_a(x^2+y^2) - \tan^{-1}\left(\frac{y}{x}\right)\log_a e$

ii) $\log_a(x^2+y^2) = 2(\alpha \ldots \beta q)$

p^2+q^2

Sol

(given) $a^{\alpha+i\beta} = (x+iy)^{p+iq}$

$e^{(\alpha+i\beta)\text{Log}\, a} = e^{(p+iq)\text{Log}(x+iy)}$

$(\alpha+i\beta)\text{Log}\, a = (p+iq)\text{Log}(x+iy)$

$(\alpha+i\beta)\left[\ln\sqrt{a^2} + i\tan^{-1}\frac{0}{a}\right] = (p+iq)\left[\ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}\right]$

$(\alpha+i\beta)\left[\ln a + i\cdot 0\right] = (p+iq)\left[\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\frac{y}{x}\right]$

$\alpha\ln a + i\beta\ln a = \left[\frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}\frac{y}{x}\right] + i\left[\frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}\right]$ — (1)

Equating Real & Imaginary Parts

$\alpha\ln a = \frac{p}{2}\ln(x^2+y^2) - q\tan^{-1}\frac{y}{x}$

$\beta\ln a = \frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}$

$\beta = \frac{q}{2}\ln(x^2+y^2) + p\tan^{-1}\frac{y}{x}$

Q8 If $\text{Log}\,\text{Sin}(x+iy) = u+iv$ show that

i) $\cosh 2y = \cos 2x + 2e^{2u}$

ii) $e^{2y} = \dfrac{\cos(x-v)}{\cos(x+v)}$

Sol $\text{Log}\,\text{Sin}(x+iy) = u+iv$ given

$$\sin(x+iy) = e^{u+iv}$$

$$\sin x\cos iy + \cos x\sin iy = e^{u}.e^{iv}$$

$$\sin x\cosh y + i\cos x\sinh y = e^{u}(\cos v + i\sin v)$$

Equating Real & Imaginary parts.

$$\sin x\cosh y = e^{u}\cos v \quad \text{——(i)}$$

$$\cos x\sinh y = e^{u}\sin v \quad \text{——(ii)}$$

Squaring & adding (i)&(ii) (To change into double angle as required)

$\because 2\cosh^2\theta - 1 = \cosh 2\theta$

$1 + 2\sinh^2\theta = \cosh 2\theta$

$$\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y = e^{2u}(\cos^2 v + \sin^2 v)$$

$$\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cosh 2y}{2}\right) + \left(\frac{1+\cos 2x}{2}\right)\left(\frac{\cosh 2y - 1}{2}\right) = e^{2u}$$

$$\frac{1-\cos 2x + \cosh 2y - \cos 2x\cosh 2y}{4} + \frac{\cosh 2y + \cos 2x\cosh 2y - 1 - \cos 2x}{4} = e^{2u}$$

$$\frac{\cancel{1}-\cancel{1}-2\cos 2x + 2\cosh 2y - \cancel{\cos 2x\cosh 2y} + \cancel{\cos 2x\cosh 2y}}{4} = e^{2u}$$

$$2(\cosh 2y - \cos 2x) = 4e^{2u}$$

$$\cosh 2y - \cos 2x = 2e^{2u}$$

$$\boxed{\cosh 2y = \cos 2x + 2e^{2u}} \quad \text{proved}$$

Dividing eq (i) by (ii)

$$\frac{\sin x\cosh y}{\cos x\sinh y} = \frac{\cos v}{\sin v}$$

$$\frac{\cosh y}{\sinh y} = \frac{\cos v\cos x}{\sin v\sin x}$$

$$\frac{e^{y}+e^{-y}}{e^{y}-e^{-y}} = \frac{\cos v\cos x}{\sin v\sin x}$$

Comp & Div. $\left(\frac{a}{b} = \frac{a+b}{a-b}\right)$

Now apply Componendo & Dividendo

$$\frac{e^{y}+\cancel{e^{-y}}+e^{y}-\cancel{e^{-y}}}{\cancel{e^{y}}+e^{-y}-\cancel{e^{y}}+e^{-y}} = \frac{\cos v\cos x + \sin v\sin x}{\cos v\cos x - \sin v\sin x}$$

$$\frac{\cancel{2}e^{y}}{\cancel{2}e^{-y}} = \frac{\cos(x-v)}{\cos(x+v)}$$

$$e^{y}e^{y} = \frac{\cos(x-v)}{\cos(x+v)}$$

$$e^{2y} = \frac{\cos(x-v)}{\cos(x+v)} \quad \text{proved}$$

(9) Show that $\operatorname{Log}(1+\cos\theta+i\sin\theta) = \ln\left(2\cos\frac{\theta}{2}\right)+i\frac{\theta}{2}$

LHS $\operatorname{Log}(1+\cos\theta+i\sin\theta)$

$$=\ln\sqrt{(1+\cos\theta)^2+\sin^2\theta}+i\tan^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$=\ln\sqrt{1+\cos^2\theta+2\cos\theta+\sin^2\theta}+i\tan^{-1}\left(\frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}\right)$$

$$=\ln\sqrt{1+1+2\cos\theta}+i\tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$=\ln\sqrt{2(1+\cos\theta)}+i\frac{\theta}{2}$$

$$=\ln\sqrt{2\left(2\cos^2\frac{\theta}{2}\right)}+i\frac{\theta}{2}$$

$$=\ln 2\cos\frac{\theta}{2}+i\frac{\theta}{2}\quad\text{proved}$$

Imp (10) Prove that $\tan^{-1}\left(\frac{x+iy}{x-iy}\right)=\frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$

Let $\alpha+i\beta=\tan^{-1}\left(\frac{x+iy}{x-iy}\right)$ —(1)

$\alpha-i\beta=\tan^{-1}\left(\frac{x-iy}{x+iy}\right)$ —(2)

Add (1)+(2) $\quad 2\alpha=\tan^{-1}\left[\dfrac{\frac{x+iy}{x-iy}+\frac{x-iy}{x+iy}}{1-\left(\cancel{\frac{x+iy}{x-iy}}\right)\left(\cancel{\frac{x-iy}{x+iy}}\right)}\right]$

$$2\alpha=\tan^{-1}(\infty)=\frac{\pi}{2}$$

$$\boxed{\alpha=\frac{\pi}{4}}$$

Subtracting (2) from (1)

$$2i\beta=\tan^{-1}\left[\dfrac{\frac{x+iy}{x-iy}-\frac{x-iy}{x+iy}}{1+\cancel{\frac{x+iy}{x-iy}}\,\cancel{\frac{x-iy}{x+iy}}}\right]$$

$$\tan 2i\beta=\frac{(x+iy)^2-(x-iy)^2}{2(x^2+y^2)}$$

$$i\tanh 2\beta=\frac{\cancel{x^2}-\cancel{y^2}+2ixy-\cancel{x^2}+\cancel{y^2}+2ixy}{2(x^2+y^2)}$$

$$i\tanh 2\beta=\frac{2ixy}{x^2+y^2}$$

$$\tanh 2\beta=\frac{2xy}{x^2+y^2}$$

$$\frac{e^{2\beta}-e^{-2\beta}}{e^{2\beta}+e^{-2\beta}}=\frac{2xy}{x^2+y^2}$$

By Comp & Dividendo

$$\Rightarrow\frac{e^{2\beta}-\cancel{e^{-2\beta}}+e^{2\beta}+\cancel{e^{-2\beta}}}{\cancel{e^{2\beta}}-e^{-2\beta}-\cancel{e^{2\beta}}-e^{-2\beta}}=\frac{2xy+x^2+y^2}{2xy-x^2-y^2}$$

$$\frac{\cancel{2}e^{2\beta}}{\cancel{-2}e^{-2\beta}}=\frac{(x+y)^2}{-(x-y)^2}$$

$$e^{2\beta}e^{2\beta}=\left(\frac{x+y}{x-y}\right)^2$$

$$e^{4\beta}=\left(\frac{x+y}{x-y}\right)^2$$

Take Square Root $\quad e^{2\beta}=\dfrac{x+y}{x-y}$

Take log $\quad 2\beta=\ln\left(\dfrac{x+y}{x-y}\right)$

$$\beta=\frac{1}{2}\ln\left(\frac{x+y}{x-y}\right)$$

Hence

$$\tan^{-1}\left(\frac{x+iy}{x-iy}\right)=\frac{\pi}{4}+\frac{i}{2}\ln\left(\frac{x+y}{x-y}\right)$$

proved.

$\boxed{1.4 - 10}$

See Pag # 85 Ex#1.4(i) TP

Q. To Prove $\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left(\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right)$

Sol. Let $\alpha + i\beta = \cos^{-1}(\cos\theta + i\sin\theta)$ —— ① (Now we find values of α & β and put in ① to get required result.)

$\therefore \cos(\alpha + i\beta) = \cos\theta + i\sin\theta$

$\cos\alpha\cos i\beta - \sin\alpha\sin i\beta = \cos\theta + i\sin\theta$

$\cos\alpha\cosh\beta - i\sin\alpha\sinh\beta = \cos\theta + i\sin\theta$

Equating Real & Imaginary parts

$\cos\alpha\cosh\beta = \cos\theta$ & $-\sin\alpha\sinh\beta = \sin\theta$

$\cosh\beta = \dfrac{\cos\theta}{\cos\alpha}$ —— ⓘⓘ & $\sinh\beta = \dfrac{\sin\theta}{-\sin\alpha}$ —— ⓘⓘⓘ

Since

$\cosh^2\beta - \sinh^2\beta = 1$ (To eliminate β)

$\left(\dfrac{\cos\theta}{\cos\alpha}\right)^2 - \left(\dfrac{-\sin\theta}{\sin\alpha}\right)^2 = 1$

$\dfrac{\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha}{\cos^2\alpha\sin^2\alpha} = 1$

$\cos^2\theta\sin^2\alpha - \sin^2\theta\cos^2\alpha = \cos^2\alpha\sin^2\alpha$

$(1-\sin^2\theta)\sin^2\alpha - \sin^2\theta(1-\sin^2\alpha) = (1-\sin^2\alpha)\sin^2\alpha$

$\sin^2\alpha - \cancel{\sin^2\theta\sin^2\alpha} - \sin^2\theta + \cancel{\sin^2\theta\sin^2\alpha} = \sin^2\alpha - \sin^4\alpha$

$\cancel{-}\sin^2\theta = \cancel{\sin^2\alpha} - \cancel{\sin^2\alpha} \cancel{+} \sin^4\alpha$

$\sin\theta = \sin^2\alpha$

$\sqrt{\sin\theta} = \sin\alpha$

$\boxed{\sin^{-1}\sqrt{\sin\theta} = \alpha}$

Now Since $\cosh^2\beta - \sinh^2\beta = 1$

$\cosh^2\beta = 1 + \sinh^2\beta$

$\cosh\beta = \sqrt{1+\sinh^2\beta}$ —— ⓘⓥ

from ⓘⓘⓘ $\sinh\beta = \dfrac{\sin\theta}{-\sin\alpha} = \dfrac{\sin\theta}{-\sqrt{\sin\theta}} = -\sqrt{\sin\theta}$

∴ ⓘⓥ becomes $\cosh\beta = \sqrt{1+(-\sqrt{\sin\theta})^2} = \sqrt{1+\sin\theta}$

$\cosh\beta + \sinh\beta = \sqrt{1+\sin\theta} + (-\sqrt{\sin\theta})$

$\dfrac{e^{\beta}+e^{-\beta}}{2} + \dfrac{e^{\beta}-e^{-\beta}}{2} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$\dfrac{2e^{\beta}}{2} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$e^{\beta} = \sqrt{1+\sin\theta} - \sqrt{\sin\theta}$

$\beta = \ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

Put values of α & β in ①

$\cos^{-1}(\cos\theta + i\sin\theta) = \sin^{-1}\sqrt{\sin\theta} + i\ln\left[\sqrt{1+\sin\theta} - \sqrt{\sin\theta}\right]$

proved

1·4–11

Q(ii) Prove that $\tan^{-1}(\cos\theta + i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

Sol Let $\alpha + i\beta = \tan^{-1}(\cos\theta + i\sin\theta)$ —(i)

$\alpha - i\beta = \tan^{-1}(\cos\theta - i\sin\theta)$ —(ii)

Adding (i)+(ii)

$\tan^{-1}(\cos\theta + i\sin\theta) + \tan^{-1}(\cos\theta - i\sin\theta) = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta}}{1-(\cos\theta+i\sin\theta)(\cos\theta - i\sin\theta)}\right] = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{2\cos\theta}{1-(\cos^2\theta + \sin^2\theta)}\right] = 2\alpha$

$\Rightarrow \tan^{-1}\left[\frac{2\cos\theta}{1-1}\right] = 2\alpha$

$\Rightarrow \tan^{-1}(\pm\infty) = 2\alpha$ as $\cos\theta > 0$ or $\cos\theta < 0$

$\Rightarrow \pm\frac{\pi}{2} = 2\alpha$

$\Rightarrow \boxed{\pm\frac{\pi}{4} = \alpha}$

Again Subtracting

$\tan^{-1}(\cos\theta + i\sin\theta) - \tan^{-1}(\cos\theta - i\sin\theta) = 2i\beta$

$\Rightarrow \tan^{-1}\left[\frac{\cancel{\cos\theta} + i\sin\theta - \cancel{\cos\theta} + i\sin\theta}{1+(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}\right] = 2i\beta$

$\Rightarrow \tan^{-1}\left[\frac{2i\sin\theta}{1+\cos^2\theta + \sin^2\theta}\right] = 2i\beta$

$\Rightarrow \tan^{-1}\left(\frac{\cancel{2}i\sin\theta}{\cancel{1+1}}\right) = 2i\beta$

$\Rightarrow i\sin\theta = \tan 2i\beta$

$\Rightarrow \cancel{i}\sin\theta = \cancel{i}\tanh 2\beta$

$\sin\theta = \frac{e^{2\beta} - e^{-2\beta}}{e^{2\beta} + e^{-2\beta}}$

Comp & Dividendo

$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{e^{2\beta} - \cancel{e^{-2\beta}} + e^{2\beta} + \cancel{e^{-2\beta}}}{\cancel{e^{2\beta}} - e^{-2\beta} - \cancel{e^{2\beta}} - e^{-2\beta}}$

$\frac{\sin\theta + 1}{\sin\theta - 1} = \frac{\cancel{2}e^{2\beta}}{-\cancel{2}e^{-2\beta}} = -e^{4\beta}$

$e^{4\beta} = \frac{1+\sin\theta}{1-\sin\theta}$

$4\beta = \ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

$\beta = \frac{1}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

Put values of α & β in (i) we get

$\tan^{-1}(\cos\theta + i\sin\theta) = \pm\frac{\pi}{4} + \frac{i}{4}\ln\left(\frac{1+\sin\theta}{1-\sin\theta}\right)$

proved.