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CH # 10

Higher Order Linear Differential Eqs

OR

Diff eqs of Ist Degree, but not of Ist order.

Linear Diff eqs with constant coefficients.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = F(x)$$

If $F(x)=0$, then Homogeneous Linear Diff. Eq.
If $F(x)\neq 0$, then NonHomogeneous Linear Diff Eq.

1) **Real and Distinct Roots** If $m_1, m_2, \ldots m_n$ are distinct real roots then General Sol is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$.

2) **Real and Equal Roots** If $m_1 = m_2$ are roots then G. Sol is $y = (c_1 + c_2 x)e^{m_1 x}$
If $m_1 = m_2 = m_3$ are real roots then G. Sol is $y = (c_1 + c_2 x + c_3 x^2)e^{m_1 x}$

3) **Imaginary & Distinct Roots** If Roots are $\alpha \pm i\beta$ then G. Sol is $y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

4) **Imaginary & Repeated Roots** If Roots are $\alpha \pm i\beta$, & $\alpha \pm i\beta$ then G. Sol $y = e^{\alpha x}\{(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x\}$

Ex 10.1

Solve

① $(9D^2 - 12D + 4)y = 0$

$9D^2 - 12D + 4 = 0$ Characteristic Eq or Auxiliary Eq.

$D = \frac{12 \pm \sqrt{144 - 4\cdot 9\cdot 4}}{18} = \frac{12 \pm \sqrt{144-144}}{18}$

$D = \frac{12}{18} + 0,\ \frac{12}{18} - 0 = \boxed{\frac{2}{3}, \frac{2}{3}}$ (case II)

$\therefore$ G. Sol is $y = (c_1 + c_2 x)e^{\frac{2}{3}x}$

② $(75D^2 + 50D + 12)y = 0$

$75D^2 + 50D + 12 = 0$ characteristic Eq

$D = \frac{-50 \pm \sqrt{2500 - }}{2(75)} = \frac{-50 \pm \sqrt{2500 - 3600}}{150}$

$= \frac{-50 \pm \sqrt{-1100}}{150} = \frac{-50 \pm \sqrt{(-1)(100)}}{150} = \frac{-50 \pm 10\sqrt{-11}}{150}$

$= -10\left(\frac{-5 \pm \sqrt{-11}}{150}\right) = \frac{-5 \pm i\sqrt{11}}{15} = \boxed{-\frac{1}{3} \pm \frac{i\sqrt{11}}{15}}$

G. Sol $y = e^{-\frac{1}{3}x}\left(c_1 \cos\frac{\sqrt{11}}{15}x + c_2 \sin\frac{\sqrt{11}}{15}x\right)$

③ $(D^3 - 4D^2 + D + 6)y = 0$

$D^3 - 4D^2 + D + 6 = 0$ Characteristic Eq.

-1	1	-4	1	6
	$\downarrow$	-1	5	-6
	1	-5	6	$\boxed{0}$

$D^2 - 5D + 6 = 0$ Depressed Eq

$(D-2)(D-3) = 0$

$\boxed{D = 2, 3 \ \&\ -1}$

G. Sol is $y = c_1 e^{-1x} + c_2 e^{2x} + c_3 e^{3x}$

④ $(D^3 + D^2 + D + 1)y = 0$

$D^3 + D^2 + D + 1 = 0$ characteristic Eq

-1	1	1	1	1
	$\downarrow$	-1	0	-1
	1	0	1	$\boxed{0}$

$D^2 + 0D + 1 = 0$

$D^2 = -1$

$D = \pm i$

G. Sol. $y =$ $\therefore D = -1, \pm i$

G. Sol is $y = c_1 e^{-1x} + e^{0x}(c_2 \cos x + c_3 \sin x)$

$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

Solution of Non Homogeneous Linear Diff Eq of order 'n'.

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) Y = F(x)$$

The solution consist of two parts

(i) Complementary Function (C.F) :- It is sol of Homogeneous L. Diff Eq i.e $(a_0 D^n + a_1 D^{n-1} + \cdots + a_n)Y = 0$. It is denoted by Y_c

(ii) Particular Integral (P.I) :- It is sol of $\frac{1}{a_0 D^n + a_1 D^{n-1} + \cdots a_n} F(x)$. It is denoted by Y_p

$\therefore$ General Sol $Y = Y_c + Y_p$

Properties of Differential Operator 'D' $= \frac{d}{dx}$

i) $D^n(ae^{bx}) = a(b)^n e^{bx}$ 'D is replaced by b'

ii) $F(D)(ae^{bx}) = aF(b)e^{bx}$ 'D is replaced by b'

iii) $D^n(e^{bx})U = e^{bx}(D+b)^n U$ b is added in D in D

iv) $F(D)e^{bx}U = e^{bx}F(D+b)U$ b is added in D

v) $\frac{1}{F(D)} ae^{bx} = \frac{ae^{bx}}{F(b)}$ 'D is replaced by b'

vi) $\frac{1}{F(D)} e^{bx} \cdot U = e^{bx} \frac{1}{F(D+b)} \cdot U$, b is added in D

vii) $\frac{1}{F(D^2)} \sin(ax) = \frac{\sin ax}{F(-a^2)}$

viii) $\frac{1}{F(D^2)} \cos(ax) = \frac{\cos ax}{F(-a^2)}$ $\Big\}$ only for D^2 — D^2 is replaced by $(-a^2)$

(ix) $\frac{1}{F(D)} \sin bx = \operatorname{Im} \frac{1}{F(D)} e^{ibx} = \operatorname{Im} \frac{e^{ibx}}{F(ib)}$

x) $\frac{1}{F(D)} \cos bx = \operatorname{Re} \frac{1}{F(D)} e^{ibx} = \operatorname{Re} \frac{e^{ibx}}{F(ib)}$

xi) $D^2 \cos bx = (-b^2) \cos bx$

xii) $D^2 \sin bx = (-b^2) \sin bx$ $\Big\}$ only for D^2

xiii) $\frac{1}{D^2} \cos bx = \frac{1}{-b^2} \cos bx$

xiv) $\frac{1}{D^2} \sin bx = \frac{1}{-b^2} \sin bx$ $\Big\}$ only for D^2

Imp Note

1) when $\frac{1}{F(D)}(ae^{bx}) = \frac{ae^{bx}}{F(b)}$ if $F(b) = 0$

then $\frac{1}{F(D)} ae^{bx} = \frac{x}{F'(D)}(ae^{bx})$

$= \frac{x}{F'(b)} ae^{bx}$ if $F'(b) = 0$

then $\frac{x}{F'(D)} ae^{bx} = \frac{x^2}{F''(D)} ae^{bx}$

$= \frac{x^2}{F''(b)} ae^{bx}$ & so on. if $F''(b) \neq 0$

$$D(\sin x) = \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{1}{D}(\sin x) = \int \sin x\, dx = -\cos x$$

B. Series. If n is -ve or fraction.

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

We apply B. Series when $F(x)$ is other than $\sin, \cos$ or e^{bx} See Q4, 5, 9,