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Exact Diff Eq. (EDE)

A diff eq of the form $M(x,y)dx + N(x,y)dy = 0$
is said to be an Exact diff^l eq if it is expressible in total diff^l i.e $d[f(x,y)]$

$\left[\; d[f(x,y)] = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy \right.$

Condition for an Exact Diff Eq.

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

M, N have 1st order continuous partial derivatives.

$M = \dfrac{\partial f}{\partial x}$, $N = \dfrac{\partial f}{\partial y}$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$, $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$

$\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y}$ $\because$ M, N have 1st order cont partial derivatives.

To Solve

1) Integrate M w.r.t 'x' keeping 'y' const
2) Add the integral w.r.t 'y' of the terms of N free from 'x'.
3) & Equate to arbitrary const.

i.e $\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = c$

Ex 9.4

① Solve $(3x^2+4xy)dx + (2x^2+2y)dy = 0$

$M = 3x^2+4xy$, $N = 2x^2+2y$

$\dfrac{\partial M}{\partial y} = 0 + 4x$ $\qquad$ $\dfrac{\partial N}{\partial x} = 4x + 0$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ So given Diff Eq is Exact.

Now $\int M\,dx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\therefore \int(3x^2+4xy)dx + \int 2y\,dy = C$

$3\dfrac{x^3}{3} + \dfrac{4x^2y}{2} + \dfrac{2y^2}{2} = c$

$x^3 + 2x^2y + y^2 = c$

② $(2xy+y-\tan y)dx+(x^2-x\tan^2 y+\sec^2 y)dy=0$

$\therefore M=2xy+y-\tan y \quad , \quad N=x^2-x\tan^2 y+\sec^2 y$

$\frac{\partial M}{\partial y}=2x+1-\sec^2 y \quad , \quad \frac{\partial N}{\partial x}=2x-\tan^2 y+0$

$=2x-\tan^2 y$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ So given diff eq is Exact

$\int M\,dx+\int(\text{terms of } N \text{ free from } x)\,dy=c$

$\int(2xy+y-\tan y)\,dx+\int \sec^2 y\,dy=c$

$\frac{2x^2y}{2}+xy-x\tan y+\tan y=c$

$x^2y+xy-x\tan y+\tan y=c$

③ $\left(\frac{x+y}{y-1}\right)dx-\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2dy=0$

$M=\frac{x+y}{y-1} \qquad N=-\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$

$N=-\frac{1}{2}\frac{(x^2+2x+1)}{(y-1)^2}$

$\frac{\partial M}{\partial y}=\frac{(y-1)(0+1)-(x+y)(1)}{(y-1)^2} \qquad \frac{\partial N}{\partial x}=\frac{-(2x+2)}{2(y-1)^2}$

$=\frac{\cancel{y}-1-x-\cancel{y}}{(y-1)^2} \qquad =\frac{-x-1}{(y-1)^2}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ $\therefore$ given diff eq is Exact.

$\int M\,dx+\int(\text{terms of } N \text{ free from } x)\,dy=c$

$\int\left(\frac{x+y}{y-1}\right)dx+\int-\frac{1}{2(y-1)^2}\,dy=c$

$\left(\frac{1}{y-1}\right)\int(x+y)\,dx+\left(-\frac{1}{2}\right)\int(y-1)^{-2}\,dy=c$

$\left(\frac{1}{y-1}\right)\left(\frac{x^2}{2}+xy\right)+\left(-\frac{1}{2}\right)\left(\frac{-1}{y-1}\right)=c$

$\frac{x^2+2xy}{2(y-1)}+\frac{1}{2(y-1)}=c$

$x^2+2xy+1=c'(y-1)$ Ans

④ $\frac{dy}{dx}=-\frac{(ax+hy)}{hx+by}$

$(hx+by)dy=-(ax+hy)dx$

$(ax+hy)dx+(hx+by)dy=0$

$M=ax+hy \qquad N=hx+by$

$\frac{\partial M}{\partial y}=0+h \qquad \frac{\partial N}{\partial x}=h$

$\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence Exact Diff Eq

$\int M\,dx+\int(\text{terms of } N \text{ free from } x)\,dy=c$

$\int(ax+hy)\,dx+\int by\,dy=c$

$a\frac{x^2}{2}+hxy+b\frac{y^2}{2}=c$

$ax^2+2hxy+by^2=c'$

⑤ $(1+\ln xy)dx+\left(1+\frac{x}{y}\right)dy=0$

$M=1+\ln xy \qquad N=1+\frac{x}{y}$

$\frac{\partial M}{\partial y}=0+\frac{1}{xy}\cdot x \qquad \frac{\partial N}{\partial x}=0+\frac{1}{y}$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M\,dx+\int(\text{terms of } N \text{ free from } x)\,dy=c$

$\int(1+\ln xy)\,dx+\int 1\cdot dy=c$

$\int dx+\int \ln xy\,dx+\int dy=c$

$x+\left[\ln xy\cdot(x)-\int\frac{1}{xy}\cdot y\cdot x\,dx\right]+y=c$

$x+x\ln xy-\int dx+y=c$

$\cancel{x}+x\ln xy-\cancel{x}+y=c$

$x\ln xy+y=c$

(6) $\dfrac{Ydx + xdy}{1-x^2y^2} + xdx = 0$

$\dfrac{Ydx}{1-x^2y^2} + \dfrac{xdy}{1-x^2y^2} + xdx = 0$

$\left(x + \dfrac{Y}{1-x^2y^2}\right)dx + \dfrac{xdy}{1-x^2y^2} = 0$

$M = x + \dfrac{Y}{1-x^2y^2}$

$\dfrac{\partial M}{\partial y} = 0 + \dfrac{(1-x^2y^2)\cdot 1 - Y(-2x^2y)}{(1-x^2y^2)^2}$

$= \dfrac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2} = \dfrac{1+x^2y^2}{(1-x^2y^2)^2}$

$N = \dfrac{x}{1-x^2y^2}$

$\dfrac{\partial N}{\partial x} = \dfrac{(1-x^2y^2)1 - x(-2xy^2)}{(1-x^2y^2)^2}$

$= \dfrac{1-x^2y^2+2x^2y^2}{(1-x^2y^2)^2} = \dfrac{1+x^2y^2}{(1-x^2y^2)^2}$

$\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ Exact Diff Eq.

$\int Mdx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\int\left(x + \dfrac{Y}{1-x^2y^2}\right)dx + \text{Nil} = C$

$\int xdx + \int \dfrac{Y\,dx}{1-x^2y^2} = C$

$\dfrac{x^2}{2} + \int \dfrac{Y/y^2}{\frac{1}{y^2} - \frac{x^2y^2}{y^2}}dx = C$

$\dfrac{x^2}{2} + \dfrac{1}{Y}\int \dfrac{dx}{(\frac{1}{y})^2 - x^2} = C$

$\dfrac{x^2}{2} + \dfrac{1}{Y}\left\{\dfrac{1}{2(\frac{1}{y})} \ln\left|\dfrac{\frac{1}{y}+x}{\frac{1}{y}-x}\right|\right\} = C$

$\dfrac{x^2}{2} + \dfrac{1}{2}\ln\left|\dfrac{1+xy}{1-xy}\right| = C$

$x^2 + \ln\left|\dfrac{1+xy}{1-xy}\right| = \bar{C}$ Ans.

(7) $(6xy+2y^2-5)dx + (3x^2+4xy-6)dy$

$M = 6xy+2y^2-5$, $N = 3x^2+4xy-6$

$\dfrac{\partial M}{\partial y} = 6x+4y$ $\quad \dfrac{\partial N}{\partial x} = 6x+4y$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int Mdx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\int(6xy+2y^2-5)dx + \int -6dy = C$

$\dfrac{6x^2}{2}y + 2xy^2 - 5x - 6y = C$

$3x^2y + 2xy^2 - 5x - 6y = C$

(8) $(Y\sec^2 x + \sec x \tan x)dx + (\tan x + 2y)dy = 0$

$M = Y\sec^2 x + \sec x \tan x$, $N = \tan x + 2y$

$\dfrac{\partial M}{\partial y} = \sec^2 x$ $\quad \dfrac{\partial N}{\partial x} = \sec^2 x$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int Mdx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\int(Y\sec^2 x + \sec x \tan x)dx + \int 2y\,dy = C$

$Y\tan x + \sec x + y^2 = C$

(9) $(Y\cos x + 2xe^y)dx + (\sin x + x^2e^y - 1)dy = 0$

$M = Y\cos x + 2xe^y$, $N = \sin x + x^2e^y - 1$

$\dfrac{\partial M}{\partial Y} = \cos x + 2xe^y$ $\quad \dfrac{\partial N}{\partial x} = \cos x + 2xe^y$

$\dfrac{\partial M}{\partial Y} = \dfrac{\partial N}{\partial x}$ Hence Exact diff eq.

$\int Mdx + \int(\text{terms of } N \text{ free from } x)dy = C$

$\int(Y\cos x + 2xe^y)dx + \int -1\,dy = C$

$Y\sin x + \cancel{2}\dfrac{x^2}{\cancel{2}}e^y - Y = C$

$Y\sin x + x^2e^y - Y = C$

(10) $(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + (xe^{xy}\cos 2x - 3)dy = 0$

Sol $M = ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x \qquad N = xe^{xy}\cos 2x - 3$

$\frac{\partial M}{\partial y} = \{ye^{xy}(x) + e^{xy}\}\cos 2x - 2\sin 2x\, e^{xy}(x)$

$= e^{xy}\{yx\cos 2x + \cos 2x - 2\sin 2x(x)\}$

$\frac{\partial N}{\partial x} = 1.e^{xy}\cos 2x + xe^{xy}(y)\cos 2x + xe^{xy}(-2\sin 2x) - 0$

$= e^{xy}(\cos 2x + xy\cos 2x - x2\sin 2x)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x)dx + \int -3dy = c$

$y\int e^{xy}\cos 2x\,dx - 2\int e^{xy}\sin 2x + 2\int x\,dx - 3\int dy = c$

$y\left[\cos 2x \frac{e^{xy}}{y} - \int -\sin 2x(2)\frac{e^{xy}}{y}dx\right] - 2\int e^{xy}\sin 2x + \cancel{2}\frac{x^2}{\cancel{2}} - 3y = c$

$\cos 2x\, e^{xy} + 2\int \sin 2x\, e^{xy}dx - 2\int e^{xy}\sin 2x + x^2 - 3y = c$

$\cos 2x\, e^{xy} + x^2 - 3y = c$

(11) Solve the initial value problems.

$(2xy - 3)dx + (x^2 + 4y)dy = 0, \quad y(1) = 2$

$M = 2xy - 3 \qquad N = x^2 + 4y$

$\frac{\partial M}{\partial y} = 2x \qquad \frac{\partial N}{\partial x} = 2x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(2xy - 3)dx + \int 4y\,dy = c$

$\cancel{2}\frac{x^2}{\cancel{2}}y - 3x + 4\frac{y^2}{2} = c$

$x^2y - 3x + 2y^2 = c$

$\because y(1) = 2$ (given)

$\therefore 2 - 3 + 8 = c$

$\boxed{7 = c}$

Hence $x^2y - 3x + 2y^2 = 7$

$\longrightarrow$ P. Sol.

(12) $(2x\cos y + 3x^2y)dx + (x^3 - x^2\sin y - y)dy = 0$

$y(0) = 2$

$M = 2x\cos y + 3x^2y \qquad N = x^3 - x^2\sin y - y$

$\frac{\partial M}{\partial y} = 2x(-\sin y) + 3x^2 \qquad \frac{\partial N}{\partial x} = 3x^2 - 2x\sin y - 0$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq.

$\int M dx + \int(\text{terms of } N \text{ free from } x)dy = c$

$\int(2x\cos y + 3x^2y)dx + \int -y\,dy = c$

$2\frac{x^2}{2}\cos y + 3\frac{x^3}{3}y - \frac{y^2}{2} = c$

$x^2\cos y + x^3y - \frac{y^2}{2} = c$

$\because y(0) = 2$

$0 + 0 - \frac{4}{2} = c$

$\boxed{-2 = c}$

$x^2\cos y + x^3y - \frac{y^2}{2} = -2$

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(13) $(3x^2y^2 - y^3 + 2x)dx + (2x^3y - 3xy^2 + 1)dy = 0 \quad , \; y(-2) = 1$

$M = 3x^2y^2 - y^3 + 2x \qquad N = 2x^3y - 3xy^2 + 1$

$\frac{\partial M}{\partial y} = 6x^2y - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x^2y - 3y^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq

$\int M dx + \int (\text{terms of } N \text{ free from } x) dy = c.$

$\int (3x^2y^2 - y^3 + 2x)dx + \int 1\, dy = c$

$3\frac{x^3}{3}y^2 - xy^3 + 2\frac{x^2}{2} + y = c$

$x^3y^2 - xy^3 + x^2 + y = c$

$\because y(-2) = 1$

$\therefore -8 + 2 + 2 + 4 + 1 = c$

$\boxed{-1 = c}$

Hence $x^3y^2 - xy^3 + x^2 + y + 1 = 0$

(15) $(4x^3 e^{x+y} + x^4 e^{x+y} + 2x)dx + (x^4 e^{x+y} + 2y)dy$

$M = 4x^3 e^{x+y} + x^4 e^{x+y} + 2x$

$\frac{\partial M}{\partial y} = 4x^3 e^{x+y}(0+1) + x^4 e^{x+y}(0+1) + 0$

$\frac{\partial M}{\partial y} = (4x^3 + x^4)e^{x+y}$

$N = x^4 e^{x+y} + 2y$

$\frac{\partial N}{\partial x} = 4x^3 e^{x+y} + x^4 e^{x+y}(1+0) + 0$

$\frac{\partial N}{\partial x} = (4x^3 + x^4)e^{x+y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence Exact Diff Eq

$\int M dx + \int (\text{terms of } N \text{ free from } x) dy = c.$

$\int (4x^3 e^{x+y} + x^4 e^{x+y} + 2x)dx + \int 2y\, dy = c$

$\int 4x^3 e^{x+y} dx + \int \underset{\text{I}}{x^4} \underset{\text{II}}{e^{x+y}} dx + \int 2x\, dx + \int 2y\, dy = c$

$\cancel{\int 4x^3 e^{x+y} dx} + \left\{ x^4 e^{x+y} - \cancel{\int 4x^3 e^{x+y} dx} \right\} + x^2 + y^2 = c$

$x^4 e^{x+y} + x^2 + y^2 = c$

$\because y(0) = 1 \implies e^{0+1}(0)^4 + 0^2 + 1^2 = c$

$\boxed{1 = c}$

$\therefore x^4 e^{x+y} + x^2 + y^2 = 1$ Ans.

(14) $\left(\frac{3-y}{x^2}\right)dx + \left(\frac{y^2 - 2x}{xy^2}\right)dy = 0$

$M = \frac{3-y}{x^2} \qquad N = \frac{y^2 - 2x}{xy^2}$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{y^2}\left[\frac{x(0-2) - (y^2 - 2x)(1)}{x^2}\right]$

$= \frac{(-2x - y^2 + 2x)}{y^2x^2}$

$\frac{\partial N}{\partial x} = \frac{-y^2}{y^2x^2} = \frac{-1}{x^2}$

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore$ Exact Diff Eq.

$\int M dx + \int (\text{terms of } N \text{ free from } x) dy = c$

$\int \left(\frac{3-y}{x^2}\right)dx + \int -\frac{2}{y^2} dy = c$

$(3-y)\int \frac{dx}{x^2} - 2\int \frac{dy}{y^2} = c$

$(3-y)\left(\frac{-1}{x}\right) - 2\left(-\frac{1}{y}\right) = c$

$\frac{-y(3-y) + 2x}{xy} = c$

$\rightarrow -y(3-y) + 2x = cxy$

$\because y(-1) = 2$

$\therefore -2 + 2(-1) = c(-1)(2)$

$-4 = c(-2)$

$\boxed{+2 = c}$

$\therefore -y(3-y) + 2x = 2xy$

$y^2 - 3y + 2x - 2xy = 0$