Exercise 7.2

Absolute Maximum

Let a function f be defined on $[a,b]$. Then f is said to have absolute maximum on $[a,b]$ if there is a number $c \in [a,b]$

s.t. $f(c) \geq f(x) \ \forall \ x \in [a,b]$

$f(c)$ is the absolute maximum value of f on $[a,b]$

Absolute Minimum

Let a function f be defined on $[a,b]$. Then f is said to have absolute minimum on $[a,b]$ if there is number $D \in [a,b]$ s.t.

$f(D) \leq f(x) \ \forall \ x \in [a,b]$.

$f(D)$ is the absolute minimum value of f on $[a,b]$.

Relative Maximum

The function f is said to have relative Maximum at $c \in]a,b[$ if there exist a number $\delta > 0$ s.t. $[c-\delta, c+\delta] \in]a,b[$ and $f(c)$ is the absolute Maximum value of the function f on $[c-\delta, c+\delta] \in]a,b[$ i.e.

$f(c) \geq f(x)$ for all $x \in [c-\delta, c+\delta]$.

$f(c)$ is called relative maximum value of f at c.

Relative Minimum

The function f is said to have a relative Minimum at $d \in]a,b[$ if there exist a number $\delta > 0$ s.t. $[d-\delta, d+\delta] \in]a,b[$ and $f(d)$ is the absolute minimum value of f on $[d-\delta, d+\delta]$ that is,

$f(d) \leq f(x) \ \forall \ x \in [d-\delta, d+\delta]$.

$f(d)$ is called the relative minimum value of f at d.

The term (relative) extreme values (extrema) is used to refer to either a relative maximum value or a relative minimum value.

Extreme mean one value and Extreme is singular. where Extrema is plural mean Max and Min. values.

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Stationary points

A critical point for f is any point c in the domain of f at which $f'(c) = 0$ or f is not differentiable at c. The critical points where $f'(x) = 0$ are called stationary points.

First Derivative Test For Relative Maximum and Relative Minimum.

f is relative Maximum at $x = c$ if

i) $f'(x) > 0$ before $x = c$

ii) $f'(x) = 0$ at $x = c$

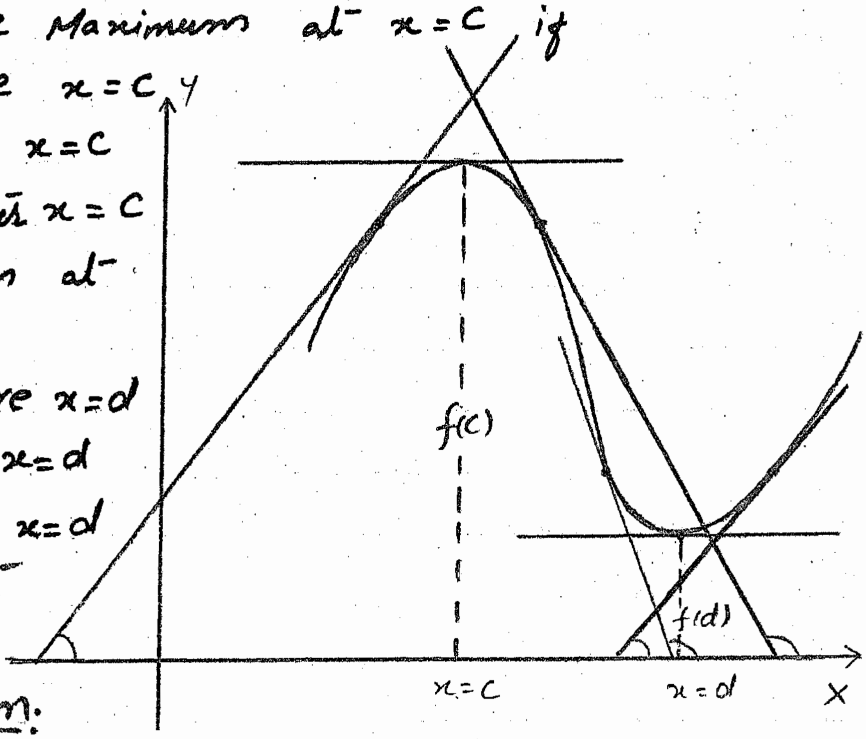
iii) $f'(x) < 0$ after $x = c$

f is relative Minimum at $x = d$ if

i) $f'(x) < 0$ before $x = d$

ii) $f'(x) \leq 0$ at $x = d$

iii) $f'(x) > 0$ after $x = d$

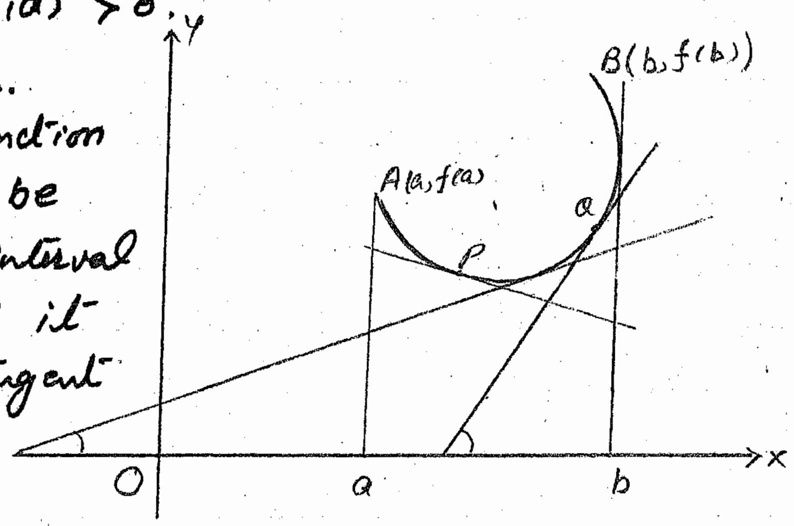


2nd Derivative Test For Relative Maximum And Relative Minimum.

A function f is Relative Maximum at $x = c$ if
$$f''(c) < 0.$$
And f is relative Minimum at $x = d$ if
$$f''(d) > 0.$$

Curve Concave Up.

The graph of a function $y = f(x)$ is said to be concave up in an interval $]a, b[$ if and only if it lies above every tangent line at the points between $(a, f(a))$ and $(b, f(b))$ on the curve.



we know that if $f''(x) > 0$ in some interval, then

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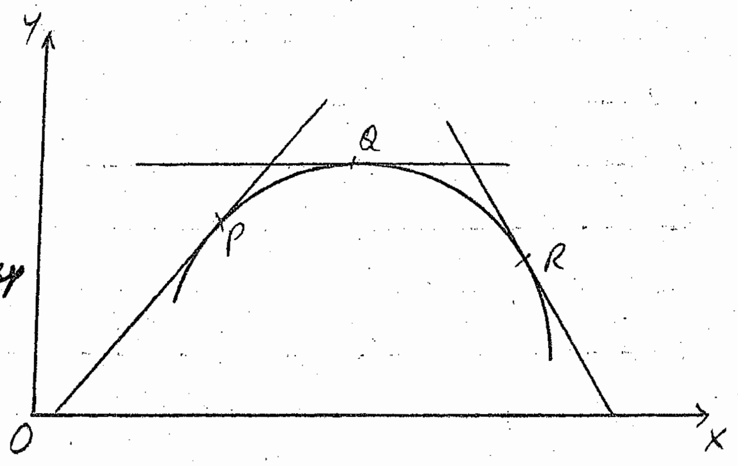
$f'(x)$ is an increasing function. Since $f'(x)$ is slope of the tangent line, and if $f'(x)$ is an increasing function, then as a point P moves from left to right along the curve, the slope of tangent line to the curve increases. Thus the curve is concave up.

Curve Concave Down:

The graph of a curve is concave down in an open interval if and only if its graph lies below every tangent line at all points of the open interval.



We know that if $f''(x) < 0$ in an interval, then $f'(x)$ is a decreasing function. In this case as point P moves from left to right along the curve, the slope of the tangent line to the curve decreases. Thus the curve is concave down.

Test For Concave Up and Concave Down:

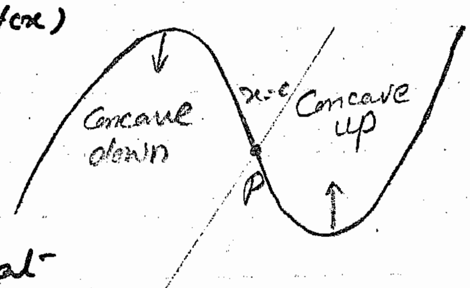
i) The curve $y = f(x)$ is concave up in $]a, b[$ if and only if $f''(x) > 0 \quad \forall\, x \in\,]a, b[$

ii) The curve $y = f(x)$ is concave down in $]a, b[$ iff $f''(x) < 0$ $\forall\, x \in\,]a, b[$

iii) The curve $y = f(x)$ is concave up at $c \in\,]a, b[$ if $f''(c) > 0$ and it is concave down at c if $f''(c) < 0$.

Point of Inflection.

Def:- A point $x = c$ on a curve $y = f(x)$ is called a point of inflection if f is concave up on one side of $x = c$ and concave down on the other side of $x = c$ and f is continuous at $x = c$



Note. For Pts. of Inflection put $f''(x) = 0$.

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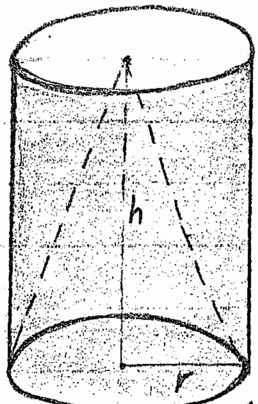

نوٹ: فرض کریں کہ ایک سلنڈر ہے جس کے دونوں اطراف کھلے ہیں۔ اس کو curved cylinder کہتے ہیں۔ اس کی لمبائی h ہے اور اس کا نچلا حصہ جو دائرہ کی شکل میں ہے یعنی ○ ۔ اگر اس دائرہ کو ہم سیدھا کریں تو اس کی شکل ایک سیدھی لائن جیسی ہو گی جیسے ______ اس کو دائرہ کا محیط کہتے ہیں یعنی Circumference ۔ دائرہ کا محیط $2\pi r$ کے برابر ہوتا ہے۔ جبکہ r دائرہ کا رداس ہے۔

پس بیلن کی سطح کا رقبہ Area of the curved surface = برابر ہوگا دائرہ کا محیط × اونچائی کے

$$S = h \times 2\pi r$$

$$S = 2\pi r h$$

اگر ہم ایک ایسا سلنڈر لیں جو نیچے سے بند ہو اور اوپر سے کھلا ہو تو اس کا رقبہ ہم مندرجہ ذیل طریقہ سے معلوم کریں گے۔

Surface Area = Area of the curved surface + Area of the bottom.

چونکہ سلنڈر کا نچلا حصہ بند ہے اور یہ دائرہ ہے اور دائرہ کا رقبہ πr^2 ہوتا ہے اور بیلن کی سطح کا رقبہ $2\pi r h$ لہذا

Surface Area of semi open cylinder = Area of the curved surface + Area of the bottom.

$$= \pi r^2 + 2\pi r h$$

اسی طرح اگر سلنڈر دونوں طرف سے بند ہو تو اس کا رقبہ مندرجہ ذیل طریقہ سے معلوم کریں گے۔

Surface Area of the closed cylinder = Area of the Top + Area of the curved surface + Area of the bottom

$$= \pi r^2 + 2\pi r h + \pi r^2$$

$$= 2\pi r^2 + 2\pi r h$$

اگر سلنڈر کا حجم معلوم کرنا ہو یعنی Volume تو اس کا طریقہ درج ذیل ہو گا

$$\text{Volume} = \text{Base area} \times \text{height}$$

$$= \pi r^2 \times h$$

$$= \pi r^2 h$$

اور اگر Cone کا Volume معلوم کرنا ہو تو اس کے لیے سلنڈر کا Volume جس کا رداس Cone کے رداس کے برابر اور height، Cone کی height کے برابر ہو تو اسے 3 سے تقسیم کر دو۔ جو نتیجہ آئے گا وہ Cone کا حجم ہو گا۔ یعنی

$$\text{Volume of the Cone} = \frac{1}{3}\pi r^2 h$$

Exercise 7.2

Locate the points of relative extrema of each of the following curve (1- 11)

Q.1. $f(x) = 2x^3 - 15x^2 + 36x + 10$ ——— I

Diff. I w.r.t. x

$\Rightarrow f'(x) = 6x^2 - 30x + 36$ ——— II

Put $f'(x) = 0$

$\Rightarrow 6x^2 - 30x + 36 = 0$

$\Rightarrow 6(x^2 - 5x + 6) = 0$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow (x-3)(x-2) = 0$

$\Rightarrow x = 3,\ 2$

Now Diff. II w.r.t. 'x'

$\Rightarrow f''(x) = 12x - 30$ ——— III

put $x = 2$ in III

$\Rightarrow f''(2) = 12(2) - 30$

$= 24 - 30 = -6$

$f''(2) < 0$

$\Rightarrow f$ is Relative Maximum at $x = 2$.

Put $x = 3$ in III

$\Rightarrow f''(3) = 12(3) - 30$

$= 36 - 30$

$= 6$

$\Rightarrow f''(3) > 0$

$\Rightarrow f$ is realtive minimum at $x = 3$.

Q.2. $f(x) = 3x^4 - 4x^3 + 5$ ——— I

I) $\Rightarrow f'(x) = 12x^3 - 12x^2$ ——— II

Put $f'(x) = 0$

$12x^3 - 12x^2 = 0$

$12x^2(x-1) = 0$

$\Rightarrow 12x^2 = 0 \quad , \quad x - 1 = 0$

$\Rightarrow x = 0 \quad , \quad x = 1$

نوٹ چونکہ x=0 رکھنے سے f''(0)=0 آتا ہے
لہٰذا ہم f'''(x) لیں اور اس میں صفر Put کردیں
گے اگر کوئی قیمت آجائے تو وہ Point of
inflection ہوگا اور اگر پھر بھی صفر
آجائے تو f''''(x) لیں گے اور اگر
جواب +ve یا -ve آئے تو اسے پھر

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2) ⇒ $f''(x) = 36x^2 - 24x$ ——— III

سوال کے مطابق Minimum یا Maximum لکھ دیں گے۔

Put $x=0$ in III

$\Rightarrow f''(0) = 0$

So f has no relative extrema at $x=0$

Put $x=1$ in III, ⇒ $f''(1) = 36(1)^2 - 24(1)$

$= 12$

$f''(1) > 12$

⇒ f has relative Minimum at $x=1$

Q.3. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$ ——— I

I) ⇒ $f'(x) = 60x^4 - 180x^3 + 120x^2$ ——— II

Put $f'(x) = 0$

$60x^4 - 180x^3 + 120x^2 = 0$

$60x^2(x^2 - 3x + 2) = 0$

⇒ $60x^2 = 0$, $x^2 - 3x + 2 = 0$

⇒ $x = 0$, $(x-1)(x-2) = 0$ ⇒ $x-1=0$, $x-2=0$

⇒ $x = 1, x = 2$

So $x = 0, 1, 2$

II) ⇒ $f''(x) = 240x^3 - 540x^2 + 240x$ ——— III

Put $x=0$ in III, ⇒

$f''(0) = 0$

i.e f has no relative extrema at $x=0$

Put $x=1$ in III, ⇒

$f''(1) = 240 - 540 + 240$

$= -60$

i.e. $f''(1) < 0$

⇒ f has relative Maximum at $x=1$

Put $x=2$ in III, ⇒

$f''(2) = 240(2)^3 - 540(2)^2 + 240(2)$

$= 1920 - 2160 + 480$

$= 240$

i.e $f''(2) > 0$

⇒ f has relative Minimum at $x=2$

Q.4. $f(x) = (x-1)(x-2)(x-3)$

$f(x) = (x^2-3x+2)(x-3)$

$f(x) = x^3-6x^2+11x-6$ ——— I

$f'(x) = 3x^2-12x+11$ ——— II

Put $f'(x) = 0$

$\Rightarrow 3x^2-12x+11=0$

By using Quadratic formula.

$$x = \frac{-(-12) \pm \sqrt{(-12)^2-4(3)(11)}}{2(3)}$$

$$x = \frac{12 \pm \sqrt{144-132}}{6}$$

$$x = \frac{12 \pm \sqrt{12}}{6} = \frac{2(6 \pm \sqrt{3})}{6} = \frac{6 \pm \sqrt{3}}{3}$$

$$\Rightarrow x = \frac{6+\sqrt{3}}{3}, \quad \frac{6-\sqrt{3}}{3}$$

II) $\Rightarrow$ $f''(x) = 6x-12$ ——— III

Put $x = \frac{6+\sqrt{3}}{3}$ in III) $\Rightarrow$

$$f''\left(\frac{6+\sqrt{3}}{3}\right) = 6\left(\frac{6+\sqrt{3}}{3}\right) - 12$$
$$= 2(6+\sqrt{3}) - 12$$
$$= 12+2\sqrt{3}-12$$
$$= 2\sqrt{3}$$

i.e $f''\left(\frac{6+\sqrt{3}}{3}\right) > 0$

$\Rightarrow$ f is relative Minimum at $x = \frac{6+\sqrt{3}}{3}$

Put $x = \frac{6-\sqrt{3}}{3}$ in III, $\Rightarrow$

$$f''\left(\frac{6-\sqrt{3}}{3}\right) = 6\left(\frac{6-\sqrt{3}}{3}\right) - 12$$
$$= 2(6-\sqrt{3}) - 12$$
$$= 12-2\sqrt{3}-12$$
$$= -2\sqrt{3}$$

i.e $f''\left(\frac{6-\sqrt{3}}{3}\right) < 0$

$\Rightarrow$ f is relative Maximum at $x = \frac{6-\sqrt{3}}{3}$

Q.5.

$$f(x) = \sin x \cos 2x$$
$$f(x) = \sin x \,(1 - 2\sin^2 x)$$
$$f(x) = \sin x - 2\sin^3 x \quad \text{------ I}$$

$1-\cos x = 2\sin^2 x/2$

$1-2\sin^2 x/2 = \cos x$

$$\text{I} \Rightarrow \quad f'(x) = \cos x - 6\sin^2 x \cos x$$
$$f'(x) = \cos x\,(1 - 6\sin^2 x) \quad \text{------ II}$$

For Maxima and Minima $f'(x) = 0$

$$\cos x\,(1 - 6\sin^2 x) = 0$$
$$\Rightarrow \cos x = 0 \quad , \quad 1 - 6\sin^2 x = 0$$
$$x = \pm\frac{\pi}{2} \quad , \quad 6\sin^2 x = 1$$
$$\sin^2 x = 1/6$$
$$\sin x = \pm\frac{1}{\sqrt{6}}$$

$$\text{II} \Rightarrow f''(x) = -\sin x(1 - 6\sin^2 x) + \cos x(-12\sin x\cos x)$$
$$f''(x) = -\sin x(1 - 6\sin^2 x) - 12\sin x\cos^2 x \quad \text{------ III}$$

Put $x = \frac{\pi}{2}$ in III

$$\Rightarrow f''\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)\left(1 - 6\sin^2\left(\frac{\pi}{2}\right)\right) - 12\sin\left(\frac{\pi}{2}\right)\cos^2\left(\frac{\pi}{2}\right).$$
$$f''\left(\frac{\pi}{2}\right) = -1(1 - 6(1)) - 0$$
$$= -1 + 6$$
$$= 5$$

i.e. $f''\left(\frac{\pi}{2}\right) > 0$

$\Rightarrow$ f is relative Minimum at $x = \frac{\pi}{2}$

Put $x = -\frac{\pi}{2}$ in III, we have

$$f''\left(-\frac{\pi}{2}\right) = -\sin\left(-\frac{\pi}{2}\right)\left(1 - 6\sin^2\left(-\frac{\pi}{2}\right)\right) - 12\sin\left(-\frac{\pi}{2}\right)\cos^2\left(-\frac{\pi}{2}\right)$$
$$= \sin\left(\frac{\pi}{2}\right)\left(1 - 6\left(-\sin\frac{\pi}{2}\right)^2\right) - 0$$
$$= \sin\left(\frac{\pi}{2}\right)\left(1 - 6\left(\sin\frac{\pi}{2}\right)^2\right)$$
$$= 1(1 - 6(1)^2)$$
$$= 1 - 6 = -5$$

i.e. $f''\left(-\frac{\pi}{2}\right) < 0$

$\Rightarrow$ f is relative Maximum at $x = -\frac{\pi}{2}$

Put $\sin x = \frac{1}{\sqrt{6}}$ in III, we have

$$f''\left(\sin^{-1}\frac{1}{\sqrt{6}}\right) = -\frac{1}{\sqrt{6}}\left(1 - 6\left(\frac{1}{\sqrt{6}}\right)^2\right) - 12\,\frac{1}{\sqrt{6}}\cos^2 x$$
$$= -\frac{1}{\sqrt{6}}(1 - 1) - \frac{12}{\sqrt{6}}\cos^2 x$$
$$= -\frac{12}{\sqrt{6}}\cos^2 x$$

چونکہ $\cos^2 x$ کا جواب ہمیشہ مثبت آئے گا اس لیے جواب کے sign میں تبدیلی نہیں ہوگی یعنی -ve سے تو $\cos^2 x$ کی value رکھنے کے بعد بھی -ve ہی آئے گا اور اگر جواب +ve ہو تو $\cos^2 x$ کی value رکھنے کے بعد جواب +ve ہی آئے گا

i.e $f''(\sin^{-1}\frac{1}{\sqrt{6}}) < 0$

Thus f has relative Maximum at $x = \sin^{-1}\frac{1}{\sqrt{6}}$

Put $\sin x = -\frac{1}{\sqrt{6}}$ in II, we have

$$f''(\sin^{-1}(-\tfrac{1}{\sqrt{6}})) = -(-\tfrac{1}{\sqrt{6}})(1-6(-\tfrac{1}{\sqrt{6}})^2) - 12(-\tfrac{1}{\sqrt{6}})\cos^2 x$$

$$= \frac{1}{\sqrt{6}}(1-\frac{6}{6}) + \frac{12}{\sqrt{6}}\cos^2 x$$

$$= \frac{1}{\sqrt{6}}(1-1) + \frac{12}{\sqrt{6}}\cos^2 x$$

$$= \frac{12}{\sqrt{6}}\cos^2 x$$

i.e $f''(\sin^{-1}(-\frac{1}{\sqrt{6}})) > 0$

Thus f has relative Minimum at $x = \sin^{-1}(-\frac{1}{\sqrt{6}})$

Q.6. $f(x) = a\sec x + b\csc x \quad (0 < a < b)$ ——— I

$$\Rightarrow f'(x) = a\sec x\tan x - b\csc x\cot x$$

$$= \frac{a}{\cos x}\cdot\frac{\sin x}{\cos x} - \frac{b}{\sin x}\cdot\frac{\cos x}{\sin x}$$

$$= \frac{a\sin x}{\cos^2 x} - \frac{b\cos x}{\sin^2 x}$$

$$f'(x) = \frac{a\sin^3 x - b\cos^3 x}{\sin^2 x\cos^2 x} \quad \text{——— (2)}$$

For extreme values Put $f'(x) = 0$

$$\Rightarrow \frac{a\sin^3 x - b\cos^3 x}{\sin^2 x\cos^2 x} = 0$$

$$a\sin^3 x - b\cos^3 x = 0$$

$$a\sin^3 x = b\cos^3 x$$

$$\frac{\sin^3 x}{\cos^3 x} = \frac{b}{a}$$

$$\tan^3 x = \frac{b}{a}$$

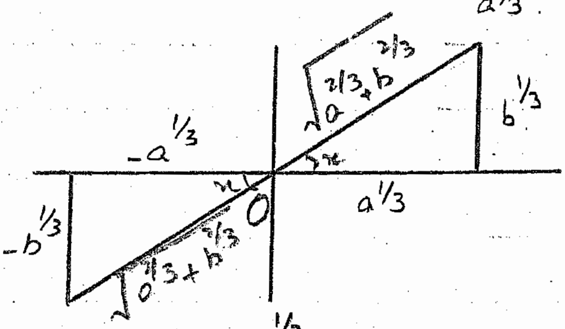
$$\tan x = (\frac{b}{a})^{1/3}$$

$$\tan x = \frac{b^{1/3}}{a^{1/3}}$$

نوٹ

چونکہ $\tan x = \frac{b^{1/3}}{a^{1/3}}$ ایک +ve رقم ہے
اور ہم جانتے ہیں کہ Tan پہلے اور تیسرے
Quadrant میں +ve ہوتا ہے اس لیے ہم
پہلے اور تیسرے Quadrant کے حساب
سے قیمتیں نکالیں گے۔

$$Tan = \frac{b^{1/3}}{a^{1/3}}$$



i.e. $$\sin x = \pm \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

and $$\cos x = \pm \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

$\because$ Tan x is +ve

$\therefore$ Both sin x and cos x have same sign. $Tan = \frac{-b^{1/3}}{-a^{1/3}}$

2) $\Rightarrow$ $$f''(x) = \frac{(\sin^2 x\cos^2 x)(3a\sin^2 x\cos x + 3b\cos^2 x\sin x)}{(\sin^2 x\cos^2 x)^2} + \text{terms involving } (a\sin^3 x - b\cos^3 x)$$

$$f''(x) = \frac{3a\sin^2 x\cos x + 3b\sin x\cos^2 x}{\sin^2 x\cos^2 x} + \text{terms involving } (a\sin^3 x - b\cos^3 x)$$

when Sin x and Cos x are both +ve

$f''(x)$ is +ve

$\Rightarrow$ f has relative Minimum at

$$\sin x = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}} \text{ and } \cos x = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

چونکہ +ve Tan x ہے اس لیے
اگر sin x اور cos x کی قیمتیں
Put کریں گے تو ایک ہی sign
والی Put کریں گے۔

And when Sin x and Cos x are both −ve

$f''(x)$ is −ve

$\Rightarrow$ f has relative Maximum at

$$\sin x = \frac{-b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}} \text{ and } \cos x = \frac{-a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$$

Q.7.

$$f(x) = \sin x\cos^2 x \qquad \text{——— I}$$

1) $\Rightarrow$ $$f'(x) = \cos x\cos^2 x - \sin x \cdot 2\cos x\sin x$$
$$= \cos x(\cos^2 x - 2\sin^2 x)$$
$$= \cos x(1 - \sin^2 x - 2\sin^2 x)$$
$$= \cos x(1 - 3\sin^2 x) \qquad \text{——— II}$$

put $f'(x) = 0$

$$\cos x(1 - 3\sin^2 x) = 0$$

$\Rightarrow$ $\cos x = 0$ or $1 - 3\sin^2 x = 0$

$x = \cos^{-1}(0)$ or $3\sin^2 x = 1$

$x = \pm\frac{\pi}{2}$ or $\sin^2 x = \frac{1}{3}$

$$\sin x = \pm\frac{1}{\sqrt{3}}$$

Now II $\Rightarrow$ $f''(x) = -\sin x(1-3\sin^2 x) + \cos x(0 - 6\sin x \cos x)$

$$= -\sin x(1-3\sin^2 x) - 6\sin x\cos^2 x$$

$$= -\sin x(1-3\sin^2 x) - 6\sin x(1-\sin^2 x)$$

<u>when $x = \frac{\pi}{2}$</u>

$$f''(\tfrac{\pi}{2}) = -\sin\tfrac{\pi}{2}\left(1-3(\sin(\tfrac{\pi}{2}))^2\right) - 6\sin(\tfrac{\pi}{2})\left(1-(\sin\tfrac{\pi}{2})^2\right)$$

$$= -1(1-3(1)^2) - 6(1)(1-(1)^2)$$

$$= -1(1-3) - 6(1-1)$$

$$= -1+3 - 0$$

$$= 2$$

i.e $f''(\frac{\pi}{2}) > 0$

$\Rightarrow$ f has relative Minimum at $x = \frac{\pi}{2}$

<u>when $x = -\frac{\pi}{2}$</u>

$$f''(-\tfrac{\pi}{2}) = -\sin(-\tfrac{\pi}{2})\left(1-3(\sin(-\tfrac{\pi}{2}))^2\right) - 6\sin(-\tfrac{\pi}{2})\left(1-(\sin(-\tfrac{\pi}{2}))^2\right)$$

$$f''(-\tfrac{\pi}{2}) = -(-1)(1-3(-1)^2) - 6(-1)(1-(-1)^2)$$

$$= 1(1-3) - 0$$

$$= -2$$

i.e $f''(-\frac{\pi}{2}) < 0$

$\Rightarrow$ f has relative Maximum at $x = -\frac{\pi}{2}$

<u>When $x = \sin^{-1}(\frac{1}{\sqrt{3}})$</u>

$$f''(\sin^{-1}\tfrac{1}{\sqrt{3}}) = -\frac{1}{\sqrt{3}}\left(1-3(\tfrac{1}{\sqrt{3}})^2\right) - 6\frac{1}{\sqrt{3}}\left(1-(\tfrac{1}{\sqrt{3}})^2\right)$$

$$= -\frac{1}{\sqrt{3}}\left(1-\frac{3}{3}\right) - \frac{6}{\sqrt{3}}\left(1-\frac{1}{3}\right)$$

$$= -\frac{1}{\sqrt{3}}(1-1) - \frac{6}{\sqrt{3}}\left(\frac{3-1}{3}\right)$$

$$= 0 - \frac{6}{3\sqrt{3}} \cdot 2$$

$$= -\frac{4}{\sqrt{3}}$$

i.e. $f''(\sin^{-1}\frac{1}{\sqrt{3}}) < 0$

$\Rightarrow$ f has relative Maximum at $x = \sin^{-1}(\frac{1}{\sqrt{3}})$

When $x = \sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

$$f''\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right) = -\left(-\frac{1}{\sqrt{3}}\right)\left(1-3\left(-\frac{1}{\sqrt{3}}\right)^2\right) - 6\left(-\frac{1}{\sqrt{3}}\right)\left(1-\left(-\frac{1}{\sqrt{3}}\right)^2\right)$$

$$= \frac{1}{\sqrt{3}}\left(1-\frac{3}{3}\right) + \frac{6}{\sqrt{3}}\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{\sqrt{3}}(1-1) + \frac{6}{\sqrt{3}}\left(\frac{3-1}{3}\right)$$

$$= 0 + \frac{6}{\sqrt{3}}\left(\frac{2}{3}\right)$$

$$= \frac{4}{\sqrt{3}}$$

i.e $f''\left(\sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right) > 0$

$\Rightarrow$ f has relative Minimum at $x = \sin^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

Q.8.

$$f(x) = e^x \cos(x-a) \quad \text{——— } I$$

$I \Rightarrow$ $f'(x) = e^x \cos(x-a) + e^x(-\sin(x-a)(1-0))$

$$f'(x) = e^x \cos(x-a) + e^x(-\sin(x-a))$$

$$f'(x) = e^x \cos(x-a) - e^x \sin(x-a)$$

Put $f'(x) = 0$ for extreme values

$$e^x \cos(x-a) - e^x \sin(x-a) = 0$$

$$e^x\left[\cos(x-a) - \sin(x-a)\right] = 0$$

$e^x \neq 0$ or $\cos(x-a) - \sin(x-a) = 0$

$$\cos(x-a) = \sin(x-a)$$

$$\frac{\sin(x-a)}{\cos(x-a)} = 1$$

$$\tan(x-a) = 1$$

$$\tan(x-a) = \tan\frac{\pi}{4} \text{ or } \tan\frac{5\pi}{4}$$

$$x-a = \frac{\pi}{4} \text{ or } x-a = \frac{5\pi}{4}$$

$$x = a+\frac{\pi}{4}, \quad x = a+\frac{5\pi}{4}$$

Put $x-a = \frac{\pi}{4}$ or $x = a+\frac{\pi}{4}$ in $f''(x)$.

$$f''(x) = -2e^x \sin(x-a)$$

$f''(x) = e^x \cos(x-a) - e^x \sin(x-a) + e^x \cos(x-a) + e^x \sin(x-a)$

$= -2e^x \sin(x-a)$

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$$f''(a+\pi/4) = -2e^{a+\pi/4}\sin(\pi/4)$$
$$= -2e^{a+\pi/4}\frac{1}{\sqrt{2}}$$

i.e. $f''(a+\pi/4) < 0$

$\Rightarrow$ f has relative Maximum at $x = a+\frac{\pi}{4}$

<u>When $x = a+\frac{5\pi}{4}$ or $x-a = \frac{5\pi}{4}$.</u>

$$f''(a+\tfrac{5\pi}{4}) = -2e^{a+5\pi/4}\sin(\tfrac{5\pi}{4})$$
$$= -2e^{a+5\pi/4}\left(-\frac{1}{\sqrt{2}}\right)$$
$$= 2e^{a+5\pi/4}\left(\frac{1}{\sqrt{2}}\right)$$

i.e. $f''(a+\frac{5\pi}{4}) > 0$

$\Rightarrow$ f has relative Minimum at $x = a+\frac{5\pi}{4}$

Q.9.

$$f(x) = x^x$$

Let $y = x^x$

$$\ln y = x\ln x \quad \text{———— I}$$

Diff (1) w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = \ln x + x\cdot\frac{1}{x}$$
$$= \ln x + 1$$
$$\frac{dy}{dx} = y(1+\ln x) \quad \text{———— A}$$
$$\frac{dy}{dx} = x^x(1+\ln x) \quad \text{———— II}$$

Put $\frac{dy}{dx} = 0$, For extremetics

$$x^x(1+\ln x) = 0$$

$x^x \neq 0$, $1+\ln x = 1$

$$\ln x = -1$$
$$x = e^{-1}$$
$$x = \frac{1}{e}$$

<u>نوٹ</u>

اگر e^{-1} ہے تو اس کا ln لینے سے اس کا جواب 1- آئے گا اسی طرح سے e^x یعنی کوئی بھی پاور ہو تو ln لینے سے جو پاور ہو گی وہی جواب آئے گا یعنی x،

مثال: $\ln e^{-1} = -1$

$\ln e^2 = 2$

$\ln e^0 = 0$

Differentiating A we get

$$\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x}\right) + \frac{dy}{dx}(1+\ln x)$$

$$= x^x\frac{1}{x} + y(1+\ln x)(1+\ln x)$$

$$= x^x\frac{1}{x} + y(1+\ln x)^2$$

$$= x^x\frac{1}{x} + x^x(1+\ln x)^2$$

Put $x = \frac{1}{e}$ or $\ln x = -1$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{1/e}\cdot\frac{1}{1/e} + \left(\frac{1}{e}\right)^{1/e}(1+(-1))^2$$

$$= \left(\frac{1}{e}\right)^{1/e}\cdot e + 0$$

$$= \left(\frac{1}{e}\right)^{1/e}\cdot e$$

i.e $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} > 0$

$\Rightarrow$ $f(x) = x^x$ has relative Minimum at $x = \frac{1}{e}$

Q.10 $f(x) = \frac{\ln x}{x}$, $0 < x < \infty$

Let $y = \frac{\ln x}{x}$ ———— I

Differentiating I w.r.t. 'x'

$$\frac{dy}{dx} = \frac{1}{x}\cdot\frac{1}{x} + \ln x\left(-\frac{1}{x^2}\right) \quad \text{(u.v for}$$

$$= \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2} \quad \text{———— II}$$

Put $\frac{dy}{dx} = 0$

$$\frac{1-\ln x}{x^2} = 0$$

$1-\ln x = 0 \Rightarrow 1 = \ln x \Rightarrow \ln x = 1$

$x = e$

Now Differentiating II, we have

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\cdot\left(-\frac{1}{x}\right) + (1-\ln x)\left(-\frac{2}{x^3}\right)$$

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$$\frac{d^2y}{dx^2} = -\frac{1}{x^3} - \frac{2(1-\ln x)}{x^3}$$

$$= \frac{-1-2(1-\ln x)}{x^3}$$

when $x = e$

$$\left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{-1-2(1-\ln e)}{(e)^3}$$

$$= \frac{-1-2(1-1)}{e^3}$$

$$= \frac{-1}{e^3}$$

i.e. $\left.\frac{d^2y}{dx^2}\right|_{x=e} < 0$

$\Rightarrow$ f has relative Maximum at $x = e$

Q.11. $r = 1 + \sin\theta$

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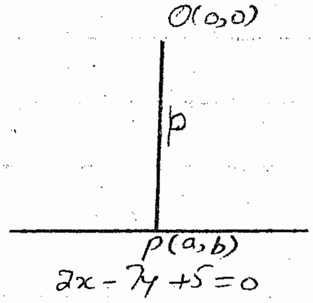
Q.12. Find the point on the straight line $2x-7y+5=0$ that is closest to the origin.



Sol: Let $P(a,b)$ be a point on the line which is closest to the origin.

Let the distance of $P(a,b)$ from the origin $= p$

Then by distance formula

$$p = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$p = \sqrt{(a-0)^2+(b-0)^2}$$

$$p = \sqrt{a^2+b^2} \quad \text{———} \quad I$$

$\because$ $P(a,b)$ lies on the line $2x-7y+5=0$ ——— II

$\therefore$ $2a-7b+5=0$

$$\Rightarrow \quad b = \frac{2a+5}{7} \quad \text{———} \quad III$$

III in I $\Rightarrow$

$$p^2 = a^2 + \left(\frac{2a+5}{7}\right)^2$$

$$p^2 = a^2 + \frac{4a^2+25+20a}{49}$$

$$49p^2 = 49a^2 + 4a^2 + 25 + 20a$$

$$49p^2 = 53a^2 + 20a + 25$$

Diff. w.r.t. 'a'

$$98p\frac{dp}{da} = 106a + 20 \quad \text{———} \quad IV$$

For extreme values put $\frac{dp}{da} = 0$

$$98p(0) = 106a + 20$$

$$106a + 20 = 0$$

$$106a = -20$$

$$a = \frac{-20}{106}$$

$$a = -\frac{10}{53} \quad \text{put in } III$$

$$7b = 2\left(-\frac{10}{53}\right) + 5$$

$$7b = -\frac{20}{53} + 5$$

$$7b = \frac{-20+265}{53}$$

$$7b = \frac{245}{53} \Rightarrow b = \frac{245}{371} \Rightarrow b = \frac{35}{53}$$

so the point is $P\left(-\frac{10}{53}, \frac{35}{53}\right)$

Now If $\frac{d^2p}{da^2}$ will +ve Then p will minimum.

Diff. II w.r.t. 'a'

$$98\left[\frac{dp}{da}\cdot\frac{dp}{da} + p\frac{d^2p}{da^2}\right] = 106$$

$$98\left[\left(\frac{dp}{da}\right)^2 + p\frac{d^2p}{da^2}\right] = 106$$

$$98\left(\frac{dp}{da}\right)^2 + 98p\frac{d^2p}{da^2} = 106$$

$$98p\frac{d^2p}{da^2} = 106 - 98\left(\frac{dp}{da}\right)^2$$

$$98p\frac{d^2p}{da^2} = 106 - 98\left[\frac{106a+20}{98p}\right]^2$$

$$98p\frac{d^2p}{da^2} = 106 - \frac{(106a+20)^2}{98p}$$

$$98p\frac{d^2p}{da^2} = 106 - \frac{1}{98p}\left[106a-20\right]^2$$

$$\frac{d^2p}{da^2} = \frac{106}{98p} - \frac{1}{(98p)^2}\left[106a-20\right]^2$$

$$\left.\frac{d^2p}{da^2}\right|_{P\left(-\frac{10}{53}, \frac{35}{53}\right)} = \frac{106}{98p} - \frac{1}{(98p)^2}\left[106\left(-\frac{10}{53}\right)-20\right]^2$$

$$= \frac{106}{98p} - \frac{1}{(98p)^2}\left[-20-20\right]^2$$

$$= \frac{106}{98p} - \frac{1}{(98p)^2}(-40)^2$$

$$= \frac{106}{98p} - \frac{1600}{(98p)^2}$$

$$\left.\frac{d^2p}{da^2}\right|_p = \frac{106}{98p} - \frac{1600}{9604p^2}$$

$$= \frac{10388p - 1600}{9604p^2}$$

i.e. $\left.\frac{d^2p}{da^2}\right|_p > 0$

$\Rightarrow$ p is minimum at $\left(-\frac{10}{53}, \frac{35}{53}\right)$

Q.13. Find the maxima and minima of the radius vectors of the curve.

$$\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta} \; ; \qquad a>0 \; , b>0.$$

$$\frac{c^4}{r^2} = a^2 \operatorname{Cosec}^2\theta + b^2 \sec^2\theta$$

$$c^4/r^2 = a^2(1+\cot^2\theta) + b^2(1+\tan^2\theta)$$

$$c^4/r^2 = a^2 + a^2\cot^2\theta + b^2 + b^2\tan^2\theta$$

$$c^4/r^2 = a^2 + b^2 + 2ab + a^2\cot^2\theta + b^2\tan^2\theta - 2ab$$

$$c^4/r^2 = (a+b)^2 + a^2\cot^2\theta + b^2\tan^2\theta - 2ab \tan\theta \cdot \frac{1}{\tan\theta}$$

$$c^4/r^2 = (a+b)^2 + a^2\cot^2\theta + b^2\tan^2\theta - 2ab \tan\theta \cot\theta.$$

$$c^4/r^2 = (a+b)^2 + (a\cot\theta - b\tan\theta)^2$$

$$\frac{r^2}{c^4} = \frac{1}{(a+b)^2 + (a\cot\theta - b\tan\theta)^2}$$

$$r^2 = \frac{c^4}{(a+b)^2 + (a\cot\theta - b\tan\theta)^2}$$

$$r = \frac{c^2}{\sqrt{(a+b)^2 + (a\cot\theta - b\tan\theta)^2}}$$

r is Max. if $a\cot\theta - b\tan\theta = 0$

$$\Rightarrow \quad r_{max} = \frac{c^2}{\sqrt{(a+b)^2 + 0}}$$

$$r_{max} = \frac{c^2}{a+b}$$

r is Min. if $a\cot\theta - b\tan\theta = \infty$

$$r_{min.} = \frac{c^2}{\sqrt{(a+b)^2 + \infty}}$$

نوٹ :-

اس سوال میں ہم a+b کو صفر کے
برابر نہیں رکھ سکتے کیونکہ یہ Contt. ہے
اور Max. بنانے کے لیے نیچے والی رقم
کو چھوٹی سے چھوٹی کرنا ہو گا۔ تو جواب
زیادہ r سے زیادہ آئے گا۔ اسی طرح Min.
بنانے کے لیے نیچے کی رقم کو بڑی سے بڑی
بنانا ہو گا جس سے جواب کم سے کم
آئے گا۔

$$r_{min} = \frac{c^2}{\infty}$$

$$r_{min.} = 0$$

Find the points of inflection of each of the following curves (14-17)

Q.14.

$$y = \frac{x^3 - x}{3x^2+1}$$

$$3x^2+1 \overline{\big) x^3 - x} \quad \tfrac{1}{3}x$$
$$\underline{-\,x^3 \mp \tfrac{1}{3}x}$$
$$-\tfrac{4}{3}x$$

Now $y = \frac{1}{3}x + \frac{-4/3\, x}{3x^2+1}$

$$y = \frac{1}{3}x - \frac{4}{3}\cdot\frac{x}{3x^2+1}$$

Diff w.r.t. 'x'

$$y' = \frac{1}{3} - \frac{4}{3}\cdot\frac{(3x^2+1)\cdot 1 - x(6x)}{(3x^2+1)^2}$$

$$= \frac{1}{3} - \frac{4}{3}\cdot\frac{3x^2+1-6x^2}{(3x^2+1)^2}$$

$$= \frac{1}{3} - \frac{4}{3}\cdot\frac{1-3x^2}{(3x^2+1)^2}$$

$$= \frac{1}{3} + \frac{4}{3}\cdot\frac{3x^2-1}{(3x^2+1)^2}$$

Diff. again w.r.t. 'x'

$$y'' = \frac{4}{3}\left[\frac{(3x^2+1)^2\cdot 6x - (3x^2-1)\cdot 2(3x^2+1)(6x)}{(3x^2+1)^4}\right]$$

$$= \frac{4}{3}\,6x(3x^2+1)\left[\frac{3x^2+1-2(3x^2-1)}{(3x^2+1)^4}\right]$$

$$= \frac{24}{3}x\left[\frac{3x^2+1-6x^2+2}{(3x^2+1)^3}\right]$$

$$= \frac{24}{3}x\left[\frac{3-3x^2}{(3x^2+1)^3}\right]$$

$$= \frac{24}{3}x\cdot 3\left(\frac{1-x^2}{(3x^2+1)^3}\right)$$

$$= 24x\left[\frac{(1-x)(1+x)}{(3x^2+1)^3}\right]$$

For pts. of inflection put $y_2 = 0$

$$24x(1-x)(1+x) = 0$$

$$x = 0,\ 1,\ -1$$

At $x=0$. $\quad y_2 = \dfrac{24x(1+x)(1-x)}{(3x^2+1)^3}$

$\Rightarrow$ y_2 is $-$ve just before $x=0$
and y_2 is $+$ve just after $x=0$
So $x=0$ is a point of inflection.
$(0,0)$ is a point of inflection.

At $x=1$.

y_2 is $+$ve just before $x=1$
and y_2 is $-$ve just after $x=1$
$\Rightarrow$ $(1,0)$ is a point of inflection.

At $x=-1$.

y_2 is $+$ve just before $x=-1$
y_2 is $-$ve just after $x=-1$
$\Rightarrow$ $(-1,0)$ is a point of inflection.

Q.15.

$x=(y-1)(y-2)(y-3)$

$x = y^3-6y^2+11y-6$ —— (1)

$$\begin{array}{r} y-1 \\ y-2 \\ \hline y^2-y \\ -2y+2 \\ \hline y^2-3y+2 \\ y-3 \\ \hline y^3-3y^2+2y \\ -3y^2+9y-6 \\ \hline y^3-6y^2+11y-6 \end{array}$$

$\dfrac{dx}{dy} = 3y^2-12y+11$

$\dfrac{d^2x}{dy^2} = 6y-12$

$\dfrac{d^3x}{dy^3} = 6$

For points of inflections put $\dfrac{d^2y}{dx^2}=0$

$6y-12=0$

$6y=12$

$\Rightarrow y=2$

Put $y=2$ in (1)

$x=(2)^3-6(2)^2+11(2)-6$

$x=0$

$(0,2)$ may be the points of inflection.
Let us check it out it.
Put $(0,2)$ in $\dfrac{d^3x}{dy^3}$

$$\Rightarrow \quad \frac{d^3x}{dy^3} \neq 0$$

So the (0,2) is a point of inflection.

Q.16. $y^2 = x(x+1)^2$ ——— I

Differentiating w.r.t. 'x'

$$2y\frac{dy}{dx} = (x+1)^2 + 2x(x+1)$$

$$2y\, dy/dx = (x+1)[x+1+2x]$$

$$dy/dx = (x+1)(3x+1)/2y$$

$\because y^2 = x(x+1)^2$

$y = \sqrt{x}\,(x+1)$

$$\frac{dy}{dx} = \frac{(x+1)(3x+1)}{2\sqrt{x}\,(x+1)}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3x+1}{\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{3x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(3\sqrt{x} + x^{-1/2}\right)$$

Differentiating again w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[3\cdot\frac{1}{2}\, x^{1/2-1} + \left(-\frac{1}{2}\, x^{-1/2-1}\right)\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{3}{2}\, x^{-1/2} - \frac{1}{2}\, x^{-3/2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\left[\frac{3}{\sqrt{x}} - \frac{1}{x^{3/2}}\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{4}\left[\frac{3x-1}{x^{3/2}}\right]$$

$$\Rightarrow \quad y'' = \frac{3x-1}{4x^{3/2}}$$

For point of inflection put $y''=0$

$$\Rightarrow \quad \frac{3x-1}{4x^{3/2}} = 0$$

$$\Rightarrow \quad 3x-1=0$$

$$\Rightarrow \quad 3x=1 \quad \Rightarrow \quad x = 1/3$$

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Put $x = 1/3$ in I

$$y^2 = \frac{1}{3}\left(\frac{1}{3}+1\right)^2$$

$$y^2 = \frac{1}{3}\left(\frac{1+3}{3}\right)^2$$

$$y^2 = \frac{1}{3}\left(\frac{4}{3}\right)^2$$

$$y^2 = \frac{1}{3}\cdot\frac{16}{9}$$

$$y^2 = \frac{16}{27}$$

$$\Rightarrow y = \pm\frac{4}{3\sqrt{3}}$$

Thus the possible points of inflection are

$$\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right) \text{ and } \left(\frac{1}{3}, -\frac{4}{3\sqrt{3}}\right)$$

If $x < \frac{1}{3}$, $y'' < 0$ and if $x > \frac{1}{3}$, $y'' > 0$

Thus $x = \frac{1}{3}$ gives points of inflection.

Hence $\left(\frac{1}{3}, \pm\frac{4}{3\sqrt{3}}\right)$

Q.17.

$$a^2y^2 = x^2(a^2-x^2) \text{ ——— I}$$

Differentiating w.r.t. 'x'

$$2a^2y\frac{dy}{dx} = 2x(a^2-x^2) + x^2(0-2x)$$

$$2a^2y\frac{dy}{dx} = 2x(a^2-x^2) - 2x^3$$

$$\frac{dy}{dx} = \frac{2x(a^2-x^2)-2x^3}{2a^2y}$$

$\because a^2y^2 = x^2(a^2-x^2)$

$\Rightarrow y = \frac{x\sqrt{a^2-x^2}}{\sqrt{a^2}}$

$$\frac{dy}{dx} = \frac{2x(a^2-x^2)-2x^3}{2a^2\frac{x\sqrt{a^2-x^2}}{\sqrt{a^2}}}$$

$$\frac{dy}{dx} = \frac{2x(a^2-x^2)-2x^3}{2ax\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \frac{2x\,(a^2-x^2)}{2ax\sqrt{a^2-x^2}} - \frac{2x^3}{2ax\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \frac{\sqrt{a^2-x^2}}{a} - \frac{x^2}{a\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \frac{a^2-x^2-x^2}{a\sqrt{a^2-x^2}}$$

$$\frac{dy}{dx} = \frac{a^2-2x^2}{a\sqrt{a^2-x^2}}$$

Again Differentiating w.r.t. 'x'

$$\frac{d^2y}{dx^2} = \frac{1}{a}\left[\frac{\sqrt{a^2-x^2}\,(0-4x) - (a^2-2x^2)\left\{\frac{1}{2}(a^2-x^2)^{-1/2}(-2x)\right\}}{(\sqrt{a^2-x^2})^2}\right]$$

$$= \frac{1}{a}\left[\frac{-4x\sqrt{a^2-x^2} - (a^2-2x^2)(-x\,(a^2-x^2)^{-1/2})}{a^2-x^2}\right]$$

$$= \frac{1}{a}\left[\frac{-4x\sqrt{a^2-x^2} + x(a^2-2x^2)/\sqrt{a^2-x^2}}{a^2-x^2}\right]$$

$$= \frac{1}{a}\left[\frac{\frac{-4(x)(a^2-x^2) + x(a^2-2x^2)}{\sqrt{a^2-x^2}}}{a^2-x^2}\right]$$

$$= \frac{1}{a}\left[\frac{-4x(a^2-x^2) + x(a^2-2x^2)}{\sqrt{a^2-x^2}\,(a^2-x^2)}\right]$$

$$= \frac{1}{a}\left[\frac{-4a^2x+4x^3+a^2x-2x^3}{(a^2-x^2)^{3/2}}\right]$$

$$= \frac{1}{a}\left[\frac{2x^3-3a^2x}{(a^2-x^2)^{3/2}}\right]$$

Put $\frac{d^2y}{dx^2} = 0$ for points of inflection.

$$\Rightarrow \frac{1}{a}\cdot\frac{2x^3-3a^2x}{(a^2-x^2)^{3/2}} = 0$$

$$2x^3-3a^2x = 0$$

$$x(2x^2-3a^2) = 0$$

$$\Rightarrow x = 0 \quad , \quad 2x^2-3a^2 = 0$$

$$2x^2 = 3a^2$$

$$x^2 = 3/2\,a^2$$

$$x = \pm a\sqrt{3/2}$$

Put $x=0$ in (1)

$$a^2y^2 = 0$$

$$\Rightarrow y=0$$

$\Rightarrow$ $(0,0)$ is possible point of inflection.

Put $x = -a\sqrt{\frac{3}{2}}$ in (1)

$$a^2y^2 = \left(-a\sqrt{3/2}\right)^2\left(a^2-\left(-a\sqrt{\tfrac{3}{2}}\right)^2\right)$$

$$a^2y^2 = a^2\frac{3}{2}\left(a^2 - \frac{3}{2}a^2\right)$$

$$a^2y^2 = \frac{3}{2}a^2\left(\frac{2a^2-3a^2}{2}\right)$$

$$a^2y^2 = \frac{3a^2}{2}\left(-\frac{a^2}{2}\right)$$

$$a^2y^2 = -\frac{3a^4}{4}$$

$$y^2 = -\frac{3a^4}{4}$$

$$y = \pm\sqrt{-\frac{3a^2}{4}}$$

which is imaginary

$\therefore$ we ignore this point

Similarly when we put $x = a\sqrt{\frac{3}{2}}$

it's answer will imaginary

Hence possible point of inflection is $(0,0)$

For $x<0$ and $x>0$ $\frac{d^2y}{dx^2}$ changes sign

therefore $(0,0)$ is a point of inflection.

Q.18. Find a and b so that the function f given by

$$f(x) = ax^3+bx^2$$

has $(1,6)$ as a point of inflection.

Soln. Here

$$f(x) = ax^3+bx^2 \quad \text{——— I}$$

Let $y = ax^3+bx^2$ ——— II

$\because$ $(1,6)$ is a point of inflection and it lies on

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the on the given curve

$\therefore \ \text{II} \Rightarrow \qquad 6 = a + b \qquad \text{———} \quad \text{III}$

Diff. II w.r.t. 'x'

$$y_1 = 3ax^2 + 2bx$$
$$y_2 = 6ax + 2b$$

For point of inflection put $y_2 = 0$

$$\Rightarrow \quad 6ax + 2b = 0$$

at $(1, 6)$

$$6a + 2b = 0 \qquad \text{———} \quad \text{IV}$$

$\text{IV} \Rightarrow \qquad b = -3a$

Put $b = -3a$ in III

$$6 = a - 3a$$
$$6 = -2a$$
$$\Rightarrow \quad a = -3$$

Put $a = -3$ in III

$$6 = -3 + b$$
$$\Rightarrow \quad b = 9$$

Hence $a = -3$ and $b = 9$

Q.19 Find the intervals in which the curve $y = 3x^5 - 40x^3 + 3$ faces.

(i) upward (ii) downward. Also find the point of inflection.

Soln: Here $y = 3x^5 - 40x^3 + 3x - 20 \qquad \text{———} \quad \text{I}$

Diff. w.r.t. 'x', we have

$$y_1 = 15x^4 - 120x^2 + 3$$
$$y_2 = 60x^3 - 240x \qquad \text{———} \quad \text{II}$$

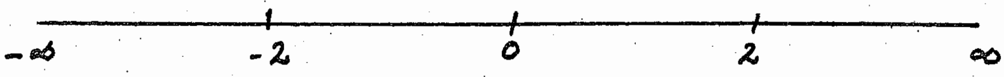
Now put $y_2 = 0$ for faces up and down.

$$y_2 = 0$$
$$60x^3 - 240x = 0$$
$$x(60x^2 - 240) = 0$$

$\Rightarrow \qquad x = 0 \quad , \quad 60x^2 - 240 = 0$

$$60x^2 = 240$$
$$x^2 = 4$$
$$x = \pm 2$$

$\Rightarrow \quad x = 0, 2, -2$



The possible open intervals are

$]-\infty, -2[\ ,\ ]-2, 0[\ ,\ ]0, 2[\ ,\ ]2, \infty[$

Now for curve concave up or down we check these intervals

For interval $]-\infty, -2[$ i.e $x < -2$

put in y_2

$$y_2 = 60(-4)^3 - 240(-4)$$
$$= -3840 + 960$$
$$= -2880$$

i.e $y_2 < 0$

$\Rightarrow$ The curve concave down in $]-\infty, -2[$

For interval $]-2, 0[$ i.e $-2 < x < 0$

$$\Rightarrow y_2 = 60(-1)^3 - 240(-1)$$
$$= -60 + 240$$
$$= 180$$

i.e $y_2 > 0$

$\Rightarrow$ The curve concave upward in $]-2, 0[$

For $]0, 2[$ i.e $0 < x < 2$

$$\Rightarrow y_2 = 60(1)^3 - 240(1)$$
$$= 60 - 240$$
$$= -180$$

i.e. $y_2 < 0$

$\Rightarrow$ The curve down in $]0, 2[$

For $]2, \infty[$ i.e $2 < x < \infty$

$$\Rightarrow y_2 = 60(3)^3 - 240(3)$$
$$= 1620 - 720$$
$$= 900$$

i.e $y_2 > 0$

$\Rightarrow$ The curve concave up in $]2, \infty[$

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For points of inflection
put $y_2 = 0$, we have

$$x = 0, 2, -2$$

put $x = 0$ in I

$$\Rightarrow \quad y = 3(0)^5 - 40(0)^3 + 3(0) - 20$$

$$\Rightarrow \quad y = -20$$

i.e. $(0, -20)$ is a possible point of inflection

put $x = 2$ in I

$$\Rightarrow \quad y = 3(2)^5 - 40(2)^3 + 3(2) - 20$$

$$= 96 - 320 + 6 - 20$$

$$= -238$$

i.e $(2, -238)$ is a possible point of inflection.

put $x = -2$

$$\Rightarrow \quad y = 3(-2)^5 - 40(-2)^3 + 3(-2) - 20$$

$$= -96 + 320 - 6 - 20$$

$$= 198$$

i.e $(-2, 198)$ is a possible point of inflection.

Diff. again II w.r.t. 'x', we have

$$y_3 = 180x^2 - 240$$

when $x = 0$ or $(0, -20)$

$$y_3 = 180(0)^2 - 240$$

$$= -240 \neq 0$$

i.e $(0, -20)$ is a point of inflection.

when $x = 2$ or $(2, -238)$

$$y_3 = 180(2)^2 - 240$$

$$= 720 - 240$$

$$= 480 \neq 0$$

i.e $(2, -238)$ is a point of inflection.

when $x = -2$, or $(-2, 198)$

$$y_3 = 180(-2)^2 - 240$$

$$= 720 - 240$$

$$= 480 \neq 0$$

i.e $(-2, 198)$ is a point of inflection.

Hence the points of inflection are
$(0, -20)$, $(2, -238)$, $(-2, 198)$

Q.20. Find the intervals in which the curve $y = (x^2+4x+5)\,e^{-x}$ faces upward or downward. Also find it's point of inflection.

Soln. Here

$$y = (x^2+4x+5)\,e^{-x} \qquad \text{--- I}$$

$$y_1 = (x^2+4x+5)\,e^{-x}(-1) + e^{-x}(2x+4)$$

$$y_1 = e^{-x}\left[-x^2-4x-5+2x+4\right]$$

$$y_1 = e^{-x}\left[-x^2-2x-1\right]$$

$$y_2 = e^{-x}\left[-2x-2\right] + e^{-x}(-1)\left[-x^2-2x-1\right]$$

$$y_2 = e^{-x}\left[-2x-2\right] - e^{-x}\left[-x^2-2x-1\right]$$

$$y_2 = e^{-x}\left[-2x-2+x^2+2x+1\right]$$

$$y_2 = e^{-x}\left[x^2-1\right] \qquad \text{--- II}$$

Put $y_2 = 0$ for faces up and down.

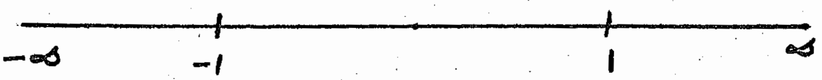
$$\Rightarrow \quad e^{-x}(x^2-1) = 0$$

$$\because e^{-x} \neq 0 \quad , \quad x^2-1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$\Rightarrow \quad x = 1, -1$$



The possible open intervals are

$$]-\infty, -1[\;,\quad]1, \infty[\;,\;]-1, 1[$$

For $]-\infty, -1[$ i.e $x < -1$

$$y_2 = e^{-(-2)}\left((-2)^2-1\right)$$

$$= e^2\,(4-1)$$

$$= 3e^2 > 0$$

$\Rightarrow$ The curve up in $]-\infty, -1[$

For $]-1, 1[$ i.e $-1 < x < 1$

$$y_2 = e^{-(0)}\left((0)^2-1\right)$$

$$= 1(-1)$$

$$= -1 < 0$$

$\Rightarrow$ The curve concave down in $]-1, 1[$

For $]1, \infty[$

$$y_2 = e^{-2}((2)^2 - 1)$$
$$y_2 = e^{-2}(4-1)$$
$$y_2 = 3e^{-2}$$
$$y_2 = 3/e^2 > 0$$

$\Rightarrow$ The curve is concave up in $]1, \infty[$

Now for points of inflection if we put $y_2 = 0$, we will get

$$x = 1, -1$$

Diff. again II w.r.t. 'x', we have

$$y_3 = e^{-x}(-1)[x^2-1] + e^{-x}(2x)$$
$$y_3 = e^{-x}[-x^2+1+2x]$$
$$y_3 = e^{-x}[-x^2+1+2x] \quad \text{———— III}$$

Put $x = 1$ in I

$$y = [(1)^2 + 4(1) + 5]e^{-1}$$
$$y = 10e^{-1}$$
$$y = 10/e$$

$\Rightarrow$ $(1, 10/e)$ is a possible point of inflection.

Now put $x = -1$ in I

$$y = [(-1)^2 + 4(-1) + 5]e^{-(-1)}$$
$$y = [1 - 4 + 5]e$$
$$y = 2e$$

$\Rightarrow$ $(-1, 2e)$ is a possible point of inflection

Now we check these possible points of inflection.

when $x = 1$, or $(1, 10/e)$

$$y_3 = e^{-1}[-(1)^2 + 1 + 2(1)]$$
$$= e^{-1}[-1 + 1 + 2]$$
$$= 2e^{-1}$$

i.e. $y_3 \neq 0$

$\Rightarrow$ $(1, 10/e)$ is a point of inflection.

Now when $x = -1$, or $(-1, 2e)$

$$y_3 = e^{-(-1)}[-(-1)^2 + 1 + 2(-1)]$$
$$= e[-1 + 1 - 2]$$

$y_3 = -2e$

i.e $y_3 \neq 0$

$\Rightarrow$ $(-1, 2e)$ is the pt. of inflection.

Hence the points of inflection are

$(1, 10/e)$ and $(-1, 2e)$

Q.21. Use calculus to show that $5x^2 - 20x + 81 > 0$ for all x.

Soln:

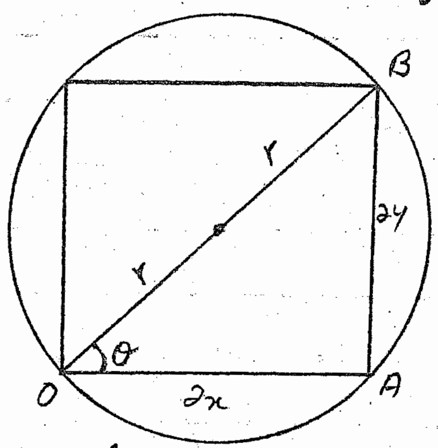
Q.22. Show that $x^4 - 4x^3 - 12x^2 + 40 > 0$

Q.23. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius 'r'.

Soln. Let $2x$ and $2y$ be the sides of a rectangle.

Then area of the triangle is given by

$$A = 2x \cdot 2y$$
$$A = 4xy \quad \text{———} \quad I$$



Now from ΔOAB,

$$(2x)^2 + (2y)^2 = (r+r)^2$$
$$4x^2 + 4y^2 = 4r^2$$
$$\Rightarrow \quad x^2 + y^2 = r^2$$

Changing the above eq I in polar form

i.e. $x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow$

$$A = 4r\cos\theta \cdot r\sin\theta$$
$$A = 4r^2\cos\theta\sin\theta.$$
$$A = 2r^2 \, 2\cos\theta\sin\theta$$
$$A = 2r^2\sin 2\theta$$

Diff. w.r.t. 'θ'

$$\frac{dA}{d\theta} = \frac{d(2r^2\sin 2\theta)}{d\theta}$$
$$= 2r^2\cos 2\theta \cdot 2$$
$$= 4r^2\cos 2\theta.$$

For extreme values

$$\frac{dA}{d\theta} = 0$$
$$4r^2\cos 2\theta = 0$$
$$\cos 2\theta = 0$$
$$2\theta = 90^\circ$$
$$\theta = 45^\circ$$

Dimension کا مطلب ہوتا ہے کہ لمبائی اور
چوڑائی معلوم کریں ۔ اس سوال میں r یعنی
radius کے بارے میں کہا گیا ہے اس لیے
یہ given ہے لہذا اسے ہم Constant
کہیں گے ۔ ویسے بھی r Constant ہی ہے ۔
اگر کسی چیز کا Area یا کد اور چیز Max.
اور Min ہو تو اس کو اوپر اور جو معلوم
کرنی ہو اس کو نیچے رکھیں ۔ پھر
Dimension والا سوال اس طرح سے حل
کیا جا سکتا ہے ۔ اور اس کو صفر کے برابر
لکھتے ہیں ۔ ہم اس سوال کو اور طریقوں
سے بھی حل کر سکتے ہیں ۔ ہم اس کی Sides
2x اور 2y لے گئے اس سے سوال
آسان ہوجاتا ہے ۔

$$\frac{d(\text{Area})}{d(\text{Dimension})} = 0 \qquad \because r = \text{Const.}$$

Now $\dfrac{d^2A}{d\theta^2} = 4r^2(-\sin 2\theta)\cdot 2$

$$= -8r^2 \sin 2(45^\circ) \qquad \because \theta = 45^\circ$$

$$= -8r^2 \sin 90^\circ$$

$$= -8r^2$$

$\Rightarrow$ The Area is maximum at $\theta = 45^\circ$

Now we find dimensions of the rectangle

i.e. $2x|_\theta = 2r\cos\theta$

$$2x|_\theta = 2r\cos 45^\circ$$

$$2x|_\theta = 2r\frac{1}{\sqrt{2}}$$

$$2x|_\theta = \sqrt{2}\, r$$

یہاں 2 کو cancel نہیں کرنا ہے
کیونکہ rectangle کی side
2x کے برابر ہے اگر اس
کو ختم کر دیں گے تو side
آدھی رہ جائے گی۔

and $2y = 2r\sin\theta$

$$2y|_\theta = 2r\sin 45^\circ$$

$$= 2r\frac{1}{\sqrt{2}}$$

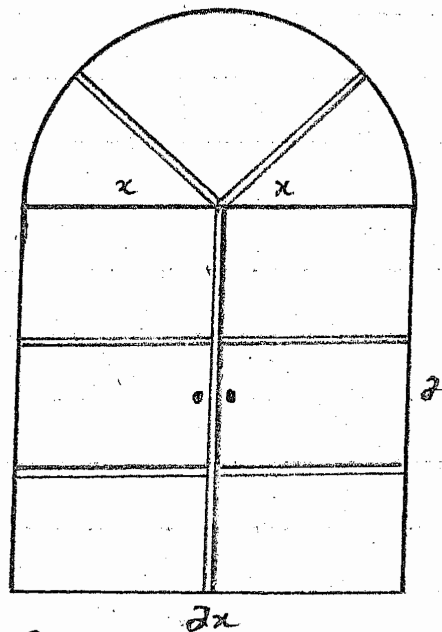
$$= \sqrt{2}\, r$$

Hence the dimensions of the rectangle are

$\sqrt{2}\, r,\ \sqrt{2}\, r.$

Q.24. A window has the shape of a rectangle surmounted by a semi circle. Find the dimensions that maximize the area of the window if it's perimeter is m meters.

Soln. Let $2x$ and $2y$ be the lengths of the sides of the rectangle. And m is the perimeter.



Then $m = 2x + 2x + 2y + 2y + \pi x$

$$m = 4x + 4y + \pi x \quad \text{—— I}$$

Area of the window

$$A = 2x\cdot 2y + \frac{\pi x^2}{2}$$

$$A = 4xy + \frac{\pi x^2}{2} \quad \text{—— (2)}$$

From I $\quad 4y = m - (4x + \pi x)$

$$y = \frac{m}{4} - x + \frac{\pi x}{4} \quad \text{—— (3)}$$

Put in (2)

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$$A = 4x\left[\frac{m}{4} - x - \frac{\pi x}{4}\right] + \frac{\pi x^2}{2}$$

$$A = mx - 4x^2 - \pi x^2 + \frac{\pi x^2}{2}$$

$$A = mx - 4x^2 - \left(\frac{2\pi x^2 - \pi x^2}{2}\right)$$

$$A = mx - 4x^2 - \frac{\pi x^2}{2}$$

Now $$\frac{dA}{dx} = m - 8x - \frac{2\pi x}{2}$$

$$\frac{dA}{dx} = m - 8x - \pi x$$

Put $\frac{dA}{dx} = 0$ for extreme values

i.e. $$m - 8x - \pi x = 0$$

$$m - x(8+\pi) = 0$$

$$x(8+\pi) = m$$

$$x = \frac{m}{8+\pi}$$

Put in (3)

$$y = \frac{1}{4}\left[m - 4x - \pi x\right]$$

$$y = \frac{1}{4}\left[m - 4\left(\frac{m}{8+\pi}\right) - \pi\left(\frac{m}{8+\pi}\right)\right]$$

$$y = \frac{1}{4}\left[\frac{m}{8+\pi}(8+\pi-4-\pi)\right]$$

$$y = \frac{1}{4}\left[\frac{m}{8+\pi}(4)\right]$$

$$y = \frac{m}{8+\pi}$$

Now $$\left.\frac{d^2A}{dx^2}\right|_x = 0 - 8 - \pi$$

$$= -8 - \pi$$

$$= -(8+\pi) < 0$$

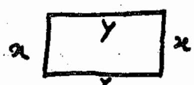
So area of the window is maximum at $x = \frac{m}{8+\pi}$

So the required dimensions are

$$2\left(\frac{m}{8+\pi}\right), \; 2\left(\frac{m}{8+\pi}\right)$$

ہم اس سوال میں ایک ایسی کھڑکی
کھڑکی کی dimensions معلوم
کرنا چاہتے ہیں جو ایک مستطیل
اور آدھے circle پر مشتمل ہے
perimeter احاطے کو کہتے ہیں اور
اس کھڑکی کا احاطہ ہم یوں معلوم
کرتے ہیں۔

مستطیل کا احاطہ برابر ہوتا ہے
اس کے چاروں اضلاع کے مجموعہ
کے برابر یعنی



احاطہ = $x + y + x + y$

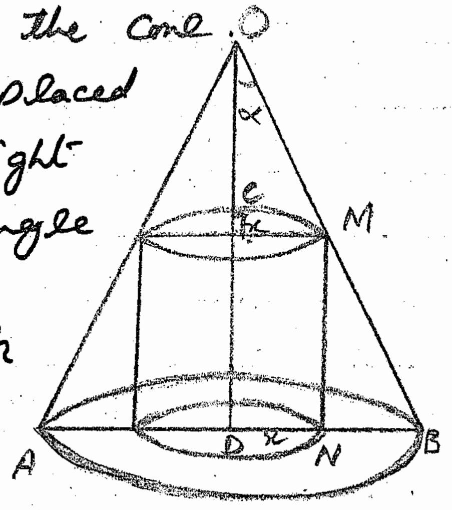
اور چونکہ پورے circle
کی لمبائی $2\pi x$ کے برابر ہوتی
ہے اس لیے آدھے circle کی
لمبائی πx کے برابر ہوگی۔
اس طرح پوری کھڑکی کا احاطہ
برابر ہوگا مستطیل کے چاروں اضلاع
کی لمبائی کا مجموعہ جمع آدھے
circle کی لمبائی یعنی

احاطہ = $x + y + x + y + 2\pi x$

اسی طرح اس کھڑکی کا ایریا برابر
ہوگا مستطیل کا ایریا جمع آدھے
circle کا Area یعنی

$$A = xy + \frac{\pi x^2}{2}$$

Q.25. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.



Soln: Let a cylinder of base radius x be placed in a cone of base radius y and height h. Let α be the semi vertical angle of the cone.

Then $DN = x$, $DB = y$, $OD = h$

From right triangle OCM

$$\frac{OC}{CM} = \cot\alpha$$

$$OC = CM\cot\alpha$$

$$OC = x\cot\alpha \quad \text{—— I}$$

Now $CD = OD - OC$

$$CD = h - x\cot\alpha \quad \text{—— II}$$

Let S be the curved surface Area of the cylinder then

$$S = 2\pi x \cdot CD$$

$$S = 2\pi x(h - x\cot\alpha)$$

$$S = 2\pi h x - 2\pi x^2\cot\alpha$$

Diff. w.r.t. 'x'

$$\frac{ds}{dx} = 2\pi h - 4\pi x\cot\alpha$$

$$\frac{d^2S}{dx^2} = -4\pi\cot\alpha.$$

For R. Max. or R. Min put $\frac{ds}{dx} = 0$

i.e. $2\pi h - 4\pi x\cot\alpha = 0$

$$2\pi h = 4\pi x\cot\alpha$$

$$h = 2x\cot\alpha$$

$$x = \frac{h}{2\cot\alpha}$$

$$x = \frac{h}{2}\tan\alpha. \quad \text{—— III}$$

$\because \frac{d^2s}{dx^2} < 0$ at $x = \frac{h}{2}\tan\alpha$

$\therefore$ S is R. Max at $x = \frac{h}{2}\tan\alpha$.

From r.t. Δ ODB

$$\frac{BD}{OD} = \tan\alpha$$

$$BD = OD\tan\alpha$$

$$BD = h\tan\alpha$$

$$y = h\tan\alpha \quad \text{——— IV}$$

from III and IV

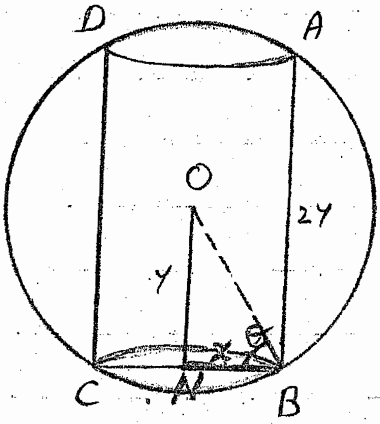
$$x = \frac{1}{2}y$$

i.e the radius of the cylinder is half the radius of the cone.

Q26. Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r.

Soln: Let a cylinder with dimension $2x$ and $2y$ be placed in the sphere of radius r as shown in the Fig.

And radius of the base of the cylinder is x. as shown in the Fig.



Height $2y = AB$

Let $O\hat{B}C = \theta$

then from r.t. Δ ONB

$$\frac{BN}{OB} = \cos\theta$$

$$x = r\cos\theta$$

and $$ON = r\sin\theta \qquad \because ON = y$$

$$\Rightarrow y = r\sin\theta$$

Let S be the surface area of the cylinder then

S = Area of the Top + Area of the middle portion + Area of the base.

$$S = \pi x^2 + 4\pi xy + \pi x^2$$
$$= 2\pi x^2 + 4\pi xy$$
$$= 2\pi\, r^2\cos^2\theta + 4\pi\, r\cos\theta\, r\sin\theta$$
$$= 2\pi r^2(\cos^2\theta + 2\sin\theta\cos\theta)$$
$$= \pi r^2(2\cos^2\theta + 2\sin 2\theta)$$
$$= \pi r^2(1 + \cos 2\theta + 2\sin 2\theta) \quad \text{—— I}$$

Now Diff. w.r.t. 'θ'

$$\frac{dS}{d\theta} = \pi r^2(0 - \sin 2\theta.2 + 2\cos 2\theta.2)$$

$$\frac{dS}{d\theta} = \pi r^2(-2\sin 2\theta + 4\cos 2\theta)$$

(For R. Max and R. Min) put $\frac{dS}{d\theta} = 0$

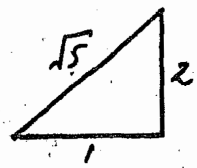
$$\pi r^2(-2\sin 2\theta + 4\cos 2\theta) = 0$$

$\pi r^2 \neq 0$, $\quad -2\sin 2\theta + 4\cos 2\theta = 0$

$$\Rightarrow \quad 2\sin 2\theta = 4\cos 2\theta$$

$$\frac{\sin 2\theta}{\cos 2\theta} = 2$$

$$\tan 2\theta = 2$$



$$\Rightarrow \quad \sin 2\theta = \frac{2}{\sqrt{5}}$$

$$\cos 2\theta = \frac{1}{\sqrt{5}}$$

Now $\quad \frac{d^2S}{d\theta^2} = \pi r^2(-2\cos 2\theta.2 + 4(-\sin 2\theta.2))$

$$= \pi r^2(-4\cos 2\theta - 8\sin 2\theta)$$

$$\left.\frac{d^2S}{d\theta^2}\right|_p = \pi r^2\left(-4.\frac{1}{\sqrt{5}} - \frac{8.2}{\sqrt{5}}\right)$$

$$= \pi r^2\left(-\frac{4}{\sqrt{5}} - \frac{16}{\sqrt{5}}\right)$$

$$= \pi r^2\left[-\left(\frac{4+16}{\sqrt{5}}\right)\right]$$

$$= \pi r^2\left(-\frac{20}{\sqrt{5}}\right) \Rightarrow \frac{d^2S}{d\theta^2} = -\frac{20}{\sqrt{5}}\pi r^2$$

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$\Rightarrow$ Surface is Maximum at $\tan 2\theta = 2$

<u>Maximum Surface Area</u>

$$S = \pi r^2\left(1 + \frac{1}{\sqrt5} + 2\cdot\frac{2}{\sqrt5}\right)$$

$$S = \pi r^2\left(\frac{\sqrt5 + 1 + 4}{\sqrt5}\right)$$

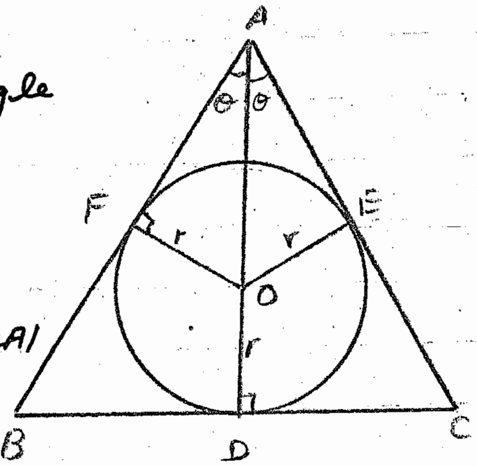
$$S = \pi r^2\left(\frac{\sqrt5 + 5}{\sqrt5}\right)$$

$$S = \pi r^2\left(\frac{\sqrt5}{\sqrt5} + \frac{5}{\sqrt5}\right)$$

$$S = \pi r^2(1 + \sqrt5)$$

Is the required greatest surface.

Q.27. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt3$.



Soln. Let ABC be an isosceles triangle with $|AB| = |AC|$

The perimeter of triangle is

$$p = |AB| + |BC| + |CA|$$

$$p = |AB| + |BC| + |AB| \qquad \because |AB| = |CA|$$

$$p = 2|AB| + |BC|$$

Now $\because |AB| = |AF| + |FB|$

$$\therefore \quad p = 2(|AF| + |FB|) + |BC|$$

and $|BC| = |BD| + |DC|$

and $|BD| = |DC|$

$$\therefore \quad |BC| = |BD| + |BD|$$

$$= 2|BD|$$

So $p = 2(|AF| + |FB|) + 2|BD|$

$\because |FB| = |BD|$ or $|DC|$

$$\therefore \quad p = 2(|AF| + |BD|) + 2|BD|$$

$$p = 2|AF| + 2|BD| + 2|BD|$$

$$p = 2|AF| + 4|BD|$$

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From Δ AOF,

$$\frac{|OF|}{|AF|} = \tan\theta ,$$

$$r = |AF| \tan\theta$$

$$\Rightarrow |AF| = r\cot\theta$$

Now From Δ ABD

$$\frac{|AD|}{|BD|} = \cot\theta$$

$$|AD| = |BD| \cot\theta$$

$$\Rightarrow |BD| = |AD| \tan\theta$$

Now $p = 2r\cot\theta + 4|AD| \tan\theta$

$$p = 2r\cot\theta + 4(|AO| + |OD|)\tan\theta$$

$$p = 2r\cot\theta + 4(|AO| + r)\tan\theta$$

Now From Δ AOF

$$\frac{|OF|}{|AO|} = \sin\theta$$

$$r\csc\theta = |AO|$$

So $p = 2r\cot\theta + 4(r\csc\theta + r)\tan\theta$

$$p = 2r\cot\theta + 4r(\csc\theta\tan\theta + \tan\theta)$$

$$p = 2r\cot\theta + 4r(\sec\theta + \tan\theta)$$

Now $\frac{dp}{d\theta} = 2r(-\csc^2\theta) + 4r(\sec\theta\tan\theta + \sec^2\theta)$

$$= -\frac{2r}{\sin^2\theta} + 4r\left(\frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos^2\theta}\right)$$

$$= -\frac{2r}{\sin^2\theta} + 4r\left(\frac{\sin\theta + 1}{\cos^2\theta}\right)$$

$$\frac{dp}{d\theta} = \frac{-2r\cos^2\theta + 4r\sin^3\theta + 4r\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

put $\frac{dp}{d\theta} = 0$ for extrema

So $\frac{-2r\cos^2\theta + 4r\sin^3\theta + 4r\sin^2\theta}{\sin^2\theta\cos^2\theta} = 0$

$$-2r\cos^2\theta + 4r\sin^3\theta + 4r\sin^2\theta = 0$$

Written by
Shahid Javed

$$-\cos^2\theta + 2\sin^3\theta + 2\sin^2\theta = 0$$

$$-(1-\sin^2\theta) + 2\sin^2\theta(\sin\theta + 1) = 0$$

$$-(1+\sin\theta)(1-\sin\theta) + 2\sin^2\theta(1+\sin\theta) = 0$$

$$(1+\sin\theta)\left[-(1-\sin\theta) + 2\sin^2\theta\right] = 0$$

$1+\sin\theta = 0$, $\qquad -1+\sin\theta + 2\sin^2\theta = 0$

$\sin\theta = -1$ $\qquad 2\sin^2\theta + \sin\theta - 1 = 0$

$\theta = \frac{3\pi}{2}, -\frac{\pi}{2}$ $\qquad 2\sin^2\theta + 2\sin\theta - \sin\theta - 1 = 0$

$$2\sin\theta(\sin\theta+1) - 1(\sin\theta+1) = 0$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0$$

$\Rightarrow 2\sin\theta - 1 = 0, \qquad \sin\theta + 1 = 0$

$\sin\theta = \frac{1}{2} \qquad \sin\theta = -1$

$\theta = \pi/6 \qquad \theta = \frac{3\pi}{2}, -\frac{\pi}{2}$

So $\theta = \frac{3\pi}{2}, +\frac{\pi}{6}, -\frac{\pi}{2}$

But $\theta = \frac{3\pi}{2}, -\frac{\pi}{2}$ is inadmissible

Hence $\theta = \pi/6$

نوٹ:- چونکہ Triangle میں آدھا زاویہ 90° یا 270° ہے تو ان زاویہ سے Triangle نہیں بنتے اور اگر زاویہ 90° ہیں اور اس کو ڈبل کریں تو تو 180° کے برابر ہوتا ہے مگر Triangle کے تمام زاویوں کا مجموعہ 180° کے برابر ہوتا ہے اس لیے ہم 90° یا 270° θ کی قیمت نہیں لیں گے۔

Now

$$\frac{dp}{d\theta} = \left[-2r\cos^2\theta + 4r\sin^3\theta + 4r\sin^2\theta\right]\frac{1}{(\sin\theta\cos\theta)^2}$$

$$\Rightarrow \frac{d^2p}{d\theta^2} = (2r\,2.\cos\theta\sin\theta + 4r.3\sin^2\theta\cos\theta + 4r.2\sin\theta\cos\theta)\cdot\frac{1}{(\sin\theta\cos\theta)^2}$$

$$+\left[-2r\cos^2\theta + 4r\sin^3\theta + 4r\sin^2\theta\right]\cdot\frac{-2(\cos\theta.\cos\theta - \sin\theta\sin\theta)}{(\sin\theta\cos\theta)^3}$$

$$\Rightarrow \frac{d^2p}{d\theta^2} = \left\{4r\sin\theta\cos\theta + 12r\sin^2\theta\cos\theta + 8r\sin\theta\cos\theta\right\}\frac{1}{(\sin\theta\cos\theta)^2}$$

$$-\left[-2r\cos^2\theta + 4r\sin^3\theta + 4r\sin^2\theta\right]\frac{(\cos^2\theta - \sin^2\theta)}{(\sin\theta\cos\theta)^2}$$

$$\left.\frac{d^2p}{d\theta^2}\right|_{\theta=\pi/6} = \left\{4r\sin 30^\circ\cos 30^\circ + 12r\sin^2 30^\circ\cos 30^\circ + 8r\sin 30^\circ\cos 30^\circ\right]\frac{1}{(\sin 30^\circ\cos 30^\circ)^2}$$

$$-\left[-2r\cos^2 30^\circ + 4r\sin^3 30^\circ + 4r\sin^2 30^\circ\right]\frac{(\cos^2 30^\circ - \sin^2 30^\circ)}{(\sin 30^\circ\cos 30^\circ)^2}$$

$$\left.\frac{d^2p}{d\theta^2}\right|_{\theta=\pi/6} = [1.732r + 2.598r + 3.464r]\frac{1}{.187} - [-1.5r + .5r + 1r]\cdot\left[\frac{.75-.25}{.187}\right]$$

$$= 41.68r - 0$$
$$= 41.68r$$

i.e $\frac{d^2p}{d\theta^2} > 0$

$\Rightarrow$ p is minimum at $\theta = \pi/6$

So minimum value of p is

$$p = 2r\cot\theta + 4r(\sec\theta + \tan\theta)$$
$$= 2r\cot 30^\circ + 4r(\sec 30^\circ + \tan 30^\circ)$$
$$= 2r\cdot\sqrt{3} + 4r\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right)$$

$$= 2r\sqrt{3} + 4r\left(\frac{2+1}{\sqrt{3}}\right)$$
$$= 2r\sqrt{3} + 4r\left(\frac{3}{\sqrt{3}}\right)$$
$$= 2r\sqrt{3} + 4r\sqrt{3}$$
$$= \sqrt{3}(2r + 4r)$$
$$= 6r\sqrt{3}$$

Is as required.

Q.28. A cone is circumscribed to a sphere of radius r. Show that when the volume of the cone is minimum, it's altitude is 4r and it's semi vertical angle is $\sin^{-1}(1/3)$.

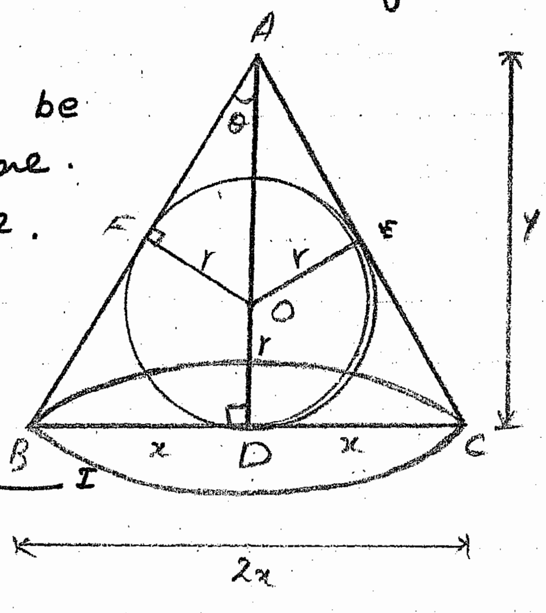
Soln: Let ABC be a cone. Let x be the radius of the base of the cone. And y be the altitude of the cone.

i.e $AD = y$

$\because |OA| = |AD| - |OD|$

$|OA| = y - r$

Now volume of the cone $= \frac{1}{3}\pi x^2 y$ ——— I



Now from ΔABD

$$\frac{BD}{AD} = \tan\theta$$

$\Rightarrow \frac{x}{y} = \tan\theta$ ——— II

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Now from Δ AOF.

$$\frac{OF}{FA} = \tan\theta$$

$$\Rightarrow \frac{r}{|AF|} = \tan\theta \quad \text{——— } III$$

Now

$$|OA|^2 = |AF|^2 + |OF|^2$$

$$(y-r)^2 = |AF|^2 + (r)^2$$

$$|AF|^2 = (y-r)^2 - (r)^2$$

$$|AF| = \sqrt{(y-r)^2 - r^2}$$

So $III \Rightarrow$

$$\frac{r}{\sqrt{(y-r)^2 - r^2}} = \tan\theta \quad \text{——— } IV$$

Now from II and IV, we get

$$\frac{x}{y} = \frac{r}{\sqrt{(y-r)^2 - r^2}}$$

$$\frac{x}{y} = \frac{r}{\sqrt{y^2 + r^2 - 2yr - r^2}}$$

$$\frac{x}{y} = \frac{r}{\sqrt{y^2 - 2yr}}$$

$$\Rightarrow x = \frac{yr}{\sqrt{y^2 - 2yr}} \qquad \text{put in } I$$

$$V = \frac{1}{3}\pi\left(\frac{yr}{\sqrt{y^2-2yr}}\right)^2 \cdot y$$

$$= \frac{1}{3}\pi\left(\frac{y^2 r}{\sqrt{y^2-2yr}}\right)^2 y$$

$$= \frac{1}{3}\pi \frac{y^2 r^2}{y^2 - 2yr} \cdot y$$

$$= \frac{1}{3}\pi \frac{y^3 r^2}{y(y-2r)}$$

$$= \frac{1}{3}\pi \frac{y^2 r^2}{y-2r}$$

Now Diff. w.r.t. y

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$$\frac{dV}{dy} = \frac{1}{3}\pi r^2\left[\frac{(y-2r)(2y) - y^2(1-0)}{(y-2r)^2}\right]$$

$$= \frac{1}{3}\pi r^2\left[\frac{2y^2-4ry-y^2}{(y-2r)^2}\right]$$

Put $\frac{dV}{dy} = 0$

$$\frac{1}{3}\pi r^2\left[\frac{2y^2-4ry-y^2}{(y-2r)^2}\right] = 0$$

$$2y^2-4yr-y^2=0$$

$$y(2y-4r-y)=0$$

$\Rightarrow$ $y=0$, $y-4r=0$

$y=4r$

So $y = 0, 4r$

$y=0$ is not admissible because cone has height.

Hence $y=4r$

Now

$$\frac{dV}{dy} = \frac{1}{3}\pi r^2\left[\frac{y^2-4ry}{(y-2r)^2}\right]$$

$$\Rightarrow \left.\frac{d^2V}{dy^2}\right|_y > 0$$

Which indicates that Volume is minimum at $y=4r$

Hence the altitude of the Cone is $4r$.

Now from Δ OAF

$$\sin\theta = \frac{OF}{OA}$$

$$\sin\theta = \frac{r}{y-r}$$

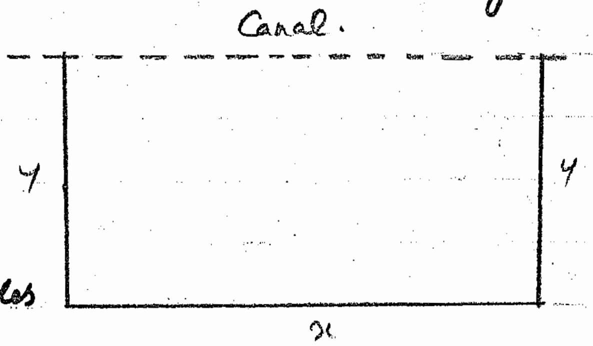
$$\sin\theta = \frac{r}{4r-4} = \frac{r}{3r} = \frac{1}{3}$$

$$\Rightarrow \sin\theta = \frac{1}{3}$$

$$\theta = \sin^{-1}\left(\frac{1}{3}\right)$$

As required.

Q.29. A farmer has 1000 meters of barbed wire with which he is to fence off there sides of a rectangular field, the fourth side being bounded by a straight canal. How can the farmer enclosed the largest field?



Soln. Let x and y be the dimension of the rectangular field, then.

$$1000 = \text{perimeter of 3 sides}$$
$$1000 = x + y + y$$
$$1000 = x + 2y$$
$$\Rightarrow \quad x = 1000 - 2y \qquad \text{———— I}$$

Now
$$A = xy$$
$$A = (1000-2y)\,y$$
$$A = 1000y - 2y^2$$

Diff. w.r.t. y
$$\frac{dA}{dy} = 1000 - 4y$$

For extrema $\frac{dA}{dy} = 0$
$$1000 - 4y = 0$$
$$4y = 1000$$
$$y = 250 \qquad \text{put in I}$$
$$x = 1000 - 2(250)$$
$$x = 500$$

Now $\left.\frac{d^2A}{dy^2}\right|_y = -4$ i.e $\frac{d^2A}{dy^2} < 0$

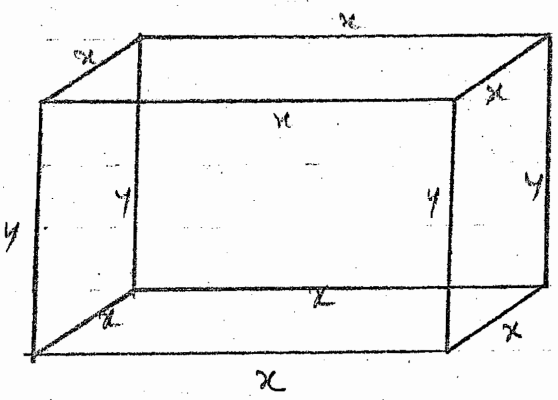
So

Area of the field is Max. at $y = 250$

Hence the dimension are 500 and 250.

Q.30: A topless rectangular box with a square base is to have a volume of 1926 cubic cm. The material for the base costs Rs.3 per square cm. and the material for the sides costs Rs.2 per square cm. What dimension should the box have to minimize it's cost?

Soln. Let length of the base be x cm and hight of each side be y cm.



Now

Volume of cylinder $= x \cdot x \cdot y$

$V = x^2 y$

But $V = 1926$

$1926 = x^2 y$ ——— I

Now Area of 4 Sides $= 4xy$ $\quad\because$ Area of 1 side $= xy$

Material for the base costs $= 3$ Rs.

Material for the side costs $= 2$ Rs.

$\therefore$ Cost of material for base area $= 3x^2$ $\quad\because$ Area of base $= x \cdot x$

Cost of material for 4 sides area $= 2 \times 4xy$

$= 8xy$

Total cost $= 3x^2 + 8xy$ ——— II

From I

$$y = \frac{1926}{x^2}$$

So Total cost $= 3x^2 + 8x \dfrac{1926}{x^2}$

$$C = 3x^2 + 15408\,x^{-1}$$

$$C = 3x^2 + 15408\,x^{-1}$$

$$\Rightarrow \frac{dC}{dx} = 6x - 15408\,x^{-2}$$

Put $\dfrac{dc}{dx} = 0$ for extrema

$$6x - 15408\,x^{-2} = 0$$

$$6x - \frac{15408}{x^2} = 0$$

$$6x^3 - 15408 = 0$$
$$6x^3 = 15408$$
$$x^3 = 2568$$
$$x = (2568)^{1/3}$$
$$x = 13.7$$

$$\Rightarrow \quad y = \frac{1926}{(13.7)^2}$$

$$y = 10.3$$

Now $\dfrac{d^2c}{dx^2} = 6 - 15408\,(-2)\,x^{-3}$

$$= 6 + 30816\,x^{-3}$$
$$= 6 + \frac{30816}{x^3}$$

$$\left.\frac{d^2c}{dx^2}\right|_x > 0$$

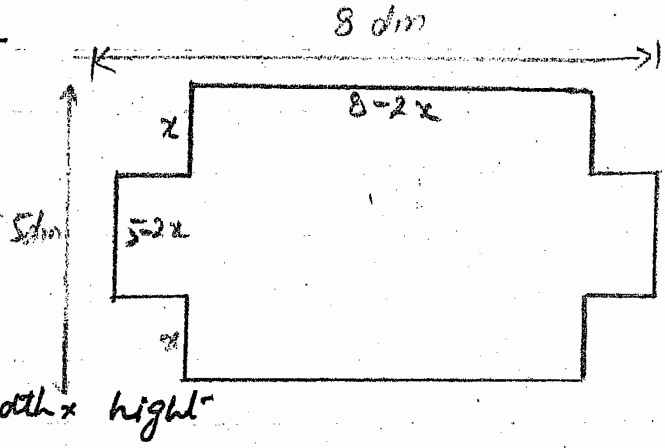
So minimize cost at $x = 13.7$ Dimension کی box

Hence dimensions are یہ ہونگی

$= 13.7 \times 13.7 \times 10.3$ cm. لمبائی x چوڑائی x اونچائی

Q.31. An open rectangular box is to be made from a sheet of cardboard 8dm. by 5dm by cutting equal squares from each corner and turningup the sides. Find the edge of the square which make the volume maximum.

Soln. Let the side of the square cut from each corner be x dm. Then the edge of the box formed by binding the sides are $8-2x$, $5-2x$ and x decimeters.



Now Volume = Length × width × hight

$$V = (8-2x)(5-2x)\,x$$
$$V = [40 - 16x - 10x + 4x^2]\,x$$
$$V = [4x^2 - 26x + 40]\,x$$
$$V = 4x^3 - 26x^2 + 40x$$

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Now $\dfrac{dV}{dx} = 12x^2 - 52x + 40$

Put $\dfrac{dV}{dx} = 0$ for extrema

$$12x^2 - 52x + 40 = 0$$
$$3x^2 - 13x + 10 = 0$$
$$3x^2 - 10x - 3x + 10 = 0$$
$$x(3x - 10) - (3x - 10) = 0$$
$$(x-1)(3x-10) = 0$$

$\Rightarrow \quad x - 1 = 0 \quad , \quad 3x - 10 = 0$

$x = 1 \quad , \quad x = 10/3$

Hence $x = 1, 10/3$

Now $\dfrac{d^2V}{dx^2} = 24x - 52$

$$\left.\frac{d^2V}{dx^2}\right|_{x=10/3} = 24 \cdot \frac{10}{3} - 52$$
$$= 80 - 52$$
$$= 28$$

i.e $\dfrac{d^2V}{dx^2} > 0$

$\Rightarrow$ The area is minimum

So $x = 10/3$ is inadmissible.

Now $\left.\dfrac{d^2V}{dx^2}\right|_{x=1} = 24(1) - 52$
$= -28$

i.e $\left.\dfrac{d^2V}{dx^2}\right|_{x=1} < 0$

$\Rightarrow$ The Volume is Max. when $x = 1$

So edge $= 1$ dm. for Max. Volume.

Q.37. A local train has 1200 passengers at a fare of Rs.2 each. For every paisa the fare is reduced, 10 more passangers ride the train. What fare should be charged to maximize the revenue?

Soln. If one paisa of fare is reduced then number of more passenger = 10

If x paisa of fare is reduced then the number of more passengers $= 10x$

Number of Passengers	:	Rate of fare (Paisa)
1200	:	200
$1200+10x$	:	$200-x$

So Revenue $= (1200+10x)(200-x)$

$$R = 240000 - 1200x + 2000x - 10x^2$$

Now $$\frac{dR}{dx} = 0 - 1200 + 2000 - 20x$$

$$\frac{dR}{dx} = -20x + 800$$

put $\frac{dR}{dx} = 0$

$$-20x + 800 = 0$$

$$20x = 800$$

$$x = 40 \text{ (paisa)}$$

Now $$\left.\frac{d^2R}{dx^2}\right|_{x=40} = -20$$

i.e $$\frac{d^2R}{dx^2} < 0$$

$\Rightarrow$ The revenue is maximum if the fare is lowered upto 40 paisa.

Thus the new fare for each person $= 200 - 40$

$= 160$ Paisa

$= 1.6$ Rs.

Hence the revenue is maximum if fare of each Person is 1.6 Rs.

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Q.33. A merchant has 200 quintals of cattle that he can sell at a profit of Rs. 500 per quintal. If the cattle gains 5 quintals per week, but the profit fall fall by Rs 10 per quintal per week. when should the cattle be sold to obtain maximum profit?

Soln. Suppose the cattle are sold after x weeks to get maximum profit.

weight of cattle = 200 quintals

Profit at 200 quintals = 500 Rs.

Weight of cattle after x weeks = $200+5x$

Profit after x weeks = $500-10x$

So the profit, when the cattle are sold is

$$P = (200+5x)(500-10x)$$

$$P = 100000 - 2000x + 2500x - 50x^2$$

$$P = 100000 + 500x - 50x^2$$

$$\Rightarrow \frac{dP}{dx} = 500 - 100x$$

$$\Rightarrow \frac{d^2p}{dx^2} = -100$$

Put $\frac{dP}{dx} = 0$

$$500 - 100x = 0$$

$$500 = 100x$$

$$\Rightarrow x = 5$$

So $\left.\frac{d^2p}{dx^2}\right|_{x=5} = -100$

i.e. $\left.\frac{d^2p}{dx^2}\right|_{x=5} < 0$

The profit is maximum after 5 weeks.

i.e. The cattle should be sold after 5 weeks to obtain maximum profit.

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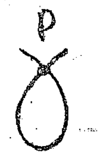
Shahid Javed

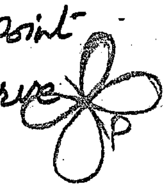
Double Point:

Def. A point P on the curve is called double point if the curve passes through P twice.



Multiple Point:

Def. A point P on the curve is called a Multiple point of intensity r if there pass r branches of the curve through P.



Singular Point:

Def. A Multiple point is also called a Singular Point.

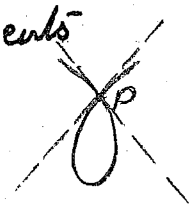
Types of Singular Point

There are three types of Singular point.

1) Nodal point
2) A cusp
3) An isolated point.

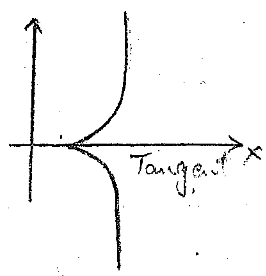
1) Nodal Point:

Nodal point is a singular if the tangents drawn at that point are real and distinct.



2) A Cusp.

A cusp point is a singular point if the tangents at this point are identical.



3) An Isolated Point:

Def. An isolated point is a singular point if the tangents at this point are imaginary.