Chapter #7

Plane Curves II

Asymptote:

Def.: An asymptote is a straight line for a given curve C,
if the distance between L and C tends to 0 as the
infinite distance is moved along L.

E.g.: 

\[ C: \ xy = 1 \]
\[ y = \frac{1}{x} \]

As \( x \to \infty \), \( y \to 0 \)
y=0 is an asymptote for curve C.
likewise for \( x = \frac{1}{y} \)
As \( y \to \infty \), \( x \to 0 \)
x=0 is also an asymptote.
The graph of the asymptote is as shown
\[ y = \frac{1}{x} \]

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>1/2</th>
<th>1/3</th>
<th>1/4</th>
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<tbody>
<tr>
<td>y</td>
<td>1</td>
<td>1/2</td>
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Types of Asymptotes:

There are three types of asymptotes:
1) Horizontal asymptote.
2) Vertical asymptote.
3) Inclined asymptote.

No. of Asymptotes:
The number of asymptotes is less
dthan or equal to the degree of the given
equation.
### How to Find the Asymptote

Arrange the given equation in descending powers of \( x \) and \( y \) like:

\[ a_n x^n + a_{n-1} x^{n-1} y + \cdots + a_1 xy + a_0 y^n + \cdots + x + y + x + y + 1 = 0 \]

**For Horizontal Asymptote:**

Equate the coefficient of highest power of \( x \) to zero if any.

**For Vertical Asymptote:**

Equate the coefficient of highest power of \( y \) to zero if any.

**Inclined Asymptote:**

Equation of inclined asymptote is \( y = mx + c \)

Put \( x = \infty \), \( y = m \) in the highest degree terms and equate to zero.

\[ \therefore f(m) = 0 \]

\[ \Rightarrow m_1, m_2, m_3, \ldots, m_n \]

**Value of \( C \):**

\[ C = -\frac{f_{n-1}(m)}{f_n(m)} \]

**Value of \( C \) in the presence of two equal Values of \( m \):**

By putting the values of \( m \) in the below formula we will get the value of \( C \).

\[ \frac{C}{a!} f^{(m)}(m) + \frac{C^2}{b!} f^{(m)}(m) + \frac{C^3}{c!} f^{(m)}(m) + \frac{C^4}{d!} f^{(m)}(m) + \cdots = 0 \]

**For three equal values of \( m \):**

We will use this formula for three equal values of \( m \) and get the value of \( C \).

\[ \frac{C^3}{3!} f^{(m)}(m) + \frac{C^2}{2!} f^{(m)}(m) + \frac{C}{1!} f^{(m)}(m) + f^{(m)}(m) = 0 \]
Asymptotes of Polar Curves:

Let \( r = f(\theta) \) be a curve.

Put \( r = a \) in the equation of the curve and find the value of \( \theta \).

Say \( \theta = \alpha, \beta, \gamma, \ldots \).

Then by using the formula

\[
P = r \sin(a - \theta)
\]

we can find the equation of the asymptote.

Where

\[
P = \lim_{\theta \to a} \frac{r^2}{\theta - \alpha, \beta, \gamma, \ldots}
\]

Available at

http://www.MathCity.org

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Exercise 7.I.

Find equations of the asymptotes of the following curves.

Q.9. \( y = \frac{(x-2)^2}{x^2} \)

The equation can be written as
\[ x^2 y = x^4 - 4x \]
\[ x^2 (y-1) + 4x - 4 = 0 \]

H.A. Coefficient of highest power of \( x = y - 1 \)
Put \( y - 1 = 0 \)
\( x = 0 \)

is an asymptote 11 to \( x \)-axis.

V.A. Coefficient of highest power of \( y = x^2 \)
Put \( x^2 = 0 \)
\( \Rightarrow x = 0, x = \infty \)

i.e. \( y \)-axis plays the role of asymptote.
Hence the required asymptotes are
\( y = 1, x = 0, x = \infty \)

Note: Number of asymptotes is at the most three which has been achieved. There is no need to look for an inclined asymptote.

Q.10. \( x^2 y = 12 (x-3) \)
\[ x^2 y = 12(x-3) \]
\[ x^2 y - 12x + 36 = 0 \]

H.A. Coefficient of highest power of \( x = y^2 \)
Put \( y^2 = 0 \)
\( \Rightarrow y = 0, y = \infty \)

i.e. \( x \)-axis plays the role of asymptote.

V.A. Coefficient of the highest power of \( y = x^2 \)
Put \( x^2 = 0 \)
\( \Rightarrow x = 0, x = \infty \)

i.e. \( y \)-axis plays the role of asymptote.
Hence the required asymptotes are
\( y = 0, y = 0, x = 0 \)
Q.8. \( \frac{dx}{dy} = x^2 + 3 \Rightarrow x^2 - 3x + 6 = 0 \)

H.A.

- Co-efficient of highest power of \( x = 1 \)

\[ \Rightarrow \text{there is no asymptote parallel to } x\text{-axis}. \]

V.A.

- Co-efficient of highest power of \( y = -2x \)

\[ \Rightarrow 2x = 0 \]

\[ \Rightarrow x = 0. \]

\[ \Rightarrow y\text{-axis plays the role of asymptote}. \]

I.A. Arrange the given in descending power of \( x \)

\[ \Rightarrow x^2 - 3x + 6 = 0 \]

Put \( x = 1 \) and \( y = m \), we have

\[ \phi_x(m) = 1 - 6m \]

Put \( \phi_x(m) = 0 \)

\[ \Rightarrow 1 - 6m = 0 \]

\[ \Rightarrow 6m = 1 \]

\[ m = \frac{1}{6} \]

Now for value of \( c \)

\[ \phi_x(m) + \phi_y(m) = 0 \]

\[ \Rightarrow -2 + \phi_y(m) = 0 \]

Put these values in (I) we have

\[ c(-2) + 0 = 0 \]

\[ -2c = 0 \]

\[ c = 0 \]

So eq. of Enclined asymptote is

\[ y = mx + c \]

\[ y = \frac{1}{2}x + 0 \]

\[ y = \frac{1}{2}x \]

Hence the required asymptote are

\( x = 0 \) and \( y = \frac{1}{2}x \).
Q.9. \[ \frac{x^3(x+y)^2}{a^2(x^2-y^2)} = a^2xy. \]

\[ x^2(x^3-3xy+y^3) + a^2x^2 - a^2xy \]

\[ x^2 - 2xy + x^2 + a^2 - a^2 - a^2 = 0 \]

I

H.A. G-efficient of highest power of \( x = 1 \)

\[ \implies \text{there is no horizontal asymptote.} \]

V.A.

G-efficient of highest power of \( y = x - a \)

\[ \begin{align*}
  &\text{Put} \quad x^2 - a^2 = 0 \\
  &\quad x = a \\
  &\quad x = -a \\
  &\quad x = a, \quad x = -a
\end{align*} \]

These are asymptotes \( \parallel \) to \( y \)-axis.

II.

Put \( x = 1 \) and \( y = m \) in the highest degree terms of (I)

\[ \begin{align*}
  &\text{I.e.} \\
  &\frac{\phi_1(m)}{\phi_2(m)} = 1 - 2m + m^2 \\
  &\frac{\phi_3(m)}{\phi_4(m)} = 1 + m^2 - 2m
\end{align*} \]

\[ \begin{align*}
  &\text{Equal to zero} \\
  &\begin{align*}
    &1 - m^2 - 2m = 0 \\
    &1 - m = 0 \quad \text{or} \quad 1 + m = 0
  \end{align*} \\
  &\implies m = 1 \\
  &\implies m = 1
\end{align*} \]

Now

\[ \frac{c}{a} \frac{\phi_2''(m)}{\phi_2'(m)} + \frac{c}{a} \frac{\phi_3'(m)}{\phi_3(m)} + \phi_4(m) = 0 \]

II

So

\[ \begin{align*}
  &\phi_2'(m) = 1 + m^2 - 2m \\
  &\phi_3'(m) = 2m - 2 \\
  &\phi_4'(m) = 2 \\
  \end{align*} \]

a-d

\[ \begin{align*}
  &\phi_2'(m) = 0 \\
  &\phi_3'(m) = 0 \\
  &\phi_4'(m) = 0
\end{align*} \]

Also

\[ \begin{align*}
  &\phi_2(2m) = a^2 - a^2m^2 - a^2m \\
  &\phi_4(m) = a^2(1 - m^2 - m)
\end{align*} \]

Put these values in II, we have

\[ \begin{align*}
  &\frac{c}{a} (2) + \frac{c}{a} (0) + a^2(1 - m^2 - m) = 0 \\
  &c^2 + a^2(1 - m^2 - m) = 0
\end{align*} \]

Put \( m = 1 \)
\[ c^2 + a^2(-1 - 1) = 0 \\
\; \quad c^2 + a^2(-1) = 0 \\
\quad c^2 = a^2 \\
\; \quad c = \pm a \\
\Rightarrow c = a \quad \text{and} \quad c = -a \\
\] 

Hence the inclined asymptotes are 
\[ y = x + a \quad \text{and} \quad y = x - a. \]

Hence the required asymptotes are 
\[ x = a, \quad x = -a, \quad y = x + a \quad \text{and} \quad y = x - a. \]

\[ \begin{align*}
(x - y)^2 (x^2 + y^2) - 10(x - y)(x^2 + 12y^2 + 2x + y^2 - 2xy) &= 0 \\
x^2 - xy - x^2 - y^2 + 10x^2 + 10y^2 + 12y^2 - 2xy &= 0 \\
4x^2 - xy - 2x^2 - y^2 + 10x^2 + 10y^2 + 12y^2 - 2xy &= 0 \\
&= 0 \quad \text{is highest power of} \ x = 1 \\
\Rightarrow & \quad \text{There is no horizontal asymptote.} \\
\end{align*} \]

V.A. Co-efficient of highest power of \( y = 1 \)
\[ \Rightarrow \text{There is no vertical asymptote.} \]

I.A. Put \( x = 1 \) and \( y = m \) in the highest powers of \( x \) and \( y \)
\[ \Rightarrow \phi_2(m) = 1 + 2m^2 + m^4 - 2m - 2m^3 \\
\] 

Put \( \phi_1(m) = 0 \\
\Rightarrow 2m^2 + m^4 - 2m - 2m^3 = 0 \\
(m^2 + 1) - 2m(m^2 + 1) = 0 \\
(m^2 + 1)(m^2 + 1) = 0 \\
(m - 1)^2 (m + 1) = 0 \\
\Rightarrow (m - 1) = 0 \quad \text{and} \quad (m + 1) = 0 \\
\Rightarrow m(m - 1) = 0 \quad \text{and} \quad m^2 = -1 \\
\Rightarrow m = 1, 1 \quad \text{Imaginary, not included.} \]

Now \[ \phi_0(m) + \phi_1(m) + \phi_2(m) = 0 \]
\begin{align*}
\phi_2(m) &= 1 + 2m^2 + m^4 - 2m - 2m^3 \\
\phi_1(m) &= 1 + 4m + 4m^3 - 2m - 6m^3 \\
\phi_0(m) &= 1 + 4m - 6m^2 \\
\end{align*}
\[ f''(m) = 4 + 12m^2 - 12m \]

Now, \[ f'(m) = -10 + 10m \]

\[ f'(m) = 10 \]

and \[ f''(m) = 12m^2 \]

Put in \( f'' \)

\[ \Rightarrow \frac{d^2}{dx^2} (4 + 12m^2 - 12m) + C(10) + 12m^2 = 0 \]

\[ \Rightarrow \frac{d^2}{dx^2} (2 + 6m^2 - 6m) + 10C + 12m^2 = 0 \]

\[ \Rightarrow 2c^4 + 6c^2m - 6c^2m + 10c + 12m^2 = 0 \]

Putting \( m = 1 \), the above eq. is:

\[ c^2 (2 + 6 - 6) + 10C + 12 = 0 \]

\[ 2c^2 + 10c + 12 = 0 \]

\[ c^2 + 5c + 6 = 0 \]

\[ (c+2)(c+3) = 0 \]

\[ \Rightarrow c+2 = 0 \quad \text{or} \quad c+3 = 0 \]

\[ \Rightarrow c = -2 \quad \text{or} \quad c = -3 \]

So the eqs. of inclined asymptotes are:

\[ y = x - 2 \quad \text{and} \quad y = x - 3 \]

Hence the required asymptotes are:

\[ y = x - 2 , \quad y = x - 3 \]

Q. 6.

\[ x^2y + xy^2 + xy + y^2 + 3x = 0 \]

Here \( x^2y + xy^2 + y^2 + xy + 3x = 0 \) \( \quad \) \( \text{v) \) \( \)

H.A. Co-efficient of highest power of \( x = y \)

Put \( y = 0 \) is an asymptote.

i.e. \( x \)-axis plays the role of asymptote.

V.A. Co-efficient of highest power of \( y = x + 1 \)

Put \( x + 1 = 0 \)

\[ \Rightarrow x = -1 \]

is an asymptote.
Inclined Asymptote

Put \( n = 1 \) and \( y = m \) in the highest degree terms of \( I \), we have

\[
\phi_3(m) = m + m^2
\]

Put \( \phi_3(m) = 0 \)

\[
\Rightarrow m + m^2 = 0
\]

\[
\Rightarrow m(1 + m) = 0
\]

\[
\Rightarrow m = 0 \quad \text{and} \quad m = -1
\]

Value of \( C \) for \( m = 0 \)

For value of \( C \) we use the formula

\[
C = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}
\]

\[
\Rightarrow \phi_3(m) = m + m^2 \quad \Rightarrow \phi'_3(m) = 1 + 2m
\]

\[
\text{Put these values in (2)}
\]

\[
C = -\frac{1 + 2m}{m^2 + m}
\]

Put \( m = 0 \), we have \( \frac{1 + 2m}{m^2 + m} \)

\[
\Rightarrow C = 0
\]

So the equation of the asymptote for \( m = 0, 0 \) and \( C = 0 \)

is

\[
y = 0
\]

which we have already found.

Value of \( C \) for \( m = -1 \)

From above

\[
c = -\frac{mt + m}{1 + 2m}
\]

Put \( m = -1 \)

\[
\Rightarrow c = -\frac{(-1)t + (-1)}{1 + 2(-1)}
\]

\[
\Rightarrow c = -\frac{1 - 1}{1 - 2}
\]

\[
\Rightarrow c = 0
\]
So eq. of asymptote is
\[ y = -x \]

Hence required asymptote are
\[ y = 0, \quad x + 1 = 0 \quad \text{and} \quad y = -x \]

Q.7
\[(x-y+1)(x-y-2)(x+y) = 8x-1\]
\[(x^2-xy-2x-xy+y^2+3y+x-y-2)(x+y) = 8x-1\]
\[(x+y)^2-2xy-x+y-2)(x+y) = 8x-1\]
\[(x^3+xy^2-2x+y^2-2x+3y+y^3-2x^2-xy^2+y^2+2y)=0\]
\[x^3+y^3-3x^2+y^2-x^2+y^2-10x-3y+1=0 \quad I\]

**Horizontal Asymptote**

- Co-efficient of highest power of \( x = 1 \)
  \[ \Rightarrow \text{No Horizontal Asymptote.} \]

**Vertical Asymptote**

- Co-efficient of highest power of \( y = 1 \)
  \[ \Rightarrow \text{No Vertical Asymptote.} \]

**Inclined Asymptote**

Put \( x = 1 \) and \( y = m \) in the highest degree terms
\[ y = x - 1 \]
\[ m = 1 \]
\[ m - 1 \]
\[ m = 1 \]

**Value of \( C \) for \( m = 1 \)**

\[ \frac{C^2}{3} \frac{y^{(m)}}{3} + \frac{C}{1} \frac{y^{(m)}}{1} + \frac{C}{3-2} \frac{y^{(m)}}{3-2} = 0 \quad I \]

\[ \Rightarrow \frac{d}{d} \frac{y^{(m)}}{1} = m^2 - m^2 + 1 \]
\[ \Rightarrow \frac{d}{d} \frac{y^{(m)}}{3} = 3m^2 - 3m - 1 \]
\[ \frac{d}{d} \frac{y^{(m)}}{1} = 6m - 2 \]
\[ f'(m) = -1 + m^2 \]

\[ f'(m) = 2m \]

and \[ f'(m) = -10 - 2m \]

Put these values in \( I \), we have

\[ \frac{x}{2} \cdot 2(3m-1) + c \cdot 2m - 10 - 2m = 0 \]

\[ \frac{x}{2} \cdot (3m-1) + 2cm - 2m = 0 \]

Put \( m = 1 \)

\[ c(3-1) + 2c - 2 - 10 = 0 \]

\[ 2c + 2c - 12 = 0 \]

\[ c^2 + c - 6 = 0 \]

\[ (c+3)(c-2) = 0 \]

\[ \Rightarrow c = -3 \quad c = 2 \]

So the eq. of asymptote when \( m = 1 \) are

\[ y = x - 3 \quad y = x + 2 \]

Value of \( c \) for \( m = -1 \)

We know that

\[ c = -\frac{\phi_k(m)}{\phi_1'(m)} \]

\[ \Rightarrow c = -\frac{m^2 - 1}{3m^2 - 2m - 1} \]

Put \( m = -1 \)

\[ \Rightarrow c = -\frac{(-1)^2 - 1}{3(-1)^2 - 2(-1) - 1} \]

\[ \Rightarrow c = 0 \]

So eq. of the asymptote when \( m = -1 \) is

\[ y = -x \]

Hence the required asymptotes are

\[ y = x - 3 \quad y = x + 2 \quad \text{and} \quad y = -x \]
\[
\begin{align*}
\text{Horizontal Asymptote} & \quad \text{Coefficient of highest power of } x = 1 \\
\quad & \quad \text{Put } y = 0 \\
\quad & \quad \text{is an asymptote.}
\end{align*}
\]

Vertical Asymptote
\[
\begin{align*}
\text{Coefficient of highest power of } y & = 1 \\
\Rightarrow & \quad \text{There is no Vertical Asymptote.}
\end{align*}
\]

Inclined Asymptote
\[
\begin{align*}
& \quad \text{Put } x = a \text{ and } y = m \text{ in the highest degree terms of } x \text{ and } y. \\
\Rightarrow & \quad \Phi(m) = m^3 + m^2 + 2m + 3 \\
& \quad \Phi'(m) = 0 \\
& \quad m^3 + 2m^2 + m = 0 \\
& \quad m(m^2 + 2m + 1) = 0 \\
\Rightarrow & \quad m = 0, \quad m = -1, -1 \\
& \text{New Value of } c \text{ for } m = 0 \\
& \quad c = -\frac{\Phi(m)}{\Phi'(m)} \\
& \quad c = 0 \\
& \text{so eq. of the asymptote is } y = 0
\end{align*}
\]

New Value of \( c \) for \( m = -1 \)
\[
\begin{align*}
& \frac{c}{2} \Phi''(m) + \frac{c}{3} \Phi'(m) + \Phi(m) = 0 \\
& \text{New } \Phi(m) = m^3 + 2m^2 + m \\
& \Phi'(m) = 3m^2 + 4m \\
& \Phi''(m) = 6m + 4
\end{align*}
\]
\[ a + d = g_1(m) = 0 \]
\[ a + d = g_2(m) = -m \]

Put these values in \( \Delta \)
\[ \Delta = 2(3m+2) + 2 \cdot 0 - m^2 = 0 \]

Put \( m = -1 \)
\[ \Rightarrow c^2 (3(-1)+2) - (-1) = 0 \]
\[ c^2 (-1) + 1 = 0 \]
\[ 1 - c^2 = 0 \]
\[ c^2 = 1 \]
\[ c = \pm 1 \]

\[ \Rightarrow c = 1, \; c = -1 \]

So eqs of the asymptotes are
\[ y = -x + 1 \; \; \; y = -x - 1 \]

Hence the required asymptotes are
\[ y = 0 \; , \; y = -x - 1 \; \; \; \text{and} \; \; y = -x + 1 \]

**Q.9.**

**Horizontal Asymptote:**

Coefficient of highest power of \( x = y \)

\[ \Rightarrow y = 0 \; \; \; \text{i.e. an asymptote.} \]

**Vertical Asymptote:**

Coefficient of highest power of \( y = 1 \)

\[ \Rightarrow \text{there is no vertical asymptote.} \]

**Oblique Asymptote:**

Put \( m = 1 \) and \( y = m \) in highest degree terms

\[ a + d = g_y(m) = m^4 + m - 2m^3 \]
\[ \Rightarrow m^4 + m - 2m^3 = 0 \]
\[ m^4 - 2m^3 + m = 0 \]
\[ m(m^3 - 2m^2 + 1) = 0 \]
Value of C for m = 1

\[
\frac{c^2}{3} \phi_3(m) + \frac{c}{3} \phi_3'(m) + \phi_3(m) = 0
\]

Now \(\phi_3(m) = m^3 - 3m^2 + m\)
\(\phi_3'(m) = 3m^2 - 4m + 1\)
\(\phi_3''(m) = 6m - 4\)
and \(\phi_3(m) = 0\)
\(\phi_3'(m) \neq 0\)
\(\phi_3''(m) = -1 - m\)

Put these values in I

\[
\frac{c^2}{2} (6m-4) + c(0) - 1 - m = 0
\]

Put \(m = 1\)

\[
\frac{c^2}{2} (6-4) - 1 - 1 = 0
\]
\[
\frac{c^2}{2} \cdot 2 - 2 = 0
\]
\[
c^2 - 2 = 0
\]
\[
c = \pm \sqrt{2}
\]

So eqs. of asymptote are
\(y = x + \sqrt{2}, y = x - \sqrt{2}\)

New Value of C for \(m = 0\)

We know that:

\[
C = - \frac{\phi_3(m)}{\phi_3'(m)}
\]

\[
C = - \frac{0}{\phi_3'(m)}
\]

\(\Rightarrow C = 0\)

So eq. of asymptote is \(y = 0\)

Hence the required asymptote are
\(y = 0, y = x + \sqrt{2}, a d y = x - \sqrt{2}\)
Q.10. \( x^2 y^2 (x^2 - y^2) = (x^2 + y^2)^3 \)
\( x^2 y^2 (x^2 + y^2) - (x^2 + y^2)^3 = 0 \)
\( x^2 + x^2 y^2 - 3x^2 y^2 - (x^2 + 3x^2 y^2 + 3x^2 y^2) = 0 \)
\( x^2 - x^2 y^2 - x^2 y^2 - x^2 y^2 = 0 \)
\( x^2 - x^2 y^2 - x^2 y^2 = 3x^2 y^2 - 3x^2 y^2 = 0 \)

**Horizontal Asymptote**
- Coefficient of highest power of \( x = y^4 - 1 \)
  - Put \( y^2 = 1 \)
  - \( y = \pm 1 \) is an asymptote.

**Vertical Asymptote**
- Coefficient of highest power of \( y = x^2 - 1 \)
  - Put \( x^2 = 1 \)
  - \( x = \pm 1 \) is an asymptote.

**Inclined Asymptote**
- Put \( x = 1 \) and \( y = m \) in the highest-degree terms.
  - \( \frac{d}{dx} (m) = m^2 - 2mx + m^2 \)
  - \( \frac{d}{dx} (m) = 0 \)
  - \( m^3 - 2m^2 + m^2 = 0 \)
  - \( m^3 - m^2 = 0 \)
  - \( m^2 = 0 \), \( m^2 = 1 \)
  - \( m = 0, 1 \)

- \( m = 0, 1 \) was already used.
  - No need to check for it.

**Value of \( C \) for \( m = 1,1 \)**

**Formula for \( C \)**

\[ \frac{C}{2!} \left( \frac{d^2}{dx^2} (m) + \frac{C}{1!} \frac{d}{dx} (m) + \frac{C}{0!} (m) \right) = 0 \]
\[ f'(m) = m^6 - 5m^3 + \lambda \\
\]
\[ g'(m) = 6m^5 - 3m^2 + \mu \\
\]
\[ h'(m) = 3m^4 - 2m^2 + \nu \\
\]
\[ ad \]
\[ f'(m) = 0 \\
\]
\[ g'(m) = 0 \\
\]
\[ ad \]
\[ g'(m) = -1 + m^6 - 3m^4 - 3m^2 \\
\]
\[ Put \text{ these values in} \]
\[ \frac{c^2}{2} (3m^6 - 24m^4 + 2) + C(0) - 1 + m^6 - 3m^4 - 3m^2 = 0 \]
\[ \text{Put} \ m = 1 \], we have
\[ \frac{c^2}{2} (30 - 24 + 2) - 1 - 1 - 3 = 0 \]
\[ \frac{c^2}{2} (0) = 8 = 0 \]
\[ 4c^2 = 8 \]
\[ c^2 = 2 \]
\[ \Rightarrow c = \sqrt{2}, -\sqrt{2} \]

So the inclined asymptotes when \( m = 1 \) and \( c = \sqrt{2}, -\sqrt{2} \) are
\[ y = a(x - b) \quad \text{and} \quad y = a(x - b) \]

Value of \( c \) for \( m = -1, 1 \)

Put \( m = -1 \) in \( II \), we have
\[ \frac{c^2}{2} (30 - 24 + 2) - 1 - 1 - 3 = 0 \]
\[ \Rightarrow c = \sqrt{2}, -\sqrt{2} \]

So eq. of asymptotes when \( m = -1 \) and \( c = \sqrt{2}, -\sqrt{2} \) are
\[ y = -x + \sqrt{2}, \quad y = -x - \sqrt{2} \]

Hence, the required asymptotes are
\[ y = \pm 1, \quad x = \pm 1, \quad x^2 + 4x + \sqrt{2}, \quad y = \pm x - \sqrt{2} \]

\[ xy^2 = (x + y)^2 \]
\[ x^2 - 2xy - y^2 = 0 \]

**Horizontal Asymptote**

Coefficient of highest power of \( x = 1 \)

\( \Rightarrow \) no horizontal asymptote.

**Vertical Asymptote**

Coefficient of highest power of \( y = x - 1 \)
\[ x = 1 \]

\[ \Rightarrow x = 1 \] is an asymptote.

**Inclined Asymptote**

Put \( x = 1 \) and \( y = m \) in the highest degree terms.

\[ \Rightarrow f(x,y) = m^2 - m + 1 \]

Put \( f(m) = 0 \)

\[ \Rightarrow m^2 - m + 1 = 0 \]

\[ m = 0, m = 1 \]

\[ \therefore \] there is no oblique asymptote.

**Horizontal Asymptote**

Coefficients of highest powers of \( x = -y - 3 \)

Put \( -y - 3 = 0 \)

\[ -y = 3 \]

\[ y = -3 \] is an asymptote.

**Vertical Asymptote**

Coefficients of highest powers of \( y = x + 1 \)

Put \( x + 1 = 0 \)

\[ \Rightarrow x = -1 \] is an asymptote.

**Oblique Asymptote**

Put \( x = 1 \) and \( y = m \) in the highest degree terms.

\[ \Rightarrow \frac{\partial f}{\partial y} = m^2 - m + 1 \]

Put \( \frac{\partial f}{\partial y} = 0 \)

\[ \Rightarrow m^2 - m + 1 = 0 \]

\[ m = 0, m = 1 \]

\[ \therefore \] when \( m = 0 \) there is no inclined asymptote.

We look for \( m = 1 \)

Value of \( c \) for \( m = 1 \)

\[ c = -\frac{\partial f}{\partial y} \]

\[ \Rightarrow 1 \]
\[ f(m) = m^2 - m \]
\[ g'(m) = 2m - 1 \]
\[ h'(m) = -3 - 2m + m^2 \]

Put these values in \( \Sigma \), we have
\[ C = -\frac{-3 - 2m + m^2}{2m - 1} \]

Put \( m = 1 \)
\[ \Rightarrow C = -\frac{-3 - 2 \cdot 1}{2 - 1} = -\frac{5}{1} = 4 \]

So eq. of inclined asymptote when \( m = 1 \) and \( C = 4 \) is
\[ y = x + 4 \]

Hence the required asymptotes are
\[ y + 3 = 0 \quad \text{and} \quad x + 1 = 0 \]

Q.13
\[ r = \frac{a}{\theta} \]
\[ \theta = 0 \]
\[ r = \frac{a}{\theta} \]
\[ \text{Diff. w.r.t. } \theta. \]
\[ \frac{dr}{d\theta} = -\frac{a}{\theta^2} \]
\[ \frac{d\theta}{d\theta} = -\frac{a}{\theta} \]

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ x \frac{dy}{dx} = \frac{r \cos \theta \frac{d}{d\theta} (r \sin \theta) - r \sin \theta \frac{d}{d\theta} (r \cos \theta)}{r^2} = \frac{r \cos \theta \sin \theta - r \sin \theta \cos \theta}{r^2} = -1 \]

\[ \text{Now } \lim_{\theta \to 0} \frac{b \frac{d}{d\theta} \theta}{ \frac{d}{d\theta} \theta} = -a \]
\[ \rho = \frac{a}{\theta} \]
\[ \lim_{\theta \to 0} \rho = -a \]

Hence the asymptote is
\[ \rho = r \sin (\theta - \theta) \]
\[ -a = r \sin (\theta - \theta) \]
\[ -a = r \sin (-\theta) \]
\[ -a = -r \sin \theta \]
\[ \begin{align*}
&f_1(m) = m^2 - m \\
&f_1'(m) = 2m - 1 \\
&f_2(m) = -3 - 2m + m^2
\end{align*} \]

Put these values in \( I \), we have
\[ C = -\frac{3 - 2m + m^2}{2m - 1} \]

Put \( m = 1 \)
\[ C = -\frac{3 - 2 \times 1}{2 - 1} = -\frac{4}{1} = 4 \]

So eq. of inclined asymptote when \( m = 1 \) and \( c = 4 \) is
\[ y = x + 4 \]

Hence the required asymptotes are
\[ y + 3 = 0, \quad x + 1 = 0 \quad \text{and} \quad y = x + 4 \]

\[ r = \frac{a}{\Theta} \]

\[ \Theta = 0 \]

\[ r e \Theta = 0 \]

Diff. wrt. \( \Theta \).
\[ \frac{dr}{d\Theta} = -\frac{a}{\Theta^2} \]
\[ \frac{d\Theta}{dh} = -\frac{a}{\Theta} \]
\[ h x \beta r^3 \]
\[ \frac{\rho d\theta}{dh} = \frac{a^2}{\Theta^2} - \frac{\Theta}{a} = -a \]

Now \( \rho = \lim_{\Theta \to 0} \Theta \frac{d\theta}{dh} \)
\[ \rho = \lim_{\Theta \to 0} -a \]
\[ \rho = -a \]

Hence the asymptote is
\[ \rho = r \sin (\theta - \Theta) \]
\[ -a = r \sin (\Theta - \Theta) \]
\[ -a = r \sin (\theta - \Theta) \]
\[ -a = r \sin \Theta \]
\[ \frac{a}{r} = \sin \Theta \quad \text{is the required asymptote.} \]
Q. 14.

\[ r = a \tan \theta \quad \longrightarrow \quad I \]

Put \( r = \infty \)

\[ \Rightarrow \infty = \frac{a}{\tan \theta} \quad \Rightarrow \quad \sqrt{\theta} = \frac{a}{\theta} \quad \Rightarrow \quad \sqrt{\theta} = 0 \quad \Rightarrow \theta = 0 \]

\[ r = a \tan \theta \quad \rightarrow \quad \theta = \theta \]

\[ \frac{dr}{d\theta} = -\frac{1}{2} a \tan^{-1} \theta \]

\[ \frac{d\theta}{d\theta} = \frac{2}{a \tan^{-1} \theta} \]

\[ \frac{dr}{d\theta} = -\frac{2}{a} \theta^{-1/2} \]

\[ \times \text{ by } r^2 \]

\[ \Rightarrow \quad r^2 \frac{d\theta}{d\theta} = \frac{a^2}{\theta} - \frac{2}{a} \theta^{3/2} \]

\[ = -\frac{2a \sqrt{\theta}}{\theta} \]

Now

\[ P = \lim_{\theta \to 0} r^2 \frac{d\theta}{d\theta} \]

\[ P = \lim_{\theta \to 0} -2a \sqrt{\theta} \]

\[ P = 0 \]

So, the equation of the asymptote is

\[ \rho = r \sin (\theta - \phi) \]

\[ 0 = r \sin (\theta - \phi) \]

\[ r \sin (\theta - \phi) = 0 \]

\[ -r \sin \theta = 0 \]

\[ r \sin \theta = 0 \]

\[ \sin \theta = 0 \]

\[ \theta = 0 \]

Q. 15.

\[ r = a \cos \theta + b \quad \longrightarrow \quad (\theta) \]

\[ r = a \cos \theta + b \]

\[ r = \frac{a}{\sin \theta} + b \sin \theta \quad \longrightarrow \quad I \]

Put \( r = \infty \)

\[ \Rightarrow \quad \sin \theta = 0 \]

\[ \theta = 0, \pi \]

\[ \text{i.e. } \theta = a, \beta = \pi \]
New Diff (r) w.r.t. \( \theta \):

\[
\frac{dr}{d\theta} = -a \cos \theta \sin \theta
\]

\[
\frac{d\theta}{dr} = \frac{1}{a \cos \theta \sin \theta}
\]

Multiply \( r \):

\[
\implies \frac{d}{dr} \left( r^2 \right) = \frac{(a + b \sin \theta)^2}{\sin^2 \theta} \cdot \frac{\sin \theta \cdot \sin \theta}{a \cos \theta}
\]

\[
= -\frac{(a + b \sin \theta)^2}{a \cos \theta}
\]

Value of \( \rho \) when \( \alpha = 0 \):

\[
\rho = \lim_{\theta \to 0} \frac{(a + b \sin \theta)^2}{a \cos \theta}
\]

\[
= -\frac{a^2}{a} = -a
\]

So eq. of the asymptote when \( \alpha = 0 \) is:

\[
\rho = r \sin (\alpha - \theta)
\]

\[
a = r \sin \theta
\]

\[
a = r \sin \theta \quad \text{is the asymptote.}
\]

Value of \( \rho \) when \( \beta = \pi \):

\[
\rho = \lim_{\theta \to -\pi} \frac{(a + b \sin \theta)^2}{a \cos \theta}
\]

\[
= -\frac{a^2}{-a} = a
\]

So eq. of the asymptote when \( \beta = \pi \) is:

\[
\rho = r \sin (\beta - \theta)
\]

\[
a = r \sin (\pi - \theta)
\]

\[
a = r \sin \theta
\]

\[
\implies r \sin \theta = a \quad \text{is the asymptote.}
\]
\[ r = 2a \sin \theta \sin \theta \]
\[ r = 2a \sin \theta \]
\[ r = \frac{2a \sin \theta}{\cos \theta} \]

Put \( r = \alpha \).

\[ \Rightarrow \alpha \cos \theta = 0 \]
\[ \theta = \frac{\pi}{2}, \frac{3\pi}{2} \]
\[ \Rightarrow \alpha = \frac{\pi}{2}, \frac{3\pi}{2} \]

\[ \text{Diff. w.r.t. } \theta \]
\[ \frac{dr}{d\theta} = 2a \left[ \frac{\cos \theta 2 \sin \theta \cos \theta - \sin^3 \theta (\sin \theta)}{\cos^3 \theta} \right] \\
= 2a \left[ \frac{2 \sin \theta \cos^2 \theta + \sin^3 \theta}{\cos^3 \theta} \right] \\
\]
\[ \text{by } r^2 \]
\[ \frac{dr}{d\theta} = \frac{2a \sin^3 \theta}{4a^2 \sin^2 \theta} \]
\[ = 2a \left[ \frac{2 \sin \theta \cos^2 \theta + \sin^3 \theta}{\cos^3 \theta} \right] \\
= \frac{2a \sin^3 \theta}{\cos^3 \theta} \\
= 2a \sin \theta \frac{\sin^3 \theta}{\cos^3 \theta} \]

Now
\[ p = \lim_{\theta \to \frac{\pi}{2}} \frac{r^2}{\theta} \]
\[ p = \lim_{\theta \to \frac{\pi}{2}} \frac{2a \sin^3 \theta}{\theta \cos^3 \theta + \sin^3 \theta} \]
\[ p = \frac{2a \sqrt{1}}{\theta \cos^3 \theta + \sin^3 \theta} \]

So eq. \( r^2 \) the asymptote when \( \alpha = \frac{\pi}{2} \) is
\[ 2a = \theta \sin (\frac{\pi}{2} - \theta) \]
\[ 2a = r \cos \theta \] is the asymptote.

Now
\[ p = \lim_{\theta \to \frac{3\pi}{2}} \frac{2a \sin^3 \theta}{\theta \cos^3 \theta + \sin^3 \theta} \]
\[ = \frac{2a (-1)^3}{\theta \cos^3 \theta + \sin^3 \theta} \]
So Eq. of the asymptote when \( \beta = \frac{3\pi}{2} \) and \( \rho = -2a \) is
\[
-2a = r \sin \left( \frac{3\pi}{2} - \phi \right)
\]
\[
-2a = -r \cos \phi
\]
\[
\Rightarrow 2a = r \cos \phi
\]
so that the required asymptote is
\[
2a = r \cos \phi.
\]
\[
r \sin 2\phi = a \cos 3\phi
\]
\[
r = \frac{a \cos 3\phi}{\sin 2\phi}
\]
Putting \( r = \rho \) in \( \Sigma \)
\[
\Rightarrow \rho = \frac{a \cos 3\phi}{\sin 2\phi}
\]
\[
\Rightarrow \sin 2\phi = 0
\]
\[
2\phi = 0, \pi
\]
\[
\phi = 0, \frac{\pi}{2}
\]
\[
\rho = 0, \frac{a \pi}{2}
\]
Difference w.r.t. \( \theta \)
\[
\frac{d\rho}{d\theta} = a \left( \frac{\sin 2\phi (-3 \sin 3\phi) - \cos 3\phi 2 \cos \phi}{\sin^2 2\phi} \right)
\]
\[
\frac{d\theta}{d\phi} = a \left( \frac{-3 \sin 3\phi \sin 3\phi - 2 \cos 3\phi \cos 2\phi}{\sin^2 3\phi} \right)
\]
\[
\Rightarrow \frac{d\rho}{d\phi} = \frac{\sin^3 2\phi}{\cos^3 3\phi} \cdot a \left[ \frac{-3 \sin 3\phi \sin 3\phi - 2 \cos 3\phi \cos 2\phi}{\sin^2 2\phi} \right]
\]
\[
\Rightarrow \frac{d\rho}{d\phi} = \frac{-3 \sin 2\phi \sin 3\phi - 2 \cos 3\phi \cos 2\phi}{a \cos^3 3\phi}
\]
\[
\Rightarrow \frac{d\rho}{d\phi} = -\frac{a \cos^3 3\phi}{3 \sin 2\phi \sin 3\phi + 2 \cos 3\phi \cos 2\phi}
\]
Value of \( \rho \) when \( \phi = 0 \)
\[
\rho = \lim_{\phi \to 0} \frac{a \cos 3\phi}{\sin 2\phi}
\]
\[
\rho = \lim_{\phi \to 0} \frac{a \cos 3\phi}{3 \sin 2\phi \sin 3\phi + 2 \cos 3\phi \cos 2\phi}
\]
\[
\rho = \frac{a((1))}{3(0)(0) + 2((1)(1))}
\]
\[
\rho = -\frac{a}{2}
\]
So eq. of the asymptote is
\[ P = r \sin (\alpha - \beta) \]
\[ - \frac{a}{2} = r \sin (\alpha - \beta) \]
\[ - \frac{a}{2} = - r \sin \theta \]
\[ 3r \sin \theta = a \quad \text{is an asymptote}. \]

Value of \( P \) when \( \beta = \frac{\pi}{3} \):
\[ \rho = \lim_{\theta \to \frac{\pi}{3}} \frac{a \cos^3 \theta}{3 \sin^2 \theta \sin \theta + 2 \cos^3 \cos \theta} \]
\[ \rho = - \frac{a \cos^3 \theta}{3 \sin^2 \frac{\pi}{3} \sin \frac{\pi}{3} + 2 \cos^3 \cos \frac{\pi}{3}} \]
\[ \rho = - \frac{a \cos^3 \theta}{3 \sin^2 \frac{\pi}{3} \sin \frac{\pi}{3} + 2 \cos^3 \cos \frac{\pi}{3}} \]
\[ \rho = - \frac{a \cos^3 \theta}{3 \sin^2 \frac{\pi}{3} \sin \frac{\pi}{3} + 2 \cos^3 \cos \frac{\pi}{3}} \]

\[ P = 0 \]
So eq. of the asymptote is
\[ P = h \sin (\beta - \theta) \]
\[ 0 = h \sin (\frac{\pi}{3} - \theta) \]
\[ 0 = h \sin \theta \]
\[ \Rightarrow \quad \cos \theta = \infty \]
\[ \cos \theta = \infty \]
\[ \theta = \frac{\pi}{2} \]

Hence, the required asymptote are
\[ 3r \sin \theta = a \quad \text{and} \quad \theta = \frac{\pi}{3} \]

\[ v = \frac{a}{1 - \cos \theta} \quad \text{and} \quad 2 \]

Put \( r = \infty \):
\[ 1 - \cos \theta = 0 \]
\[ \cos \theta = 1 \]
\[ \theta = 0 \quad \text{i.e.} \ a = 0 \]

Diff. 2 w.r.t. \( \theta \):
\[ \frac{d\theta}{d\theta} = \frac{a (1 - \cos \theta)^2 (0 + \sin \theta)}{(1 - \cos \theta)^2 (0 + \sin \theta)} \]
\[ = - \frac{a}{(1 - \cos \theta)^2} (0 + \sin \theta) = - \frac{a \sin \theta}{(1 - \cos \theta)^2} \]
\[
\frac{d}{dx} \frac{1}{\sqrt{a^2 - \sin^2 \theta}} = \frac{(a^2 - \sin^2 \theta)^{-\frac{3}{2}} \cdot \cos \theta}{a^2 - \alpha^2}
\]

\[
\frac{d}{d\theta} \frac{1}{\sin \theta} = -\frac{1}{\sin \theta}
\]

\[
\frac{d^2}{d\theta^2} \frac{1}{\sin \theta} = -\frac{a}{\sin^3 \theta}
\]

**Value of \( P \) when \( \alpha = 0 \):**

\[
P = \lim_{\theta \to 0} r^2 \frac{d\theta}{d\theta} = 0
\]

\[
P = \lim_{\theta \to 0} \frac{a}{\sin \theta}
\]

\[
P = \infty
\]

\text{as} \quad r \text{ tends to} \infty

\text{There is no asymptote for the given curve.}

\[
r \sin \theta = a
\]

\[
r = \frac{a}{\sin \theta}
\]

At \( r = \infty \):

\[
\sin \theta = 0
\]

\[\theta = \frac{\pi}{2}, k \pi\] \text{where } k = 0, 1, 2, 3, \ldots

\[\theta = \frac{\pi}{n}, \text{ i.e. } \alpha = \frac{\pi}{n}\]

**Differentiate w.r.t. \( \theta \):**

\[
\frac{d}{d\theta} \sin \theta = -\frac{a}{\sin \theta} \cos \theta
\]

\[
\frac{d}{d\theta} \sin \theta = -\frac{a}{\sin \theta} \cos \theta
\]

\[\Rightarrow \frac{d^2}{d\theta^2} \sin \theta = -\frac{a}{\sin \theta} \cos \theta
\]

\[\Rightarrow \frac{d^2}{d\theta^2} \sin \theta = -\frac{a}{\sin \theta} \cos \theta
\]

**Applying \( \lim_{\theta \to k\pi}\):**

\[
\lim_{\theta \to k\pi} r^2 \frac{d\theta}{d\theta} = \lim_{\theta \to k\pi} \frac{a}{\sin \theta}
\]
\[ \rho = -\frac{a}{n\sin^2 \kappa \bar{\kappa}} \]

\[ = \frac{-a}{\sin^2 \bar{\kappa} \bar{\kappa}} \]

So eq. (7) the asymptote is

\[ \rho = r \sin (\kappa - \theta) \]

\[ -a \frac{n \sec \kappa}{\sin^2 \bar{\kappa} \bar{\kappa}} = r \sin (\kappa - \theta) \]

\[ -a \frac{n \sec \kappa}{\sin^2 \bar{\kappa} \bar{\kappa}} = -r \sin (\kappa - \theta) \]

\[ -a \frac{n \sec \kappa}{\sin^2 \bar{\kappa} \bar{\kappa}} = r \sin (\kappa - \theta) \]

\[ -a \frac{n \sec \kappa}{\sin^2 \bar{\kappa} \bar{\kappa}} = -r \sin (\kappa - \theta) \]

\[ r (e^\alpha - 1) = a (e^\beta - 1) \]

\[ r = \frac{a(e^\beta - 1)}{e^\alpha - 1} \]

Let \( r = 0 \)

\[ \Rightarrow \quad e^\alpha - 1 = 0 \]

\[ \Rightarrow \quad e^\alpha = 1 \]

\[ \Rightarrow \quad e^\beta = e^0 \]

\[ \Rightarrow \quad \alpha = 0 \quad \text{or} \quad \beta = 0 \]

Diff. (1) w.r.t. \( \alpha \)

\[ \frac{dr}{d\alpha} = a \left[ \frac{(e^\alpha - 1)(e^\beta) - (e^\alpha)(e^\beta)}{(e^\alpha - 1)^2} \right] \]

\[ = a \left[ \frac{e^\alpha(e^\beta - 1) - e^\beta}{(e^\alpha - 1)^2} \right] \]

\[ \times \frac{e^\alpha}{(e^\alpha - 1)^2} \]

\[ \Rightarrow \quad \frac{1}{a^2} \cdot \frac{dr}{d\alpha} = \frac{(e^\beta - 1)}{a^2 (e^\alpha - 1)^2} \left( \frac{2ae^\alpha}{a^2 (e^\alpha - 1)^2} \right) \]

\[ = -\frac{2ae^\alpha}{a^2 (e^\alpha - 1)^2} \]

\[ \Rightarrow \quad \frac{dr}{d\alpha} = -\frac{a(e^\beta - 1)}{2e^\alpha} \]

Now

\[ \rho = \lim_{\alpha \to 0} \frac{dr}{d\alpha} \]
\[
\rho = - \frac{a(e^{\theta})^4}{2e^\theta} \quad \text{as} \quad \rho \to 0
\]

So eq. 8 the asymptote is

\[
\rho = a (\sin(\theta - \phi))
\]

\[-2a = a \sin(\theta - \phi)
\]

\[2a = a \sin \theta]

is the required asymptote.

\[
\rho^2 \sin \theta = a^2
\]

\[
\Rightarrow \quad \rho = \frac{a^2}{\sin \theta}
\]

At \(r = \infty\)

\[
\Rightarrow \quad \sin \theta = 0
\]

\[\theta = k \frac{\pi}{2}
\]

where \(k = 0, 1, 2, 3, \ldots \)

\[
\theta = k \frac{\pi}{2}
\]

**Diff. (1) w.r.t. \(\theta\):**

\[
\frac{dr}{d\theta} = - \frac{a^2}{\sin^2 \theta} \cos \theta
\]

\[
\frac{d\rho}{d\theta} = - \frac{a^2 \cos \theta}{\sin \theta}
\]

\[
\frac{d\rho}{d\theta} = - \frac{a^2 \cos \theta}{\sin \theta}
\]

\[
\Rightarrow \quad \frac{d\rho}{d\theta} = - \frac{a^2 \cos \theta}{\sin \theta}
\]

\[
x \text{ by } a
\]

\[
\Rightarrow \quad \frac{d\rho}{d\theta} = - \frac{a^2 \cos \theta}{\sin \theta}
\]

\[\rho^2 \frac{d\theta}{d\rho} = - \frac{\sin \theta}{\cos \theta}
\]

\[\rho^2 \frac{d\theta}{d\rho} = - \frac{\sin \theta}{\cos \theta}
\]

\[\rho = \frac{a}{(\sin \theta)^{1/2}}
\]

\[\rho = \frac{a}{(\sin \theta)^{1/2}}
\]

**Conclusion:**

\[
\rho = \frac{a}{(\sin \theta)^{1/2}}
\]
\[ \frac{d^2 \theta}{d\lambda^2} = -\frac{a \left( \sin \theta \right)^{1-n}}{\cos \theta} \]

Now
\[ p = \lim_{\theta \to \pi \over 2} \frac{d \theta}{d\lambda} \]
\[ = \lim_{\theta \to \pi \over 2} \frac{a (\sin n \theta)^{n-1}}{\cos \theta} \]
\[ = \lim_{\theta \to \pi \over 2} \frac{a (\sin n \theta)^{n-1}}{\cos \theta} \]
\[ = 0 \]
So eq. 7 asymptote is
\[ p = a \sin (n \theta - \phi) \]
\[ c = r \sin (\pi \over 2 - \phi) \]
\[ \Rightarrow \sin (\theta - \phi) = 0 \]
\[ \frac{\pi \over 2 - \phi}{\pi \over 2} = 0 \]
\[ \theta = \frac{\pi \over 2} \]
\[ i's an asymptote. \]
\[ r^2 \sin^2 \theta = a^2 \sin 2\theta \]
\[ r = a \sqrt{2} \]
\[ \rho = \infty \]
\[ \Rightarrow \sin \theta = 0 \]
\[ \theta = 0, \pi \]
\[ \beta = \pi \]

**Riff. (i) w.r.t. \'\theta'**
\[ dr \frac{d\theta}{dr} = a \left\{ \frac{-2 \sin 2\theta \cos \theta + \cos 2\theta \sin 2\theta}{\sin^2 \theta} \right\} \]
\[ \frac{dr}{d\theta} = -\frac{a}{dr} \left\{ \frac{2 \sin \theta \cos \theta + \cos \theta \sin \theta}{\sin^2 \theta} \right\} \]
\[
x \text{by } \frac{1}{h^2} \Rightarrow \frac{1}{h^2} \frac{dh}{d\theta} = -\sin\theta - \frac{x}{2r} \left( \frac{2\sin\theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta} \right)
\]

\[
\frac{1}{h^2} \frac{dh}{d\theta} = \frac{1}{\partial \alpha^2} \left( \frac{2\sin \theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta} \right)
\]

\[
\frac{1}{h^2} \frac{dh}{d\theta} = -\frac{1}{\partial \alpha^2} \left( \frac{2\sin \theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta} \right)
\]

\[
\frac{1}{h^2} \frac{dh}{d\theta} = -\frac{1}{\partial \alpha^2} \left( \frac{2\sin \theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta} \right)
\]

\[
\frac{1}{h^2} \frac{dh}{d\theta} = -\frac{2\sin \theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta} \frac{\sin \theta}{\partial \alpha^2}
\]

\[
\frac{1}{h^2} \frac{dh}{d\theta} = -\frac{2\sin \theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta}
\]

\[
\frac{1}{h^2} \frac{dh}{d\theta} = -\frac{2\sin \theta \sin 2\theta + \cos \theta \cos 2\theta}{\sin \theta}
\]

Now \text{ Value of } \rho \text{ when } \alpha = 0
\]

\[
\rho = \lim_{\alpha \to 0} \frac{2\alpha \tan \frac{\theta}{2} \theta}{\sin \theta \sin \theta + \cos \theta \cos \theta}
\]

\[
\rho = -\lim_{\alpha \to 0} \frac{2\alpha \tan \frac{\theta}{2} \theta}{\sin \theta \sin \theta + \cos \theta \cos \theta}
\]

\[
\Rightarrow \rho = 0
\]

So eq. (8) asymptotic is
\[
\rho = r \sin (\pi - \theta)
\]

\[
0 = r \sin (0 - \theta)
\]

\[
\rho = -r \sin \theta
\]

\[
\Rightarrow \rho = 0
\]

So eq. (8) asymptotic is
\[
0 = r \sin (\pi - \theta)
\]

\[
0 = r \sin \theta
\]

\[
\Rightarrow \theta = 0 \text{ is an asymptote}
\]

\[
\text{Now Value of } \rho \text{ when } \theta = \pi
\]

\[
\rho = \lim_{\theta \to \pi} \frac{2\alpha \tan \frac{\theta}{2} \theta}{\sin \theta \sin \theta + \cos \theta \cos \theta}
\]

\[
\rho = -\lim_{\theta \to \pi} \frac{2\alpha \tan \frac{\theta}{2} \theta}{\sin \theta \sin \theta + \cos \theta \cos \theta}
\]

\[
\Rightarrow \rho = 0
\]

So eq. (8) asymptotic is
\[
0 = r \sin (\pi - \theta)
\]

\[
0 = r \sin \theta
\]

\[
\Rightarrow \theta = 0 \text{ is an asymptote}
\]