Some Useful results:

Eq. of the tangent at any point \( P(x_1, y_1) \) to the parabola is

\[
11. \quad y = \frac{2a}{x + x_1}
\]

eq 10. \quad y = \frac{2a}{x + x_1}

Eq. of the tangent at any point \( P(x_1, y_1) \) to the ellipse is

\[
\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1
\]

In Parabola \( x = at^2, y = 2at \) is the parametric form

\[ (at^2, 2at) = T \]

Exercise 6.2

Find equations of tangent and normal to each of the following curves at the indicated point \((P, \pm a)\).

\( \#1 \)

\[ y^2 = 4ax \quad \text{at} \quad (a, -2a) \]

The given eq. is

\[ y^2 = 4ax \quad \text{(1)} \]

Diff. w.r.t. \( x' \)

\[
dy \quad dy = 4a \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}
\]

Now

\[
\frac{dy}{dx} \bigg|_{(a, -2a)} = \frac{2a}{-2a} = -1
\]

Eq. of tangent at any point \( P(x, y) \) is

\[ y - y_1 = \frac{dy}{dx} \bigg|_{(x, y_1)} \cdot (x - x_1) \]

Now

\[ \text{Eq. of tangent at} \quad (a, -2a) \quad \text{is} \]

\[ (y + 2a) = -1(x - a) \]

\[ y + 2a = -x + a \]

\[ x + y + a = 0 \]

Is the required eq. of Tangent.

Now Eq. of the normal at any point \( P(x_1, y_1) \) is

\[ \frac{dy}{dx} \bigg|_{(x, y)} = \frac{-a}{x + x_1} \]
Eq of the normal at \((y + 2a)\) is
\[(y + 2a) = \frac{-1}{\frac{1}{t}} (x - a)\]
\[x - y - 3a = 0\]
Is the required eq. of the normal.

The given eq. is
\[xy = e^{2t}\]
\[\frac{dx}{dt} = \frac{e^{2t}}{x}\]
Diff. w.r.t. \(x^t\)
\[y + x \frac{dy}{dx} = 0\]
\[\Rightarrow x \frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = -\frac{y}{x}\]
Now
\[\frac{dy}{dx} = \frac{-\frac{y}{t}}{\frac{x}{t}} = \frac{-e^{2t}}{x} \cdot \frac{1}{t} = \frac{-1}{t^2}\]

Eq. of the tangent at \((ct, \frac{c}{t})\) is
\[y - \frac{c}{t} = -\frac{t}{t^2} (x - ct)\]
\[t^2y - t^2c = -x + ct\]
\[x + t^2 - 2ct = 0\]
\[x + yt^2 = 2ct\]

Eq. of normal at \((ct, \frac{c}{t})\) is
\[y - \frac{c}{t} = t^2 (x - ct)\]
\[y t^2 - c = t^2 u - ct^4\]
\[x t^3 - y t^2 - ct^4 + c = 0\]
\[x t^3 - y = \frac{ct^4 - c}{t}\]
\[x t^3 - y t^2 = c (t^4 - 1)\]
\[ Q.5 \]
\[ 2(x^2 + y^3) - ay^2 = 0 \quad \Rightarrow \quad x = \frac{a}{2} \]

The eq. is
\[ x(x^2 + y^3) - ay^2 = 0 \quad \tag{iii} \]
\[ x^3 + xy^2 - ay^2 = 0 \quad \tag{iv} \]

Diff. w.r.t. \( x \),
\[
3x^2 + y^2 + 2xy \frac{dy}{dx} - 2ay \frac{dy}{dx} = 0
\]
\[
3x^2 + y^2 + (2xy - 2ay) \frac{dy}{dx} = 0
\]
\[
\Rightarrow \frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy - 2ay} \quad \tag{v} \]

For value of \( y \) we put the value of \( x \) in eq. (iii)
\[
\left( \frac{a}{2} \right)^3 + \left( \frac{a}{2} \right) \frac{a}{2} - ay^2 = 0
\]
\[
\frac{a^3}{8} + \frac{a^3}{8} - ay^2 = 0
\]
\[
\frac{a^3}{4} = ay^2
\]
\[
y^2 = \frac{a^3}{4}
\]
\[
\Rightarrow y = \pm \frac{a^{3/2}}{2}
\]

\[ \Rightarrow \] we have to find the eq. of tangent and normal at \((\frac{a}{2}, \frac{a^{3/2}}{2})\) and \((\frac{a}{2}, -\frac{a^{3/2}}{2})\).

Now we find the eq. of tangent and normal at \((\frac{a}{2}, \frac{a^{3/2}}{2})\)

\[ \Rightarrow \frac{dy}{dx} \bigg|_{(\frac{a}{2}, \frac{a^{3/2}}{2})} = -\frac{3\left( \frac{a}{2} \right)^3 + \left( \frac{a}{2} \right) \frac{a}{2}}{2\left( \frac{a}{2} \right) \frac{a^{3/2}}{2} - 2a\left( \frac{a^{3/2}}{2} \right)}
\]
\[
= -\frac{\frac{3a^3}{8} + \frac{a^3}{8}}{\frac{a^{3/2}}{2} - 2a\left( \frac{a^{3/2}}{2} \right)}
\]
\[
= \frac{a^3}{4} \cdot \frac{1}{a^{3/2}} = 2
\]
Eq. of Tangent: \( \gamma - \gamma_1 = \left( \frac{dy}{dx} \right)_p (x - x_1) \)

\[
(\gamma - \gamma_1) = a (x - \gamma_1)
\]

\[
\gamma - \gamma_1 = \frac{dy}{dx} (x - \gamma_1)
\]

\[
\Rightarrow \frac{dy}{dx} - \gamma + \gamma_1 - a = 0
\]

\[
\Rightarrow \frac{dy}{dx} - \gamma_1 = 0
\]

\[
\Rightarrow 4x - dy - a = 0
\]

**Is the required Eq. of Tangent.**

Eq. of Normal: \( \gamma - \gamma_1 = -\left( \frac{dx}{dy} \right)_p (x - x_1) \)

\[
\gamma - \gamma_1 = -\frac{1}{\frac{dx}{dy}} (x - x_1)
\]

\[
\frac{dy}{dx} - a = -x + \gamma_1
\]

\[
x + dy - \frac{3a}{2} = 0
\]

**Is the required Eq. of Normal.**

Eq. of Tangent and Normal at \( (\alpha_1, -\gamma_1) \)

\[
i \Rightarrow \left( \frac{dy}{dx} \right) = -\frac{3(\gamma_1) + (-\gamma_1)^2}{3(\gamma_1)(-\gamma_1) - 2a(-\gamma_1)}
\]

\[
= -\frac{3\alpha_1 + a^2}{4 - a^2 + a^2}
\]

\[
= -\frac{a^2 x}{a^2}
\]

\[
= -2
\]

**Eq. of Tangent: \( \gamma - \gamma_1 = \left( \frac{dx}{dy} \right)_p (x - x_1) \)**

\[
(\gamma + \gamma_1) = -2 (x - \gamma_1)
\]
\[ x + y = 2x \]
\[ 4x + 2y = a \]

Hence the Eq. of Tangent at \((\theta, -\frac{a}{\sin \theta})\) is
\[ 4x + 2y = a \]

Eq. of Normal:
\[ \frac{y - y_1}{(x - x_1)} = -\frac{1}{(\frac{dy}{dx})} \]
\[ (y + \frac{a}{\sin \theta}) = 2(x - \frac{a}{\sin \theta}) \]
\[ y + \frac{a}{\sin \theta} = 2x - 2a \]
\[ \overline{2x - y - a} = 0 \]
\[ \overline{2x - y} = \frac{a}{\sin \theta} \]
\[ 4x - 2y = 3a \]

Is the required Eq. of Normal.

Using,
\[ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{d y}{dx}\right) = \frac{d}{dx}\left(\frac{c}{\cos \theta} - \frac{c}{\sin \theta}\right) \]
\[ \frac{d^2y}{dx^2} = \frac{c}{\sin \theta} - \frac{c}{\cos \theta} \]

Dividing both sides by \(x^2\), we have,
\[ \frac{1}{x^2} + \frac{1}{x^2} = \frac{c}{\sin \theta} \]
\[ \frac{x^2 + y^2}{x^2} = \frac{c}{\sin \theta} \]
\[ \frac{x^2}{\sin \theta} = \frac{c}{\cos \theta} \]

Diff. w.r.t. \(x\),
\[ -2x^3 - 2y^3 \frac{dy}{dx} = 0 \Rightarrow -2x^3 = y^3 \frac{dy}{dx} \]
\[ \Rightarrow \frac{dy}{dx} = -\frac{x^3}{y^3} = \frac{-y^3}{x^3} \]

\[ \left(\frac{dy}{dx}\right)_{(\theta, \frac{-a}{\sin \theta})} = \frac{(\cos \theta)^3}{(\sin \theta)^3} = \frac{c^3}{\sin^3 \theta} \]
\[ (dy) \quad m = \frac{\cos \theta}{\sin \theta} \]

Eq. of the Tangent:
\[ y - \frac{c}{\sin \theta} = m(x - \frac{c}{\sin \theta}) \]
\[ y - \frac{c}{\sin \theta} = \frac{\cos \theta}{\sin \theta} \left(x - \frac{c}{\cos \theta}\right) \]
\[ y \sin \theta - c \sin \theta = -\cos \theta \left(x - \frac{c}{\cos \theta}\right) \]
\[ y \sin^2 \theta - \cos^2 \theta = c \cos^2 \theta - x \cos^3 \theta \]
\[ x \cos^2 \theta + y \sin^2 \theta = c (\cos^2 \theta + \sin^2 \theta) \]
\[ x \cos^3 \theta + y \sin^3 \theta = c \]

Is the required Eq. of Tangent.  

Eq. of Normal: \[ y - y_1 = -\frac{1}{m} (x - x_1) \]

\[
\begin{align*}
(\gamma - \frac{c}{\sin \theta}) &= \frac{\sin^2 \theta}{\cos^2 \theta} (x - \frac{c}{\cos \theta}) \\
\gamma \cos^2 \theta - \frac{c \cos^2 \theta}{\sin \theta} &= \sin^2 \theta (x - \frac{c}{\cos \theta}) \\
\gamma \cos^2 \theta - \frac{c \cos^2 \theta}{\sin \theta} &= \frac{x \sin^2 \theta}{\cos \theta} - c \sin^3 \theta \\
\gamma \sin^2 \theta - \gamma \cos^2 \theta &= \frac{c \sin^3 \theta}{\cos \theta} - \frac{c \cos^3 \theta}{\sin \theta} \\
\gamma \sin^3 \theta - \gamma \cos^2 \theta &= \frac{c \sin^3 \theta}{\cos \theta} - \frac{c \cos^3 \theta}{\sin \theta} \\
\gamma \sin^2 \theta - \gamma \cos^2 \theta &= -\frac{2c \sin \theta}{\sin^2 \theta} = -2c \cot \theta \\
\gamma \sin^3 \theta - \gamma \cos^2 \theta &+ 2c \cot \theta = 0
\end{align*}
\]

Is the required Eq. of Normal.

\[
\begin{align*}
\frac{c \sin^4 \theta - c \cos^4 \theta}{\cos \theta \sin \theta} &= \frac{c (\sin^4 \theta - \cos^4 \theta)}{\cos \theta \sin \theta} \\
&= \frac{c [(\sin^2 \theta + \cos^2 \theta)(\sin^2 \theta - \cos^2 \theta)]}{\cos \theta \sin \theta} \\
&= \frac{c (\sin^2 \theta - \cos^2 \theta)}{\cos \theta \sin \theta} \\
&= \frac{-c (\cos \theta \cos \theta - \sin \theta \sin \theta)}{\cos \theta \sin \theta} = \frac{c \cos^2 \theta}{\cos \theta \sin \theta} \\
&= -\frac{2c \cos \theta}{\sin \theta} \geq -2c \cot \theta
\end{align*}
\]
Find the points where the tangent is \(11\) to the \(x\)-axis and where it is \(11\) to the \(y\)-axis for each of the given curves. (Ps. 7):

\[ x^2 + y^3 = a^3 \quad \text{(i)} \]

Differentiate w.r.t. \(x\):

\[ 3x^2 + 3y^2 \frac{dy}{dx} = 0 \]

\[ \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \quad \text{(ii)} \]

If the tangent is \(11\) to \(x\)-axis then \(\frac{dy}{dx} = 0\)

\[ \Rightarrow -\frac{x^2}{y^2} = 0 \]

\[ \Rightarrow y^2 = 0 \]

Put in (i):

\[ y = a \]

Hence the tangent is \(11\) to \(x\)-axis at \((0, a)\).

Now if the tangent is \(11\) to \(y\)-axis then \(\frac{dx}{dy} = \infty\)

\[ \Rightarrow \frac{dx}{dy} = 0 \]

\[ \Rightarrow x = a \]

Hence the tangent is \(11\) to \(y\)-axis at \((a, 0)\).

Available at http://www.MathCity.org
\[ x^3 + y^3 = 3axy \quad \text{(ii)} \]

Differentiate w.r.t. \( x \):
\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3a \left[ y + x \frac{dy}{dx} \right] \]
\[ 3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax \frac{dy}{dx} \]
\[ (3y^2 - 3ax) \frac{dy}{dx} = 3ay - 3x^2 \]
\[ \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} \]

For tangent parallel to x-axis:
\[ \frac{dy}{dx} = 0 \]
\[ ay - x^2 = 0 \]
\[ y^2 - ax = 0 \]
\[ ay = x^2 \]
\[ y = \frac{x^2}{a} \quad \text{(iii)} \]

Put in (ii):
\[ x^3 + \left( \frac{x^2}{a} \right)^3 = 3a - x \cdot \frac{x^2}{a} \]
\[ x^3 + \frac{x^6}{a^3} = 3x \]
\[ \frac{x^6}{a^3} = 3x^3 \]
\[ x^3 = 2a^3 \]
\[ x = \sqrt[3]{2a} \]

Put in (iii):
\[ y = \left( \frac{\sqrt[3]{2a}}{a} \right) \]
\[ y = \left( \frac{2}{3} \right) a \]

5. The tangent is parallel to x-axis at \( \left( \frac{\sqrt[3]{2a}}{3}, \frac{2}{3} a \right) \)

Available at http://www.MathCity.org
Now for tangent parallel to $y$-axis.
\[ \frac{dy}{dx} = \infty \]
\[ \frac{ax - y}{x^2 - ay} = \infty \]
\[ \Rightarrow \frac{ax}{x^2 - ay} = 0 \]
\[ \Rightarrow x = ay \]
\[ \Rightarrow x = \frac{y^2}{a} \quad \text{(iii)} \]

Put in (ii):
\[ \left(\frac{y}{a}\right)^3 + y^3 = 3a \cdot \frac{x^2}{a} \cdot y \]
\[ \Rightarrow \frac{y^6}{a^3} + y^3 = 3a \frac{x^2}{a} \cdot y \]
\[ \Rightarrow \frac{y^6}{a^3} = 3a \frac{x^2}{a} \cdot y \]
\[ \Rightarrow \frac{y^6}{a^3} = 3a \frac{x^2}{a} \cdot \frac{y}{a} \]
\[ \Rightarrow \frac{y}{a} = 2a \quad \text{(iv)} \]

Put in (iii):
\[ x = \frac{(2a)^3}{a} \]
\[ x = 8a \quad \text{(v)} \]

Hence tangent will parallel to $y$-axis at $\left( \frac{1}{8}a, \frac{3}{8}a \right)$

(ii) $3x^2 + 12xy + 4y^2 = 1$

Diff. w.r.t $x'$
\[ 6x + 12y + 12x \frac{dy}{dx} + 8y \frac{dy}{dx} = 0 \]
\[ (12x + 8y) \frac{dy}{dx} = -6x + 12y \]
\[ \frac{dy}{dx} = \frac{6x + 12y}{12x + 8y} \]

The tangent will parallel to $x$-axis if
\[ \frac{dy}{dx} = 0 \]
\[ \Rightarrow \frac{6x + 12y}{12x + 8y} = 0 \Rightarrow 6x + 12y = 0 \]
\[ 25x + 6y = 0 \]
\[ x = -\frac{6}{25} y \quad \text{(ii)} \]

\textbf{Put in (i)}

\[ 25\left(-\frac{6}{25}y\right)^2 + 10\left(-\frac{6}{25}y\right)y + 4y^2 = 1 \]
\[ \frac{36}{25}y^2 - \frac{12}{25}y^2 + 4y^2 = 1 \]
\[ y^2\left(\frac{36 - 12 + 100}{25}\right) = 1 \]
\[ y^2 = \frac{4}{8} \]
\[ y = \pm \frac{\sqrt{2}}{2} \quad \text{(iii)} \]

\textbf{Put in (ii)}

\[ x = -\frac{6}{25}\left(\pm \frac{\sqrt{2}}{2}\right) \quad x = -\frac{6}{25} \left(\pm \frac{\sqrt{2}}{2}\right) \]
\[ x = -\frac{3}{20}, \quad x = \frac{3}{20} \]

\textit{Hence the tangent is parallel to x-axis at (-3/20, 5/8) and (3/20, -5/8).}

\textbf{Now the tangent will parallel to y-axis if}

\[ \frac{dy}{dx} = \infty \]
\[ \frac{12y + 9x}{10x + 6y} = \infty \]
\[ 12y + 9x = 0 \]
\[ 3x + 2y = 0 \]
\[ y = -\frac{3x}{2} \quad \text{(iv)} \]

\textbf{Put in (i)}

\[ 25\left(-\frac{3x}{2}\right)^2 + 12\left(-\frac{3x}{2}\right) + 4\left(-\frac{3x}{2}\right)^2 = 1 \]
\[25x^2 - 18x^2 + 9 = 1\]
\[16x^2 = 1\]
\[x^2 = \frac{1}{16}\]
\[x = \pm \frac{1}{4}\] \(\text{--- (iv)}\)

Put in (iv)
\[y = -\frac{3}{2} \left( \frac{1}{4} \right) = -\frac{3}{8}\]
\[y = -\frac{3}{2} \left( -\frac{1}{4} \right) = \frac{3}{8}\]

Hence the tangent is parallel to \(y\)-axis at \(\left(\frac{1}{4}, -3/8\right), \left(-\frac{1}{4}, 3/8\right)\)

**Q.8.** If \(P = a \cos \theta + y \sin \theta\) touches the curve
\[
\left(\frac{x}{a}\right)^{n-1} + \left(\frac{y}{b}\right)^{n-1} = 1
\]
Prove that \(P^n = (a \cos \theta)^n + (b \sin \theta)^n\).

**Proof.**
The given eq. is \((\frac{x}{a})^{n-1} + (\frac{y}{b})^{n-1} = 1\) \(--- (i)\)

Diff. w.r.t. \(x\)
\[
\frac{n}{n-1} \left(\frac{x}{a}\right)^{n-2} \cdot \frac{1}{a} + \frac{n}{n-1} \left(\frac{y}{b}\right)^{n-2} \cdot \frac{1}{b} \frac{dy}{dx} = 0
\]
\[
\frac{n}{n-1} \left[ \frac{1}{a} \left(\frac{x}{a}\right)^{n-1} + \frac{1}{b} \left(\frac{y}{b}\right)^{n-1} \frac{dy}{dx} \right] = 0
\]
\[
\frac{1}{a} \left(\frac{x}{a}\right)^{\frac{1}{n-1}} + \frac{1}{b} \left(\frac{y}{b}\right)^{\frac{1}{n-1}} \frac{dy}{dx} = 0
\]
\[
\frac{1}{b} \left(\frac{y}{b}\right)^{\frac{1}{n-1}} \frac{dy}{dx} = -\frac{1}{a} \left(\frac{x}{a}\right)^{\frac{1}{n-1}}
\]
\[
\frac{dy}{dx} = -\frac{\frac{1}{b} \left(\frac{y}{b}\right)^{\frac{1}{n-1}}}{\frac{1}{a} \left(\frac{x}{a}\right)^{\frac{1}{n-1}}}
\]

Let \(P(x, y)\) be any point on the curve.
\[
\frac{dy}{dx} \bigg|_{(x_i, y_i)} = -\frac{1}{b} \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} - \frac{1}{a} \left( \frac{x_i}{y_i} \right)^{\frac{1}{n-1}}
\]

Thus the eq. of the tangent at \( P(x_i, y_i) \) is given by

\[
y - y_i = \frac{dy}{dx} \bigg|_P (x - x_i)
\]

\[
\Rightarrow \quad y - y_i = -\frac{1}{b} \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} (x - x_i)
\]

Multiplying by \( \frac{b}{1 - \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}}} \)

\[
\Rightarrow \quad \left( \frac{x_i}{b} \right) \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} - \left( \frac{y_i}{b} \right) \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} = -\left( \frac{x_i}{a} \right) \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} + \left( \frac{x_i}{a} \right) \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}}
\]

\[
\left( \frac{x_i}{b} \right) \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} - \left( \frac{y_i}{b} \right) \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} = -\left( \frac{x_i}{a} \right) \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} + \left( \frac{x_i}{a} \right) \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}}
\]

\[
\Rightarrow \quad \frac{x_i}{a} \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} + \frac{y_i}{b} \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} = \left( \frac{x_i}{a} \right) \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} + \left( \frac{y_i}{b} \right) \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}}
\]

\[
\frac{x_i}{a} \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} + \frac{y_i}{b} \left( \frac{y_i}{x_i} \right)^{\frac{1}{n-1}} = 1 \quad \text{(2)}
\]

Comparing from \( 1 \)

\[
\begin{align*}
\text{But} \quad \rho &= \frac{1}{a} \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} + \frac{1}{b} \left( \frac{y_i}{b} \right)^{\frac{1}{n-1}} \quad \text{(3)} \\
\text{Comparing (2) and (3)}
\end{align*}
\]

\[
\frac{1}{a} \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} = \frac{1}{b} \left( \frac{y_i}{b} \right)^{\frac{1}{n-1}} = \frac{1}{\rho}
\]

\[
\Rightarrow \quad \frac{1}{a} \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} = \frac{1}{\rho} = \Rightarrow \quad \frac{1}{a} \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} = \frac{1}{\rho} = \frac{1}{\rho}
\]

\[
\Rightarrow \quad \frac{\rho}{\left( \frac{x_i}{a} \right)^{\frac{1}{n-1}}} = \frac{1}{\rho} \quad \Rightarrow \quad \frac{\rho}{\left( \frac{x_i}{a} \right)^{\frac{1}{n-1}}} = \frac{1}{\rho} = \frac{1}{\rho}
\]

\[
\Rightarrow \quad \rho \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} = \rho \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} = \frac{1}{\rho} = \frac{1}{\rho}
\]

\[
\Rightarrow \quad \rho \left( \frac{x_i}{a} \right)^{\frac{1}{n-1}} = \left( a \cos \theta \right)^n \quad \text{(5)}
\]

\[
\text{and similarly}
\]

\[
\rho \left( \frac{y_i}{b} \right)^{\frac{1}{n-1}} = \left( b \sin \theta \right)^n \quad \text{(6)}
\]
\[
\left( a + b \right)^n = \sum \binom{n}{k} a^{n-k} b^k
\]

\[
p^n \left( \frac{x}{a} \right)^\frac{n}{a} + p^n \left( \frac{y}{b} \right)^\frac{n}{b} = \left( a \cos \theta \right)^n + \left( b \sin \theta \right)^n
\]

\[
p^n \left( \frac{x}{a} \right)^\frac{n}{a} + p^n \left( \frac{y}{b} \right)^\frac{n}{b} = \left( a \cos \theta \right)^n + \left( b \sin \theta \right)^n
\]

\[
p^n (1) = \left( a \cos \theta \right)^n + \left( b \sin \theta \right)^n
\]

\[
p^n = \left( a \cos \theta \right)^n + \left( b \sin \theta \right)^n
\]

Is as required.

\#9. The tangent at any point on the curve \( y^3 + y = 2x^3 \) makes intercepts \( p \) and \( q \) on the coordinate axes. Show that:

\[
\frac{p}{q} = \frac{-3}{4}, \quad \frac{q}{p} = \frac{3}{4}
\]
Alternative Method

The given curve is
\[ \frac{x^3}{y^3} = \frac{2a^3}{b^3} \]  \hspace{1cm} (i)

Eq. of the tangent on the curve at any point \( P(x, y) \) is
\[ x.x_1 + y.y_1 = 2a^3 \]  \hspace{1cm} (ii)

\( x \) -intercept with \( x \)-axis:

When the curve intercepts with \( x \)-axis then \( y = 0 \)

\[ \Rightarrow \frac{x}{x_1} = \frac{2a^3}{x_1} \]

The curve makes intercept with \( x \)-axis at \( P \)

\[ P = \frac{2a^3}{x_1} \Rightarrow x_1 = \frac{2a^3}{p} \]

\( y \) -intercept with \( y \)-axis:

When the curve intercepts with \( y \)-axis then \( x = 0 \)

\[ \Rightarrow \frac{y}{y_1} = \frac{2a^3}{y_1} \]

The curve makes intercept with \( y \)-axis at \( Q \)

\[ Q = \frac{2a^3}{y_1} \Rightarrow y_1 = \frac{2a^3}{y_1} \]

\( P(x_1, y_1) \) lies on the curve

\[ x_1^3 + y_1^3 = 2a^3 \]
\[ (a^2 b^{3/2})^3 + (\frac{1}{2} a^{3/2} b^{3/2})^3 = 2a^3 \]

\[ \frac{\frac{\sqrt{2}}{2} a^{3/2}}{b^{3/2}} + \frac{\frac{\sqrt{3}}{2} a^{3/2}}{b^{3/2}} = 2a^3 \]

\[ \Rightarrow p^{-3/2} + q^{-3/2} = \frac{2a^3}{a^{3/2} b^{3/2}} \]

\[ \Rightarrow p^{-3/2} + q^{-3/2} = \frac{2a^{3/2}}{a^{1/2} b^{1/2}} \]

As required.

**Angles of intersection of two curves:**

**Definition:** The angle between two curves at their common point \( P \) is defined as the angle between the tangents at this point.

Find the angle of intersection of the given curves (10-12)

1. The parabolas \( y = 4ax \) and \( x = 4by \) at the point other than \((0,0)\).

The given curves are

\[ y^2 = 4ax \quad \text{(i)} \]
\[ x^2 = 4by \quad \text{(ii)} \]
\[ i) \Rightarrow x = \frac{y^2}{4a} \quad \text{(iii)} \]

Put in (iii)

\[ \left( \frac{y^2}{4a} \right)^2 = 4by \]
\[ \frac{y^4}{16a^2} = 4by \]
\[ y^4 = 64a^2 by \]
\[ y^4 - 64a^2 by = 0 \Rightarrow y(y^3 - 64a^2 b) = 0 \]
\[ y = 0 \quad \text{or} \quad y^3 - 6a^2b = 0 \quad \Rightarrow \quad y = \frac{b}{a^2} \]

\[ y = 4a^2b^3 \]

**Case I:** If \( y = 0 \), put in (iii)

\[ x = 0 \]

\[ \Rightarrow \quad (0,0) \text{ is the point of intersection.} \]

**Case II:** If \( y = 4a^2b^3 \)

\[ x = \left(4a^2b^3\right)^{\frac{1}{3}} \]

\[ = \frac{16}{9}b^3 \] \[ = \frac{4}{9}a^3 \]

\[ \Rightarrow \quad (4a^3b^3, \frac{4}{9}a^3b^3) \text{ is the point of intersection.} \]

By the given condition we find the angle of intersection at \((4a^3b^3, \frac{4}{9}a^3b^3)\).

\[ \text{Diff. (i) w.r.t. } x', \quad \text{Diff. (ii) w.r.t. } x' \]

\[ \frac{dy}{dx} = 4a \quad \quad \frac{dx}{dx} = 4b \quad \frac{dy}{dx} = \frac{x}{2b} \]

\[ m_1 = \frac{dy}{dx} \bigg|_{x = \frac{a^3}{2b}} \quad m_2 = \frac{dy}{dx} \bigg|_{x = \frac{a^3}{2b}} \quad m_3 = \frac{4a^3b^3}{ab} \]

\[ m_1 = \frac{a^3}{2b^3} \quad m_2 = \frac{2a^{\frac{3}{2}}}{b^{\frac{3}{2}}} \quad m_3 = \frac{2a^{\frac{3}{2}}}{b^{\frac{3}{2}}} \]

Let \( \theta \) be the angle between the tangents at

\( (4a^3b^3, \frac{4}{9}a^3b^3) \).

Then

\[ \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} \]

\[ \tan \theta = \frac{\frac{a^3}{2b^3} - \frac{2a^{\frac{3}{2}}}{b^{\frac{3}{2}}}}{1 + \frac{a^3}{2b^3} \cdot \frac{2a^{\frac{3}{2}}}{b^{\frac{3}{2}}}} \]

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\[ \tan \theta = \left| \frac{\frac{a}{\sqrt{3}} - \frac{4a}{2b\sqrt{3}}}{1 + \frac{2a^2}{2b^2}} \right| \]

\[ \tan \theta = \left| \frac{-\frac{3a}{2\sqrt{3}}}{\frac{2ab}{2b^2}} \right| \]

\[ \tan \theta = \frac{\frac{3a}{\sqrt{3}} \cdot \frac{2b}{2b^2}}{2b^2} \quad \frac{2a^3 + 2a^3}{2b^2} \]

\[ \tan \theta = \frac{\frac{3a}{\sqrt{3}} b^{\frac{1}{3}}}{2 (2a^2 + b^2)} \]

\[ \Rightarrow \quad \theta = \tan^{-1} \left\{ \frac{\frac{3a}{\sqrt{3}} b^{\frac{1}{3}}}{2 (2a^2 + b^2)} \right\} \]

We have to find the angle of intersection between the curves:

\[ x^2 - y^2 = a^2 \quad \text{(i)} \]

\[ x^2 + y^2 = a^2 \sqrt{2} \quad \text{(ii)} \]

Let \((x, y)\) be the pt. of intersection of \((i) \text{ and } (ii)\)

Differentiate \((i)\) w.r.t. \(x\):

\[ 2x - 2y \frac{dy}{dx} = 0 \quad \Rightarrow \quad 2x + 2y \frac{dy}{dx} = 0 \]

\[ m_1 = \frac{dy}{dx} = \frac{x}{y} \quad m_2 = \frac{dy}{dx} = -\frac{x}{y} \]

Let \(\theta\) be the angle of intersection of \((i) \text{ and } (ii)\), then

\[ \tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \]

\[ \tan \theta = \left| \frac{x/y + x/y}{1 - x^2/y^2} \right| \]
\[ \tan \theta = \frac{3 x/y}{y^2 - x^2} = \frac{2 x y}{y^3 - x^3} \quad (A) \]

Now we find the coordinate \((x, y)\) from (i) and (iii): 
\[ x^2 - y^2 = a^2 \quad \Rightarrow \quad x^2 = y^2 + a^2 \quad (iii) \]

(ii) Put in (iii): 
\[ \begin{align*}
  y^2 + a^2 + y^2 &= a^2 \sqrt{2} \\
  a^2 + a^2 &= a^2 \sqrt{2} \\
  a^2 &= a^2 (\sqrt{2} - 1) \\
  y^2 &= \frac{a^2}{2} (\sqrt{2} - 1) \quad (iv) \\
\end{align*} \]

Put (iv) in (iii): 
\[ \begin{align*}
  x^2 &= \frac{a^2}{2} (\sqrt{2} - 1) + a^2 \\
  x^2 &= \frac{a^2}{2} (\sqrt{2} - 1 + 2) \\
  x^2 &= \frac{a^2}{2} (\sqrt{2} + 1) \quad (v) \\
\end{align*} \]

(iii) \( x (iv) \Rightarrow \)
\[ \begin{align*}
  x^2 - y^2 &= \frac{a^2}{2} (\sqrt{2} + 1) - \frac{a^2}{2} (\sqrt{2} - 1) \\
  x^2 - y^2 &= \frac{a^4}{4} \left( 2 - 1 \right) = \frac{a^4}{4} \\
  x y &= \pm \left( \frac{a^2}{2} \right) \quad (vi) \\
\end{align*} \]

Now \( i) \Rightarrow \)
\[ \begin{align*}
  x^2 - y^2 &= a^2 \\
  \gamma = \frac{a^2 - a^2}{-a^2} \quad (vii) \\
\end{align*} \]

Put (vi) and (vii) in \((A)\)
\[ \begin{align*}
  \tan \theta &= \frac{3 \left( \pm \frac{a^2}{2} \right)}{-a^2} \\
  &= \pm \frac{a^2}{-a^2} \\
  &= \frac{a^2}{-a^2} \quad ( - a^2 ) \\
\end{align*} \]
\[\Theta = \tan^{-1}(\pm 1)\]
\[\Theta = \frac{\pi}{2}\]
\[\tan \Theta = 1\]

\[y^2 = ax\quad \text{and} \quad x^3 + y^3 = 3axy\]

Here \(y^2 = ax\quad \text{--- (i)}\)
\[x^3 + y^3 = 3axy\quad \text{--- (ii)}\]

\[b = \frac{x}{a} \quad \text{--- (iii)}\]

\[\begin{align*}
\int & (y^2 + 3) = 3a\left(\frac{y^2}{2}\right) + (y) \\
\frac{y^6}{a^3} + y^3 &= 3y^3 \\
\frac{y^6}{a^3} &= 2y^3 \\
y^6 &= 2y^3a^3 \\
y^3 &= 2y^3a \\
y^6 - 2y^3a^3 &= 0 \\
y^3(y^3 - 2a^3) &= 0
\end{align*}\]

\[\Rightarrow \quad y = 0, \quad y^3 = 2a^3, \quad y = \sqrt[3]{2a^3}\]

Case I: if \(y = 0\) \quad put \(y = 0\) \quad in (iii)
\[x = 0\]
\[\Rightarrow \quad \text{the point of intersection is (0, 0)}\]

Case II: if \(y = \sqrt[3]{2a^3}\) \quad put \(y = \sqrt[3]{2a^3}\) \quad in (iii)
\[x = a\]
\[x = \sqrt[3]{2a^3} \quad \text{--- (iv)}\]
\[x = \frac{a^{\frac{7}{3}}}{a}\]
[\[x = 2\sqrt[3]{a}\quad \text{--- (v)}\]
\[\Rightarrow \quad \text{the point of intersection is (2a, 2a)}\]

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First we find the angle of intersection at \((0,0)\)

\[
\text{Diff. (i) w.r.t. } x' \\
\frac{dy}{dx} = a \\
\frac{dy}{dx} = \frac{a}{\frac{dy}{dx}} \quad \text{(iv)} \\
m_1 = \left. \frac{dy}{dx} \right|_{(0,0)} = a \\
m_1 = \infty \\
\text{Diff. (ii) w.r.t. } x' \\
3x^2 + 3y^2 \frac{dy}{dx} = 3ay + 3ax \frac{dy}{dx} \\
(\gamma - ax) \frac{dy}{dx} = ay - x^2 \\
\frac{dy}{dx} = \frac{ay - x^2}{\gamma - ax} \quad \text{(v)} \\
m_2 = \left. \frac{dy}{dx} \right|_{(0,0)} = 0 \\
\text{i.e. The angle of intersection is undefined at } (0,0)\).

Now we find the angle of intersection at \((\frac{\gamma}{2}, \frac{\gamma}{2})\).

\[
\text{(vi)} \Rightarrow \quad m_1 = \left. \frac{dy}{dx} \right|_{p} = \frac{a}{\frac{\gamma}{2} - \frac{\gamma}{2}a} = \frac{1}{2} \\
\text{(vii)} \Rightarrow \quad m_2 = \left. \frac{dy}{dx} \right|_{p} = \frac{a(\frac{\gamma}{2}) - (\frac{\gamma}{2})^2}{(\frac{\gamma}{2}a)^2 - a(\frac{\gamma}{2})} = a
\]

Let \(\theta\) be the angle of intersection.

\[
\tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2}
\]
\[
\tan \theta = \frac{m_1 (\frac{a}{b} - m_2)}{m_2 (\frac{b}{a} + m_1)}
\]
\[
= \frac{\frac{a}{b} - m_1}{\frac{m_1}{b} + m_2} = \frac{1}{\frac{b}{a} + 1}
\]
\[
\tan \theta = -\frac{1}{\frac{b}{a} + 1} = -2 \tan \gamma
\]
\[
\gamma = \frac{\pi}{4}
\]
\[
\theta = 68.21^\circ
\]

Is the required angle of intersection.

**Q.13** Find the condition that the curves \(ax^2 + by^2 = 1\) and \(a_1 x^2 + b_1 y^2 = 1\) should intersect orthogonally.

Here \(ax^2 + by^2 = 1\) \(\quad \text{(i)} \)

\[a_1 x^2 + b_1 y^2 = 1 \quad \text{(ii)} \]

**Diff w.r.t. \(x\)**

\[2ax + aby \frac{dy}{dx} = 0 \quad \text{Diff. w.r.t. } x \]

\[2a_1 x + b_1 y \frac{dy}{dx} = 0 \]

\[\frac{dy}{dx} = \frac{-ax}{by} \quad \text{and} \quad \frac{dy}{dx} = \frac{-a_1 x}{b_1 y} \]

\(\therefore\) the curves cut orthogonally

\[m_1 m_2 = -1 \]

\[\frac{2ax}{by} - \frac{a_1}{b_1 y} = -1 \]

\[\frac{a_1 x^2}{b_1 y^2} = -1 \quad \text{(iii)} \]

\(\therefore \) (iii) \(\Rightarrow\)

\[(a-a_1) x^2 + (b-b_1) y^2 = 0 \]

\[\frac{x^2}{y^2} = \frac{(b-b_1)}{(a-a_1)} \quad \text{(iv)} \]

**Put in** (iii)

\[+ \frac{a_1}{b_1} \cdot \frac{-{b-b_1}}{a-a_1} = -1 \]

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\[
\frac{a - a_1}{a} = \frac{b - b_1}{b}
\]
\[
\frac{a - a_1}{a_1} = \frac{b - b_1}{bb_1}
\]
\[
\frac{a_1 - a}{a_1} = \frac{b_1 - b}{bb_1}
\]
\[
\frac{1}{a} - \frac{1}{a_1} = \frac{1}{b} - \frac{1}{b_1}
\]
\[
\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}
\]

Is the required condition.

Q# 14. Show that the pedal eq. of the ellipse
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
is
\[
\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{y^2}{a^2b^2}
\]

Soln.

Here the eq. of an ellipse is
\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]  ---- (i)

Diff. w.r.t. \('x'\)
\[
\frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0
\]
\[
\frac{dy}{dx} = -\frac{a^2b^2}{b^2x} = -\frac{a^2y}{b^2y}
\]
\[
\frac{dy}{dx} = -\frac{b^2x}{a^2y}
\]

Let \(P(x,y)\) be the point where the tangent is drawn.

Then
\[
\frac{dy}{dx} = -\frac{b^2x}{a^2y}
\]

If \(p\) is the length of the distance of the tangent from \(O(0,0)\) then
\[
p = \sqrt{\left(y_1 - y_2\right)^2 + \left(x_1 - x_2\right)^2}
\]
\[ p = \frac{\gamma_i - \gamma_i}{\sqrt{1 + \left(-\frac{b\gamma_i}{a\gamma_i}\right)^2}} \]
\[ p = \frac{\gamma_i + \frac{b^2\gamma_i}{a\gamma_i}}{\sqrt{a^2\gamma_i^2 + b^2\gamma_i^2}} = \frac{\frac{a\gamma_i^2 + b^2\gamma_i^2}{a\gamma_i}}{\sqrt{a^2\gamma_i^2 + b^2\gamma_i^2}} \quad (2) \]

From (1)
\[ \frac{\gamma_i^2 + \gamma_i}{a^2 + b^2} = 1 \]
\[ \Rightarrow \quad b^2\gamma_i^2 + a^2\gamma_i = a^2b^2 \quad (3) \]

Put in (2)
\[ p = \frac{a^2b^2}{a^2b^2 + \frac{\gamma_i^2}{a^2} + \frac{\gamma_i^2}{b^2}} \]
\[ p^2 = \frac{1}{\frac{\gamma_i^2}{a^2} + \frac{\gamma_i^2}{b^2}} \quad (4) \]

We know that
\[ \frac{r_i}{a^2} = \frac{\gamma_i^2}{a^2} + \frac{\gamma_i^2}{b^2} \quad (5) \]
\[ \frac{r_i}{a^2} = \frac{\gamma_i^2}{a^2} + \frac{\gamma_i^2}{b^2} - \frac{\gamma_i^2}{b^2} \]
\[ \frac{r_i}{a^2} = \frac{\gamma_i^2}{a^2} + \frac{\gamma_i^2}{b^2} \quad (5) \]
\[ \Rightarrow \quad \frac{r_i}{a^2} = 1 + \frac{\gamma_i^2}{a^2 - \frac{b^2}{a^2}} \]
\[ \Rightarrow \quad \frac{r_i}{a^2} = \gamma_i^2 \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \]
\[ \Rightarrow \quad \frac{r_i}{a^2} = \gamma_i^2 \left( \frac{1}{a^2} - \frac{1}{b^2} \right) \]
\[ \frac{r^2 - a^2}{a^2} = \gamma^2 \left( \frac{b^2 - a^2}{r^2 b^2} \right) \]

\[ \frac{b^2 (r^2 - a^2)}{b^2 - a^2} = \gamma^2 \]

\[ \rho \text{ at } \theta = \frac{r^2}{\gamma^2} \]

\[ r^2 = \gamma^2 + \frac{b^2 (r^2 - a^2)}{b^2 - a^2} \]

\[ \gamma^2 = \frac{b^2}{b^2 - a^2} \]

\[ \gamma^2 = \frac{r^2 b^2 - r^2 a^2 - b^2 r^2 + b^2 a^2}{b^2 - a^2} \]

\[ \gamma^2 = \frac{a^2 (b^2 - r^2)}{b^2 - a^2} \]

\[ A \Rightarrow \]

\[ \frac{1}{P^2} = \frac{\gamma^2}{a^4 + \frac{b^4}{b^4}} \]

\[ \frac{1}{P^2} = \frac{a^2 (b^2 - r^2)}{b^2 - a^2} + \frac{b^2 (r^2 - a^2)}{b^2 - a^2} \]

\[ \frac{1}{P^2} = \frac{b^2 - r^2}{a^4 (b^2 - a^2)} + \frac{r^2 - a^2}{b^2 (b^2 - a^2)} \]

\[ \frac{1}{P^2} = \frac{b^2 - r^2}{b^2 - a^2} + \frac{r^2 - a^2}{a^2 b^2 (b^2 - a^2)} \]

\[ \frac{1}{P^2} = \frac{(b^2 - a^2) (b^2 - a^2) - r^2 (b^2 - a^2)}{a^2 b^2 (b^2 - a^2)} \]

\[ \frac{1}{P^2} = \frac{(b^2 - a^2) (b^2 - a^2) - r^2 (b^2 - a^2)}{a^2 b^2 (b^2 - a^2)} \]
\[
\frac{1}{b^2} = \frac{b^2 + x^2 - r^2}{a^2 b^2}
\]

\[
\frac{1}{b^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}
\]

Is as required.

**Q15.** Show that the pedal equation of the curve

\[c^2 (x^2 + y^2) = x^2 y^2\]

Is

\[
\frac{1}{b^2} + \frac{3}{y^2} = \frac{1}{c^2}
\]

**Solt.** Here

\[c^2 (x^2 + y^2) = x^2 y^2 \quad \text{(i)}\]

\[\div \text{by } c^2 \text{ we have}
\]

\[
\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{c^2} \quad \text{(ii)}
\]

Differentiating, we have

\[\frac{2}{x^3} \frac{dy}{dx} + (-2 \cdot \frac{1}{x^3}) = 0
\]

\[
\frac{2}{x^3} \frac{dy}{dx} = \frac{2}{x^3}
\]

\[\Rightarrow \frac{dy}{dx} = -\frac{2y^3}{x^3}
\]

\[\frac{dx}{dy} = -\frac{y^3}{x^3}
\]

Let \(p(x, y)\) be any point on the curve where the tangent drawn. Then the slope of the tangent at that point is given by

\[\frac{dy}{dx} = -\frac{y^3}{x^3}
\]

If \(p\) is the length of the line on the tangent from \((0, 0)\), then

\[p = \frac{|y_1 - x_1 y_1'|}{\sqrt{1 + y_1^2}}
\]

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\[ p = \frac{\gamma_i + \gamma_i (\gamma_i^2)}{\sqrt{1 + (\gamma_i^2)}^2} \]
\[ p = \frac{\gamma_i + \gamma_i^2}{\sqrt{1 + \gamma_i^2}} \]
\[ p = \frac{\gamma_i + \gamma_i^2}{\gamma_i^2} \]
\[ p = \frac{\gamma_i^3 \gamma_i^3 + \gamma_i^3}{\gamma_i^3 + \gamma_i^3} \]
\[ p = \frac{\gamma_i^3 + \gamma_i^3}{\gamma_i^3 + \gamma_i^3} \]
\[ p = \frac{\gamma_i^3}{\gamma_i^3 + \gamma_i^3} \]
\[ p = \frac{1}{\gamma_i^3 + 1/\gamma_i^3} \]

\[ \beta = \frac{\gamma_i^3}{\gamma_i^3 + 1/\gamma_i^3} \]

Using the identity
\[ a^3 + b^3 = (a + b)(a^2 - ab + b^2) \]

\[ p = \frac{1/\gamma_i^3 + 1/\gamma_i^3}{\sqrt{(1/\gamma_i^3 + 1/\gamma_i^3)^3 - 3(1/\gamma_i^3)(1/\gamma_i^3)(1/\gamma_i^3 + 1/\gamma_i^3)}} \]

\[ p = \frac{1/C^2}{\sqrt{(1/C^2)^3 - 3/C^2}} \cdot \frac{1/C^2}{1/C^2} \]

\[ b^4 = \frac{1/C^4}{1/C^4 - \frac{2}{C^2 \gamma_i^2 \gamma_i^2}} \]
\[ \frac{1}{p^2} = \frac{1/c^6 - \frac{3}{c^c x^l y^l}}{1/c^6} \]
\[ \frac{1}{p^2} = c^4 \left( \frac{1}{c^6} - \frac{3}{c^c x^l y^l} \right) \]
\[ \frac{1}{p^2} = c^4 \left( \frac{1}{c^6} - \frac{3 c^c}{x^l y^l} \right) \]

(iii)

We know that \( r^2 = x^l + y^l \)
\[ \frac{r^2}{x^l y^l} = \frac{1}{x^l} + \frac{1}{y^l} \]
\[ \frac{r^2}{x^l y^l} = \frac{1}{c^c} \]
\[ \frac{r^2}{x^l y^l} = \frac{1}{r^2} \]

Put in (iii) , we have
\[ \frac{1}{p^2} = \frac{1}{c^c} - \frac{3}{r^2} \]
\[ \frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^c} \]

Is as required.

Q#16. Show that from any point three normals can be drawn to a parabola \( y^2 = 4ax \) and the sum of squares of the three normals is zero.

\[ \text{We know that the eq. of the normal to the parabola} \]
\[ y^2 = 4ax \]
\[ \text{is} \]
\[ y = mx - am^3 \]
\[ \Rightarrow am^3 + 2am - mx + y = 0 \]
\[ am^3 + am^2 + (2a - x)m + y = 0 \]
\[ \text{This eq. is cubic in m} \]
\[ \text{We have three values of m from this eq.} \]
m being slope of the normal, we can say that three normals can be drawn to this parabola.

If \( m_1, m_2, \text{ and } m_3 \) are the slopes of this eq.

then we know that

\[
m_1 + m_2 + m_3 = -\frac{a - 2\alpha + \alpha^2}{\alpha - \alpha^2} \cdot \frac{a - 2\alpha + \alpha^2}{\alpha - \alpha^2}
\]

\[
a^2 + bx + c = 0
\]

If \( \alpha \& \beta \) are the roots

then \( \alpha + \beta = -\frac{b}{\alpha} \)

\[\alpha \beta = \frac{c}{\alpha}\]

Similarly for

\[a^2 + bx^2 + c = 0\]

\[\alpha + \beta = -\frac{b}{\alpha} \]

\[\alpha \beta = \frac{c}{\alpha}\]

Hence the sum of the slopes of the three normals is zero.

Q. 17. Show that tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix. (To understand this question see example).

Sam at the focal chord (AB)

of the parabola

\[y^2 = 4ax\]  

having the focus at

\[F(a, 0)\]

let \(A(at_1^2, at_1)\) and \(B(at_2^2, at_2)\) be the extremities of a focal chord.

Then we know that if \( (at_1^2, at_1) \text{ and } (at_2^2, at_2)\) be the extremities of a focal chord then

\[t_1t_2 = -1\]  

Also the equations of the tangent at \(A\) and \(B\) are

\[t_1y = x + at_1^2 \quad \Rightarrow \quad x - t_1y + at_1^2 = 0\]  

\[t_2y = x + at_2^2 \quad \Rightarrow \quad x - t_2y + at_2^2 = 0\]  

Now

\[m_1 = \text{slope of the tangent at } A = \frac{dx}{dy}\]  

\[m_2 = \text{slope of the tangent at } B = \frac{dx}{dy}\]  

\[m_3 = \text{slope of the tangent at } C = \frac{dx}{dy}\]
\[ m_1 = \frac{1}{-t_1} = \frac{1}{t_1} \]

\[ m_2 = \text{slope of the tangent at } B = -\frac{1}{-t_2} = \frac{1}{t_2} \]

Now \[ m_1 m_2 = \frac{1}{t_1} \cdot \frac{1}{t_2} = \frac{1}{t_1 t_2} \]

\[ m_2 = -1 \quad \text{from (ii)} \]

\[ \Rightarrow \text{The tangent at } A \text{ is } L \text{ to the tangent at } B. \]

Multiplying (iii) by \( t_2 \)

\[ t_2 x - t_2 y + a t_2 t_2 = 0 \quad (v) \]

Multiplying (iv) by \( t_1 \)

\[ t_1 x - t_1 y + a t_1 t_1 = 0 \quad (vi) \]

\( \times (vi) \Rightarrow \]

\[ (t_2 - t_1) x + a (t_1 t_2 - t_1 t_2) = 0 \]

\[ (t_2 - t_1) x + a t_1 t_2 (t_1 - t_2) = 0 \]

\[ (t_2 - t_1) x - a t_1 t_2 (t_2 - t_1) = 0 \]

\[ (x - a t_1) (t_2 - t_1) = 0 \]

\[ \Rightarrow x - a t_1 t_2 = 0 \]

\[ \Rightarrow x - at_{12} = 0 \quad \text{from (iii)} \]

\[ \Rightarrow x + a = 0 \]

Hence the tangents at the ends of a focal chord of a parabola intersect at right angle on the directrix.

**Prove**. (a) show that the tangent at the vertex of a diameter of a parabola is parallel to the chord bisected by the diameter.

Soh, let us consider the case of the parabola \[ y^2 = 4ax \quad (i) \]

Let \( AB \) be the one of the \( n \) chords whose eq. is \[ y = mx + c \]

we know that the equation of the diameter is \[ y = \pm \frac{2a}{y} \quad \text{where } m \text{ is the slope of the chord.} \]

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If the pt. \(C(x_1, y_1)\) lies on the diam\(\bar{\text{e}}\), then
\[
\gamma = \frac{2a}{x_1} \quad \quad \text{(i)}
\]
\[
\Rightarrow m = \frac{2a}{y_1} \quad \quad \text{(ii)}
\]
which is the slope of \(AB\).

We know that the equation of tangent at any point \(P\) of the parabola is
\[
y = 2ax
\]
New eq. of the tangent at \(C(x_1, y_1)\) is
\[
y_1 = 2a(x + x_1)
\]
\[
y_1 = 2ax + 2ax_1
\]
\[
2ax - y_1 - 2ax_1 = 0 \quad \quad \text{(iii)}
\]
At \(m = \text{slope of the tangent at } C = \frac{2a}{y_1} \quad \quad \quad \text{(iv)}
\]
\[
m_1 = -\frac{2a}{y_1} \quad \quad \text{(v)}
\]
Eq. \(B\) and \(iv \Rightarrow \)
\[
m = m_1
\]
\[
\Rightarrow \text{Tangent at } C \text{ is parallel to the chord } AB.
\]
(b) Prove that the tangents at the (vertices) of any chord of a parabola meet on the diameter which bisects the chord.

Let us consider the chord \(AB\) of the parabola
\[
y^2 = 4ax \quad \quad \text{(i)}
\]
Let the coordinates of \(A\) and \(B\) be \((x_1, y_1)\) and \((x_2, y_2)\) respectively.

Now equations of the tangents at \(A\) and \(B\) will
\[
y_1 = 2a(x + x_1)
\]
\[
y_2 = 2a(x + x_2)
\]
Subtracting
\[
y_1 - y_2 = 2a(x + x_1) - 2a(x + x_2)
\]

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\[
\begin{align*}
\gamma(x, y) &= \alpha x + 2\alpha x' - 2\alpha x'' - \alpha x'''
\gamma'(x, y) &= 2\alpha (x' - x''')
\Rightarrow 
\gamma &= \frac{2\alpha (x' - x''')}{\gamma(x, y)} \\
\Rightarrow 
\gamma &= \frac{2\alpha x'}{\gamma(x, y)}
\end{align*}
\]

If \(m\) is the slope of \(AB\), then
\[
m = \frac{\gamma - \gamma'}{x' - x''}
\]
Put in (i)
\[
\Rightarrow \frac{1}{m} = \frac{x'' - x'}{x' - x''}
\]
which is the eq. of the diameter.
Hence the tangents at the ends of any chord of a parabola meet on the diameter which bisects the chord.

Q. 19. Find the condition that straight line
\[
a x + my + n = 0
\]
may touch the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\). Also find the coordinates of the point of contact.

\[
\Rightarrow \text{Line } = a x + my + n = 0 \\
\Rightarrow \text{Ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
We know that the eq. of the tangent at \(P(x_1, y_1)\) is
\[
\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1
\]
(3)
\(\Rightarrow (a)\) and (3) are the tangents at the same point.
\(\Rightarrow\) Comparing (1) and (3)
\[
\frac{x}{a^2} = \frac{y}{b^2} = \frac{-1}{m}
\]
\[
\frac{x_1}{a^2} = \frac{y_1}{b^2} = \frac{-1}{m} \quad \text{(4)}
\]
4)\[ \frac{x_1}{a^2} = -\frac{1}{n} \]
\[ x_1 = -\frac{a^2}{n} \]

\[ \frac{y_1}{b^2} = -\frac{1}{n} \]
\[ y_1 = -\frac{b^2}{n} \]

Hence the point of contact is \((-\frac{a^2}{n}, -\frac{b^2}{n})\).

The point lies on the line
\[ l (-\frac{a^2}{n} + m (-\frac{b^2}{n}) - n = 0 \]
\[ -\frac{a^2}{n} + (-\frac{b^2 m}{n}) + n = 0 \]
\[ -\frac{a^2}{n} - \frac{m b^2}{n} + n = 0 \]
\[ \Rightarrow n = \frac{a^2}{b^2} + m b^2 \]
\[ n^2 = a^2 b^2 + m b^2 \]

is the required condition.

\( \textbf{Q.}\) Show that the condition that normals at that point's \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\) on the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) may be concurrent is

\[ \begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0 \]

\textbf{Sol.} The given eq. of the ellipse is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{(1)} \]

We know that the eq. of the normal at \((x_1, y_1)\) is

\[ \frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 - b^2 \]

\[ \text{by } x_1 y_1 \]
\[ a^2 y_1 + b^2 x_1 = (a^2 - b^2) x_1 y_1 \quad \text{(2)} \]
Similarly the equations of the normals at \((x_1, y_1)\) and \((x_2, y_2)\) are:

\[a^2 x^2 + b^2 y^2 = (a^2 - b^2)(x_1 y_2 - y_1 x_2) \quad (3)\]

\[a^3 x^3 + b^3 y^3 = (a^2 - b^2)(x_2 y_3 - y_2 x_3) \quad (4)\]

\[\Rightarrow \text{The condition is}\]

\[
\begin{vmatrix}
  a^2 x^2 & b^2 y^2 & (a^2 - b^2) x_1 y_2 - y_1 x_2 y_3 \\
  a^3 x^3 & b^3 y^3 & (a^2 - b^2) x_2 y_3 - y_2 x_3 y_1 \\
  a^4 x^4 & b^4 y^4 & (a^2 - b^2) x_3 y_1 - y_3 x_1 y_2 \\
\end{vmatrix} = 0
\]

\[a^2 b^2 (a^2 - b^2) \begin{vmatrix}
  y_1 & x_1 & y_1 x_1 y_2 - x_1 y_1 x_2 \\
  y_2 & x_2 & y_2 x_2 y_3 - x_2 y_2 x_3 \\
  y_3 & x_3 & y_3 x_3 y_1 - x_3 y_3 x_1 \\
\end{vmatrix} = 0
\]

\[a^2 b^2 (a^2 - b^2) \begin{vmatrix}
  y_1 & x_1 & y_1 x_1 y_2 - x_1 y_1 x_2 \\
  y_2 & x_2 & y_2 x_2 y_3 - x_2 y_2 x_3 \\
  y_3 & x_3 & y_3 x_3 y_1 - x_3 y_3 x_1 \\
\end{vmatrix} = 0
\]

\[\Rightarrow \begin{vmatrix}
  x_1 & y_1 & x_1 y_1 \\
  x_2 & y_2 & x_2 y_2 \\
  x_3 & y_3 & x_3 y_3 \\
\end{vmatrix} = 0
\]

Is the required condition.

Q.\# 3! If a tangent to the ellipse \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) with centre \(C\), meets the major and minor axes in \(T\) and \(t\) prove that \(\frac{a^2}{CT^2} + \frac{b^2}{Ct^2} = 1\).

**Soln.**

The eq. of the ellipse is \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) \(\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)\)

The eq. of the tangent at any point \((x_1, y_1)\) of the ellipse is \(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1\) \(\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)\)

Intersection of tangent with major axis! :-
Put in (2)

\[ x \frac{x_1}{a^2} + 0 = 1 \]

\[ \Rightarrow x_1 = \frac{a^2}{x} \Rightarrow x = \frac{a^2}{x_1} \]

\[ \Rightarrow \text{The coordinates of } P \text{ are } (\frac{a^2}{x_1}, 0) \]

\[ x \text{ of tangent with minor axis} \]

\[ \Rightarrow k = 0 \]

\[ \text{Put in (2)} \]

\[ 0 + \frac{yy_1}{b^2} = 1 \]

\[ \Rightarrow y = \frac{b^2}{y_1} \]

\[ \Rightarrow \text{The coordinates of } T \text{ are } (0, \frac{b^2}{y_1}) \]

Now

\[ CT = (\frac{a^2}{x_1} - 0) + (0 - 0) \Rightarrow CT = \frac{a^2}{x_1} \]

Squaring \( (CT)^2 = \frac{a^4}{x_1^2} \) \hspace{1cm} (3)

and \( cT = \frac{b^2}{y_1} \)

\( (cT)^2 = \frac{b^4}{y_1^2} \) \hspace{1cm} (4)

3) \Rightarrow \frac{x_1^2}{a^2} = \frac{a^2}{cT^2} \hspace{1cm} (5)

4) \Rightarrow \frac{y_1^2}{b^2} = \frac{b^2}{cT^2} \hspace{1cm} (6)

3 + 4 \Rightarrow \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{a^2}{cT^2} + \frac{b^2}{cT^2}

\[ \Rightarrow \frac{a^2}{cT^2} + \frac{b^2}{cT^2} = 1 \]

is as required.
Show that the locus of the point of intersection of tangents at two points on the ellipse
\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

where \( \Delta \) is the difference of the eccentric angles of two points.

**Solution**

The ellipse is \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) \( \hspace{1cm} (1) \)

Difference in eccentric angles = \( 2\Delta \)

Target: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = \Delta^2 \)

Let \( P \left( a\cos\theta, b\sin\theta \right) \) and \( Q \left( a\cos\phi, b\sin\phi \right) \)

be any points on \( (1) \) with \( \theta \) and \( \phi \) as the eccentric angles of

Point \( P \) and \( Q \).

Then eq. of the tangent at \( P \left( a\cos\theta, b\sin\theta \right) \) to the

ellipse is

\[ \frac{x}{a} \cos\theta + \frac{y}{b} b\sin\theta = 1 \] \( \hspace{1cm} (2) \)

\[ \frac{b\cos\theta}{a} x + \frac{a\sin\theta}{b} y = 1 \]

Like wise the equation of tangent at \( Q \)

\[ \frac{a\cos\phi}{b} x + \frac{b\sin\phi}{a} y = 1 \] \( \hspace{1cm} (3) \)

Intersection of the tangents\( b \)

\[ \frac{a\cos\phi}{b} x + \frac{b\sin\phi}{a} y = 1 \] \( \hspace{1cm} (4) \)

\[ \frac{x}{a} = \frac{\sin\theta - \sin\phi}{\sin(e - \alpha)} \]

\[ \frac{y}{b} = \frac{\cos\theta - \cos\phi}{\sin(e - \alpha)} \]

\[ e - \alpha = \frac{\Delta}{2} \]

\[ \frac{x^2}{a^2} = \frac{\sin^2(e - \alpha)}{\sin^2\alpha} \]

\[ \frac{y^2}{b^2} = \frac{\sin^2(e - \alpha)}{\sin^2\alpha} \]

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\[(\sin \theta - \sin \beta)^2 + (\cos \theta - \cos \beta)^2 \]
\[= \frac{\sin^2 2\lambda}{\sin^2 2\lambda} \cdot \frac{(\sin \theta - \sin \beta)^2 + (\cos \theta - \cos \beta)^2}{\sin^2 2\lambda} \]
\[= \frac{\sin^2 \theta + \sin^2 2\lambda - 2 \sin \theta \sin \beta \sin \theta + \cos^2 \theta + \cos^2 2\lambda - 2 \cos \theta \cos \beta}{\sin^2 2\lambda} \]
\[= \frac{2 - 2(\cos \theta \cos \beta + \sin \theta \sin \beta)}{\sin^2 2\lambda} \]
\[= \frac{2(1 - \cos(\theta - \beta))}{\sin^2 2\lambda} \]
\[= \frac{2 \cdot 2 \sin^2 2\lambda}{4 \sin^2 2\lambda \cos^2 2\lambda} \]
\[= \frac{1}{\cos^2 2\lambda} \]

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = \sec^2 2\lambda \]

Is as required.

**Q2.** Show that the sum of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle and product of the lengths of perpendiculars is equal to square on the semi-minor axis.

**Soln.**

The ellipse is

\[\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\]

We know that the square of the tangent at any point on the ellipse can be written in the form
\[ y = mx + \sqrt{a^2 + b^2} \]  \hspace{1cm} (i)

\( \text{Slope of the tangent} = m \)

\( \text{Slope of the perpendicular to the tangent} = -\frac{1}{m} \)

Eq. of the perpendicular from \( (ae, c) \) to the tangent is

\[ (y - c) = -\frac{1}{m} (x - ae) \]

\[ \Rightarrow \quad my = -x + ae \]

\[ \Rightarrow \quad x + my = ae \]  \hspace{1cm} (ii)

Elimination of \( m \) \:

\[ y - mx = \sqrt{a^2m^2 + b^2} \]  \hspace{1cm} (iii)

Squaring and adding (ii) and (iii)

\[ (x + my)^2 + (y - mx)^2 = a^2e^2 + a^2m^2 + b^2 \]

\[ 1 + m^2)(x^2 + y^2) = a^2e^2 + a^2m^2 + b^2 \]

\[ (1 + m^2)(x^2 + y^2) = a^2e^2 + a^2m^2 + b^2 \]

\[ (1 + m^2)(x^2 + y^2) = a^2(1 + m^2) \]

\[ x^2 + y^2 = a^2 \]  \hspace{1cm} (iv)

Let product of \( FN \) and \( FM \) on the tangent = \( b^2 \)

Then \( i) \Rightarrow \quad xm - y + \sqrt{a^2m^2 + b^2} = 0 \)

Now \( |FN| = \text{Distance of F from the tangent} \)

\[ = \frac{|mae - \sqrt{a^2m^2 + b^2}|}{\sqrt{m^2 + 1}} \]

\[ = \frac{|mae + \sqrt{a^2m^2 + b^2}|}{\sqrt{m^2 + 1}} \]

Similarly

\[ |FM| = \frac{|-mae + \sqrt{a^2m^2 + b^2}|}{\sqrt{m^2 + 1}} \]

\[ |FN| \cdot |FM| = \frac{(mae + \sqrt{a^2m^2 + b^2})(\sqrt{a^2m^2 + b^2} - mae)}{m^2 + 1} \]

\[ |FN| \cdot |FM| = \frac{\sqrt{a^2m^2 + b^2} - mae}{m^2 + 1} \]
For ellipse

\[ \frac{a^2}{m^2} + b^2m^2(b^2-a^2) = \frac{a^2m^2 + b^2m^2(b^2-a^2)}{1+m^2} \]

\[ = \frac{a^2m^2 + b^2m^2(b^2-a^2)}{1+m^2} \]

\[ = \frac{b^2}{1+m^2} \]

1\,FN/1\,FM = \frac{b^2}{1+m^2}

(iii) In a circle having the centre at (0,0) and radius \( a \) and is the auxiliary circle, the product of the lengths of perpendiculars is equal to the square of the semi-minor axis.

(ii) #24. Prove that the area enclosed by the parallelogram formed by the tangents at the ends of the conjugate diameters of an ellipse is constant.

Let \( A'B' \) be the conjugate diameter then

\[ P = (a \cos \theta + b \sin \theta) \]

\[ Q = (-a \sin \theta, b \cos \theta) \]

\( ABCD \) is a parallelogram due to the tangents drawn at the pts. \( P, Q, P', Q' \).

Now, area of the parallelogram \( ABCD = \sqrt{\text{Area of the parallelogram}} \)

\[ \text{Area of the parallelogram } ABCD = \frac{1}{2} \left| 0 \right| \]

\[ \text{Vector area of the parallelogram } = \begin{vmatrix} \sin \theta & \cos \theta \ \end{vmatrix} \]

\[ = \hat{k} (-ab \sin \theta - ab \cos \theta) \]

\[ = -ab (\sin \theta + \cos \theta) \hat{k} \]

\[ = -ab \hat{k} \]

Required area of the parallelogram \( OPCA = \sqrt{ab} \).

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Given in (i)
\[ \Rightarrow \text{Area of the parallelogram } ABCD = 4(ab) \]
\[ a \cdot 4ab = 4a^2b \]
\[ 4a^2b = \text{Constant} \]

Hence the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

Q. #25. The hyperbolas \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) and \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \)
are said to conjugate to each other. If \( e \) and \( e' \) are eccentricities of a
hyperbola and its conjugate, prove that
\[ \frac{1}{e^2} + \frac{1}{e'^2} = 1 \]

\[ \text{Soln: The given equations of the hyperbolas are} \]
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (i) \]
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad (ii) \]

If \( e \) is the eccentricity of \( (i) \) then we know that
\[ \frac{c}{a} = \sqrt{e^2 - 1} \]
\[ \Rightarrow \]
\[ b^2 = a^2(e^2 - 1) \quad (iii) \]

Now (ii) =
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \]

If \( e' \) is the eccentricity of this hyperbola then
\[ \frac{c'}{a} = \sqrt{e'^2 - 1} \]
\[ \Rightarrow \]
\[ b'^2 = a^2(e'^2 - 1) \quad (iv) \]
\[ (iii) \times (iv) \]
\[ a^2b^2 = a^2b'^2 \]
\[ \frac{a^2b^2}{a^2b'^2} = (e^2 - 1)(e'^2 - 1) \]

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\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

**III.** Show that the asymptotes of the hyperbola

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

and the lines drawn from any point on the hyperbola parallel to the asymptotes form a parallelogram of constant area \( \frac{ab}{2} \).

\[ y = \pm \frac{b}{a} x \]

**Solu.**

Here hyperbola is \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

and eq. of the asymptotes is \[ y = \pm \frac{b}{a} x \]

Let \( P = (x_0, y_0) \) be any pt. on the hyperbola.

Let the line drawn at \( P \parallel \) to the asymptotes meet the asymptotes at \( P' \) and \( Q' \).

Eq. of line through \( P \) or drawn from \( P \parallel \) to the
asymptote: \( y = \frac{b}{a}x \)

- The asymptote and line drawn from \( P \) are \( \perp \).
- The slope of the asymptote and line are equal:
  \( i.e. \) the slope of line = \( \frac{b}{a} \)

Now eq. of the line drawn from \( P \) is

\( y = \frac{b}{a}x + \frac{b}{a}(x - a \sec \theta) \) \( \quad (2) \)

Also eq. of the asymptote is

\( y = -\frac{b}{a}x \) \( \quad (3) \)

\( P \) in the pt. of intersection eq (2) and (3)

Put (3) in (2)

\[ \Rightarrow -\frac{b}{a}x + \frac{b}{a}(x - a \sec \theta) = \frac{b}{a}x + \frac{b}{a} \]

\[ b \sec \theta - b \tan \theta = \frac{b}{a}x + \frac{b}{a} \]

\[ \Rightarrow x = \frac{a}{b}(\sec \theta - \tan \theta) \] \( \quad (4) \)

Put (4) in (1)

\[ y = -\frac{b}{a} \left( \frac{a}{b}(\sec \theta - \tan \theta) \right) \]

\[ y = -\frac{b}{a}(\sec \theta - \tan \theta) \]

Hence the coordinates of \( P \) are

\( \left( \frac{a}{b}(\sec \theta - \tan \theta), -\frac{b}{a}(\sec \theta - \tan \theta) \right) \)

**Now**

The vector area of the \( \text{llm} \) OPQ is:

\[ \begin{vmatrix} x & y & 1 \\ \frac{b}{a}(\sec \theta - \tan \theta) & -\frac{b}{a}(\sec \theta - \tan \theta) & 0 \\ \frac{a}{b}(\sec \theta - \tan \theta) & -\frac{b}{a}(\sec \theta - \tan \theta) & 1 \end{vmatrix} \]

\[ = \left[-\frac{ab}{a^2}(\sec \theta - \sec \theta \tan \theta) - \frac{ab}{a^2}(\sec \theta \tan \theta - \sec \theta \tan \theta) \right] \]

\[ = \left[-\frac{ab}{a^2}(\sec \theta - \sec \theta \tan \theta - \sec \theta \tan \theta + \sec \theta \tan \theta) \right] \]

\[ = -\frac{ab}{a^2} \]

Hence area of the \( \text{llm} \) OPQ is:

\[ \sqrt{\left(-\frac{ab}{a^2}\right)^2} = \frac{ab}{a^2} \]

i.e. the area of \( \text{llm} \) OPQ = \( \frac{ab}{a^2} \)

Is as required.
Draw a normal to the curve at the point 't'.

Note: The point \( (ct, \frac{ct^2}{2}) \) on \( xy = c^2 \) is referred to as the point 't'.

Proof: The equation of the rectangular hyperbola is

\[ xy = c^2 \quad (i) \]

We find the equation of the normal at the point \( (ct, \frac{ct^2}{2}) \).

From (i),

\[ \frac{dy}{dx} = -\frac{c^2}{x^2} \]

\[ \frac{dy}{dx} \bigg|_{x=ct} = -\frac{c^2}{(ct)^2} = -\frac{1}{t^2} \]

The equation of the normal

\[ y - \frac{ct^2}{2} = -\frac{1}{t^2} (x - ct) \]

\[ y - \frac{ct^2}{2} = \frac{t(x - ct)}{t^2} \]

\[ t(x - ct) + t^2y - ct^2 = x - ct^4 \]

\[ t^3x + ty + c - ct^2 = 0 \quad (ii) \]

If this normal meets the hyperbola at \( t' (ct', \frac{ct'^2}{2}) \), then

\[ t'^3 + ct' - ct^2 = 0 \]

\[ t'^3 - t' + tct^4 = 0 \]

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\[ ct' t^2 + ct - ct' t^2 - ct = 0 \]
\[ ct' (t' t^2 - 1) - ct (t' t^2 + 1) = 0 \]
\[ (ct' - ct) (t' t^2 + 1) = 0 \]
\[ c (t' t^2 - 1) = 0 \]

\[ \Rightarrow \quad t' t^2 + 1 = 0 \quad \Rightarrow \quad t' \neq t \]
\[ t' t^2 = -1 \]
\[ \text{or} \quad t' t = -1 \]

Is as required.

Q#23. Prove that if \( P \) is any point on a hyperbola with foci \( F_1 \) and \( F_2 \), the tangent at \( P \) bisects the angle \( F_1 PF_2 \).

\[ \text{Soh.} \]

Consider the hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{(i)} \]

having the pt.
\[ P(a \sec \theta, b \tan \theta) \]
on it.

\[ \frac{dx}{a^2} - \frac{dy}{b^2} \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \]

\[ m_1 = \text{slope of the tangent at} \ P = (dy/dx) \]
\[ m_1 = -\frac{b^2 \sec \theta}{a^2 \tan \theta} \]

\[ m_1 = \frac{b \sec \theta}{a \tan \theta} \quad \text{(ii)} \]

\[ m_2 = \text{slope of} \quad \overrightarrow{PF} = \frac{y_2 - y}{x_2 - x_1} = \frac{b \sec \theta - 0}{a \sec \theta - ae} \]

\[ m_2 = \frac{b \sec \theta}{a \sec \theta - ae} \]

Now \[ \tan \quad \text{angle} \]
\[ = \frac{m_2 - m_1}{1 + mm_1} \]
\[
\tan \theta = \frac{b \tan \phi - a \sec \phi}{a \sec \phi - b \tan \phi}
\]

\[
\tan \theta = \frac{1 + \frac{b^2 \sec \phi}{a \sec \phi - b \tan \phi}}{a \sec \phi - b \tan \phi} - \frac{a \tan \phi - b \sec \phi}{a \sec \phi - b \tan \phi}
\]

\[
\tan \theta = \frac{(a \sec \phi - ae)(a \tan \phi) + b \tan \phi b \sec \phi}{(a \sec \phi - ae)(a \tan \phi)}
\]

\[
\sin \theta = \frac{ab \tan \phi + ab e \sec \phi a \sec \phi}{(a^2 + b^2) \sec \theta \tan \phi - a \tan \phi b \sec \phi}
\]

\[
\sin \theta = \frac{(-ab + ab e \sec \phi)}{a \sec \phi \tan \theta}
\]

\[
\sin \theta = \frac{ab(e \sec \phi - 1)}{a e \sec \phi(e \sec \phi - 1)}
\]

\[
\sin \theta = \frac{b}{ae \tan \phi}
\]

\[
\Rightarrow \alpha = \sin^{-1} \left( \frac{b}{ae \tan \phi} \right)
\]

Similarly, for \( \beta \), only replace \( ae \) by \(-ae\) to get:

\[
\beta = \tan^{-1} \left( \frac{b}{ae \tan \phi} \right)
\]

\[
\Rightarrow \alpha = \beta
\]

Which is the required.
Q# 29. Find an equation of a tangent to the hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
in the form
\[ \frac{x}{a} \coth \theta - \frac{y}{b} \sinh \theta = 1 \]
show that the product of lengths of perpendiculars on it from the foci is constant.

Sub.
The given hyperbola is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \)

- P(a \cosh \theta, b \sinh \theta) pt. lies on it.
- Eq. of tangent at P is given by
  \[ \frac{x}{a} \cosh \theta - \frac{y}{b} \sinh \theta = 1 \]
  \[ \Rightarrow \frac{x}{a} \cosh \theta - \frac{y}{b} \sinh \theta = 1 \]

is the required eq. of the tangent.

Now set \( d_1 \) and \( d_2 \) are the perpendicular distances from F1, F2 to the tangent line, then from (i)

\[ bx \cosh \theta - ay \sinh \theta - ba = 0 \]

Now
\[ d_1 = |FA1| = \text{Distance of F(a, e, 0) from the tangent line} \]
\[ d_1 = \frac{|bae \cosh \theta - 0 - ab|}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}} \]
Now \( d_2 = |FB| = \text{Distance of } f(-a,0) \text{ from the tangent line} \)

\[
d_2 = \frac{|b \cdot e \cdot \cosh \theta - a \cdot ab|}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}
\]

Hence

\[
d_1 = \frac{ab(e \cdot \cosh \theta - 1)}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}
\]

\[
d_2 = \frac{ab(e \cdot \cosh \theta + 1)}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}
\]

Now

\[
d_1/d_2 = \frac{ab(e \cdot \cosh \theta - 1) \cdot ab(e \cdot \cosh \theta + 1)}{\sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta} \cdot \sqrt{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta}}
\]

\[
= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{b^2 \cosh^2 \theta + a^2 \sinh^2 \theta} \Rightarrow \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{b^2 \cosh^2 \theta + a^2 \cosh^2 \theta - a^2}
\]

\[
= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{a^2 e^2 \cosh^2 \theta - a^2}
\]

\[
= \frac{a^2 b^2 (e^2 \cosh^2 \theta - 1)}{a^2 (e^2 \cosh^2 \theta - 1)}
\]

\[
= \frac{b^2}{a^2}
\]

\text{constant}.

It as required.
Find an equation of a normal to the hyperbola
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]
in the form
\[ \frac{ax}{\sec \theta} + \frac{by}{\tan \theta} = a^2 + b^2. \]

Prove that the normal is external bisector of the angles between the focal distances of its foot.

Soh.

The eq. of the hyperbola is
\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]  \( (i) \)

Diff. \( (i) \) w.r.t. \( x \)

\[ \frac{2x}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0 \]

\[ \frac{dy}{dx} = \frac{bxy}{a^2y} \]

Then the slope of the normal is \( -\frac{a \sin \theta}{b} \) and equation of the normal at \( P \) is

\[ y - b \tan \theta = -\frac{a \sin \theta}{b} (x - a \sec \theta) \]

\[ by - b^2 \tan \theta = -ax \sin \theta + a^2 \sin \theta \]

\[ by - b^2 \tan \theta = -ax \sin \theta + a^2 \tan \theta \]

\[ by + ax \sin \theta = a^2 \tan \theta + b^2 \tan \theta \]

\[ by + ax \sin \theta \cdot \sec \theta = \frac{a^2 + b^2}{\tan \theta} \]

\[ by + \frac{ax \sin \theta}{\tan \theta} = a^2 + b^2 \]

Is the required eq. of normal.
Additional Work

Question. Define homogeneous equation of degree 2. Prove that this eq. always represents two straight lines. Find the angle b/w these two straight lines also deduce a condition so that the straight lines are parallel.

ii) perpendicular.

Solution.

Def. The eq. of the form
\[ ax^2 + 2hxy + by^2 = 0 \]
where \( a, h, \) and \( b \) are not simultaneously zero, is called homogeneous eq. of degree 2.

Now we show that this eq. always represents two straight lines.

\[ \therefore \text{Homogeneous eq. of Degree 2 is} \]
\[ ax^2 + 2hxy + by^2 = 0 \]
\[ \div by^2 \rightarrow \frac{x^2}{y} + \frac{2hx}{y} + \frac{ax^2}{by^2} = 0 \]
which is quadratic in \( \frac{x}{y} \). So by using quadratic formula, we have

\[ \frac{x}{y} = \frac{-2h \pm \sqrt{4h^2 - 4ab}}{2b} \]
\[ \frac{y}{x} = \frac{-2h \pm \sqrt{h^2 - ab}}{2} \]
\[ \frac{x}{y} = \frac{-h \pm \sqrt{h^2 - ab}}{b} \]
\[ \frac{y}{x} = \frac{-h \pm \sqrt{h^2 - ab}}{b} \]

So the given homogeneous eq. of Degree 2 will represent two straight lines which are given by
\[ \gamma = \frac{-h + \sqrt{h^2 - ab}}{b} \quad \text{(2)} \]

\[ \gamma = \frac{-h - \sqrt{h^2 - ab}}{b} \quad \text{(3)} \]

Now from (2) and (3) it is clear that the slopes of the lines are

\[ m_1 = \frac{-h + \sqrt{h^2 - ab}}{b} \]

\[ m_2 = \frac{-h - \sqrt{h^2 - ab}}{b} \]

if \( \theta \) is the required angle then we know that

\[ \tan \theta = \frac{1}{m_1 m_2} \]

\[ = \frac{-h + \sqrt{h^2 - ab} + h + \sqrt{h^2 - ab}}{b} \]

\[ = \frac{b^2 (-h + \sqrt{h^2 - ab})(-h - \sqrt{h^2 - ab})}{b^2} \]

\[ = \frac{2b \sqrt{h^2 - ab}}{b^2} \]

\[ = \frac{2b \sqrt{h^2 - ab}}{b^2 + h^2 - (h^2 - ab)} \]

\[ = \frac{2b \sqrt{h^2 - ab}}{b(a + b)} \]

\[ \tan \theta = \frac{2 \sqrt{h^2 - ab}}{a + b} \quad \text{(4)} \]

Hence \( \theta = \tan^{-1} \left( \frac{2 \sqrt{h^2 - ab}}{a + b} \right) \)

Now if \( \tan \theta = 0 \)

\[ \Rightarrow \tan 0 = 0 \]
Put in (v) \[ \frac{\sqrt{h^2 - ab}}{a+b} = \tan 9\theta. \]

Is the required condition for lines to be parallel.

Now if \( \theta = 9\theta \)

\[ \Rightarrow \tan 9\theta = \infty \]

Put in (v)

\[ \frac{\sqrt{h^2 - ab}}{a+b} = \infty \]

\[ \Rightarrow a + b = 0 \]

is the required condition for perpendicularity.

**Question** (Theorem 6.15)

Show that the general eq. of 2nd in \( x \) and \( y \)
always represents a conic section.

**Ans.** We know that the general eq. of 2nd degree in \( x, y \)
is given by

\[ ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad (1) \]

where \( a, b, h \) and \( b \) are not simultaneously zero.

If the axis of coordinates rotate through an angle \( \theta \)
then we know that the new coordinates are

\[ x = x' \cos \theta - y' \sin \theta \]
\[ y = x' \sin \theta + y' \cos \theta \]

So from (1) we have

\[ a(x' \cos \theta - y' \sin \theta)^2 + 2h(x' \cos \theta - y' \sin \theta)(x' \sin \theta + y' \cos \theta) + b(x' \sin \theta + y' \cos \theta)^2 + 2g(x' \cos \theta - y' \sin \theta) + 2f(x' \sin \theta + y' \cos \theta) + c = 0 \]

\[ a(x'^2 \cos^2 \theta - 2x'y' \cos \theta \sin \theta + y'^2 \sin^2 \theta) + 2h(x'^2 \cos \theta \sin \theta + y'^2 \cos \theta \sin \theta) + b(x'^2 \sin^2 \theta + y'^2 \cos^2 \theta) + 2g(x' \cos \theta - y' \sin \theta) + 2f(x' \sin \theta + y' \cos \theta) + c = 0 \]

Available at http://www.MathCity.org
\[ a x^2 + ay^2 - 2xy = a \sin \theta \sin \phi + \sin \theta \cos \theta + \sin \phi \cos \phi - \sin \psi \cos \psi \]

\[ a x^2 - 2y^2 + b x^2 \sin \theta + b y^2 \sin \phi - 2x y \cos \theta \cos \phi - 2x y \sin \theta \sin \phi = 0 \]

\[ x^2 (a \cos \theta + b \sin \theta \sin \phi) + y^2 (a \sin \theta - b \cos \theta \cos \phi) + 2xy (b \sin \theta \cos \phi - c \cos \theta - d \sin \phi) = 0 \]

\[ x^2 (a \cos \theta + b \sin \theta \sin \phi) + 2x y [h (a \cos \theta - b \sin \theta) - (a - b) \sin \phi \cos \phi] + y^2 (a \sin \theta - b \cos \theta \cos \phi) + 2xy (f \cos \theta - g \sin \phi) + c = 0 \]

\[ \Rightarrow x^2 (a \cos \theta + b \sin \theta \sin \phi) + 2x y [h (a \cos \theta - b \sin \phi) - (a - b) \sin \phi \cos \phi] + y^2 (a \sin \theta - b \cos \theta \cos \phi) + 2xy (f \cos \theta - g \sin \phi) + c = 0 \]

Now

\[ A = a \cos \theta + b \sin \theta \cos \phi + b \sin \phi \]
\[ H = h (a \cos \theta - b \sin \phi) - (a - b) \sin \phi \cos \phi \]
\[ B = a \sin \theta - b \cos \theta \cos \phi + b \cos \phi \]
\[ C = g \cos \theta + f \sin \phi \]
\[ F = f \cos \theta - g \sin \phi \]

So (2) can be written as

\[ A x^2 + B xy + C y^2 + F x + G y + C = 0 \]

If we vanish \( \sin \) involving \( x \) in (2) then we have

\[ \Rightarrow \]

\[ A x^2 + B xy + C y^2 + F x + C = 0 \]

By (2) we have

\[ A = h (a \cos \theta - b \sin \phi) - (a - b) \sin \phi \cos \phi \]
\[ A = a \cos \theta - b \sin \phi \cos \phi + b \sin \phi \]

\[ \Rightarrow \]

\[ \sin \phi = -\frac{2h}{a - b} \]

\[ \cos \phi = \frac{2h}{a - b} \]

\[ \Rightarrow \]

\[ \tan \phi = \frac{a - b}{2h} \]

And then (4) can be written as

\[ A x^2 + B xy + C y^2 + F x + C = 0 \]
\[ A x^2 + 2Gxy + B y^2 + 2Fy' = -C \]
\[ A (x^2 + \frac{2G}{A}x + \frac{G^2}{A^2}) + B (y^2 + \frac{2F}{B}y + \frac{F^2}{B^2}) = -C \]
\[ A \left( x' + \frac{G}{A} \right)^2 - \frac{G^2}{A^2} + B \left( y' + \frac{F}{B} \right)^2 - \frac{F^2}{B^2} = -C \]
\[ A \left( x' + \frac{G}{A} \right)^2 + B \left( y' + \frac{F}{B} \right)^2 = \frac{G^2}{A^2} + \frac{F^2}{B^2} - C \]  

**Case I:** If \( MB \neq 0 \)

\[ \Rightarrow A \neq 0, \ B \neq 0 \]

so from (6), we have

\[ A x' + B y' = C \]  

where \( X = x' + \frac{G}{A}, \ Y = y' + \frac{F}{B} \)

and

\[ C = \frac{G^2}{A} + \frac{F^2}{B} - C \]  

(17) \( \div \) by \( C \), we have

\[ \frac{x^2}{C/A} + \frac{y^2}{C/B} = 1 \]

\[ \frac{x^2}{(\sqrt{C/A})^2} + \frac{y^2}{(\sqrt{C/B})^2} = 1 \]  

(18)

Now may discuss the following cases.

**Case 1:** If \( \frac{C}{A} \) and \( \frac{C}{B} \) are we and also

\[ \frac{C}{A} = \frac{C}{B} \text{ then (18) will represent a} \]

circle of the form

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]
Case (b): If \( \frac{x}{A} \) and \( \frac{y}{B} \) both are finite and \( \frac{A}{B} \neq \frac{C}{D} \), then (2) represents an ellipse.

Case (c): If both \( \frac{x}{A} \) and \( \frac{y}{B} \) are infinite, then (2) will represent an imaginary ellipse.

Case (d): If \( \frac{x}{A} \) and \( \frac{y}{B} \) are of opposite sign, then (2) will represent a hyperbola.

Case (e): If \( C = 0 \), then (2) can be written as

\[
Ax^2 + By^2 = 0
\]

which is homogeneous equation of degree 2 will represent two straight lines.

Case 2: If \( AB = 0 \), then \( A = 0 \) or \( B = 0 \) or \( A = 0 \), \( B = 0 \).

Now we discuss these cases.

Case (a): Both \( A \) and \( C \) cannot be zero because if so then from (5)

\[
0 + 2Cy + 2Fy + C = 0
\]

\[
2y^2 + 2Fy + C = 0
\]

which is linear and so it is a contradiction.

Case (b): If \( A < 0 \), \( B > 0 \), then (5) can be written as

\[
A x^2 + 2Cy + 2Fy + C = 0
\]

\[
A (x^2 + \frac{2Cy}{A} + \frac{2Fy}{A}) = -2Cy - C
\]

\[
A (x^2 + \frac{2y}{\frac{A}{B}} x + \frac{2y}{\frac{A}{B}} - \frac{A^2}{B^2}) = -2Fy - C
\]

\[
A \left( x^2 + \frac{y}{B} \right)^2 - \frac{A^2}{B^2} = -2Fy - (C)
\]

\[
A \left( x^2 + \frac{y}{B} - \frac{A^2}{B^2} \right) = -2Fy - C
\]

\[
A (x^2 + \frac{y}{B}) = -2Fy + \frac{A^2}{B^2} - C
\]

\[
A (x^2 + \frac{y}{B}) = -2Fy + \frac{A^2 - AC}{B}
\]
\[ A(x' + \frac{C}{A})^2 = \frac{2F}{\lambda} \left( y' - \frac{C^2 AC}{2\lambda F} \right) \]
\[ (x' + \frac{C}{A})^2 = \frac{2F}{\lambda} \left( y' - \frac{C^2 AC}{2\lambda F} \right) \]
\[ (x' + \frac{C}{A})^2 = \frac{4F}{\lambda} \left( y' - \frac{C^2 AC}{2\lambda F} \right) \]
\[ \Rightarrow x^2 = -\frac{4}{\lambda} \frac{F}{\lambda} y^2 \]

*Where*
\[ x = x' - \frac{C}{A}, \quad y = y' - \frac{C^2 AC}{2\lambda F} \]

*which represent a parabola.*

**Corol.**

If \( A \neq 0, \ B = 0 \) in addition to this, \( F \) is also equal to zero.

So \( 5) \Rightarrow A x^2 + 2 C x + C = 0 \)

Then this will represent two straight lines.

**Corol.**

If \( A = 0, \ B \neq 0 \) then

\( 5) \Rightarrow \)
\[ B y^2 + 2 C x' + 2 F y' + C = 0 \]
\[ B (y' + \frac{2F}{B}) \]
\[ = \frac{C}{B^2} x' - C \]
\[ B (y' + \frac{2F}{B}) y' = -\frac{C}{B} x' - C \]
\[ B (y' + \frac{2F}{B}) \]
\[ = -\frac{C}{B} x' - C \]
\[ B (y' + \frac{2F}{B}) \]
\[ = -\frac{C}{B} \left( x' - \frac{F^2 - 8BC}{2B^2} \right) \]
\[ \Rightarrow (y' + \frac{2F}{B}) \]
\[ = -\frac{C}{B} \left( x' - \frac{F^2 - 8BC}{2B^2} \right) \]
\[ \Rightarrow y' = -\frac{C}{B} x' \]
where \( y = \frac{1}{2} \frac{f^2 - bc}{a^2} \)

which represents a parabola.

Case 4. If \( A = 0 \), \( B = 0 \) and \( C = 0 \) then

\[ S = y^2 + 2xy + x^2 = 0 \]

then this represents two straight lines and it is quintic in \( y \).

From the above discussion we see that the case the general equation of 2nd degree always represents a conic section.

In above article we may discuss the note.

Note 3

\[ h^2 - AB = 0. \]

\[ h(C^2 - 4a^2) - 4(c+b)(a-b) \]

\[ = hC^2 - 4a^2 - ahC + ab^2 \]

\[ = hC^2 - a(C + b) \]

\[ - hC^2 + ahC + ab^2 \]

\[ = h^2 - ab \]

Similarly, we may show that

\[ A + B = a + b \]

These are called invariants of rotation.

Note 2. The invariants of rotation provide a rule to identify the conic, which is as follows.

1) If \( h^2 - ab > 0 \) then conic will be hyperbola.

2) If \( h^2 - ab < 0 \) then conic will be an ellipse.

3) If \( h^2 - ab = 0 \) then conic will be a parabola.
Theorem. Consider that \( F_1(c, 0), F_2(-c, 0) \) are the foci of an ellipse such that the sum of the distances of all points on the ellipse to the foci is \( 2a \) then prove that the equation of the ellipse is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

**Proof:** Consider the point \((x, y)\) on the ellipse then given that

\[
\begin{align*}
|PF_1| + |PF_2| &= 2a \\
|PF_1| &= \sqrt{(x - c)^2 + y^2} \\
|PF_2| &= \sqrt{(x + c)^2 + y^2}
\end{align*}
\]

Put in (1)

\[
\begin{align*}
\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} &= 2a \\
\sqrt{x^2 + c^2 - 2cx + y^2} + \sqrt{x^2 + c^2 + 2cx + y^2} &= 2a \\
x^2 + c^2 - 2cx + y^2 &= \left(2a - \sqrt{x^2 + c^2 + 2cx + y^2}\right)^2 \\
x^2 + c^2 - 2cx + y^2 &= 4a^2 + x^2 + c^2 + 2cx + y^2 - 2a \sqrt{x^2 + c^2 + 2cx + y^2} \\
-2cx &= 4a^2 - 2cx = -2(2a) x^2 + c^2 + 2cx + y^2 \\
-2cx - 4a^2 - 2cx &= -4a^2 x^2 + c^2 + 2cx + y^2 \\
-4(3x + d) &= -4a \sqrt{x^2 + c^2 + 2cx + y^2} \\
cx + a^2 &= a \sqrt{x^2 + c^2 + 2cx + y^2} \\
c^2 x^2 + a^4 + 2a^2 cx &= a^2 x^2 + a^2 c^2 + 2ca^2 x + a^4 \\
a^2 x^2 + a^4 x^2 + a^4 y^2 + a^2 c^2 - a^4 &= 0 \\
(a^2 - c^2) x^2 + a^2 y^2 &= a^2 a^2 c^2 \\
(a^2 - c^2) x^2 + a^2 y^2 &= a^2 (a^2 - c^2)
\]

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
Theorem. Consider that $F_1 (c,0)$ and $F_2 (-c,0)$ are the foci of a hyperbola $H$. Difference between the distances from any arbitrary pt. on this hyperbola to the foci is $2a$. Then prove that eq. of the hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Proof: Prove $a$ and $b$.

Questions.

Find the equation of the tangent and equation of the normal to the following curves at the given pts.:

i) $x^2 + y^2 = a^2$ at $(x_1, y_1)$

ii) $y^2 = 4ax$ at $(x_1, y_1)$

iii) $y^2 = 4ax$ at $(at^2, 2at)$

iv) $x^2 = 4ay$ at $(x_1, y_1)$

v) $x^2 = 4ay$ at $(2at, at^2)$

vi) $x^2 + y^2 + 2fy + c = 0$ at $(x_1, y_1)$

vii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(x_1, y_1)$

viii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \sin \theta, b \cos \theta)$

ix) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$

x) $\frac{x}{a} - \frac{y}{b} = 1$ at $(x_1, y_1)$

xi) $\frac{x}{a} - \frac{y}{b} = 1$ at $(a \cos \theta, b \sin \theta)$

xii) $\frac{x}{a} - \frac{y}{b} = 1$ at $(a \csc \theta, b \sec \theta)$

Questions.

Find the equation of the tangent and normal at the given pts.:

i) $x = at^2, y = 2at$

ii) $x = a \cos \theta, y = b \sin \theta$

iii) $x = a \sec \theta$, $y = b \tan \theta$

iv) $x = a \csc \theta, y = b \sinh \theta$
Theorem

The pure distance of a pt. $P$ on the parabola from its focus is the same as distance of $P$ from the directrix i.e. $1PF = 1PA$

Proof:

Let $P(at^2, 2at)$ be a pt. on the parabola $y^2 = 4ax$ having the focus $F(a,0)$ and directrix $xx + a = 0$.

Then $1PF = (at^2 - a)^2 + (2at - a)^2$

$= a^2t^4 - 2at^2 + 4at^2 - 2a^2t + a^2$

$= a^2t^4 + a^2 + 2a^2t^2$

$= (a + at^2)^2$

$\Rightarrow 1PF = \frac{a + at^2}{\sqrt{1 + a^2}}$ \hspace{3cm} (1)

$1PA = \text{Distance of the pt. } P(at^2, 2at) \text{ from the line } xx + a = 0$

$= \frac{|at^2 + a|}{\sqrt{1 + a^2}}$ \hspace{3cm} (2)

(1) and (2) $\Rightarrow$

$1PF = 1PA$.

As required.

Theorem: If the tangent and the normal at a pt. $P$ of a parabola meet $x$-axis at $A$ and $B$ respectively, then prove that

$1PF = 1PA \times 1FB$ where $F$ is the focus.

Proof:

At the tangent and normal at the pt. $P(at^2, 2at)$ of the parabola $y^2 = 4ax$ meet $x$-axis at $A$ and $B$ resp. $F(a,0)$ is the focus of the parabola.

\textbf{Diagram}
We know that equation of the tangents at \( P \) in
\[ x - ty + at^2 = 0 \]
for the coordinates of \( A \) put \( y = 0 \) then
\[ x - 0 + at^2 = 0 \]
\[ x = -at^2 \]
\[ \Rightarrow \text{Coordinates of } A \ (-at^2, 0) \]
Similarly equation of the normal at \( P \) is
\[ tx + ty - at - at^3 = 0 \]
For the coordinates of \( B \) put \( y = 0 \), then
\[ tx + 0 - at - at^3 = 0 \]
\[ tx = at + at^3 \]
\[ x = \frac{at}{t^2} + at^2 \]
\[ \Rightarrow \text{Coordinates of } B \ are \ (\frac{at}{t^2}, at^2) \]

Now
\[ |FP| = \sqrt{\left( a - \frac{at^2}{t^2} \right)^2 + \left( \frac{at}{t^2} - 0 \right)^2} \]
\[ = \frac{a + at^4}{t^2} \hspace{1cm} (1) \]
\[ |FA| = \sqrt{\left( \frac{at}{t^2} - 0 \right)^2 + \left( 0 - 0 \right)^2} \]
\[ = \frac{a + at^4}{t^2} \hspace{1cm} (2) \]
\[ |FB| = \sqrt{\left( \frac{at}{t^2} + at^2 \right)^2 + \left( 0 - 0 \right)^2} \]
\[ = \frac{\sqrt{a + at^4}}{t^2} \hspace{1cm} (3) \]

(1), (2) and (3) \[ |FP| = |FA| = |FB| \]
as required.

**Diameter of a Parabola.**
The locus of the middle pts. of parallel chords of a
parabola is called the
**Diameter of the Parabola.**
Consider the parabola
\[ y^2 = 4ax \quad (1) \]

Let \( AB \) be one of the parallel chords.

We suppose that the coordinates of \( A \) and \( B \) are \((x_1,y_1)\) and \((x_2,y_2)\) respectively. Let \( C(h,k) \) be the mid-point of this chord.

Consider that the equation of this chord is
\[ y = mx + c \]
\[ x = \frac{y - c}{m} \quad (2) \]

Put in (1)
\[ y^2 = 4a \cdot \frac{y - c}{m} \]
\[ my^2 = 4ay - 4ac \]
\[ my^2 - 4ay - 4ac = 0 \]

Which is quadratic in \( y \) thus if \( y_1 \) and \( y_2 \) are the roots of this eq. then
\[ y_1 + y_2 = \frac{-4a}{m} \quad (a + b)x + c = 0 \]
\[ y_1y_2 = \frac{4a}{m} \]
\[ \frac{y_1 + y_2}{2} = \frac{2a}{m} \quad (3) \]

\( C(h,k) \) is the mid pt. of \( AB \)
\[ h = \frac{x_1 + x_2}{2} \quad \text{and} \quad k = \frac{y_1 + y_2}{2} \]

Put in (3)
\[ k = \frac{2a}{m} \]

Hence equation of the diameter
\[ y = \frac{2a}{m} \]
Diameter of an ellipse:

The locus of the middle point of parallel chords of an ellipse is called diameter of the ellipse.

Equation of the Diameter of an Ellipse:

Consider that AB is one of the parallel chords of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) (1)

denote A and B has coordinates \((x_1, y_1)\) and \((x_2, y_2)\) resp.

\( C(h, k) \) is the mid point of \( AB \). Then \( h = \frac{x_1 + x_2}{2} \) and \( k = \frac{y_1 + y_2}{2} \)

we consider that eq. of the chord AB is \( y = mx + C \) (2)

Put in (1)

\[
\frac{x^2}{a^2} + \frac{(mx+C)^2}{b^2} = 1
\]

\[
b^2x^2 + a^2(mx+C)^2 = a^2b^2
\]

\[
b^2x^2 + a^2m^2x^2 + a^2Cx + a^2Cmx = -a^2b^2 = 0
\]

\[
(b^2+a^2m^2)x^2 + 2amCax + a^2C^2 - a^2b^2 = 0
\]

which is quadratic in \( x \).

\( \therefore \) if \( x_1 \) and \( x_2 \) are the roots of the equation then

\[
x_1 + x_2 = \frac{-2amC}{b^2+a^2m^2}
\]

\[
x_1x_2 = \frac{-a^2C^2}{b^2+a^2m^2}
\]

\[
=> h = \frac{-mc}{b^2+a^2m^2}
\]

\( C(h, k) \) lies on AB whose equation is \( y = mx + C \)
Then \( K = mh + C \)

\[ K - mh = C \]

Put this value of \( C \) in (4)

\[ h = -\frac{ma^2(K - mh)}{b^2 + a^2m^2} \]

\[ h(b^2 + a^2m^2) = -ma^2(h - mh) \]

\[ h^2 + ha^2m^2 = -ma^2k + ma^2h \]

\[ h^2 = -ma^2k \]

\[ k = -\frac{h}{ma^2} \]

Thus the equation \( \frac{y^2}{a^2} \) the diameter is

\[ y = -\frac{b^2}{ma^2}x \]

**Theorem.**

1) Prove that distance of a pt. \( P \) on an ellipse from the
focius = \( c \) times its distance from the corresponding
directrix.

2) Also prove that

\[ \frac{PF}{PF'} = \text{Constant} \]

**Proof:**

1) Let the pt. \( P(a\cos\phi, b\sin\phi) \) on the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \] \hspace{1cm} (1)

having the focius \((ae, 0)\)

and \( F'(-ae, 0) \), where \( al -
O(0, 0) = \text{directrix} \).

Also distance

\[ PF = \sqrt{(x - ae)^2 + y^2} \]

Now \( PF^2 = (ae - a\cos\phi)^2 + (0 - b\sin\phi)^2 \)

\[ = a^2e^2 + a^2\cos^2\phi - 2ae\cos\phi + b^2\sin^2\phi \]

\[ = a^2\cos^2\phi + b^2 - 2ae\cos\phi + b^2(1 - \cos^2\phi) \]

\[ = a^2\cos^2\phi + \frac{b^2 - a^2}{a^2} - 2ae\cos\phi + b^2 - b^2 \cos^2\phi \]

\[ = (a^2 - b^2) \cos^2\phi + \frac{b^2 - a^2}{a^2} - 2ae\cos\phi \] \hspace{1cm} (2)

\[ = (a^2 - b^2) \cos^2\phi + a^2 - 2a^2e\cos\phi \]

\[ = a^2 + \frac{b^2 - a^2}{a^2} - 2a^2e\cos\phi \]

\[ = a^2(c\cos\phi + 1 - 2ae\cos\phi) \] \hspace{1cm} (3)
\[ (PF')^2 = a^2 (e \cos \theta - 1)^2 \]

\[ (PF)^2 = a^2 (e \cos \theta - 1)^2 \quad (3) \]

Now equation of the directrix is

\[ x = e \quad \frac{e}{x} = a \quad ex = a \quad ex-a = 0 \quad (4) \]

Now \[ (PA) = \text{Distance of the pt. P from the directrix.} \]

\[ (PA) = \frac{a(1-e \cos \theta)}{e + a} \]

\[ (PA) = a(1-e \cos \theta) \]

\[ e(1-P) = a \quad (5) \]

\[ e(1-P) = a \quad (6) \]

Hence distance of pt. P on an ellipse from the focus = e times its distance from the corresponding directrix.

\[ (PF')^2 = a^2 (1-e \cos \theta)^2 \quad (5) \]

\[ (PF)^2 = a^2 (1-e \cos \theta)^2 \quad (6) \]

Similarly \[ (PF') = a(1+e \cos \theta) \quad (7) \]

(iii) Similarly \[ (PF) + (PF') = a(1-e \cos \theta) + a(1+e \cos \theta) \]

\[ = 2a \]

\[ = \text{Length of the major axis of the ellipse.} \]
Question.

Find the locus of the intersection of normals to the parabola \( y^2 = 4ax \) inclined at right angles to each other.

Solution.

The equation of the parabola is

\[
y^2 = 4ax
\]

Then the equation of the normal is

\[
y = mx - 2am - am^3
\]

If the pt. \((x_1, y_1)\) lies on it then

\[
y_1 = mx_1 - 2am - am^3
\]

\[
\Rightarrow \quad am^3 + am^2 + (2a - x_1)m + y_1 = 0 \quad (2)
\]

Now if \(m_1, m_2\) and \(m_3\) are the roots then

\[
m_1m_2m_3 = (-1)^3 \frac{y_1}{a} \quad (3)
\]

Hence given that two of the normals are 1

\[
m_1m_2 = -1 \quad \text{put } m(3)
\]

\[
m_3 = \frac{y_1}{a}
\]

\[
\Rightarrow \quad m_3 = \frac{y_1}{a}
\]

But \(m_3\) is the root of (2)

\[
\quad \frac{y_1}{a} + (2a - x_1) \frac{y_1}{a} + y_1 = 0
\]

\[
\frac{y_1}{a} + (2a - x_1) \frac{y_1}{a} + y_1 + a^2y_1 = 0
\]

\[
\frac{y_1}{a} + 3a^2y_1 - ax_1y_1 + a^2y_1 = 0
\]

\[
\frac{y_1}{a} + 3a^2y_1 - ax_1y_1 = 0
\]

Thus the locus of the pt. \((x, y)\) is

\[
y^2 + 3a^2y - axy = 0
\]

\[
y^2 + 3a^2y - axy = 0
\]

\[
y(3a^2 - ax) = 0
\]

\[
\Rightarrow \quad y = 0, \quad y^2 = ax - 3a^2
\]

\[
\Rightarrow \quad y = 0, \quad y^2 = ax - 3a^2
\]

as required.
Question: Show that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.

Solution:

Consider that the focal chord of the parabola \( y^2 = 4ax \). We know that the coordinates of the extremities are as shown.

Differentiating (1), we have

\[
2y \frac{dy}{dx} = 4a \\
\Rightarrow \quad \frac{dy}{dx} = 2\frac{a}{y} = m_1
\]

Now,

\[
m_1 = \text{Slope of the tangent at } A
\]

\[
m_1 = \frac{dy}{dx} \bigg|_{(at^2, at)} = \frac{2a}{2at} = \frac{1}{t}
\]

Now, \( m_2 = \text{Slope of the tangent at } B \)

\[
m_2 = \frac{dy}{dx} \bigg|_{\left(\frac{a}{t^2}, -\frac{a}{t}\right)} = \frac{2a}{-2at} = \frac{-1}{t}
\]

At \( m_2 = \text{Slope of the normal at } B \)

\[
- \frac{1}{m_2} = - \frac{1}{-t} = \frac{1}{t} = m_3
\]

From (3) and (5),

\[
m_1 = m_3
\]

\( \Rightarrow \) The tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.
Question

Plane that for tangent to a parabola intersect on the direction and the chord joining the pts. of contact
passes through the focus.

Solution

we know that the equation of the tangent to the parabola

\[ y^2 = 4ax \]

is

\[ ty = x + at^2 \]

\[ x - ty - at^2 = 0 \] \hspace{1cm} (1)

\[ m = \text{slope of the tangent} = \frac{a - \text{off.} y}{a - \text{off.} x} \]

\[ = \frac{1}{t} \]

Now \[ m = \text{slope of the tangent} \text{ for } t = \frac{1}{t} \]

\[ = -t \]

and so the equation of the tangent is

\[ x + \frac{t}{2} y + \frac{a}{2} t = 0 \] \hspace{1cm} (replace by \(-\frac{1}{t}\))

\[ t^2 x + ty + a = 0 \] \hspace{1cm} (a)

\[ x - ty + at^2 = 0 \]

\[ t^2 x + ty + a = 0 \]

\[ \Rightarrow (1 + t^2)(x + a) = 0 \]

\[ \Rightarrow x + a = 0 \] which is the equation of the directrix.

Hence, the linear tangent is a parabola intersect on directrix.
Now points of contact are
\[ A \left( at^2, 2at \right) \] and \[ B \left( \frac{5t}{3}, \frac{2a}{3} \right) \]

Eq. of the line passing through \( A \) and \( B \) is
\[ \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \]

Putting the values
\[ x = 2at \]
\[ y = \frac{-at^2 - 2at}{a - at^2} \]
\[ y - 2at = \frac{t^2}{a - at^2} (x - at^2) \]

\[ \Rightarrow y - 2at = \frac{-2a(1 + t^2)}{t} \frac{t^2}{a(1 - t^2)} (x - at^2) \]

\[ \Rightarrow y - 2at = \frac{-2t}{(1 - t^2)(1 + t^2)} (x - at^2) \]

Now the focus is \((a, 0)\) put in (4)
\[ 0 - 2at = \frac{-2t}{(1 - t^2)(1 + t^2)} (a - at^2) \]
\[ -2at = \frac{-2t}{(1 - t^2)(1 + t^2)} (a - at^2) \]
\[ -2at = a - at^2 \]
\[ \Rightarrow \] the chord joining \( A \) and \( B \) passes through the focus.

Question: If the tangent at \( P \) of a parabola meets the directrix at \( K \), then prove that \( PK \) is right-angled, where \( F \) is the focus.

Solution:
We know that the eq.
Of the tangent to the parabola at \( P \) is
\[ y = x + at^2 \]
but at \( K \), \( x = a \)
put in (1)
\[-\alpha - t \gamma + a \alpha^2 = 0\]
\[y = \frac{a (t^2 - 1)}{t} \Rightarrow \text{the coordinate of the point \( P \)}\]
\[(-a, \frac{a(t^2 - a)}{t})\]
Now \( m_1 = \text{slope of } PR = \frac{(t^2 - a) \cdot o - 2at}{a \cdot t - t} = \frac{-2at}{a \cdot t - t} = \frac{-2at}{a - t}\)
Also
\[m_2 = \text{slope of } PK = \frac{\frac{a(t^2 - a)}{t} - 0}{\frac{-2at}{a - t}} = \frac{t^2 - a}{-2at} \cdot \frac{a - t}{t} = \frac{a - at^2}{-2at} = \frac{a - at^2}{2at} \]
Consider
\[m_1 \cdot m_2 = \frac{-2at}{a - t} \cdot \frac{a - at^2}{2at} = \frac{-a}{t} \cdot \frac{a - at^2}{2at} = \frac{-a \cdot 2at}{2at} = \frac{-a}{t} \cdot \frac{a - at^2}{2at} = \frac{-1}{-}\]
\[\Rightarrow \frac{PK}{K^2} \cdot \frac{PK}{K^2} = 90^\circ \]

**Question:** If \( I \) is the tangent line at \( Pm(x,y) \) of a parabola \( y^2 = 4ax \) and if \( K \) is a line through \( P \parallel \) to \( x \)-axis. Show that the measure of the angle \( b/w \) \( K \) and \( I \) is equal to the measure of the angle between \( PA \) and \( PK \) defined.

Consider the tangent \( I \) at the pt. \((x_1, y_1)\) of the parabola \( y^2 = 4ax \) meet \( x \)-axis at \( A \). Now

\[|PF| = |FL| \quad (\text{Reflection property})\]
\[\Rightarrow \triangle AEP \text{\ is isosceles} \]
\[\Rightarrow \alpha = \beta \quad (i)\]
Also by elementary geometry

β = 0

Put in (1) ⇒

\[ x = 0 \]

as required.

**Question.**

If \( \alpha, \beta, \gamma \) are the eccentric angles of vertices of a \( \triangle \) inscribed in an ellipse, find the Area.

**Solution.**

Consider the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]

If \( A, B, C \) are the vertices of the said \( \triangle \), the eccentric angles of the vertices are given.

\[ \Rightarrow \]

The coordinates of these values will be as shown.

\[ \therefore \]

Area of \( \triangle ABC \) is given by

\[
\Delta = \frac{1}{2} \begin{vmatrix}
\alpha \cos \phi & b \sin \phi_1 & 1 \\
\alpha \cos \phi_2 & b \sin \phi_2 & 1 \\
\alpha \cos \phi_3 & b \sin \phi_3 & 1 \\
\end{vmatrix}
\]

\[ = \frac{1}{2} \left[ a \cos \phi \left( b \sin \phi_2 - b \sin \phi_3 \right) - b \sin \phi \left( a \cos \phi_2 - a \cos \phi_3 \right) \right] \]

Simplify to get result.

(Additional work is over)