**Tangent and Normal**

Let \( y = f(x) \) be the curve, and \( P(x_1, y_1) \) be any point on it.

\[
\begin{align*}
\frac{dy}{dx} &= f'(x) \\
\left( \frac{dy}{dx} \right)_{P(x_1,y_1)} &= f'(x_1)
\end{align*}
\]

**Tangent.** If a line touches the curve at the pt. \( P(x_1, y_1) \), then this line is called Tangent to the curve at the pt. \( P(x_1, y_1) \). The equation of the Tangent at this point to the curve \( y = f(x) \) is given by

\[
y - y_1 = m \cdot (x - x_1)
\]

Similarly \( y - y_1 = f'(x_1) \cdot (x - x_1) \)

**Normal.** A line passing through the pt. \( P(x_1, y_1) \) to the Tangent at the pt. \( P(x_1, y_1) \) to the curve is called the Normal at that pt. \( P \).

The equation of the Normal at the pt. \( P(x_1, y_1) \) to the curve \( y = f(x) \) is given by

\[
y - y_1 = -\frac{1}{f'(x_1)} \cdot (x - x_1)
\]

**Example 7.** Find the equation of the normal to the parabola \( y^2 = 4ax \) in the form \( y = mx - am^3 \)

**Soln.**

\[
y^2 = 4ax
\]

\[
\frac{dy}{dx} = 4a \\
\Rightarrow \frac{dx}{dy} = \frac{a}{y}
\]

\[
\frac{dy}{dx} \bigg|_{P(x_1,y_1)} = \frac{a}{y_1}
\]

**Slope of the normal** \( = -\frac{1}{\frac{a}{y_1}} = \frac{y_1}{a} \) (say)

http://www.MathCity.org
\[ y = 2ax \quad \Rightarrow \quad y_1 = -2am \quad \text{(2)} \]

\[ (x_1, y_1) \text{ lies on } y^2 = 4ax \]

\[ y_1^2 = 4ax_1 \quad \text{(3)} \]

Put \( y = -2am \) in (3)

\[ 4a^2m^2 = 4ax_1 \]

\[ \Rightarrow \quad x_1 = am^2 \quad \text{(4)} \]

\[ P(x_1, y_1) = P(am^2, -2am) \]

Eq. of the normal

\[ y - (-2am) = m(x - am^2) \]

\[ y + 2am = mx - am^3 \]

\[ y = mx - 2am - am^3 \]

It is as required.

**Parametric form of Parabola:**

\[ y^2 = 4ax \]

\[ x = at^2, \quad y = 2at \]

\( x = at^2, \quad y = 2at \) are the equations representing the parabola in parametric form. Where 't' is called the parameter.

**Parametric form of Ellipse:**

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

\[ x = a \cos \theta, \quad y = b \sin \theta \] are the parametric equations of the ellipse.

**Parametric form of Hyperbola:**

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \]

\[ x = a \sec \theta; \quad y = b \tan \theta \]

\[ \text{or} \quad x = a \cosh \theta; \quad y = b \sinh \theta \]

are the parametric equations of the hyperbola.
Example #8

Show that the points \((at, at^2)\) always lies on the parabola \(y = 4ax\). Find the condition that the chord joining the points \((at_1, at_1^2)\) and \((at_2, at_2^2)\) may be a focal chord. Find an equation of the tangent to the parabola at \((at, at^2)\).

Solution:

Let \(P(at_1, at_1^2)\) and \(Q(at_2, at_2^2)\).

Using two points formula:

\[
\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}
\]

Putting \((at_1, at_1^2)\) and \((at_2, at_2^2)\)

\[
\frac{y-at_1^2}{at_2-at_1} = \frac{x-at_1^2}{at_2-at_1}
\]

\[
\frac{y-at_1^2}{at_2-at_1} = \frac{x-at_1^2}{at_2-at_1}
\]

\[
\frac{y-at_1^2}{at_2-at_1} = \frac{x-at_1^2}{at_2-at_1}
\]

Focus: \((0, 0)\), put \(x = a\), \(y = 0\) in (1)

\[
\frac{a-at_1^2}{at_2-at_1} = \frac{x-at_1^2}{at_2-at_1}
\]

\[
\frac{a-at_1^2}{at_2-at_1} = \frac{x-at_1^2}{at_2-at_1}
\]

\[
\frac{a-at_1^2}{at_2-at_1} = \frac{x-at_1^2}{at_2-at_1}
\]
\[-t_i = \frac{1 - t_i^2}{t_i + t_i^2}\]

\[-t_i^2 - t_i t_i = 1 - t_i^2\]

\[-t_i t_i = 1\]

\[t_i, t_i = -1\]

Is the required condition for the chord PA to be focal chord.

Now equation of the tangent at \(P(\alpha t^2, 2\alpha t)\) to the parabola \(y^2 = 4\alpha x\) is

\[y - y_i = m(x - x_i)\] \(\text{(1)}\)

\[m = \text{slope of tangent at } P(\alpha t^2, 2\alpha t)\]

\[m = \frac{dy}{dx} = \frac{d(\alpha t)}{d(\alpha t^2)} = \frac{2\alpha t}{\alpha t} = \frac{2}{t}\]

\[
\Rightarrow \text{ we have to find the equation of tangent at } P(\alpha t^2, 2\alpha t).
\]

\[
\text{put } x_i = \alpha t^2, \; y_i = 2\alpha t \Rightarrow \frac{dy}{dx} = \frac{1}{t} \text{ in (2) we have}
\]

\[y - 2\alpha t = \frac{1}{t}(x - \alpha t^2)\]

\[ty - 2\alpha t = x - \alpha t^2\]

\[yt = x - \alpha t + 2\alpha t^2\]

\[\Rightarrow yt = x + \alpha t^2\]

Note: \(x = \alpha t^2, \; y = 2\alpha t\) are called the parametric equations of the parabola \(y^2 = 4\alpha x\). The point \((\alpha t^2, 2\alpha t)\) is also referred to as point "\(t\)" on the parabola.
Pedal Equation:

The pedal equation is an equation in \( p \) and \( r \) where 'r' is the distance of any point 'P' on the curve from O and \( p \) is the distance of O from the tangent at P.

Let \( P(x_1, y_1) \) be any point on the curve \( y = f(x) \). Then \( r = |OP| = \sqrt{x_1^2 + y_1^2} \) by distance formula.

\[
\begin{align*}
    r &= x_1^2 + y_1^2 \quad (1) \\
    P \text{ lies on the curve } f(x) \quad (2) \\
    \text{equation of the tangent at } P(x_1, y_1): \\
    (y - y_1) &= f'(x_1)(x - x_1) \\
    \nu &= f(x_1) \quad (3) \\
    \nu &= f'(x_1) \\
    \eta &= x - x_1 \\
    \eta &= y - y_1 \quad \text{from eq (1) to (3) will give us the equation in } p \text{ and } r \text{ called the pedal equation.}
\end{align*}
\]

Now, \( p = \text{distance of } O(0,0) \text{ from tangent line.} \)

\[
p = \frac{|f'(x_1)(0) - 0 + y_1 - x_1 f(x_1)|}{\sqrt{(f(x_1))^2 + 1}}
\]

The distance of point \( P(x_1, y_1) \) from line \( ax + by + c = 0 \) is given by

\[
p = d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}
\]

\[
\text{Elimination of } x_1 \text{ and } y_1
\]
Example #9

IF the tangent at any point \( Q \) of the parabola meets \( y \)-axis at \( A, \) then prove that
\[
\hat{PAF} = 90^\circ
\]
where \( P \) is any point on the parabola, \( A \) is point on \( y \)-axis and \( F \) is focus.

Proof

Consider that the tangent at the point \( P(at^2, 2at) \) of the parabola,
\[ y^2 = 4ax \]
meets \( y \)-axis at the point \( A. \)

\( P(a, 0) \) is the focus of this parabola.

We know that the equation of the tangent at the point \( (at^2, 2at) \) is
\[ x - ty + at^2 = 0 \]
for the coordinates of \( A, \) put \( x = 0 \)
then
\[ 0 - 2at + at^2 = 0 \]
\[ at^2 - 2at + 0 = 0 \]
\[ y = at \]
the coordinates of \( A(0, at) \)

\[ m_1 = \text{slope of } (PA) = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2at - at}{2at^2 - at^2} = \frac{at}{at^2} = \frac{1}{t} \]

by\[
m_2 = \text{slope of } (FA) = \frac{a - at}{0 - 2at} = \frac{-at}{a} = -t
\]

Now \[ m_1 m_2 = \frac{1}{t} \cdot -t = -1 \]

\[ \Rightarrow \overrightarrow{PA} \perp \overrightarrow{FA} \]
\[ \Rightarrow \hat{PAF} = 90^\circ \]
Example 10

Find the locus of the middle points of a system of parallel chords of the parabola \( y = 4ax \).

Soln. Given \( y = 4ax \)

Consider a system of chords of the parabola.

Let \( m \) be their slopes

and \( y = mx + c \) \( \cdots (2) \)

be the equation of the chord \( PA \), representative of the chords.

Further let \( R = (h,k) \)

be the middle point of the chord \( PA \) not on the parabola.

Now \( P \) from \( (2) \) in \( (1) \)

\[ (mx + c)^2 = 4ax \]

\[ m^2x^2 + 2mcx + c^2 = 4ax \]

\[ m^2x^2 + 2(mc - 2a)x + c^2 = 0 \] \( \cdots (3) \)

\( \Delta \) is quadratic in \( x \);

the values of \( x \) obtained from \( (3) \) will give as the \( x \)-coordinates of \( P \) and \( Q \).

\( R \) is the mid pt. of \( P \) \& \( Q \).

\[ h = \frac{x_1 + x_2}{2} \]

\[ h = \frac{-2(mc - 2a)}{m^2} \]

\[ h = \frac{-(mc - 2a)}{m^2} \]

\( \cdots (4) \)

\[ \Rightarrow \; R = (h,k) \] lies on \( (2) \)

\[ \Rightarrow \; h = mh + c \]

\[ \Rightarrow \; c = k - mh \] \( \cdots (5) \)

http://www.MathCity.org
\[ \text{(5) im}(y) = \gamma \Rightarrow h = \frac{-(m(k - mh) - 2a)}{m^2} \]

\[
\begin{align*}
hm^2 &= -(mk - m^2h) - 2a) \\
m^2h &= -mk + m^2h + 2a \\
m^2h - m^2h &= -mk + 2a \\
o &= -mk + 2a \\
vh &= 2a/m \\
k &= \frac{2a}{m} \\
\end{align*}
\]

\[ i.e. \text{ the pt. } R(h, k) \text{ lies on the locus } \gamma = \frac{2a}{m} \]
\[ x^2 \Omega \gamma = \frac{2a}{m} \text{ and } y^2 = 4ax \]
\[ \text{Put } \gamma = \frac{2a}{m} \text{ in } \gamma^2 = 4ax \]
\[ \frac{4a^2}{m^2} = 4ax \]
\[ \frac{a}{m^2} = x \]
\[ \begin{align*}
\frac{a}{m^2} &= x \\
x &= \frac{a}{m^2} 
\end{align*} \]

\[ \text{Point } \left( \frac{a}{m^2}, \frac{2a}{m} \right) \text{ is the locus.} \]

\[ \text{Note: } \text{The locus of the middle points of parallel chords of a parabola is called a diameter of the parabola.} \]

**Auxiliary Circle.**

Ref. "The circle constructed on the major axis of the ellipse as a diameter is called the auxiliary circle."

At \( P(x, y) \) be any point on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \). Draw \( PM \perp ax \)-axis.

Produce \( PM \) so that it meets the circle at \( Q \). Join \( QA \) to \( Q \).

http://www.MathCity.org
Let \( \alpha \not\equiv \theta \) then we call \( \theta \) as the eccentric angle of \( P \).

From \( \triangle OMQ \):

\[
\frac{x}{a} = \cos \theta
\]

\[
x = a \cos \theta
\]

Put \( x = a \cos \theta \) in \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \)

\[
\Rightarrow \frac{a^2 \cos^2 \theta}{a^2} + \frac{y^2}{b^2} = 1
\]

\[
\cos^2 \theta + \frac{y^2}{b^2} = 1
\]

\[
\frac{y^2}{b^2} = 1 - \cos^2 \theta
\]

\[
\frac{y^2}{b^2} = \sin^2 \theta
\]

\[
\gamma^2 = b^2 \sin^2 \theta
\]

\[
\gamma = b \sin \theta
\]

\[
\gamma = b \sin \theta
\]

i.e. \( P(x, y) = (a \cos \theta, b \sin \theta) \)

**Theorem.**

Show that the locus of the middle points of a system of \( n \) chords of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is

\[
\gamma = -\frac{b^2}{a^2 m} x
\]

where \( m \) is the slope of the \( n \) chords.

*The parallel chords*
have slope \( m \) so that the equation to any one of them, say \( PQ \) is

\[ y = mx + c \]

The straight line \( (1) \) meets the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

at points whose abscissae are given by

\[ \frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1 \]

\[ \Rightarrow \quad b^2x^2 + a^2(mx + c)^2 = a^2b^2 \]

\[ \Rightarrow \quad b^2x^2 + a^2(m^2x^2 + 2mcx + c^2) = a^2b^2 \]

\[ \Rightarrow \quad b^2x^2 + a^2m^2x^2 + 2a^2mcx + a^2c^2 = a^2b^2 \]

\[ \Rightarrow \quad x^2(b^2 + a^2m^2) + 2a^2mcx + a^2c^2 - a^2b^2 = 0 \]

i.e. \( x^2(\frac{a^2m^2}{b^2} + \frac{2a^2mc}{b^2} + \frac{a^2c^2}{b^2}) = 0 \)

Let the roots of this equation be \( x_1, x_2 \). Then \( x_1, x_2 \) are the abscissae of \( P \) and \( Q \). Let \( M(h,k) \) be the middle point of \( PQ \). Then by using them of the roots we have

\[ h = \frac{x_1 + x_2}{2} = \frac{-2a^2mc}{2a^2m^2b^2 + a^4b^2} = \frac{-a^2mc}{a^2m^2 + b^2} \]

(2)

\[ M(h,k) \] lies on (1).

\[ k = mh + c \]

\[ c = k - mh \]  

(3)

in (2)

\[ h = \frac{a^2m(h-mh)}{a^2m^2 + b^2} \]

\[ a^2m^2h + b^2h = a^2mk + a^2m^2h \]

\[ b^2h = a^2mk \]

\[ h = \frac{a^2m}{a^2m} \]

i.e. the pt. \( M(h,k) \) lies on the locus, \( y = \frac{a^2m}{a^2m} \).
**Diameters of an Ellipse**

**Def:** The locus of the middle pts. of a system of 11 chords of an ellipse is called a diameter of an ellipse.

**Conjugate Diameters**

**Def:** Two diameters of an ellipse are called conjugate if each bisects chord 11 to the other.

**Result for Conjugate Diameters:**

For Conjugate diameters the product of their slopes is \( \frac{-b^2}{a^2} \)

**Theorem**

If \( CP \) and \( CD \) are semiconjugate diameters of an ellipse with center \( C \), show that:

i) The eccentric angles of \( P \) and \( D \) differ by a right-angle.

ii) \( CP^2 + CD^2 = a^2 + b^2 \) is a constant.

iii) The locus of the point of intersection of tangents at \( P \) and \( D \) is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2(\cos \theta, \sin \theta)
\]

**Proof.** Let \( O \) and \( O' \) be the eccentric angles of \( P \) and \( D \) where \( CO \) and \( CO' \) are the semi conjugate diameters.

Then \( P = (a \cos \theta, b \sin \theta) \), \( D = (a \cos \theta', b \sin \theta') \)

http://www.MathCity.org
Slope of CP = \( \frac{b \sin \theta - 0}{a \cos \theta - 0} \)

= \( \frac{b \sin \theta}{a \cos \theta} \)

Similarly, slope of \( \mathcal{Q} \) \( \mathcal{P} \) = \( \frac{b \sin \theta}{a \cos \theta} \)

\( \mathcal{C} \) \( \mathcal{P} \) and \( \mathcal{Q} \) are semi conjugate diameters.

\[ \frac{b \sin \theta}{a \cos \theta} \times \frac{b \sin \theta}{a \cos \theta} = -\frac{b^2}{a^2} \]

\[ \Rightarrow \frac{\sin \theta \sin \theta}{\cos \theta \cos \theta} = -1 \]

\[ \Rightarrow \sin \theta \sin \theta + \cos \theta \cos \theta = 0 \]

\[ \cos (\theta - \theta') = 0 \]

\( \Rightarrow \theta - \theta' = 90^\circ \)

\( \therefore \theta = \theta' + 90^\circ \)

i.e. \( \theta \) \( \theta' \) are equal to each other.

Deduction:

\[ \Rightarrow \theta = \theta' + 90^\circ \]

\[ \Rightarrow D = \left( a \cos (\mathcal{P} + 90^\circ), b \sin (\mathcal{Q} + 90^\circ) \right) \]

= \( (-a \sin \theta, b \cos \theta) \)

\( \therefore \)

Target:

\[ cP^2 + cQ^2 = a^2 + b^2 \]

\[ IC^2 = \\left| \left( a \cos \theta - 0 \right)^2 + \left( b \sin \theta - 0 \right)^2 \right| \]

\[ = a^2 \cos^2 \theta + b^2 \sin^2 \theta \]

\( \therefore \)

Now:

\[ IC^2 = \\left( -a \sin \theta - 0 \right)^2 + \left( b \cos \theta - 0 \right)^2 \]

\[ = a^2 \sin^2 \theta + b^2 \cos^2 \theta \]

\( \therefore \)

\[ IC^2 + IC^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta + a^2 \sin^2 \theta + b^2 \cos^2 \theta \]

\[ = a^2 + b^2 \]

\( \therefore \) Proved.
iii) We know that the Eq of ellipse is

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

It can be written as

\[ \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \]

New tangent at P is

\[ \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \]  \hspace{1cm} (i)

Tangent at D is

\[ \frac{x (-a \sin \theta)}{a} + \frac{y (b \cos \theta)}{b} = 1 \]  \hspace{1cm} (ii)

Equating and adding (i) and (ii)

\[ \frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \]

Proved.