EXERCISE 2.6 (NEW BOOK)

Function of several variables

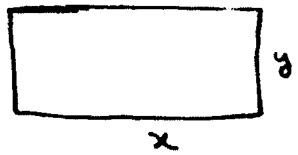
If $z = f(x, y)$

then z is called a fn. of two independent variables x & y.

Ex

Area of a rectangle

$$A = xy$$



Here A is a fn. of two variables x & y.

Similarly if

$$z = f(x, y, w)$$

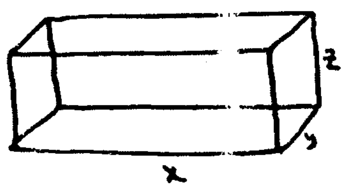
then z is called a fn. of three variables x, y & w

Ex

The volume of a rectangular parallelepiped with dimensions x, y & z is

$$V = xyz$$

Here V is a fn. of three variables x, y & z.



Similarly we can define fn. of several variables

Limit of a function

A function $z = f(x,y)$ is said to tend to a limit l as $(x,y) \to (a,b)$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x,y) - l| < \epsilon \quad \text{whenever} \quad |x-a| < \delta, \; |y-b| < \delta$$

for all pts. (x,y) other than (a,b)

We write it as

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

Note

If in $\lim_{(x,y)\to(a,b)} f(x,y)$

we get two or more different values as $(x,y) \to (a,b)$ along different paths then

$\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

This path may be a line or plane curve through the pt. (a,b).

Partial derivatives

Let $z = f(x,y)$

$$z+\delta z = f(x+\delta x, y)$$

$$\delta z = f(x+\delta x, y) - f(x,y)$$

Dividing both sides by δx

$$\frac{\delta z}{\delta x} = \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

$$\lim_{\delta x \to 0} \frac{\delta z}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x,y)}{\delta x}$$

If the limit on R.H.S. exists as a finite & definite quantity than it is called partial derivative of z (or of f) w.r.t. x & is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

So
$$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$$

Similarly
$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$$

By def.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

& $$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

Partial derivatives of higher orders:

Let $z = f(x,y)$ then

higher derivatives of $f(x,y)$ are

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x) = (f_x)_x = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x) = (f_x)_y = f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y) = (f_y)_x = f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y) = (f_y)_y = f_{yy}$$

Implicit function

A function f of two variables of the form

$$f(x,y) = 0$$

is called an implicit fn.

Now in this function

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

or $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

EXERCISE 2.6 (NEW BOOK)

Exercise 2.5 72

Evaluate the given limit (Problems 1–5):

Q1 $\lim_{(x,y)\to(0,0)} \dfrac{5-x^2}{4+x+y}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{5-x^2}{4+x+y}$$

$$= \frac{\lim_{(x,y)\to(0,0)} (5-x^2)}{\lim_{(x,y)\to(0,0)} (4+x+y)}$$

$$= \frac{5-0}{4+0+0}$$

$$l = \frac{5}{4}$$

Q2 $\lim_{(x,y)\to(1,-1)} e^{-xy}$

Sol.

Let l be the req. limit then

$$l = \lim_{(x,y)\to(1,-1)} e^{-xy}$$

$$= e^{(-1)(-1)}$$

$$= e^{1}$$

$$l = e$$

Q3 $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{xy}\sin xy}{xy}$ 73

Sol:- Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy}$$

$$= \lim_{(x,y)\to(0,0)} \left[e^{xy} \cdot \frac{\sin xy}{xy} \right]$$

$$= \left[\lim_{(x,y)\to(0,0)} e^{xy} \right] \left[\lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} \right]$$

$$= e^{0} \cdot 1$$

$$= 1 \cdot 1$$

$$l = 1$$

Q4 $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2}$

Sol:- Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

Put $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right]$

$\Rightarrow x^2+y^2 = r^2$ As $(x,y)\to(0,0)$, $r \to 0$

So above limit becomes

$$l = \lim_{r\to 0} \frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2}$$

$$l = \lim_{r \to 0} \frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}$$

$$= \lim_{r \to 0} r(\cos^3\theta - \sin^3\theta)$$

$$= 0(\cos^3\theta - \sin^3\theta)$$

$$l = 0$$

Q5 $\lim\limits_{(x,y)\to(2,2)} \dfrac{x^3 - 2xy + 3x^2 - 2y}{x^2y + 4y^2 - 6x^2 + 24y}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(2,2)} \frac{x^3 - 2xy + 3x^2 - 2y}{x^2y + 4y^2 - 6x^2 + 24y}$$

$$= \frac{\lim\limits_{(x,y)\to(2,2)}(x^3 - 2xy + 3x^2 - 2y)}{\lim\limits_{(x,y)\to(2,2)}(x^2y + 4y^2 - 6x^2 + 24y)}$$

$$= \frac{8 - 8 + 12 - 4}{8 + 16 - 24 + 48}$$

$$= \frac{8}{48}$$

$$l = \frac{1}{6}$$

In problems 6 – 10, show that the given limit does not exist

Q6 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

Sol. let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

If we show that $f(x,y) = \dfrac{xy}{x^2+y^2}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y) \longrightarrow (0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{x(mx)}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{m}{1+m^2}\right)$$

which is different for different values of m

So the given limit does not exist.

Q7 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

If we show that $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^2(1-m^2)}{x^2(1+m^2)}$$

$$= \lim_{(x,y)\to(0,0)} \left(\frac{1-m^2}{1+m^2}\right)$$

which is different for different values of m

So the given limit does not exist.

Q8 $\lim\limits_{(x,y)\to(0,0)} \dfrac{ax^2+by}{cy^2+dx}$

Sol:- let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{ax^2+by}{cy^2+dx}$$

If we show that $f(x,y) = \dfrac{ax^2+by}{cy^2+dx}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \left(\frac{ax^2+bmx}{cm^2x^2+dx}\right)$$

$$l = \lim_{(x,y)\to(0,0)} \left(\frac{ax+bm}{cm^2x+d}\right)$$

$$= \frac{0+bm}{0+d}$$

$$l = \frac{bm}{d}$$

which is different for different values of m

So the given limit does not exist.

Q9 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^4+y^4}$

Sol. Let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^4+y^4}$$

If we show that $f(x,y) = \dfrac{(x^2+y^2)^2}{x^4+y^4}$ approaches to different values as $(x,y) \to (0,0)$ from different directions then we say that limit does not exist.

Let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{(x^2+m^2x^2)^2}{x^4+m^4x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4(1+m^2)^2}{x^4(1+m^4)}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(1+m^2)^2}{1+m^4}$$

viz different for different values of m 78

So the given limit does not exist.

Q10 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Sol: let l be the req. limit then

$$l = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

If we show that $f(x,y) = \dfrac{xy^2}{x^2+y^4}$ approaches to different values as $(x,y)\to(0,0)$ from different directions then we say that limit does not exist.

Let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{x\,m^2x^2}{x^2+m^4x^4}$$

$$l = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^4x^2}$$

So $l \to 0$ as $(x,y)\to(0,0)$

Hence along every st. line through origin

$f(x,y) \longrightarrow (0,0)$

Next

Suppose $(x,y)\to(0,0)$ along $x = y^2$ then

$$l = \lim_{(x,y)\to(0,0)} \frac{y^2\cdot y^4}{y^4+y^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{y^4}{2y^4}$$

$$l = \lim_{(x,y)\to(0,0)} \frac{1}{2}$$ 79

So $l \to \frac{1}{2}$ as $(x,y)\to(0,0)$ along parabola $x = y^2$

Hence the limit does not exist.

Q11 Let $f(x,y) = \dfrac{xy^2}{x^3+y^3}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Show that f is not continuous at the origin.

Sol: Given function is

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^3+y^3} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 0$ —— (given)

Now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^3+y^3}$$

let $(x,y)\to(0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x\,m^2x^2}{x^3+m^3x^3}$$

$$= \lim_{(x,y)\to(0,0)} \frac{m^2}{1+m^3}$$

viz different for different values of m

So $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist

Hence given function is not continuous at $(0,0)$

Q12 Find a such that the function

$$f(x,y) = \frac{3xy}{\sqrt{x^2+y^2}} \quad \text{if } (x,y) \neq (0,0)$$

$$= a \quad \text{if } (x,y) = (0,0)$$

is Continuous at $(0,0)$.

Sol:- Given function is

$$f(x,y) = \begin{cases} \dfrac{3xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ a & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = a$ (given)

Now $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3xy}{\sqrt{x^2+y^2}}$

let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{3x(mx)}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{3mx}{\sqrt{1+m^2}}$$

So $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Since $f(x,y)$ is Continuous at $(0,0)$

So $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$\Rightarrow \quad 0 = a$

or $\underline{a = 0}$

Q13 Let $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Examine the Continuity of f at $(0,0)$.

Do $f_x(0,0)$ & $f_y(0,0)$ exist.

Sol. Given function is

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 0$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

Let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+m^3x^3}{x^2+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x+m^3x}{1+m^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(1+m^3)x}{1+m^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Since $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

So $f(x,y)$ is Continuous at $(0,0)$

Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h-0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h}$$

$$= \lim_{h\to 0} (1)$$

So $f_x(0,0) = 1$

&

$$f_y(0,0) = \lim_{K\to 0} \frac{f(0,K)-f(0,0)}{K}$$

$$= \lim_{K\to 0} \frac{K-0}{K}$$

$$= \lim_{K\to 0} \frac{K}{K}$$

$$= \lim_{K\to 0} 1$$

$f_y(0,0) = 1$

So both $f_x(0,0)$ & $f_y(0,0)$ exist

Q14 let $f(x,y) = \dfrac{x^2y}{x^4+y^2}$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Prove that f is not Continuous at $(0,0)$.

Do $f_x(0,0)$ & $f_y(0,0)$ exist

Sol. Given fn. is

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Here $f(0,0) = 0$

Now

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

let $(x,y) \to (0,0)$ along the line $y = mx$ then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(mx)}{x^4+m^2x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{mx}{x^2+m^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

But if $(x,y) \to (0,0)$ along the Curve $x^2 = y$

then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2(x^2)}{x^4+x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x^4}{2x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$$

So the unique limit does not exist.

Hence f is not continuous at $(0,0)$

Now

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h}$$

$$= \lim_{h\to 0} 0$$

$$= 0$$

$$\& \quad f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k}$$

$$= \lim_{k\to 0} 0$$

$$= 0$$

So both $f_x(0,0)$ & $f_y(0,0)$ exist.

Find the first order partial derivatives of the given function (Problems 15 – 22).

Q15 $\quad f(x,y) = x^{y^2}$

Sol. Given

$$f(x,y) = x^{y^2}$$

Diff. partially w.r.t. x & y

$$f_x = y^2 \cdot x^{y^2-1}$$

$$f_y = x^{y^2} \cdot \ln x \cdot 2y$$

Q16 $f(x,y) = e^{x^2+y^2}$

Sol:- Given

$$f(x,y) = e^{x^2+y^2}$$

Diff. partially w.r.t. x & y

$$f_x = e^{x^2+y^2} \cdot 2x$$

$$= 2x\, e^{x^2+y^2}$$

$$f_y = e^{x^2+y^2} \cdot 2y$$

$$= 2y \cdot e^{x^2+y^2}$$

Q17 $f(x,y) = \tan^{-1}(y/x)$

Sol.

Given $f(x,y) = \tan^{-1}(y/x)$

Diff. partially w.r.t. x & y

$$f_x = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}(y/x)$$

$$= \frac{1}{1+\frac{y^2}{x^2}} \cdot -\frac{y}{x^2}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{-y}{x^2}$$

$$f_x = \frac{-y}{x^2+y^2}$$

& $$f_y = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial y}(y/x)$$

$$= \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$f_y = \frac{x}{x^2+y^2}$$

Q18 $f(x,y) = \tan^{-1}(x+y)$

Sol. Given

$$f(x,y) = \tan^{-1}(x+y)$$

Diff. partially w.r.t. x & y

$$f_x = \frac{1}{1+(x+y)^2} \cdot \frac{\partial}{\partial x}(x+y)$$

$$= \frac{1}{1+(x+y)^2} \cdot 1$$

$$f_x = \frac{1}{1+(x+y)^2}$$

& $f_y = \frac{1}{1+(x+y)^2} \cdot \frac{\partial}{\partial y}(x+y)$

$= \frac{1}{1+(x+y)^2} \cdot 1$

$f_y = \frac{1}{1+(x+y)^2}$

Q19 $f(x,y) = e^{ax}\sin by$

Sol. Given

$f(x,y) = e^{ax}\sin by$

Diff. partially w.r.t. x & y

$f_x = (e^{ax} \cdot a)(\sin by)$

$= a e^{ax} \cdot \sin by$

& $f_y = e^{ax} \cdot \cos by \cdot b$

$= b e^{ax} \cos by$

Q20 $f(x,y) = \ln(x^2+y^2)$

Sol. Given

$f(x,y) = \ln(x^2+y^2)$

Diff. partially w.r.t. x & y

$f_x = \frac{1}{x^2+y^2} \cdot \frac{\partial}{\partial x}(x^2+y^2)$

$= \frac{1}{x^2+y^2}(2x)$

$$f_x = \frac{2x}{x^2+y^2}$$

& $$f_y = \frac{1}{(x^2+y^2)} \cdot \frac{\partial}{\partial y}(x^2+y^2)$$

$$= \frac{1}{x^2+y^2} \cdot 2y$$

$$= \frac{2y}{x^2+y^2}$$

<u>Q21</u> $f(x,y) = \ln\left[\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$

<u>Sol:-</u> Given

$$f(x,y) = \ln\left(\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right)$$

$$f(x,y) = \ln\left(\sqrt{x^2+y^2}-x\right) - \ln\left(\sqrt{x^2+y^2}+x\right)$$

Diff. partially w.r.t. x & y

$$f_x = \frac{1}{\left(\sqrt{x^2+y^2}-x\right)} \cdot \left(\frac{1}{2\sqrt{x^2+y^2}} \cdot 2x - 1\right) - \frac{1}{\left(\sqrt{x^2+y^2}+x\right)} \cdot \left(\frac{1}{2\sqrt{x^2+y^2}} \cdot 2x + 1\right)$$

$$= \frac{1}{\left(\sqrt{x^2+y^2}-x\right)}\left(\frac{x}{\sqrt{x^2+y^2}} - 1\right) - \frac{1}{\left(\sqrt{x^2+y^2}+x\right)}\left(\frac{x}{\sqrt{x^2+y^2}} + 1\right)$$

$$= \frac{\left(x-\sqrt{x^2+y^2}\right)}{\left(\sqrt{x^2+y^2}-x\right)\left(\sqrt{x^2+y^2}\right)} - \frac{\left(x+\sqrt{x^2+y^2}\right)}{\left(\sqrt{x^2+y^2}+x\right)\left(\sqrt{x^2+y^2}\right)}$$

$$f_x = \frac{-1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2}} = -\frac{2}{\sqrt{x^2+y^2}}$$

$$\& \; f_y = \frac{1}{\left(\sqrt{x^2+y^2}-x\right)}\cdot\left[\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y\right] - \frac{1}{\left(\sqrt{x^2+y^2}+x\right)}\cdot\left[\frac{1}{2\sqrt{x^2+y^2}}\cdot 2y\right]$$

$$= \frac{y}{\left(\sqrt{x^2+y^2}-x\right)\left(\sqrt{x^2+y^2}\right)} - \frac{y}{\left(\sqrt{x^2+y^2}+x\right)\left(\sqrt{x^2+y^2}\right)}$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{1}{\sqrt{x^2+y^2}-x} - \frac{1}{\sqrt{x^2+y^2}+x}\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{\cancel{\sqrt{x^2+y^2}}+x-\cancel{\sqrt{x^2+y^2}}+x}{\left(\sqrt{x^2+y^2}-x\right)\left(\sqrt{x^2+y^2}+x\right)}\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}}\left[\frac{2x}{\cancel{x^2}+y^2-\cancel{x^2}}\right]$$

$$= \frac{2xy}{\sqrt{x^2+y^2}\cdot y^2}$$

$$= \frac{2x}{y\sqrt{x^2+y^2}}$$

Q22 $\quad f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Sol. Given

$$f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \left(x^2+y^2+z^2\right)^{-1/2}$$

Diff. partially w.r.t. x, y & z

$$f_x = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\cdot\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)$$

$$= \frac{-1}{2\left(x^2+y^2+z^2\right)^{3/2}}\cdot 2x$$

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$$f_x = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

Now

$$f_y = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot \frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= \frac{-1}{2(x^2+y^2+z^2)^{3/2}} \cdot 2y$$

$$f_y = -\frac{y}{(x^2+y^2+z^2)^{3/2}}$$

&

$$f_z = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot \frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2z$$

$$f_z = -\frac{z}{(x^2+y^2+z^2)^{3/2}}$$

Find the second order partial derivatives (Problems 23 – 26):

Q23 e^{x-y}

Sol. Let $z = e^{x-y}$ ——— (1)

Diff. (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = e^{x-y} \cdot 1$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x-y} \cdot 1$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = e^{x-y}}$$

Now $\frac{\partial z}{\partial y} = e^{x-y} \cdot (-1)$

$$\partial z/\partial y = -e^{x-y}$$

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$$\frac{\partial^2 z}{\partial y^2} = -e^{x-y}\cdot(-1)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = e^{x-y}}$$

Now

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-e^{x-y}\right)$$

$$= -e^{x-y}\cdot 1$$

$$\boxed{\frac{\partial^2 z}{\partial x\partial y} = -e^{x-y}}$$

Q24 $\quad \dfrac{x+y}{x-y}$

Sol.

let $z = \dfrac{x+y}{x-y}$ ——— ①

Diff. ① partially w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{(x-y)\cdot 1-(x+y)\cdot 1}{(x-y)^2}$$

$$= \frac{\cancel{x}-y-\cancel{x}-y}{(x-y)^2}$$

$$\frac{\partial z}{\partial x} = -\frac{2y}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = -2y\cdot\frac{-2}{(x-y)^3}$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{4y}{(x-y)^3}}$$

Now Diff. ① partially w.r.t. y

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$$\frac{\partial z}{\partial y} = \frac{(x-y).1-(x+y).(-1)}{(x-y)^2}$$

$$= \frac{x-\cancel{y}+x+\cancel{y}}{(x-y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{2x}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2x.\frac{-2}{(x-y)^3}(-1)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = \frac{4x}{(x-y)^3}}$$

Now

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2x}{(x-y)^2}\right)$$

$$= 2\left[\frac{(x-y)^2.1-x.2(x-y)}{(x-y)^4}\right]$$

$$= 2\left(\frac{x^2-\cancel{2xy}+y^2-2x^2+\cancel{2xy}}{(x-y)^4}\right)$$

$$= 2\left[\frac{y^2-x^2}{(x-y)^4}\right]$$

$$= \frac{-2(x^2-y^2)}{(x-y)^4}$$

$$= \frac{-2(x-y)(x+y)}{(x-y)^4}$$

$$\boxed{\frac{\partial^2 z}{\partial x\partial y} = \frac{-2(x+y)}{(x-y)^3}}$$

Q25 e^{x^y} 93

Sol.

Let $z = e^{x^y}$ ——— ①

Diff. ① partially w.r.t. x

$$\frac{\partial z}{\partial x} = e^{x^y} \cdot y x^{y-1}$$

$$\frac{\partial z}{\partial x} = y x^{y-1} \cdot e^{x^y}$$

$$\& \quad \frac{\partial^2 z}{\partial x^2} = y\left[x^{y-1} \cdot e^{x^y} \cdot y x^{y-1} + e^{x^y} \cdot (y-1) x^{y-2}\right]$$

$$= y\left[y x^{2y-2} \cdot e^{x^y} + (y-1) \cdot x^{y-2} \cdot e^{x^y}\right]$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x^y}\left[y^2 \cdot x^{2y-2} + y(y-1) \cdot x^{y-2}\right]$$

Now Diff. ① w.r.t. y

$$\frac{\partial z}{\partial y} = e^{x^y} \cdot x^y \ln x$$

$$\frac{\partial^2 z}{\partial y^2} = \ln x \cdot \frac{\partial}{\partial y}\left(e^{x^y} \cdot x^y\right)$$

$$= \ln x\left[e^{x^y} \cdot x^y \ln x + x^y \cdot e^{x^y} \cdot x^y \ln x\right]$$

$$\frac{\partial^2 z}{\partial y^2} = e^{x^y}\left[x^y (\ln x)^2 + x^{2y} \cdot (\ln x)^2\right]$$

Now

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(e^{x^y} \cdot x^y \ln x\right)$$

$$= e^{x^y} \cdot x^y \cdot \frac{1}{x} + e^{x^y} \cdot \ln x \cdot y x^{y-1} + x^y \cdot \ln x \cdot e^{x^y} \cdot y x^{y-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{x^y}\left[x^{y-1} + y x^{y-1} \cdot \ln x + y x^{2y-1} \cdot \ln x\right]$$

Q26 $\tan(\tan^{-1}x + \tan^{-1}y)$

Sol.

Let $z = \tan(\tan^{-1}x + \tan^{-1}y)$

or $z = \dfrac{\tan(\tan^{-1}x) + \tan(\tan^{-1}y)}{1 - \tan(\tan^{-1}x)\cdot\tan(\tan^{-1}y)}$

$$z = \frac{x+y}{1-xy} \quad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial z}{\partial x} = \frac{(1-xy)(1) - (x+y).(-y)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+\cancel{xy}+y^2}{(1-xy)^2}$$

$$\frac{\partial z}{\partial x} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = (1+y^2).\frac{-2}{(1-xy)^3}.(-y)$$

$$\boxed{\frac{\partial^2 z}{\partial x^2} = \frac{2y(1+y^2)}{(1-xy)^3}}$$

Diff. (1) partially w.r.t. y

$$\frac{\partial z}{\partial y} = \frac{(1-xy).1 - (x+y)(-x)}{(1-xy)^2}$$

$$= \frac{1-\cancel{xy}+x^2+\cancel{xy}}{(1-xy)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = (1+x^2).\frac{-2}{(1-xy)^3}.(-x)$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = \frac{2x(1+x^2)}{(1-xy)^3}}$$

Now

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1+x^2}{(1-xy)^2}\right)$$

$$= \frac{(1-xy)^2 \cdot 2x - (1+x^2) \cdot 2(1-xy)(-y)}{(1-xy)^4}$$

$$= \frac{2x(1-xy) + 2y(1+x^2)}{(1-xy)^3}$$

$$= \frac{2x - \cancel{2x^2y} + 2y + \cancel{2x^2y}}{(1-xy)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{2(x+y)}{(1-xy)^3}$$

In problems 27 – 32, verify that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

Q27 $f(x,y) = e^{xy} \cos(bx+c)$ —— ①

Diff. ① partially w.r.t. x

$$\frac{\partial f}{\partial x} = e^{xy} \cdot -\sin(bx+c) \cdot b + \cos(bx+c) \cdot e^{xy} \cdot y$$

$$\frac{\partial f}{\partial x} = e^{xy}\left[y\cos(bx+c) - b\sin(bx+c)\right]$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y} \cdot e^{xy}\left[y\cos(bx+c) - b\sin(bx+c)\right]$$

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$$= e^{xy}\left[\cos(bx+c)\right] + \left[y\cos(bx+c) - b\sin(bx+c)\right] \cdot e^{xy} \cdot x$$

$$\frac{\partial^2 f}{\partial y \partial x} = e^{xy}\left[\cos(bx+c) + xy\cos(bx+c) - bx\sin(bx+c)\right] \text{ ——— } (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = x e^{xy} \cdot \cos(bx+c)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x e^{xy} \cos(bx+c)\right]$$

$$= x e^{xy} \cdot -\sin(bx+c) \cdot b + x\cos(bx+c) \cdot e^{xy} \cdot y + e^{xy} \cdot \cos(bx+c) \cdot 1$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{xy}\left[\cos(bx+c) + xy\cos(bx+c) - b\sin(bx+c)\right] \text{ ——— } (B)$$

From (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q28 $f(x,y) = \ln(e^x + e^y)$

Sol.- Given

$$f(x,y) = \ln(e^x + e^y) \text{ ——— } (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{(e^x + e^y)} \cdot e^x$$

$$\frac{\partial f}{\partial x} = \frac{e^x}{(e^x + e^y)}$$

Now

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{e^x}{(e^x+e^y)}\right)$$

$$= e^x \cdot \frac{-1}{(e^x+e^y)^2} \cdot e^y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{e^{x+y}}{(e^x+e^y)^2} \quad \text{——— } (A)$$

Now diff. (1) w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{(e^x+e^y)} \cdot e^y$$

$$\frac{\partial f}{\partial y} = \frac{e^y}{(e^x+e^y)}$$

Now

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{e^y}{(e^x+e^y)}\right)$$

$$= e^y \cdot \frac{-1}{(e^x+e^y)^2} \cdot e^x$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{e^{x+y}}{(e^x+e^y)^2} \quad \text{——— } (B)$$

from (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q29 $f(x,y) = \ln\left(\frac{x^2+y^2}{xy}\right)$ 98

Sol. Given

$$f(x,y) = \ln\left(\frac{x^2+y^2}{xy}\right)$$

$$f(x,y) = \ln(x^2+y^2) - \ln(xy) \quad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}\cdot 2x - \frac{1}{xy}\cdot y$$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{2x}{x^2+y^2} - \frac{1}{x}\right]$$

$$= 2x \cdot \frac{-1}{(x^2+y^2)^2}\cdot 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{4xy}{(x^2+y^2)^2} \quad \text{——— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y - \frac{1}{xy}\cdot x$$

$$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{2y}{x^2+y^2} - \frac{1}{y}\right]$$

$$= 2y \cdot \frac{-1}{(x^2+y^2)^2}\cdot 2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{4xy}{(x^2+y^2)^2} \quad \text{———} \quad (B)$$

from (A) + (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q30 $\quad f(x,y) = x^y + y^x$

Sol.- Given

$$f(x,y) = x^y + y^x \quad \text{———} \quad (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = y x^{y-1} + y^x . \ln y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[y x^{y-1} + y^x \ln y \right]$$

$$= y.(x^{y-1}.\ln x) + x^{y-1}.1 + y^x . \frac{1}{y} + \ln y . x y^{x-1}$$

$$\frac{\partial^2 f}{\partial y \partial x} = x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y) \quad \text{———} \quad (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = x^y \ln x + x y^{x-1}$$

$$\frac{\partial^2 f}{\partial x . \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x^y \ln x + x y^{x-1} \right]$$

$$= x^y . \frac{1}{x} + \ln x . y x^{y-1} + x . y^{x-1} . \ln y + y^{x-1} . 1$$

$= x^{y-1} + \ln x \cdot y x^{y-1} + x y^{x-1} \ln y + y^{x-1}$ 100

$= x^{y-1}(1 + y\ln x) + y^{x-1}(x\ln y + 1)$

$\dfrac{\partial^2 f}{\partial x \partial y} = x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y)$ ——— (B)

from (A) + (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q31 $f(x,y) = \sin^{-1}\left(\frac{x}{y}\right)$

Sol:- Given

$f(x,y) = \sin^{-1}(x/y)$ ——— (1)

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot \frac{1}{y}$$

$$= \frac{1}{\frac{\sqrt{y^2-x^2}}{y}} \cdot \frac{1}{y}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{1}{y}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y} \frac{1}{\sqrt{y^2-x^2}}$$

$$= -\frac{1}{2}(y^2-x^2)^{-3/2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\frac{y}{(y^2-x^2)^{3/2}} \quad \text{———} \quad (A)$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(x/y)^2}} \cdot -\frac{x}{y^2}$$

$$= \frac{1}{\sqrt{1-x^2/y^2}} \cdot \frac{-x}{y^2}$$

$$= \frac{y}{\sqrt{y^2-x^2}} \cdot \frac{-x}{y^2}$$

$$\frac{\partial f}{\partial y} = -\frac{x}{y\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{-x}{y\sqrt{y^2-x^2}}\right]$$

$$= -\frac{1}{y} \cdot \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{y^2-x^2}}\right)$$

$$= -\frac{1}{y}\left[\frac{\sqrt{y^2-x^2} \cdot 1 - x \cdot \frac{(-2x)}{2\sqrt{y^2-x^2}}}{y^2-x^2}\right]$$

$$= -\frac{1}{y}\left[\frac{(y^2-x^2)+x^2}{(y^2-x^2)\sqrt{y^2-x^2}}\right]$$

$$= -\frac{1}{y}\left[\frac{y^2}{(y^2-x^2)^{3/2}}\right]$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{y}{(y^2-x^2)^{3/2}} \quad \text{———} \quad (B)$$

From (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Q32 $f(x,y) = \dfrac{xy}{\sqrt{1+x^2+y^2}}$

Sol.- Given

$$f(x,y) = \frac{xy}{\sqrt{1+x^2+y^2}} \quad \text{———} \quad (1)$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = y.\frac{\partial}{\partial x}\left[\frac{x}{\sqrt{1+x^2+y^2}}\right]$$

$$= y\left[\frac{\sqrt{1+x^2+y^2}\,.\,1 - x.\frac{1}{2\sqrt{1+x^2+y^2}}.2x}{(1+x^2+y^2)}\right]$$

$$= y\left[\frac{\sqrt{1+x^2+y^2} - \frac{x^2}{\sqrt{1+x^2+y^2}}}{(1+x^2+y^2)}\right]$$

$$= y\left[\frac{1+\cancel{x^2}+y^2-\cancel{x^2}}{(1+x^2+y^2)^{3/2}}\right]$$

$$\frac{\partial f}{\partial x} = \frac{y+y^3}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left[\frac{(y+y^3)}{(1+x^2+y^2)^{3/2}}\right]$$

$$= \frac{(1+x^2+y^2)^{3/2}.(1+3y^2) - (y+y^3).\frac{3}{\cancel{2}}(1+x^2+y^2)^{1/2}.\cancel{2}y}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2).(1+3y^2) - 3(y^2+y^4)}{(1+x^2+y^2)}$$

$$= \frac{1+x^2+y^2+\cancel{3y^2}+3x^2y^2+\cancel{3y^4}-\cancel{3y^2}-\cancel{3y^4}}{(1+x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^3} \quad \text{——— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = x.\frac{\partial}{\partial y}\left[\frac{y}{\sqrt{1+x^2+y^2}}\right]$$

$$= x.\left[\frac{\sqrt{1+x^2+y^2}\cdot 1 - y\cdot\frac{1}{\cancel{2}\sqrt{1+x^2+y^2}}\cdot\cancel{2}y}{(1+x^2+y^2)}\right]$$

$$= x\left[\frac{\sqrt{1+x^2+y^2} - \frac{y^2}{\sqrt{1+x^2+y^2}}}{(1+x^2+y^2)}\right]$$

$$= x\left[\frac{1+x^2+\cancel{y^2}-\cancel{y^2}}{(1+x^2+y^2)^{3/2}}\right]$$

$$\frac{\partial f}{\partial y} = \frac{x+x^3}{(1+x^2+y^2)^{3/2}}$$

Now

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{x+x^3}{(1+x^2+y^2)^{3/2}}\right]$$

$$= \frac{(1+x^2+y^2)^{3/2}\cdot(1+3x^2)-(x+x^3)\cdot\frac{3}{\cancel{2}}(1+x^2+y^2)^{1/2}\cdot\cancel{2}x}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)^{3/2}\cdot(1+3x^2)-3x(x+x^3)(1+x^2+y^2)^{1/2}}{(1+x^2+y^2)^3}$$

$$= \frac{(1+x^2+y^2)(1+3x^2)-3x(x+x^3)}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1+x^2+y^2+\cancel{3x^2}+\cancel{3x^4}+3x^2y^2-\cancel{3x^2}-\cancel{3x^4}}{(1+x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1+x^2+y^2+3x^2y^2}{(1+x^2+y^2)^{3/2}} \quad \text{——— (B)}$$

From (A) & (B)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Show that each of the following functions satisfies Laplace's eq. $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ (Problems 33-36)

Q33 $f(x,y) = \sin x . \sinh y$

Sol Given

$$f(x,y) = \sin x . \sinh y \quad \text{——— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \cos x . \sinh y$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin x . \sinh y \quad \text{——— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \sin x \cosh y$$

$$\frac{\partial^2 f}{\partial y^2} = \sin x \sinh y \quad \text{——— (B)}$$

Adding (A) & (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

1.5

Q34 $f(x,y) = e^{-x}\cos y$

Sol:- Given

$$f(x,y) = e^{-x}\cos y \quad \text{--------- (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = -e^{-x}\cos y$$

$$\frac{\partial^2 f}{\partial x^2} = -e^{-x}(-1).\cos y$$

$$\frac{\partial^2 f}{\partial x^2} = e^{-x}\cos y \quad \text{--------- (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = -e^{-x}.\sin y$$

$$\frac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y \quad \text{--------- (B)}$$

Adding (A) & (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Q35 $f(x,y) = \ln\sqrt{x^2+y^2}$

Sol. Given

$$f(x,y) = \ln\sqrt{x^2+y^2}$$

$$f(x,y) = \ln(x^2+y^2)^{1/2}$$

$$f(x,y) = \frac{1}{2}\ln(x^2+y^2) \quad \text{--------- (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{2}.\frac{1}{x^2+y^2}.2x$$

$$\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y^2).1 - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \quad \text{———— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{2} \cdot \frac{1}{(x^2+y^2)} \cdot 2y$$

$$\frac{\partial f}{\partial y} = \frac{y}{x^2+y^2}$$

Now

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2).1 - y.2y}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} \quad \text{———— (B)}$$

Adding (A) + (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Q36 $f(x,y) = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

Sol. Given

$$f(x,y) = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right) \quad \text{———— (1)}$$

Diff. (1) partially w.r.t. x

$$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{2xy}{x^2-y^2}\right)$$ 107

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \cdot 2y\left[\frac{(x^2-y^2).1 - x(2x)}{(x^2-y^2)^2}\right]$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2} \cdot 2y\left[\frac{x^2-y^2-2x^2}{\cancel{(x^2-y^2)^2}}\right]$$

$$= \frac{2y(-x^2-y^2)}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{-2y(x^2+y^2)}{x^4+y^4+2x^2y^2}$$

$$= \frac{-2y(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x} = \frac{-2y}{(x^2+y^2)}$$

$$\frac{\partial^2 f}{\partial x^2} = (-2y) \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4xy}{(x^2+y^2)^2} \quad \text{———— (A)}$$

Now diff. (1) partially w.r.t. y

$$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}} \cdot 2x\left[\frac{(x^2-y^2).1 - y(-2y)}{(x^2-y^2)^2}\right]$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2+4x^2y^2} \cdot 2x\left[\frac{x^2-y^2+2y^2}{(x^2-y^2)^2}\right]$$

$$= \frac{2x(x^2+y^2)}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{2x(x^2+y^2)}{x^4+y^4+2x^2y^2}$$

$$= \frac{2x(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x \cdot \frac{-1}{(x^2+y^2)^2} \cdot 2y$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{4xy}{(x^2+y^2)^2} \quad \text{------ (B)}$$

Adding (A) + (B)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

Q37 If $f(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$ then show that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

Sol:- Given

$$f(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$$

Diff. partially w.r.t. y

$$\frac{\partial f}{\partial y} = x^2 \cdot \frac{1}{1+y^2/x^2} \cdot \frac{1}{x} - \left[y^2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{y^2} + \tan^{-1}(x/y) \cdot 2y\right]$$

$$= x^2 \cdot \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} - \left[\frac{-xy^2}{y^2+x^2} + 2y\tan^{-1}(x/y) \right]$$ 109

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}(x/y)$$

$$= \frac{x^3+xy^2}{x^2+y^2} - 2y\tan^{-1}(x/y)$$

$$= \frac{x(x^2+y^2)}{(x^2+y^2)} - 2y\tan^{-1}(x/y)$$

$$\frac{\partial f}{\partial y} = x - 2y\tan^{-1}(x/y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[x - 2y\tan^{-1}(x/y) \right]$$

$$= 1 - 2y \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= 1 - \frac{2}{1+\frac{x^2}{y^2}}$$

$$= 1 - \frac{2y^2}{y^2+x^2}$$

$$= \frac{y^2+x^2-2y^2}{y^2+x^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2-y^2}{x^2+y^2}$$

<u>Q38</u> If $f(x,y) = \dfrac{x^2+y^2}{x+y}$, prove that

$$(f_x - f_y)^2 = 4(1 - f_x - f_y)$$

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Sol. Given

$$f(x,y) = \frac{x^2+y^2}{x+y}$$

Diff. partially w.r.t. x & y

$$f_x = \frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2}$$

$$f_x = \frac{x^2+2xy-y^2}{(x+y)^2}$$

Now

$$f_y = \frac{(x+y).2y-(x^2+y^2).1}{(x+y)^2}$$

$$= \frac{2xy+2y^2-x^2-y^2}{(x+y)^2}$$

$$f_y = \frac{y^2+2xy-x^2}{(x+y)^2}$$

Now

$$f_x - f_y = \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$= \frac{x^2+\cancel{2xy}-y^2-y^2-\cancel{2xy}+x^2}{(x+y)^2}$$

$$= \frac{2x^2-2y^2}{(x+y)^2}$$

$$= \frac{2(x-y)(x+y)}{(x+y)^2}$$

$$f_x - f_y = \frac{2(x-y)}{(x+y)}$$

Sq. both sides

$$(f_x - f_y)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

Now

Consider $1 - f_x - f_y$

$$= 1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{y^2+2xy-x^2}{(x+y)^2}$$

$$= \frac{(x+y)^2 - (x^2+2xy-y^2) - (y^2+2xy-x^2)}{(x+y)^2}$$

$$= \frac{\cancel{x^2}+\cancel{2xy}+y^2-\cancel{x^2}-\cancel{2xy}+\cancel{y^2}-\cancel{y^2}-2xy+x^2}{(x+y)^2}$$

$$= \frac{x^2-2xy+y^2}{(x+y)^2}$$

$$= \frac{(x-y)^2}{(x+y)^2}$$

So

$$4(1-f_x-f_y) = \frac{4(x-y)^2}{(x+y)^2}$$

Hence $(f_x - f_y)^2 = 4(1-f_x-f_y)$

Q39 Show that the function $f(x,y) = \sin(xy)$ satisfies the diff. eq.

$$x^2 f_{xx} - y^2 f_{yy} = 0$$

Sol: Given

$$f(x,y) = \sin(xy) \quad \text{——— (1)}$$

Diff (1) partially w.r.t. x

$$f_x = \cos(xy)\cdot y$$

$$f_x = y\cos(xy)$$

$$f_{xx} = y\cdot -\sin(xy)\cdot y$$

$$f_{xx} = -y^2\sin(xy)$$

Now diff. (1) partially w.r.t. y

$$f_y = \cos(xy)\cdot x$$

$$f_y = x\cos(xy)$$

$$f_{yy} = x\cdot -\sin(xy)\cdot x$$

$$f_{yy} = -x^2\sin(xy)$$

Now

Consider $x^2 f_{xx} - y^2 f_{yy}$

$$= x^2\left(-y^2\sin(xy)\right) - y^2\left(-x^2\sin(xy)\right)$$

$$= -x^2\cancel{y^2}\sin(xy) + x^2\cancel{y^2}\sin(xy)$$

$$= 0$$

Q40 Let $f(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$

<u>Sol.</u> Given

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$$f(x,y) = \begin{cases} x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Now

$$f_x = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right) + \tan^{-1}(y/x) \cdot 2x - y^2 \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{1}{y}$$

$$= \cancel{x^2} \cdot \frac{x^2}{x^2+y^2} \cdot \left(\frac{-y}{\cancel{x^2}}\right) + 2x\tan^{-1}(y/x) - y^2 \cdot \frac{y^2}{y^2+x^2} \cdot \frac{1}{y}$$

$$= -\frac{x^2y}{x^2+y^2} + 2x\tan^{-1}(y/x) - \frac{y^3}{x^2+y^2}$$

$$= 2x\tan^{-1}(y/x) - \frac{y}{(\cancel{x^2+y^2})}\left[\cancel{x^2+y^2}\right]$$

$$\boxed{f_x = 2x\tan^{-1}(y/x) - y}$$

Now

$$f_y = x^2 \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} - \left[\cancel{y^2} \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \frac{-x}{\cancel{y^2}} + \tan^{-1}\frac{x}{y} \cdot 2y\right]$$

$$= \frac{x}{\frac{x^2+y^2}{x^2}} + \frac{xy^2}{y^2+x^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= \frac{x}{(\cancel{x^2+y^2})}(\cancel{x^2+y^2}) - 2y\tan^{-1}\frac{x}{y}$$

$$\boxed{f_y = x - 2y\tan^{-1}\frac{x}{y}}$$

Now

$$f_{xy} = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}$$

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$$= \lim_{h\to 0} \frac{-h-0}{h}$$

$$= \lim_{h\to 0} (-1)$$

$$f_{xy}(0,0) = -1$$

Now

$$f_{yx} = \lim_{K\to 0} \frac{f_y(K,0)-f_y(0,0)}{K}$$

$$= \lim_{K\to 0} \frac{K-0}{K}$$

$$= \lim_{K\to 0} (1)$$

$$f_{yx}(0,0) = 1$$

So $f_{xy}(0,0) \neq f_{yx}(0,0)$

Q41 (a)

Let $f(x,y,z) = x^3+3yz+\sin(xyz)$

Prove that $f_{xyz} = f_{zxy}$

Sol- Given

$$f(x,y,z) = x^3+3yz+\sin(xyz)$$

then

$$f_x = 3x^2 + yz\cos(xyz)$$

$$f_{xy} = z\left[y\cdot -\sin(xyz)\cdot xz + \cos(xyz)\cdot 1\right]$$

$$f_{xy} = z\cos(xyz) - xyz^2\sin(xyz)$$

$$f_{xyz} = z\cdot -\sin(xyz)\cdot xy + \cos(xyz)\cdot 1 - xy\left[z^2\cos(xyz)\cdot xy + \sin(xyz)\cdot 2z\right]$$

$$f_{xyz} = \cos(xyz) - xyz\sin(xyz) - x^2y^2z^2\cos(xyz) - 2xyz\sin(xyz)$$

$$f_{xyz} = \cos(xyz) - 3xyz\sin(xyz) - x^2y^2z^2\cos(xyz) \quad \text{——— (A)}$$

Now

$$f_z = 3y + xy\cos(xyz)$$

$$f_{zx} = y\left[x \cdot -\sin(xyz) \cdot yz + \cos(xyz) \cdot 1\right]$$

$$f_{zx} = y\cos(xyz) - xy^2z\sin(xyz)$$

$$f_{zxy} = y \cdot -\sin(xyz) \cdot xz + \cos(xyz) \cdot 1 - xz\left[y^2\cos(xyz) \cdot xz + \sin(xyz) \cdot 2y\right]$$

$$= -xyz\sin(xyz) + \cos(xyz) - x^2y^2z^2\cos(xyz) - 2xyz\sin(xyz)$$

$$f_{zxy} = \cos(xyz) - 3xyz\sin(xyz) - x^2y^2z^2\cos(xyz) \quad \text{——— (B)}$$

From (A) & (B)

$$f_{xyz} = f_{zxy}$$

(b) If $f(x,y,z,w) = \dfrac{xy}{z+w}$, Show that

$$f_{xyzw} = \frac{2}{(z+w)^3}$$

Sol:- Given

$$f(x,y,z,w) = \frac{xy}{z+w}$$

Diff. partially w. r. t. x

$$f_x = \frac{y}{z+w}$$

$$f_{xy} = (f_x)_y$$

$$f_{xy} = \frac{1}{z+w}$$

$$f_{xyz} = (f_{xy})_z$$

$$f_{xyz} = \frac{-1}{(z+w)^2}$$

Now

$$f_{xyzw} = (f_{xyz})_w$$

$$= \frac{(-1)(-2)}{(z+w)^3}$$

$$= \frac{2}{(z+w)^3}$$

In problems 42–45, find $\frac{dy}{dx}$

Q42 $\quad y^2 + x^2 y + ax^4 = 0$

Sol.

let $f(x,y) = y^2 + x^2 y + ax^4$ ——— (1)

Diff. (1) partially w.r.t. x & y

$$f_x = 2xy + 4ax^3$$

$$f_y = 2y + x^2$$

Now $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$\frac{dy}{dx} = -\frac{2xy+4ax^3}{2y+x^2}$$

Q43 $\quad 3x^2 - y^2 + x^3 = 0$

Sol.

Let $f(x,y) = 3x^2 - y^2 + x^3$ ——— ①

Diff. ① partially w.r.t. x & y

$f_x = 6x + 3x^2$

$f_y = -2y$

Now $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$= -\dfrac{6x+3x^2}{-2y}$

$= \dfrac{6x+3x^2}{2y}$

Q44 $\quad x^2 + xy + y^2 + ax + by = 0$

Sol.

Let $f(x,y) = x^2 + xy + y^2 + ax + by = 0$ ——— ①

Diff. ① partially w.r.t. x & y

$f_x = 2x + y + a$

$f_y = x + 2y + b$

then $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y}$

$= -\dfrac{2x+y+a}{x+2y+b}$ — Ans.

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Q45 $x^2 + xy^2 + \sin y = 0$

Sol. let $f(x,y) = x^2 + xy^2 + \sin y$ —— (1)

Diff. (1) partially w.r.t. x & y

$$f_x = 2x + y^2$$

$$f_y = 2xy + \cos y$$

Now $$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$\frac{dy}{dx} = -\frac{2x+y^2}{2xy+\cos y}$$

End of Ch-2

Thanks to mighty God.
