Section A

1- a) Suppose that a group G has only one element a of order 2. Show that, for all $x \in G$, $ax = xa$. 5

b) Let G be a group and $a, b \in G$. Show that the orders of ab and ba are equal. 5

2- a) Prove that every subgroup of a cyclic group is cyclic. 5

b) Let H and K be two finite subgroups (of a group G) whose orders are relatively prime. Prove that $H \cap K = \{e\}$. 5

3- a) For permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 2 & 6 & 4 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 2 & 6 & 5 \end{pmatrix}$ Show that . 5

b) Prove that every cyclic permutation can be expressed as a product of transposition. 5

Section B

4- a) Let V be a vector space over a field F. Then, if av = 0 then either a = 0 or v = 0. 5

b) For what value of k will the vector $(1, -2, k)$ in R^3 be a linear combination of the vectors $(3, 0, -2)$ and $(2, -1, -5)$? 5

5- a) Find a basis and dimension of the subspace W of R^4 spanned by $(1, 4, -1, 3)$, $(2, 1, -3, -1)$ and $(0, 2, 1, -5)$. 5

b) Define Linearly Dependent Vectors of a vector space over a field. Show that the vectors $(1 - i, i)$ and $(2, -1 + i)$ in C^2 are linearly dependent. 5

6- a) If A is a matrix over R and $AA^T = 0$, show that A = 0. 5

b) Find the Inverse of the Matrix $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$ 5

7- a) Solve the following system of equations 5

$$5x_1 + 5x_2 - x_3 = 0$$
$$10x_1 + 5x_2 + 2x_3 = 0$$
$$5x_1 + 15x_2 - 9x_3 = 0$$

b) Show that the transformation $T : R^3 \to R^2$ defined by $T(x_1, x_2, x_3) = (|x_1|, 0)$ is not linear. 5

8- a) Prove that $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$ 5

b) Let A and B be distinct $n \times n$ matrices with real entries. If $AB^2 = BA^2$ and $A^3 = B^3$, show that $A^2 + B^2$ is not invertible. 5