Operations Research:

It started in England during world war II.

What is operations Research?

Operations Research deals with problems concerning the best utilization of limited resources.

The subject developed as a result of efforts by Scientists from different disciplines for obtaining the best utilization of limited resources. In OR the constraints (شرائط) on resources are formed and their utility is expressed as mathematical function and this mathematical function is then solved by using some algebraic technique.

This subject was founded by British Scientists during the World War II as a result of their efforts for optimum (best) allocation (تقسیم) of limited military resources.

Mathematical Model:-

Mathematical Model is a representation of a real life system using mathematical symbols which are related by appropriate mathematical functions.

A typical OR mathematical model is usually organized as follows

maximize or minimize (objective function)
Subject to (constraints)

Linear Programming (LP): Linear programming (LP) is a mathematical modeling technique designed to optimize (موزوں) the usage of limited resources.

[illegible]

objectives - that we need to optimize (maximize or minimize)

2

Objective Function:-

It is a measure of the effectiveness of the system as a mathematical function in terms of its decision variables (e.g.)
If the objective function is maximize the total profit, the objective function must specify the profit in terms of decision variables.

Decision Variables:-

these are the unknowns to be determined from the solution of the model.

Solution Space:-

It is the space enclosed by the given constraints, upon which the objective function is to be subjected.

Optimum Solution:-

It is the point in the solution space that extremizes the given objective function.

Standard Form:-

A Linear programming model is said to be in the standard form if it has the following properties.

1) All the Constraints are equations with non-negative right hand side.
2) All Variables are non-negative.
3) the objective function may be of the maximization or the minimization type.

A sol. of the model is feasible if it satisfies all the constraints. It is optimal if in addition to being feasible it yields the best (maximum or minimum) value of the objective function. Best feasible sol. is optimum.

Example: Write the following LP model in Standard form

LP model has three basic properties
1- proportionality
2- Additivity
3- Certainty.

Minimize $Z = 2x_1 + 3x_2$

subject to

$-x_1 + x_2 = 10$

$-2x_1 + 3x_2 \le -5$

$7x_1 - 4x_2 \le 6$

x_1 is unrestricted, $x_2 \ge 0$

The following changes may be effected

(i) Multiply 2nd constraint by (-1) and subtract a surplus variable $S_2 \ge 0$ from the left side.

(ii) Add a slack variable $S_3 \ge 0$ to the left side of 3rd Constraint.

(iii) Substitute $x_1 = x_1' - x_1''$

where $x_1', x_1'' \ge 0$.

then the model in Standard form is

minimize $Z = 2x_1' + 3x_2 - 2x_1''$

Subject to

$-x_1' + x_1'' + x_2 = 10$

$2x_1' - 2x_1'' - 3x_2 - S_2 = 5$

$7x_1' - 7x_1'' - 4x_2 + S_3 = 6$

$x_1', x_1'', x_2, S_2, S_3 \ge 0$

Convertion of maximization to minimization.

The maximization of a function $f(x_1, x_2, \ldots x_n)$ is equivalent to the minimization of $-f(x_1, x_2, \ldots x_n)$ in the sence that both problems yield the same optimal values of $x_1, x_2, \ldots$, and x_n.

Linear Programming Formulation and Graphical Solution

Example:- Reddy Mikks produces both interior and exterior paints from two raw materials, M_1 and M_2 the following table provides the basic data of the problem

	Tons of raw material per ton of		
	Exterior paint	interior paint	Maximum daily availability (tons)
Raw material, M_1	6	4	24
Raw material, M_2	1	2	6
Profit per ton ($1000)	5	4	

A market survey restricts the maximum daily demand of interior paint to 2 tons.
Additionally, the daily demand for interior paint can not exceed that of exterior paint by more than 1 ton. Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximize the total daily profit.

Sol. Let

x_1 = tons produced daily of exterior paint

x_2 = tons produced daily of interior paint

~~Using these designation~~, the next task of is to construct the objective function. A logical objective for the Company is to increase as much as possible (ie. maximize) the total daily profit ~~(in thousands of dollars)~~ from both exterior and interior paints. Letting z represents the total daily profit (in thousands of dollars), we get

$$z = 5x_1 + 4x_2$$

then the objective of the Company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

the last element of the model deals with the Constraints that restricts raw materials usage and demand. the raw materials restrictions are expressed verbally as

$$\begin{pmatrix}\text{usage of a raw material}\\ \text{by both paints}\end{pmatrix} \le \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

میری فنی سیٹ
ایک ٹن کیلیے درکار 111, = 6 ٹن
6x_1 = " " " "

Hence from the data of the problem

usuage of raw material $M_1 = 6x_1 + 4x_2$ tons
" " " $M_2 = 1x_1 + 2x_2$ tons

because the daily availabilities of raw materials M_1 and M_2 are limited to 24 and 6 tons, respectively, so the associated restrictions are given as

$6x_1 + 4x_2 \le 24$ (Raw material M_1)
$x_1 + 2x_2 \le 6$ (" " M_2)

there are two types of demand restrictions: (1) Maximum daily demand of interior paint is limited to 2 tons, and (2) excess (زیادتی) of daily production of interior paint over that of exterior paint is at most 1 ton.

the first restriction is straight forward and is expressed as $x_2 \le 2$

the 2nd restriction can be translated to state that the difference b/w the daily production of interior paint and exterior paints, $x_2 - x_1$, does not exceed 1 ton i.e, $x_2 - x_1 \le 1$

An implicit (or "understood-to-be") restriction on the model is that the variables x_1 and x_2 must not be negative, we thus add the non negativity restrictions, $x_1 \ge 0$, $x_2 \ge 0$, to account for this requirement. (non negativity restrictions are essential for the development of the solution algorithm of the LP model.)

the complete Reddy Mikks model is written as

Maximize $z = 5x_1 + 4x_2$

subject to

$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

(the sol. of this problem is on page no. ~~8~~ of ~~Notes~~) 36 of Book.

Example: (from old book)

The Reddy Mikks Company owns a small paint factory that produces both interior and exterior house paints for whole sale distribution. two basic raw materials, A and B, are used to manufacture the paint. the maximum availability of A is 6 tons a day; that of B is 8 tons a day. the daily requirements of the raw materials per ton of interior and exterior paints are summarized in the following table:

	tons of raw material per ton of paint		
	Exterior paint	Interior	Max. availability
Raw material A:	1	2	6
" " B	2	1	8

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 (one) ton. the survey also shows that the maximum demand for interior is limited to 2 tons daily.

the whole sale price per ton is \$ 3000 for the exterior paint and \$2000 for interior paint.

How much interior and exterior paints should the company produce to maximize gross income?

Solution, Step 1. Let x_1 and x_2 be the tons produced daily of exterior and interior paint, respectively. then the mathematical model of the problem is:

$$\text{Maximize } z = 3000x_1 + 2000x_2$$

subject to

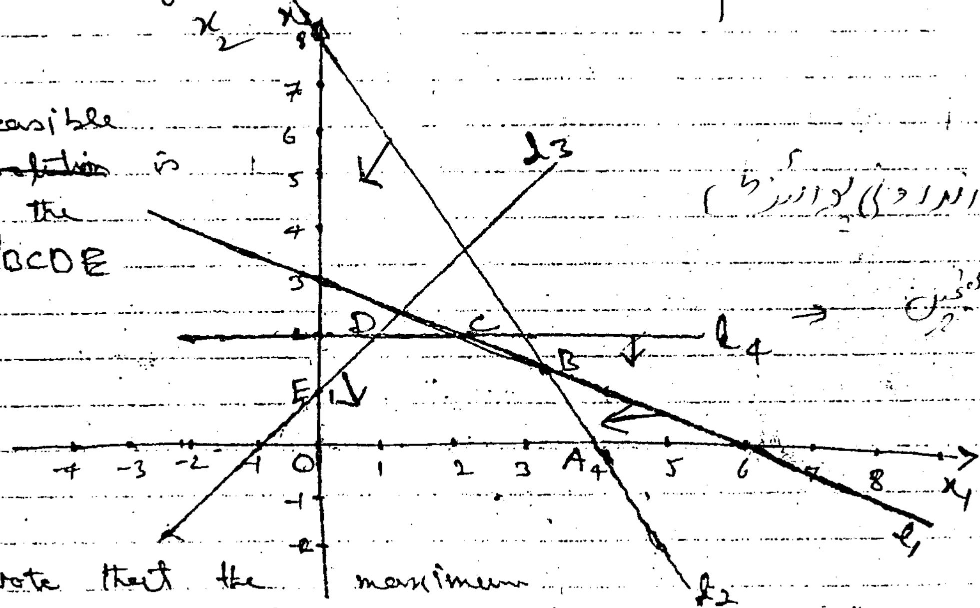
$$x_1 + 2x_2 \leq 6$$
$$2x_1 + x_2 \leq 8$$
$$x_2 - x_1 \leq 1$$
$$x_2 \leq 2 \quad ; \quad x_1, x_2 \geq 0$$

Step II Change all the Constraints into equations:-

$2x_1+4x_2=12$ $2x_1+x_2=8$ $3x_2=4$ $x_2=4/3$ $x_1+8/3=6$ $x_1=6-8/3$ $x_1=\frac{10}{3}$	$x_1+2x_2=6$	$\rightarrow l_1$	Intercept form is $\frac{x_1}{6}+\frac{x_2}{3}=1$
	$2x_1+x_2=8$	$\rightarrow l_2$	$\frac{x_1}{4}+\frac{x_2}{8}=1$
	$x_2-x_1=1$	$\rightarrow l_3$	$\frac{x_1}{-1}+\frac{x_2}{1}=1$
	$x_2=2$ $x_1,x_2\geq 0$	$\rightarrow l_4$	$x_2=2$

Draw the lines l_1, l_2, l_3 and l_4 corresponding to these equations in the (x_1, x_2) plane.

the feasible solution ~~solution~~ is given by the area OABCDE



Step 3. Note that the maximum value of Z can occur at one of the corner points O, A, B, C, D & E of the feasible solution space

So Now we find the coordinates of these points

O = (0,0)

A = intersection of l_2 and x_1-axis = (4,0)

B = 〃 〃 of l_1 and l_2 = $(\frac{10}{3}, \frac{4}{3}) = (3\frac{1}{3}, 1\frac{1}{3})$

C = 〃 〃 l_1 and l_4 = (2,2)

D = 〃 〃 l_3 & l_4 = (1, 2)

E = 〃 〃 l_3 and x_2-axis = (0,1)

Now we evaluate the objective function at these corner points.

Z at $O(0,0) = 0$ (Also solved on page No. 2)

Z at $A(4,0) = 12000$

Z at $B(10/3, 4/3) = \frac{38000}{3} = 12666\frac{2}{3}$

Z at $C(2,2) = 10,000$

Z at $D(1,2) = 7,000$

Z at $E(0,1) = 2,000$

We observe that the maximum value of Z is $12666\frac{2}{3}$ and occurs at $B(3\frac{1}{3}, 1\frac{1}{3})$. Thus, the daily production should be $3\frac{1}{3}$ tons of exterior paint and $1\frac{1}{3}$ tons of interior paint to obtain the maximum revenue of $12\frac{2}{3}$ thousands dollars.

آدمی کو A, B, اور C chems کی ضرورت ہے جو کہ liquid اور dry فارم میں موجود ہیں ملتی ہیں علیحدہ علیحدہ

Ex. A person requires 10, 12, 12 units of chemicals A, B, C respectively. A liquid product contains 5, 2 and 1 units of chemicals A, B, and C respectively per jar; and a dry product contains 1, 2, and 4 unit of A, B and C respectively per carton (ڈبہ). If the liquid product costs \$3 per jar and the dry product costs \$2 per carton; how many of each should be purchased to minimize the cost and meet the requirements?

Solution:

Primarily, we put the given information in the table as below:

	unit/jar	unit/carton	units required
Chemical A	5	1	10
" B	2	2	12
" C	1	4	12

Cost = \$3 per jar & 2\$ per carton

Let x_1, x_2 be the number of jars and no. of cartons purchased. Then the mathematical model of the problem is:

Minimize $Z = 3x_1 + 2x_2$

Subject to

$$5x_1 + 1x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$1x_1 + 4x_2 \geq 12; \quad x_1, x_2 \geq 0$$

Now, we will solve the problem by graphical method. ~~(as follows)~~

Step I Change all the constraints into eqs.

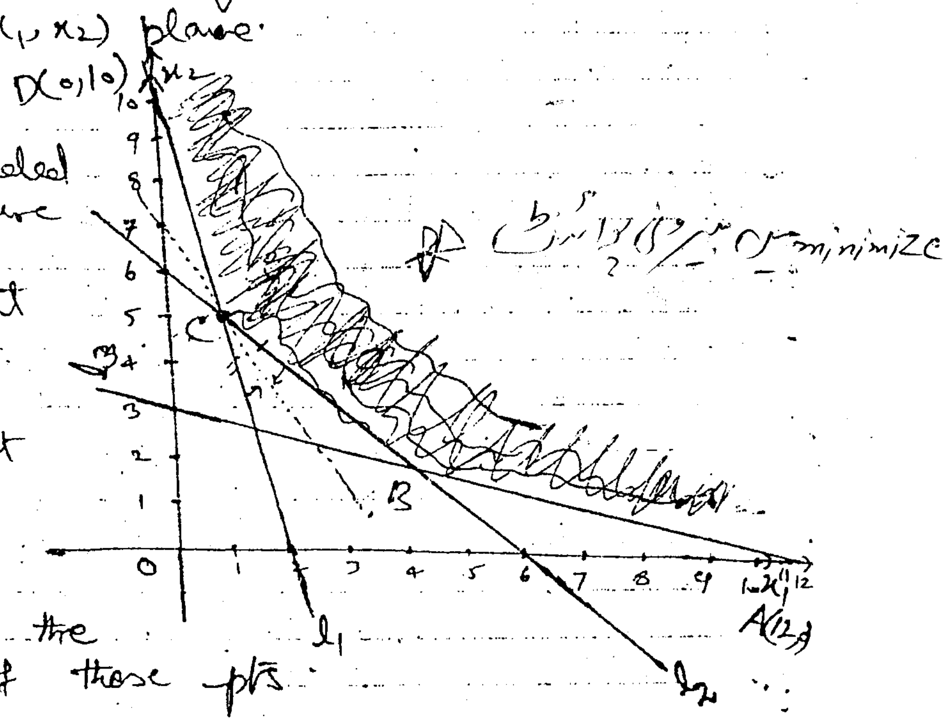
$5x_1 + x_2 = 10 \rightarrow l_1$ Intercept form is $\frac{x_1}{2} + \frac{x_2}{10} = 1$

$x_1 + x_2 = 6 \rightarrow l_2$ $\frac{x_1}{6} + \frac{x_2}{6} = 1$

$x_1 + 4x_2 = 12 \rightarrow l_3$ $\frac{x_1}{12} + \frac{x_2}{3} = 1$

Draw the lines l_1, l_2 & l_3 corresponding to these equations in the (x_1, x_2) plane.

The feasible region is shaded in the figure



Step 2. Note that the minimum value of Z can occur at one of the corner pts A, B, C, & D.

Now we find the coordinates of these pts.

A = (12, 0)

B = intersection of l_3 & l_2 = (4, 2)

C = " " l_1 & l_2 = (1, 5)

D = (0, 10)

Now we evaluate the objective function at these points.

Z at A(12, 0) = 36 $

Z at B(4, 2) = 16 $

Z at C(1, 5) = 13 $

Z at D(0, 10) = 20 $

Thus, Z is minimum at C(1, 5) and has value 13. Hence 1 jar and 5 cartons should be purchased to give the minimum cost of 13 $.

Example: A bank has two types of branches. A satellite branch employs 3 people, requires \$ 100,000 to construct and open, and generates an average daily revenue of \$ 10,000. A full service branch employs 6 people, requires \$ 140,000 to construct and open, and generates an average daily revenue of \$ 18,000. The bank has set up \$ 2.98 million available to open new branches, and has decided to limit the new branches to a maximum of 25. If the bank further decides to hire at most 120 new employs, how many branches of each type should the bank open in order to maximize the average daily revenue?

Solution: The given problem can be summarized in the following table as.

	Satellite branch	full service branch	Max. availability
employs	3	6	120
cost	100,000	140,000	2,980000
branches	1	1	25
daily revenue	\$10,000	18,000	Maximize

Sol. Let x and y denote the number of satellite and full service branches, resp. Then the mathematical model of the problem is

$$\text{Maximize } z = 10{,}000x + 18000y$$

Subject to

$$100{,}000x + 140{,}000y \le 2980{,}000 \Rightarrow 5x + 7y \le 149$$

$$x + y \le 25$$

$$3x + 6y \le 120 \Rightarrow x + 2y \le 40$$

$$x, y \geq 0$$

Step 1: Change all the constraints into eqs.

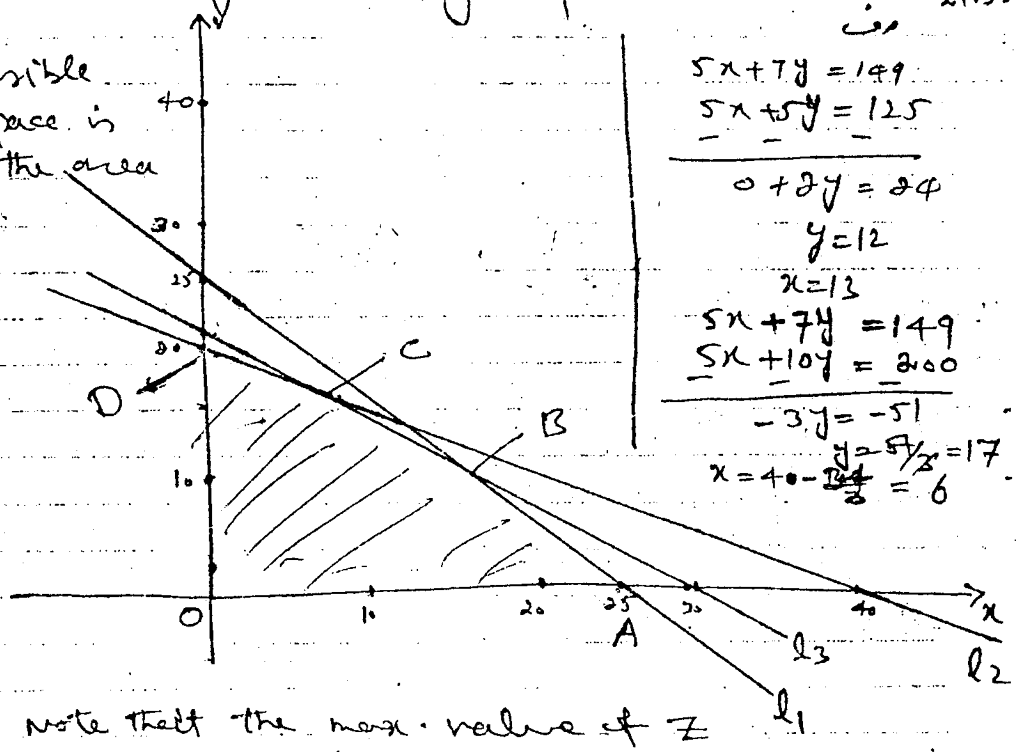
$x + y = 25 \rightarrow l_1$ Intercept form $\frac{x}{25} + \frac{y}{25} = 1$

$x + 2y = 40 \rightarrow l_2$ $\frac{x}{40} + \frac{y}{20} = 1$

$5x + 7y = 149 \rightarrow l_3$ $\frac{x}{149/5} + \frac{y}{149/7} = 1$

Step 2. Draw the lines corresponding to these equations in the (x, y) plane.

the feasible solution space is given by the area OABCD.



150
30 29.8
21.3

$5x + 7y = 149$
$5x + 5y = 125$
$0 + 2y = 24$
$y = 12$
$x = 13$

$5x + 7y = 149$
$5x + 10y = 200$
$-3y = -51$
$y = 51/3 = 17$
$x = 40 - 34 = 6$

Step 3.

Note that the max. value of Z can occur at one of the corner points O, A, B, C, & D of the feasible sol. space. Now we find the coordinates of these pts:

$O = (0, 0)$

$A = (25, 0)$

$B =$ Intersection of l_1 & $l_3 = (13, 12)$

$C =$ " " l_2 & $l_3 = (6, 17)$

$D = (0, 20)$

Now we evaluate the objective function at these corner pts.

$Z = 10000x + 18000y$ at $O(0,0) = 0$

Z at $A(25,0) = 250,000$

Z at $B(13,12) = 130000 + 216000$

$= 346000$

Z at $C(6,17) = 60000 + 306000$

$= 366000$

Z at $D(0,20) = 36000$

The calculation shows that the value of Z is maximum at $C(6,17)$

Hence, we conclude that 6 satellite and 17 full service branches should be opened to obtain a maximum revenue of \$ 366000.

Example:- A lake contains two types fish, I and II. The owner provides two types of food, A and B, for these fishes. Species I require 2 units of food A and 4 units of food B; and species II require 5 units of food A and 2 units of food B. If the owner has 800 units of each food, find the maximum number of fish that the lake can support.

Sol. Let x_1, x_2 be the number of fish of type I and II respectively then the mathematical problem is

Maximize $Z = x_1 + x_2$

subject to

$$2x_1 + 5x_2 \le 800$$

$$4x_1 + 2x_2 \le 800$$

$$x_1, x_2 \ge 0$$

Now we solve the problem by Graphical method

Step I:- Change all the constraints into equations

$$2x_1 + 5x_2 = 800 \quad \rightarrow l_1$$

$$4x_1 + 2x_2 = 800 \quad \rightarrow l_2$$

Intercept form is

$$\frac{x_1}{400} + \frac{x_2}{160} = 1$$

$$\frac{x_1}{200} + \frac{x_2}{400} = 1$$

$$4x_1 + 10x_2 = 1600$$

$$4x_1 + 2x_2 = 800$$

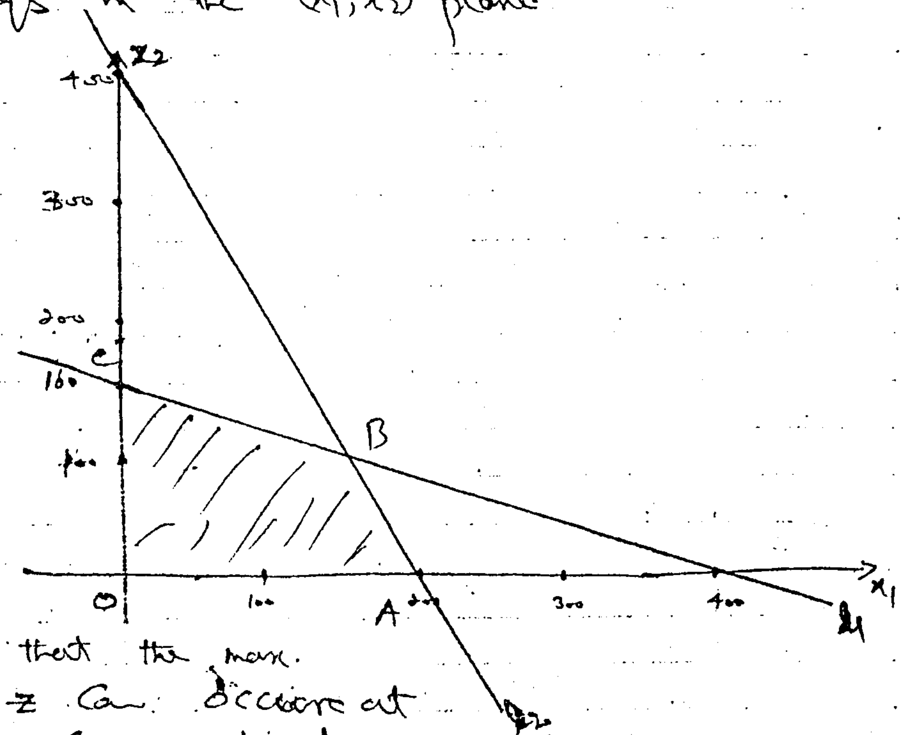
$$8x_2 = 800$$

$$x_2 = 100$$

$$x_1 = 150$$

Draw the lines l_1 and l_2 corresponding to these eqs in the (x_1, x_2) plane

the feasible sol. space is given by the area OABC.



Step 3:- Note that the max. value of z can occur at one of the corner points O, A, B, C.
So, now we find the coordinates of these points.

$O = (0,0)$
$A = (200, 0)$
$B =$ Intersection of l_1 & $l_2 = (150, 100)$
$C = (0, 160)$

Now we evaluate the objective function z at these corner pts

z at $O(0,0) = 0$
z at $A(200,0) = 200$
z " $B(150, 100) = 250$
z at $C(0, 160) = 160$

the calculation shows that the value of z is max. at B(150, 100). Hence we conclude that there should be 150 fish of sort I and 100 fish of sort II to obtain 250, the maximum no. of fish in the lake.

[redacted] A Candidate wishes to use a Combination of radio and television advertisement in his campaign. Research has shown that each one minute spot on TV reaches 0.09 million people and each one minute spot on radio reaches 0.006 million. the Candidate feels that he must reach at least 2.16 million people, and he must buy a total of atleast 80 minutes of advertisement. How many minutes of each medium should be used to minimize the costs if television costs \$500 per minute and radio costs \$100 per minute?

Solution:-

Let x_1 and x_2 be the number of minutes of radio and television bought. then the mathematical model of the problem is

$$\text{Minimize } Z = 100x_1 + 500x_2$$

Subject to

$$0.006x_1 + 0.09x_2 \geq 2.16$$

$$x_1 + x_2 \geq 80$$

$$\Rightarrow \quad 6x_1 + 90x_2 \geq 2160$$

$$\Rightarrow \quad x_1 + 15x_2 \geq 360$$

Now we solve the problem by Graphical method.

Step 1. Change all the Constraints into equations

$$x_1 + 15x_2 = 360 \rightarrow l_1$$

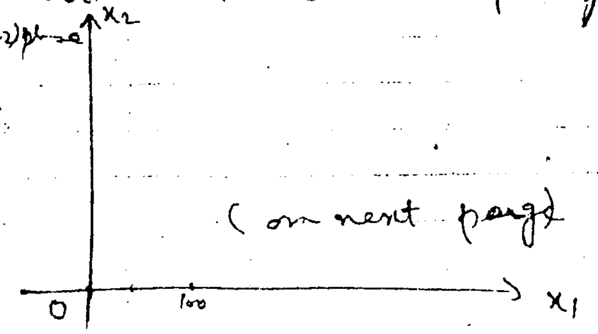
$$x_1 + x_2 = 80 \rightarrow l_2$$

Intercept form

$$\frac{x_1}{360} + \frac{x_2}{24} = 1$$

$$\frac{x_1}{80} + \frac{x_2}{80} = 1$$

Draw these lines l_1 & l_2 Corresponding to these eqs in (x_1, x_2) plane

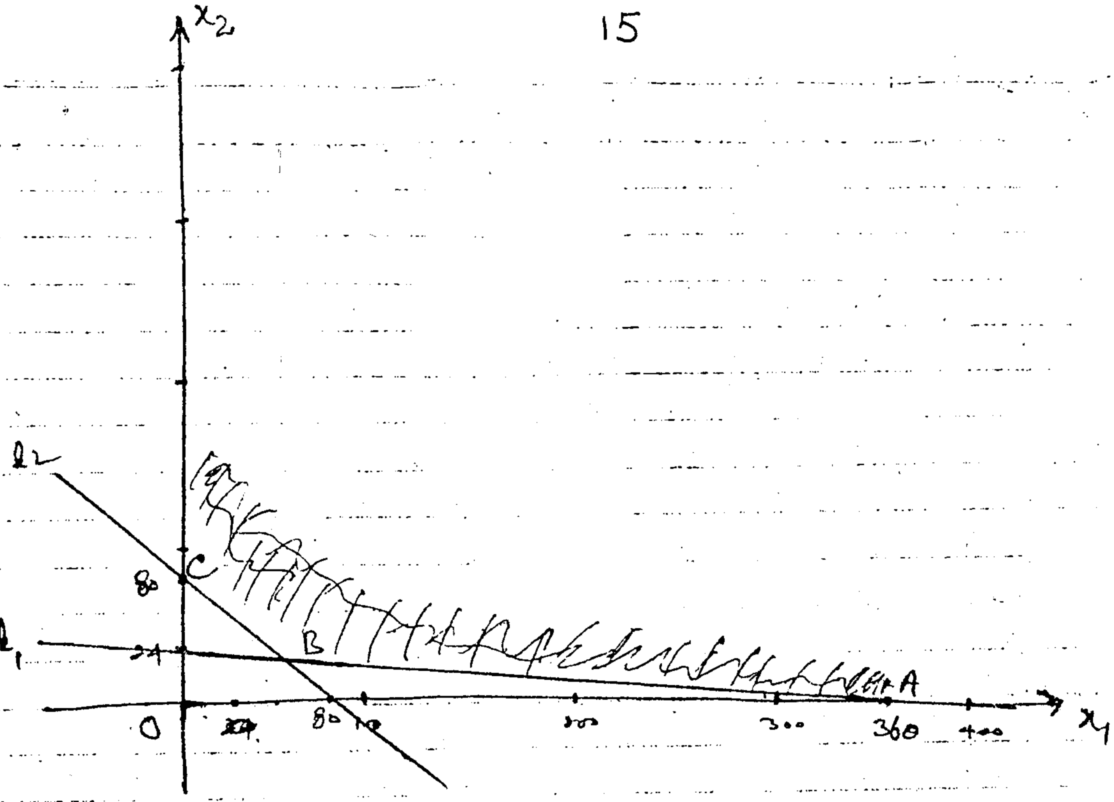


$1 + x_2 = 380$

$x_2 = 20$

$x_1 = 60$

(on next page)



the feasible solution space is shaded ~~on~~ in fig

<u>Step 2.</u> Note that the minimum value of Z can occur at any of these corner pts ~~O~~, A, B, C
Now we find the coordinates of these pts

~~O(0,0)~~ ~~A=(6,0)~~
A = (360, 0)
B = Intersection of l_1 & l_2 = (60, 20)
C = (0, 80)

Now we evaluate the value of Z at these points.

~~Z at O(0,0) = 0~~
Z at A(360, 0) = 36000
Z at B(60, 20) = 16000
Z at C(0, 80) = 40,000

the computations reveals that Z has minimum value at B (60, 20), Hence, we conclude that 60 minutes of radio and 20 minutes of TV should be bought to pay the minimum cost of 16000 $.

F.

Example:- Three products are produced to three different operations. the time required per unit of each product, the daily capacity of the operation under profit per unit sold of each product are as follows.

Time/unit (minutes)

operation	Product 1	Product 2	Product 3	operation capacity minutes/day.
1	1	2	1	430
2	3	0	2	460
3	2	4	0	420
Profit/unit ($)	3	2	6	

formulate a model to determine the optimum daily production for the three products that maximize the profit.

Solution:- Let x_1, x_2 and x_3 be the number of daily units produced of product 1, 2 and 3 resp. then the mathematical model of the problem is

$$\text{maximize } Z = 3x_1 + 2x_2 + 6x_3$$

Subject to

$$x_1 + 2x_2 + x_3 \le 430$$

$$3x_1 + 2x_3 \le 460$$

$$2x_1 + 4x_2 \le 420$$

$$x_1, x_2, x_3 \ge 0$$

* Since it is not possible to produce negative quantities, So we impose non-negativity condition on x_1, x_2, and x_3 *.

Example:- the required daily batch of feed mix for a broiler diet is 100 pounds. the diet must Contain

1. At least 0.8% but not more than 1.2% Calcium;
2. At least 22% protien;
3. At least 5% fiber

the mean ingrediant weet include lime-stone, Corn, Soyabeam Meal with the following compositions.

Ingrediant	Calcium	Protien	Fiber	Cost/pound ($)
	pounds of			
Lime-Stone	0.380	0.00	0.00	0.0164
Corn	0.001	0.09	0.02	0.0645
Soyabean Meal	0.002	0.50	0.08	0.1250

Formulate an LP model for determining feed mix which can be prepared at the least Cost.

Solution:- Let x_1, x_2 and x_3 describe the amounts of Lime-stone, Corn and Soyabean meal respectively. then the desired LP model is:

Minimize $Z = 0.0164\, x_1 + 0.0645\, x_2 + 0.1250\, x_3$

Subject to

$$0.380 x_1 + 0.001 x_2 + 0.002 x_3 \geq 0.8$$
$$0.380 x_1 + 0.001 x_2 + 0.002 x_3 \leq 1.2$$
$$0.09 x_2 + 0.50 x_3 \geq 22$$
$$0.02 x_2 + 0.08 x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0$$

~~Example: Ozark farms uses at least 800 lb of~~ special feed daily. the special feed is a mixture of corn and soybean meal with the following compositions.

	lb per lb of feedstuff		
Feedstuff	Protien	Fiber	Cost ($/lb)
Corn	.09	.02	.30
Soybean meal	.60	.06	.90

the dietary requirements of the special feed stipulates at least 30% protein and at most 5% fiber. ozark farms wishes to determine the daily minimum-cost feed mix.

Sol.

because the feed mix consists of corn and soybean meal, the decision variables of the model are defined as

x_1 = lb of corn in the daily mix

x_2 = " soyabean meal " "

the objective function seeks to minimize the total daily cost (in dollars) of the feed mix and is thus expressed as

$$\text{minimize } z = .3x_1 + .9x_2$$

the constraints of the model must reflect the daily amount needed and the dietery requirements. Because ozark farms need 800 lb of feed a day, the associated constraint can be expressed as

$$x_1 + x_2 \geq 800.$$

the protein dietary requirement constraint is developed next. the amount of protein included x_1 lb of corn and x_2 lb of Soyabean is $(0.09x_1 + 0.60x_2)$ lb. this quantity should equal atleast 30% of the total feed mix (x_1+x_2) lb. thus yielding

$$0.09x_1 + .6x_2 \geq .3(x_1+x_2)$$

In a similar fashion, the fiber constraint is constructed as

$$0.02x_1 + 0.06x_2 \leq .05(x_1+x_2)$$

the preceding constraints are simplified by grouping all the coefficients of the x_1 and x_2 on the left-hand side of each inequality. The complete model thus becomes.

$$\text{minimize } z = .3x_1 + .9x_2$$

Subject to

$$x_1 + x_2 \geq 800$$

$$.21x_1 - .30x_2 \leq 0$$

$$.03x_1 - .01x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

For Sol See book. (Page 19)

Example:- A person requires 10, 12, 12 units of Chemicals A, B, C respectively. ~~A B C~~ A liquid Contains 5, 2, & 1 units of A, B & C per jar (مرطبان) and a dry product Contains 1, 2, 4 units of A, B, C per Carton (card board box = علبة). If the liquid product Costs \$3 per jar and dry product \$2 per Carton. How many of each should be purchased to minimize the Cost and meet the requirements.

Sol:- Primarily, we put the problem in tableau form as under

	liquid Units/jar	dry Units/Carton	unit require
Chemical A	5	1	10
" B	2	2	12
" C	1	4	12

Let x_1, x_2 denotes the No. of Jars and no. of Cartons purchased

then the mathematical model of our problem is

Minimize $Z = 3x_1 + 2x_2$

Subject to

$$5x_1 + x_2 \geq 10$$
$$2x_1 + 2x_2 \geq 12$$
$$x_1 + 4x_2 \geq 12 \qquad , x_1, x_2 \geq 0$$

we will solve this problem graphically as Change the Constraints in eqs.

$\Rightarrow$ Minimize $Z = 3x_1 + 2x_2$

Subject to

$5x_1 + x_2 = 10 \rightarrow l_1$

$2x_1 + 2x_2 = 12 \rightarrow l_2$

$x_1 + 4x_2 = 12 \rightarrow l_3$

Intercept form is

$\frac{x_1}{2} + \frac{x_2}{10} = 1$

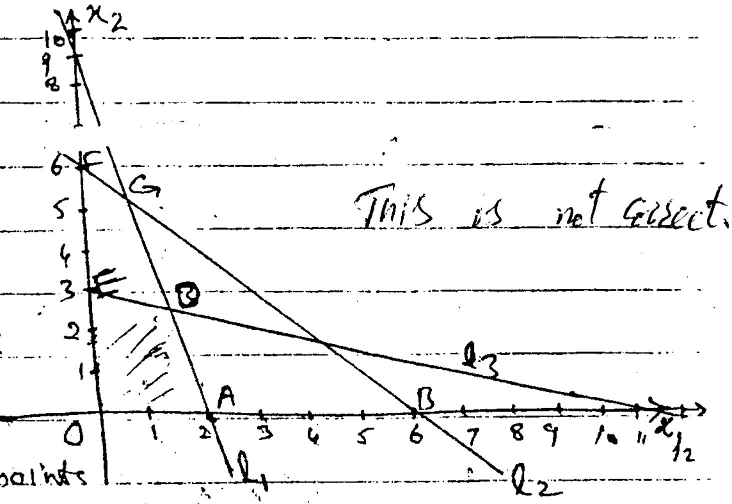
$\frac{x_1}{6} + \frac{x_2}{6} = 1$

$\frac{x_1}{12} + \frac{x_2}{3} = 1$

$x_1, x_2 \geq 0$

WRONG

Draw the lines l_1, l_2 & l_3 Corresponding to these equations in x_1, x_2 plane.



Now the feasible sol. space is given by the area OADE and Note that the minimum value of Z Can occur at one of the Corner points O, A, D & E of the feasible Solution Space.

This is not correct.

ie, O, A, D & E are the feasible points.

Now we determine the Coordinates of the Corner points of The Solution Space

i.e. O(0,0), A(2,0), E(0,3)

& D = Intersection of l_1 & l_3

$D = (1\frac{9}{19}, 2\frac{12}{19})$

$Z = 3x_1 + 2x_2$

Now value of Z at A(2,0) = 6

" " Z at $D(\frac{28}{19}, \frac{50}{19}) = \frac{3\times 28}{19} + \frac{2\times 50}{19}$

$= \frac{1}{19}(84+100)$

$= \frac{184}{19}$

$= 9\frac{13}{19}$

~~value of Z at E(0,3) = 6~~

$5x_1 + x_2 = 10$

$5x_1 + 24x_2 = 60$

$-19x_2 = -50$

$x_2 = \frac{50}{19}$

$x_2 = 2\frac{12}{19}$

So

$5x_1 + \frac{50}{19} = 10$

$5x_1 = 10 - \frac{50}{19}$

$5x_1 = \frac{140}{19}$

$x_1 = \frac{28}{19}$

$x_1 = 1\frac{9}{19}$

Transition from Graphical to algebraic sol.

The zero $n-m$ variable are nonbasic variables.

ie if we set $n-m$ variable equal to zero

The remaining m variables are basic variable and their sol. is basic sol. m = no. of Eq. n = no. of variables $m < n$

Max. no. of Corner pts. is $C^n_m = \frac{n!}{m!(n-m)!}$

Simplex Method:- (or the general Sol. technique)

Algorithm:- طریقِ کار

The algorithm for the Simplex method is as below.

1) Convert the given problem in the Standard mathematical form using slack (or surplus) variables $S_1, S_2, \dots, S_n$

and

$$Z - x_i = 0 \qquad 1 \leq i \leq n$$

2) Write the objective function and the equalities in the form of a Table i.e. find the starting basic sol.

3) Select an entering variable using the optimality condition. Stop if there is no entering variable.

4) Select a leaving variable using the feasibility condition.

5) Determine the new basic sol. by using the appropriate Gauss – Jordan Computations. and Go to 3.

The Simplex method Computations are iterative, in the sense that fixed Conditions and Computations are applied to the Current tableau to produce the next tableau. We Thus refer to The Successive tableaus as iteration and Stop the iteration when all the Coeff. in Z-eq. becomes non-negative.

Optimality Condition:-

the entering variable in a maximization (minimization) problem is the non-basic variable having the most negative (positive) coefficient in the z-row.

The optimum is reached at the iteration where all the z-row coefficients of the non-basic variables are nonnegative (non-positive).

Feasibility Condition:-

For both the maximization and the minimization problems, the leaving variable is the basic variable associated with the smallest non-negative ratio i.e. Take the ratios of the values in the sol. column to the constraint coefficients under the entering variable.

Degeneracy:-

Degenerate بگڑ جانا، خراب ہو جانا

~~In the application of the~~

Special Cases in Simplex Method Application.

There are four special cases that arises in the application of the simplex method.

1) Degeneracy
2) Alternative optima
3) Unbounded Solutions.
4) Nonexisting or Infeasible Solutions.

Now we will give here the theoretical explanation for the

~~reason these situations arise and he provides~~ a practical ~~explanation for the reason~~ interpretation of what these special results could mean in a real-life problem.

1) Degeneracy.

In the application of the feasibility condition of the simplex method a tie for the minimum ratio may be broken arbitrarily for the purpose of determining the leaving variable.

When this happens, one or more of the basic variables will be zero in the next iteration. In this case, the new solution is degenerate. نقطه نظر - نقطه نگاه - زاویه نگاه.

ii- From the practical stand-point (~~power of seeing (into some problem etc) with the mind~~ (بصيرة)) this happens if the model has atleast one redundant (زائد) constraint. Degeneracy has two implications 1- cycling or circling

Unfortunately, there are no reliable techniques for idenfying redundant constraints directly from the tableau. (عمل)

(So to be able to provide more insight (power of seeing (into some problem etc) with the mind) into the practical and theoretical impacts of degeneracy, we consider an example.

Also the graphical illustration should

Basic variable (S_1, S_2, S_3 ....)
Non basic " = x_1, x_2

enhance the understanding of ideas underlying this special situation.)

<u>Example 3.5.1</u> (Degenerate optimal Solution)

Maximize $z = 3x_1 + 9x_2$

subject to

$x_1 + 4x_2 \le 8$ $\qquad \frac{x_1}{8} + \frac{x_2}{2} = 1$

$x_1 + 2x_2 \le 4$ $\qquad \frac{x_1}{4} + \frac{x_2}{2} = 1$

$x_1, x_2 \geq 0$

The given model expressed in Standard form is

~~$x_1 + 4x_2 + S_1$~~

Maximize $z = 3x_1 + 9x_2$

Subject to

$x_1 + 4x_2 + S_1 = 8$

$x_1 + 2x_2 + S_2 = 4$

$x_1, x_2, S_1, S_2 \geq 0$

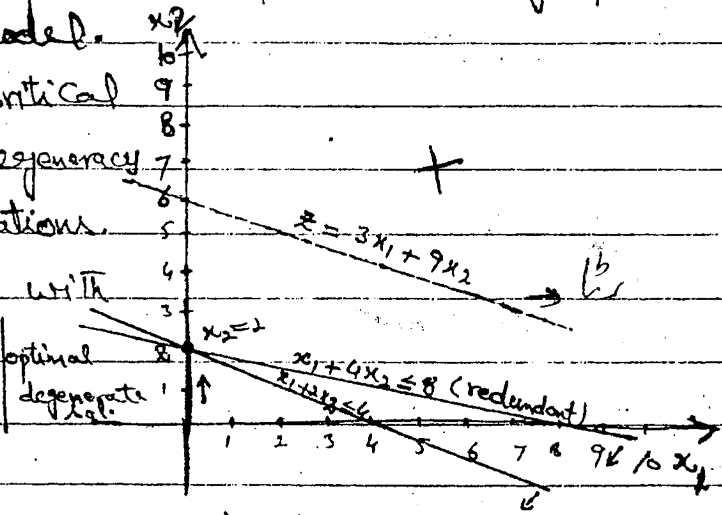
The Standard form Can be expressed in a tabular form as. [illegible]

Iteration	Basic	x_1	x_2	S_1	S_2	Solution
0	z	-3	-9	0	0	0
Introduce x_2 drop S_1	S_1	1	[4]	1	0	8 $\quad 8/4 = 2$
	S_2	1	2	0	1	4 $\quad 4/2 = 2$ tie
1	z	$-3/4$	0	$9/4$	0	18
x_1 enters	x_2	$1/4$	1	$1/4$	0	2
& S_2 leaves	S_2	[1/2]	0	$-1/2$	1	0
2	z	0	0	$3/2$	$3/2$	18
optimum	x_2	0	1	$\frac{1}{2}$	$-1/2$	(2)
	x_1	1	0	-1	2	0

Entering variable column is pivot column and Leaving variable ~~column~~ row as pivot row. Intersection of pivot row and pivot column is pivot element.

In the starting iteration, S_1 and S_2 tie for the leaving variable. this is the reason that the basic variable S_2 has a zero value in iteration 1, thus resulting in a degenerate basic solution. The optimum is reached after an additional iteration is carried out.

What is the practical implication of degeneracy? Look at the fig 1 which provides the graphical solution to the model.



From the theoritical stand point, degeneracy has two implications. The first deal with the phenomenon of cycling.

If you look at iteration 1 and 2 in the tableaux, you will find that the objective value has not improved ($Z = 18$). it is thus conceivable that the simplex procedure would repeat the same sequence of iterations, never improving the objective value and never terminating the computations.

the 2nd theoretical point arises in the examination of iterations 1 and 2. is that Both iterations (although differing in classifying the variables as basic and non basic) yield identical values of all the variables and objective value, namely

$x_1 = 0$; $x_2 = 2$, $S_1 = 0$, $S_2 = 0$, $Z = 18$

<u>Note</u>. It is not possible to stop the computation at iteration 1 when degeneracy first appears, even though it is not optimum. because the solution may be temporarily degenerate.

Alternative optima :-

When the objective function is parallel to a binding constraint (i.e. a constraint that is satisfied as an equation by the optimal sol.), the objective function will assume the same optimal value at more than one solution point. For this reason they are called alternative optima.

Now we give an example which has infinite no. of solutions

Example:- 3.5.2 (Infinity of solutions).

Maximize $Z = 2x_1 + 4x_2$

Subject to

$$x_1 + 2x_2 \leq 5$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

the given model expressed in standard form is

Max. $Z = 2x_1 + 4x_2$

S.to $x_1 + 2x_2 + S_1 = 5$

$x_1 + x_2 + S_2 = 4$

$x_1, x_2, S_1, S_2 \geq 0$.

the above model can be expressed in tableau form as

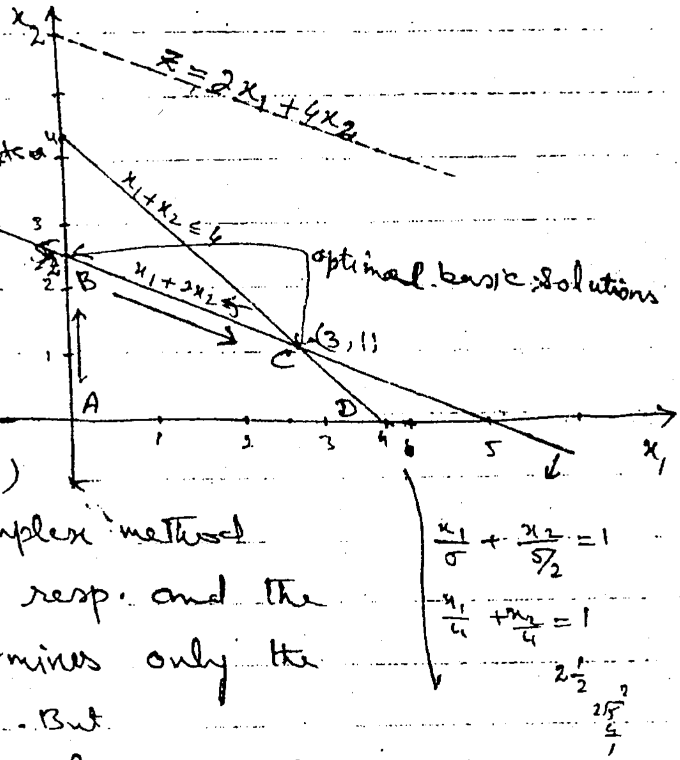
Basic	x_1	x_2	S_1	S_2	Sol.	iteration
Z	-2	-4	0	0	0	0
S_1	1	[2]	1	0	5	x_2 enters
S_2	1	1	0	1	4	S_1 leaves
Z	0	0	2	0	10	1
x_2	$1/2$	1	$1/2$	0	$5/2$	optimum x_1 enter
S_2	$1/2$	0	$-1/2$	1	$3/2$	S_2 leave
Z	0	0	2	0	10	2
x_2	0	1	1	-1	1	alternative
x_1	1	0	-1	2	3	optimum.

Iteration 1 gives the optimum $x_1 = 0$, $x_2 = 5/2$ and $Z = 10$. and the iteration 2 letting x_1 to enter the basic solution, which will force S_2 to leave. and this result in the new solution i.e. alternative optimal sol. at $x_1 = 3$, $x_2 = 1$ & $Z = 10$

The the two optimal solutions are $(0, 5/2)$ & $(3, 1)$

Now the question arises that how do we know from this tableau that the alternative optima exists? For this look at the coefficients of the non basic variables in the Z-equation of iteration 1 the coefficients of non basic x_1 is zero, indicating that x_1 can enter the basic solution without changing the value of Z, but causing a change in the values of the variables.

Also fig. 1 demonstrate graphically how alternative optima can arise in the LP model when the objective function is parallel to a binding constraint.



Any point on the line segment BC represents an alternative optimum with the same objective value.

also note that the two optimal solutions $(0, 5/2)$ & $(3,1)$ obtained in simplex method coincides with B & C resp. and the simplex method determines only the two corner points. But

Mathematically, we can find all the points $(\hat{x}_1, \hat{x}_2)$ on the line segment BC as a non-negative weighted average of the points B and C. Thus, given $0 \le \alpha \le 1$ and

B: $x_1 = 0$, $x_2 = 5/2$

C: $x_1 = 3$, $x_2 = 1$

then all the points on the line segment BC are given by

$\hat{x}_1 = \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha$

$\hat{x}_2 = \alpha(5/2) + (1-\alpha)(1) = 1 + \frac{3}{2}\alpha$

when $\alpha = 0$, $(\hat{x}_1, \hat{x}_2) = (3, 1)$, which is point C, when $\alpha = 1$ $(\hat{x}_1, \hat{x}_2) = (0, 5/2)$ which is pt. B. For values of α b/w 0 & 1 $(\hat{x}_1, \hat{x}_2)$ lies b/w B and C.

Unbounded Solutions:-

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraints, meaning that the solution space is unbounded in at least one direction. As a result, the objective value may increase (maximization case) or decrease (minimization case) indefinitely. In this case both the solution space and the optimum objective value are unbounded.

Example:- 3.5.3 (Unbounded objective value)

$$\text{Maximize } z = 2x_1 + x_2$$

Subject to

$$x_1 - x_2 \le 10 \qquad \frac{x_1}{10} + \frac{x_2}{-10} = 1$$

$$2x_1 \le 40$$

$$x_1, x_2 \ge 0$$

Sol. Standard form is

$$\max. \; Z = 2x_1 + x_2$$

Subject to

$$x_1 - x_2 + S_1 = 10$$

$$2x_1 + S_2 = 40$$

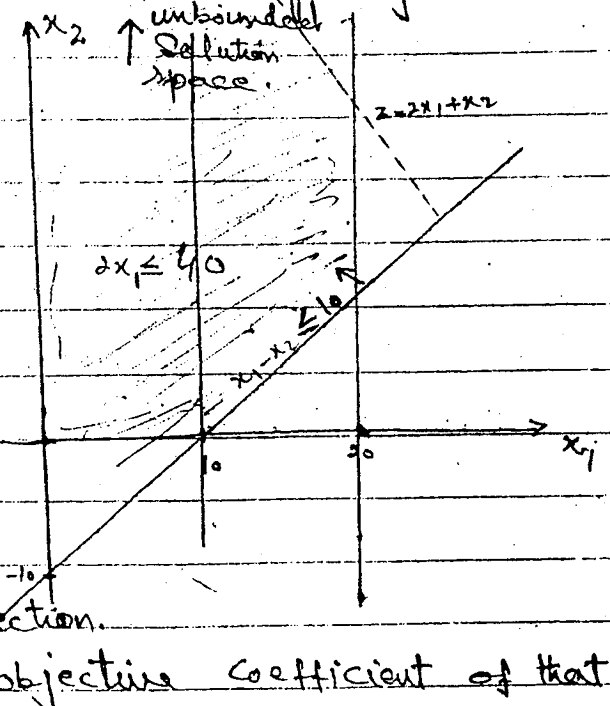
$$S_1, S_2, x_1, x_2 \ge 0$$

Its tableau is

Basic	x_1	x_2	S_1	S_2	Sol.	Ratio
Z	-2	-1	0	0	0	
S_1	1	-1	1	0	10	10
S_2	2	0	0	1	40	20

Now in the starting tableau, ~~both~~ since x_1 has the most -ve coefficient, it is normally selected as the entering variable. However, all the constraint coefficients under x_2 are negative or zero, meaning that x_2 can be increased indefinitely without violating any of the constraints. There is no leaving variable. Because each unit increase in x_2 will increase Z by 1, an infinite increase in x_2 will also result in an infinite increase in Z. Thus the problem has no bounded solution. This result can be seen in fig. The solution space is unbounded in the direction of x_2 and the value of Z can be increased indefinitely.

The rule for recognizing the unboundedness is as follows. If at any iteration the constraint coefficients of any non-basic variable are non-positive, then the solution space is unbounded in that direction.



If, in addition, the objective coefficient of that variable is negative in the case of maximization or positive in the case of minimization, then the objective value also is unbounded.

<u>Note</u>. This is all due to poorly constructed model.

Infeasible Solutions

If all the Constraints in a L.P model Can not be satisfied simultaneously. Then the model has no feasible solution, The situation Can occur only if all the Constraints are not of the type "≤" with non negative right hand sides, because the slacks provides a feasible solution.

For other types of Constraints, we use artificial variables. (Although the artificials are penalized in the objective function to force them to zero at the optimum— this Can occur only if the model has a feasible space. otherwise, at least one artificial variable will be +ve in the optimum iteration.

also In case of phase I) If r is not zero then we have no feasible solution (artificial variable).

From the practical stand-point, an infeasible space points to the possibility that the model is not formulated Correctly.

Example:- 3.5.4 (Infeasible solution space).

$$\text{Maximize } z = 3x_1 + 2x_2$$

Subject to

$$2x_1 + x_2 \le 2 \qquad \frac{x_1}{1} + \frac{x_2}{2} = 1$$

$$3x_1 + 4x_2 \ge 12 \qquad \frac{x_1}{4} + \frac{x_2}{3} = 1$$

$$x_1, x_2 \ge 0$$

the given model expressed in standard form is

$$\text{Max. } z = 3x_1 + 2x_2$$

s-to

$$2x_1 + x_2 + S_1 = 2$$

$$3x_1 + 4x_2 - S_2 = 12$$

$$x_1, x_2, S_1, S_2 \ge 0,$$

Penalty

adding Artificial variables

Max. $Z = 3x_1 + 2x_2 - MR_1$
s.to
$2x_1 + x_2 + S_1 = 2$
$3x_1 + 4x_2 - S_2 + R_1 = 12$
$x_1, x_2, S_1, S_2, R_1 \geq 0$

OR $Z = 3x_1 + 2x_2 - M(12 - 3x_1 - 4x_2 + S_2)$
$Z = (3 + 3M)x_1 + (2 + 4M)x_2 - MS_2 - 12M$

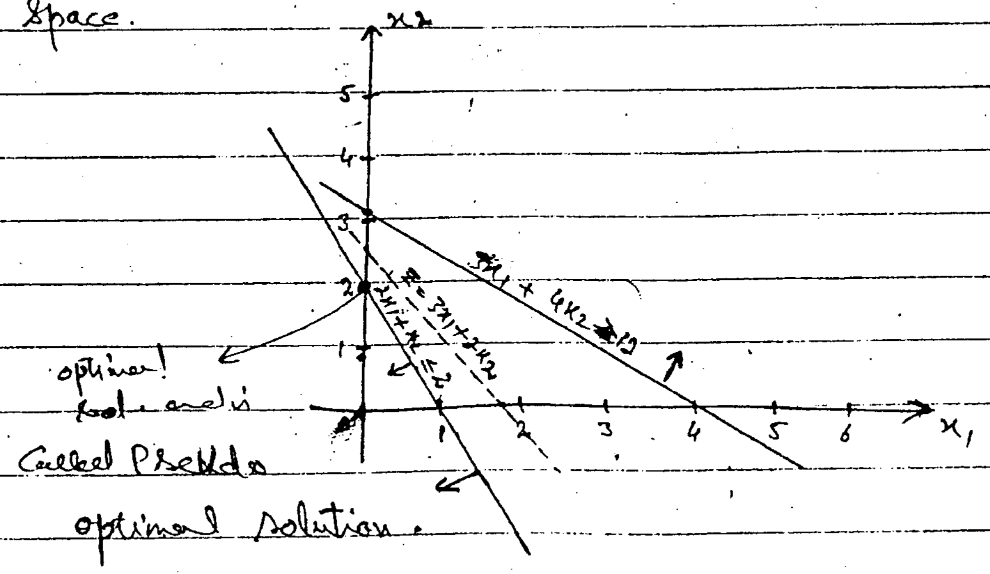
In tableaux form

iteration	Basic	x_1	x_2	S_2	S_1	R_1	Sol.	
0	Z	-3-3M	-2-4M	M	0	0	-12M	
x_2 enters, S_1 leaves	S_1	2	1	0	1	0	2	$\frac{2}{1}=2$
	R_1	3	4	-1	0	1	12	$\frac{12}{4}=3$
1	Z	1+5M	0	M	2+4M	0	4-4M	
optimum	x_2	2	1	0	1	0	2	
	R_1	-5	0	-1	-4	1	4	

Now the iteration 1 is the optimum iteration because their is no more negative coeff. in objective row. (Maximization problem)

So the optimum iteration 1 shows that the artificial variable R_1 is +ve (=4) which indicates that the problem is infeasible.

The fig. 1 demonstrates the infeasible solution space.



Ex. max. $Z = 3x_1 + 4x_2$ (degenerate sol)

Subject to

$$x_1 + 4x_2 \leq 8$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

standard

Sol. the given problem expressed in standard form is

$$\max. \ Z = 3x_1 + 4x_2$$

Subject to

$$x_1 + 4x_2 + S_1 = 8$$

$$x_1 + 2x_2 + S_2 = 4$$

$$x_1, x_2, S_1, S_2 \geq 0$$

the above standard form can be expressed in tabular form as

Basic	x_1	x_2 ↓	S_1	S_2	Sol.	ratio
Z	-3	-4	0	0	0	
S_1	1	[4]	1	0	8	$8/4 = 2$
S_2	1	2	0	1	4	$4/2 = 2$

Since the minimum ratio for leaving variable tie have So the new sol. will degenerate.

So x_2 enters & S_1 leaves

Basic	x_1	x_2	S_1	S_2	Sol.	ratio
Z	-2	0	1	0	8	
x_2	$1/4$	1	$1/4$	0	2	8
S_2	$1/2$	0	$-1/2$	1	0	$= 0$

Here x_1 enters and S_2 leaves.

Basic	x_1	x_2	S_1	S_2	Sol.
Z	0	0	-1	4	8
x_2	0	1	$1/2$	$-1/2$	2
x_1	1	0	-1	2	0

$x_1 = 0, \quad x_2 = 2, \quad Z = 8$

Ex. $\text{Min } Z = x_1 - 3x_2 - 2x_3$ most −ve ⇒ maximize

subject to   most +ve ⇒ minimize

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \geq 0$$

Sol. The given problem expressed in standard form is

$$\text{Min } Z = x_1 - 3x_2 - 2x_3$$

Subject to

$$3x_1 - x_2 + 2x_3 + S_1 = 7$$
$$-2x_1 + 4x_2 + S_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + S_3 = 10$$
$$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

The above standard form can be expressed in tabular form as.

Basic	x_1	↓ x_2	x_3	S_1	S_2	S_3	Sol	Ratio
Z	-1	3	2	0	0	0	0	
S_1	3	-1	2	1	0	0	7	
← S_2	-2	[4]	0	0	1	0	12	$\frac{12}{4}=3$
S_3	-4	3	8	0	0	1	10	$\frac{10}{3}=3\frac{1}{3}$

Basic	x_1	x_2	x_3 ↓	S_1	S_2	S_3	Sol.	Ratio
Z	$\frac{1}{2}$	0	2	0	$-3/4$	0	-9	
S_1	$5/2$	0	2	1	$1/4$	0	10	5
x_2	$-\frac{1}{2}$	1	0	0	$1/4$	0	3	
← S_3	$-5/2$	0	(8)	0	$-3/4$	1	1	$\frac{1}{8}$

↓ x_3 enters and S_3 leaves.

Basic	x_1 ↓	x_2	x_3	S_1	S_2	S_3	Sol	Ratio
Z	$9/8$	0	0	0	$-9/16$	$-1/4$	$-37/4$	
S_1	$25/8$	0	0	1	$7/16$	$-1/4$	$\frac{39}{4}$	$78/25$
x_2	$-1/2$	1	0	0	$1/4$	0	3	–
x_3	$-5/16$	0	1	0	$-3/32$	$\frac{1}{8}$	$1/8$	–

x_1 enters and S_1 leaves.

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
Z	0	0	0	$-9/25$	$-18/25$	$-4/25$	$-319/25$
x_1	1	0	0	$8/25$	$7/50$	$-2/25$	$78/25$
x_2	0	1	0	$4/25$	$8/25$	$-1/25$	$114/25$
x_3	0	0	1	$1/10$	$-1/20$	$1/10$	$11/10$

Hence $x_1 = \frac{78}{25}$, $x_2 = \frac{114}{25}$, $x_3 = \frac{11}{10}$, $Z = \frac{-319}{25}$

Ex. Consider the Constraints

$$x_1 + 7x_2 + 7x_4 \le 46$$

$$3x_1 - x_2 + x_3 + 2x_4 \le 8$$

$$2x_1 + 3x_2 - x_3 + x_4 \le 10$$

Solve the problems by Simplex method.

(i) Max. $Z = 2x_1 + x_2 - 3x_3 - 2x_4$

(ii) " $Z = -2x_1 + 6x_2 + 3x_3 - 2x_4$

(iii) " $Z = 3x_1 - x_2 + 3x_3 + 4x_4$

(iv) Min $Z = 5x_1 - 4x_2 + 6x_3 + 8x_4$

(v) " $Z = 3x_1 + 6x_2 - 2x_3 + 4x_4$.

Example 2.2 - 1

	A	B	Profit
interior paint	4	2	$ 4,000
Exterior paint	6	1	$ 5,000
Availability	24	6	

✓ **Graphic Solution:-**

Let x_1 and x_2 be the tons of exterior and interior paints produced. The mathematical model is then

Step 1. Maximize $z = 5000\, x_1 + 4000\, x_2$

subject to

$$6x_1 + 4x_2 \le 24$$
$$1x_1 + 2x_2 \le 6$$
$$x_2 \le 2$$
$$-x_1 + x_2 \le 1$$
$$x_1,\ x_2 \ge 0$$

Step 2. Change the constraints into equations

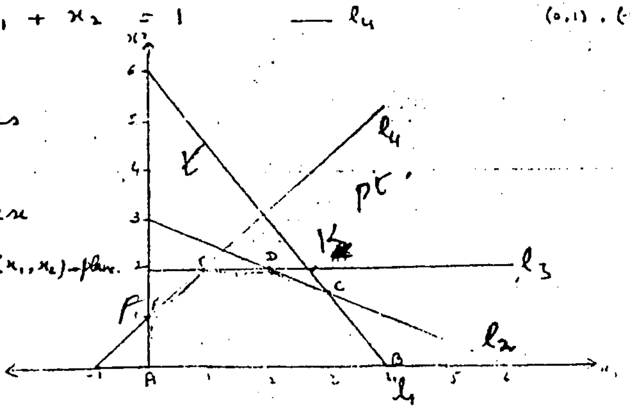
$3x_1 + 2x_2 = 12$ — l_1 (0,6), (4,0)

$x_1 + 2x_2 = 6$ — l_2 (0,3), (6,0)

$x_2 = 2$ — l_3

$-x_1 + x_2 = 1$ — l_4 (0,1), (-1,0)

Draw the lines l_1, l_2, l_3 and l_4 corresponding to these equations in the (x_1, x_2)-plane.



The feasible solution space is given by the area ABCDEF.

Step 3. Note that the maximum value of Z can occur at one of the corner points A, B, C, D, E and F of the feasible solution space. Now we determine the coordinates of the corner points of the solution space.

A = (0,0)
B = intersection of l_1 & x_1-axis = (4,0)
C = " " l_1 & l_2 = (3, 3/2)
D = " " l_2 & l_3 = (2, 2)
E = " " l_3 & l_4 = (1, 2)
F = " " l_4 & x_2-axis = (0, 1)

Step 4.
Now we Evaluate the objective profit at each corner.

Z at A(0,0) = 0
Z at B(4,0) = 20,000
Z at C(3,3/2) = 21,000
Z at D(2,2) = 18,000
Z at E(1,2) = 13,000
Z at F(0,1) = 4,000

Clearly Z attains maximum value at C(3, 3/2) and the maximum value (profit) is 21,000 ($).

If the we take the objective function as $Z = 5x_1 + 4x_2$ then the answer C(3, 3/2) max profit Z = 21, will be in ($1000) Thousands of dollar.

The Simplex method :-

Standard form :- If given $x \leq y$

then we take $x + z = y$

where z is **Slack Variable.**

If $x \geq y$

$\Rightarrow x - S = y$

where S is **Surplus Variable**

If $x_3 \leq 0$ (unrestricted variable $x_3 = 0$)

Put $x_3 = -x_3'$

In equations Right side should be non-negative.

The Simplex Algorithm

Example 3.3-1 ✓

Solve the following LP model using simplex method.

Maximize $Z = 5x_1 + 4x_2$

subject to

$$3x_1 + 2x_2 \leq 12$$

$$x_1 + 2x_2 \leq 6$$

$$x_2 \leq 2$$

$$-x_1 + x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Solution :-

the given model expressed in <u>standard form</u> is

Maximize $Z = 5x_1 + 4x_2$, $Z - 5x_1 - 4x_2 = 0$

subject to

$$3x_1 + 2x_2 + S_1 = 12$$

$$x_1 + 2x_2 + S_2 = 6$$

$$x_2 + S_3 = 2$$

$$-x_1 + x_2 + S_4 = 1$$

$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

The above standard form can be expressed in a tabular form as:

Basic	z	x_1	x_2	S_1	S_2	S_3	S_4	Solution	
z	1	-5	-4	0	0	0	0	0	
S_1	0	3	2	1	0	0	0	12	$12/3=4$
S_2	0	1	2	0	1	0	0	6	$6/1=6$
S_3	0	0	1	0	0	1	0	2	-
S_4	0	-1	1	0	0	0	1	1	-

This Tableau provides us with a starting basic solution in which, $x_1 = x_2 = 0$, x_1, x_2 non-basic variables & $S_1 = 12$, $S_2 = 6$, $S_3 = 2$, $S_4 = 1$ & S_1, S_2, S_3, S_4 Basic variables

Now we form another table by taking x_1 as Entering variable (The variable having most negative value) and S_1 as Leaving variable (The variable whose ratio is smaller)

Basic	z	x_1	x_2 ↓	S_1	S_2	S_3	S_4	Solution	
z	1	0	$-2/3$	$5/3$	0	0	0	20	
x_1	0	1	$2/3$	$1/3$	0	0	0	4	6
S_2 ←	0	0	$\boxed{4/3}$	$-1/3$	1	0	0	2	$3/2$
S_3	0	0	1	0	0	1	0	2	2
S_4	0	0	$5/3$	$1/3$	0	0	1	5	3

Now x_2 = Entering variable, S_2 = Leaving variable

Basic	z	x_1	x_2	S_1	S_2	S_3	S_4	Solution
z	1	0	0	$3/2$	$1/2$	0	0	(21)
x_1	0	1	0	$1/2$	$-1/2$	0	0	3 ✓
x_2	0	0	1	$-1/4$	$3/4$	0	0	$3/2$ ✓
S_3	0	0	0	$1/4$	$-3/4$	1	0	$1/2$
S_4	0	0	0	$3/4$	$-5/4$	0	1	$5/2$

$S_1 = S_2 = 0$, Maximum value of $z = 21$ at $x_1 = 3$, $x_2 = 3/2$

∵ all the coefficients in the z-equation are non-negative, so it yields the optimal solution.

EX Maximize $x_0 = 4x_1 + 3x_2$

subject to

$$2x_1 + 3x_2 \le 6$$
$$-3x_1 + 2x_2 \le 3$$
$$2x_2 \le 5$$
$$2x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$

Sol:- Change the constraints into equations

$2x_1 + 3x_2 = 6$ — l_1 • (0,2), (3,0)

$-3x_1 + 2x_2 = 3$ — l_2 (0, 3/2), (-1,0)

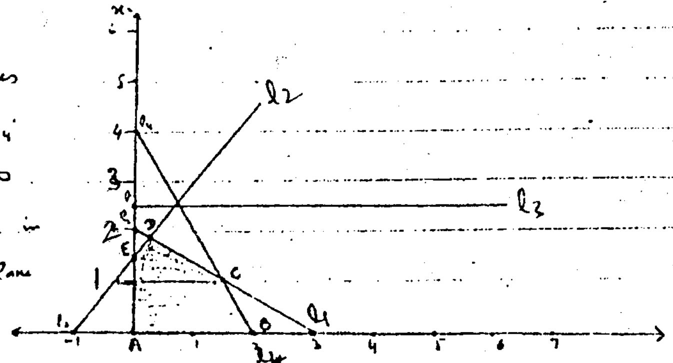
$2x_2 = 5$ — l_3

$2x_1 + x_2 = 4$ — l_4 (0,4), (2,0)

$6x_1 + 9x_2 = 18$
$-6x_1 + 4x_2 = 6$
$13x_2 = 24$

$4x_1 + 6x_2 = 12$
$-9x_1 + 6x_2 = 9$
$13x_1 = 3$
$x_1 = 3/13$
$6 - 6/13 = 3x_2$
$x_2 = 24/13$

Draw the lines l_1, l_2, l_3 and l_4 corresponding to these equations in the (x_1, x_2) plane



The feasible solution space is given by the area ABCDE. The maximum value of x_0 will lie at one of the corner points of ABCDE.

Now we find coordinates

A = (0,0) , C = (intersection of l_1 & l_4) = (3/2, 1)

B = (2,0) D = intersection of l_2 & l_1 = (3/13, 24/13)

E = (0, 3/2)

Now we find maximum value of x_0 at each corner point

$12 + 72 = 84$

x_0 at A(0,0) = 0 , x_0 at B(2,0) = 8

x_0 at C(3/2, 1) = 9 , x_0 at D(3/13, 24/13) = 84/13

x_0 at E(0, 3/2) = 9/2

Maximum value is 9 at (3/2, 1)

By Simplex Method :-

The given model expressed in Standard form is

Maximize $x_0 = 4x_1 + 3x_2$

subject to

$$2x_1 + 3x_2 + S_1 = 6$$

$$-3x_1 + 2x_2 + S_2 = 3$$

$$2x_2 + S_3 = 5$$

$$2x_1 + x_2 + S_4 = 4$$

$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

The standard form can be expressed in a tabular form as:

Iteration	Basic	x_1	x_2	S_1	S_2	S_3	S_4	solution	
0	x_0	[-4]	-3	0	0	0	0	0	
introduce x_1	S_1	2	3	1	0	0	0	6	6/2 = 3
leap S_4	S_2	-3	2	0	1	0	0	3	3/-3 [illegible]
	S_3	0	2	0	0	1	0	5	[illegible]
	S_4	2	1	0	0	0	1	4	4/2 = 2
1	x_0	0	-1	0	0	0	2	8	
introduce x_2	S_1	0	[2]	1	0	0	-1	2	2/2 = 1
leap S_1	S_2	0	7/2	0	1	0	3/2	9	18/7 = 2.6
	S_3	0	2	0	0	1	0	5	5/2 = 2.5
	x_1	1	1/2	0	0	0	1/2	2	4
2	x_0	0	0	1/2	0	0	3/2	9	
	x_2	0	1	1/2	0	0	-1/2	1	
	S_2	0	0	-7/4	1	0	13/4	(5/2) 11/2	
	S_3	0	0	-1	0	1	+1	3	
	x_1	1	0	-1/4	0	0	3/4	3/2	

- Optimal solution is 9 at (3/2, 1)

Ex: Maximize $x_0 = 3x_1 + 2x_2$

subject to

$$x_1 + 2x_2 \le 6$$
$$2x_1 + x_2 \le 8$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Sol:- Change the constraints into equations

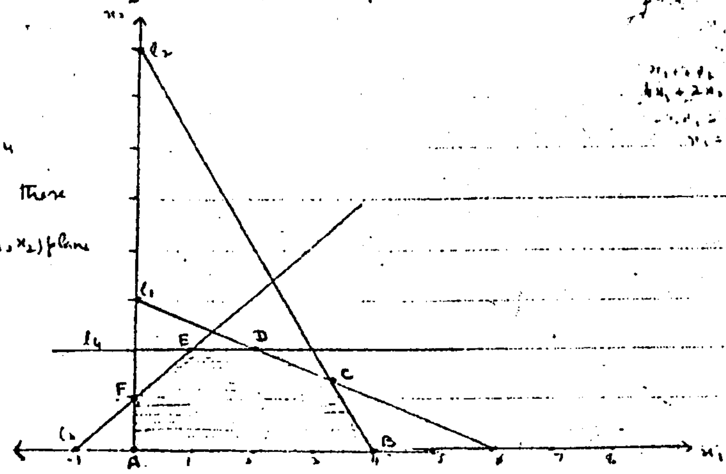
$x_1 + 2x_2 = 6$ — l_1 — $(0,3), (6,0)$

$2x_1 + x_2 = 8$ — l_2 — $(0,8), (4,0)$

$-x_1 + x_2 = 1$ — l_3 — $(0,1), (-1,0)$

$x_2 = 2$ — l_4

Draw lines l_1, l_2, l_3 & l_4 corresponding to these equations in (x_1, x_2) plane



The feasible solution space is the area ABCDEF.

Now we find the coordinates of corner points.

$A = (0,0)$, $B = (4,0)$, C = intersection of l_1 & l_2 = $(10/3, 4/3)$

D = (intersection of l_1 & l_4) = $(2,2)$, E = intersection of l_3 & l_4 = $(1,2)$, $F = (0,1)$

Now we find value of x_0 at these points

x_0 at $A(0,0) = 0$, x_0 at $B(4,0) = 12$, x_0 at $C(10/3, 4/3) = 38/3$

x_0 at $D(2,2) = 10$, x_0 at $E(1,2) = 7$, x_0 at $F(0,1) = 2$

Maximum value of Z is $38/3$ at $C(10/3, 4/3)$.

Simplex Method:

The given model in Standard form is

Max $x_0 = 3x_1 + 2x_2$

subject to

$$x_1 + 2x_2 + S_1 = 6$$

$$2x_1 + x_2 + S_2 = 8$$

$$-x_1 + x_2 + S_3 = 1$$

$$x_2 + S_4 = 2$$

$$x_1, x_2, S_1, S_2, S_3, S_4 \geq 0$$

The standard form can be expressed in tabular form as:

Iteration	Basic	x_1	x_2	S_1	S_2	S_3	S_4	Solution	
0	z_0	-3	-2	0	0	0	0	0	
introduce x_1	S_1	1	2	1	0	0	0	6	6
drop S_2	S_2	2	1	0	1	0	0	8	4
	S_3	-1	1	0	0	1	0	1	
	S_4	0	1	0	0	0	1	2	
1	x_0	0	$-1/2$	0	$3/2$	0	0	12	
introduce x_2	S_1	0	$3/2$	1	$-1/2$	0	0	2	
drop S_1	x_1	1	$1/2$	0	$1/2$	0	0	4	8
	S_3	0	$3/2$	0	$1/2$	1	0	5	
	S_4	0	1	0	0	0	1	2	
2	x_0	0	0	$1/3$	$4/3$	0	0	$38/3$	
	x_2	0	1	$2/3$	$-1/3$	0	0	$4/3$	
	x_1	1	0	$-1/3$	$2/3$	0	0	$10/3$	
	S_3	0	0	-1	1	1	0	(7) 3	
	S_4	0	0	$-2/3$	$1/3$	0	1	$2/3$	

Example :- Three products are produced through three different operations. The time required per unit of each product, the daily capacities of the operations and the profit per unit sold of each product are as follows:

Operations	Product 1	Product 2	Product 3	Operation capacities
1	1	2	1	430
2	3	0	2	460
3	2	4	0	420
Profit / unit	\$ 3	\$ 2	\$ 5	

Formulate a model to determine the optimum daily production for the three products that maximizes profit.

Artificial Starting Solution:-

If all the constraints in the original problem are of "≤" type, then the slack variables are used for a starting solution.

But if the problem includes constraints of the type "≥" or "=" then the natural question arises, How can we find a starting basic solution for models that involve '≥' and '=' constraints. In this case the artificial variables are introduced to obtain a starting basic sol.
These are the variables that assume the role of slacks at the 1st iteration, only to be disposed of at a later iteration.
The following two methods are derived for this purpose.

i) The M-technique or Method of penalty.
ii) The two phase method.

i) The M-method i.e. M-technique method.

The M-Method starts with the LP in the standard form. For any equation i that does not have a slack, we augment (يزيد) an artificial variable R_i. Such a variable then becomes part of the starting basic solution. However, because artificials are extraneous (غريبة) to the LP model, we assign them a penalty in the objective function to force them to zero level at a later iteration of the Simplex algorithm. ($M \to \infty$)

Given M is a sufficiently large positive value, the variable R_i is penalized in the objective function using $-MR_i$ in the case of maximization and $+MR_i$ in the case of minimization. Because of this penalty, the nature of the optimization process will logically attempt to drive R_i to zero level during the course of the Simplex Method

Artificial Starting Solution :-

M-Technique or Method of Penalty

Example :- Minimize $Z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

$0+0 \neq 3$, $0+0 \neq 6$ both are not true so R_1 & R_2 are added in first two eqns, third satisfies e.g. $0+0 \leq 4$

Sol:- The model in Standard form is

Minimize $Z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 - x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

[illegible]

The model after adding artificial variables becomes

Minimize $Z = 4x_1 + x_2 + MR_1 + MR_2$ $\because M$ is very large

subject to

$$3x_1 + x_2 + R_1 = 3$$
$$4x_1 + 3x_2 - x_3 + R_2 = 6$$
$$x_1 + 2x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

artificial is variable + surplus variable.

Before writing the model in tabular form the objective equation must be expressed in terms of non-basic variables only.

slack & artificial become basic

$$Z = 4x_1 + x_2 + MR_1 + MR_2$$
$$= 4x_1 + x_2 + M(3 - 3x_1 - x_2) + M(6 - 4x_1 - 3x_2 + x_3)$$
$$= (4 - 7M)x_1 + (1 - 4M)x_2 + Mx_3 + 9M$$

or $Z - (4-7M)x_1 - (1-4M)x_2 - Mx_3 = 9M$

∵ M is very large So $-M$ is most negative.

 ↕ and $7M$ is most +ve.

Iteration	Basic	x_1	x_2	x_3	R_1	R_2	x_4	Solution	
0	Z	$-4+7M$	$-1+4M$	$-M$	0	0	0	$9M$	
Introduce x_1 & drop R_1	R_1	3	1	0	1	0	0	3	$3/3=1$ ✓
	R_2	4	3	-1	0	1	0	6	$6/4=3/2$
	x_4	1	2	0	0	0	1	4	$4/1=4$
1	Z	0	$\frac{1+5M}{3}$	$-M$	$\frac{4-7M}{3}$	0	0	$4+2M$	
Introduce x_2 & drop R_2	x_1	1	$1/3$	0	$1/3$	0	0	1	3
	R_2	0	$5/3$	-1	$-4/3$	1	0	2	$6/5$ ✓
	x_4	0	$5/3$	0	$-1/3$	0	1	3	$9/5$
2	Z	0	0	$1/5$	$8/5-M$	$-\frac{1}{5}-M$	0	$18/5$	
Introduce x_3 drop x_4	x_1	1	0	$1/5$	$3/5$	$-1/5$	0	$3/5$	≥ 3
	x_2	0	1	$-3/5$	$-4/5$	$3/5$	0	$6/5$	
	x_4	0	0	1	1	-1	1	1	1 ✓
3	Z	0	0	0	$\frac{7}{5}-M$	$-M$	$-1/5$	$17/5$	
	x_1	1	0	0	$2/5$	0	$-1/5$	$2/5$	
	x_2	0	1	0	$-1/5$	0	$3/5$	$9/5$	
	x_3	0	0	1	1	-1	1	1	

✓ $(\frac{2}{5}, \frac{9}{5})$ $17/5$

Another Simple Algorithm of The M-technique

Step 1:- Express the problem in Standard form.

Step II:- i Add nonnegative artificial variables to the left side of each Constraint of the equation Corresponding to constraints of the type $\geq$ or $=$

ii, Assign a very large per unit penalty to these artificial variables in the objective function to ensure that the artificial variables are zero in the final solution.

Note such a penalty will be designated by $-M$ for Maximization and $+M$ for minimization problems, $M > 0$)

Step III:- Before putting the problem in tabular form the objective equation is to be expressed in terms of the non-basic variables only, (So that the right side Columns render the ~~Starting solution directly.~~)

This is done by using the Constraint equations to eliminate artificial variables from the objective equation.

The Two-Phase Method.

Phase I Add artificial variables (after expressing the problem in standard form) to obtain a starting solution. Then form a new objective function that seeks the minimization of the sum of the artificial variables subject to constraints of the original problem.

Note: If the minimum value of the new objective function is zero, the problem has a feasible solution. Otherwise, the problem has no feasible solution.

Phase II Use the optimal basic solution of phase I as a starting solution for the original problem.

Example: Minimize $Z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solution. Express the problem in standard form

Minimize $Z = 4x_1 + x_2$

subject to

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 - S_2 = 6$$

$$x_1 + 2x_2 + S_3 = 4$$

$$x_1, x_2, S_2, S_3 \geq 0$$

$Z = 4(3/5) + \frac{6}{5}$

Phase I

Minimize $r = R_1 + R_2$

subject to

$$3x_1 + x_2 + R_1 = 3$$

$$4x_1 + 3x_2 - S_2 + R_2 = 6$$

$$x_1 + 2x_2 + S_3 = 4$$

$$x_1, x_2, S_2, S_3, R_1, R_2 \geq 0$$

Before putting the problem in tabular form the objective function must be expressed in terms of non-basic variables only.

$r_0 = R_1 + R_2 = (3 - 3x_1 - x_2) + (6 - 4x_1 - 3x_2 + S_2)$

$= -7x_1 - 4x_2 + S_2 + 9$

The starting tableau is now

Basic	x_1	x_2	S_2	R_1	R_2	S_3	Solution
r_0	7	4	−1	0	0	0	9
R_1	3	1	0	1	0	0	3
R_2	4	3	−1	0	1	0	6
S_3	1	2	0	0	0	1	4

Introduce x_1 and drop R_1

Basic	x_1	x_2	S_2	R_1	R_2	S_3	Solution	
r_0	0	5/3	−1	−7/3	0	0	2	
x_1	1	1/3	0	1/3	0	0	1	3
R_2	0	5/3	−1	−4/3	1	0	2	6/5
S_3	0	5/3	0	−1/3	0	1	3	9/5

Introduce x_2 and drop R_2

Basic	x_1	x_2	S_2	R_1	R_2	S_3	Solution
r_0	0	0	0	−1	−1	0	0
x_1	1	0	1/5	3/5	−1/5	0	3/5
x_2	0	1	−3/5	−4/5	3/5	0	6/5
S_3	0	0	1	1	−1	1	1

Since the minimum value of r_0 is zero, thus the problem has a feasible solution and we proceed to phase II

Phase II Eliminate from the optimal solution tableau of phase I columns corresponding to artificial variables and replace the objective eqn by the objective equation of the original problem.

Basic	x_1	x_2	S_2	S_3	R.H.S.
Z	−4	−1	0	0	0
x_1	1	0	1/5	0	3/5
x_2	0	1	−3/5	0	6/5
S_3	0	0	1	1	1

New Z row = old Z-row + (4 × x_1-row + 1 × x_2 row)

~~[illegible] coefficients [illegible] Z row [illegible] x_1, x_2~~

To obtain a starting basic solution the objective row must be expressed in the standard form

Basic	x_1	x_2	S_2	S_3	solution
Z	0	0	1/5	0	18/5
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
S_3	0	0	1	1	1

Introduce S_2 & drop S_3

Basic	x_1	x_2	S_2	S_3	Solution
Z	0	0	0	-1/5	17/5
x_1	1	0	0	-1/5	2/5
x_2	0	1	0	3/5	9/5
S_2	0	0	1	1	1

$x_1 = 2/5$, $x_2 = 9/5$, $S_2 = 1$ & $Z = 17/5$ Ans.

Differences between M-technique and Two-Phase Method

M-technique	Two-Phase Method
(i) M-technique uses modified objective function with penalty for each artificial variable	(i) Two-phase method uses two objective functions. the original objective function and another that is sum of the artificial variables.
(ii) M-Technique solves the problem in one step.	(ii) Two-phase method solves the problem in two stages.
(iii) M-technique may lead to incorrect solution because of error propagation due to large value of M.	(iii) Two-phase method does not have this draw back.
(iv) M-Technique is not very suitable for computer implementation.	(iv) Two-phase method is better suitable for computer implementation.

Practically Two-phase method is used and M-method ~~is never used~~ in practice.

~~Ex. Consider the following LP.~~

Minimize $z = x_1 + 3x_2 - 2x_3$

Subject to

$$3x_1 - x_2 + 2x_3 \le 7$$

$$-2x_1 + 4x_2 \le 12$$

$$-4x_1 + 3x_2 + 8x_3 \le 10$$

$$x_1, x_2, x_3 \ge 0$$

Solve the problem by the Simplex method, where the entering variable is the non-basic variable with the most +ve objective coefficient.

<u>Sol.</u> We put the data in the tabular form:

basic	x_1	x_2 ↓	x_3	S_1	S_2	S_3	<u>Sol.</u>	Ratio
z	-1	3	2	0	0	0	0	-
S_1	3	-1	2	1	0	0	7	-
← S_2	-2	4	0	0	1	0	12	3
S_3	-4	3	8	0	0	1	10	10/3

x_2 (being most +ve) enters & S_2 (bearing least ratio) leaves.

basic	x_1	x_2	x_3 ↓	S_1	S_2	S_3	<u>Sol.</u>	
z	1/2	0	2	0	-3/4	0	-9	-
S_1	5/2	0	2	1	1/4	0	10	5
x_2	-1/2	1	0	0	1/4	0	3	
S_3	-5/2	0	8	0	-3/4	1	1	1/8

Introduce x_3 and leaves S_3

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol
Z	$9/8$	0	0	0	$-9/16$	$-1/4$	$-37/4$
S_1	$25/8$	0	0	1	$7/16$	$-1/4$	$3/4$
x_2	$-1/2$	1	0	0	$1/4$	0	3
x_3	$-5/16$	0	1	0	$-3/32$	$1/8$	$1/8$

Introduce x_1 and drop S_1

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol
Z	0	0	0	$-9/25$	$-18/25$	$-4/25$	$-12\frac{19}{25}$
x_1	1	0	0	$8/25$	$7/50$	$-2/25$	$3\frac{3}{25}$
x_2	0	1	0	$4/25$	$3/25$	$-1/25$	$4\frac{14}{25}$
x_3	0	0	1	$1/10$	$-1/20$	$1/10$	$1\frac{1}{10}$

Hence $x_1 = 3\frac{3}{25}$; $x_2 = 4\frac{14}{25}$; $x_3 = 1\frac{1}{10}$ and $Z = -12\frac{19}{25}$.

Ex. Consider the following set of Constraints:

$$x_1 + 7x_2 + 3x_3 + 7x_4 \le 46$$

$$3x_1 - x_2 + x_3 + 2x_4 \le 8$$

$$2x_1 + 3x_2 - x_3 + x_4 \le 10$$

Solve the problem by the Simplex method assuming that the objective function is given as follows:-

(a) Maximize $Z = 2x_1 + x_2 - 3x_3 + 5x_4$

(b) Maximize $Z = -2x_1 + 6x_2 + 3x_3 - 2x_4$

(c) Maximize $Z = 3x_1 - x_2 + 3x_3 + 4x_4$

(d) Minimize $Z = 5x_1 - 4x_2 + 6x_3 + 3x_4$

(e) Minimize $Z = 3x_1 + 6x_2 - 2x_3 + 4x_4$

Solutions: (a) $x_1 = 0$; $x_2 = 1\frac{5}{7}$; $x_3 = 0$; $x_4 = 4\frac{6}{7}$ and $Z = 26$

(b) $x_1 = 0$; $x_2 = 11/5$; $x_3 = 51/5$; $x_4 = 0$ and $Z = 219/5$

~~(c) $x_1 = 0$; $x_2 = 11/5$; $x_3 = 51/5$; $x_4 = 0$ and $Z = 142/5$~~

(d) Solution: the starting tableau is

basic	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Sol.
Z	-5	4	-6	-8	0	0	0	0
S_1	1	7	3	7	1	0	0	46
S_2	3	-1	1	2	0	1	0	8
S_3	2	3	-1	1	0	0	1	10

Introduce x_2 and leaves S_3.

basic	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Sol.
Z	$-23/3$	0	$-14/3$	$-28/3$	0	0	$-4/3$	$-40/3$
S_1	$-11/3$	0	16	$14/3$	1	0	$-7/3$	$68/3$
S_2	$11/3$	0	$2/3$	$7/3$	0	1	$1/3$	$34/3$
x_2	$2/3$	1	$-1/3$	$1/3$	0	0	$1/3$	$10/3$

Since all the Coefficients of Z-eq are non-positive So the optimum solution is obtained as.

$$x_1 = 0;\quad x_2 = 10/3;\quad x_3 = 0 = x_4;\ \text{and}\ Z = -40/3$$

(e) Solution is $x_1 = x_2 = x_4 = 0$; $x_3 = 8$ and $Z = -16$

Ex. Solve the following problem by using x_4, x_5 and x_6 for the starting basic (feasible) solution.

$$\text{Maximize } Z = 3x_1 + x_2 + 2x_3$$

Subject to

$$12x_1 + 3x_2 + 6x_3 + 3x_4 = 9$$
$$8x_1 + x_2 - 4x_3 + 2x_5 = 10$$
$$3x_1 - x_6 = 0$$
$$x_1, \ldots, x_6 \geq 0$$

Solution: the above mathematical model can be expressed in a tabular form as.

basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol.	Ratio
z	-3	-1	-2	0	0	0	0	
x_4	12	3	6	3	0	0	9	$3/4$
x_5	8	1	-4	0	2	0	10	$5/4$
x_6	3	0	0	0	0	-1	0	—

Introduce x_1 and x_4

basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol.
z	0	$-1/4$	$-1/2$	$3/4$	0	0	$9/4$
x_1	1	$1/4$	$1/2$	$1/4$	0	0	$3/4$
x_5	0	-1	-8	-2	2	0	4
x_6	0	$-3/4$	$-3/2$	$-3/4$	0	-1	$-9/4$

Introduce x_3 and drop x_1

basic	x_1	x_2	x_3	x_4	x_5	x_6	Sol.
z	1	0	0	1	0	0	3.
x_3	2	$1/2$	1	$1/2$	0	0	$3/2$
x_5	16	3	0	2	2	0	16
x_6	3	0	0	0	0	-1	0.

Hence, the optimal is

$x_1 = 0 = x_2$; $x_3 = 3/2$ and $z = 3$

Q:4 Consider the following set of Constraints:

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Solve the problem for each of the following objective functions.

(a) Maximize $Z = 2x_1 + 3x_2 - 5x_3$

(b) Minimize $Z = 2x_1 + 3x_2 - 5x_3$

(c) Maximize $Z = x_1 + 2x_2 + x_3$

(d) Minimize $Z = 4x_1 - 8x_2 + 3x_3$

Solution :- the Constraints in standard form are

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 - x_4 = 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Now after adding artificial variables the model becomes

(a) Maximize $Z = 2x_1 + 3x_2 - 5x_3 - MR_1 - MR_2$

s.t

$$x_1 + x_2 + x_3 + R_1 = 7$$

$$2x_1 - 5x_2 + x_3 - x_4 + R_2 = 10$$

$$x_1, x_2, x_3, x_4, R_1, R_2 \geq 0$$

before writing the model in tabular form the objective equation must be expressed in terms of non basic variables only.

$$Z = 2x_1 + 3x_2 - 5x_3 - M(7 - x_1 - x_2 - x_3) - M(10 - 2x_1 + 5x_2 - x_3 + x_4)$$

$$\Rightarrow Z = (2 + 3M)x_1 + (3 - 4M)x_2 - (5 - 2M)x_3 - Mx_4 - 17M$$

Now the above mathematical can be expressed in Standard form as.

Basic	x_1	x_2	x_3	x_4	R_1	R_2	Sol.	Ratio
Z	$-(2+3M)$	$-(3-4M)$	$(5-2M)$	M	0	0	$-17M$	
R_1	1	1	1	0	1	0	7	$\frac{7}{1}=7$
← R_2	2	-5	1	-1	0	1	10	$\frac{10}{2}=5$

Introduce x_1 and drop R_2.

Basic	x_1	x_2	x_3	x_4	R_1	R_2	Sol.
Z	0	$-(8+\frac{7}{2}M)$	$(6-\frac{1}{2}M)$	$-(1+\frac{1}{2}M)$	0	$\frac{1}{2}(2+3M)$	$(10-2M)$
R_1	0	$7/2$	$1/2$	$1/2$	1	$-1/2$	2
x_1	1	$-5/2$	$1/2$	$-1/2$	0	$1/2$	5

Introduce x_2 and drop R_1

Basic	x_1	x_2	x_3	x_4	R_1	R_2	Sol.
Z	0	0	$50/7$	$1/7$	$\frac{16+7M}{7}$	$\frac{-1+7M}{7}$	$102/7$
x_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-1/7$	$4/7$
x_1	1	0	$6/7$	$-1/7$	$1/7$	$\frac{1}{7}$	$\frac{45}{7}$

Since none of non basic variables has negative coefficients so the optimum is

$x_1 = 45/7$; $x_2 = 4/7$; and $Z = 102/7$

(b) **Ans is** $x_1 = 3$; $x_2 = 0$; $x_3 = 4$ and $Z = -14$

(c) **Ans is** $x_1 = 45/7$; $x_2 = 4/7$; $x_3 = 0$ and $Z = 53/7$.

d) **Ans is** $x_1 = 45/7$; $x_2 = 4/7$; $x_3 = 0$ and $Z = 148/7$.

Q:7. Consider the problem.

$$\text{Maximize } Z = x_1 + 5x_2 + 3x_3$$

Subject to

$$x_1 + 2x_2 + x_3 = 3$$

$$2x_1 - x_2 = 4$$

$$x_1, x_2, x_3 \geq 0$$

the variable x_3 plays the role of a slack - hence, we donot need an artificial variable in the first Constraint. However, in the 2nd Constraint, an artificial variable is needed. Use this starting solution (ie, x_3 in the first Constraint and R_2 in the 2nd Constraint) to solve the system.

Sol. After adding the artificial variables R_2 in the second Constraint, we have

$$\text{Maximize } Z = x_1 + 5x_2 + 3x_3 - MR_2$$

Subject to

$$x_1 + 2x_2 + x_3 = 3$$

$$2x_1 - x_2 + R_2 = 4$$

$$x_1, x_2, x_3, R_2 \geq 0$$

Since x_3 and R_2 are basic variables So before writing the model in tabular form the objective equation must be expressed in terms of non-basic variables only. ie

$$Z = x_1 + 5x_2 + 3(3 - x_1 - 2x_2) - M(4 - 2x_1 + x_2)$$

$$Z = (-2 + 2M)x_1 + (-1 - M)x_2 + (9 - 4M)$$

$$\Rightarrow Z + (2 - 2M)x_1 + (1 + 1M)x_2 = 9 - 4M$$

~~Thus, the starting tableau is variables of use x_1, x_2~~

Basic	x_1	x_2	x_3	R_2	Sol. objective
z	$2-2M$	$1+M$	0	0	$9-4M$
x_3	1	2	1	0	3
R_2	2	-1	0	1	4

Non-basic ... variables

Introduce x_1 and drop R_2

Basic	x_1	x_2	x_3	R_2	Sol.
z	0	2	0	$-1+M$	5
x_3	0	$5/2$	1	$-1/2$	1
x_1	1	$-1/2$	0	$1/2$	2

Hence the optimal is $x_1 = 2$; $x_2 = 0$, $x_3 = 1$ and $z = 5$

Q: Consider the problem

$$\text{Maximize } z = 2x_1 + 4x_2 + 4x_3 - 3x_4$$

Subject to

$$x_1 + x_2 + x_3 = 4$$

$$x_1 + 4x_2 + x_4 = 8$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solve the problem using x_3 and x_4 for the starting basic feasible solution. (Do not use any artificial variables)

Solution:- Since x_3 and x_4 are the basic variables. So the objective function in terms of non basic variables is

$$z = 2x_1 + 4x_2 + 4(4 - x_1 - x_2) - 3(8 - x_1 - 4x_2)$$

$$\Rightarrow z = 2x_1 + 4x_2 + 16 - 4x_1 - 4x_2 - 24 + 3x_1 + 12x_2$$

$$z = x_1 + 12x_2 - 8$$

$$\Rightarrow z - x_1 - 12x_2 = -8.$$

tableau form is

basic	x_1	x_2 ↓	x_3	x_4	Soln.
Z	-1	-12	0	0	-8
x_3	1	1	1	0	4
← x_4	1	4	0	1	8

Introduce x_2 and leaves x_4

basic	x_1	x_2	x_3	x_4	Soln.
Z	2	0	0	3	16
x_3	$3/4$	0	1	$-1/4$	2
x_2	$1/4$	1	0	$1/4$	2

Since none of the non-basic variables has a negative coefficient, so this tableau leads to the optimum sol.

$x_1=0$; $x_2=2$; $x_3=2$; $x_4=0$ and $Z=16$

Q:6. Solve the following problem using x_3 and x_4 as starting basic feasible variables. Do not use any artificial variables.

Minimize $Z = 3x_1 + 2x_2 + 3x_3$

Subject to

$$x_1 + 4x_2 + x_3 \geq 7$$

$$2x_1 + x_2 + x_4 \geq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Sol. The given problem expressed in standard form is: Min $Z = 3x_1 + 2x_2 + 3x_3$

s.t. $x_1 + 4x_2 + x_3 - x_5 = 7$; $2x_1 + x_2 + x_4 - x_6 = 10$

$x_1, x_2, \ldots, x_6 \geq 0$

~~the objective function expressed in terms of~~
non basic variables is

$$Z = 3x_1 + 2x_2 + 3(-x_1 - 4x_2 + x_5 + 7)$$

$$\therefore \quad Z = 0x_1 - 10x_2 + 3x_5 + 21$$

$$Z - 0x_1 + 10x_2 - 3x_5 = 21$$

The starting tableau hence becomes:

basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln
Z	0	10	0	0	−3	0	21
x_3	1	4	1	0	−1	0	7
x_4	2	1	0	1	0	−1	10

Solve yourself.

Ex: Consider the following Constraints

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Solve by using the two-phase method, assuming that the objective function is given as follows

(a) Maximize $Z = 2x_1 + 3x_2 - 5x_3$

(b) Minimize $Z = 2x_1 + 3x_2 - 5x_3$

(c) Maximize $Z = x_1 + 2x_2 + x_3$

(d) Minimize $Z = 4x_1 - 8x_2 + 3x_3$

Solution:- the Constraints in standard form are

$$x_1 + x_2 + x_3 = 7$$

$$2x_1 - 5x_2 + x_3 - S = 10$$

$$x_1, x_2, x_3, S \geq 0$$

Phase I:- Minimize $r_0 = R_1 + R_2$

Subject to

$$x_1 + x_2 + x_3 + R_1 = 7$$

$$2x_1 - 5x_2 + x_3 - S + R_2 = 10$$

$$x_1, x_2, x_3, S, R_1, R_2 \geq 0$$

Before writing the problem in tabular form the objective function must be expressed in terms of non-basic variables only

$$\text{Minimize } r_0 = (7 - x_1 - x_2 - x_3) + (10 - 2x_1 + 5x_2 - x_3 + S)$$

$$\text{Min } r_0 = -3x_1 + 4x_2 - 2x_3 + S + 17$$

$$\Rightarrow r_0 + 3x_1 - 4x_2 + 2x_3 - S = 17$$

R_2	2	-5	1	-1	0	1	10

Introduce x_1 and leaves R_2

basic	x_1	x_2	x_3	S	R_1	R_2	Sol.
r_0	0	$7/2$	$1/2$	$1/2$	0	$-3/2$	2
R_1	0	$7/2$	$1/2$	$1/2$	1	$-1/2$	2
x_1	1	$-5/2$	$1/2$	$-1/2$	0	$1/2$	5

x_2 enters and R_1 leaves

basic	x_1	x_2	x_3	S	R_1	R_2	Sol.
r_0	0	0	0	0	-1	-1	0
x_2	0	1	$1/7$	$1/7$	$2/7$	$-1/7$	$4/7$
x_1	1	0	$6/7$	$-1/7$	$5/7$	$1/7$	$45/7$

Since the minimum value of r_0 is zero. So the problem has a feasible solution and we proceed to phase II.

Note: Since the constraints are common to all the four parts, so the phase I is also common to all the four ~~cases~~ parts.

So Phase II (for part (a)). We eliminate the column corresponding to the artificial variables from the optimal sol. of phase I, and replace the objective equation by the objective equation of the original problem.

basic	x_1	x_2	x_3	S	Soln.
Z	-2	-3	5	0	0
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7

In order to obtain a starting basic sol the objective row must be ~~correspond~~ expressed in standard form as:

basic	x_1	x_2	x_3	S	Soln.
Z	0	0	50/7	1/7	102/7
x_2	0	1	1/7	1/7	4/7
x_1	1	0	6/7	-1/7	45/7

Since we have to maximize z and are seeking for most -ve coeff in the z-eq. which is not found, so the tableau is optimal and is

$x_1 = 45/7$; $x_2 = 4/7$ and $Z = 102/7$

Similarly (solve 2nd phase) for b, c, & d part.

(b) Ans is (same as a.)

(c) $x_1 = 45/7$; $x_2 = 4/7$ & $Z = 53/7$

(d) $x_1 = 45/7$; $x_2 = 4/7$; $x_3 = 0$ & $Z = 148/7$

Ex. Consider the following linear program

Maximize $Z = 2x_1 + 4x_2 + 4x_3 - 3x_4$

s.t.

$x_1 + x_2 + x_3 = 4$

$x_1 + 4x_2 + x_4 = 8$

$x_1, x_2, x_3, x_4 \geq 0$

next article

by using x_3 and x_4 as the starting variables, the optimum tableau is given by

basic	x_1	x_2	x_3	x_4	sol.
Z	2	0	0	3	16
x_3	3/4	0	1	-1/4	2
x_2	1/4	1	0	1/4	2

Write the dual problem and find its Solution from the optimal primal tableau.

Sol. the primal in standard form is given in the statement. the dual, therefore, is given as follows.

Minimize $w = 4y_1 + 8y_2$

s.t

$y_1 + y_2 \geq 2$

$y_1 + 4y_2 \geq 4$

$y_1 \geq 4$

$y_2 \geq -3$

the primal comprises x_3 and x_4 as the starting variables. their respective coefficients in the optimal primal tableau are 0 & 3. and the dual constraints associated with x_3 & x_4 are $y_1 \geq 4$ & $y_2 \geq -3$.

thus, $y_1 - 4 = 0 \Rightarrow y_1 = 4$ Hence $y_1 = 4$; $y_2 = 0$

$y_1 + 3 = 3 \Rightarrow y_1 = 0$ with $w = 16$. Ans.

[illegible] <u>Duality And Sensitivity Analysis.</u>

The LP model we develop for a [illegible] situation is referred to as the <u>Primal</u> problem.

Every LP problem has a second problem associated with it. One problem is called Primal and the other is called dual. The dual problem is constructed from the primal expressed in the standard form, as follows

1) If the primal seeks maximization then the dual seeks minimization and vice versa.

2) Each constraint in the primal problem corresponds to a variable in the dual problem

3) The right hand side elements of the constraints in the primal are coefficients of the respective variables in the objective function of the dual problem.

4) Each variable in the primal corresponds to a constraint in the dual problem.

5) the constraint coefficiente of a primal variable form the left side coefficients of the corresponding dual constraint.

6) If the dual seeks maximization then its constraints will be of the type '≤' and if it seeks minimization then the constraints will be of the type ≥.

7) the objective coefficients of a primal variables become the right hand side of the corresponding dual constraint.

Note. A dual variable corresponding to an equality constraint in the primal is unrestricted in sign.

Conversely, when a primal variable is unrestricted in sign, its dual constraint is an equation.

Example. 4.2.1

Primal

Maximize $z = 5x_1 + 12x_2 + 4x_3$

subject to

$$x_1 + 2x_2 + x_3 \le 10$$

$$2x_1 - x_2 + 3x_3 = 8$$

$$x_1, x_2, x_3 \ge 0$$

Sol. Primal in standard form is

Max. $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$

Subject to — Dual variable

$$x_1 + 2x_2 + x_3 + x_4 = 10 \rightarrow y_1$$

$$2x_1 - x_2 + 3x_3 = 8 \rightarrow y_2$$

$$x_1, x_2, x_3, x_4 \ge 0$$

Dual

Minimize $w = 10y_1 + 8y_2$

subject to

$$y_1 + 2y_2 \ge 5$$

$$2y_1 - y_2 \ge 12$$

$$y_1 + 3y_2 \ge 4$$

$$y_1 + 0y_2 \ge 0 \Rightarrow y_1 \ge 0$$

y_2 is unrestricted

Example:- Primal

Minimize $z = 5x_1 + 12x_2 + 4x_3$

Subject to

$x_1 + 2x_2 + x_3 \leq 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \geq 0$

Sol. Primal in Standard form is

Min. $z = 5x_1 + 12x_2 + 4x_3 + 0x_4$

Subject to

$x_1 + 2x_2 + x_3 + x_4 = 10 \longrightarrow y_1$

$2x_1 - x_2 + 3x_3 = 8 \longrightarrow y_2$

$x_1, x_2, x_3, x_4 \geq 0$

dual

Maximization $w = 10y_1 + 8y_2$

Subject to

$y_1 + 2y_2 \leq 5$

$2y_1 - y_2 \leq 12$

$y_1 + 3y_2 \leq 4$

$y_1 \leq 0$

y_2 is unrestricted

Again dual in Standard form is

put $y_1 = -y_1'$ & $y_2 = y_2' - y_2''$, $y_2', y_2'' \geq 0$

~~Max. $w = 10y_1 + 8y_2$~~

~~subject to~~

~~$y_1 + 2y_2 + S_1 = 5$~~

y_2 و [illegible] ≥ 0 و y_1

[illegible] restrict

$y_2 = y_2' - y_2''$. $y_1 = -y_1'$

So Max. $w = -10y_1' + 8y_2' - 8y_2''$

Subject to

$-y_1' + 2y_2' - 2y_2'' + S_1 = 5$

$-2y_1' - y_2' + y_2'' + S_2 = 12$

$-y_1' + 3y_2' - 3y_2'' + S_3 = 4$

$y_1' \geq 0$, $y_2', y_2'', S_1, S_2, S_3, S_4 \geq 0$

Dual of dual is — *The dual of the dual yield the original primal.*

~~Min.~~ $Z = 5x_1 + 12x_2 + 4x_3$

subject to

solve yourself

$-x_1 - 2x_2 - x_3 \geq -10$

$2x_1 - x_2 + 3x_3 \geq 8$

$-2x_1 + x_2 - 3x_3 \geq -8$

$x_1 \geq 0$

$x_2 \geq 0$

$x_3 \geq 0$

or.

Min. $Z = 5x_1 + 12x_2 + 4x_3$

subject to

$x_1 + 2x_2 + x_3 \leq 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \geq 0$

Ex. 4.2.2 Min. $Z = 15x_1 + 12x_2$

subject to

$x_1 + 2x_2 \geq 3$

$2x_1 - 4x_2 \leq 5$

$x_1, x_2 \geq 0$

Primal in standard form is

Min. $Z = 15x_1 + 12x_2 + 0x_3 + 0x_4$

subject to

$x_1 + 2x_2 - x_3 = 3 \longrightarrow y_1$

$2x_1 - 4x_2 + x_4 = 5 \longrightarrow y_2$

$x_1, x_2, x_3, x_4 \geq 0$

Dual

Max. $W = 3y_1 + 5y_2$

Subject to

$y_1 + 2y_2 \le 15$

$2y_1 - 4y_2 \le 12$

i.e. +ve of R.H.S. $-y_1 \le 0$ or $y_1 \ge 0$

$y_2 \le 0$ {y_1, y_2 unrestricted (redundant)}

Ex. Primal

Min. $Z = 5x_1 - 2x_2$

Subject to

$-x_1 + x_2 \ge -3$ or $x_1 - x_2 \le 3$

$2x_1 + 3x_2 \le 5$ $x_1, x_2 \ge 0$

Primal in standard form is

Min. $Z = 5x_1 - 2x_2 + 0x_3 + 0x_4$

Subject to

dual variables:

y_1 | $+x_1 - x_2 + x_3 = 3$

y_2 | $2x_1 + 3x_2 + x_4 = 5$, $x_1, x_2, x_3, x_4 \ge 0$

Dual

Max. $W = 3y_1 + 5y_2$

Subject to

$y_1 + 2y_2 \le 5$

$-y_1 + 3y_2 \le -2$

or $y_1 - 3y_2 \ge 2$

$-y_1 \le 0$

$y_2 \le 0$

Ex. Primal

$$\text{Min. } Z = 5x_1 + 6x_2$$

subject to

$$x_1 + 2x_2 = 5$$

$$-x_1 + 5x_2 \geq 3$$

$$4x_1 + 7x_2 \leq 8$$

$$x_2 \geq 0$$

x_1 unrestricted.

Primal in standard form is

$$\text{Min. } Z = 5x_1 + 6x_2 + 0x_3 + 0x_4$$

Subject to

$$x_1 + 2x_2 = 5$$

Since x_1 is unrestricted so put $x_1 = x_1' - x_1''$ to make $x_1 \geq 0$

Thus Min. $Z = 5x_1' - 5x_1'' + 6x_2 + 0x_3 + 0x_4$

Subject to

$$x_1' - x_1'' + 2x_2 = 5 \quad \rightarrow y_1$$

$$-x_1' + x_1'' + 5x_2 - x_3 = 3 \quad \rightarrow y_2$$

$$4x_1' - 4x_1'' + 7x_2 + x_4 = 8 \quad \rightarrow y_3$$

$$x_3, x_4, x_1', x_1'', x_2 \geq 0$$

Dual Max. $W = 5y_1 + 3y_2 + 8y_3$

subject to

$$y_1 - y_2 + 4y_3 = 5$$

$$2y_1 + 5y_2 + 7y_3 \leq 6$$

$$-y_2 \leq 0 \Rightarrow y_2 \geq 0$$

$$y_3 \leq 0$$

y_1 is unrestricted

Example 4.2.3: Primal

Maximize $z = 5x_1 + 6x_2$

Subject to

$x_1 + 2x_2 = 5$

$-x_1 + 5x_2 \geq 3$

$4x_1 + 7x_2 \leq 8$

x_1 unrestricted, $x_2 \geq 0$

Primal in standard form is

put $x_1 = x_1' - x_1''$

Max. $z = 5x_1' - 5x_1'' + 6x_2$

Subject to

	Dual variables
$x_1' - x_1'' + 2x_2 = 5$	y_1
$-x_1' + x_1'' + 5x_2 - x_3 = 3$	y_2
$4x_1' - 4x_1'' + 7x_2 + x_4 = 8$	y_3

$x_1', x_1'', x_2, x_3, x_4 \geq 0$

Dual Min $w = 5y_1 + 3y_2 + 8y_3$

Subject to

$y_1 - y_2 + 4y_3 \geq 5$

$2y_1 + 5y_2 + 7y_3 \geq 6$

$-y_2 \geq 0 \Rightarrow y_2 \leq 0$

$y_3 \geq 0$

y_1 unrestricted.

Relation Ship between the optimal primal and dual Solutions.

The primal and dual problems are so closely related that the optimal solution of one problem ~~Can be secured directly~~ automatically Provides ~~(without further computations) from the optimal simplex tableau of the other problem.~~ the optimal sol. to the other.

this result is based ~~based~~ on the following properties.

Property I. At any simplex iteration of the primal or the dual.

$$\begin{pmatrix}\text{objective Coefficients} \\ \text{of variable } j \text{ in} \\ \text{one problem}\end{pmatrix} = \begin{pmatrix}\text{Left-hand side minus} \\ \text{right hand side} \\ \text{of Constraint } j \text{ in} \\ \text{The other problem.}\end{pmatrix}$$

j is a starting variable.
variable … starting variable

This property is symmetrical w.r.t both the primal and the dual problems.
Property I Can be used to determine the optimal solution of one problem directly from the ~~other~~ optimal tableau of the other.

Property II

For any pair of feasible primal and dual solutions

$$\begin{pmatrix}\text{objective value in} \\ \text{the maximization} \\ \text{problem}\end{pmatrix} \leq \begin{pmatrix}\text{objective value in} \\ \text{the minimization} \\ \text{problem}\end{pmatrix}$$

At the optimum the relationship holds as a strict equation.

(Note: These results could be advantageous computationally if the computations associated with the solved problems is considerably less than those associated with the other problem. For example, if a model has 100 variables and 500 constraints, then it is advantageous computationally to solve the dual because it has only 100 constraints.)

Now to see the use of these properties we have an example.

Example: 4.3.1 (Primal dual relationship)

Primal

Maximize $z = 5x_1 + 12x_2 + 4x_3$

Subject to

$$x_1 + 2x_2 + x_3 \le 10$$

$$2x_1 - x_2 + 3x_3 = 8$$

$$x_1, x_2, x_3 \ge 0$$

Sol. primal in standard form is

Maximize $z = 5x_1 + 12x_2 + 4x_3$

Subject to

$$x_1 + 2x_2 + x_3 + x_4 = 10 \quad \rightarrow y_1$$

$$2x_1 - x_2 + 3x_3 = 8 \quad \rightarrow y_2$$

$$x_1, x_2, x_3, x_4 \ge 0.$$

~~Dual~~ ~~Minimize $w = 10y_1 + 8y_2$~~

~~subject to~~

~~$y_1 + 2y_2 \ge 5$~~

~~$2y_1 - y_2 \ge 12$~~

$y_1 + 3y_2 \ge 4$

$y_1 \ge 0$, ~~y_2 is unrestricted.~~

Solution of primal

Add Artificial variables (using M technique)

$$\text{Maximize } Z = 5x_1 + 12x_2 + 4x_3 - MR$$

Subject to

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

$$2x_1 - x_2 + 3x_3 + R = 8 \qquad x_1, x_2, x_3, x_4, R \geq 0, \; M > 0$$

the objective function in terms of non basic variables is $\text{Max } Z = 5x_1 + 12x_2 + 4x_3 - M(8 - 2x_1 + x_2 - 3x_3)$

OR. $\text{Max. } Z = (5 + 2M)x_1 + (12 - M)x_2 + (4 + 3M)x_3 - 8M$

Now the following tableaux provide the simplex iterations for the primal problem.

Basic	x_1	x_2	x_3 ↓	x_4	R	Sol.	Ratio
Z	$-5-2M$	$-12+M$	$-4-3M$	0	0	$-8M$	
x_4	1	2	1	1	0	10	10
← R	2	-1	3	0	1	8	$8/3$

x_3 enters because M is very large and $-M$ is very negative and R leaves

Basic	x_1	x_2 ↓	x_3	x_4	R	Sol.	Ratio
Z	$-7/3$	$-40/3$	0	0	$4/3+M$	$32/3$	
← x_4	$+1/3$	$7/3$	0	1	$-2/3$	$22/3$	
x_3	$2/3$	$-1/3$	1	0	$1/3$	$8/3$	

x_2 enters and x_4 leaves

Basic	x_1	x_2	x_3	x_4	R	Sol.	Ratio
Z	$-3/7$	0	0	$40/7$	$-4/7+M$	$368/7$	
x_2	$1/7$	1	0	$3/7$	$-1/7$	$22/7$	22
x_3	$5/7$	0	1	$1/7$	$2/7$	$26/7$	$26/5$

Note this is not the optimal sol because we still have the -ve coeff of x_1 in the objective row and the sol will be optimum only when all the coeff of z row (objective row) will become non negative. (Max)

So to make the sol optimum

x_1 enters → x_3 leaves

basic	x_1	x_2	x_3	x_4	R	Sol.
Z	0	0	3/5	29/5	$-\frac{2}{5}+M$	274/5
x_2	0	1	-1/5	2/5	-1/5	12/5
x_1	1	0	7/5	1/5	2/5	26/5

∴ $x_1 = \frac{26}{5}$, $x_2 = 12/5$, $x_3 = 0$ and $z = 274/5$

Now we solve the dual.

So Dual in Standard form is

y_2 is unrestricted, put $y_2 = y_2' - y_2''$

Minimize $W = 10y_1 + 8y_2' - 8y_2''$

Subject to

$$y_1 + 2y_2' - 2y_2'' - y_3 = 5$$

$$2y_1 - y_2' + y_2'' - y_4 = 12$$

$$y_1 + 3y_2' - 3y_2'' - y_5 = 4$$

$$y_1, y_2', y_2'', y_4, y_5 \geq 0$$

adding artificial variables we get

$$\text{Min } W = 10y_1 + 8y_2' - 8y_2'' + MR_1 + MR_2 + MR_3$$

Subject to

$$y_1 + 2y_2' - 2y_2'' - y_3 + R_1 = 5$$

$$2y_1 - y_2' + y_2'' - y_4 + R_2 = 12$$

$$y_1 + 3y_2' - 3y_2'' - y_5 + R_3 = 4$$

$$y_1, y_2', y_2'', y_3, y_4, y_5, R_1, R_2, R_3 \geq 0$$

Now $W = 10y_1 + 8y_2' - 8y_2'' + M(5 - y_1 - 2y_2' + 2y_2'' + y_3)$
$+ M(12 - 2y_1 + y_2' - y_2'' + y_4) + M(4 - y_1 - 3y_2' + 3y_2'' + y_5)$

or $W = (10 - 4M)y_1 + (-8 - 4M)y_2' + (-8 + 4M)y_2''$
$+ My_3 + My_4 + My_5 + 21M$

On tableau form

Basic	y_1	↓ y_2'	y_2''	y_3	y_4	y_5	R_1	R_2	R_3	Sol.
W	$-10+4M$	$-8+4M$	$8-4M$	$-M$	$-M$	$-M$	0	0	0	$21M$
R_1	1	2	−2	−1	0	0	1	0	0	5 $\frac{5}{2}$
R_2	2	−1	1	0	−1	0	0	1	0	12
← R_3	1	3	−3	0	0	−1	0	0	1	4 $\frac{4}{3}$
W	$\frac{-22+8M}{3}$	0	0	$-M$	$-M$	$-\frac{8+M}{3}$	0	0	$\frac{8-4M}{3}$	$\frac{32}{3}+47M$
R_1	$\frac{1}{3}$	0	0	−1	0	$\frac{2}{3}$	1	0	$-\frac{2}{3}$	$\frac{7}{3}$
R_2	$\frac{7}{3}$	0	0	0	−1	$-\frac{1}{3}$	0	1	$\frac{1}{3}$	$\frac{40}{3}$
y_2'	$\frac{1}{3}$	1	−1	0	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{4}{3}$

$\frac{8-4M}{3} - M$

$-\frac{8-4M}{3} - M$

$-\frac{8}{3} + \frac{4M}{3} - 3M$

y_1 enters and y_2' leaves

Basic	y_1	y_2'	y_2''	y_3	y_4	y_5	R_1	R_2	R_3	Sol
W	0	22-8M	-22+8M	-M	-M	-10+3M	0	0	10-4M	40+5M
R_1	0	-1	1	-1	0	1	1	0	-1	1
R_2	0	-7	7	0	-1	2	0	1	-2	4
y_1	1	3	-3	0	0	-1	0	0	1	4

y_2'' enters while R_2 leaves

Basic	y_1	y_2'	y_2''	y_3	y_4	y_5	R_1	R_2	R_3	Sol.
W	0	0	0	-M	$\frac{-22+M}{7}$	$\frac{-26+5M}{7}$	0	$\frac{22-8M}{7}$	$\frac{26-12M}{7}$	$\frac{368+3M}{7}$
R_1	0	0	0	-1	$\frac{1}{7}$	$\frac{5}{7}$	1	$-\frac{1}{7}$	$-\frac{5}{7}$	$\frac{3}{7}$
y_2''	0	-1	1	0	$-\frac{1}{7}$	$\frac{2}{7}$	0	$\frac{1}{7}$	$-\frac{2}{7}$	$\frac{4}{7}$
y_1	1	0	0	0	$-\frac{3}{7}$	$-\frac{1}{7}$	0	$-\frac{3}{7}$	$\frac{1}{7}$	$\frac{40}{7}$

y_5 enters and R_1 leaves:

Basic	y_1	y_2'	y_2''	y_3	y_4	y_5	R_1	R_2	R_3	Sol.
W	0	0	0	$-\frac{26}{5}$	$-\frac{12}{5}$	0	$\frac{26-5M}{5}$	$\frac{12-5M}{5}$	-M	$\frac{274}{5}$
y_5	0	0	0	$-\frac{7}{5}$	$\frac{1}{5}$	1	$\frac{7}{5}$	$-\frac{1}{5}$	-1	$\frac{3}{5}$
y_2''	0	-1	1	$+\frac{9}{5}$	$-\frac{1}{5}$	0	$-\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{2}{5}$
y_1	1	0	0	$-\frac{1}{5}$	$-\frac{2}{5}$	0	$\frac{2}{5}$	$\frac{-16}{35}$	0	$\frac{29}{5}$

$\therefore$ $y_1 = \frac{29}{5}$, $y_2' = 0$, $y_2'' = \frac{2}{5}$, $y_5 = \frac{3}{5}$, $W = \frac{274}{5}$

So $y_1 = \frac{29}{5}$, $y_2 = y_2' - y_2'' = -\frac{2}{5}$ & $W = \frac{274}{5}$

Which is the sol^n of the dual problem

Now we find the same sol. of dual using the property 1 & (2) as

Now since after adding the artificial variable, we have the standard form of primal is

$$\text{Max. } Z = 5x_1 + 12x_2 + 4x_3 - MR$$

subject to

$$x_1 + 2x_2 + x_3 + x_4 = 10$$

$$2x_1 - x_2 + 3x_3 + R = 8 \quad \leftarrow \text{Technique M}$$

حل كيا

$$x_1, x_2, x_3, x_4, R \geq 0$$

and then by Simplex method we get the the following optimum ~~primal~~ sol. tableau

Basic	x_1	x_2	x_3	x_4	R	Sol.
Z	0	0	$3/5$	$29/5$	$-2/5+M$	$274/5$
x_2	0	1	$-1/5$	$2/5$	$-1/5$	$12/5$
x_1	1	0	$7/5$	$1/5$	$2/5$	$26/5$

Now since the corresponding dual is

$$\text{Minimize } W = 10y_1 + 8y_2$$

subject to

$$y_1 + 2y_2 \geq 5 \quad \rightarrow x_1$$

$$2y_1 - y_2 \geq 12 \quad \rightarrow x_2 \quad \text{[illegible] Unrestricted}$$

$$y_1 + 3y_2 \geq 4 \quad \rightarrow x_3$$

$$y_1 \geq 0 \quad \rightarrow x_4$$

$$y_2 \geq -M \quad \rightarrow R$$

Now find the sol. of dual without calculating !

Applying property I to the starting solution variables x_4 and R in optimal iteration, we obtain the following information

z-eq. Coefficient of x_4 & R is $29/5$ & $-2/5+M$ & associated dual Constraints is $y_1 \geq 0$ & $y_2 \geq -M$

So the property I

(optimal z-eq. coeff of starting variable in primal) = (Difference b/w left and right side of the dual constraint corresponding with starting variable)

implies that

$$\frac{29}{5} = y_1 - 0 \quad \& \quad -2/5 + M = y_2 + M$$

$$\Rightarrow \quad y_1 = 29/5 \quad \& \quad y_2 = -2/5$$

which is the same which was obtained in the independent sol. of dual problem.

Now using property II we find the value of objective function for the dual problem i.e. for optimal primal & dual solution.

(optimal objective value in the maximization problem) = (optimal objective value in the minimization problem)

$\Rightarrow$ $z = W$

$\Rightarrow$ $W = 274/5$ which is the same as obtained in the independent sol of the dual problem

Also because of the symmetry of property 1 with respect to the primal problem, a similar application to the starting variables in the optimal dual tableau will automatically yield the optimal primal solution, $x_1 = 26/5$, $x_2 = 12/5$ & $x_3 = 0$

X

(The applications of property I to the starting variables always results in easy to solve equations because ~~eq~~ each eq. involves exactly one variable.

Nothing, however should prevent us from using any two of the other primal variables i.e. x_1, x_2 & x_3, to generate the desired equations.

For example, at the optimal tableau, Property I equations associated with [illegible] x_3 are resp.

$$y_1 + 2y_2 - 5 = 0$$

$$y_1 + 3y_2 - 4 = 3/5$$

where 0 is objective Coeff. of x_1; 3/5 is objective Coeff. of x_3

on solving:

$$-y_2 - 1 = -3/5$$

$$y_2 = 3/5 - 1$$

$$y_2 = -2/5 \quad \& \quad y_1 = 5 - 2y_2$$

$$y_1 = 5 + 4/5$$

$$y_1 = 29/5$$

i.e.

The sol. of these two equations still yields the same optimal dual values $y_1 = 29/5$, $y_2 = -2/5$. However the equations are not as simple as those associated with x_4 & R.)

Ex. Consider

$$\text{Max. } z = 5x_1 + 2x_2 + 3x_3$$

s. to

$$x_1 + 5x_2 + 2x_3 = 30$$

$$x_1 - 5x_2 - 6x_3 \le 40 \qquad x_1, x_2, x_3 \ge 0$$

The optimal solution is given as

Basic	x_1	x_2	x_3	R	x_4	Sol
z	0	23	7	5+M	0	150
x_1	1	5	2	1	0	30
x_2	0	-10	-8	-1	1	10

Write dual problem and find its optimal solution from the optimal primal tabalue

Solution:-

The primal in standard form is

$$\text{max. } Z = 5x_1 + 2x_2 + 3x_3 - MR$$

subject to

$$x_1 + 5x_2 + 2x_3 + R = 30$$

$$x_1 - 5x_2 - 6x_3 + x_4 = 40$$

$$x_1, x_2, x_3, x_4, R \ge 0$$

The dual problem is

$$\text{Min. } W = 30y_1 + 40y_2$$

subject to

$$y_1 + y_2 \ge 5$$

$$5y_1 - 5y_2 \ge 2$$

$$2y_1 - 6y_2 \ge 3$$

$$y_1 \ge -M \rightarrow (R)$$

$$y_2 \ge 0 \rightarrow (x_4)$$

the starting variables for primal are R and x_4 & their respective coefficients in the optimal primal tableau are $5+M$ & 0

and the dual constraints associated with R and x_4 are

$y_1 \geq -M$, $y_2 \geq 0$

Hence $y_1 - (-M) = 5+M$, $y_2 - 0 = 0$

$y_1 + M = 5+M$

$y_1 = 5$, $y_2 = 0$

For these values of $y_1 = 3$ & $y_2 = 0$, $W = 150$

Ex. Consider Max $Z = x_1 + 5x_2 + 3x_3$

subject to ~~$x_1 + 2x_2 + x_3 = 3$~~ $x_1 + 2x_2 \leq 3$, $x_1 + 2x_2 \leq 3$

$2x_1 - x_2 = 4$, $x_1, x_2, x_3 \geq 0$

using a starting sol. consisting of x_3 & an artificial variable R in the 2nd constraint, we obtain the optimal tableau.

Basic	x_1	x_2	x_3	R	Sol
Z	0	2	0	$-1+M$	5
x_3	0	$5/2$	1	$-1/2$	1
x_1	1	$-1/2$	0	$1/2$	2

solve yourself.

on next page

Write the dual problem and its optimal solution from this optimal primal tableau, i.e. $W = 5$, $y_1 = 3$, $y_2 = -1$

Sol. The primal in standard form is

Max. $Z = x_1 + 5x_2 + 3x_3 - MR$ ~~$Z = -2x_1 - 3x_2 + 9 - MR$~~

subject to

$$x_1 + 2x_2 + x_3 = 3$$

$$2x_1 - x_2 + R = 4$$

$$x_1, x_2, x_3, R \geq 0$$

The dual problem is

$$\text{Min. } W = 3y_1 + 4y_2$$

subject to

$$y_1 + 2y_2 \geq +1$$

$$2y_1 - y_2 \geq +5$$

$$y_1 \geq 3$$

$$y_2 \geq -M$$

Now the starting variables for primal are x_3 & R, & their respective coefficients in the optimal primal tableau are 0 & $-1+M$

and the dual constraints associated with x_3 & R are

$$y_1 \geq 0, \quad y_2 \geq -M$$

Hence

$$y_1 - 0 = 3, \quad y_2 + M = -1 + M$$

$$y_1 = 3, \quad y_2 = -1$$

and for these values of y_1 & y_2 $W = 5$

Ans

Dual Simplex Method

this method starts better than optimal but infeasible and moves to achieve feasibility while maintaining the optimality.

Example:-

The optimum cannot occur with z strictly less than w i.e. ($z < w$)

Minimize $z = 2x_1 + x_2$

subject to

$$3x_1 + x_2 \geq 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

Solution: step 1 Convert all the Constraints to the inequalities of the type '≤' and add slack variables, So the problem is

Min. $z = 2x_1 + x_2$

S.to

$$-3x_1 - x_2 + x_3 = -3$$

$$-4x_1 - 3x_2 + x_4 = -6$$

$$x_1 + 2x_2 + x_5 = 3$$

$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

The problem can be expressed in a tabular form as

} Leaving [illegible]

Basic	x_1	x_2 ↓	x_3	x_4	x_5	Sol.
z	-2	-1	0	0	0	0
x_3	-3	-1	1	0	0	-3
← x_4	-4	[-3]	0	1	0	-6
x_5	1	2	0	0	1	3

The solution provided in the above tableau is optimal but not feasible. We shall use feasibility and optimality conditions to generate a new solution from the above tableau.

Feasibility Condition:- or (Dual feasibility condition)

The leaving variable is the basic variable having the most -ve value in the Solution Column, which is in this Case x_4. If all the basic variables are non-negative, the algorithm ends.

Optimality Condition or (Dual optimality Condition)

take the ratio of the L.H.S Coefficients of the z-equation to the Corresponding Coefficients in the equation associated with the leaving variable. Ignore the ratioes with +ve or zero denominators. The entering variable is the non basic variable associated with the smallest ratio if the problem is minimization or the smallest absolute value of the ratio if the problem is maximization.

If all the denominators are +ve or zero then the problem has no feasible sol.

So selecting the entering variable, we have

Variable	x_1	x_2	x_3	x_4	x_5
z-eq	-2	-1	0	0	0
x_4-eq	-4	-3	0	1	0
Ratio	1/2	1/3	$\frac{0}{0}$ ∵ zero eqt	$\frac{0}{1}$ +ve	$\frac{0}{0}$ zero

So the entering variable is x_2.

and x_4 is the leaving variable so the new tableau is

Basic	x_1 ↓	x_2	x_3	x_4	x_5	Sol.
z	$-2/3$	0	0	$-1/3$	0	2
← x_3	$-5/3$	0	1	$-1/3$	0	-1
x_2	$4/3$	1	0	$-1/3$	0	2
x_5	$-5/3$	0	0	$2/3$	1	-1

the new sol. is also optimal but infeasible. Now for $x_3 = x_5 = -1$, Assume that x_3 is the leaving variable. Thus take ratio, we obtain the tableau. ~~for enter~~ x_1

Basic	x_1	x_2	x_3	x_4	x_5	Sol.
z	0	0	$-2/5$	$-1/5$	0	$12/5$
x_1	1	0	$-3/5$	$+1/5$	0	$3/5$
x_2	0	1	$4/5$	$3/5$	0	$6/5$
x_5	0	0	-1	1	1	0

the tableau is now optimal as well as feasible. the optimal sol. is

$$x_1 = 3/5, \quad x_2 = 6/5, \quad z = 12/5$$

Ex. Solve the following problem by dual Simplex method

$$\text{Min } Z = 2x_1 + 3x_2$$

Subject to

$$2x_1 + 3x_2 \le 30$$

$$x_1 + 2x_2 \ge 10, \quad x_1, x_2 \ge 0$$

Sol. Convert all the constraints to inequalities of the type '≤' and add slack variables, so the problem is

$$\text{Min } Z = 2x_1 + 3x_2$$

s. to

$$2x_1 + 3x_2 + x_3 = 30$$

$$-x_1 - 2x_2 + x_4 = -10$$

$$x_1, x_2, x_3, x_4 \ge 0$$

In tableau is

Basic	x_1	x_2 ↓	x_3	x_4	Sol.
Z	-2	-3	0	0	0
x_3	2	3	1	0	30
← x_4	-1	-2	0	1	-10

x_4 leaves & x_2 enters.

Basic	x_1	x_2	x_3	x_4	Sol.
Z	$-1/2$	0	0	$-3/2$	15
x_3	$1/2$	0	1	$3/2$	15
x_2	$1/2$	1	0	$-1/2$	5

which is an optimal sol. So

$x_1 = 0, \quad x_2 = 5, \quad Z = 15$ Ans.

Ex. Solve by dual Simplex Method

$$\text{Min } z = 5x_1 + 6x_2$$

s. to

$$x_1 + x_2 \geq 2$$

$$4x_1 + x_2 \geq 4, \quad x_1, x_2 \geq 0$$

Sol. Convert all the Constraints into inequalities of the type '≤' and add slack variables, So the problem is

$$\text{Min. } Z = 5x_1 + 6x_2$$

s. to

$$-x_1 - x_2 + x_3 \leq -2$$

$$-4x_1 - x_2 + x_4 \leq -4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

In tableau form is

Basic	x_1 ↓	x_2	x_3	x_4	Sol.
Z	-5	-6	0	0	0
x_3	-1	-1	1	0	-2
← x_4	-4	-1	0	1	-4

x_4 leaves while x_1 enters.

Basic	x_1	x_2	x_3	x_4 ↓	Sol.
Z	0	$-19/4$	0	$-5/4$	5
← x_3	0	$-3/4$	1	$-1/4$	-1
x_1	1	$1/4$	0	$-1/4$	1

x_3 leaves while x_4 enters.

Basic	x_1	x_2	x_3	x_4	Sol.
Z	0	-1	-5	0	10
x_4	0	3	-4	1	4
x_1	1	1	-1	0	2

which is optimal & feasible sol. So

$x_1 = 2$, $x_2 = 0$, $Z = 10$ Ans

4.4 Economic Interpretation of Duality:-

Primal Maximize $z = \sum_{j=1}^{n} c_j x_j$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad ; \quad i = 1, 2, \ldots, m$$

$$x_j \ge 0 \quad ; \quad j = 1, 2, \ldots, n$$

Dual Minimize $w = \sum_{i=1}^{m} b_i y_i$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j \quad ; \quad j = 1, 2, \ldots, n$$

$$y_i \ge 0 \quad ; \quad i = 1, 2, \ldots, m$$

Property II

$$z = \sum_{j=1}^{n} c_j x_j \le \sum_{i=1}^{m} b_i y_i = w$$

At optimal iteration

$$z = w = \sum_{i=1}^{m} b_i y_i$$

or

profit \$ = $\sum_{i=1}^{m}$ (units of resource i)(\$ worth per unit of resource i)

Problem set 4.4.(a) Q.3

Production	leather	labour	Price
jacket	8	12	\$ 350
handbag	2	5	\$ 120
Availability	1200 m^2	1850 hrs	

Sol Let x_1, x_2 be the nos of jackets and handbags produced, respectively.

Then the LP Model is

Maximize $z = 350x_1 + 120x_2$

subject to

$$8x_1 + 2x_2 \le 1200$$
$$12x_1 + 5x_2 \le 1850$$
$$x_1, x_2 \ge 0$$

Primal in Standard form

Maximize $z = 350x_1 + 120x_2$

subject to

$8x_1 + 2x_2 + S_1 = 1200$

$12x_1 + 5x_2 + S_2 = 1850$

$x_1, x_2, S_1, S_2 \geq 0$

Iteration	Basic	x_1	x_2	S_1	S_2	Solution	
0	z	-350	-120	0	0	0	
Enter x_1	S_1	(8)	2	1	0	1200	1200/8 = 150
Leave S_1	S_2	12	5	0	1	1850	1850/12 = 154.1
1	z	0	-32.5	43.75	0	52500	
Enter x_2	x_1	1	1/4	1/8	0	150	600
Leave S_2	S_2	0	(2)	-3/2	1	50	25
2	z	0	0	19.375	16.25	53312.5	
	x_1	1	0	-1/16	1/8	701/4	
	x_2	0	1	-3/4	1/2	25	

Dual is Minimize $w = 1200y_1 + 1850y_2$

subject to

$8y_1 + 12y_2 \geq 350$

$2y_1 + 5y_2 \geq 120$

$y_1, y_2 \geq 0$

According to Property I

$19.375 = y_1 - 0$

$\Rightarrow y_1 = 19.375$

& $16.25 = y_2 - 0$

$\Rightarrow y_2 = 16.25$

Thus we have

$(19.375, 16.25)$

The Company should not pay more than 19.375/m² for leather and 16.25 / hours for Labour.

c_j = profit per unit of activity j

a_{ij} = rate of consumption of resource i per unit economic activity j

$z_j = \sum_{i=1}^{m} a_{ij} y_i$ is the cost of all resources needed to produce 1 unit of activity j

Example 4.4-2

	operation 1	operation 2	operation 3	Profit
Trains	1	3	1	3 $
Trucks	2	0	4	2 $
Cars	1	2	0	5 $
Availability	430	460	420	

Primal	Dual
Maximize $z = 3x_1 + 2x_2 + x_3$	Minimize $w = 430y_1 + 460y_2 + 420y_3$
subject to	subject to
$x_1 + 2x_2 + x_3 \le 430$	$y_1 + 3y_2 + y_3 \ge 3$
$3x_1 + x_3 \le 460$	$2y_1 + 4y_3 \ge 2$
$x_1 + 4x_2 \le 420$	$y_1 + 2y_2 \ge 5$
$x_1, x_2, x_3 \ge 0$	$y_1, y_2, y_3 \ge 0$
$x_1 = 0,\ x_2 = 100;\ x_3 = 230$	$y_1 = 1,\ y_2 = 2,\ y_3 = 0$
$z = \$1350$	

Sol.

Toy trains will become attractive economically only if $z_1 < c_1$

Let x_1, x_2, x_3 represent the proportions by which the unit times of the three operations are reduced. Then

$z_1 = 1\cdot(1-r_1)y_1 + 3(1-r_2)y_2 + 1\cdot(1-r_3)y_3$

Thus we need to find r_1, r_2 & r_3 such that

$(1-r_1)y_1 + 3(1-r_2)y_2 + (1-r_3)y_3 < 3$

Taking $y_1 = 1,\ y_2 = 2,\ y_3 = 0$, we get

$(1-r_1) + 6(1-r_2) < 3$

$1 - r_1 + 6 - 6r_2 < 3$

$7 - r_1 - 6r_2 < 3$

$-r_1 - 6r_2 < 3-7$

$-(r_1 + 6r_2) < -4$

$\therefore\ r_1 + 6r_2 > 4$

One possible solution is $r_1 = 0.5$, $r_2 = 0.6$

Then $r_1 + 6r_2 = 0.5 + 3.6 = 4.1 > 4$

Problem set 4.4(b) Q-1

r_2 = reduction per minute of operation 2 $= \frac{3-1.25}{3} = \frac{1.75}{3} = 0.583$

Train is profitable if

$r_1 + 6r_2 > 4$

or $r_1 + 6\left(\frac{1.75}{3}\right) > 4$

or $r_1 + 3.5 > 4$

or $r_1 > 4 - 3.5 = 0.5$

Hence a reduction of 0.5 minutes per unit usage of operation 1 will make the trains just profitable.

Repeated X 94

Dual Simplex Method :-

This method starts better than optimal but infeasible and moves to achieve feasibility while maintaining the optimality.

Example :-

Min $Z = 2x_1 - 3x_2$ if $x_1 = x_2 = 0$ then not optimal.

Minimize $Z = 2x_1 + x_2$

subject to

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3$

$x_1, x_2 \geq 0$

Sol:-

Convert all constraints to inequalities of the type '≤' and then add slack variables.

Minimize $Z = 2x_1 + x_2$

subject to

$-3x_1 - x_2 + x_3 = -3$

$-4x_1 - 3x_2 + x_4 = -6$

$x_1 + 2x_2 + x_5 = 3$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

The problem can be expressed in the tableau form as:

Basic	x_1 ↓	x_2	x_3	x_4	x_5	Solution
z	-2	-1	0	0	0	0
x_3	-3	-1	1	0	0	-3
← x_4	-4	-3	0	1	0	-6
x_5	1	2	0	0	1	3
	$2/4 = 1/2$	$1/3$	—	—	—	

The solution provided in the above table is optimal but not feasible we shall use feasibility and optimality conditions to generate a new solution from the above tableau.

Optimality Conditions:-

Take the ratio of the L.H.S. coefficients of the z-equation to the corresponding coefficients in the equation associated with the leaving variable. Ignore the ratios with +ve or zero denominators. The entering variable is the non-basic variable with the smallest ratio if the problem is minimization, or the smallest absolute value of the ratios if the problem is maximization. If all the denominators are zero or +ve, the problem has no feasible solution.

After selecting the leaving and the entering variable, row operation are applied to obtain the next solution.

drop x_4 & introduce x_2

Basic	x_1	x_2	x_3	x_4	x_5	solution
z	-2/3	0	0	-1/3	0	2
x_3	-5/3	0	1	1/3	0	-1
x_2	4/3	1	0	-1/3	0	2
x_5	-5/3	0	0	-2/3	1	-1

drop x_3 & introduce x_1

Basic	x_1	x_2	x_3	x_4	x_5	solution
z	0	0	-2/5	-1/5	0	12/5
x_1	1	0	-3/5	1/5	0	3/5
x_2	0	1	4/5	-3/5	0	6/5
x_5	0	0	-1	1	1	0

CH-5 The Transportation Problems

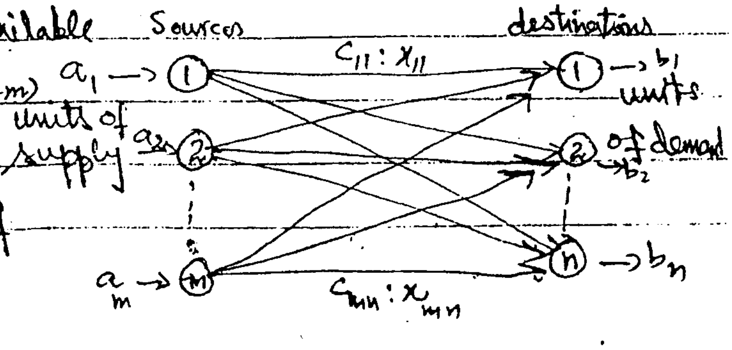
Transportation Model:-

The Transportation model is an important class of LP models. In this class the model seeks the minimization of the cost of the transportation of a commodity from a number of sources to several destinations. For example a product may be transported from factories to retail stores.

(i.e. It deals with the situation in which a commodity is shipped from sources (i.e. factories) to destinations (i.e. warehouses). The objective is to determine the amounts shipped from each source to each destination that minimize the shipping cost while satisfying both the supply limits and the demand requirements.

The model assumes that the shipping cost on a given route is directly proportional to the number of units shipped on that route. The general problem is represented as in fig.)

Suppose there are m sources and n destinations. Let a_i be the number of units of supply available at source i $(i=1,2,\dots m)$ and let b_j be the number of



~~units demanded at destination j (j=1,2,—,n)~~

Let c_{ij} be the per unit transportation cost on route (i,j) joining source i and the destination j. the objective is to determine the number of units transported from source i to destination j such that the net transportation cost is minimized.

Model:

Let x_{ij} be the number of units shipped from source i to destination j. thus LP model is

Destinations

SOURSE	1	2	— —	n	SUPPLY
1	c_{11} x_{11}	c_{12} x_{12}	— — —	c_{1n} x_{1n}	a_1
2	c_{21} x_{21}	c_{22} x_{22}	— —	c_{2n} x_{2n}	a_2
:			: :		:
m	c_{m1} x_{m1}	c_{m2} x_{m2}	— —	c_{mn} x_{mn}	a_m
Demands	b_1	b_2	— .	b_n	

$$\text{minimize } x_0 = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \cdot c_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; \quad i = 1,2,3,—,m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; \quad j = 1,2,3,--,n$$

$$x_{ij} \geq 0$$

Example. Consider the case when $m=2$, $n=3$ i.e. 2 sources and 3 destination then the model is

$$\min x_0 = \sum_{i=1}^{2} \sum_{j=1}^{3} x_{ij} c_{ij}$$

subject to

$$\sum_{j=1}^{3} x_{ij} = a_i \quad ; \quad i = 1,2$$

$$\sum_{i=1}^{2} x_{ij} = b_j \quad ; \quad j = 1,2,3$$

$\Rightarrow$ Min $x_0 = c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{21}x_{21} + c_{22}x_{22} + c_{23}x_{23}$

Subject to $\sum_{j=1}^{3} x_{ij} = a_i$; $i = 1, 2$

$x_{11} + x_{12} + x_{13} = a_1$

$x_{21} + x_{22} + x_{23} = a_2$

and

$\sum_{i=1}^{2} x_{ij} = b_j$; $j = 1, 2, 3$

$\Rightarrow$

$x_{11} + x_{21} = b_1$

$x_{12} + x_{22} = b_2$

$x_{13} + x_{23} = b_3$

where

$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23} \geq 0$

(In the tableau form, this problem can be represented as

	x_0	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	Sol
obj-eq.	1	$-c_{11}$	$-c_{12}$	$-c_{13}$	$-c_{21}$	$-c_{22}$	$-c_{23}$	0
Source Constraints	0	1	1	1	0	0	0	a_1
	0	0	0	0	1	1	1	a_2
Destination Constraints	0	1	0	0	1	0	0	b_1
	0	0	1	0	0	1	0	b_2
	0	0	0	1	0	0	1	b_3

)

(The Transportation model can be solved by regular Simplex method. However its special property is offer more convenient solution procedure, because the above tableau

~~does not offer an obvious starting solution~~ So we shall use the following array permutation to (we) develop the transportation technique.)

Destinations

SOURCE	1	2	3	SUPPLY
1	x_{11} [c_{11}]	x_{12} [c_{12}]	x_{13} [c_{13}]	a_1
2	x_{21} [c_{21}]	x_{22} [c_{22}]	x_{23} [c_{23}]	a_2
DEMANDS	b_1	b_2	b_3	

Balancing of the Transportation Model

Consider the constraints of the model

$$(1) \leftarrow \sum_{j=1}^{n} x_{ij} = a_i \quad ; \quad i = 1, 2, \ldots m$$

$$(2) \leftarrow \sum_{i=1}^{m} x_{ij} = b_j \quad ; \quad j = 1, 2, \ldots n$$

From (1) and (2) we obtain.

Using (1) and (2) in L.H.S

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i \longrightarrow (3)$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} b_j \longrightarrow (4)$$

from (3) and (4), we have

R.H.S [illegible] L.H.S [illegible]

$$\sum_{i=1}^{m} a_i = \sum_{j=1} b_j$$

This relation implies that the supply at all sources must be equal to the demand at all destinations. In real life problem, the restriction need not be satisfied always.

(The model is said to be balanced if supply at all sources is equal to the demand at all destinations, otherwise the model is said to be unbalanced.

In case of an unbalanced model in which demand exceeds supply, we introduce a dummy source that supplies the amount $\left(\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i\right)$ for balancing the model.

If there is a surplus supply then a dummy destination is used to absorb the surplus quantity $\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$.

Note that the per unit transportation cost from the dummy source to all the destinations is zero, because nothing is actually shipped from the dummy source.

Likewise, the per unit transportation cost from all sources to dummy destination is zero.

The Transportation Algorithm or Technique

the transportation algorithm follows the exact steps of the Simplex method However instead of using the regular Simplex tableau, we take advantage of the special structure of the transportation model to present the algorithm in a more convenient form.

the basic steps for transportation technique are as below.

Step I: Determine a starting basic feasible sol.

Step II: Use the optimality condition of the Simplex method to determine the entering variable from among all the non basic variables. If the optimality condition is satisfied, stop, otherwise, go to step 3.

Step III: Use the feasibility condition of the Simplex method to determine the leaving variable from among all the current basic variables, and find the new basic sol. Return to step 2.

Determination of the Starting Solution.

A general transportation model with m sources and n destinations has m+n constraint equations, one for each source and each destination. However the transportation model is always balanced (i.e. Sum of supply = Sum of the demand), one of these equations must be redundant. thus, the model has m+n-1 independent constraint equations, which means that the starting basic solution consists of m+n-1 basic variables.

there are three methods available to obtain the starting solution.

1) NorthWest-Corner Method
2) Least-Cost Method
3) Vogel approximation method.

the difference among the three methods is the "quality" of the starting basic solution they produce, in the sense that a better starting solution yields a smaller objective value. In general, the Vogel approximation method yields the best starting basic solution, and the North west Corner method yields the worst.

North-West Corner Method :- (or Top left)

the method starts at the north-west corner cell of the tableau (variable x_{11})

متعین کرنا - مقرر کرنا - تخصیص کرنا - مخصوص کرنا

Step I :- Allocate as much as possible to the selected cell, and adjust the associated amounts of supply and demand by subtracting the allocated amount.

Step II :- Cross out the row or column with zero supply or demand to indicate that no further assignments can be made in that row or column. If both the row and column net to zero simultaneously, cross out one only, and leave a zero supply (demand) in the uncrossed row (column)

Step III :- If exactly one row or column is left uncrossed out, stop. otherwise, move to the cell to the right if a column has just been crossed ~~out~~ or ~~the one~~ below if a row has been crossed out. and Go to Step 1.

Route

Example:- Find the starting solution for the problem (by North west Corner Method)

DESTINATION

SOURCE	1	2	3	4	SUPPLY
	x_{11} [10]	x_{12} [0]	x_{13} [20]	x_{14} [11]	15
	x_{21} [12]	x_{22} [7]	x_{23} [9]	x_{24} [20]	25
	x_{31} [0]	x_{32} [14]	x_{33} [16]	x_{34} [18]	5
DEMAND	5	15	15	10	

سب سے پہلے دیکھنا ہے کہ اگر سوال Balance ہے یا نہیں اگر نہیں
تو اسے Balance بنا لیں۔

Sol. Here total Supply = total demand = 45.
So the model is balanced.

(i) take $x_{11}=5$, it crossed out C_1 and leaves 10 units in R_1

(ii) take $x_{12}=10$ it crossed out R_1 and leaves 5 units in C_2

	1	2	3	4	
	$x_{11}=5$	$x_{12}=10$			15/10/0
		$x_{22}=5$	$x_{23}=15$	$x_{24}=5$	25/20/5 0
				$x_{34}=5$	5
	5	15/5/0	15/0	10/5	

(iii) take $x_{22}=5$, it crossed out C_2 and leaves 20 units in R_2.

(iv) take $x_{23}=15$, it cross out C_3 and leaves 5 units in R_2.

(v) take $x_{24}=5$, it crossed out R_2 and leaves 5 units in C_4

(vi) take $x_{34}=5$, This crossed out R_3 and leaves 0 unit in ~~C_4~~ R_3

Since only one row R_3 is remained uncrossed, the procedure ends and we obtain the following sol.

جن x_{ij} کی قیمتیں نہیں نکالیں وہ صفر کے برابر ہوں گی

[10] $x_{11}=5$	[0] $x_{12}=10$	[20] $x_{13}=0$	[11] $x_{14}=0$
[12] $x_{21}=0$	[7] $x_{22}=5$	[9] $x_{23}=15$	[20] $x_{24}=5$
[0] $x_{31}=0$	[14] $x_{32}=0$	[16] $x_{33}=0$	[18] $x_{34}=5$

and the associated transportation Cost is

$x_0 = x_{11}c_{11} + x_{12}c_{12} + x_{22}c_{22} + x_{23}c_{23} + x_{24}c_{24} + x_{34}c_{34}$

$= (5)(10) + (10)(0) + (5)(7) + (15)(9) + (5)(20) + (5)(18)$

$= 50 + 0 + 35 + 135 + 100 + 90$

$x_0 = 410$ Ans.

the north west cell, we start by assigning as much as possible to the cell with the smallest unit cost (ties are broken arbitrarily). We then cross out the satisfied row or column, and adjust the amounts of supply and demand accordingly. If both a row and a column are satisfied simultaneously, only one is crossed out, the same as in the north west corner method. Next, we always look for the uncrossed-out cell with the smallest unit cost and repeat the process untill we are left at the end with exactly one uncrossed-out row or column.

#

Ex. Find the starting solution for the problem (by least cost method)

Destinations

Source	1	2	3	4	Supply
1	10 x_{11}	0 x_{12}	20 x_{13}	11 x_{14}	15
2	12 x_{21}	7 x_{22}	9 x_{23}	20 x_{24}	25
3	0 x_{31}	14 x_{32}	16 x_{33}	10 x_{34}	5
Demand	5	15	15	10	

Solution:- (i) In the given table, x_{12} and x_{31} are the variables associated with the smallest unit cost ($c_{12} = 0 = c_{31}$). Breaking the ties arbitrarily, take $x_{12} = 15$; which satisfies both R_1 and C_2. Crossing out C_2 arbitrarily and adjust the supply in R_1 as zero.

(ii) Now x_{31} has the smallest unit cost ($c_{31} = 0$) so take $x_{31} = 5$ which satisfies both R_3 and C_1. Cross out R_3 and adjust zero ~~Supply~~ demand in C_1

(iii) Now x_{23} has the smallest unit cost ($c_{23} = 9$) So take $x_{23} = 15$, this cross out C_3 and leaves 10 units supply in R_2.

(iv) Now x_{11} has the smallest unit cost ($c_{11} = 10$). So take $x_{11} = 0$ and cross out C_1

(v) Now x_{14} has the smallest unit cost ($c_{14} = 11$). Choose $x_{14} = 0$ and cross out R_1

(vi) Now x_{24} has the smallest unit cost ($c_{24} = 20$). So take $x_{24} = 10$ and cross out R_2.

So the following table is the ~~resulting~~ starting sol

$x_{11} = 0$ [10]	$x_{12} = 15$ [0]	[20]	$x_{14} = 0$ [11]	15 0
[12]	[7]	$x_{23} = 15$ [9]	$x_{24} = 10$ [20]	25 10
$x_{31} = 5$ [0]	[14]	[16]	[18]	5 0
5	15	15		

and the associated transportation cost is

$$x_0 = (0)(10) + (15)(0) + (5)(0) + (10)(20) + (15)(9)$$
$$= 335 \text{ Ans}$$

Vogel's Approximation Method :- (VAM)

VAM is an improved version of the least - cost method that generally produces better starting solutions.

The steps of the procedure are as follows:

Step 1:- Evaluate a penalty for each row and each column by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column).

Step 2:- Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. If a row and a column are satisfied simultaneously, only one of the two is crossed out, and the remaining row (column) is assigned zero supply (demand). Any row or column with zero supply and demand should not be used in computing future penalties.

Step III:- (a) If exactly one row or column with zero supply or demand remains uncrossed out, stop.

(b) If one row (column) with positive supply (demand) remains uncrossed out, determine the basic variables in the row (column) by the least cost method. Stop.

C) If all the uncrossed out rows or columns have (remaining) zero supply and demand, determine the zero basic variables by the least cost method. Stop.

(d) otherwise go to step I.

Ex. Find a starting basic solution for the given problem using VAM

Destination

Source	1	2	3	4	Supply
1	10	0	20	11	15
2	12	7	9	20	25
3	0	14	16	18	5
Demand	5	15	15	10	

Sol:- step 1 Evaluate a penalty for each row and column

DESTINATION

Source	1	2	3	4	Supply	Row penalty
1	10	0	20	11	15	10
2	12	7	9	20	25	2
3	0, $x_{31}=5$	14	16	18	5 0	14 ✓
Demand	5 0	15	15	10		
Column penalty	10	7	7	7		

(ii) Here R_3 has the largest penalty $(=14)$ and $c_{31}=0$ is the least unit cost in this row. So take $x_{31}=5$ which satisfies both R_3 and C_1, so cross out C_1 arbitrarily C_1 and adjust zero supply in R_3.

[illegible] Penalty [illegible] Supply.
110 [illegible]

III ~~Re- Compute the penalties~~

	2	3	4	Supply	Row penalty
1	0	20	11	15	11
2	7	15 (9)	20	~~25~~ 10	2
3	14	16	18	0	—
Demand	15	15	10		
Column penalty	7	11	9		

Source

Here R_1 and C_3 have largest penalty (= 11). we select C_3 arbitrarily and ~~assign value~~ $C_{23} (= 9)$ is the least unit cost in this column. So take $x_{23} = 15$, this crossed out C_3 and leaves 10 units in R_2.

(iv) Recompute the penalties.

	2	4	Supply	Row penalty
1	0	11	15	11
2	7 (10)	20	~~10~~ 0	13
3	14	18	0	—
Demand	~~15~~ 5	10		
Column penalty	7	9		

Here R_2 has the largest penalty (= 13) and $C_{22} (= 7)$ is the ~~smallest~~ least unit cost in this row. So we take $x_{22} = 10$ and this cross out R_2 and leaves 5 unit of demand in C_2.

(v) Since only one row with +ve supply is left uncrossed, so we determine the basic variables of this row using least cost method.

$C_{12} (=0)$ is the least cost in this row. So we take $x_{12} = 5$ and, this cross out C_2 and leaves 10 units of supply in R_1

	2	4	Supply
1	[0] 5	[11] 10	15/0
3	[14]	[18] 0	0
Demand	5	10/0	

Now select $x_{14} = 10$ and cross out R_1. It leaves '0' units of demand in C_4

Finally take $x_{34} = 0$.

the required starting solution is

DESTINATION

SOURCE	1	2	3	4	Supply
1	[10]	[0] 5	[20]	[11] 10	15
2	[12]	[7] 10	[9] 15	[20]	25
3	[0] 5	[14]	[16]	[18] 0	5
Demand.	5	15	15	10	

and the associated transportation cost is

$x_c = 5 \times 0 + 5 \times 0 + 10 \times 7 + 15 \times 9 + 10 \times 11 + 18 \times 0$

$= 70 + 135 + 110$

$= 315$ Ans.

<u>Remarks</u>- We have seen that the same problem of minimization, solved by three different techniques, presented the following starting solutions:-

1) By North-West Corner method : min Cost = 410

2) By least Cost Method : min. Cost = 335

3) by Vogel's Approximation method (VAM) ; min. Cost = 315

All these solutions are feasible but may not be the optimal ones. (In order to improve these starting basic feasible solutions, we employ the method of multipliers. This method provides a basis for the determination of entering variable. Before giving the elaboration of this method, we give some definitions and before this we have some examples of VAM and L.C.M.)

Example 5.3-3:

Find the starting solution for the problem by Least Cost Method:

Source	1	2	3	4	Supply
1	[10] x_{11}	[2] $x_{12}=15$	[20] x_{13}	[11] x_{14}	15/0
2	[12] x_{21}	[7] $x_{22}=0$	[9] $x_{23}=15$	[20] $x_{24}=10$	25/10
3	[4] $x_{31}=5$	[14] $x_{32}=$	[16] $x_{33}=$	[18] $x_{34}=5$	10/5
Demand	5	15/0	15	15/10	

Sol: In the given table, x_{12} is the variable associated with the smallest unit cost $c_{12}=(2)$. So take $x_{12}=15$ which satisfies both row 1 and column 2 simultaneously. We arbitrarily cross out R_1 and leave '0' units of demand in C_2

② Now x_{31} has the smallest unit cost $c_{31}(=4)$. So take $x_{31}=5$ and this cross out C_1 and leaves 5 units of supply in R_3.

③ Now x_{22} has the smallest unit cost $c_{22}(=7)$ so take $x_{22}=0$, this cross out C_2 and does ~~leaves 15 units of~~ not change the supply in R_2.

④ Now x_{23} has the smallest unit cost $c_{23}(=9)$. So take $x_{23}=15$, this cross out C_3 and leaves 10 units of supply in R_2

⑤ Now x_{34} has the smallest unit cost $c_{34}(=18)$, So take $x_{34}=5$ which satisfies R_3. So this cross out R_3 and leaves 10 units of demand in C_4.

and finally take $x_{23}=10$.

the required starting solution is

[10]	[2] 15	[20]	[11]	15
[12]	[7] 0	[9] 15	[20] 10	25
[4] 5	[14]	[16]	[18] 5	10
5	15	15	15	

and the associated transportation cost is

$z = 5\times4 + 15\times2 + 0\times7 + 15\times9 + 10\times20 + 18\times5$

$= 20 + 30 + 0 + 135 + 200 + 90$

$= 475$ Ans

this question is solved in the middle-pages precisely.

Ex Solve the following unbalanced transportation problem by using VAM to find the starting solution. The demand at destination 1 must be supplied from source 4.

Solve yourself [illegible]

				Supply
	5	1	0	20
	3	2	4	10
	7	5	2	15
	9	6	0	15
DEMANDS	5	10	15	

Solution: Here

$60 = \text{total supply} \neq \text{total demand} = 30$

So the model is unbalanced.

First, we balance the problem as

Source \ Destinations	1	2	3	4	Supply
1	5	1	0	0	20
2	3	2	4	0	10
3	7	5	2	0	15
4	9 (5)	6	0	0	~~15~~ 10
DEMAND	5	10	15	30	

At first we observe the restriction of a (i.e. given in (i))

(i) Since the demand at destination 1 must be supplied from source 4; so we assign $x_{41} = 5$. This crossed out c_1 and leaves 10 units of supply in R_4.

(ii) Now we use VAM to generate a starting sol. Evaluate a penalty for each row and column for the new table

	2	3	4	Supply	Row penalty
1	1	0	0	20	1 − 0 = 1
2	2	4	0 (10)	10	2 − 0 = 2
3	5	2	0	15	2 − 0 = 2
4	6	0	0	10	6 − 0 = 6
Demand	10	15	~~30~~		
Column Penalty	1	2	—		

: after '0' next smallest is '6'

(ii) Here R_2 and R_3 both have the largest penalty (=2) we arbitrarily choose R_2, $C_{24}=0$ is the least unit cost in this row, so we take $x_{24}=10$, this crossed out R_2 and leaves 20 units of demand in C_4.

(iii) Re-compute the penalties for the uncrossed rows and columns :-

	2	3	4	Supply	Row Penalty
1	1 (10)	0	0	~~20~~ 10	0
3	5	2	0	15	2
4	6	0	0	10	0
DEMAND	10	15	20		
Column penalty	4	0	0		

Here C_2 has the largest penalty (=4) and C_{12} is the least unit cost in this column so take $x_{12}=10$ this leaves 10 units of supply in R_1 and crossed out C_2.

(iv) Re-compute the penalties.

	3	4	Supply	Row Penalty
1	0	0	10	0
3	2	0 (15)	~~15~~	2
4	0	0	10	0
DEMAND	15	~~20~~ 5		
Column penalty	2	0		

Here C_3 and R_3 both have the largest penalty (=2) we arbitrarily choose R_3. Now since $C_{34}(=0)$ is the least unit cost in this row so we assign $x_{34}=15$ which crossed out R_3 and leaves 5 unit of demand in C_4.

(V)

Method of Multipliers:- this method is used to optimize the starting solution (i.e) After finding a starting solution, we can use this method (the method of multipliers) for a possible improvement in the solution. usually VAM is employed to obtain starting solution, for this method provides a better starting solution than the others.

Determination of entering variable:- the starting solution obtained in the example of North west corner method is "in the tableau form" as follows:

	1	2	3	4	Supply
1	[10] 5	[0] 10	[20]	[11]	15
2	[12]	[7] 5	[9] 15	[20] 5	25
3	[0]	[14]	[16]	[18] 5	5
Demand	5	15	15	10	

cost = 410

the determination of the entering variable from among the current non basic variables (those that are not part of the starting basic solution) is done by using the method of multipliers (by computing the non-basic Coefficients in the z-row).

In this method we associate the multipliers u_i and v_j with each row i and each Column j of the transportation tableau.

For each current basic variable x_{ij}, these multipliers are to satisfy the following equation —

$$u_i + v_j = c_{ij} \quad \text{for each basic } x_{ij}$$

Since here we have 6 basic variables so, we get 6 equations in 7 unknowns and Ingeneral if there are $(m+n-1)$ basic variables, we get $(m+n-1)$ equations in $(m+n)$ unknowns.

$u_1, u_2, u_3, v_1, v_2, v_3, v_4$ (7 unknowns)

The values of the multipliers can be determined from these equations by arbitrarily setting $u_1 = 0$, and then solving for the remaining variables, as shown below

Row U — Column V

Basic variable	(u,v) equation	Sol.
x_{11}	$u_1 + v_1 = 10$	$\Rightarrow u_1 = 0 \rightarrow v_1 = 10$
x_{12}	$u_1 + v_2 = 0$	$u_1 = 0 \rightarrow v_2 = 0$
x_{22}	$u_2 + v_2 = 7$	$v_2 = 0 \rightarrow u_2 = 7$
x_{23}	$u_2 + v_3 = 9$	$u_2 = 7 \Rightarrow v_3 = 2$
x_{24}	$u_2 + v_4 = 20$	$u_2 = 7 \Rightarrow v_4 = 13$
x_{34}	$u_3 + v_4 = 18$	$v_4 = 13 \Rightarrow u_3 = 5$

i.e. $u_1 = 0 \Rightarrow u_2 = 7$, $u_3 = 5$, $v_1 = 10$, $v_2 = 0$, $v_3 = 2$, $v_4 = 13$

next, we use the computed values of u_i and v_j to evaluate the non basic variables by computing

$$\bar{x}_{ij} = u_i + v_j - c_{ij} \quad \text{for each nonbasic } x_{ij}$$

the results of these evaluations are shown in the following table.

Z-Row

Non basic variables	Corresponding Coefficients $u_i + v_j - c_{ij}$
x_{21}	$u_2 + v_1 - c_{21} = 7 + 10 - 12 = 5$
x_{31}	$u_3 + v_1 - c_{31} = 5 + 10 - 0 = 15$
x_{32}	$u_3 + v_2 - c_{32} = 5 + 0 - 14 = -9$
x_{33}	$u_3 + v_3 - c_{33} = 5 + 2 - 16 = -9$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 2 - 20 = -18$
x_{14}	$u_1 + v_4 - c_{14} = 0 + 13 - 11 = 2$

Optimality Condition:

Since the transportation model seeks to minimize cost, the entering variable is the one having the most +ve coefficient in the Z-row. Thus, x_{31} is the entering variable in this case because the coefficient of x_{31} is 15 which is most +ve.

x (the preceding computations are usually done directly on the transportation tableau (as shown in the below tableau) meaning that it is not necessary really to write the (u,v)-equations explicitly. Instead, we start by setting $u_1 = 0$, then we can compute v-values of all the columns that have basic variables in row 1. Next, we compute u_2 based on the (u,v)-equation of basic x_{22}. Now given u_2, we can compute v_3 and v_4 which finally leads us to determining u_3.

Once all the u's and v's are determined, we can evaluate the non basic variables by computing $u_i + v_j - c_{ij}$ for each non basic x_{ij}. These evaluations are shown in the following table in the South West corner of each cell.

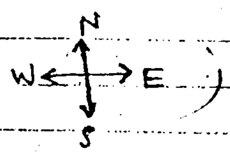
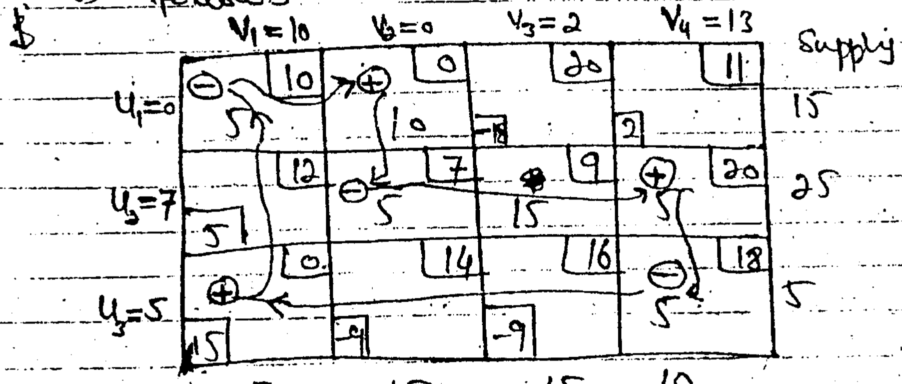


table — Nonbasic variable [illegible]

	$V_1=10$	$V_2=0$	$V_3=2$	$V_4=13$	Supply
$U_1=0$	[10] 5	[0] 10	[20] [-18]	[11] [2]	15
$U_2=7$	[12] [5]	[7] 5	[9] 15	[20] 5	25
$U_3=5$	[0] [15]	[14] [-9]	[16] [-9]	[18] 5	5
Demand	5	15	15	10	

Here x_{31} is the entering variable. Now, we are to determine the leaving variable amongst the basic variables. For this purpose, we construct the loop as follows



Loop:- A loop consists of successive horizontal and vertical connected segments only. (no diagonals are allowed) whose end points must be basic variables except for the two segments starting and ending at the non-basic variables with which it is associated (ie) (each corner of the loop, with the exception of that in the entering variable cell, must coincide with a current basic variable.) the above table shows loop for x_{31}. It is remarkable that exactly one loop exists for a given entering variable.

Also it is immaterial (غير مهم) if the loop is traced in a clockwise or counterclockwise direction.

~~Optimality~~ → corrected by SHAKAIB AKRAM

Feasibility Condition:- the leaving variable is selected as the variable having the smallest value from among the corner variables of the loop associated with entering variable, which will decrease when the entering variable increases above zero.

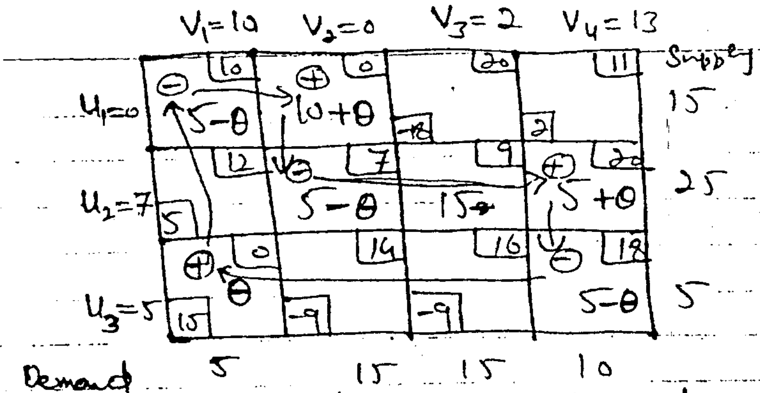
explanation:- the leaving variable is determined in the following manner. the selection of x_{31} as the entering variable means that we want to ship through this route because it reduces the total shipping cost. (because unit cost is less company totally) What is the most we can shipp through the new route ?.

We observe in the previous table that if we shipp an amount θ through route (3,1) i.e. $x_{31} = \theta$. then the maximum value of θ is determined based on two conditions

(i) the supply limits and the demand requirements remain satisfied.

(ii) No negative shipments are allowed any of the routes.

these two conditions are determined the maximum value of θ (non-negative) and the leaving variable by constructing the loop as



After Constructing the loop we see that the new values of the variables then remain non negative if

$x_{11} = 5 - \theta \geq 0$

$x_{22} = 5 - \theta \geq 0$

$x_{34} = 5 - \theta \geq 0$

if $10 - \theta \geq 0$ then 5 is max.

$\Rightarrow$ the Corresponding maximum value of θ is 5, which occurs when x_{11}, x_{22} and x_{34} all reach zero level, Because only one Current basic Variable must leave the basic solution, so we can choose either x_{11} or x_{22} or x_{34} as the leaving variable. We arbitrarily choose x_{11} to leave the solution.

The selection of $x_{31} (= 5)$ as the entering variable and x_{11} as the leaving variable require adjusting the values of basic variables at the Corners of the closed loop. Because each unit shipped through route (3,1) reduces the shipping Cost.

So the new sol. is

Note $\bar{c}_{ij} = u_i + v_j - c_{ij}$ are the Coefficients of non basic variables.

	1	2	3	4	Supply
1	[10] [illegible]	[0] 15	[20]	[11]	15
2	[12]	[7] 0	[9] 15	[20] 10	25
3	[0] 5	[14]	[16]	[18] 0	5
Demand	5	15	15	10	

new cost = 335

the new sol. in above table costs 75 less than the previous one.

<u>iteration II</u>

the new non-basic variables are tested for the possibility of improving the current solution. A solution is optimal if the new evaluation $u_i + v_j - c_{ij}$ for all nonbasic x_{ij} are non positive i.e. -ve.

again Associate multipliers to each row and column as earlier and determine the leaving and the entering variables.

	$V_1 = 0$	$V_2 = 0$	$V_3 = 2$	$V_4 = 13$	Supply
$u_1 = 0$	[10] -5	[0] ⊖ 15	[20] -18	[11] ⊕ 2	15
$u_2 = 7$	[12] 0	[7] ⊕ 0	[9] 15	[20] ⊖ 10	25
$u_3 = 5$	[0] 5	[14] -9	[16] -9	[18] 0	5
Demand	5	15	15	10	

using optimality condition, we observe that x_{14} is the entering variable. the loop corresponding to x_{14} indicates that x_{12} and x_{24} are the basic variables that will decrease when x_{14} increases above its current zero value. Since ~~both of these~~ x_{24} has smaller value so x_{24} is the leaving variable.

therefore the new table is

	1	2	3	4	Supply
1	[10]	[0] 5	[20]	[11] 10	15
2	[12]	[7] 10	[9] 15	[20]	25
3	[0] 5	[14]	[16]	[18] 0	5
Demand	5	15	15	10	

A sol. is optimum if the new evaluation $u_i + v_j - c_{ij}$ for all non basic x_{ij} are non +ve i.e. -ve.
$\Rightarrow$ If all the coefficients of the non-basic variables are non-positive.

3rth iteration:-

Associate a multiplier to each row and column and proceed as ~~follows earlier~~

$v_1 = -7 \quad v_2 = 0 \quad v_3 = 2 \quad v_4 = 11$

	$v_1=-7$	$v_2=0$	$v_3=2$	$v_4=11$
$u_1 = 0$	[10] (-17)	[0] 5	[20] (-18)	[11] 10
$u_2 = 7$	[12] (-12)	[7] 10	[9] 15	[20] (-2)
$u_3 = 7$	[0] 5	[14] (-7)	[16] (-7)	[18] 0

$u_3 + v_4 = 18$
$u_3 = 18 - 11$
$= 7$

Now since all the non basic variables corresponds to non positive coefficients so the optimal solution is achieved as follows

Ship 5 units from source 1 to destination 2
" 10 " " 1 " " 4
" 10 " " 2 " " 2
" 15 " " 2 " " 3
" 5 " " 3 " " 1

and the optimal shipping cost $= 5\times 0 + 10\times 11 + 10\times 7 + 15\times 9 + 5\times 0 + 18\times 0$
$= 110 + 70 + 135$
$= 315$ Ans

Ex. Solve the following transportation model whose starting are degenerate. Use the north-west-corner method to find the starting sol. (the numbers in the box give c_{ij})

(a)

0	2	1	5
2	1	5	10
2	4	3	5
5	5	10	

no. of basic variables = 5

6 eqs

Solution:- First, we find the starting solution using northwest-corner method.

	1	2	3	Supply
1	5 [0]	0 [2]	[1]	5 0
2	[2]	5 [1]	[5]	10 5
3	[2]	[4]	5 [3]	5
Demand	5	5	10	

Cost = 45

i. assign $x_{11}=5$, this crosses out C_1 and leaves zero supply in R_1

(ii) assign $x_{12}=0$, this crosses out R_1

(iii) assign $x_{22}=5$, this crosses out C_2 and leaves 5 units of supply in R_2

(iv) assign $x_{23}=5$, this crosses out R_2 and leaves 5 units of demand in C_3 and finally

(v) assign $x_{33}=5$. which leaves zero demand and supply in R_3 & C_3.

Thus a feasible starting solution is obtained.

(1) Now associate a multiplier to each row and column, and choose the entering and leaving variables as follows.

[illegible] consult [illegible] Basic [illegible] multipliers [illegible]

	$v_1=0$	$v_2=2$	$v_3=6$	Supply
$u_1=0$	[0] 5	[2] ⊖ 0	[1] ⊕ [5]	5
$u_2=-1$	[2] [-3]	[1] ⊕ 5	[5] ⊖ 5	10
$u_3=-3$	[2] [-5]	[4] [-5]	[3] 5	5
Demand	5	5	10	

[illegible] (115) [illegible]

Since x_{13} has the most +ve coefficient so ie ($x_{13}=5$) so x_{13} is the entering variable and the loop construction shows that $x_{12}=0$ is the leaving variable.

So the new table is

[illegible] Leaving variable [illegible] Entering variable [illegible] Table [illegible] Basic [illegible] Non basic variables

	[0] 5	[2]	[1] 0
	[2]	[1] 5	[5] 5
	[2]	[4]	[3] 5

Cost = 45

(ii) Again associate a multiplier to each row and column and construct the loop as follows [illegible]

	$v_1=0$	$v_2=-3$	$v_3=1$	Supply
$u_1=0$	[0] ⊖ 5	[2] [-5]	[1] ⊕ 0	5
$u_2=4$	[2] ⊕ [2]	[1] 5	[5] ⊖ 5	10
$u_3=2$	[2] [0]	[4] [-5]	[3] 5	5
Demand	5	5	10	

The most +ve coefficient corresponds to x_{21}. So x_{21} is the entering variable as the loop construction shows that $x_{11}=5$ and $x_{23}=5$ ties for the leaving variable. So we arbitrarily choose $x_{11} (=5)$ as the leaving variable. So the new table is

			Supply
[0]	[2]	5 [1]	5
5 [2]	5 [1]	0 [5]	10
[3]	[4]	5 [3]	5
Demand 5	5	10	

Cost = 35

(iii) Again associate a multiplier to each row and column and compute the coefficients of non basic variables.

	$v_1 = 2$	$v_2 = -3$	$v_3 = 1$	Supply
$u_1 = 0$	[0] −2	[2] −5	5 [1]	5
$u_2 = 4$	5 [2]	5 [1]	0 [5]	10
$u_3 = 2$	[2] −2	[4] −5	5 [3]	5
Demand	5	5	10	

Since all the coefficients of non-basic variables are non-positive.
So the optimal solution is obtained as

$x_{13} = 5$, $x_{21} = 5$, $x_{22} = 5$, $x_{33} = 5$

and the optimal transportation cost = 35 Ans.

جو ان Basic variables [illegible]

Problem Set 5.3b:- In the given Transportation models, use north west corner method to find the starting solution. then, determine, the optimum solution.

(a)

\$0	\$2	\$1	6
\$2	\$1	\$5	9
\$2	\$4	\$3	5
5	5	10	

Sol:- First we find the starting sol. using north west corner method i.e.,

	1	2	3	Supply
1	[0] 5	[2] 1	[1]	6, 1
2	[2]	[1] 4	[5] 5	9, 5
3	[2]	[4]	[3] 5	5
	5	5, 4	10	

(i) assign $x_{11}=5$, this crosses out C_1 and leaves 1 unit of supply in R_1

(ii) assign $x_{12}=1$, this crosses out R_1 and leaves 4 units of demand in C_2.

(iii) assign $x_{22}=4$, this crosses out C_2 and leaves 5 unit of supply in R_2

(iv) assign $x_{23}=5$ and $x_{33}=5$, thus a starting feasible sol is obtained as:

3			5	5 cost
Demand	5	5	10	

ii, Associate a multiplier to each row and Column and Choose the entering and leaving variables.

	$V_1=0$	$V_2=2$	$V_3=6$	Supply
$U_1=0$	[0] 5	[2] ⊖ 1	[1] ⊕ (2) →5	6
$U_2=-1$	[2] (-3)	[1] ⊕ 4	[5] ⊖ 5	9
$U_3=-3$	[2] (-5)	[4] (-5)	[3] (0) 5	5
Demand	5	5	10	

Since x_{13} has the most +ve Coefficient i.e. $(\bar{c}_{13}=2)$ So x_{13} is the entering variable and the loop Construction shows that $x_{12}(1)$ is the leaving variable.

So the new table is

	1	2	3	Supply
1	[0] 5	[2]	[1] 1	6
2	[2]	[1] 5	[5] 4	9
3	[2]	[4]	[3] 5	5
Demand	5	5	10	

(ii) Again associate a multiplier to each row and Column and Compute the Coefficients of non basic variables.

	$V_1 = 0$	$V_2 = -3$	$V_3 = 1$	Supply
$U_1 = 0$	[0] ⊖ 5	[2] (−5)	[1] ⊕ 1	6
$U_2 = 4$	[2] ⊕ (2)	[1] 5	[5] ⊖ 4	9
$U_3 = 2$	[2] (0)	[4] (−5)	[3] 5	5
demand	5	5	10	

Now since x_{21} has the most +ve Coeff. is $(\bar{c}_{21} = 2)$, So x_{21} is the entering variable and the loop construction show that $x_{23} = 4$ leaves the sol.

So new table is

	1	2	3	Supply
1	[0] 1	[2]	[1] 5	6
2	[2] 4	[1] 5	[5]	9
3	[2]	[4]	[3] 5	5
Demand	5	5	10	

(iii) Associate a multiplier to each row and Column and Compute the Coefficients of non basic variables

	$v_1=0$	$v_2=-1$	$v_3=1$	Supply
$u_1=0$	[0] 1	[2] (−3)	[1] 5	6
$u_2=2$	[2] 4	[1] 5	[5] (−2)	9
$u_3=2$	[2] (0)	[4] (−5)	[3] 5	5
Demand	5	5	10	

Since all the coefficients of non-basic variables are non-positive
So the optimal solution obtained is

$x_{11}=1$, $x_{13}=5$, $x_{21}=4$, $x_{22}=5$, $x_{33}=5$

and the optimal transportation cost

$= 1\times0 + 5\times1 + 4\times2 + 5\times1 + 5\times3$

$= 33$ \$

(b)

0	4	2	8
2	3	4	5
1	2	0	6
7	6	6	

Sol. First, we find the starting sol by using north west corner method i.e.

	1	2	3	
1	[0] 7	[4] 1	[2] —	81
2	[2] —	[3] 5	[4] 0	5 0
3	[1] —	[2] —	[0] 6	6
	7	65	6	

(i) Assign $x_{11}=7$, this crosses out C_1 and leaves 1 unit of supply in R_1

(ii) Assign $x_{12}=1$, this crosses out R_1 and leaves 5 unit of demand in C_2

(iii) Assign $x_{22}=5$, this crosses out C_2 and leaves zero supply in R_2.

(iv) Assign $x_{23}=0$ and $x_{33}=6$. the starting sol obtained is as ~~follows~~.

7 [0]	1 [4]	[2]	8
[2]	5 [3]	0 [4]	5
[1]	[2]	6 [0]	6
7	6	6	

Cost = 19

(2) Now Associate a multiplier to each row and column and choosing the entering and leaving variables as

	$v_1=0$	$v_2=4$	$v_3=5$
$u_1=0$	7 [0]	⊖ 1 [4]	⊕ 3 [2]
$u_2=-1$	−3 [2]	⊕ 5 [3]	⊖ 0 [4]
$u_3=-5$	−6 [1]	−3 [2]	6 [0]

Since x_{13} has the most +ve coefft. So x_{13} enters the sol. and the loop construction shows that $x_{23}(=0)$ leaves the sol.

The new table is

[0] 7	[4] 1	[2] 0
[2]	[3] 5	[4]
[1]	[2]	[0] 6

Cost = 19

(vi) Again associate multipliers ..

	$V_1=0$	$V_2=4$	$V_3=2$
$u_1=0$	[0] 7	[4] 1.	[2] 0,
$u_2=-1$	[2] box: -3	[3] 5	[4] box: -3
$u_3=-2$	[1] box: -3	[2] box: 0	[0] 6

Since all the coefficients of non-basic variables are non +ve.
So optimal sol achieved is
$x_{11}=7$, $x_{12}=1$, $x_{13}=0$, $x_{22}=5$, $x_{33}=6$

and the associated optimal cost is = 19

(c) In the transportation models in ~~table~~ the given table, use north-west-corner method to find the starting solution, then determine the optimum solution.

—	3	5	4
7	4	9	7
1	8	6	19
5	6	19	

Sol.: First, we find the starting solution using north-west-corner method.

	1	2	3	supply
1	[M] 4	[3]	[5]	4
2	[7] 1	[4] 6	[9]	~~7~~ ~~6~~ 0
3	[1]	[8] 0	[6] 19	19
demand	~~5~~ 1	~~6~~ 0	19	

(Since there is no route from source 1 to destination 1 so we take $c_{11} = M$)

i) Assign $x_{11} = 4$, this crosses out R_1 and leaves 1 unit of demand in C_1.
ii) Assign $x_{21} = 1$, this crosses out C_1 and leaves 6 units of supply in R_2.
iii) Assign $x_{22} = 6$, this crosses out R_2 and leaves zero unit of demand in C_2.
iv) Assign $x_{32} = 0$, this cross out C_2 and leaves 19 units in R_3.

vi) Finally take $x_{33} = 19$.

Thus the starting solution obtained is

				Supply
	4 [M]	[3]	[5]	4
	1 [7]	6 [4]	[9]	7
	[1]	0 [8]	19 [6]	19
Demand	5	6	19	

iv) Now associate a multiplier to each row and each column and find the entering and leaving variables.

	$V_1 = M$	$V_2 = M-3$	$V_4 = M-5$	Supply
$U_1 = 0$	⊖ 4 [M]	⊕ M-6 [3]	M-10 [5]	4
$U_2 = 7-M$	⊕ 1 [7]	⊖ 6 [4]	-7 [9]	7
$U_3 = 11-M$	10 [1]	0 [8]	19 [6]	19
Demand	5	6	19	

Since M is very large so the most +ve coefficient corresponds to x_{12}. So x_{12} is the entering variable and the loop construction shows that x_{11} is the leaving variable.

So new table is

[M]	4 [3]	[5]	4
5 [7]	2 [4]	[9]	7
[1]	0 [8]	19 [6]	19
5	6	19	

(ii) Again associate penalties and compute coefficients of non-basic variables as.

	$V_1 = 6$	$V_2 = 3$	$V_3 = 8$	Supply
$u_1 = 0$	[M] 6-M	[3] 4	[5] -4	4
$u_2 = 1$	[7] ⊖ 5	[4] ⊕ 2	[9] -7	7
$u_3 = 5$	[1] 10 ⊕	[8] ⊖ 0	[6] 19	19
Demand	5	6	19	

Since x_{31} has the most +ve coefficient so x_{31} is the entering variable and the loop construction shows that x_{32} is the leaving variable.

So the new table is

				Supply
	[M]	[3] 4	[5]	4
	[7] 5	[4] 2	[9]	7
	[1] 0	[8]	[6] 19	19
Demand	5	6	19	

(iii) Again associate a multiplier to each row and each column and compute the coefficients of non-basic variables.

	$V_1 =$	$V_2 = 3$	$V_3 = 11$	Supply
$u_1 = 0$	[M] 6-M	[3] 4 ⊖	[5] 6 ⊕	4
$u_2 = 1$	[7] 5 ⊖	[4] -2 ⊕	[9] 3	7
$u_3 = 5$	[1] ⊕ 0	[8] -10	[6] ⊖ 19	19
Demand	5	6	19	

Since x_{13} has the most +ve Coefficient. So it enters the sol. and the loop Construction indicates that $x_{12} = 4$ is the leaving variable.

So New table with associated multipliers is

	$V_1 = 0$	$V_2 = 3$	$V_3 = 5$	
$u_1 = 0$	[M] −M	[3] −6	[5] 4	4
$u_2 = 7$	[7] 1 (−)	[4] 6	[9] 3 (+)	7
$u_3 = 1$	[1] (+) 4	[8] −11	[6] 15 (−)	19
	5	6	19	

⇒ x_{23} enters and $x_{21} = 1$ leaves

So new table with associated multipliers is

	$V_1 = 0$	$V_2 = 0$	$V_3 = 5$	
$u_1 = 0$	[M] −M	[3] −3	[5] 4	4
$u_2 = 4$	[7] −3	[4] 6	[9] 1	7
$u_3 = 1$	[1] 5	[8] −7	[6] 14	19
	5	6	19	

Since all the ~~values~~ Coefficients of the non-basic variables are non positive.

So the optimal sol is

$x_{13} = 4$, $x_{22} = 6$, $x_{22} = 1$, $x_{31} = 5$, $x_{33} = 14$

and the associated optimum Trans-portation Cost is

$= 4 \times 5 + 6 \times 4 + 1 \times 9 + 5 \times 1 + 14 \times 6$

$= 20 + 24 + 9 + 5 + 84$

$= 142$ \$ Ans.

Q:2:- In the following problem, the total demand exceeds the total supply. Suppose that the penalty costs per unit of unsatisfied demand are 5 \$, 3 \$ and 2 \$ for destinations 1, 2 and 3 resp. Determine the optimum solution

5	1	7	10
6	4	6	80
3	2	5	15
75	20	50	

Sol:- our model is not balanced because total demand = 145 ~~and~~ total supply = 105

So we add dummy supply (to balance the model) of 40 units, and using the given instructions we reconstruct the model as follows:

	1	2	3	
1	5	1	7	10
2	6	4	6	80
3	3	2	5	15
4	5	3	2	40
	75	20	50	

* [illegible] 5,3,2 [illegible] 0,0,0 [illegible]

Now we employ VAM to ~~employ a~~ obtain a starting solution, for this purpose, we evaluate a penalty for each row and Column, as under

	1	2	3	Supply	R.P
1	5	1 (10)	7	~~10~~ 0	4
2	6	4	6	80	2
3	3	2	5	15	1
4	5	3	2	40	1
Demand	75	~~20~~ 10	50		
Column Penalty	2	1	3		

(i) Here R_1 has the largest penalty (= 4) and c_{12} (= 1) is the least unit cost in this row. So we assign $x_{12} = 10$, this crosses out R_1 and leaves 10 units of demand in C_2.

(ii) Re-compute the penalties

	1	2	3	Supply	R.P
2	6	4	6	80	2
3	3	2	5	15	1
4	5	3	2 (40)	~~40~~	1
Demand	75	10	~~50~~ 10		
C.P.	2	1	3		

Here C_3 has the largest penalty and c_{43} (=2) is the least unit cost in this Column so we assign $x_{43} = 40$

, this crosses out R_4 and leaves 10 units of demand in C_3.

(iii) Re-compute the penalties:

	1	2	3	Supply	R.P
2	[6]	[4]	[6]	80	2
3	[3] 15	[2]	[5]	15	1
Demand	~~75~~ 60	10	10		
C.P	3	2	1		

Here C_1 has the largest penalty and c_{31} (=3) is the least unit cost in this column. So we assign $x_{31} = 15$; this crosses out 15 and leaves 60 units of demand in C_1.

(iv) Now since only one row with the supply is left uncrossed. So we determine the remaining basic variables of this row by using least cost method as

	1	2	3	
2	[6] 60	[4] ~~10~~ 10	[6] ~~15~~ 10	~~80~~ ~~75~~ 60
	60	10	10	

We assign $x_{22} = 10 = x_{23}$ and $x_{21} = 60$.

Hence, the starting solution is

	1	2	3	Supply
1	[5]	[1] 10	[7]	10
2	[6] 60	[4] 10	[6] 10	80
3	[3] 15	[2]	[5]	15
4	[5]	[3]	[2] 40	40
Demand	75	20	50	

Now we try to to improve this solution by using the method of multipliers.

i, Associate a multiplier to each row and column and compute the coefficients of non-basic variables.

	$v_1=3$	$v_2=1$	$v_3=3$	Supply
$u_1=0$	[5] -2	[1] 10	[7] -4	10
$u_2=3$	[6] 60	[4] 10	[6] 10	80
$u_3=0$	[3] 15	[2] -1	[5] -2	15
$u_4=-1$	[5] -3	[3] -3	[2] 40	40
	75	20	50	

Since non of the non-basic variables has a +ve coefficient. So the optimal sol. is

$x_{12}=10$, $x_{21}=60$, $x_{22}=10$, $x_{23}=10$

$x_{31}=15$, $x_{43}=40$ and the associated optimal transportation cost = 595 \$

Note. the same optimal sol. can be found by finding the starting sol using Northwest-Corner method and the least cost method.

Ex. If in problem 2, there are no penalty costs, but that the demand at destination 3 must be satisfied completely. Find the optimal sol.

Sol:- Bearing in mind the instructions available, we reconstruct the table as below:

	1	2	3	
1	[5]	[1] 10	[7]	10
2	[6] 30	[4]	[8] 50	80
3	[3] 5	[2] 10	[5]	~~15~~ 5
4	[0] 40	[0]	[M]	40
	75 35 30	~~20~~ 10	50	

We have used least cost method for obtaining a starting solution.

Now we try to improve this solution by using method of multipliers.

i. Associate a multiplier to each row and column and find the coefficients of non basic variables.

	$V_1 = 2$	$V_2 = 1$	$V_3 = 2$	Supply
$u_1 = 0$	[5] -3	[1] 10	[7] -5	10
$u_2 = 4$	[6] 30 ⊖	[4] 1 ⊕ (x_{22})	[6] 50	80
$u_3 = 1$	[3] ⊕ 5	[2] ⊖ 10	[5] -2	15
$u_4 = -2$	[0] 40	[0] -1	[M] -M	40
Demand	75	20	50	

Since x_{22} has the most +ve Coeff. So x_{22} is the entering variable and the loop construction shows that x_{32} is the leaving variable.

So new table is (Also associate multipliers)

	$V_1 = 3$	$V_2 = 1$	$V_3 = 3$	Supply
$u_1 = 0$	[5] -2	[1] 10	[7] -4	10
$u_2 = 3$	[6] 20	[4] 10	[6] 50	80
$u_3 = 0$	[3] 15	[2] -1	[5] -2	15
$u_4 = -3$	[0] 40	[0] -2	[M] -M	40
Demand	75	20	50	

Since all the Coefficients of non-basic variables are non positive. So the optimal Solution achieved is

$x_{12} = 10$, $x_{21} = 20$, $x_{22} = 10$, $x_{23} = 50$, $x_{31} = 15$, $x_{41} = 40$

and the associated optimal transportation cost is

$$= 10 \times 1 + 20 \times 6 + 10 \times 4 + 50 \times 6 + 15 \times 3 + 40 \times 0$$
$$= 10 + 120 + 40 + 300 + 45$$
$$= 515 \text{ Ans.}$$

check

Q:4:- In the unbalanced transportation problem given below, If a unit from a source is not shipped out (to any destination), a storage cost ~~must be~~ is incurred at the rate of 5$ 4$ and 3$ per unit for sources 1, 2, and 3, resp. If additionally all the supply at source 2 must be shipped out completely to make room for a new product, determine the optimum shipping schedule.

1	2	1	20
3	4	5	40
2	3	3	30
30	20	20	

Sol. Bearing in mind the instructions we reconstruct the ~~total~~ transportation model as;

1	1	2	1	5	20
2	3	4	5	4	40
3	2	3	3	3	30
	30	20	20	20	

Keeping in view the restriction that all the supply at source 2 must be shipped out completely:

i) First, we employ least cost method to find the starting sol. as

	1	2	3	4	Supply
1	[1]	[2]	[1] 20	[5]	20
2	[3] 0	[4] 20	[5] 0	[4] 20	40
3	[2] 30	[3]	[3]	[3]	30
Demand	30	20	20	20	

ii) Now associate multipliers.

	$v_1=-1$	$v_2=0$	$v_3=1$	$v_4=0$	Supply
$u_1=0$	[1] -2	[2] -2	[1] 20	[5] -5	20
$u_2=4$	[3] 0 (+)	[4] 20	[5] 0 (−)	[4] 20	40
$u_3=3$	[2] 30 (−)	[3] 0	[3] 1 (+)	[3] 0	30
	30	20	20	20	

x_{33} enters and x_{23} leaves

So new table is

[1]	[2]	[1] 20	[5]
[3] 0	[4] 20	[5]	[4] 20
[2] 30	[3]	[3] 0	[3]

Again associate Multipliers

	$V_1=0$	$V_2=1$	$V_3=1$	$V_4=1$	Supply
$u_1=0$	1 / -1	2 / -1	1 / 20	5 / -4	20
$u_2=3$	3 / 0	4 / 20	5 / -2	4 / 20	40
$u_3=2$	2 / 30	3 / 0	3 / 0	3 / 0	30
demand	30	20	20	20	

Since all the coefficients of non basic variables are non +ve so the optimal sol achieved is

$x_{13}=20$, $x_{21}=0$, $x_{22}=20$, $x_{24}=20$, $x_{31}=30$, $x_{33}=0$

and the associated optimal transportation cost $= 20\times1 + 0\times3 + 20\times4 + 20\times4 + 30\times2 + 0\times3$

$= 20 + 80 + 80 + 60$

$= 240$ $ Ans.

Excercise:- Solve each of the following transportation models by using the north west-corner method, least cost method and Vogel's approximation method to obtain the starting sol. Compare the computations.

(d)

1	2	6	7
0	4	2	12
3	1	5	11
10	10	10	

Sol: i, By North West - Corner method

Destination (Source rows; cost in brackets)

Source	1	2	3	Supply
1	7 [1]	[2]	[6]	7
2	3 [0]	9 [4]	[2]	~~12~~ 9
3	[3]	1 [1]	10 [5]	11
Demand	~~10~~ 3	~~10~~ 1	10	

(i) We assign $x_{11} = 7$, this crosses out R_1 and leaves 3 units of demand in C_1

(ii) assign $x_{21} = 3$, this crosses out C_1 and leaves 9 units of supply in R_2

(iii) Assign $x_{22} = 9$, this crosses out R_2 and leaves 1 unit of demand in C_2

(iv) Finally, take $x_{32} = 1$, this leaves 10 units in R_3.

(v) Finally take $x_{33} = 10$.

Since only one row is remained uncrossed so the procedure ends and we obtain the following starting sol.

7 [1]	[2]	[6]
3 [0]	9 [4]	[2]
[3]	1 [1]	10 [5]

and the associated transportation cost $= 7\times1 + 3\times0 + 9\times4 + 1\times1 + 10\times5 = 7 + 36 + 1 + 50$
$= 94$

(ii) Using Least Cost Method:-

	1	2	3	Supply
1	[4]	[2]	[6] 7	7
2	[0] 10	[4]	[2] 2	~~12~~ 2
3	[3]	[1] 10	[5] 1	~~11~~ 1
Demand	10	10	10	

i) Assign $x_{21} = 10$, this crosses out C_1 and leaves 2 units of supply in R_2

ii) Assign $x_{32} = 10$, this crosses out C_2 and leaves 1 unit of supply in R_2

(iii) Finally, assign $x_{23} = 2$, $x_{33} = 1$ and $x_{13} = 7$ thus the starting sol obtained is given below

[4]	[2]	[6] 7	7
[0] 10	[4]	[2] 2	12
[3]	[1] 10	[5] 1	11
10	10	10	

and the associated transportation cost is $= 7 \times 6 + 10 \times 0 + 2 \times 2 + 10 \times 1 + 5 \times 1$

$= 42 + 4 + 10 + 5$

$= 61$

(iii) using Vogel's Approximation method.

	1	2	3	Supply
1	1	2	6	7
2	0	4	2	12
3	3	1	5	11
Demand	10	10	10	

Sol: i, Evaluate a penalty for each row and each column

	1	2	3	Supply	R.P
1	1	2	6	7	1
2	0	4	2 (10)	~~12~~ 2	2
3	3	1	5	11	2
Demand	10	10	10		
C.P	1	1	3		

Here C_3 has the largest penalty (= 3) and C_{23} (= 2) is the least unit cost in this Column. So we assign $x_{23} = 10$, this crosses out C_3 and leaves 2 units of supply in R_2

ii, Re-Compute the penalties;

	1	2		R.P
1	1	2	7	1
2	0 (2)	4	2	4
3	3	1	11	2
	~~10~~ 8	10		
C.P	3	1		

Here R_2 has the largest penalty (= 4) and C_{21} (= 0) is the least unit cost in this row. So we assign $x_{21} = 2$, which Crosses out R_2 and leaves 8 unit in C_1

(iii) Re-Compute the penalties

Here R_3 and C_1 both have the largest penalties (= 2), we arbitrarily take C_1 and C_{11} (= 1) is the least unit cost in this Column. So we assign $x_{11} = 7$, this crosses out R_1 and leaves 1 unit of demand in C_1

	1	2		R.P
1	[1] 7	[2] —	7	1
3	[3]	[1]	11	2
	~~8~~ 1	10		
C.P	2	1		

(iv) Now since only one row with +ve supply ~~and demand~~ so we determine the remaining basic variables in this row by using Least Cost method. ($C_{32} = 1$) is the least Cost in this row so assign $x_{32} = 10$ and Cross out C_2 ~~So assign~~ this leaves 1 unit of supply in R_3 and finally assign $x_{31} = 1$.

	1	2	
3	[3] 1	[1] 10	11
	1	10	

Hence the Starting Sol obtained is

	1	2	3	
1	[1] 7	[2]	[6]	7
2	[0] 2	[4]	[2] 10	12
3	[3] 1	[1] 10	[5]	11
	10	10	10	

and the associated transportation cost is

$= 7\times1 + 2\times0 + 10\times2 + 3\times1 + 10\times1$

$= 7 + 20 + 3 + 10$

$= 40$ Ans

Comparison:-

We have seen that the same problem of minimization solved by three methods and presented the following starting sol.

(i) By N.W.C Method : min Cost = 94

(ii) by L.C M : min Cost = 61

(iii) by VAM : " = 40

We see that VAM provides the best starting sol while the N.W.C.M provides the worst one.

(b)

5	1	8	12
2	4	0	14
3	6	7	4
9	10	11	

Solve similarly

Ex. Find the starting solution in the following problem by the (a) North west Corner method, (b) the least Cost method, (c) Vogel's approximation method obtain the optimal sol by using the best starting Solution.

10	20	5	7	10
13	9	12	8	20
4	15	7	9	30
14	7	1	0	40
3	12	5	19	50
60	60	20	10	

Sol: First we find the starting sol by using

(a) North-West-Corner Method (the starting sol is)

Destination

Source	1	2	3	4	Supply
1	10 [10]	[20]	[5]	[7]	10
2	20 [13]	[9]	[12]	[8]	20
3	30 [4]	0 [15]	[7]	[9]	~~30~~
4	[14]	40 [7]	[11]	[0]	40
5	[3]	20 [12]	20 [5]	10 [19]	50
Demand	~~60~~ ~~50~~ 30	~~60~~ 20	20	10	

Hence the starting sol is

$x_{11}=10,\quad x_{21}=20,\quad x_{31}=30,\quad x_{32}=0$

$x_{42}=40,\quad x_{52}=20,\quad x_{53}=20,\quad x_{54}=10$

and the associated transportation cost is

$= 10\times 10 + 20\times 13 + 30\times 4 + 0 + 40\times 7 + 20\times 12 + 20\times 5 + 10\times 19$

$= 100 + 260 + 120 + 280 + 240 + 100 + 190$

$= 1290$

(b) (Using Least Cost method)

Source	1	2	3	4	Supply
1	[10]	10 [20]	[5]	[7]	10
2	[13]	20 [9]	[12]	[8]	20
3	10 [4]	20 [15]	[7]	[9]	~~30~~ 20
4	[14]	10 [7]	20 [11]	10 [0]	~~40~~ ~~30~~ 10
5	50 [3]	[12]	[5]	[19]	50
	~~60~~ 10	~~60~~ ~~50~~ 30	20	10	

Hence the starting sol is

$x_{12} = 10$, $x_{22} = 20$, $x_{31} = 10$, $x_{32} = 20$, $x_{42} = 10$

$x_{43} = 20$, $x_{44} = 10$, $x_{51} = 50$

and the associated transportation cost is

$= 10\times20 + 20\times9 + 10\times4 + 20\times15 + 10\times7 + 20\times1 + 10\times0 + 50\times3$

$= 200 + 180 + 40 + 300 + 70 + 20 + 150$

$= 960$

c) by using VAM.

i) Evaluate a penalty to each row and Column.

	1	2	3	4	Supply	R.P
1	16	20	5	7	10	2
2	13	9	12	8	20	1
3	4	15	7	9	30	3
4	14	7	1	0 (10)	~~40~~ 30	1
5	3	12	5	19	50	2
Demand	60	60	20	10		
C.P	1	2	4	7		

Here C_4 has the largest penalty and $C_{44} (=0)$ is the least unit cost in this column so we assign $x_{44} = 10$, and this crosses out C_4 and leaves 30 units of supply in R_4.

(ii) Re compute the penalties.

	1	2	3	Supply	R.P
1	10	20	5	10	5
2	13	9	12	20	1
3	4	10	7	30	3
4	14	7	1 (20)	~~30~~ 10	6 ✓
5	3	12	5	50	2
Demand	60	60	20		
C.P	1	2	4		

$x_{43} = 20$

	1	2	Supply	R.P.
1	10	20	10	10
2	13	9	20	4
3	4 (30)	15 ~~30~~	30	11 ✓
4	14	7	10	7
5	3	12	50	9
Demand	~~60~~ 30	~~60~~ ~~30~~		
C.P	1	2		

$x_{31} = 30$

	1	2	Supply	R.P
1	[10] 10	[20] ~~[illegible]~~	10	10 ✓
2	[13]	[9]	20	4
4	[14]	[7]	10	7
5	[3]	[12]	50	9
Demand	~~30~~ 20	~~60~~ [illegible]		
C.P	7	2		

$x_{12} = 10$

	1	2	Supply	R.P
2	[13]	[9] 20	20	4
4	[14]	[7] 10	10	7
5	[3] 20	[12] 30	~~50~~ 30	9
Demand	20	30		
C.P	10 ✓	2		

So the starting sol. obtained is

	1	2	3	4	
1	[10] 10	[20]	[5]	[7]	10
2	[13]	[9] 20	[12]	[18]	20
3	[4] 30	[15]	[7]	[9]	30
4	[14]	[7] 10	[11] 20	[0] 10	40
5	[3] 20	[12] 30	[5]	[19]	50
	60	60	20	10	

and the associated transportation Cost is

$= 10\times10 + 20\times9 + 30\times4 + 10\times7 + 20\times1 + 0 + 20\times3 + 30\times12$

$= 100 + 180 + 120 + 70 + 20 + 60 + 360$

$= 9/0$. So VAM provides the best starting sol.

Now we try to Improve this starting sol by the method of Multipliers

	$V_1 = 10$	$V_2 = 19$	$V_3 = 13$	$V_4 = 12$	Supply
$u_1 = 0$	[10] 10 ⊖	[20] −1	[5] 8 ⊕	[7] 5	10
$u_2 = -10$	[13] −13	[9] 20	[12] −9	[8] −6	20
$u_3 = -6$	[4] 30	[15] −2	[7] 0	[9] −3	30
$u_4 = -12$	[14] −16	[7] 10 ⊕	[1] 20 ⊖	[0] 10	40
$u_5 = -7$	[3] 20 ⊕	[12] 30 ⊖	[5] 1	[19] −14	50
Demand	60	60	20	10	

x_{13} enters and x_{11} leaves the sol. —

So the new table is

	$V_1 =$ [illegible]	$V_2 = 11$	$V_3 = 5$	$V_4 = 4$	Supply
$u_1 = 0$	[10] −8	[20] −9	[5] 10	[7] −3	10
$u_2 = -2$	[13] −13	[9] 20	[12] −9	[8] −6	20
$u_3 = 2$	[4] 30	[15] −2	[7] 0	[9] −3	30
$u_4 = -4$	[14] −16	[7] 20 ⊕	[1] 10 ⊖	[0] 10	40
$u_5 = 1$	[3] 30	[12] 20 ⊖	[5] 1 ⊕	[19] −14	50
Demand	60	60	20	10	

x_{53} enters and x_{43} leaves the sol.

So the new solution is (again associate multipliers)

	$V_1 = 3$	$V_2 = 12$	$V_3 = 5$	$V_4 = 5$
$u_1 = 0$	[10] −7	[20] −8	[5] 10	[7] −2
$u_2 = -3$	[13] −13	[9] 20	[12] −10	[8] −6
$u_3 = 1$	[4] 30	[15] −2	[7] −1	[9] −3
$u_4 = -5$	[14] −16	[7] 30	[1] −1	[0] 10
$u_5 = 0$	[3] 30	[12] 10	[5] 10	[19] −14

Now since all the coefficients of non basic variables are non-positive so the optimal solution obtained is

$x_{13} = 10$, $x_{22} = 20$, $x_{31} = 30$, $x_{42} = 30$

$x_{44} = 10$, $x_{52} = 10$, $x_{53} = 10$, $x_{51} = 30$

and the associated optimal transportation cost

$= 10 \times 5 + 20 \times 9 + 30 \times 4 + 30 \times 7 + 0 + 30 \times 3 + 10 \times 12 + 10 \times 5$

$= 50 + 180 + 120 + 210 + 90 + 120 + 50$

$= 820$ Ans

→ solve mean to find the optimal sol.

Ex:- Solve the following unbalanced Transportation problem using VAM to find the starting solution, the demand at destination 1 must be shipped from Source 4.

5	1	0	20
3	2	4	10
7	5	2	15
9	6	0	15
5	10	15.	

Sol. Here the model is not balanced because total supply = 60 ≠ total demand = 30

So, first, we balance the problem we introduce a dummy destination to absorb the surplus supply of 30 i.e.

	1	2	3	4	
1	[5]	[1]	[0]	[0]	20
2	[3]	[2]	[4]	[0]	10
3	[7]	[5]	[2]	[0]	15
4	[9] 5	[6]	[0]	[0]	~~15~~ 10
	5	10	15	30	

(i) Since the demand at destination 1 must be shipped from Source 4 so to observe the restriction we assign $x_{41} = 5$ ~~which~~ this crosses out C_1 and leaves a 10 unit of supply ~~at~~ in R_4.

(ii) Now we use the VAM (for the new table) to obtained the starting sol.

	2	3	4	Supply	Row Penalty
1	[1]	[0]	[0]	20	1−0 = 1
2	[2]	[4]	[0]	10	2
3	[5]	[2]	[0]	15	2
4	[6] ~~~	[0] 10	[0] ~~~	10	6−0 = 6
Demand	10	~~15~~ 5	30		
Column penalty	1	2−0 = 2	—		

	2	3	4	Supply	R.P
1	[1]	[0]	[0]	20	1
2	[2]	[4]	[0]	10	2
3	[5] ~~~	[2] ~~~	[0] 15	15	2 ✓
Demand	10	5	~~30~~ 15		
C.P	1	2	—		

	2	3	4	Supply	R.P
1	[1]	[0] 5	[0]	~~20~~ 15	1
2	[2]	[4]	[0]	10	2
Demand	10	5	15		
C.P	1	4 ✓	—		

	2	4	Supply	R.P
1	[1] 10	[0] 5	~~15~~	1
2	[2]	[0] 10	10	2
Demand	10	~~15~~ 5		
C.P	1			

[illegible]

Since we are confined to satisfy the demand at destination 1 from source 4, so for the time being, we ignore first column from our computations.

The starting solution hence is

	2 ($v_1=1$)	3 ($v_2=0$)	4 ($v_3=0$)	
$u_1=0$ 1	[1] 10	[0] 5	[0] 5	20
$u_2=0$ 2	[2] (−1)	[4] (−4)	[0] 10	10
$u_3=0$ 3	[5] (−4)	[2] (−2)	[0] 15	15
$u_4=0$ 4	[6] (−5)	[0] 10	[0] 0	10
	10	15	30	

[illegible] ignore

Since all the coefficients of non basic variables are non-positive so the optimal solution obtained is

$x_{41}=5$, $x_{12}=10$, $x_{13}=5=x_{14}$, $x_{24}=10$; $x_{34}=15$

$x_{43}=10$

So the optimal transportation cost is

$(5)(9)+(10)(1)+(5)(0)+(5)(0)+(10)(0)+(15)(0)+(10)(0) = 55$ **Ans:**

Ex.* In the unbalanced Transportation problem given, if a unit from source i is not shipped out (to one of the destinations); a storage cost must be incurred. Let the storage costs per unit sources 1, 2, and 3 be 5, 4, and 3. If, in addition, all the supply at source 2 must be shipped out to make room for a new product, find the optimal sol.

1	2	1	20
0	4	5	40
2	3	3	30
30	20	20	

Sol:- In compliance with the information available we reconstruct the table as below.

	1	2	3	4	
1	1	2	1	5	20
2	0	4	5	4	40
3	2	3	3	3	30
	30	20	20	20	

Keeping in view the restriction that all the supply at source 2 must be shipped out, We ~~obtain~~ employ least cost method to obtain a starting solution.

	1	2	3	4	Supply
1	[1]	[2]	[1] 20	[5]	20
2	[0] 30	[4] 10	[5]	[4]	~~40~~ 10
3	[2]	[3] 10	[3] 0	[3] 20	~~30~~ 10
D	~~30~~ 0	~~20~~ 10	~~20~~ 0	20	

Cost = 150.

Now we try to improve this sol. as:

	$v_1=-3$	$v_2=1$	$v_3=1$	$v_4=1$	Supply
$u_1=0$	[1] (-4)	[2] (-1)	[1] 20	[5] (-4)	20
$u_2=3$	[0] 30	[4] 10	[5] (-1)	[4] (0)	40
$u_3=2$	[2] (-3)	[3] 10	[3] 0	[3] 20	30
Demand	30	20	20	20	

Since ~~non~~ all the coefficients of non-basic variables are non +ve so optimal sol is

$x_{13}=20$, $x_{21}=30$, $x_{22}=10$, $x_{32}=10$, $x_{33}=0$

$x_{34}=20$

and the transportation cost is

$= 20+0+40+30+0+60$

$= 150$ Ans

Ex. Show by the method of multipliers that the Solution in the table is optimal.

	1	2	3	4
1	[0] 1	[2]	[1]	[1]
2	[3]	[0]	[0] 10	[2]
3	[0]	[0] 5	[3]	[2]
4	[4]	[2]	[0]	[0] 5

Solution:— Since there must be $m+n-1 = 4+4-1$ basic variables, so we assume that $x_{22} = x$; = 7
$x_{31} = x'$; $x_{43} = x''$ to meet this requirement.

Now using Method of multipliers, associate a multiplier to each row and Column and Compute the Coefficients of non-basic variables.

	$v_1 = 0$	$v_2 = 0$	$v_3 = 0$	$v_4 = 0$
$u_1 = 0$	[0] 1	[2] -2	[1] -1	[1] -1
$u_2 = 0$	[3] -3	[0] x^*	[0] 10	[2] -2
$u_3 = 0$	[0] x'	[0] 5	[3]	[2]
$u_4 = 0$	[4] -4	[2] -2	[0] x''	[0] 5

Since all the Coefficients of non-basic variables are non +ve so the given sol is optimal.

Basic Coefficient ...

The Assignment Model.

Consider the situation of assigning m jobs/workers to n machines. A job i ($i=1,2,3,--m$), when assigned to machine j ($j=1,2,--n$), then the element C_{ij} represents the cost of assigning workers i to machine j.

The Transportation Model:-

(Consider the situation of assigning m workers to n' jobs. A worker i ($i=1,2,--n$) when assigned a job j ($j=1,2,--n$), then the element C_{ij} represents the cost of assigning worker i to job j ($i,j=1,2,--n$).)

(the objective is to assign the workers to jobs (one job per worker) at the least cost. this problem is called assignment problem.) and the model corresponds to this problem is called A.M. the formulation of the assignment problem is actually a special case of the transportation model in which workers represents the sources, and the jobs represent the destinations. the supply amount at each source, and the demand amount at each destination exactly equals 1 (ie) $a_i, b_j = 1$

The cost of assigning workers to jobs ~~if any~~ is C_{ij}. If a worker can not be assigned to a certain job, then the corresponding C_{ij} is taken equal to M, where M is a very large +ve number.

حل کرنے سے پہلے ماڈل کو Balance کرنا ضروری ہے

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(In fact the assignment model can be solved directly as a regular transportation model. before the model can be solved by the transportation technique. it is necessary first to balance the problem by adding the dummy workers or jobs etc. ~~there~~ is no loss of generality in assuming that the no. of workers always equal to the number of jobs.

the Assignment model can be solved directly by Hungarian Method.)

Simplex Explanation of the Hungarian Method

the assignment model can be expressed mathematically as follows.

~~Let c_{ij} be the~~

$$x_{ij} = \begin{cases} 1, & \text{if worker } i \text{ assigned to job } j \\ 0, & \text{otherwise} \end{cases}$$

Let c_{ij} be the cost of assigning worker i to job j, ~~and define~~.
then the LP model is given as

$$\text{Minimize} \quad x_0 = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad , \quad i, j = 1, 2, \ldots n$$

$$\sum_{i=1}^{n} x_{ij} = 1$$

Where x_{ij} is either zero or 1.

Remark:-

The optimal solution of the assignment model remains unchanged by adding (or subtracting) a constant to any row / column of the cost matrix.

Proof:-

Let p_i and q_j are subtracted from the ith row and jth column, then the new cost elements becomes:

$$c'_{ij} = c_{ij} - p_i - q_j$$

thus the new objective function is

$$x'_0 = \sum_i \sum_j c'_{ij} x_{ij}$$

$$= \sum_{i} \sum_{j} c_{ij} x_{ij} - \sum_i p_i \sum_j x_{ij} - \sum_j q_j \sum_i x_{ij}$$

$$= \sum_i \sum_j c_{ij} x_{ij} - \sum_i p_i - \sum_j q_j$$

$$= x_0 - \text{constant} \quad \because \left(\sum_j x_{ij} = \sum_i x_{ij} = 1 \right)$$

Hence the new objective function differs from the original one by a constant. the optimum values of x_{ij} must be the same in both the cases.

Mow کاٹنا (گھاس) Chore گھریلو کام / روزمرہ کا معمولی کام farm

The Hungarian Method :- (to solve the Assignment Model)

: the assignment ~~problem~~ will be solved by Hungarian Method as

Step 1:- For the original cost matrix, identify each rows minimum, and subtract it from all the entries of the row.

Step 2 For the matrix resulting from Step 1, identify each column's minimum, and subtract it from all the entries of the column

Step 3:- identify the optimal assignment as the ~~one~~ associated with the zero elements of the matrix obtained in Step 2.

Sol. as the feasible

We will explain this with the help of an example.

Example:- 5.4-1. (Read from book)

the assignment model is

	Mow	Paint	wash
John	15 $	10 $	9 $
Karen	$ 9	$ 15	$ 10
terri	$ 10	$ 12	$ 8

Mow, Paint, Wash are jobs while the John, Karen, terri are workers wanted to get job. Cost is given in the table

Let p_i and q_j be row i and column j minimum costs.

Step 1.

15	10	9	$p_1 = 9$
9	15	10	$p_2 = 9$
10	12	8	$p_3 = 8$

Now, we subtract the row minimum from each respective row to obtain the reduced matrix as

	Mow	Paint	Wash
John	6	1	0
Karen	0	6	1
terri	2	4	0

$q_1 = 0 \quad q_2 = 1 \quad q_3 = 0$

Step 2

the application of step 2 yields the column minimums. Subtracting these values from the respective columns, we get the reduced matrix.

	Mow	Paint	Wash
John	6	[0]	0
Karen	[0]	5	1
terri	2	3	[0]

the cells with squares of zero entries provide the optimum solution.

this means that John gets to paint the garage, Karen gets to Mow the lawn, and terri gets to wash the family cars.

the total cost to Mr. Klyne is = ~~9+10+8=27~~ 10+9+8 \$

this amount also will always equal to

$$\Sigma p + \Sigma q = (p_1+p_2+p_3) + (q_1+q_2+q_3) = (9+9+8) + (0+1+0) = 27 \$$$

Now we give a justification of this result:

the given steps of the Hungarian method work well for the preceeding example because the zero entries in the final matrix happen to produce a feasible assignment (in the sence that each child is assigned exactly one chore)

(Feasibility Condition:
Here the solution is Feasible only if each worker is assigned exactly one job.)

Now, in some cases, the zeroes created by steps 1 and 2 may not yield a feasible solution directly. In this case, further steps are needed to find the optimal feasible assignment

these steps are as under.

Step 2(a) If no feasible assignment (with all zero entries) can be secured from steps 1 and 2 then

(i) Draw the minimum number of horizontal and vertical lines in the last reduced matrix that will cover all the zero entries.

(ii) Select the smallest uncovered element, and subtract it from every uncovered element; then add it to every element at the intersection of any two lines.

(iii) If no feasible assignment can be found among the resulting zero entries, Repeat step 2a otherwise, go to step 3 to determine the optimal assignment.

the following example demonstrates this situation.

Suppose that the situation discussed in example 5.4-1 is extended to four children and four chores. table summarize the cost elements of the problem

Child \ Chores	1	2	3	4
1	1	4	6	3
2	9	7	10	9
3	4	5	11	7
4	8	7	8	5

For the application of step 1 and 2 identify ~~the~~ each rows (column) minimum cost and subtract it from all entries of the respective row (column)

Child \ Chores	1	2	3	4	
1	0	3	5	2	$P_1 = 1$
2	2	0	3	2	$P_2 = 7$
3	0	1	7	3	$P_3 = 4$
4	3	2	3	0	$P_5 = 5$

0	3	2	2
2	0	0	2
0	1	4	3
3	2	0	0
$q_1=0$	$q_2=0$	$q_3=3$	$q_4=0$

the locations of the zero entries do not allow assigning one chore to each child (which means a feasible assignment to the zero) elements is not possible.

For example, if we assign child 1 to chore 1, then column 1 will be eliminated, and child 3 will not have a zero entry in the remaining three columns.

Now this obstacle can be accounted by using Step 3 explained before.

Draw the minimum number of lines which will cover all the zero entries.

	1	2	3	4
1	0	3	2	2
2	2	0	0	2
3	0	1	4	3
4	3	2	0	0

intersection

Now the smallest uncovered element is 1. This element is subtracted from all the uncovered elements and added to the intersection elements. The other entries remain unaltered. The new matrix hence, is

Child \ Chore	1	2	3	4
1	[0]	2	1	1
2	3	0	[0]	2
3	0	[0]	3	2
4	4	2	0	[0]

~~Hence the opt~~ This yields the

the optimal assignment (1,1) (2,3) (3,2), (4,4) i.e. the optimal solution calls for assigning child 1 to chore 1, child 2 to chore 3, child 3 to chore 2, and child 4 to chore 4.

the associated optimal cost is $1+10+5+5 = \$21$.

the same cost is also determined by summing the P_i's the q_j's and the element that was subtracted after the covering lines were constructed i.e.

$$(1+7+4+5)+(0+0+3+0)+(1) = 21\ \$$$

Remark ⇒ Which also shows that the optimal sol remains unchanged ~~by subtracting P_i's and q_j's from the original Matrix~~

Problem 5.4(a): Solve the assignment model.

(a)

3	8	2	10	3
8	7	2	9	7
6	4	2	7	5
8	4	2	3	5
9	10	6	9	10

Sol: i. identify each row's minimum cost and subtract it from all entries of the row.

1	6	0	8	1	$P_1 = 2$
6	5	0	7	5	$P_2 = 2$
4	2	0	5	3	$P_3 = 2$
6	2	0	1	3	$P_4 = 2$
3	4	0	3	4	$P_5 = 6$

(ii) Now identity each columns

0	4	0	7	0
5	3	0	6	4
3	0	0	4	2
5	0	0	0	2
2	2	0	2	3

$q_1 = 1 \quad q_2 = 2 \quad q_3 = 0 \quad q_4 = 1 \quad q_5 = 1$

this solution is not feasible, for the tie exists between R_2 and R_5.

So Draw the minimum number of lines and select the smallest uncovered element.

0	4	0	7	0
5	3	0	6	4
3	0	0	4	2
5	0	0	0	2
2	2	0	2	3

the smallest uncovered element is 2. Subtract this element from all the uncovered entries and add it in elements corresponding to intersection element of lines.

So the new table is

	1	2	3	4	5
1	0	4	2	7	[0]
2	3	1	[0]	4	2
3	3	[0]	2	4	2
4	5	0	2	[0]	2
5	[0]	0	8	0	1

صف يا كالم فيه صفر واحد نرسم حوله square

this table yields optimal assignment (1, 5), (2, 3), (3, 2), (4, 4), (5, 1) and the associated optimal cost is

$$\Sigma p + \Sigma q + 2 = (2+2+2+2+6) + (1+2+0+1+1) + 2$$
$$= 21 \text{ \$}$$

(b)

3	9	2	3	7
6	1	5	6	6
9	4	7	10	3
2	5	4	2	1
9	6	2	4	5

Sol: identify each rows minimum cost and subtract it from all entries of the row.

1	7	0	1	5	$P_1 = 2$
5	0	4	5	5	$P_2 = 1$
6	1	4	7	0	$P_3 = 3$
1	4	3	1	0	$P_4 = 1$
7	4	0	2	3	$P_5 = 2$

Now identify each columns minimum cost and subtract it from all entries of the column.

	1	2	3	4	5
1	[0]	7	0	[0]	5
2	4	[0]	4	4	5
3	5	1	4	6	[0]
4	[0]	4	3	[0]	0
5	6	4	[0]	1	3
	$q_1 = 1$	$q_2 = 0$	$q_3 = 0$	$q_4 = 1$	$q_5 = 0$

this table yields optimal assignments (1,1) (2,2), (3,5), (4,4), (5,3).

Another optimal assignment is (1,4), (2,2), (3,5), (4,1), (5,3)

and the associated optimal cost is

$\Sigma P + \Sigma q = (2+1+3+1+2) + (1+0+0+1+0)$

$= 11$ Ans

Q1 Consider the problem of assigning four operators to four machines. The the assignment costs in dollar are given. operator 1 can not be assigned to machine 3, operator 3 can not be assigned to machine 4.
Find the optimal assignment.

Machines

Operators	1	2	3	4
1	5	5	–	2
2	7	4	2	3
3	9	3	5	–
4	7	2	6	7

Sol

3	3	–	0	$P_1 = 2$
5	2	0	1	$P_2 = 2$
6	0	2	–	$P_3 = 3$
5	0	4	5	$P_4 = 2$

0	3	–	0
2	2	0	1
3	0	2	–
2	0	4	5
$q_1=3$	$q_2=0$	$q_3=0$	$q_4=0$

0	3	—	0
2	2	0	1
3	0	2	—
2	0	4	5

the smallest uncovered element is 2, so the new table is

	1	2	3	4
1	0	5	—	[0]
2	2	4	[0]	1
3	1	[0]	0	—
4	[0]	0	2	3

this table yeilds the optimal assignment which is (1, 4), (2,3), (3,2), (4,1) and the optimal cost is

$$\Sigma p + \Sigma q + 2 = (2+2+3+2) + (3+0+0+0) + 2$$

$$= 14 \$$$

Ans

Q2: Jo Shop needs to assign four jobs it received to 4 workers. the varying skills of the workers give rise to varying costs for performing the jobs. table given below summarizes the cost data of the assignments. the data indicate that worker 1 can not work on job 3, and worker 3 can not work on job 4. Determine the optimal assignment.

	1	2	3	4
1	50	50	—	20
2	70	40	20	30
3	90	30	50	—
4	70	20	60	70

Sol. (Same as in Previous Q)

optimal ~~Assignment~~ Cost

$= \sum p + \sum q + 20 = (20+20+30+20) + (30+0+0+0) + 20$

$= 140$ \$ AC

The Transshipment Model:-

In some transportation problems we consider the possibility of shipping a commodity through intermediate or transient nodes before reaching the final destination. this case is referred to as the transshipment problem.

In the transshipment model, the supply and the demand nodes of the model are then divided into

Pure supply nodes

Pure demand nodes

Transshipment nodes

The amount of supply and demand at these nodes are computed as:

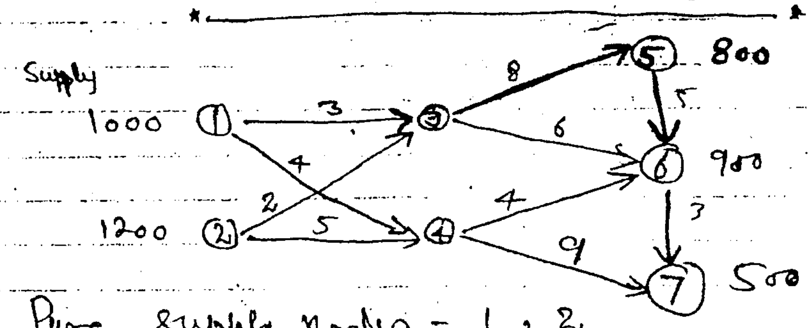
Supply at pure supply node = original supply

Demand at pure demand node = original demand

Supply at a transshipment node = original supply + Buffer

Demand at a transshipment node = original demand + Buffer amount

the buffer amount should be sufficiently large to allow the entire original supply (or demand) units to pass through the transshipment nodes and it is usually equal to total supply (or demand).



Pure supply nodes = 1, 2

" Demand " = 7

Transshipment nodes = 3, 4, 5, 6

Buffer Stock = 2200

the ~~transportation~~ transhipment model now becomes:

Demand nodes

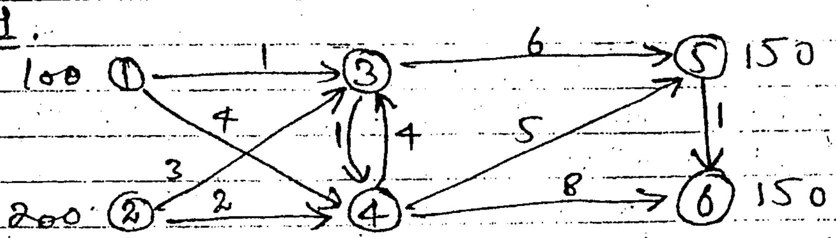
	3	4	5	6	7	
1	3	4	M	M	M	1000
2	2	5	M	M	M	1200
supply nodes 3	0	7	8	6	M	2200
4	M	0	M	4	9	2200
5	M	M	0	5	M	2200
6	M	M	M	0	3	2200
	2200	2200	3000	3100	500	

جہاں route نہیں وہاں M لکھیں یا خالی چھوڑ دیں

supply کو 1,2 Demand Nodes

Problem Set 5.5 (a) (Read from book)

Q:1.



Sol. Pure supply nodes = 1, 2

" demand " = 6

transhipment nodes = 3, 4, 5

Buffer Stock = 100 + 200 = 300

Transshipment model can be converted to a regular transportation model using the idea of a buffer.

the transshipment model is

	3	4	5	6	
1	1	4	M	M	100
2	3	2	M	M	200
3	0	1	6	M	300
4	3	0	5	8	300
5	M	M	0	1	300
	300	300	450	150	

Now find the sol of the model.

First we find the starting solution by using North west Corner method.

	3	4	5	6	
1	(1) 100	(4)	(M)	(M)	100
2	(3) 200	(2)	(M)	(M)	200
3	(0) 0	(1) 300	(6)	(M)	300
4	(3)	(0) 0	(5) 300	(8)	300
5	(M)	(M)	(0) 150	(1) 150	300
	300	300	450	150	

Now we try to improve this sol. by the method of multipliers.

So associate a multiplier to each row and column and find the coefficients of non basic variables.

	0 / $v_1 = 1$	4 / $v_2 = 2$	5 / $v_3 = 7$	6 / $v_4 = 8$	
1 $u_1 = 0$	(1) 100	(4) [-2]	(M) [–]	(M) –	100
2 $u_2 = 2$	(3) 200 ⊖	(2) [2] ⊕	(M) –	(M) –	200
3 $u_3 = -1$	(0) 0 ⊕	(1) 300 ⊖	(6) [0]	(M) –	300
4 $u_4 = -2$	(3) [-4]	(0) 0	(5) 300	(8) [-2]	300
5 $u_5 = -7$	(M) [–]	(M) –	(0) 150	(1) 150	300
	300	300	450	150	

x_{22} enters and x_{23} leaves the sol,

So new table is

	$v_1 = 1$	$v_2 = 2$	$v_3 = 7$	$v_4 = 8$	
$u_1 = 0$	(1) 100	(4) [-2]	(M) –	(M) –	100
$u_2 = 0$	(3) [-2]	(2) 200	(M) –	(M) –	200
$u_3 = -1$	(0) 200	(1) 100	(6) [0]	(M) –	300
$u_4 = -2$	(3) [-4]	(0) 0	(5) 300	(8) [-2]	300
$u_5 = -7$	(M) –	(M) –	(0) 150	(1) 150	300
	300	300	450	150	

Since all the coefficients of non basic variables are non positive. So this table yeilds the optimal solution.

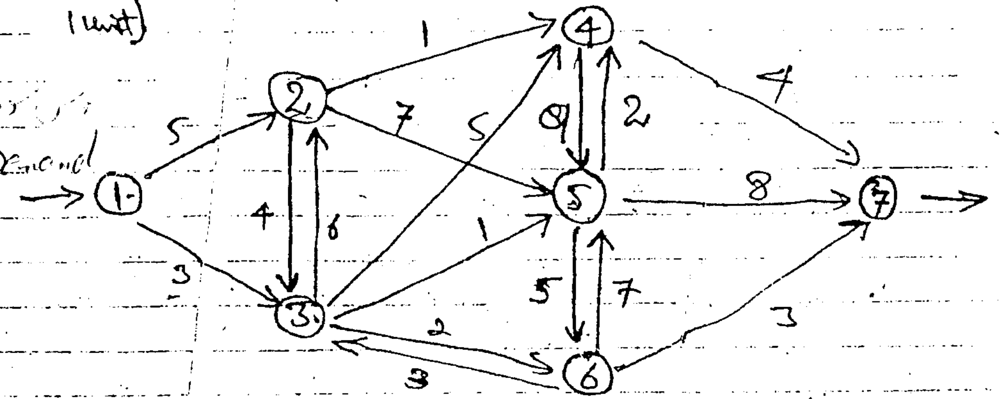
$x_{13} = 100$, $x_{24} = 200$, $x_{33} = 200$, $x_{34} = 100$

, $x_{44} = 0$, $x_{45} = 300$, $x_{55} = 150$, $x_{56} = 150$

and the associated transportation cost is

Cost $= 100 \times 1 + 200 \times 2 + 200 \times 0 + 100 \times 1 + 0 \times 0 + 300 \times 5$
$+ 150 \times 0 + 150 \times 1 = 100 + 400 + 100 + 1500 + 150$
$= 2250$ \$

Q:6 Find the shortest route between nodes 1 and 7 of the given network by formulating the problem as a transshipment model.
the distance between the different nodes are shown on the network.
(Hint: Assume that node 1 has a net supply of 1 unit, and node 7 has a net demand of 1 unit)



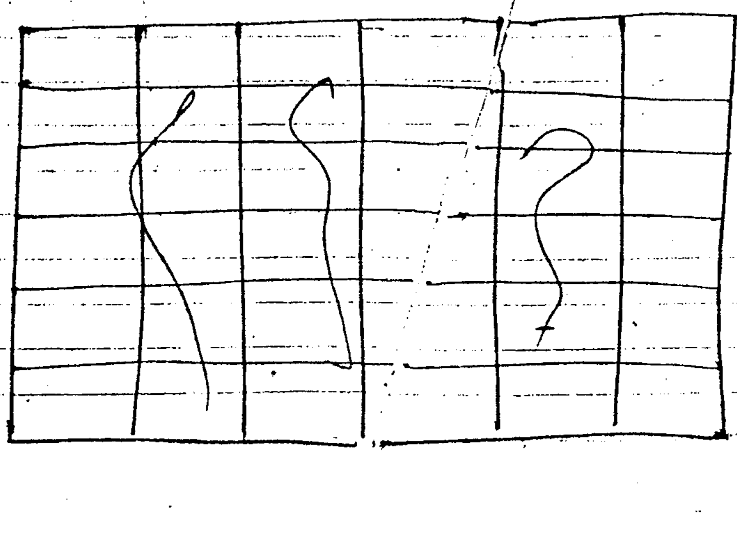
Sol. The given problem can be formulated as a transshipment model as follows:

pure supply nodes = 1

" demand nodes = 7

Transshipment ~~model as follows~~ nodes = 2,3,4,5,6

Since the supply and demand is 1 unit at each node so we can proceed this as an assignment model, as



on next page

	2	3	4	5	6	7
1	5	3	[illegible]	–	–	–
2	0	4	1	7	–	–
3	6	0	5	1	2	–
4	–	–	0	9	–	4
5	–	–	2	0	5	8
6	–	3	–	7	0	3

'identify each row's (column) minimum cost and subtract it from all the entries of the corresp-ding row (column) [illegible] take $p_1 = 3$, $q_7 = 3$ then the new cost matrix is

	2	3	4	5	6	7
1	2	0	–	–	–	–
2	0	4	1	7	–	–
3	6	0	5	1	2	–
4	–	–	0	9	–	1
5	–	–	2	0	5	5
6	–	3	–	7	0	0

$p_1 = 3$

$q_7 = 3$

this sol. is not feasible, for the tie exist b/w R_1 & R_3.

'1' is the smallest uncovered elmt

Now we have

	2	3	4	5	6	7
1	1	0	–	–	–	–
2	0	5	1	7	–	–
3	5	0	4	0	1	–
4	–	–	0	9	–	1
5	–	–	2	0	5	5
6	–	4	–	7	0	0

Again sol. is not feasible, for the tie exists b/w R_2 & R_5. So again cross all the zeroes with minimum no of lines

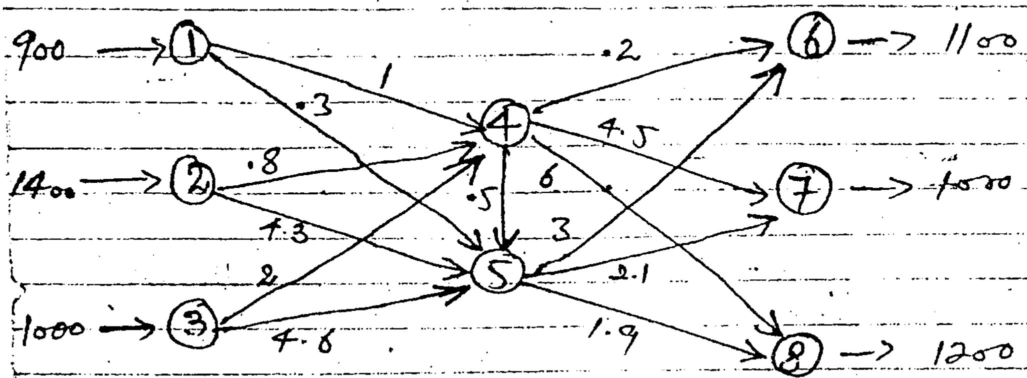
again '1' is the least uncovered entry so the new table is

	2	3	4	5	6	7
1	0	[0]	–	–	–	–
2	[0]	6	1	7	–	–
3	4	0	3	0	[0]	–
4	–	–	[0]	10	–	1
5	–	–	1	[0]	4	4
6	–	5	–	8	0	[0]

So the optimal assignment is (1,3), (2,2) (3,6), (4,4), (5,5), (6,7)

So the shortest route is $1 \rightarrow 3 \rightarrow 6 \rightarrow 7$

Q:3:- the network in fig below shows the routes for shipping cars from three plants (node 1, 2, and 3) to three dealers (nodes 6 to 8) by way of two distribution centres (nodes 4 and 5). The shipping costs per car in ($ 100) are shown on the arcs:



Solve the problem as a transshipment model and find the optimum solution.

Solution:-

pure supply nodes = 1, 2, 3

" demand " = 6, 7, 8

transshipment nodes = 4, 5

Buffer Stock = 3300

the transshipment model is

	4	5	6	7	8	Supply
1	1	.3	—	—	—	900
2	.8	4.3	—	—	—	1400
3	2	4.6	—	—	—	1000
4	0	.5	.2	4.5	6	3300
5	—	0	3	2.1	1.9	3300
Demand	3300	3300	1100	1000	1200	

the Starting Solution: by using northwest Corner method is

	4	5	6	7	8	
1	(1) 900	(.3)	–	–	–	900
2	(.8) 1400	(4.3)	–	–	–	1400
3	(2) 1000	(4.6)	–	–	–	1000
4	(0) 0	(.5) 3300	(.2) 0	(4.5)	(6)	3300
5	–	(0)	(3) 1100	(2.1) 1000	(1.9) 1200	3300
	~~1000~~ ~~2400~~ 3300	3300	1100	1000	1200	

Now we try to improve the sol. by the method of multipliers.

	$V_1 = 1$	$V_2 = 1.5$	$V_3 = 1.2$	$V_4 = 3$	$V_5 = .1$	
$u_1 = 0$	(1) 900	(.3) / 1.2	–	–	–	900
$u_2 = -2$	(.8) 1400	(4.3) / -3	–	–	–	1400
$u_3 = 1$	(2) 1000	(4.6) / -2.1	–	–	–	1000
$u_4 = -1$	(0) 0	(.5) ⊖ 3300	(.2) ⊕ 0	(4.5) / -5.2	(6) / -6.9	3300
$u_5 = 1.8$	–	(0) 33 ⊕	(3) ⊖ 1100	(2.1) 1000	(1.9) 1200	3300
	3300	3300	1100	1000	1200	

x_{55} ~~leav~~ enters and x_{56} leaves the sol.

	$V_1 = 1$	$V_2 = 1.5$	$V_3 = 1.2$	$V_4 = 3.6$	$V_5 = 3.4$
$u_1 = 0$	[1] 900 ⊖	[.3] ⊕ 1.2	—	—	—
$u_2 = -.2$	[.8] 1400	[4.3] -3	—	—	—
$u_3 = 1$	[2] 1000	[4.6] -2.1	—	—	—
$u_4 = -1$	[0] 0 ⊕	[.5] 2200 ⊖	[.2] 1100	[4.5] -1.9	[6] -3.6
$u_5 = -1.5$	—	[0] 1100	[3] -3.3	[2.1] 1000	[1.9] 1200

x_{15} enters and x_{14} leaves the sol.

So the new table is

	$V_1 = -.2$	$V_2 = .3$	$V_3 = 0$	$V_4 = 2.4$	$V_5 = 2.2$
$u_1 = 0$	[1] -1.2	[.3] 900	—	—	—
$u_2 = 1$	[.8] 1400	[4.3] -3.0	—	—	—
$u_3 = 2.2$	[2] 1000	[4.6] -2.1	—	—	—
$u_4 = .2$	[0] 900	[.5] 1300	[.2] 1100	[4.5] -1.9	[6] -3.6
$u_5 = -.3$	—	[0] 1100	[3] -3.3	[2.1] 1000	[1.9] 1200

Since all the coefficients of non basic variables are non positive, so the optimal solution obtained is $x_{15} = 900$, $x_{24} = 1400$, $x_{34} = 1000$, $x_{44} = 900$, $x_{45} = 1300$, $x_{46} = 1100$, $x_{55} = 1100$, $x_{57} = 1000$, $x_{58} = 1200$ and the corresponding optimal transhipment cost

= 8640 \$ Ans.

Network Models:

Network Minimization:-

Network minimization deals with the determination of the branches that can join all the nodes of a network such that the lengths of the choosen branches are minimized. this minimum network is called a minimal spanning tree.

(A typical application occurs in the creation of a network of paved roads that links several rural towns, where the road between two towns may pass through one or more other towns. the most economical design of the roads system calls for minimizing the total miles of paved roads, a result that is achieved by implementing the minimal spanning tree algorithm.)

Minimal Spanning Tree Algorithm.

the procedure for determing a minimal spanning tree is as follows.

begin with any node and join it to its closest node in the network. the resulting two nodes form a "connected set" and the remaining nodes "unconnected set". Next, choose a node from the unconnected set that is closest to any node in the connected set, and add it to the connected set. the process will be repeated untill the connected set includes all the nodes of the network. Note we define

C_K = set of nodes that have been ~~perman~~ permanently connected at iteration K of the algorithm.

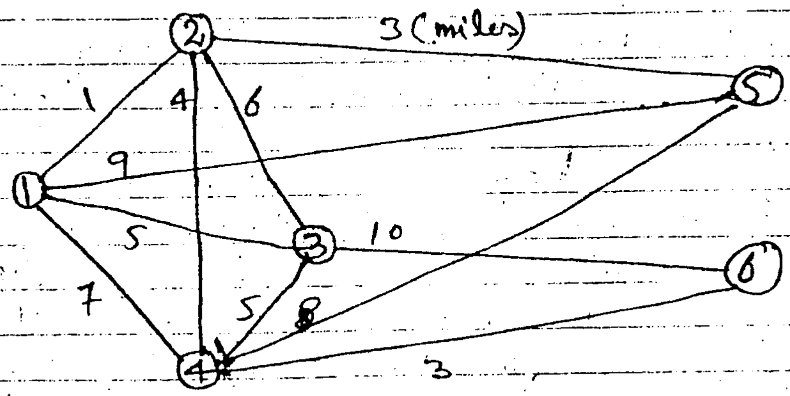
A connected network is such that every two distinct nodes are linked by at least one path.

and

$\bar{C}_K$ = Set of nodes as yet to be Connected permanently i.e. ~~set of yet disconnected nodes~~

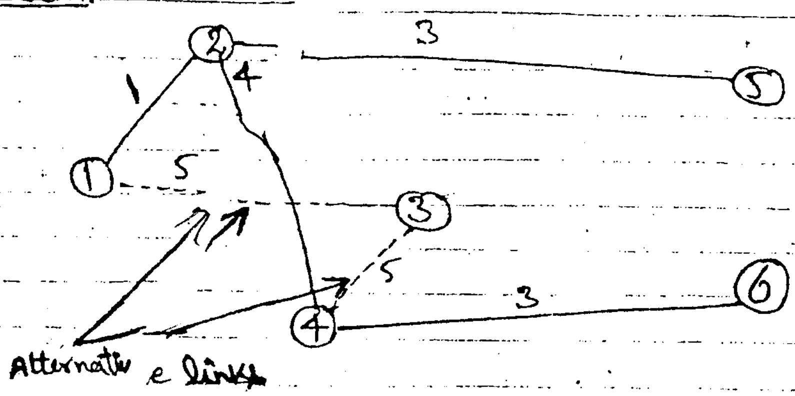
We illustrate the process with the help of an example.

Example:- The midwest TV Cable Company is in the process of providing cable service to five new housing development areas. The cable system network is summarized in the fig. below. The cable miles are shown on each branch. Node 1 represents the TV station and the remaining nodes represent the five new housing areas. A missing branch b/w two nodes implies that it is prohibitively expensive or physically impossible to connect the associated areas.



It is desired to determine the most economical cable network i.e. "the links that will result in the use of minimum cable miles while guaranteeing that all areas are connected (directly or indirectly)" to the cable TV station.

Solution:-



Procedure

I_1: Start from node 1.

$C_1 = \{1\}$, $\bar{C}_1 = \{2, 3, 4, 5, 6\}$

I_2: Connect node 1 to node 2.

$C_2 = \{1, 2\}$, $\bar{C}_2 = \{3, 4, 5, 6\}$

I_3: Connect node 2 to node 5

$C_3 = \{1, 2, 5\}$, $\bar{C}_3 = \{3, 4, 6\}$

I_4: Connect node 2 to node 4.

$C_4 = \{1, 2, 4, 5\}$, $\bar{C}_4 = \{3, 6\}$

I_5: Connect node 4 to node 6

$C_5 = \{1, 2, 4, 5, 6\}$, $\bar{C}_5 = \{3\}$

I_6: Connect node 4 to 3 or 1 to 3

$C_6 = \{1, 2, 3, 4, 5, 6\}$, $\bar{C}_6 = \phi$

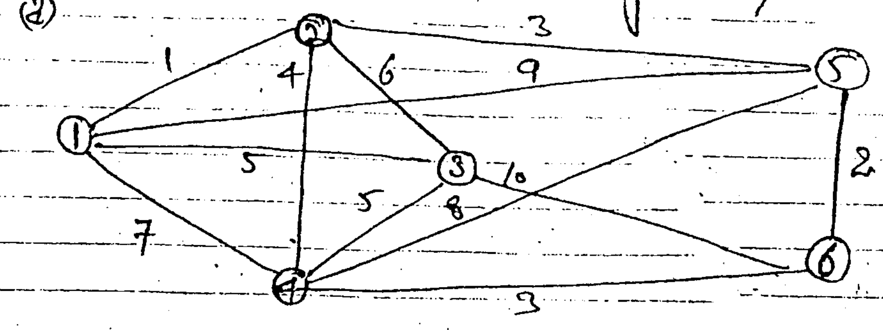
So Answer is (1 — 2), (2 — 5), (2 — 4), (4 — 6), (1 — 3) or (4 — 3) and the total milage of the spanning tree is

$1 + 3 + 4 + 5 + 3 = 16$ (miles)

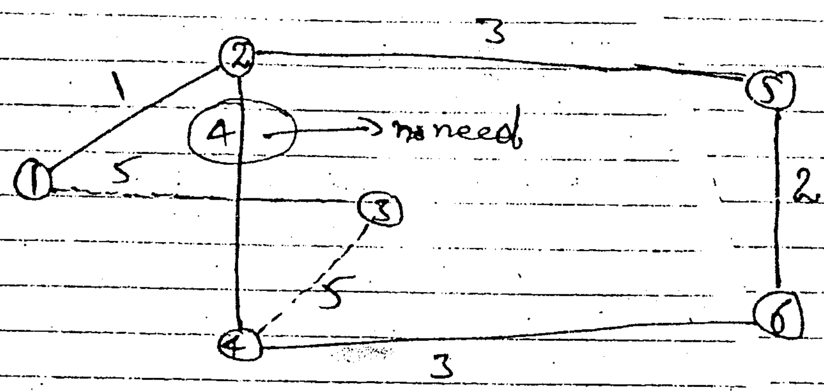
Ans

Ex. Find the minimal spanning tree of the network in the following.

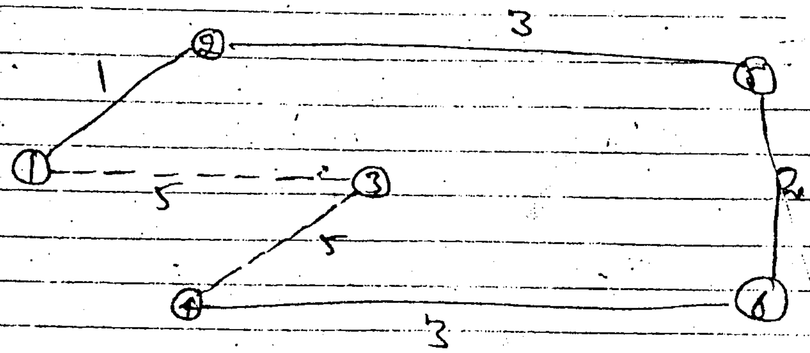
(a)



Solution:



So



I_1: Start from node 1.

$C_1=\{1\}$, $\bar{C}_1=\{2,3,4,5,6\}$

I_2: Connect node 1 to node 2

$C_2=\{1,2\}$, $\bar{C}_2=\{3,4,5,6\}$

I_3: Connect node 2 to node 5

$C_3=\{1,2,5\}$, $\bar{C}_3=\{3,4,6\}$

I_4: Connect node 5 to node 6

$C_4 = \{1, 2, 5, 6\}$, $\bar{C}_4 = \{3, 4\}$

I_5: Connect node 6 to node 4

$C_5 = \{1, 2, 4, 5, 6\}$, $\bar{C}_5 = \{3\}$

I_6: Connect node 3 to node 1 or 4

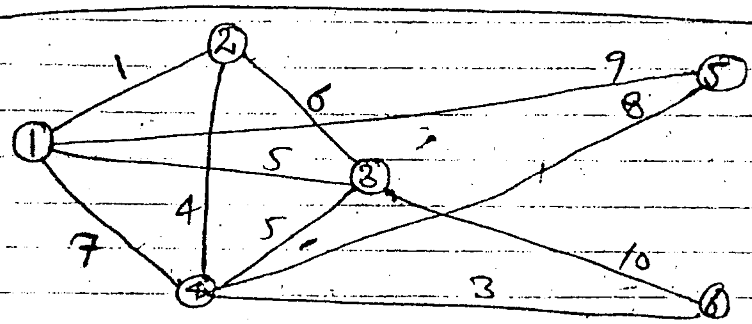
$C_6 = \{1, 2, 3, 4, 5, 6\}$, $\bar{C}_6 = \phi$

Answer is

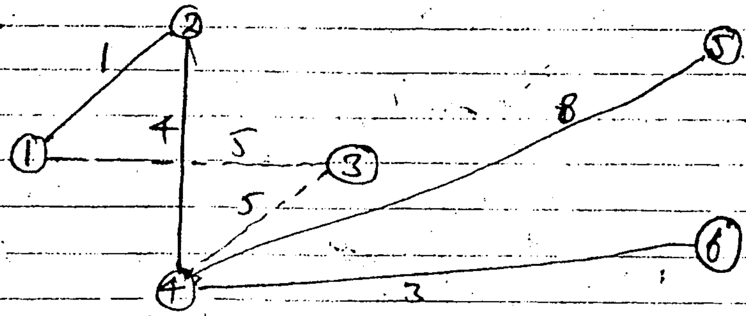
(1 — 2) , (2 — 5) , (5 — 6), (6 — 4) , (4 — 3) or (1 — 3)
and the total milage of the spanning tree is

1+3+2+3+5 = 14

(b)

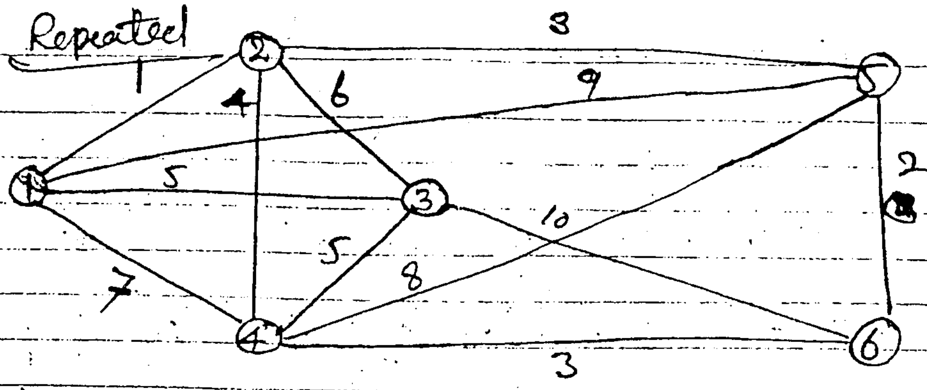


Sol:

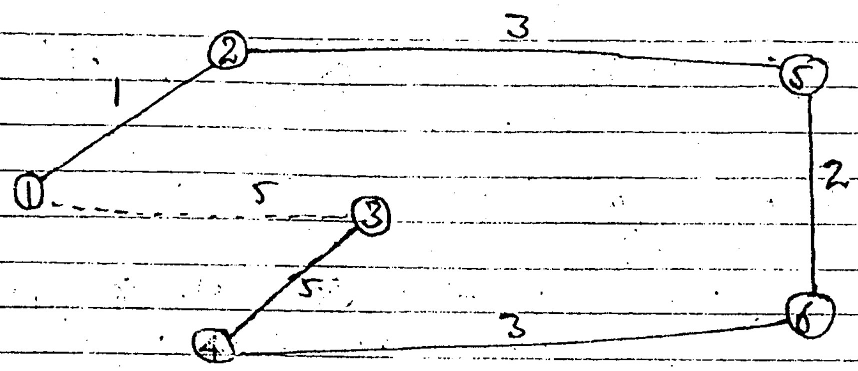


the procedure is same as before.
Connect (1,2) (2,4) , (4,6) ,(4,5) and finally (1,3) or (4,3).

(c) Repeated

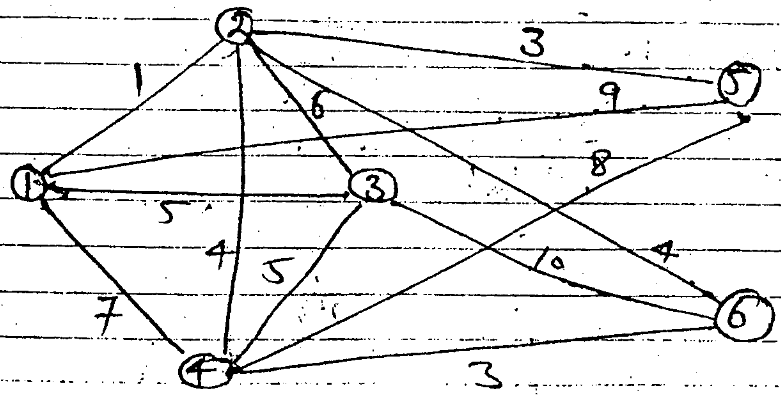


Solution:-

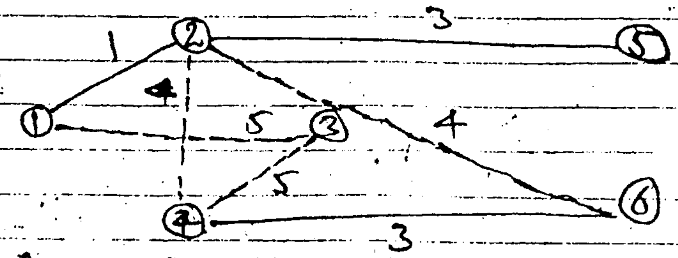


Connect 1 to 2, 2 to 5, 5 to 6,
6 to 4, 4 to 3 or 1 to 3.

(c)

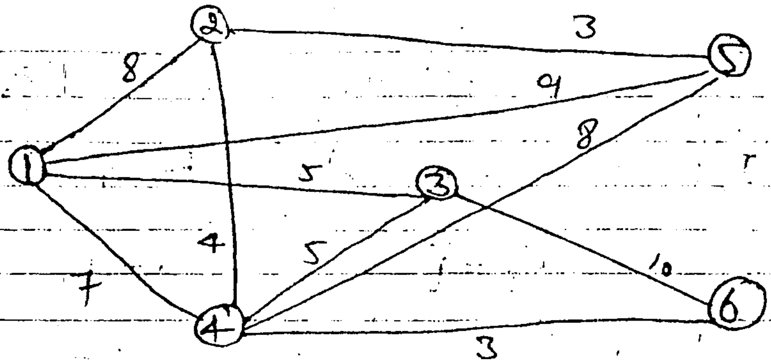


Sol

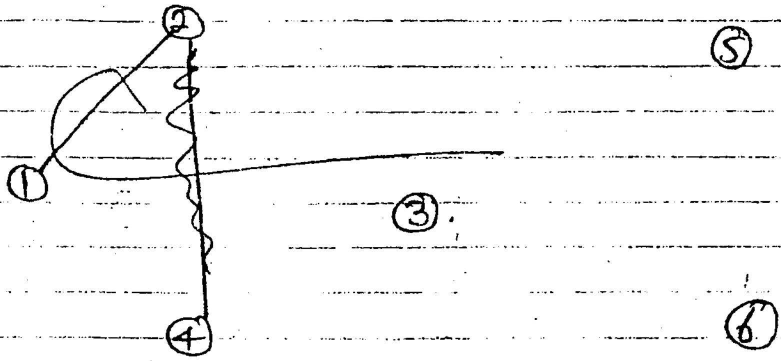


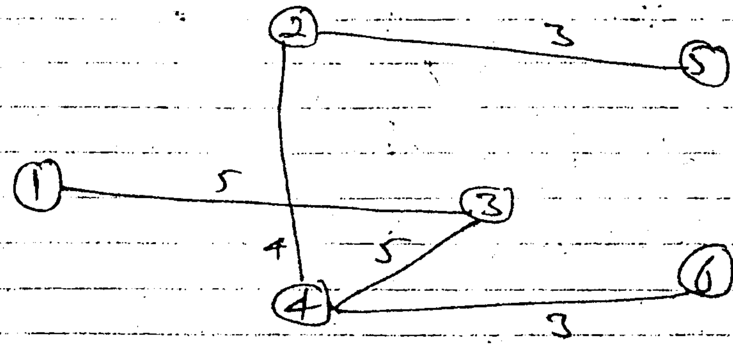
total mileage of spanning tree is 1+3+4+5+3 = 16

(d)



Sol





I_1: Start from node 1.

$C_1 = \{1\}$, $\bar{C}_1 = \{2, 3, 4, 5, 6\}$

I_2: Connect node 1 to node 3

$C_2 = \{1, 3\}$, $\bar{C}_2 = \{2, 4, 5, 6\}$

I_3: Connect node 3 to node 4.

$C_3 = \{1, 3, 4\}$, $\bar{C}_3 = \{2, 5, 6\}$

I_4: Connect node 4 to node 6

$C_4 = \{1, 3, 4, 6\}$, $\bar{C}_4 = \{2, 5\}$

I_5: Connect node 4 to node 2.

$C_5 = \{1, 2, 3, 4, 6\}$ $\bar{C}_5 = \{5\}$

I_6: Connect node 2 to node 5.

$C_6 = \{1, 2, 3, 4, 5, 6\}$, $\bar{C}_6 = \phi$

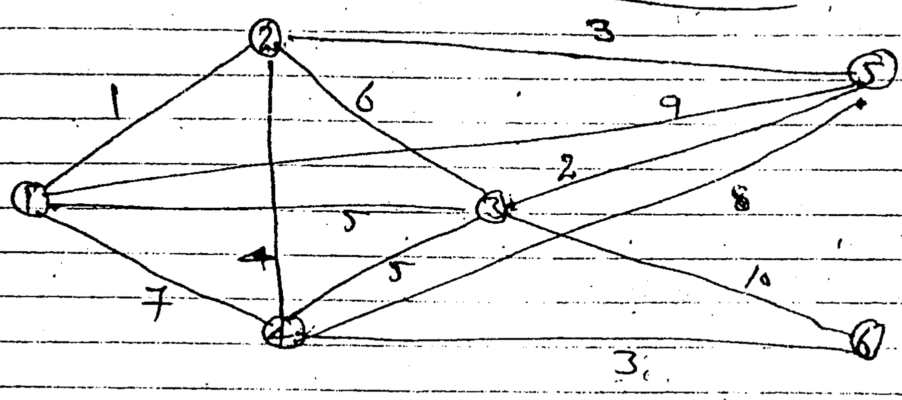
Ans is

(1 − 3), (3 − 4), (4 − 6), (4 − 2), (2 − 5)

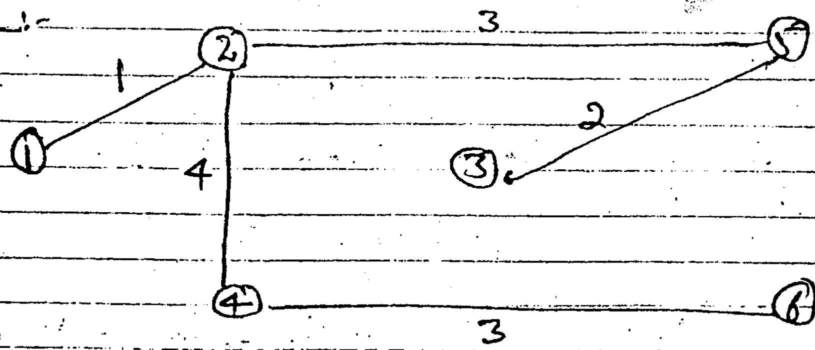
the total mileage of the spanning tree is

$5 + 5 + 3 + 4 + 3 = 20$

(e)



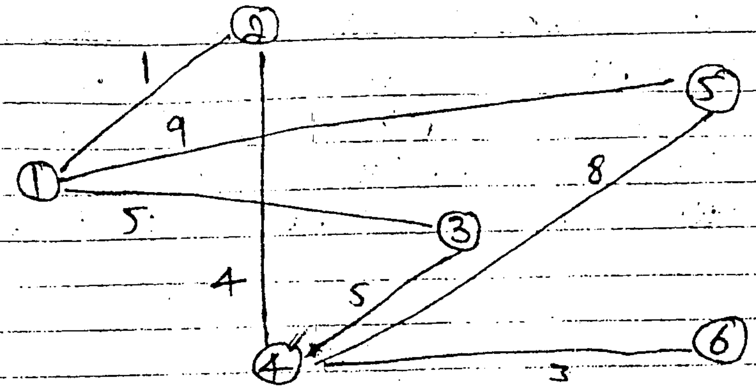
Solution :-



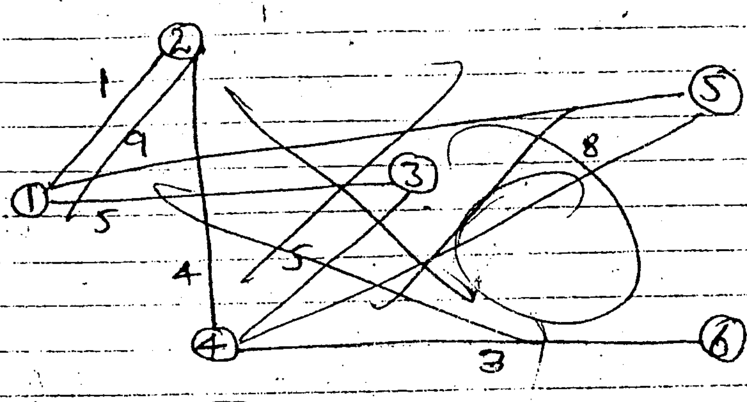
Connect (1, 2), (2, 5), (5, 3), (2, 4) and (4, 6)

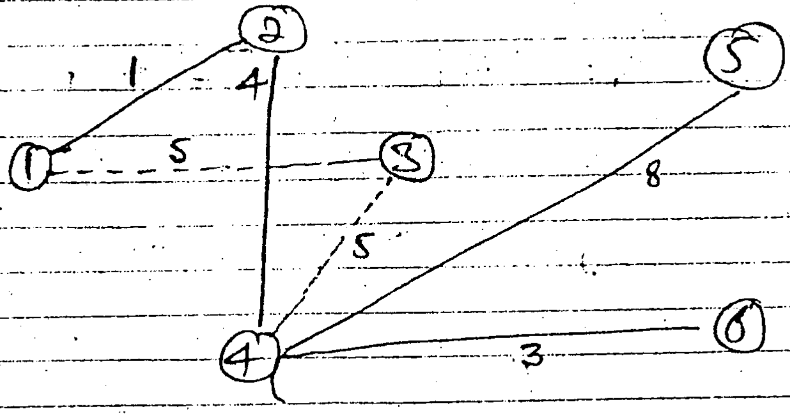
and the total mileage is 13.

(f)



Sol.

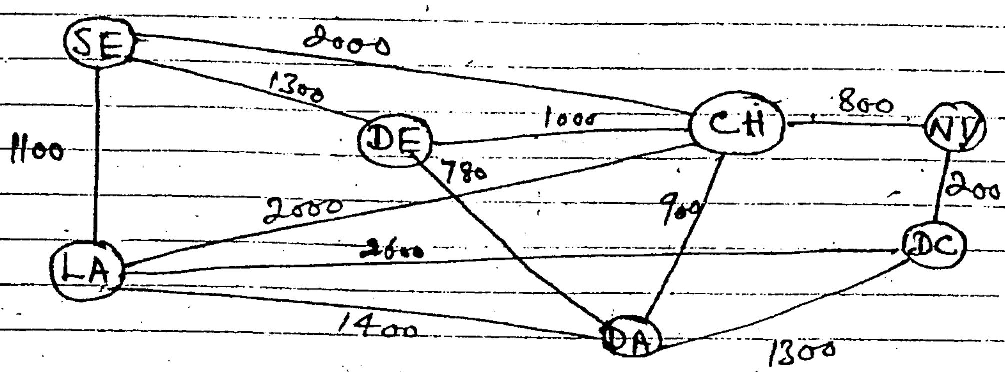




Connect 1 to 2, 2 to 4, 4 to 6, 4, 5
4 to 3 or 1 to 3

and the total mileage of the spanning tree
is 1+4+5+3+8 = 21

Ex. Suppose that it is desired to establish a cable communication network that links the major cities shown in the figure. Determine how the cities are connected such that the total used cable milage is minimized.



Solution:-

Iteration 1: Start from LA.

$C_1 = \{LA\}$, $\bar{C}_1 = \{SE, DE, DA, CH, NY, DC\}$

Iteration 2: Connect LA to SE.

$C_2 = \{LA, SE\}$, $\bar{C}_2 = \{DE, DA, CH, NY, DC\}$

Iteration 3: Connect SE to DE.

$C_3 = \{LA, SE, DE\}$, $\bar{C}_3 = \{DA, CH, NY, DC\}$

Iteration 4: Connect DE to DA

$C_4 = \{LA, SE, DE, DA\}$, $\bar{C}_4 = \{CH, NY, DC\}$

Iteration 5: Connect DA to CH

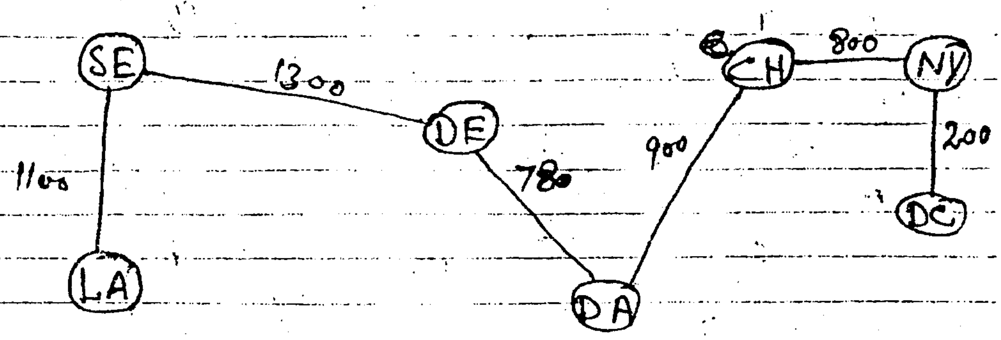
$C_5 = \{LA, SE, DE, DA, CH\}$, $\bar{C}_5 = \{NY, DC\}$

Iteration 6: Connect CH to NY

$C_6 = \{LA, SE, DE, DA, CH, NY\}, \quad \bar{C}_6 = \{DC\}$

Iteration 7: Conect NY to DC

$C_7 = \{LA, SE, DE, DA, CH, NY, DC\}, \quad \bar{C}_7 = \phi$



So the answer is (LA — SE), (SE — DE), (DE — DA), (DA — CH), (CH — NY), (NY — DC)

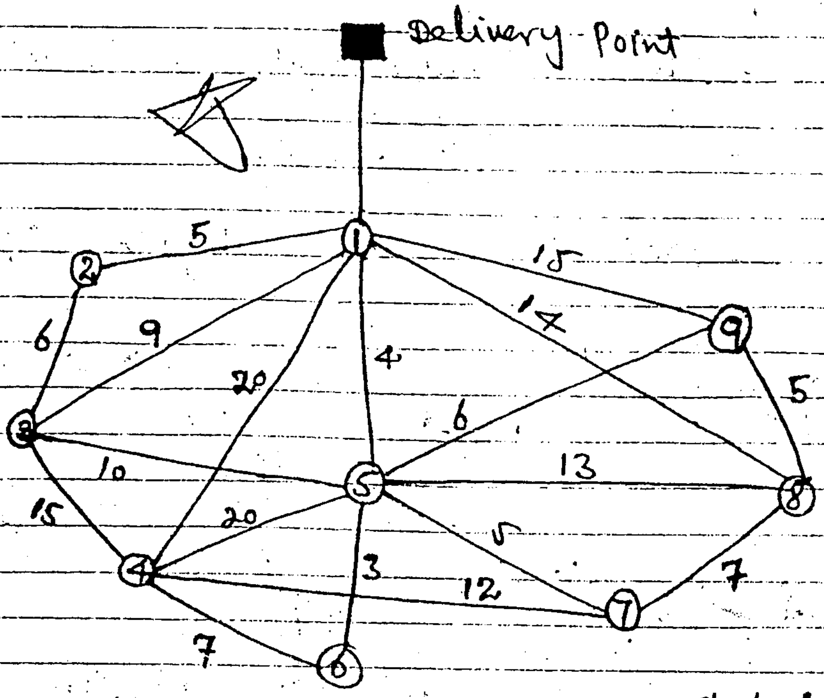
and the total milage of the spanning tree is

$1100 + 1300 + 780 + 900 + 800 + 200 = 5080$

Note:- for all such questions the procedure should be write in this way.

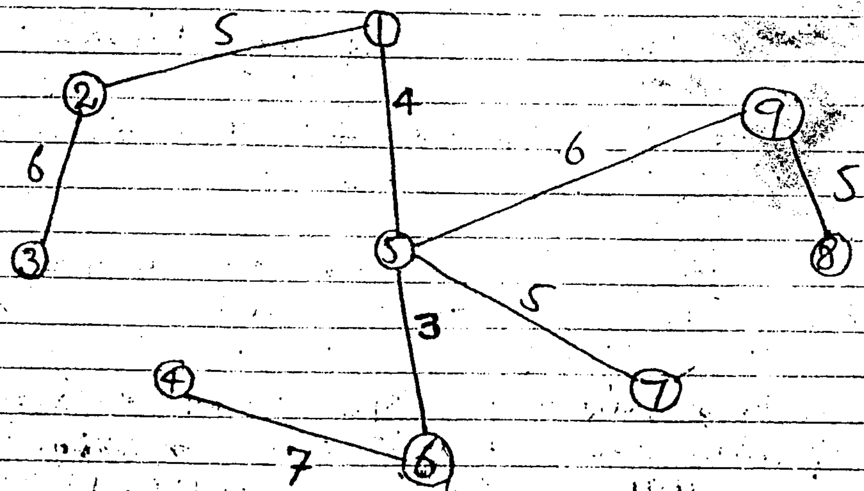
Q. 4. the following figure shows the milage of the feasible links connecting nine offshore natural gas wellheads with an inshore delievery point. Since the location of well head 1 is the closest to shore; it is equipped with sufficiant pumping and storage capacity to pump the out-put of the remaining eight wells to the delievery point.
Determine the minimum pipe line network

that links the well heads to the delivery point



Sol

start from 1



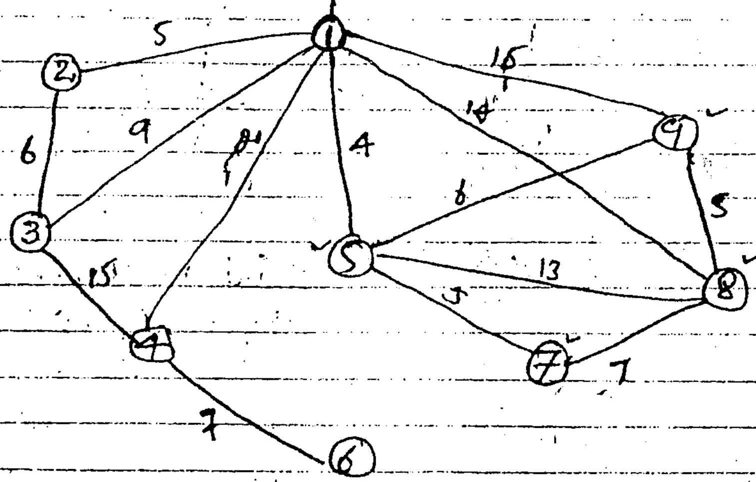
Connect (1,5), (5,6), (1,2), (5,7), (2,3), (5,9)
(9,8), (6,4),

and the total mileage is 41

Q.5:- In the previous problem '4' suppose that the well heads can be divided into two groups depending on gas pressure: a high pressure group that includes wells 2,3,4 and 6, and a low pressure group that includes wells 5,7,8, and 9. Because of the pressure difference, it is not possible to link the wellheads of the two groups. At the same time, both groups must be connected to the delivery point through well head 1. Determine the minimum pipline network for this situation.

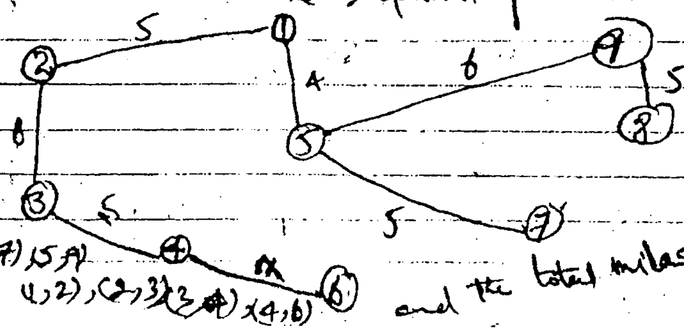
Sol.

Bearing in mind that the members of high pressure group (including wells 2,3,4 and 6) can not be linked to



any members of the low pressure group (including wells 5,7, 8&9), we reconstruct the fig. as above:

Thus the minimum spanning tree is



Connect (1,5), (5,7), (5,9) (9,8), (1,2), (2,3), (3,4), (4,6) and the total milage = 53

Acyclic Networks:-

A net work is Called to be acyclic if there are no chains Connecting a node to itself

Shortest - Route Algorith For acyclic Network

We Shall use the following notations

d_{ij} = distance b/w adjacent nodes i and j.

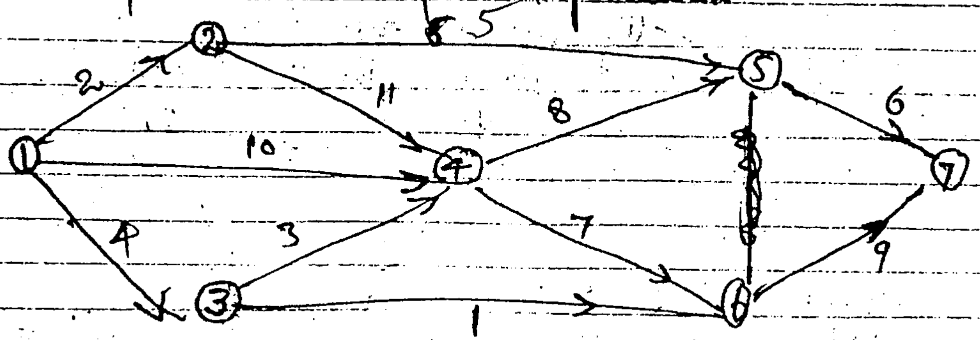
U_j = Shortest distance from node 1 to node j.

the formula for Computing the U_j is

$U_j = \min_i$ { distance b/w node j and imediately proceeding node i + Shortest distance from node 1 to node i }

$= \min_i \{ d_{ij} + U_i \}$; i is a node imediately proceeding node j.

Ex: Find the shortest route from node 1 to node 7 in the following network.



Sol. We are to calculate U_7

Stage 1: $U_1 = 0$

Stage 2: $U_2 = d_{12} + U_1$

$U_2 = 2 + 0 = 2$

$U_3 = d_{13} + U_1 = 4 + 0 = 4$

Stage 3: $U_4 = \min\{d_{14} + U_1,\ d_{24} + U_2,\ d_{34} + U_3\}$

$= \min\{10 + 0,\ 11 + 2,\ 3 + 4\}$

$= \min\{10,\ 13,\ 7\}$

$U_4 = 7$

Stage 4:

$U_5 = \min\{d_{25} + U_2,\ d_{45} + U_4\}$

$= \min\{5 + 2,\ 8 + 7\}$

$= \min\{7,\ 15\}$

$U_5 = 7$

Stage 5:

$U_6 = \min\{d_{46} + U_4,\ d_{36} + U_3\}$

$= \min\{7 + 7,\ 1 + 4\}$

$U_6 = 5$

Stage 6: $U_7 = \min\{d_{67} + U_6,\ d_{57} + U_5\}$

$= \min\{9 + 5,\ 6 + 7\} = 13$

Thus, the min. distance from node 1 to node 7 is 13 units and follows the route 1→2→5→7

Example:-

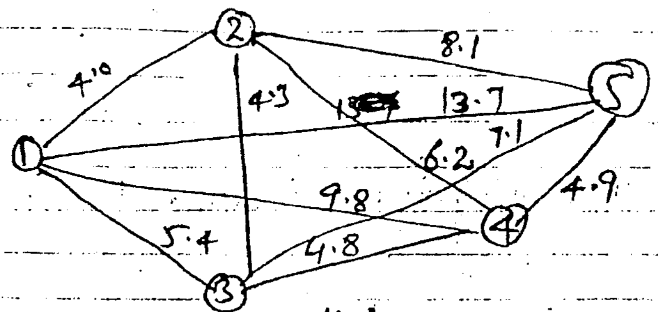
A Car renting Company is developing a replacement plan for its car fleet for a 5 year (1996 to 2000) planning horizon.

A Car must be in Service at least 1 Year but must be replaced after three Years. the following table summarizes the replacement Cost per Car (in the thousand of dollars) as a function of time and the number of Years in operation. the Cost includes purchasing, salvage operating and maintenance.

	1	2	3	4	5
1		4.0	5.4	9.8	13.7
2			4.3	6.2	8.1
3				4.8	7.1
4					4.9

Sol. Each year is represented by a node. the length of an arc joining two nodes equals the associated replacement Cost. the problem reduces to determine the Shortest route from node 1 to node 5 i.e. we are to Compute U_5.

the given data Can be expressed as follows network.



$u_5 = ?$ For this we proceeds as follows.

~~Step 1~~

Stage 1: $u_1 = 0$

Stage 2: $u_2 = d_{12} + u_1 = 4.0 + 0$

$u_2 = 4.0$

~~$u_3 = d_{13} + u_1 = 5.4 + 0$~~

~~$u_3 = 5.4$~~

Stage 3: $u_3 = \min\{d_{23} + u_2,\ d_{13} + u_1\}$

$= \min\{4.3 + 4.0,\ 5.4 + 0\}$

$= \min\{8.3,\ 5.4\}$

$u_3 = 5.4$

Stage 4: $u_4 = \min\{d_{34} + u_3,\ d_{24} + u_2,\ d_{14} + u_1\}$

$= \min\{4.8 + 5.4,\ 6.2 + 4.0,\ 9.8 + 0\}$

$= \min\{10.2,\ 10.2,\ 9.8\}$

$u_4 = 9.8$

Q: (Solution of transhipment model is necessary or not to do)

a: 5.4a Q: 3 & 4 , 5.3b Q: 4

Stage 5: $u_5 = \min\{d_{15}+u_1, d_{25}+u_2, d_{35}+u_3, d_{45}+u_4\}$

$= \min\{13.7+0, 8.1+4.0, 7.1+5.4, 4.9+9.8\}$

$= \min\{13.7, 12.1, 12.5, 14.7\}$

$u_5 = 12.1$

thus, the shortest route is 12.1 and and follows the route $1 \to 2 \to 5$.

this reveals that each car should be replaced in year 2 and discarded in year 5.

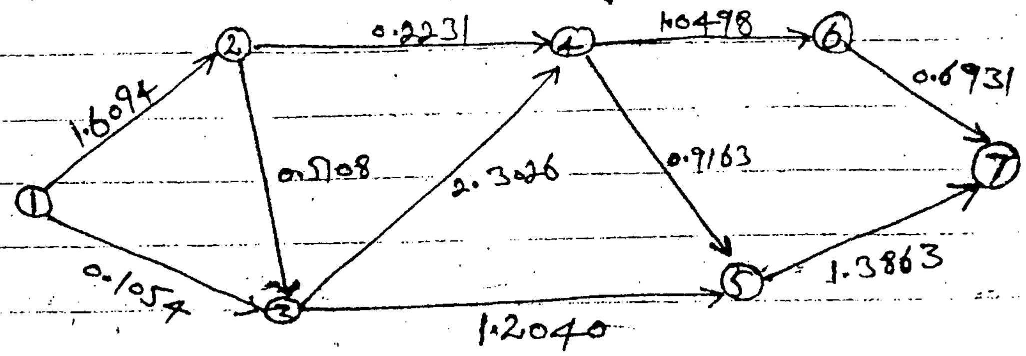
Note

In solution of transportation model check

(i) supply = demand

(ii) no. of variables m+n-1

Ex. Find the shortest route from node 1 to node 7 in the following network.



Solution:-

Stage 1: $u_1 = 0$

$u_2 = d_{12} + u_1 = 1.6094 + 0$

$\Rightarrow u_2 = 1.6094$

Stage 2: $u_3 = \min\{d_{13} + u_1, d_{23} + u_2\}$

$= \min\{0.1054 + 0, 0.5108 + 1.6094\}$

$= \min\{0.1054, 2.1202\}$

$u_3 = 0.1054$

Stage 3: $u_4 = \min\{d_{24} + u_2, d_{34} + u_3\}$

$= \min\{0.2231 + 1.6094, 2.3026 + 0.1054\}$

$= \min\{1.8325, 2.4080\}$

$u_4 = 1.8325$

Stage 4: $u_5 = \min\{d_{35} + u_3, d_{45} + u_4\}$

$= \min\{1.2040 + 0.1054, 0.9163 + 1.8325\}$

$u_5 = 1.3094$

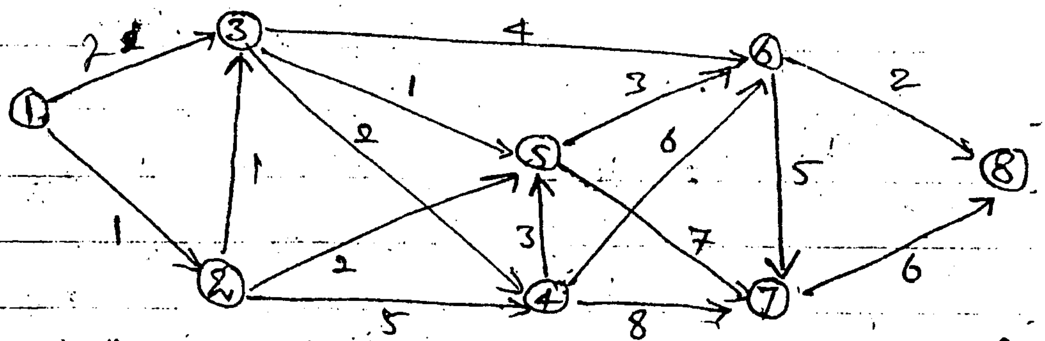
~~$U_{6,7} = \min\{$~~

Stage 5:
$$U_6 = d_{46} + U_4$$
$$U_6 = 1.0498 + 1.8325$$
$$U_6 = 2.8823$$

Stage 6:
$$U_7 = \min\{d_{57} + U_5,\ d_{67} + U_6\}$$
$$= \min\{1.3863 + 1.3094,\ 0.6931 + 2.8823\}$$
$$U_7 = 2.6957$$

Hence the shortest route is 2.6957 and the route is $1 \to 3 \to 5 \to 7$

Ex. the network in the following figure represents the distances in miles between various cities i, $i = 1, 2, \dots, 8$



Find the shortest routes between the following pairs of cities. (a) City 1 and City 8.

Solution:

Stage 1: $U_1 = 0$

Stage 2: $U_2 = d_{12} + U_1 = 1 + 0 = 1$
$$U_2 = 1$$

Stage 3: $u_3 = \min\{d_{13}+u_1,\ d_{23}+u_2\}$

$= \min\{2+0,\ 1+1\}$

$u_3 = 2$

Stage 4: $u_4 = \min\{d_{24}+u_2,\ d_{34}+u_3\}$

$= \min\{5+1,\ 2+2\}$

$u_4 = 4$

Stage 5: $u_5 = \min\{d_{35}+u_3,\ d_{25}+u_2,\ d_{45}+u_4\}$

$= \min\{1+2,\ 2+1,\ 3+4\}$

$= \min\{3, 3, 7\}$

$u_5 = 3$

Stage 6: $u_6 = \min\{d_{36}+u_3,\ d_{56}+u_5,\ d_{46}+u_4\}$

$= \min\{4+2,\ 3+3,\ 6+4\}$

$= \min\{6, 6, 10\}$

$u_6 = 6$

Stage 7: $u_7 = \min\{d_{47}+u_4,\ d_{57}+u_5,\ d_{67}+u_6\}$

$u_7 = \min\{8+4,\ 7+3,\ 5+6\}$ 1,2

$= \min\{12, 10, 11\}$ 3,2 / 3,5

$u_7 = 10$ 6

Stage 8: $u_8 = \min\{d_{68}+u_6,\ d_{78}+u_7\}$

$= \min\{2+6,\ 6+10\}$.

$u_8 = 8$

~~Thus the net mileage is 8 and the shortest~~ routes are

$1 \to 2 \to 5 \to 6 \to 8$

$1 \to 3 \to 5 \to 6 \to 8$

$1 \to 2 \to 3 \to 6 \to 8$ [illegible]

$1 \to 3 \to 6 \to 8$ [illegible]

$1 \to 2 \to 5 \to 6 \to 8$

(b) City 1 and city 6

Solution:- Same as in case (a) but only upto node 6 i.e., $U_6 = 6$ is the ans and thus the shortest routes are

$1 \to 2 \to 5 \to 6$

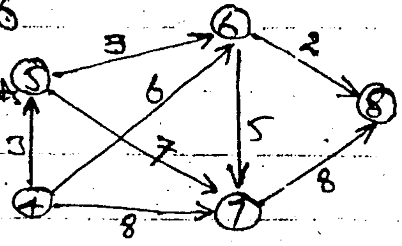
$1 \to 3 \to 5 \to 6$

$1 \to 2 \to 3 \to 6$

$1 \to 3 \to 6$

$1 \to 2 \to 5 \to 6$

(c) City 4 and City 8



Solution:-

Stage 1:- $U'_4 = 0$ → Here U'_4 represents the distance of 4 to 4.

Stage 2: $U'_5 = d'_{45} + U'_4$

$= 3 + 0$

$U'_5 = 3$

Stage 3: $U'_6 = \min\{ d'_{56} + U'_5 ,\ d'_{46} + U'_4 \}$

$= \min\{ 3+3 ,\ 6+0 \}$

$U'_6 = 6$

~~Stage 4: $U'_7 = \min\{d'_{47} + U'_4, d_{57} + U'_5\}$~~

$U'_7 = \min\{8+0, 7+3\}$

$U'_7 = 8$

54 — 6 ⟷ 8

Stage 5:

$U'_8 = \min\{d'_{68} + U'_6, d'_{78} + U'_7\}$

$U'_8 = \min\{2+6, 8+8\}$

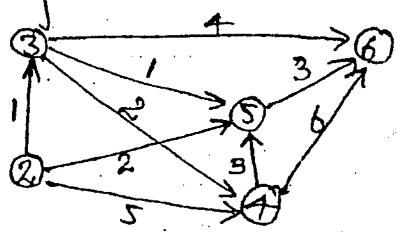
$U'_8 = 8$

Hence the net mileage is 8 and the Shortest routes are 4 → 5 → 6 → 8

4 → 6 → 8

(d) City 2 and city 6.

Sol:-



Stage 1: $U''_2 = 0$

where U''_2 represent the distance of node 2 from itself.

Stage 2: $U''_3 = $ ~~min~~ $U''_2 + d_{12}$

$= 1+0$

$U''_3 = 1$

Stage 3: $U''_4 = \min\{d_{24} + U''_2, d_{34} + U''_3\}$

$= \min\{5+0, 2+1\}$

$U''_4 = 3$

Stage 4: $U''_5 = \min\{d_{25} + U''_2, d_{35} + U''_3, d_{45} + U''_4\}$

$U''_5 = \min\{2+0, 1+1, 3+3\} = 2$

~~$U_6'' = \min\{ d_{36} + U_3'', d_{46} + U_4'', d_{56} + U_5'' \}$~~

$= \min\{4 + 1, 6+3, 3+2\}$

$U_6'' = 5$

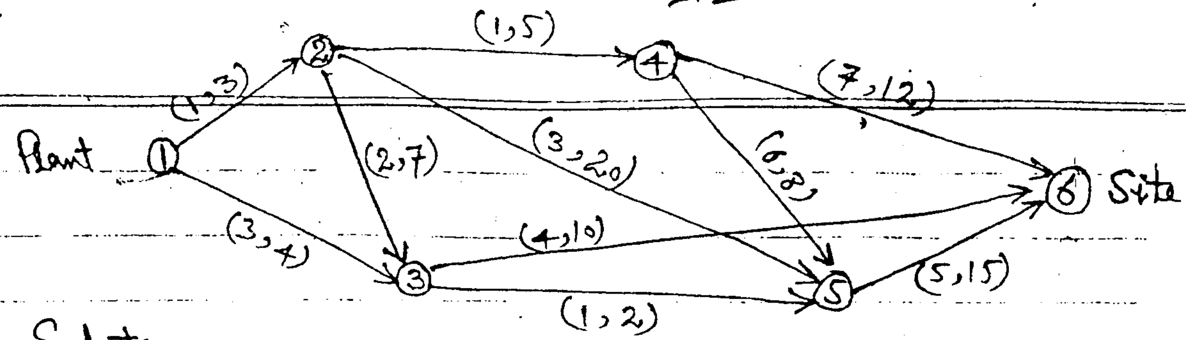
Hence the net milage is 5 and the Shortest ~~routes~~ are

$2 \longrightarrow 3 \longrightarrow 6$

$2 \longrightarrow 3 \longrightarrow 5 \longrightarrow 6$

$2 \longrightarrow 5 \longrightarrow 6$

Ex. A truck must deliver Concrete from a ready-mix plant to a Construction site. the network in the following figure represents the available routes between the plants and site. Each route is designated with two pieces of information (d, t), where d is the length of the route and t is the time the trucks takes to cross the road segment. the speed of the truck ~~is~~ on each segment is decided by the Condition of the road as well as the number and durations of the stop lights. What is the best route from plant to site?



Solutions-

Firstly, we compute the minimum distance between the plant and the site as below

Stage 1: $U_1 = 0$;

Stage 2: $U_2 = d_{12} + U_1 = 1 + 0 = 1$;

Stage 3: $U_3 = \min\{d_{13} + U_1, \ d_{23} + U_2\}$

$= \min\{3 + 0, \ 2 + 1\}$

$U_3 = 3$

Stage 4: $U_4 = d_{24} + U_2 = 1 + 1 = 2$;

Stage 5: $U_5 = \min\{d_{35} + U_3, \ d_{25} + U_2, \ d_{45} + U_4\}$

$= \min\{1 + 3, \ 3 + 1, \ 6 + 2\}$

$U_5 = 4$

Stage 6:- $U_6 = \min\{d_{56} + U_5, \ d_{36} + U_3, \ d_{46} + U_4\}$

$= \min\{5 + 4, \ 4 + 3, \ 7 + 2\}$

$U_6 = 7$

thus, the minimum distance (=7) is followed by the routes.

$1 \to 2 \to 3 \to 6$

$1 \to 3 \to 6$

Now, we calculate the minimum time as follows.

Stage 1: $\bar{U}_1 = 0$

~~Stage 2:- $\bar{u}_2 = t_{12} + \bar{u}_1 = 3 + 0 = 3$;~~

Stage 3:- $\bar{u}_3 = \min\{t_{13} + \bar{u}_1, \; t_{23} + \bar{u}_2\}$

$\bar{u}_3 = \min\{4 + 0, \; 7 + 3\}$

$\bar{u}_3 = 4$

Stage 4: $\bar{u}_4 = t_{24} + \bar{u}_2 = 5 + 3 = 8$;

Stage 5:-

$\bar{u}_5 = \min\{t_{35} + \bar{u}_3, \; t_{25} + \bar{u}_2, \; t_{45} + \bar{u}_4\}$

$= \min\{2 + 4, \; 20 + 3, \; 8 + 8\}$

$\bar{u}_5 = 6$;

Stage 6:-

$\bar{u}_6 = \min\{t_{46} + \bar{u}_4, \; t_{56} + \bar{u}_5, \; t_{36} + \bar{u}_3\}$

$\bar{u}_6 = \min\{12 + 8, \; 15 + 6, \; 10 + 4\}$

$\bar{u}_6 = 14$

Hence, the minimum time = 14 units followed by the route

$1 \longrightarrow 3 \longrightarrow 6$

Since the route Common to both d and t is $1 \longrightarrow 3 \longrightarrow 6$, which must be followed to avoid over-travelling and squandering of time

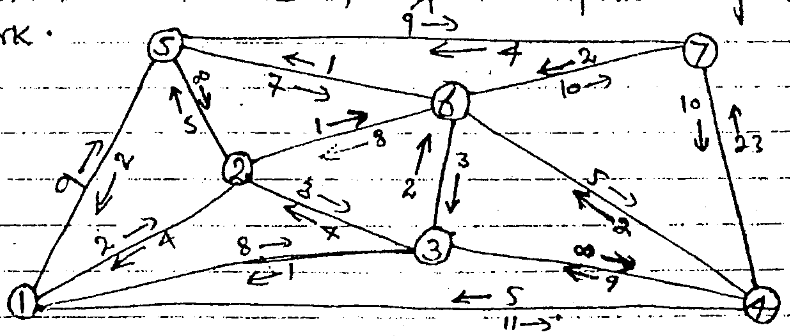
ایسا سوال دیکھئے صفحہ (184) نمبر 1/2 دیکھا جائے۔

The Shortest-Route Problem viewed as a Transshipment Model:-

The shortest-route network can be formulated as a transshipment model with one source and one destination. The supply at the source is one and the demand at destination is also one unit. The one unit will flow from the source to the destination through the admissible routes. The problem is formulated as a transshipment model in which every node of the network except the source node and the demand node act as a source and destination. The objective is to minimize the distance travelled from the source to the destination.

With the help of an example, we illustrate the method.

Example: Determine the shortest distance between the nodes 1 and 7 of the following (cyclic) network.



Solution:-

The following is the transportation model associated with the given shortest-route problem. Since supply at each source and demand at each destination is one unit, technique of assignment problem is most commensurate to attack the problem.

	2	3	4	5	6	7
1	2	8	11	9	x	x
2	0	3	x	5	1	x
3	4	0	x	x	2	x
4	x	9	0	x	2	23
5	x	x	x	0	7	9
6	8	3	5	1	0	10

$P_1 = 2$

where $x >> 0$ so that $x \to +\infty$

	2	3	4	5	6	7
1	0	6	9	7	x	x
2	0	3	x	5	1	x
3	4	0	x	x	2	x
4	x	9	0	x	2	23
5	x	x	x	0	7	9
6	8	3	5	1	0	10

$P_1 = 2$

x میں سے کسی کو بھی آگے لکھنے کی ضرورت نہیں

$q_7 = 9$

	2	3	4	5	6	7
1	0	6	9	7	x	x
2	0	3	x	5	1	x
3	4	0	x	x	2	x
4	x	9	0	x	2	14
5	x	x	x	0	7	0
6	8	3	5	1	0	1

Since a feasible assignment to the zero elements is not possible, so we draw a minimum no. of lines to cross out all zeroes. the smallest uncrossed element is '1' in this case, subtract it from every uncrossed element and add to every element at the intersection of two lines. the new matrix thus obtained is

	2	3	4	5	6	7
1	0	5	8	6	x	x
2	0	2	x	4	0	x
3	5	0	x	x	2	x
4	x	9	0	x	2	14
5	x	x	x	0	7	0
6	0	3	5	1	0	1

A feasible assignment to zero element is still not possible. We, therefore, repeat the above process and have the following cost matrix

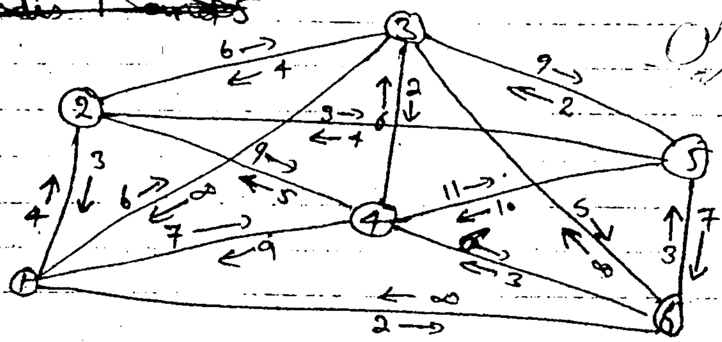
	2	3	4	5	6	7
1	[0]	4	7	5	x	x
2	0	1	x	3	[0]	x
3	6	[0]	x	x	3	x
4	x	9	[0]	x	3	14
5	x	x	x	[0]	8	[0]
6	0	2	4	[0]	0	[0]

This table provides the optimal sol. i.e. $1 \to 2 \to 6 \to 7$

However there is an alternative solution. Which is $1 \to 2 \to 5 \to 7$.

Exercise: Express the shortest route problem of the network in the following figure as a transshipment model, assuming that the shortest route is to be found between the following pair of nodes

~~(a) Nodes 1 and 5~~



(a). Nodes 1 and 5

Sol. The given problem expressed in transshipment model is as follows

	2	3	4	5	6
1	4	6	7	x	2
2	0	6	9	3	x
3	4	0	2	9	5
4	5	6	0	11	6
~~5~~	~~4~~	~~2~~	~~10~~	~~0~~	~~7~~
6	x	x	3	3	0

$P_1 = 2$

→ not taken because here node 5 is pure (demand) demand node

So

	2	3	4	5	6
1	2	4	5	x	0
2	0	6	9	3	x
3	4	0	2	9	5
4	5	6	0	11	6
6	x	x	3	3	0

$q_5 = 3$

	2	3	4	5	6
1	2	4	5	x	0
2	0	6	9	0	x
3	4	0	2	6	5
4	5	6	0	8	6
6	x	x	3	0	0

the least uncrossed element is 2.

	2	3	4	5	6
1	$\boxed{\cancel{0}}$	2	3	x	$\boxed{0}$
2	$\boxed{0}$	6	9	2	x
3	4	$\boxed{0}$	2	8	7
4	5	6	$\boxed{0}$	10	8
6	x	x	1	$\boxed{0}$	0

[illegible] Result [illegible]
[illegible]
حل [illegible]

A feasible assignment is possible and is

1 → 6 → 5

(b) Nodes 6 and 3

	1	2	3	4	5
1	0	4	6	7	x
2	3	0	6	9	3
4	9	5	6	0	11
5	x	4	2	10	0
6	x	x	x	3	3

$P_6 = 3$

$v_3 = 2$

	1	2	3	4	5
1	[0]	4	4	7	x
2	3	[0]	4	9	3
4	9	5	4	[0]	11
5	x	4	[0]	10	0
6	x	x	x	10	[0]

A feasible assignment to zero element is $6 \rightarrow 5 \rightarrow 3$

(c) Nodes 2 and 6

	1	3	4	5	6
1	0	6	7	x	2
2	3	6	9	3	x
3	x	0	2	9	5
4	9	6	0	11	6
5	x	2	10	0	7

$P_2 = 3$

$q_6 = 2$

	1	3	4	5	6
1	[0]	6	7	x	[0]
2	[0]	3	6	0	x
3	x	[0]	2	9	3
4	9	6	[0]	11	4
5	x	2	10	[0]	5

$2 \rightarrow 1 \rightarrow 6$ is a feasible assignment to zero.

Anjum Iqbal

Pure integer problem is defined to have all integer variables.
Mixed integer problem is defined to have both continuous and integer variables.
Special constraints ~~are~~ added to the sol. space to produce an optimum integer extreme points are called cuts.

Section III

INTEGER LINEAR PROGRAMMING (ILP)

Def. Integer linear programming is the programming in which some or all the variables are restricted to integer ~~to integer~~ (or discrete) values. these are commonly reffered to as the mixed or pure integer programming.

METHODS OF INTEGER PROGRAMMING.

Integer programming methods can be categorized as 1. Cutting Methods 2. Search methods.

THE FRACTIONAL (PURE INTEGER) ALGORITHM:-

A basic requisition for the application of this algorithm is that all the coefficients and the right hand side constants of each constraint must be integer. For example the constraint

$$x_1 + \frac{1}{3}x_2 \le \frac{13}{2}$$

must be transformed to

$$6x_1 + 2x_2 \le 39$$

We first demonstrate the idea by a graphical example, and then show how the cuts are implemented algebraically.

Example:-

Demonstrate graphically how the cutting plane algorithm may be used to solve the following ILP.

$$\text{Maximize } z = 7x_1 + 10x_2$$

subject to

$$-x_1 + 3x_2 \le 6$$

$$7x_1 + x_2 \le 35$$

$$x_1, x_2 \ge 0 \text{ and integer}$$

the cutting plane algorithm modifies the solution space by ~~adding cuts~~ successively adding specially constructed constraints (called cuts) that produce an optimum integer extreme point.

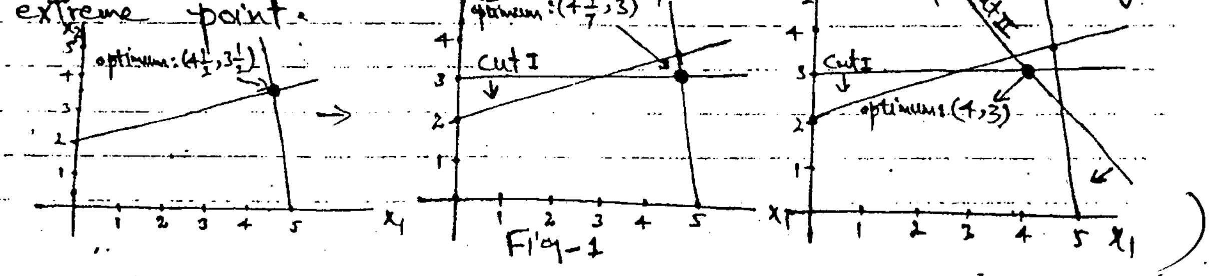


Fig-1

(Fig 1 gives an example of two such cuts.
Initially, we start with the continuous LP optimum $(x_1, x_2) = (4\frac{1}{2}, 3\frac{1}{2})$ and $z = 66\frac{1}{2}$. next we add Cut I, which produces the continuous LP optimum $(x_1, x_2) = (4\frac{4}{7}, 3)$ with $z = 62$. then, we add Cut II, which together with Cut I and the original constraints, produces the LP optimum $(x_1, x_2) = (4, 3)$ and $z = 58$. the last solution is all integer as desired.

the added cuts do not eliminate any of the original feasible integer points, but must pass through at least one feasible or infeasible integer point. these are basic requirements of any cut.

In general, it may take any (finite) numbers of cuts to reach the desired all-integer extreme point. Indeed, the number of cuts needed to produce the desired integer solution appears to be independent of the size of the problem. in the sense that a problem with a small number of variables and constraints may require more cuts than a larger problem.

ALGEBRAIC DEVELOPMENT OF CUTS:-

The cutting plane algorithm starts by solving the continuous LP problem. In the optimum LP tableau, we select one of the rows, called source rows, for which the basic variable is non integer. the desired cut is constructed from the fractional components of the coefficients of the source row. For this reason, it is referred to as the fractional cut.

We now develop the fractional cut.

the problem is solved as a regular ~~Simp~~ LP programming. If the optimal solution is integer, there is nothing more to be done. otherwise the secondary constraints (cuts), which will force the solution to be integer values are developed as follows:

Let the final optimal tableau for the linear program is given by

Basic	x_0	x_1	x_i	x_m	w_1	w_j	w_n	Soln.
x_0	1	0	0	0	$\bar{c}_1$	$\bar{c}_j$	$\bar{c}_n$	β_0
x_1	0	1	0	0	α_1^1	α_1^j	α_1^n	β_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	0	0	1	0	α_i^1	α_i^j	α_i^n	β_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	0	0	0	1	α_m^1	α_m^j	α_m^n	β_m

The variables x_i $(i = 1, 2, \dots, m)$ represents the basic variables, while the variables w_j $(j = 1, 2, \dots, n)$ are the non basic variables.

Consider the ith equation where the basic variable x_i assumes a non-integer value i.e.

$x_i = \beta_i - \sum_{j=1}^{n} \alpha_i^j w_j$, β_i being non integer. Any such equation will be reffered to as the source row.

Let $\beta_i = [\beta_i] + f_i$ (f_i is the fractional part)

and

$$\alpha_i^j = [\alpha_i^j] + f_{ij}$$

then $0 < f_i < 1$ and $0 \le f_{ij} < 1$ i.e. each of the non integer coefficients are factored into integer and fractional components, provided that the fractional component is strictly positive. for example $\frac{3}{2} = (1 + \frac{1}{2})$, $-\frac{7}{3} = -3 + \frac{2}{3}$

the source row now becomes

$$x_i = [\beta_i] + f_i - \sum_{j=1}^{n} \left[[\alpha_i^j] + f_{ij} \right] w_j$$

$$\Rightarrow \quad x_i = [\beta_i] + \sum_{j} [\alpha_i^j] w_j = f_i - \sum_{j=1} \alpha_i^j$$

$$x_i - [\beta_i] + \sum_{j=1}^{n} [\alpha_i^j] w_j = f_i - \sum_{j=1}^{n} f_{ij} w_j$$

If all the variables w_j and x_i are to be integers, the L.H.S must be integer. It follows that the R.H.S must also be integer.

(Given $f_{ij} \geq 0$ and $w_j \geq 0$ for all i and j, it follows that $\sum_{j=1}^{n} f_{ij} w_j \geq 0$. Hence

$$f_i - \sum_{j=1}^{n} f_{ij} w_j \leq f_i$$

$$\Rightarrow f_i - \sum_{j=1}^{n} f_{ij} w_j \leq f_i < 1 \quad \because f_i < 1$$

$$\Rightarrow f_i - \sum_{j=1}^{n} f_{ij} w_j < 1$$

But the R.H.S must be integer (less than 1), so

$$f_i - \sum_{j=1}^{n} f_{ij} w_j \leq 0$$

$$\Rightarrow \sum_{j=1}^{n} f_{ij} w_j - f_i \geq 0$$

or

$$\left(S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i\right)$$

Where S_i is a non-negative slack variable which must be integer. this Constraint equation defines the so-called fractional cut. from the last tableau $w_j = 0$ and thus $S_i = -f_i$, which is infeasible. this means that the new constraint is not satisfied by the given solution.

the dual simplex method can, then, be used to clear the infeasibility, which is equivalent to cutting off the solution space towards the optimal integer solution.

the new tableau after adding the fractional cut will become.)

Example: ① Consider the problem

$$\text{Maximize } z = 7x_1 + 9x_2$$

subject to

$$-x_1 + 3x_2 \le 6$$

$$7x_1 + x_2 \le 35$$

x_1, x_2 non-negative integers.

Solution: Firstly, we solve the problem by regular Simplex method as follows

Basic	x_1	x_2	x_3	x_4	Soln.	Ratio
z	-7	-9	0	0	0	
x_3	-1	3	1	0	6	$\frac{6}{3} = 2 \rightarrow$ smaller
x_4	7	1	0	1	35	$\frac{35}{1} = 35$

x_2 enters and x_3 leaves.

Basic	x_1	x_2	x_3	x_4	Soln.
z	-10	0	3	0	18
x_2	$-1/3$	1	$1/3$	0	2
x_4	$\frac{22}{3}$	0	$-1/3$	1	33

اگر پہلے cut سے جواب آ جائے
اگر تو دوسرے cut سے
جواب integer میں ہو جاتا

x_1 enters and x_4 leaves

Basic	x_1	x_2	x_3	x_4	Soln.
z	0	0	$28/11$	$15/11$	63
x_2	0	1	$7/22$	$\frac{1}{22}$	$7/2 = 3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$\frac{99}{22} = \frac{9}{2} = 4\frac{1}{2}$

this last tableau yields the optimal continous solution. Since the solution is non-integers, a fractional cut must be added to the tableau. (In general, any of the constraint equations can be selected to generate a cut. However, as a rule, one usually chooses the equation corresponding to)

Basic	x_0	x_1	x_i	x_m	w_1 — w_j — w_n	S_i	Soln.
x_0	1	0	0	0	$\bar{c}_1$ — $\bar{c}_j$ — $\bar{c}_n$	0	β_0
x_1	0	1	0	0	α_1^1 -- α_1^j — α_1^n	0	β_1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_i	0	0	1	0	α_i^1 -- α_i^j — α_i^n	0	β_i
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_m	0	0	0	1	α_m^1 -- α_m^j — α_m^n	0	β_m
S_i	0	0	0	0	$-f_{i1}$ -- $-\alpha_{ij}$ — $-\alpha_{im}$	1	$-f_i$

If the new solution after applying the dual simplex method is integer, the process ends. otherwise, a new fractional cut is constructed from the resulting tableau and the dual simplex method is employed to clear the infeasibility. the procedure is repeated until an integer solution is achieved.

this algorithm is reffered to as the fractional method.

Main Steps (Fractional Cut method):-

(i) Convert the coefficients of the constraints and the right hand side of the constraint equations with integers by multiplying with ~~GCD~~ LCM of the denominators on both sides.

(ii) Solve the problem by simplex method. If solution is in integers, the process ends.

(iii) If the solution is not in integers, define the cut and add it to the optimal last tableau of the regular simplex method. Solve the problem by dual simplex method.

the solution is still in non-integers,
So a new cut must be introduced
x_1-equ has the max $f_i (= 4/7)$
So consider the x_1-equation; this gives.

$$x_1 + \frac{1}{7}x_4 - \frac{1}{7}S_1 = 4 + 4/7$$

$$\Rightarrow x_1 + (0+\frac{1}{7})x_4 + (-1+\frac{6}{7})S_1 = 4 + 4/7$$

Hence the corresponding fractional cut is $-\frac{1}{7} = -1+1-\frac{1}{7} = -1+\frac{6}{7}$

$$S_2 = \frac{1}{7}x_4 + \frac{6}{7}S_1 - 4/7$$

$$\Rightarrow S_2 - \frac{1}{7}x_4 - \frac{6}{7}S_1 = -4/7$$

Adding this constraint to the last tableau we get

basic	x_1	x_2	x_3	x_4	S_1	S_2	Sol.
Z	0	0	0	1	8	0	59
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$1/7$	$-1/7$	0	$4\frac{4}{7} = \frac{32}{7}$
x_3	0	0	1	$1/7$	$-22/7$	0	$11/7$
S_2	0	0	0	$-1/7$	$-6/7$	1	$-4/7$

Again we apply dual simplex method. S_2 leaves the sol. and x_4 enters. then the new tableau adopts the form.

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Solution
Z	0	0	0	0	2	7	55
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
x_3	0	0	1	0	-4	1	1
x_4	0	0	0	1	6	-7	4

(this tableau yields the integral solution.
Hence, the optimal is
$x_1 = 4$, $x_2 = 3$, and $Z = 55$ Ans

~~(maximum f_i) In the current problem, both the~~ equations have the same value of f_i ($=\frac{1}{2}$), so either one may be used.

Consider the x_2 equation. This gives

x_i are basic variable while w_j are the non-basic variables.

$$x_2 + \frac{7}{22}x_3 + \frac{1}{22}x_4 = \frac{7}{2}$$

or

w_j and x_i are to be integers.

$$x_2 + (0+\tfrac{7}{22})x_3 + (0+\tfrac{1}{22})x_4 = 3 + \tfrac{1}{2}$$

Hence the corresponding fractional cut is

& $f_{ij} \ge 0$, $w_j \ge 0$

$$S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i$$

S_i is nonnegative slack variable; f_{ij} fractions ~~integers~~

$$S_1 = \frac{7}{22}x_3 + \frac{1}{22}x_4 - \frac{1}{2}$$

Now by adding this in the previous optimal tableau the following new tableau is obtained.

f_i is the fractional part.

Basic	x_1	x_2	x_3	x_4	S_1	Soln
z	0	0	$\frac{28}{11}$	$\frac{15}{11}$	0	63
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$\frac{7}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$\frac{9}{2}$
S_1	0	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$

Now we use the dual simplex method.

S_1 has the most −ve value ($=-\frac{1}{2}$), so it leaves the solution. Now we take the ratios of the L.H.Sides of the z row with S_1 (ignoring the zero and +ve denominators)

For x_3: $\frac{28}{11} \div -\frac{7}{22} = \frac{28}{11} \times -\frac{22}{7} = -8$,

For x_4: $\frac{15}{11} \div -\frac{1}{22} = -30$

x_3 has the least absolute ratio, so x_3 enters the solution. The new tableau hence is

Basic	x_1	x_2	x_3	x_4	S_1	Soln
z	0	0	0	1	+8	59
x_2	0	1	0	0	+1	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$4\frac{4}{7}$
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	$+\frac{11}{7} = 1+\frac{4}{7}$

Ex. 2.– Solve by Fractional algorithm.

Maximize $Z = 4x_1 + 6x_2 + 2x_3$

Subject to

$$4x_1 - 4x_2 \le 5$$

$$-x_1 + 6x_2 \le 5$$

$$-x_1 + x_2 + x_3 \le 5$$

x_1, x_2, x_3 non negative integers.

Solution:

Firstly we solve the problem by regular Simplex method as follows.

Basic	x_1	x_2 ↓	x_3	x_4	x_5	x_6	Soln.
Z	-4	-6	-2	0	0	0	0
x_4	4	-4	0	1	0	0	5
← x_5	-1	6	0	0	1	0	5
x_6	-1	1	1	0	0	1	5

x_2 enters and x_5 leaves.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
Z	-5	0	-2	0	1	0	5
x_4	$10/3$	0	0	1	$2/3$	0	$25/3$
x_2	$-1/6$	1	0	0	$1/6$	0	$5/6$
x_6	$-5/6$	0	1	0	$-1/6$	1	$25/6$

Introduce x_1 and drop x_4.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
Z	0	0	-2	$3/2$	2	0	$\frac{35}{2}$
x_1	1	0	0	$3/10$	$1/5$	0	$5/2$
x_2	0	1	0	$\frac{1}{20}$	$1/5$	0	$\frac{15}{12} = 5/4$
x_6	0	0	1	$1/4$	0	1	$\frac{75}{12} = \frac{25}{4}$

Introduce x_3 and drop x_6

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln
Z	0	0	0	2	2	2	30
x_1	1	0	0	$3/10$	$1/5$	0	$5/2 = 2+\frac{1}{2}$
x_2	0	1	0	$1/20$	$1/5$	0	$5/4 = 1+\frac{1}{4}$
x_3	0	0	1	$1/4$	0	1	$\frac{25}{4} = 6+\frac{1}{4}$

this tableau is optimal, but the solution is non-integers.

In this problem max $f_i (= \frac{1}{2})$ corresponds to x_1-equation. So

$$S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i \text{ leads to}$$

$$S_1 = \frac{3}{10} x_4 + \frac{1}{5} x_5 - \frac{1}{2}$$

After adding it to the last tableau, we got

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln
Z	0	0	0	2	2	2	0	30
x_1	1	0	0	$3/10$	$1/5$	0	0	$5/2$
x_2	0	1	0	$1/20$	$1/5$	0	0	$5/4$
x_3	0	0	1	$1/4$	0	1	0	$25/4$
S_1	0	0	0	$-3/10$	$-1/5$	0	1	$-1/2$

Now using dual Simplex method S_1 leaves the sol. and x_4 enter. the new tableau is.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Sol.
Z	0	0	0	0	$2/3$	2	$\frac{20}{3}$	$80/3$
x_1	1	0	0	0	0	0	1	2
x_2	0	1	0	0	$1/6$	0	$1/6$	$7/6 = 1+\frac{1}{6}$
x_3	0	0	1	0	$-1/6$	1	$5/6$	$35/6 = 5+\frac{5}{6}$
x_4	0	0	0	1	$2/3$	0	$-10/3$	$5/3 = 1+\frac{2}{3}$

~~x_3 eq. has max f_i ($= 5/6$), so consider the x_3 eq. in the last table~~

$$x_3 - \frac{1}{6}x_5 + x_6 + 5/6\, S_1 = 35/6$$

integer coeff. and we take only fractional coeff.

$$\Rightarrow \quad x_3 + \left(-1 + \frac{5}{6}\right)x_5 + x_6 + 5/6\, S_1 = 5 + \frac{5}{6}$$

Fractional component is strictly positive.

thus $S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i$ yields

$$S_2 = \frac{5}{6}x_5 + 5/6\, S_1 - 5/6$$

Adding this to the last tableau we get

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Solu.
Z	0	0	0	0	2/3	2	20/3	0	80/3
x_1	1	0	0	0	0	0	1	0	2
x_2	0	1	0	1	1/6	0	1/6	0	7/6
x_3	0	0	1	0	-1/6	1	5/6	0	35/6
x_4	0	0	0	1	2/3	0	-10/3	0	5/3
S_2	0	0	0	0	-5/6	0	+5/6	1	-5/6

Now using dual Simplex methods, S_2 leaves and x_5 to enters

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Solu.
Z	0	0	0	0	0	2	6	4/5	26
x_1	1	0	0	0	0	0	1	0	2
x_2	0	1	0	0	0	0	0	1/5	1
x_3	0	0	1	0	0	1	1	-1/5	6
x_4	0	0	0	1	0	0	-4	+4/5	1
x_5	0	0	0	0	1	0	1	$-\frac{6}{5}$	1

this tableau yields the integral solution.

Hence, the optimal is

$x_1 = 2$, $x_2 = 1$, $x_3 = 6$, and $Z = 26$

(Note that the rounded optimal solution is

$x_1 = 3$, $x_2 = 1$, $x_3 = 6$ & $Z = 30$)

ﺟﺎﻛﺒﺎ ﻣﻦ ... (Simplex method) Rounded optimal sol.

Ex. 3: Solve by the fractional algorithm

Maximize $Z = 3x_1 + x_2 + 3x_3$

subject to

$$-x_1 + 2x_2 + x_3 \leq 4$$
$$4x_2 - 3x_3 \leq 2$$
$$x_1 - 3x_2 + 2x_3 \leq 3 \; ; \; x_1, x_2, x_3 \geq 0 \text{ and integers}$$

Compare the rounded optimal solution and the integer optimal solution.

Solution:

Firstly, we solve the problem by regular Simplex method as follows:

Basic	x_1 ↓	x_2	x_3	x_4	x_5	x_6	Soln.
Z	-3	-1	-3	0	0	0	0
x_4	-1	2	1	1	0	0	4
x_5	0	4	-3	0	1	0	2
← x_6	1	-3	2	0	0	1	3

Introduce x_1 and drop x_6

Basic	x_1	x_2 ↓	x_3	x_4	x_5	x_6	Soln.
Z	0	-10	3	0	0	3	9
x_4	0	-1	3	1	0	1	7
← x_5	0	4	-3	0	1	0	2
x_1	1	-3	2	0	0	1	3

Introduce x_2 and drop x_5

Basic	x_1	x_2	x_3 ↓	x_4	x_5	x_6	Soln.
Z	0	0	$-9/2$	0	$5/2$	3	14
← x_4	0	0	$9/4$	1	$1/4$	1	$15/2$
x_2	0	1	$-3/4$	0	$1/4$	0	$1/2$
x_1	1	0	$-1/4$	0	$3/4$	1	$9/2$

Introduce x_3 and leaves x_4.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
Z	0	0	0	2	3	5	29
x_3	0	0	1	$4/9$	$1/9$	$4/9$	$10/3 = 3\frac{1}{3}$
x_2	0	1	0	$1/3$	$1/3$	$1/3$	3
x_1	1	0	0	$1/9$	$7/9$	$10/9$	$16/3 \;\; 5\frac{1}{3}$

~~(This tableau is optimal, but the solution is non-integer.~~
Thus the rounded optimal solution is
$x_3 = 3$; $x_2 = 3$, $x_1 = 5$; $Z = 29$, which is infeasible
because it does not satisfy the 2nd constraint.)

In this tableau, x_1-equation is

$$x_1 + \frac{1}{9}x_4 + \frac{7}{9}x_5 + \frac{10}{9}x_6 = 5 + \frac{1}{3}$$

$$\Rightarrow \quad x_1 + \left(0 + \frac{1}{9}\right)x_4 + \left(0 + \frac{7}{9}\right)x_5 + \left(1 + \frac{1}{9}\right)x_6 = 5 + \frac{1}{3}.$$

Hence $S_i = \sum_{j=1}^{n} f_{ij}\, w_j - f_i$ yields

$$S_1 = \frac{1}{9}x_4 + \frac{7}{9}x_5 + \frac{1}{9}x_6 - \frac{1}{3}$$

after adding this constraint in the last optimal tableau, we get

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln.
Z	0	0	0	2	3	5	0	29
x_3	0	0	1	4/9	1/9	4/9	0	10/3
x_2	0	1	0	1/3	1/3	1/3	0	3
x_1	1	0	0	1/9	7/9	10/9	0	16/3
S_1	0	0	0	-1/9	-7/9	-1/9	1	-1/3

Now we, use dual simplex method,
S_1 leaves and x_5 enters

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln
Z	0	0	0	11/7	0	32/7	27/7	194/7
x_3	0	0	1	3/7	0	3/7	1/7	23/7 = 3 + 2/7
x_2	0	1	0	2/7	0	2/7	3/7	20/7 = 2 + 6/7
x_1	1	0	0	0	0	1	1	5
x_5	0	0	0	1/7	1	1/7	-9/7	3/7

the x_2-eq^n is

$$x_2 + \frac{2}{7}x_4 + \frac{2}{7}x_6 + \frac{3}{7}S_1 = 2 + \frac{6}{7}.$$

Thus $S_i = \sum_{j=1}^{n} f_{ij}\, w_j - f_i$ yields

$$S_2 = \frac{2}{7}x_4 + \frac{2}{7}x_6 + \frac{3}{7}S_1 - \frac{6}{7}$$

				$1/7$		$1/7$	$1/7$	0	$23/7$
x_2	0	1	0	$2/7$	0	$2/7$	$3/7$	0	$2\frac{0}{7}$
x_1	1	0	0	0	0	1	1	0	5
x_5	0	0	0	$1/7$	1	$1/7$	$-9/7$	0	$3/7$
S_2	0	0	0	$-\frac{2}{7}$	0	$-\frac{2}{7}$	$-3/7$	1	$-\frac{6}{7}$

Again use dual Simplex method
S_2 leaves and x_4 enters

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Sol.
Z	0	0	0	0	0	3	$3/2$	$11/2$	23
x_3	0	0	1	0	0	[illegible]	$-1/2$	$+3/2$	2
x_2	0	1	0	0	0	0	0	1	2
x_1	1	0	0	0	0	1	1	0	5
x_5	0	0	0	0	1	0	$-7/2$	$1/2$	0
x_4	0	0	0	1	0	1	$3/2$	$-\frac{7}{2}$	3

Thus the required integral solution is

$x_1 = 5$, $x_2 = 2 = x_3$; $Z = 23$

THE MIXED ALGORITHM :-

Let x_k be an integer variable of the mixed problem. Consider the x_k-equation in the optimal Continuous Solutions, given by

$$x_k = \beta_k - \sum_{j=1}^{n} \alpha_k^j w_j \quad \text{(Source row)}$$

$$= [\beta_k] + f_k - \sum_{j=1}^{n} \alpha_k^j w_j$$

where $\beta_k = [\beta_k] + f_k$

or

$$x_k - [\beta_k] = f_k - \sum_{j=1}^{n} \alpha_k^j w_j$$

For x_K to be integer, X

either $x_K \le [\beta_K]$ → ①

or $x_K \ge [\beta_K] + 1$ → ②

Using the source row, eq ① becomes

$$f_K - \sum_{j=1}^{n} \alpha_K^j w_j \le 0$$

or $$\sum_{j=1}^{n} \alpha_K^j w_j \ge f_K \quad \rightarrow ③$$

and (2) becomes

$$f_K - \sum_{j=1}^{n} \alpha_K^j w_j \ge 1$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_K^j w_j \le f_K - 1 \quad \rightarrow ④$$

Note that only one of the equations ③ and ④ can hold at a time.

Let

J^+ = set of subscripts j for which $\alpha_K^j \ge 0$

J^- = " " " " $\alpha_K^j < 0$

Then

(3) ⇒ $$\sum_{j \in J^+} \alpha_K^j w_j \ge f_K \quad \rightarrow ⑤$$

④ ⇒ $$\sum_{j \in J^-} (-\alpha_K^j) w_j \le f_K - 1$$

⇒ $$\sum_{j \in J^-} \alpha_K^j w_j \ge f_K - 1$$

⇒ $$\left(\frac{f_K}{f_K - 1}\right) \sum_{j \in J^-} \alpha_K^j w_j \ge f_K \quad \rightarrow ⑥$$

⑤ and ⑥ can be combined into one constraint as

$$\sum_{j \in J^+} \alpha_K^j w_j + \left(\frac{f_K}{f_K - 1}\right) \sum_{j \in J^-} \alpha_K^j w_j - f_K \ge 0$$

this constraint can now be put in the form.

$$\left(S_K = \sum_{j\in J^+} \alpha_K^j w_j + \left(\frac{f_K}{f_K-1}\right)\sum_{j\in J^-} \alpha_K^j w_j - f_K\right)$$

or

$$\left(S_K - \left\{\sum_{j\in J^+} \alpha_K^j w_j + \left(\frac{f_K}{f_K-1}\right)\sum_{j\in J^-} \alpha_K^j w_j\right\} = -f_K\right)$$

this equation is the required mixed cut. Since all $w_j = 0$ at the current optimal tableau, it follows that the above cut is infeasible. The dual simplex method is thus used to clear the infeasibility.

Example:-(18)

Maximize $Z = 7x_1 + 9x_2$

subject

$$-x_1 + 3x_2 \le 6$$

$$7x_1 + x_2 \le 35$$

x_1 non-negative integer ; $x_2 \ge 0$

Solution:- Firstly, we solve the problem by regular simplex method as follows (as in Ex: 1) optimal sol is:

Basic	x_1	x_2	x_3	x_4	Soln.
Z	0	0	28/11	15/11	63
x_2	0	1	7/22	1/22	7/2
x_1	1	0	-1/22	3/22	9/2

Assuming Now since x_1 is restricted to integer value, so we consider x_1-equation of above tableau.

$$x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = \frac{9}{2} = 4 + \frac{1}{2}$$

then the mixed cut is given by

$$S_1 = \left\{\sum_{j\in J^+} \alpha_1^j w_j + \left(\frac{f_1}{f_1-1}\right)\sum_{j\in J^-} \alpha_1^j w_j\right\} - f_1$$

or,

$$S_1 = \left\{\frac{3}{22}x_4 + \left(\frac{1/2}{1/2-1}\right)\left(-\frac{1}{22}x_3\right)\right\} - 1/2$$

or

$$S_1 = \frac{3}{22}x_4 + \frac{1}{22}x_3 - \frac{1}{2}$$

or $S_1 - \frac{1}{22}x_3 - \frac{3}{22}x_4 = -\frac{1}{2}$

Adding this constraint to the last tableau, we have

Basic	x_1	x_2	x_3	x_4	S_1	Soln.
Z	0	0	28/11	15/11	0	63
x_2	0	1	7/22	1/22	0	7/2
x_1	1	0	-1/22	3/22	0	9/2
S_1	0	0	-1/22	-3/22	1	-1/2

Now, we apply the dual simplex method.

S_1 leaves and x_4 enters the solution

Basic	x_1	x_2	x_3	x_4	S_1	Soln.
Z	0	0	23/11	0	10	58
x_2	0	1	10/33	0	1/3	10/3
x_1	1	0	$-\frac{1}{11}$	0	1	4
x_4	0	0	$\frac{1}{3}$	1	$-\frac{22}{3}$	$\frac{11}{3} = 3\frac{2}{3}$

This tableau yields the requisite optimal solution as

$$x_1 = 4 ; \; x_2 = 10/3 ; \; Z = 58$$

Ex(20) Maximize $Z = 4x_1 + 6x_2 + 2x_3$

Subject to

$$4x_1 - 4x_2 \le 5$$
$$-x_1 + 6x_2 \le 5$$
$$-x_1 + x_2 + x_3 \le 5$$
$$x_1, x_2, x_3 \ge 0$$

Solve the following problem by the mixed algorithm assuming that x_1 and x_3 are the only integer variables

x_2	0	1	0	1/20	1/5	0	5/4
x_3	0	0	1	1/4	0	1	$25/4 = 6 + \frac{1}{4}$

Since x_1 and x_3 are restricted to integer values.

Consider the x_1-equation

$$x_1 + \frac{3}{10}x_4 + \frac{1}{5}x_5 = 5/2 = 2 + \frac{1}{2}$$ [illegible]

The mixed cut is, hence

$$S_i - \left\{ \sum_{j \in J^+} a_i^j w_j + \left(\frac{f_i}{f_i - 1}\right) \sum_{j \in J^-} a_i^j w_j \right\} = -f_i$$

i.e. $S_1 - \left\{ \frac{3}{10}x_4 + \frac{1}{5}x_5 + 0 \right\} = -1/2$

$$\Rightarrow S_1 - \frac{3}{10}x_4 - \frac{1}{5}x_5 = -1/2$$

Adding this Constraint to the last tableau we have

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln.
z	0	0	0	2	2	2	0	30
x_1	1	0	0	3/10	1/5	0	0	5/2
x_2	0	1	0	1/20	1/5	0	0	5/4
x_3	0	0	1	1/4	0	1	0	25/4
S_1	0	0	0	-3/10	-1/5	0	1	-1/2

Now we use the dual simplex method.

S_1 leaves and x_4 enters the sol.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Sol.
z	0	0	0	0	8/3	2	$+\frac{20}{3}$	$\frac{80}{3}$
x_1	1	0	0	0	0	0	1	2
x_2	0	1	0	0	1/6	0	1/6	7/6
x_3	0	0	1	0	-1/6	1	5/6	35/6
x_4	0	0	0	1	$\frac{2}{3}$	0	$-\frac{10}{3}$	$\frac{5}{3}$

Now Consider the x_3 equation

$$x_3 - \frac{1}{6}x_5 + x_6 + \frac{5}{6}S_1 = 35/6$$

[illegible] tableau values [illegible] S_2

Ex. Solve the following problem by the mixed algorithm assuming that x_1 and x_2 are the only integer variables.

Maximize $z = 3x_1 + x_2 + 3x_3$

subject to

$$-x_1 + 2x_2 + x_3 \leq 4$$

$$4x_2 - 3x_3 \leq 2$$

$$x_1 - 3x_2 + 2x_3 \leq 3$$

$$x_1, x_2, x_3 \geq 0$$

BRANCH-AND-BOUND METHOD: — very lengthy

(In this method, the questions problem is first solved as a continuous model. Suppose that x_r is a variable which is restricted to integer value, whose optimum continuous value x_r^* is fractional. The range $[x_r^*] < x_r < [x_r^*] + 1$ cannot include any feasible integer solution. As a consequence, a feasible integer value of x_r must satisfy one of two conditions:

$x_r \leq [x_r^*]$ or $x_r \geq [x_r^*] + 1$

We obtain two mutually exclusive problems by adding each of the above constraints to the original problem. The original problem is then said to be branched into two sub problems.

Now, each sub problem may be solved as a linear program. If its optimal is feasible w.r.t the integer problem, its solution is recorded as the best one so far available. Otherwise, the sub-problem must be ~~partioned~~ partitioned into two sub-problems. When a better integer feasible solution is obtained for any sub problem, it replaces the one at hand.

The efficiency of the computation is enhanced by introducing the notation of 'Bounding'. According to this concept, if the continuous optimum solution of a sub-problem yields a worse objective value than the one associated with the best available integer solution, then there is no need to explore the problem any further.)

Example:- Maximize $z = 2x_1 + 3x_2$

Subject to

$$5x_1 + 7x_2 \leq 35$$

$$4x_1 + 9x_2 \leq 36$$

x_1, x_2 are non-negative integers.

Solution:-

Firstly, we solve the problem by regular simplex method as follows.

Basic	x_1	x_2 ↓	x_3	x_4	Soln
z	-2	-3	0	0	0
x_3	5	7	1	0	35
← x_4	4	9	0	1	36

Introduce x_2 and drop x_4

Basic	x_1	x_2	x_3	x_4	Soln
z	$-2/3$	0	0	$1/3$	12
x_3	$17/9$	0	1	$-7/9$	7
x_2	$4/9$	1	0	$1/9$	4

Introduce x_1 and drop x_3

Basic	x_1	x_2	x_3	x_4	Soln
z	0	0	$6/17$	$1/17$	$246/17$
x_1	1	0	$9/17$	$-\frac{7}{17}$	$\frac{63}{17}$
x_2	0	1	$\frac{-4}{17}$	$5/17$	$\frac{40}{17}$

Both the variables x_1, x_2 have fractional values in the optimum continuous solution. We select x_2, arbitrarily, for branching then two problems are created by the restrictions

$$x_2 \le \left[\frac{40}{17}\right] \text{ and } x_2 \ge \left[\frac{40}{17}\right] + 1$$

[illegible]

$$\Rightarrow \quad x_2 \le 2 \quad \text{and} \quad x_2 \ge 3$$

then the subproblems are

<u>Sub-Problem 1</u>: $x_2 \le 2 \Rightarrow x_2 + S_1 = 2$

Also from the last tableau

$$x_2 - \frac{4}{17} x_3 + \frac{5}{17} x_4 = \frac{40}{17}$$

$$\Rightarrow 2 - S_1 - \frac{4}{17} x_3 + \frac{5}{17} x_4 = \frac{40}{17}$$

[illegible] x_2

$$\Rightarrow S_1 + \frac{4}{17} x_3 - \frac{5}{17} x_4 = 2 - \frac{40}{17} = \frac{-6}{17}$$

Adding this constraint to the optimal continuous tableau of original problem, we get.

Basic	x_1	x_2	x_3	x_4	S_1	Soln
Z	0	0	6/17	1/17	0	246/17
x_1	1	0	9/17	-7/17	0	63/17
x_2	0	1	-4/17	5/17	0	40/17
S_1	0	0	4/17	-5/17	1	-6/17

Sub Program 2: $-x_2 \leq -3$

$x_2 \geq 3 \Rightarrow -x_2 + S_2 = -3$

Eliminating x_2, we obtain $x_2 = S_2 + 3$

$$x_2 - \frac{4}{17}x_3 + \frac{5}{17}x_4 = 40/17$$

$$S_2 + 3 - \frac{4}{17}x_3 + \frac{5}{17}x_4 = \frac{40}{17}$$

$$S_2 - \frac{4}{17}x_3 + \frac{5}{17}x_4 = -\frac{11}{17}$$

Adding this constraint to the last tableau, we obtain,

Basic	x_1	x_2	x_3	x_4	S_2	Soln.
Z	0	0	6/17	1/17	0	246/7
x_1	1	0	9/17	-7/17	0	63/17
x_2	0	1	-4/17	5/17	0	40/17
S_2	0	0	-4/17	5/17	1	-11/17

Now, we select the subproblem 1 to arbitrarily for investigation and use dual simplex method.

Introduce x_4 and leaves S_1

Basic	x_1	x_2	x_3	x_4	S_1	Soln.
Z	0	0	2/5	0	1/5	72/5
x_1	1	0	1/5	0	-7/5	21/5
x_2	0	1	0	0	1	2
x_4	0	0	-4/5	1	-17/5	6/5

Since x_1 has fractional value, it is now selected for branching and two sub-problems are further proliferated by introducing each of the constraints

$$x_1 \leq \left[\frac{21}{5}\right] \quad \text{and} \quad x_1 \geq \left[\frac{21}{5}\right] + 1 \rightarrow \text{Continuous value}$$

$$x_1 \leq 4 \quad \text{and} \quad x_1 \geq 5$$

Sub-problem 3 (1.d):-

$$x_1 \le 4 \Rightarrow x_1 + S_3 = 4$$

Also from the last tableau

$$x_1 + \tfrac{1}{5}x_3 - \tfrac{7}{5}S_1 = \tfrac{21}{5}$$

$$\Rightarrow 4 - S_3 + \frac{1}{5}x_3 - \frac{7}{5}S_1 = \frac{21}{5}$$

$$\Rightarrow S_3 - \frac{1}{5}x_3 + \frac{7}{5}S_1 = -\tfrac{1}{5}$$

adding this Constraint to the optimal Continuous tableau of problem 1, we have.

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln.
z	0	0	2/5	0	1/5	0	72/5
x_1	1	0	1/5	0	-7/5	0	21/5
x_2	0	1	0	0	1	0	2
x_4	0	0	-4/5	1	-17/5	0	6/5
S_3	0	0	-1/5	0	7/5	1	-1/5

Subproblem (1.e):-

$$x_1 \ge 5 \Rightarrow -x_1 + S_4 = -5$$

Eliminating x_1, we obtain

$$S_4 + \tfrac{1}{5}x_3 - \tfrac{7}{5}S_1 = -\tfrac{4}{5}$$

Including this Constraint to the optimal Continuous tableau of probl 1, we have.

Basic	x_1	x_2	x_3	x_4	S_1	~~S_4~~	S_4	Soln
z	0	0	2/5	0	1/5	~~72/5~~	0	72/5
x_1	1	0	1/5	0	-7/5	~~7/5~~	0	2/5
x_2	0	1	0	0	1	~~2~~	0	2
x_4	0	0	-4/5	1	-17/5	~~6/5~~	0	6/5
S_4	0	0	1/5	0	-7/5		1	-4/5

We select sub-program 3 arbitrarily for investigation.

Introduce x_3 and drop S_3

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln
Z	0	0	0	0	3	2	14
x_1	1	0	0	0	0	1	4
x_2	0	1	0	0	1	0	2
x_4	0	0	0	1	-9	-4	2
x_3	0	0	1	0	-7	-5	1

Since this solution is in integers, it provides us a lower bound $Z = 14$. We still need to investigate sub problems 2 and 4 for potential of improvement in the solution.

Sub problem 2: Introduce x_3 and drop S_2

Basic	x_1	x_2	x_3	x_4	S_2	Soln
Z	0	0	0	1/34	3/2	27/2
x_1	1	0	0	1/4	9/4	9/2
x_2	0	1	0	0	-1	3
x_3	0	0	1	$-\frac{5}{4}$	$-\frac{17}{4}$	$\frac{11}{4}$

This solution is less than the lower bound. So there is no need to investigate it further.

Sub problem (4):

Introduce S_1 and drop S_4

Basic	x_1	x_2	x_3	x_4	S_1	S_4	Soln
Z	0	0	3/7	0	0	1/7	100/7
x_1	1	0	0	0	0	-1	5
x_2	0	1	1/7	0	0	5/7	10/7
x_4	0	0	-9/7	1	0	-17/7	22/7
S_1	0	0	-1/7	0	1	-5/7	$\frac{4}{7}$

As the solution yielded by this subproblem is fractionally higher than the lower bound, any branching from this subproblem cannot yield a better value than the lower bound across the all coefficients of the objective function are integers. It follows that the soln is

$Z = 14$, $x_1 = 4$, $x_2 = 2$ yeilded by sub-problem 3.

Ex. Solve the following problem by branch and bound algorithm:

$$\text{Maximize } z = x_1 + x_2$$
$$\text{Subject to } 2x_1 + 5x_2 \le 16$$
$$6x_1 + 5x_2 \le 30$$
x_1, x_2 non-negative integers

Soln.

Firstly, we solve the problem by regular Simplex method as follows.

Basic	x_1 ↓	x_2	x_3	x_4	Soln.
z	-1	-1	0	0	0
x_3	2	5	1	0	16
← x_4	6	5	0	1	30

Introduce x_1 and leaves x_4

Basic	x_1	x_2 ↓	x_3	x_4	Soln.
z	0	$-1/6$	0	$1/6$	5
← x_3	0	$\frac{10}{3}$	1	$-1/3$	6
x_1	1	$5/6$	0	$1/6$	5

Introduce x_2 and leave x_3

Basic	x_1	x_2	x_3	x_4	Soln.
z	0	0	$1/20$	$3/20$	$53/10$
x_2	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	$9/5$
x_1	1	0	$-1/4$	$1/4$	$7/2$

We arbitrarily select x_2 for branching then two problems are created by the restrictions.

$$x_2 \le \left[\frac{9}{5}\right], \quad x_2 \ge \left[\frac{9}{5}\right] + 1$$
$$\Rightarrow \quad x_2 \le 1, \quad x_2 \ge 2$$

then the subproblems are

Sub-Problem 1:- $x_2 \le 1 \Rightarrow x_2 + S_1 = 1$.

from the last table the x_2-eqn is

$$x_2 + \frac{3}{10}x_3 - \frac{1}{10}x_4 = 9/5$$

$\Rightarrow \quad 1 - S_1 + \frac{3}{10}x_3 - \frac{1}{10}x_4 = 9/5$

$\Rightarrow \quad S_1 - \frac{3}{10}x_3 + \frac{1}{10}x_4 = -4/5$

Adding this Constraint to the last tableau we have.

Basic	x_1	x_2	x_3	x_4	S_1	Soln
Z	0	0	$\frac{1}{20}$	$3/20$	0	$53/10$
x_2	0	1	$3/10$	$-1/10$	0	$9/5$
x_1	1	0	$-1/4$	$1/4$	0	$7/2$
S_1	0	0	$-3/10$	$1/10$	1	$-4/5$

Now we solve this programme.
Introduce x_3 and drop S_1

Basic	x_1	x_2	x_3	x_4	S_1	Soln
Z	0	0	0	$1/6$	$1/6$	$31/6$
x_2	0	1	0	0	1	1
x_1	1	0	0	$1/6$	$-5/6$	$25/6$
x_3	0	0	1	$-\frac{1}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$

Here x_1 has non integer value. So two further sub-problems are formed as

$x_1 \le \left[\frac{25}{6}\right] \quad , \quad x_1 \ge \left[\frac{25}{6}\right] + 1$

$x_1 \le 4 \quad , \quad x_1 \ge 5$

<u>Sub-program (1a):-</u>

$x_1 \le 4 \Rightarrow x_1 + S_2 = 4$

and Also

$x_1 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$

$\Rightarrow \quad 4 - S_2 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$

$\Rightarrow \quad S_2 - \frac{1}{6}x_4 + 5/6\, S_1 = -1/6$

Adding this restriction to the last tableau, we have.

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Soln.
Z	0	0	0	$1/6$	$1/6$	0	$31/6$
x_2	0	1	0	0	1	0	1
x_1	1	0	0	$1/6$	$-5/6$	0	$25/6$
x_3	0	0	1	$-1/3$	$-1/3$	0	$8/3$
S_2	0	0	0	$-1/6$	$5/6$	1	$-1/6$

Now we solve this problem by dual S. method as introduce x_4 and leaves S_2 so

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Soln.
Z	0	0	0	0	1	1	5
x_2	0	1	0	0	1	0	1
x_1	1	0	0	0	0	1	4
x_3	0	0	1	0	-5	-2	3
x_4	0	0	0	1	-5	-6	1

this tableau yields an integer sol. as $x_1 = 4$; $x_2 = 1$ and $Z = 5$

we record this sol and try for the possible improvement in the sol.

Sub-Problem (1.b) $\quad x_1 \geq 5$

$$\Rightarrow -x_1 + S_3 = -5$$

Also $\quad x_1 = S_3 + 5$

$$x_1 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$$

$$\Rightarrow \quad S_3 + 5 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$$

$$S_3 + \frac{1}{6}x_4 - 5/6\, S_1 = \frac{25}{6} - 5$$

$$= -5/6$$

So Adding the Contraint in the tableau of Continuous optimal table of Subproblem 1. we have

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln
Z	0	0	0	$1/6$	$1/6$	0	$31/6$
x_2	0	1	0	0	1	0	1
x_1	1	0	0	$1/6$	$-5/6$	0	$25/6$
x_3	0	0	1	$-1/3$	$-1/3$	0	$8/3$
S_3	0	0	0	$1/6$	$-5/6$	1	$-5/6$

~~Now we explore this problem~~

Introduces ~~x_3~~ S_1 and leaves S_3

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln.
Z	0	0	0	$1/5$	0	$1/5$	5
x_2	0	1	0	$\frac{1}{5}$	0	$-6/5$	0
x_1	1	0	0	0	0	-1	5
x_3	0	0	1	-1	0	-4	6
S_1	0	0	0	$-\frac{1}{5}$	1	$-6/5$	1

It is also an integer solution with the same of value of Z. Thus

$x_1 = 5$, $x_2 = 0$, $Z = 5$ is the alternative optima

Now ~~to~~ we try to explore the subproblem 2.

Sub-problem 2: $x_2 \geq 2 \Rightarrow$

$$-x_2 + S_4 = -2$$

Also from the original optimal tableau the x_2-eq. is

$$x_2 + \frac{3}{10}x_3 - \frac{1}{10}x_4 = 9/5$$

$$\Rightarrow S_4 + 2 + \frac{3}{10}x_3 - \frac{1}{10}x_4 = 9/5$$

$$\Rightarrow S_4 + \frac{3}{10}x_3 - \frac{1}{10}x_4 = -1/5$$

So Adding this constraint to the optimal table ~~of original function~~ obtained by S-Method

Basic	x_1	x_2	x_3	x_4	S_4	Soln.
Z	0	0	$1/20$	$3/20$	0	$53/10$
x_2	0	1	$3/10$	$-1/10$	0	$9/5$
x_1	1	0	$-1/4$	$1/4$	0	$7/2$
S_4	0	0	$3/10$	$-1/10$	1	$-1/5$

Introducing x_4 and drop S_4

Basic	x_1	x_2	x_3	x_4	S_4	Sol
Z	0	0	1/2	0	3/2	5
x_2	0	1	0	0	-1	2
x_1	1	0	1/2	0	5/2	3
S_4	0	0	-3	1	-10	2

this tableau also gives the same value of Z. thus we have three alternative optima (4, 1), (5, 0), (3, 2) with $Z = 5$

Ex. Solve the following problem by branch and bound method

$$\text{Maximize } Z = 3x_1 + x_2 + 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 \le 4$$
$$4x_2 - 3x_3 \le 2$$
$$x_1 - 3x_2 + 2x_3 \le 3$$
$$x_2 \ge 0 \text{ and } x_1, x_3 \text{ non-negative integers}$$

Soln. We firstly solve the problem by regular simplex method as follows (optimal sol. is)

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
Z	0	0	0	2	3	5	29
x_2	0	1	0	1/3	1/3	1/3	3
x_1	1	0	0	1/9	7/9	10/9	16/3
x_3	0	0	1	4/9	1/9	4/9	10/3

Here x_1, x_3 have assumed non-integer values. we arbitrary choose x_3. Thus

$$x_3 \le \left[\frac{10}{3}\right] \quad \text{and} \quad x_3 \ge \left[\frac{10}{3}\right] + 1$$

i.e,

$$x_3 \le 3 \quad \text{and} \quad x_3 \ge 4$$

Hence the two sub proplems are Supproblem 1 and Sup problem 2.

Sub-problem 1:

$$x_3 \le 3 \Rightarrow x_3 + S_4 = 3$$

Also

$$x_3 + \frac{4}{9}S_1 + \frac{1}{9}S_2 + \frac{4}{9}S_3 = 10/3$$

Eliminating x_3 we get

$$S_4 - \frac{4}{9}S_1 - 1/9\, S_2 - \frac{4}{9}S_3 = -1/3$$

Including this constraint to the last table we have,

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	Sol.
Z	0	0	0	2	3	5	0	29
x_2	0	1	0	1/3	1/3	1/3	0	3
x_1	1	0	0	1/9	7/9	10/9	0	16/3
x_3	0	0	1	4/9	1/9	4/9	0	10/3
S_4	0	0	0	-4/9	-1/9	-4/9	1	-1/3

Subproblem 2:-

$$x_3 \ge 4 \Rightarrow -x_3 + S_5 = -4$$

Also

$$x_3 + \frac{4}{9}S_1 + \frac{1}{9}S_2 + \frac{4}{9}S_3 = 10/3$$

Eliminating x_3, we get

$$S_5 + \frac{4}{9}S_1 + \frac{1}{9}S_2 + \frac{4}{9}S_3 = -2/3$$

Including this constraint to the last tableau of the original ~~constraint~~ problem, we get

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_5	Sol.
Z	0	0	0	2	3	5	0	29
x_2	0	1	0	1/3	1/3	1/3	0	3
x_1	1	0	0	1/9	7/9	10/9	0	16/3
x_3	0	0	1	4/9	1/9	4/9	0	10/3
S_5	0	0	0	4/9	1/9	4/9	1	-2/3

Now we arbitrarily select sub-problem 1 for investigation. We use dual simplex method:

S_5 leaves and S_1 enters.

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	Solution
Z	0	0	0	0	5/2	3	9/2	55/2
x_2	0	1	0	0	1/4	0	3/4	11/4
x_1	1	0	0	0	3/4	1	1/4	21/4
x_3	0	0	1	0	0	0	1	3
S_1	0	0	0	1	4	1	-9/4	3/4

Now since x_1 has fractional value so it is now selected for further branching. Thus

$$x_1 \leq \left[\frac{21}{4}\right] \quad \text{and} \quad x_1 \geq \left[\frac{21}{4}\right] + 1$$

$$\Rightarrow \quad x_1 \leq 5 \quad \text{and} \quad x_1 \geq 6$$

then the two subproblems are

i, Subproblem (1.2) for $x_1 \leq 5$ and Sub-problem (1.3) for $x_1 \geq 6$

Sub-problem (1.2) $x_1 \leq 5 \Rightarrow x_1 + S_6 = 5$

Also $x_1 + \frac{3}{4} S_2 + S_3 + \frac{1}{4} S_4 = 21/4$

Eliminating x_1, we get

$$S_6 - \frac{3}{4} S_2 - S_3 - \frac{1}{4} S_4 = -\frac{1}{4}$$

Adding this constraint to the last tableau of Sub program 1, we have

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	S_6	Sol.
Z	0	0	0	0	5/2	3	9/2	0	55/2
x_2	0	1	0	0	1/4	0	3/4	0	11/4
x_1	1	0	0	0	3/4	1	1/4	0	21/4
x_3	0	0	1	0	0	0	1	0	3
S_1	0	0	0	1	4	1	-9/4	0	3/4
S_6	0	0	0	0	-3/4	-1	-1/4	1	-1/4

Now we investigate Sub-problem (1-2) drop S_6 and enters S_3

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	S_6	Sol
Z	0	0	0	0	1/4	0	15/4	3	107/4
x_2	0	1	0	0	1/4	0	3/4	0	11/4
x_1	1	0	0	0	0	0	0	1	5
x_3	0	0	1	0	0	0	1	0	3
S_1	0	0	0	1	13/4	0	-5/2	1	1/2
S_3	0	0	0	0	3/4	1	1/4	-1	1/4

As x_1 and x_3 are integers, So this solution is acceptable. So we record this sol as the best available:

$x_1 = 5$; $x_2 = 11/4$; $x_3 = 3$ and $Z = \frac{107}{4}$

Now we form the Sub-problem (1-b) and explore it as

Sub-problem (1b):

$$x_1 \geq 6 \Rightarrow -x_1 + S_7 = -6$$

Also $x_1 + 3/4\, S_2 + S_3 + 1/4\, S_4 = 21/4$

Eliminating x_1, we get

$S_7 + 3/4\, S_2 + S_3 + 1/4\, S_4 = -3/4$

Including this restriction to the last tableau of Subprogram 1, we get

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	S_7	Soln.
Z	0	0	0	0	6/2	3	9/2	0	55/2
x_2	0	1	0	0	1/4	0	3/4	0	11/4
x_1	1	0	0	0	3/4	1	1/4	0	21/4
x_3	0	0	1	0	0	0	1	0	3
S_1	0	0	0	1	4	1	-9/4	0	3/4
S_7	0	0	0	0	3/4	1	1/4	1	-3/4

We notice that sub-problems 2 and (1b) ~~without~~ at all never improving the solution because ∃ no entering variable.

Thus we exclude them from our exploration process and accept

$x_1 = 5$; $x_2 = 11/4$; $x_3 = 3$ and $Z = 107/4$

as the optimal Solution.

X Repeated

Fig-1 gives an example of two such cuts. Initially, we start with the continous LP optimum $(x_1, x_2) = (4\frac{1}{2}, 3\frac{1}{2})$ and $z = 66\frac{1}{2}$. next we add Cut I, which produces the continous LP optimum $(x_1, x_2) = (4\frac{4}{7}, 3)$ with $z = 62$. then, we add Cut II, which together with Cut I and the original constraints, produces the LP optimum $(x_1, x_2) = (4, 3)$ and $z = 58$. the last solution is all integer as desired.

the added cuts do not eliminate any of the original feasible integer points, but must pass through at least one feasible or infeasible integer point. these are basic requirements of any cut.

In general, it may take any (finite) number of cuts to reach the desired all-integer extreme point. Indeed, the number of cuts needed to produce the desired integer solution appears to be independent of the size of the problem in the sense that a problem with a small number of variables and constraints may require more cuts than a larger problem.

ALGEBRAIC DEVELOPMENT OF CUTS:-

The cutting plane algorithm starts by solving the continous LP problem. In the optimum LP tableau, we select one of the rows, called source rows for which the basic variable is non integer. the desired cut is constructed from the fractional components of the coefficients of the source row. For this reason, it is reffered to as the fractional cut.

we now develop the fractional cut. the problem is solved as a regular ~~Simp~~ LP programming. If the optimal solution is integer, there is nothing more to be done. otherwise the secondary constraints (cuts), which

x_0	1	0	0	0	$\cdots$ c_1	$\bar{c}_j$	$\bar{c}_n$	β_0
x_1	0	1	0	0	α_1^1	α_1^j	α_1^n	β_1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	0	0	1	0	α_i^1	α_i^j	α_i^n	β_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	0	0	0	1	α_m^1	α_m^j	α_m^n	β_m

The variables x_i $(i=1,2,\ldots,m)$ represents the basic variables, while the variables w_j $(j=1,2,\ldots,n)$ are the non basic variables.

Consider the ith equation where the basic variable x_i assumes a non-integer value i.e.

$x_i = \beta_i - \sum_{j=1}^{n} \alpha_i^j w_j$, β_i being non integer. Any such equation will be reffered to as the source row.

Let $\beta_i = [\beta_i] + f_i$ (f_i is the fractional part)

and

$$\alpha_i^j = [\alpha_i^j] + f_{ij}$$

then $0 < f_i < 1$ and $0 \le f_{ij} < 1$ i.e, each of the non integer coefficients are factored into integer and fractional components, provided that the fractional component is strictly positive. for example $\frac{5}{2} = (2+\frac{1}{2})$, $-\frac{7}{3} = -3 + \frac{2}{3}$

the source row now becomes

$$x_i = [\beta_i] + f_i - \sum_{j=1}^{n} \left([\alpha_i^j] + f_{ij} \right) w_j$$

$$\Rightarrow \quad x_i ~~- [\beta_i] + \sum_j [\alpha_i^j] w_j = f_i - \sum_{j=1}^{n} \alpha_i^j~~$$

$$x_i - [\beta_i] + \sum_{j=1}^{n} [\alpha_i^j] w_j = f_i - \sum_{j=1}^{n} f_{ij} w_j$$

If all the variables w_j and x_i are to be integers, the L.H.S must be integer. It follows that the R.H.S must also be integer.

given $f_{ij} \geq 0$ and $w_j \geq 0$ for all i and j, it follows that $\sum_{j=1}^{n} f_{ij} w_j \geq 0$. Hence

$$f_i - \sum_{j=1}^{n} f_{ij} w_j \leq f_i$$

$$\Rightarrow \quad f_i - \sum_{j=1}^{n} f_{ij} w_j \leq f_i \leq 1 \quad \because f_i < 1$$

$$\Rightarrow \quad f_i - \sum_{j=1}^{n} f_{ij} w_j < 1$$

But the R.H.S must be integer (less than 1), so

$$f_i - \sum_{j=1}^{n} f_{ij} w_j \leq 0$$

$$\Rightarrow \quad \sum_{j=1}^{n} f_{ij} w_j - f_i \geq 0$$

or

$$S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i$$

Where S_i is a non-negative slack variable which must be integer. this constraint equation defines the so-called fractional cut. From the last tableau $w_j = 0$ and thus $S_i = -f_i$, which is infeasible. this means that the new constraint is not satisfied by the given solution.

the dual simplex method can, then, be used to clear the infeasibility, which is equivalent to cutting off the solution space towards the optimal integer solution.

the new tableau after adding the fractional cut will become.

Basic	x_0	x_1	x_i	x_m	w_1	w_j	w_n	S_i	Soln.
x_0	1	0	0	0	$\bar{c}_1$ --	$\bar{c}_j$ --	$\bar{c}_n$	0	β_0
x_1	0	1	0	0	α_1^1 --	α_1^j --	α_1^n	0	β_1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_i	0	0	1	0	α_i^1 --	α_i^j --	α_i^n	0	β_i
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
x_m	0	0	0	1	α_m^1 --	α_m^j --	α_m^n	0	β_m
S_i	0	0	0	0	$-f_{i1}$ --	$-f_{ij}$ --	$-\alpha_{im}$	1	$-f_i$

If the new solution after applying the dual Simplex method is integer, the process ends. otherwise a new fractional cut is constructed from the resulting tableau and the dual simplex method is employed to clear the inpeasibility. the procedure is repeated until an integer solution is achieved.

this algorithm is reffered to as the fractional method.

Main Steps (Fractional Cut method):-

(i) Convert the Coefficients of the Constraints and the right hand side of the Constraint equations with integers by multiplying with G.C.D of the denominators on both sides.

(ii) Solve the problem by Simplex method. If solution is in integers, the process ends.

(iii) If the solution is not in integers, define the Cut and add it to the optimal last tableau of the regular Simplex method. Solve the problem by dual Simplex method.

Example: ① Consider the problem

Maximize $Z = 7x_1 + 9x_2$

subject to

$$-x_1 + 3x_2 \le 6$$

$$7x_1 + x_2 \le 35$$

x_1, x_2 non-negative integers.

Solution: Firstly, we solve the problem by regular simplex method as follows

Basic	x_1	x_2	x_3	x_4	Soln.
Z	-7	-9	0	0	0
x_3	-1	3	1	0	6
x_4	7	1	0	1	35

x_2 enters and x_3 leaves.

Basic	x_1	x_2	x_3	x_4	Soln.
Z	-10	0	3	0	18
x_2	$-1/3$	1	$1/3$	0	2
x_4	$\frac{22}{3}$	0	$-1/3$	1	33

x_1 enters and x_4 leaves

Basic	x_1	x_2	x_3	x_4	Soln.
Z	0	0	$28/11$	$15/11$	63
x_2	0	1	$7/22$	$\frac{1}{22}$	$7/2 = 3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$\frac{99}{22} = \frac{9}{2} = 4\frac{1}{2}$

this last tableau yields the optimal continuous solution. Since the solution is non-integers, a fractional cut must be added to the tableau. In general, any of the constraint equations can be selected to generate a cut. However, as a rule, one usually chooses the equation corresponding to

~~maximum f_i In the current problem, both the~~ equations have the same value of f_i $(=\frac{1}{2})$, so either one may be used.

Consider the x_2-equation. this gives

$$x_2 + \frac{7}{22}x_3 + \frac{1}{22}x_4 = \frac{7}{2}$$

or

$$x_2 + (0+\tfrac{7}{22})x_3 + (0+\tfrac{1}{22})x_4 = 3 + \tfrac{1}{2}$$

Hence the corresponding Fractional Cut is

$$S_i = \sum_{j=1}^{n} f_{ij}\, w_j - f_i$$

ie,

$$S_1 = \frac{7}{22}x_3 + \frac{1}{22}x_4 - \frac{1}{2}$$

Now by adding this in the previous optimal tableau the following new tableau is obtained

Basic	x_1	x_2	x_3	x_4	S_1	Soln
z	0	0	$\frac{28}{11}$	$\frac{15}{11}$	0	63
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$\frac{7}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$\frac{9}{2}$
S_1	0	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$

Now we use the dual Simplex method. S_1 has the most -ve value $(=-\frac{1}{2})$, so it leaves the solution. Now we take the ratios of the L.H.Sides of the z row with S_1 (ignoring the zero and +ve denominators)

For x_3: $\frac{28}{11} \div -\frac{7}{22} = \frac{28}{11} \times -\frac{22}{7} = -8$,

For x_4: $\frac{15}{11} \div -\frac{1}{22} = -30$, x_3 has the least absolute ratio, so x_3 enters the Solution. The new tableau hence is

Basic	x_1	x_2	x_3	x_4	S_1	Soln
z	0	0	0	1	+8	59
x_2	0	1	0	0	+1	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$4\frac{4}{7}$
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	$+\frac{11}{7} = 1+\frac{4}{7}$

The solution is still in non-integers,
So a new cut must be introduced
x_1-equ. has the max $f_i (= 4/7)$
So consider the x_1-equation, this gives.

$$x_1 + \frac{1}{7}x_4 - \frac{1}{7}S_1 = 4 + 4/7$$

$$\Rightarrow x_1 + (0 + \frac{1}{7})x_4 + (-1 + \frac{6}{7})S_1 = 4 + 4/7$$

Hence the corresponding fractional cut is

$$S_2 = \frac{1}{7}x_4 + \frac{6}{7}S_1 - 4/7$$

$$\Rightarrow S_2 - \frac{1}{7}x_4 - \frac{6}{7}S_1 = -4/7$$

Adding this constraint to the last tableau we get

basic	x_1	x_2	x_3	x_4	S_1	S_2	Sol.
Z	0	0	0	1	8	0	59
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$1/7$	$-1/7$	0	$4\frac{4}{7} = \frac{32}{7}$
x_3	0	0	1	$1/7$	$-22/7$	0	$11/7$
S_2	0	0	0	$-1/7$	$-6/7$	1	$-4/7$

Again we apply dual simplex method. S_2 leaves the sol. and x_4 enters. Then the new tableau adopts the form.

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Solution
Z	0	0	0	0	2	7	55
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
x_3	0	0	1	0	-4	1	1
x_4	0	0	0	1	6	-7	4

This tableau yields the integral solution.
Hence, the optimal is $x_1 = 4$, $x_2 = 3$, and $Z = 55$

Ex. 2:- Solve by Fractional algorithm.

Maximize $Z = 4x_1 + 6x_2 + 2x_3$

Subject to

$4x_1 - 4x_2 \le 5$

$-x_1 + 6x_2 \le 5$

$-x_1 + x_2 + x_3 \le 5$

x_1, x_2, x_3 non negative integers

Solution:-

Firstly we solve the problem by regular Simplex method as follows.

Basic	x_1	x_2 ↓	x_3	x_4	x_5	x_6	Soln.
Z	-4	-6	-2	0	0	0	0
x_4	4	-4	0	1	0	0	5
← x_5	-1	6	0	0	1	0	5
x_6	-1	1	1	0	0	1	5

x_2 enters and x_5 leaves:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
Z	-5	0	-2	0	1	0	5
x_4	$10/3$	0	0	1	$2/3$	0	$25/3$
x_2	$-1/6$	1	0	0	$1/6$	0	$5/6$
x_6	$-5/6$	0	1	0	$-1/6$	1	$25/6$

Introduce x_1 and drop x_4

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
Z	0	0	-2	$3/2$	2	0	$\frac{35}{2}$
x_4	1	0	0	$3/10$	$1/15$	0	$5/2$
x_2	0	1	0	$\frac{1}{20}$	$1/5$	0	$\frac{15}{12} = 5/4$
x_6	0	0	1	$1/4$	0	1	$\frac{75}{12} = \frac{25}{4}$

Introduce x_3 and drop x_6

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln
Z	0	0	0	2	2	2	30
x_1	1	0	0	3/10	1/5	0	$5/2 = 2+\frac{1}{2}$
x_2	0	1	0	1/20	1/5	0	$5/4 = 1+\frac{1}{4}$
x_3	0	0	1	1/4	0	1	$\frac{25}{4} = 6+\frac{1}{4}$

this tableau is optimal, but the Solution is non-integers.

In this problem max $f_i (= (\frac{1}{2})$ Corresponds to x_1 - equation. So

$$S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i \text{ leads to}$$

$$S_1 = \frac{3}{10} x_4 + \frac{1}{5} x_5 - \frac{1}{2}$$

After adding it to the last tableau, we got

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln
Z	0	0	0	2	2	2	0	30
x_1	1	0	0	3/10	1/5	0	0	5/2
x_2	0	1	0	1/20	1/5	0	0	5/4
x_3	0	0	1	1/4	0	1	0	25/4
S_1	0	0	0	-3/10	-1/5	0	1	-1/2

Now using dual Simplex method S_1 leaves the sol. and x_4 enters.

the new tableau is.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Sol.
Z	0	0	0	0	2/3	2	20/3	80/3
	0	0	0	0	0	0	1	2
x_1	1	0	0	0	1/6	0	1/6	$7/6 = 1+\frac{1}{6}$
x_2	0	1	0	0	-1/6	1	5/6	$35/6 = 5+\frac{5}{6}$
x_3	0	0	1	0	2/3	0	-10/3	$5/3 = 1+\frac{2}{3}$
x_4	0	0	0	1				

thus $S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i$ yields

$$S_2 = \frac{5}{6} x_5 + 5/6\, S_1 - 5/6$$

Adding this to the last tableau we get

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Soln.
Z	0	0	0	0	2/3	2	20/3	0	80/3
x_1	1	0	0	0	0	0	1	0	2
x_2	0	1	0	0	1/6	0	1/6	0	7/6
x_3	0	0	1	0	-1/6	1	5/6	0	35/6
x_4	0	0	0	1	2/3	0	-10/3	0	5/3
S_2	0	0	0	0	-5/6	0	+5/6	1	-5/6

Now using dual Simplex methods, S_2 leaves and x_5 enters

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Soln
Z	0	0	0	0	0	2	6	4/5	26
x_1	1	0	0	0	0	0	1	0	2
x_2	0	1	0	0	0	0	0	1/5	1
x_3	0	0	1	0	0	1	1	-1/5	6
x_4	0	0	0	1	0	0	-4	+4/5	1
x_5	0	0	0	0	1	0	1	$-\frac{6}{5}$	1

this tableau yields the integral solution.
Hence, the optimal is

$x_1 = 2$, $x_2 = 1$, $x_3 = 6$, and $Z = 26$

Note that the rounded optimal solution is

$x_1 = 3$, $x_2 = 1$, $x_3 = 6$ & $Z = 30$

Ex. 3. Solve by the fractional algorithm

Maximize $z = 3x_1 + x_2 + 3x_3$

subject to

$-x_1 + 2x_2 + x_3 \le 4$

$4x_2 - 3x_3 \le 2$

$x_1 - 3x_2 + 2x_3 \le 3$; $x_1, x_2, x_3 \ge 0$ and integers.

Compare the rounded optimal solution and the integer optimal solution.

Solution:

Firstly, we solve the problem by regular Simplex method as follows:

Basic	x_1 ↓	x_2	x_3	x_4	x_5	x_6	Soln.
z	-3	-1	-3	0	0	0	0
x_4	-1	2	1	1	0	0	4
x_5	0	4	-3	0	1	0	2
← x_6	1	-3	2	0	0	1	3

Introduce x_1 and drop x_6

Basic	x_1	x_2 ↓	x_3	x_4	x_5	x_6	Soln.
z	0	-10	3	0	0	3	9
x_4	0	-1	3	1	0	1	7
← x_5	0	4	-3	0	1	0	2
x_1	1	-3	2	0	0	1	3

Introduce x_2 and drop x_5

Basic	x_1	x_2	x_3 ↓	x_4	x_5	x_6	Soln.
z	0	0	$-9/2$	0	$5/2$	3	14
← x_4	0	0	$9/4$	1	$1/4$	1	$15/2$
x_2	0	1	$-3/4$	0	$1/4$	0	$1/2$
x_1	1	0	$-1/4$	0	$3/4$	1	$9/2$

Introduce x_3 and leaves x_4.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
z	0	0	0	2	3	5	29
x_3	0	0	1	$4/9$	$1/9$	$4/9$	$10/3 = 3\frac{1}{3}$
x_2	0	1	0	$1/3$	$1/3$	$1/3$	3
x_1	1	0	0	$1/9$	$7/9$	$10/9$	$16/3$ $5\frac{1}{3}$

this tableau is optimal, but the solution is non-integers.
Thus the rounded optimal solution is

$x_3 = 3;\ x_2 = 3,\ x_1 = 5;\ Z = 29$, which is infeasible because it does not satisfy the 2nd constraint.

In this tableau, x_1-equation is

$$x_1 + \frac{1}{9}x_4 + \frac{7}{9}x_5 + \frac{10}{9}x_6 = 5 + \frac{1}{3}$$

$$\Rightarrow x_1 + (0 + \tfrac{1}{9})x_4 + (0 + \tfrac{7}{9})x_5 + (1 + \tfrac{1}{9})x_6 = 5 + \frac{1}{3}.$$

Hence $S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i$ yields

$$S_1 = \frac{1}{9}x_4 + \frac{7}{9}x_5 + \frac{1}{9}x_6 - \frac{1}{3}$$

after adding this constraint in the last optimal tableau, we get

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln.
Z	0	0	0	2	3	5	0	29
x_3	0	0	1	4/9	1/9	4/9	0	10/3
x_2	0	1	0	1/3	1/3	1/3	0	3
x_1	1	0	0	1/9	7/9	10/9	0	16/3
S_1	0	0	0	-1/9	-7/9	-1/9	1	-1/3

now we use dual simplex method,

S_1 leaves and x_5 enters

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln
Z	0	0	0	11/7	0	32/7	27/7	~~194~~/7
x_3	0	0	1	3/7	0	3/7	1/7	23/7 = 3 + 2/7
x_2	0	1	0	2/7	0	2/7	3/7	20/7 = 2 + 6/7
x_1	1	0	0	0	0	1	1	5
x_5	0	0	0	1/7	1	1/7	-9/7	3/7

the x_2-eq. is

$$x_2 + \frac{2}{7}x_4 + \frac{2}{7}x_6 + \frac{3}{7}S_1 = 2 + \frac{6}{7}.$$

Thus $S_i = \sum_{j=1}^{n} f_{ij} w_j - f_i$ yields

$$S_2 = \frac{2}{7}x_4 + \frac{2}{7}x_6 + \frac{3}{7}S_1 - \frac{6}{7}.$$

After adding this constraint to the last tableau we have:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Sol.
z	0	0	0	$11/7$	0	$32/7$	$27/7$	0	$194/7$
x_3	0	0	1	$3/7$	0	$-3/7$	$1/7$	0	$23/7$
x_2	0	1	0	$2/7$	0	$2/7$	$3/7$	0	$20/7$
x_1	1	0	0	0	0	1	$1/7$	0	5
x_5	0	0	0	$1/7$	1	$1/7$	$-9/7$	0	$3/7$
S_2	0	0	0	$-\frac{2}{7}$	0	$-\frac{2}{7}$	$-3/7$	1	$-\frac{6}{7}$

Again use dual simplex method
S_2 leaves and x_4 enters

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	S_2	Sol.
z	0	0	0	0	0	3	$3/2$	$11/2$	23
x_3	0	0	1	0	0	$1/2$	$-1/2$	$+3/2$	2
x_2	0	1	0	0	0	0	0	1	2
x_1	1	0	0	0	0	1	1	0	5
x_5	0	0	0	0	1	0	$-7/2$	$1/2$	0
x_4	0	0	0	1	0	1	$3/2$	$-\frac{7}{2}$	3

Thus the required integeral solution is
$x_1 = 5,\ x_2 = 2 = x_3\ ;\ Z = 23$

THE MIXED ALGORITHM:-

Let x_k be an integer variable of the mixed problem. Consider the x_k-equation in the optimal continous solutions, given by

$$x_k = \beta_k - \sum_{j=1}^{n} \alpha_k^j w_j \quad \text{(source row)}$$

$$= [\beta_k] + f_k - \sum_{j=1}^{n} \alpha_k^j w_j$$

where $\beta_k = [\beta_k] + f_k$

or

$$x_k - [\beta_k] = f_k - \sum_{j=1}^{n} \alpha_k^j w_j$$

for x_k to be integer

either $x_k \le [\beta_k]$ → ①

or $x_k \ge [\beta_k] + 1$ → ②

Using the source row, eq ① becomes

$$f_k - \sum_{j=1}^{n} \alpha_k^j w_j \le 0$$

or $$\sum_{j=1}^{n} \alpha_k^j w_j \ge f_k \longrightarrow ③$$

and (2) becomes

$$f_k - \sum_{j=1}^{n} \alpha_k^j w_j \ge 1$$

$$\Rightarrow \sum_{j=1}^{n} \alpha_k^j w_j \le f_k - 1 \longrightarrow ④$$

Note that only one of the equations ③ and ④ can hold at a time.

Let

J^+ = set of subscripts j for which $\alpha_k^j \ge 0$

J^- = " " " " $\alpha_k^j < 0$

Then

$$③ \Rightarrow \sum_{j \in J^+} \alpha_k^j w_j \ge f_k \longrightarrow ⑤$$

$$④ \Rightarrow \sum_{j \in J^-} (-\alpha_k^j) w_j \le f_k - 1$$

$$\Rightarrow \sum_{j \in J^-} \alpha_k^j w_j \ge f_k - 1$$

$$\Rightarrow \left(\frac{f_k}{f_k - 1}\right) \sum_{j \in J^-} \alpha_k^j w_j \ge f_k \longrightarrow ⑥$$

⑤ and ⑥ can be combined into one constraint as

$$\sum_{j \in J^+} \alpha_k^j w_j + \left(\frac{f_k}{f_k - 1}\right) \sum_{j \in J^-} \alpha_k^j w_j - f_k \ge 0$$

this constraint can now be put in the form.

$$S_K = \sum_{j \in J^+} \alpha_K^j w_j + \left(\frac{f_K}{f_K - 1}\right) \sum_{j \in J^-} \alpha_K^j w_j - f_K$$

OR

$$S_K - \left\{\sum_{j \in J^+} \alpha_K^j w_j + \left(\frac{f_K}{f_K - 1}\right) \sum_{j \in J^-} \alpha_K^j w_j\right\} = -f_K$$

this equation is the required mixed cut. Since all $w_j = 0$ at the current optimal tableau, it follows that the above cut is infeasible. The dual simplex method is thus used to clear the infeasibility.

Example:-(18)

Maximize $Z = 7x_1 + 9x_2$

subject

$$-x_1 + 3x_2 \le 6$$
$$7x_1 + x_2 \le 35$$

x_1 non-negative integer ; $x_2 \geq 0$

Solution:- Firstly, we solve the problem by regular simplex method as follows (as in Ex: 1) - optimal sol is:

Basic	x_1	x_2	x_3	x_4	Soln.
Z	0	0	28/11	15/11	63
x_2	0	1	7/22	1/22	7/2
x_1	1	0	-1/22	3/22	9/2

Now since only x_1 is restricted to integer value, so we consider x_1 equation of above tableau.

$$x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = \frac{9}{2} = 4 + \frac{1}{2}$$

then the mixed cut is given by

$$S_1 = \left\{\sum_{j \in J^+} \alpha_1^j w_j + \left(\frac{f_1}{f_1 - 1}\right) \sum_{j \in J^-} \alpha_1^j w_j\right\} - f_1$$

i.e.,

$$S_1 = \left\{\frac{3}{22}x_4 + \left(\frac{1/2}{1/2 - 1}\right)\left(-\frac{1}{22}x_3\right)\right\} - 1/2$$

or

$$S_1 = \frac{3}{22}x_4 + \frac{1}{22}x_3 - \frac{1}{2}$$

or $S_1 - \frac{1}{22}x_3 - \frac{3}{22}x_4 = -\frac{1}{2}$.

Adding this Constraint to the last tableau, We have

Basic	x_1	x_2	x_3	x_4	S_1	Soln.
Z	0	0	28/11	15/11	0	63
x_2	0	1	7/22	1/22	0	7/2
x_1	1	0	-1/22	3/22	0	9/2
S_1	0	0	-1/22	-3/22	1	-1/2

Now, We apply the dual Simplex method.

S_1 leaves and x_4 enters the solution

Basic	x_1	x_2	x_3	x_4	S_1	Soln.
Z	0	0	23/11	0	10	58
x_2	0	1	10/33	0	1/3	10/3
x_1	1	0	$-\frac{1}{11}$	0	1	4
x_4	0	0	$\frac{1}{3}$	1	$-\frac{22}{3}$	$\frac{11}{3} = 3\frac{2}{3}$

This tableau yeilds the requisite optimal solution as

$$x_1 = 4\ ;\ x_2 = 10/3\ ;\ Z = 58$$

Ex(2) Maximize $Z = 4x_1 + 6x_2 + 2x_3$

Subject to

$$4x_1 - 4x_2 \leq 5$$
$$-x_1 + 6x_2 \leq 5$$
$$-x_1 + x_2 + x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Solve the following problem by the mixed algorithm assuming that x_1 and x_3 are the only integer variables

Solution: We know (from ex. 1) that the optimal tableau for this problem is:

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Soln.
z	0	0	0	2	2	2	30
x_1	1	0	0	$3/10$	$1/5$	0	$5/2 = 2+\frac{1}{2}$
x_2	0	1	0	$1/20$	$1/5$	0	$5/4$
x_3	0	0	1	$1/4$	0	1	$25/4 = 6+\frac{1}{4}$

Since x_1 and x_3 are restricted to integer values.

Consider the x_1-equation

$$x_1 + \frac{3}{10}x_4 + \frac{1}{5}x_5 = 5/2 = 2 + \frac{1}{2}$$

The mixed cut is, hence

$$S_1 - \left\{ \sum_{j\in J^+} a_1^j w_j + \left(\frac{f_1}{f_1-1}\right) \sum_{j\in J^-} a_1^j w_j \right\} = -f_1$$

i.e. $S_1 - \{\frac{3}{10}x_4 + \frac{1}{5}x_5 + 0\} = -\frac{1}{2}$

$$\Rightarrow S_1 - \frac{3}{10}x_4 - \frac{1}{5}x_5 = -\frac{1}{2}$$

Adding this constraint to the last tableau we have

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Soln.
z	0	0	0	2	2	2	0	30
x_1	1	0	0	$3/10$	$1/5$	0	0	$5/2$
x_2	0	1	0	$1/20$	$1/5$	0	0	$5/4$
x_3	0	0	1	$1/4$	0	1	0	$25/4$
S_1	0	0	0	$-3/10$	$-1/5$	0	1	$-1/2$

Now we use the dual simplex method

S_1 leaves and x_4 enters the sol.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	S_1	Sol.
z	0	0	0	0	$8/3$	2	$+\frac{20}{3}$	$\frac{80}{3}$
x_1	1	0	0	0	0	0	1	2
x_2	0	1	0	0	$1/6$	0	$1/6$	$7/6$
x_3	0	0	1	0	$-1/6$	1	$5/6$	$35/6$
x_4	0	0	0	1	$\frac{2}{3}$	0	$-\frac{10}{3}$	$\frac{5}{3}$

Now consider the x_3 equation

$$x_3 - \frac{1}{6}x_5 + x_6 + \frac{5}{6}S_1 = 35/6$$

Ex. Solve the following problem by the mixed algorithm assuming that x_1 and x_3 are the only integer variables.

Maximize $z = 3x_1 + x_2 + 3x_3$

Subject to

$$-x_1 + 2x_2 + x_3 \le 4$$

$$4x_2 - 3x_3 \le 2$$

$$x_1 - 3x_2 + 2x_3 \le 3$$

$$x_1, x_2, x_3 \ge 0$$

BRANCH AND BOUND METHOD:

In this method, the problem is first solved as a continuous model. Suppose that x_r is a variable which is restricted to integer value, whose optimum continuous value x_r^* is fractional. The range $[x_r^*] < x_r < [x_r^*] + 1$ cannot include any feasible integer solution. As a consequence, a feasible integer value of x_r must satisfy one of two conditions:

$x_r \leq [x_r^*]$ or $x_r \geq [x_r^*] + 1$

We obtain two mutually exclusive problems by adding each of the above constraints to the original problem. The original problem is then said to be branched into two sub-problems.

Now, each sub problem may be solved as a linear program. If its optimal is feasible w.r.t the integer problem, its solution is recorded as the best one so far available. Otherwise, the sub-problem must be ~~partioned~~ partitioned into two sub-problems. When a better integer feasible solution is obtained for any sub-problem, it replaces the one at hand.

The efficiency of the computations is enhanced by introducing the notation of 'Bounding'. According to this concept, if the continuous optimum solution of a sub-problem yields a worse objective value than the one associated with the best available integer solution, then there is no need to explore the problem any further.

Example:- Maximize $Z = 2x_1 + 3x_2$
Subject to
$$5x_1 + 7x_2 \leq 35$$
$$4x_1 + 9x_2 \leq 36$$
x_1, x_2 are non-negative integers.

Solution:- Firstly, we solve the problem by regular Simplex method as follows.

Basic	x_1	x_2 ↓	x_3	x_4	Soln
z	-2	-3	0	0	0
x_3	5	7	1	0	35
← x_4	4	9	0	1	36

Introduce x_2 and drop x_4

Basic	x_1	x_2	x_3	x_4	Soln
z	$-2/3$	0	0	$1/3$	12
x_3	$17/9$	0	1	$-7/9$	7
x_2	$4/9$	1	0	$1/9$	4

Introduce x_1 and drop x_3

Basic	x_1	x_2	x_3	x_4	Soln
z	0	0	$6/17$	$1/17$	$246/17$
x_1	1	0	$9/17$	$-\frac{7}{17}$	$\frac{63}{17}$
x_2	0	1	$\frac{-4}{17}$	$5/17$	$\frac{40}{17}$

Both the variables x_1, x_2 have fractional values in the optimum continuous solution. We select x_2, arbitrarily, for branching then two problems are created by the restrictions

$$x_2 \leq \left[\frac{40}{17}\right] \text{ and } x_2 \geq \left[\frac{40}{17}\right] + 1$$

$$\Rightarrow \quad x_2 \leq 2 \quad \text{and} \quad x_2 \geq 3$$

then the subproblems are

<u>Sub-Problem 1</u>: $x_2 \leq 2 \Rightarrow x_2 + S_1 = 2$

Also from the last tableau

$$x_2 - \frac{4}{17} x_3 + \frac{5}{17} x_4 = \frac{40}{17}$$

$$\Rightarrow 2 - S_1 - \frac{4}{17} x_3 + \frac{5}{17} x_4 = \frac{40}{17}$$

$$\Rightarrow \quad S_1 + \frac{4}{17} x_3 - \frac{5}{17} x_4 = 2 - \frac{40}{17} = \frac{-6}{17}$$

Adding this constraint to the optimal continuous tableau of original problem, we get

Basic	x_1	x_2	x_3	x_4	S_1	Solu
Z	0	0	6/17	1/17	0	246/17
x_1	1	0	9/17	-7/17	0	63/17
x_2	0	1	-4/17	5/17	0	40/17
S_1	0	0	4/17	-5/17	1	-6/17

Sub Program 2:

$$x_2 \geq 3 \Rightarrow -x_2 + S_2 = -3$$

Eliminating x_2, we obtain

$$x_2 - \frac{4}{17}x_3 + \frac{5}{17}x_4 = 40/17$$

$$S_2 + 3 - \frac{4}{17}x_3 + \frac{5}{17}x_4 = \frac{40}{17}$$

$$S_2 - \frac{4}{17}x_3 + \frac{5}{17}x_4 = \frac{-11}{17}$$

Adding this constraint to the last tableau, we obtain

Basic	x_1	x_2	x_3	x_4	S_2	Solu.
Z	0	0	6/17	1/17	0	246/7
x_1	1	0	9/17	-7/17	0	63/17
x_2	0	1	-4/17	5/17	0	40/17
S_2	0	0	-4/17	5/17	1	-11/17

Now, we select the subproblem 1 to arbitrarily for investigation and use dual simplex method.

Introduce x_4 and leaves S_1

Basic	x_1	x_2	x_3	x_4	S_1	Solu.
Z	0	0	2/5	0	1/5	72/5
x_1	1	0	1/5	0	-7/5	21/5
x_2	0	1	0	0	1	2
x_4	0	0	-4/5	1	-17/5	6/5

Since x_1 has fractional value, it is now selected for branching and two sub-problems are further proliferated by introducing each of the constraints

$$x_1 \leq \left[\frac{21}{5}\right] \quad \text{and} \quad x_1 \geq \left[\frac{21}{5}\right] + 1$$

$$x_1 \leq 4 \quad \text{and} \quad x_1 \geq 5$$

Sub-problem (1.2) :-

$$x_1 \le 4 \quad \Rightarrow \quad x_1 + S_3 = 4$$

Also from the last tableau

$$x_1 + \tfrac{1}{5} x_3 - \tfrac{7}{5} s_1 = \tfrac{21}{5}$$

$$\Rightarrow \quad 4 - S_3 + \frac{1}{5} x_3 - \tfrac{7}{5} S_1 = \frac{21}{5}$$

$$\Rightarrow \quad S_3 - \frac{1}{5} x_3 + \frac{7}{5} S_1 = -\tfrac{1}{5}$$

adding this Constraint to the optimal Continuous tableau of problem 1, we have.

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln.
z	0	0	2/5	0	1/5	0	72/5
x_1	1	0	1/5	0	-7/5	0	21/5
x_2	0	1	0	0	1	0	2
x_4	0	0	-4/5	1	-17/5	0	6/5
S_3	0	0	-1/5	0	7/5	1	-1/5

Subproblem (1.3) :-

$$x_1 \ge 5 \quad \Rightarrow \quad -x_1 + S_4 = -5$$

Eliminating x_1, we obtain

$$S_4 + \frac{1}{5} x_3 - \tfrac{7}{5} S_1 = -\tfrac{4}{5}$$

Including this Constraint to the optimal Continous tableau of probl 1, we have

Basic	x_1	x_2	x_3	x_4	S_1	~~S_4~~	S_4	Soln
z	0	0	2/5	0	1/5	~~72/5~~	0	72/5
x_1	1	0	1/5	0	-7/5	~~7/5~~	0	2/5
x_2	0	1	0	0	1	~~2~~	0	2
x_4	0	0	-4/5	1	-17/5	~~6/5~~	0	6/5
S_4	0	0	1/5	0	-7/5		1	-4/5

We select sub-program 3 arbitrarily for investigation.

Introduce x_3 and drop S_3

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln
Z	0	0	0	0	3	2	14
x_1	1	0	0	0	0	1	4
x_2	0	1	0	0	1	0	2
x_4	0	0	0	1	-9	-4	2
x_3	0	0	1	0	-7	-5	1

Since this solution is in integer, it provides us a lower bound $Z = 14$. We still need to investigate sub problems 2 and 4 for potential of improvement in the solution.

Sub problem 2: Introduce x_3 and drop S_2

Basic	x_1	x_2	x_3	x_4	S_2	Soln
Z	0	0	0	1/34	3/2	27/2
x_1	1	0	0	1/4	9/4	9/2
x_2	0	1	0	0	-1	3
x_3	0	0	1	$-\frac{5}{4}$	$-\frac{17}{4}$	$\frac{11}{4}$

This solution is less than the lower bound. So there is no need to investigate it further.

Sub problem 1:

Introduce S_1 and drop S_4

Basic	x_1	x_2	x_3	x_4	S_1	S_4	Soln
Z	0	0	3/7	0	0	1/7	100/7
x_1	1	0	0	0	0	-1	5
x_2	0	1	1/7	0	0	5/7	10/7
x_4	0	0	-9/7	1	0	-17/7	22/7
S_1	0	0	-1/7	0	1	-5/7	$\frac{4}{7}$

As the solution yielded by this subproblem is fractionally higher than the lower bound, any branching from this subproblem cannot yield a better value than the lower bound across the all coefficients of the objective function are integers. It follows that the soln. is

$Z = 14, \quad x_1 = 4, \quad x_2 = 2$ yielded by sub-problem 3.

Ex. Solve the following problem by branch and bound algorithm.

Maximize $z = x_1 + x_2$
Subject to $2x_1 + 5x_2 \le 16$
$6x_1 + 5x_2 \le 30$
x_1, x_2 non-negative integers

Soln.

Firstly, we solve the problem by regular Simplex method as follows.

Basic	x_1 ↓	x_2	x_3	x_4	Soln.
z	-1	-1	0	0	0
x_3	2	5	1	0	16
← x_4	6	5	0	1	30

Introduce x_1 and leaves x_4

Basic	x_1	x_2 ↓	x_3	x_4	Soln.
z	0	$-1/6$	0	$1/6$	5
← x_3	0	$10/3$	1	$-1/3$	6
x_1	1	$5/6$	0	$1/6$	5

Introduce x_2 and leave x_3

Basic	x_1	x_2	x_3	x_4	Soln.
z	0	0	$1/20$	$3/20$	$53/10$
x_2	0	1	$\frac{3}{10}$	$-\frac{1}{10}$	$9/5$
x_1	1	0	$-1/4$	$1/4$	$7/2$

We arbitrarily select x_2 for branching then two problems are created by the restrictions.

$$x_2 \le \left[\frac{9}{5}\right] \quad , \quad x_2 \ge \left[\frac{9}{5}\right] + 1$$

$$\Rightarrow \quad x_2 \le 1 \quad , \quad x_2 \ge 2$$

then the subproblems are

Sub-Problem 1:-

$$x_2 \le 1 \quad \Rightarrow \quad x_2 + S_1 = 1$$

from the last table the x_2-eqn is

$$x_2 + \frac{3}{10} x_3 - \frac{1}{10} x_4 = 9/5$$

$$\Rightarrow 1 - S_1 + \frac{3}{10}x_3 - \frac{1}{10}x_4 = 9/5$$

$$\Rightarrow S_1 - \frac{3}{10}x_3 + \frac{1}{10}x_4 = -4/5$$

Adding this Constraint to the last tableau we have.

Basic	x_1	x_2	x_3	x_4	S_1	Soln
Z	0	0	$\frac{1}{20}$	$3/20$	0	$53/10$
x_2	0	1	$3/10$	$-1/10$	0	$9/5$
x_1	1	0	$-1/4$	$1/4$	0	$7/2$
S_1	0	0	$-3/10$	$1/10$	1	$-4/5$

Now we solve this program &
introduce x_3 and drop S_1

Basic	x_1	x_2	x_3	x_4	S_1	Soln
Z	0	0	0	$1/6$	$1/6$	$31/6$
x_2	0	1	0	0	1	1
x_1	1	0	0	$1/6$	$-5/6$	$25/6$
x_3	0	0	1	$-\frac{1}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$

Here x_1 has non integer value. So two further sub-problems are formed as

$$x_1 \le \left[\frac{25}{6}\right] \quad , \quad x_1 \ge \left[\frac{25}{6}\right] + 1$$

$$x_1 \le 4 \quad , \quad x_1 \ge 5$$

Sub-program (1a):- $x_1 \le 4 \Rightarrow x_1 + S_2 = 4$

and Also

$$x_1 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$$

$$\Rightarrow 4 - S_2 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$$

$$\Rightarrow S_2 - \frac{1}{6}x_4 + 5/6\, S_1 = -1/6$$

Adding this restriction to the last tableau, we have.

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Soln.
Z	0	0	0	1/6	1/6	0	31/6
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1/6	-5/6	0	25/6
x_3	0	0	1	-1/3	-1/3	0	8/3
S_2	0	0	0	-1/6	5/6	1	-1/6

Now we solve this problem by dual S. method as introduce x_4 and leaving S_2 so

Basic	x_1	x_2	x_3	x_4	S_1	S_2	Soln.
Z	0	0	0	0	1	1	5
x_2	0	1	0	0	1	0	1
x_1	1	0	0	0	0	1	4
x_3	0	0	1	0	-5	-2	3
x_4	0	0	0	1	-5	-6	1

this tableau yields an integer sol. as $x_1 = 4$; $x_2 = 1$ and $Z = 5$

We record this sol and try for the possible improvement in the sol.

Sub-Problem (1.b) $\quad x_1 \geq 5$

$$\Rightarrow -x_1 + S_3 = -5$$

Also

$$x_1 = S_3 + 5$$

$$x_1 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$$

$$\Rightarrow S_3 + 5 + \frac{1}{6}x_4 - 5/6\, S_1 = 25/6$$

$$S_3 + \frac{1}{6}x_4 - 5/6\, S_1 = \frac{25}{6} - 5 = -5/6$$

So Adding the Constraint on the tableau of continuous optimal table of subproblem 1. we have

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln
Z	0	0	0	1/6	1/6	0	31/6
x_2	0	1	0	0	1	0	1
x_1	1	0	0	1/6	-5/6	0	25/6
x_3	0	0	1	-1/3	-1/3	0	8/3
S_3	0	0	0	1/6	-5/6	1	-5/6

~~Now we explore this problem~~

Introduce S_1 and leaves S_3

Basic	x_1	x_2	x_3	x_4	S_1	S_3	Soln.
Z	0	0	0	1/5	0	1/5	5
x_2	0	1	0	$\frac{1}{5}$	0	6/5	0
x_1	1	0	0	0	0	−1	5
x_3	0	0	1	−1	0	−4	6
S_1	0	0	0	$-\frac{1}{5}$	1	−6/5	1

It is also an integer solution with the same of value of Z. Thus $x_1 = 5$, $x_2 = 0$, $Z = 5$ is the alternative optimum.

Now we try to explore the subproblem 2.

Sub-problem 2: $x_2 \geq 2 \Rightarrow$

$$-x_2 + S_4 = -2$$

Also from the original optimal table, the x_2-eq. is

$$x_2 + \frac{3}{10} x_3 - \frac{1}{10} x_4 = 9/5$$

$$\Rightarrow S_4 + 2 + \frac{3}{10} x_3 - \frac{1}{10} x_4 = 9/5$$

$$\Rightarrow$$

$$S_4 + \frac{3}{10} x_3 - \frac{1}{10} x_4 = -1/5$$

So Adding this constraint to the optimal table ~~of original function~~ obtained by S-Method

Basic	x_1	x_2	x_3	x_4	S_4	Soln.
Z	0	0	1/20	3/20	0	53/10
x_2	0	1	3/10	−1/10	0	9/5
x_1	1	0	−1/4	1/4	0	7/2
S_4	0	0	3/10	−1/10	1	−1/5

Introducing x_4 and drop S_4

Basic	x_1	x_2	x_3	x_4	S_4	Sol
Z	0	0	1/2	0	3/2	5
x_2	0	1	0	0	-1	2
x_1	1	0	1/2	0	5/2	3
S_4	0	0	-3	1	-10	2

this tableau also gives the same value of Z. thus we have three alternative optima (4, 1), (5, 0), (3, 2) with Z = 5

Ex. Solve the following problem by branch and bound method

Maximize $Z = 3x_1 + x_2 + 3x_3$
subject to

$$-x_1 + 2x_2 + x_3 \leq 4$$
$$4x_2 - 3x_3 \leq 2$$
$$x_1 - 3x_2 + 2x_3 \leq 3$$

$x_2 \geq 0$ and x_1, x_3 -- non-negative integer

Soln.

We firstly solve the problem by regular simplex method as follows (optimal sol. is)

Basic	x_1	x_2	x_3	S_1	S_2	S_3	Sol.
Z	0	0	0	2	3	5	29
x_2	0	1	0	1/3	1/3	1/3	3
x_1	1	0	0	1/9	7/9	10/9	16/3
x_3	0	0	1	4/9	1/9	4/9	10/3

Here x_1, x_3 have assumed non-integer values, we arbitrary choose x_3. Thus

$$x_3 \leq \left[\frac{10}{3}\right] \quad \text{and} \quad x_3 \geq \left[\frac{10}{3}\right] + 1$$

i.e.,

$$x_3 \leq 3 \quad \text{and} \quad x_3 \geq 4$$

Hence the two sub-problems are Subproblem 1 and Sub problem 2.

Including this constraint to the last table we have,

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	Sol.
Z	0	0	0	2	3	5	0	29
x_2	0	1	0	1/3	1/3	1/3	0	3
x_1	1	0	0	1/9	7/9	10/9	0	16/3
x_3	0	0	1	4/9	1/9	4/9	0	10/3
S_4	0	0	0	-4/9	-1/9	-4/9	1	-1/3

<u>Sub-problem 2:-</u>

$$x_3 \geq 4 \Rightarrow -x_3 + S_5 = -4$$

Also

$$x_3 + \frac{4}{9} S_1 + \frac{1}{9} S_2 + \frac{4}{9} S_3 = 10/3$$

Eliminating x_3, we get

$$S_5 + \frac{4}{9} S_1 + \frac{1}{9} S_2 + \frac{4}{9} S_3 = -2/3$$

Including this constraint to the last tableau of the original problem, we get

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_5	Sol.
Z	0	0	0	2	3	5	0	29
x_2	0	1	0	1/3	1/3	1/3	0	3
x_1	1	0	0	1/9	7/9	10/9	0	16/3
x_3	0	0	1	4/9	1/9	4/9	0	10/3
S_5	0	0	0	4/9	1/9	4/9	1	-2/3

Now we arbitrarily select sub-program 1 for investigation; we use dual simplex method:

S_5 leaves and S_1 enters

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	Solution
Z	0	0	0	0	5/2	3	9/2	55/2
x_2	0	1	0	0	1/4	0	3/4	11/4
x_1	1	0	0	0	3/4	1	1/4	21/4
x_3	0	0	1	0	0	0	1	3
S_1	0	0	0	1	4	1	-9/4	3/4

Now since x_1 has fractional value, so it is now selected for further branching. Thus

$$x_1 \leq \left[\frac{21}{4}\right] \quad \text{and} \quad x_1 \geq \left[\frac{21}{4}\right]+1$$

$$\Rightarrow \quad x_1 \leq 5 \quad \text{and} \quad x_1 \geq 6$$

then the two subproblems are

i, Sub-problem (1.2) for $x_1 \leq 5$ and Sub-problem (1.3) for $x_1 \geq 6$

Sub-problem (1.2) $x_1 \leq 5 \Rightarrow x_1 + S_6 = 5$

Also $x_1 + \frac{3}{4}S_2 + S_3 + \frac{1}{4}S_4 = 21/4$

Eliminating x_1, we get

$$S_6 - \frac{3}{4}S_2 - S_3 - \frac{1}{4}S_4 = -\frac{1}{4}$$

Adding this constraint to the last tableau of sub program 1, we have

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	S_6	Sol
Z	0	0	0	0	5/2	3	9/2	0	55/2
x_2	0	1	0	0	1/4	0	3/4	0	11/4
x_1	1	0	0	0	3/4	1	1/4	0	21/4
x_3	0	0	1	0	0	0	1	0	3
S_1	0	0	0	1	4	1	-9/4	0	3/4
S_6	0	0	0	0	-3/4	-1	-1/4	1	-1/4

Now we investigate Sub-problem (1.2) drop S_6 and enters S_3

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	S_6	Sol
Z	0	0	0	0	1/4	0	15/4	3	107/4
x_2	0	1	0	0	1/4	0	3/4	0	11/4
x_1	1	0	0	0	0	0	0	1	5
x_3	0	0	1	0	0	0	1	0	3
S_1	0	0	0	1	13/4	0	-5/2	1	1/2
S_3	0	0	0	0	3/4	1	1/4	-1	1/4

As x_1 and x_3 are integers, So this solution is acceptable. So we record this sol as the best available.

$x_1 = 5$; $x_2 = 11/4$; $x_3 = 3$ and $Z = \frac{107}{4}$

Now we form the sub-problem (1-b) and explore it as

Sub-problem (1b):

$$x_1 \geq 6 \Rightarrow -x_1 + S_7 = -6$$

Also $x_1 + \frac{3}{4}S_2 + S_3 + \frac{1}{4}S_4 = \frac{21}{4}$

Eliminating x_1, we get

$\therefore S_7 + \frac{3}{4}S_2 + S_3 + \frac{1}{4}S_4 = -\frac{3}{4}$

Including this restriction to the last tableau of Subprogram 1, we get

Basic	x_1	x_2	x_3	S_1	S_2	S_3	S_4	S_7	Soln.
Z	0	0	0	0	5/2	3	9/2	0	55/2
x_2	0	1	0	0	1/4	0	3/4	0	11/4
x_1	1	0	0	0	3/4	1	1/4	0	21/4
x_3	0	0	1	0	0	0	1	0	3
S_1	0	0	0	1	4	1	-9/4	0	3/4
S_7	0	0	0	0	3/4	1	1/4	1	-3

We notice that sub-problems 2 and (1b) are ~~all~~ never improving the solution because altering variable.

Thus we exclude them from our exp process and ~~~~ accept.

$x_1 = 5$; $x_2 = \frac{11}{4}$; $x_3 = 3$ and z

as the optimal solution.