MULTIVARIABLE
CALCULUS

A COLLECTION OF LECTURES DELIVERED AT
UNIVERSITY OF EDUCATION
JAUHARABAD SUB-CAMPUS

WRITTEN BY
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MULTIVARIABLE CALCULUS

WHAT IS CALCULUS?

ICE-BREAKING SESSION:

- Have you ever wondered how we can understand the speed of a moving object at any instant of time?
- Did you know that Calculus can help us predict future trends by analyzing patterns in data?
- Picture a scenario where you could optimize a process and obtain best possible solution.
- Have you ever thought about how bridges and skyscrapers are designed to withstand different forces?
- Imagine you can understand the behavior of biological systems, like how population change over time.
- What if you could analyze the behavior of financial markets or predict economic trends?

WHAT IS CALCULUS?

- **Calculus is a branch of Mathematics dealing with rates of change and accumulation.**
  - Provides tool for studying rates of change.
  - Helps us analyze how quantities accumulate over time/space.
  - Important in the fields like Physics, Engineering, and Economics.
- **Calculus is a bridge between Algebra and Geometry.**
  - Connects algebraic equation with respective geometric shapes.
  - Allows us to manipulate equations to view their visual implications.

REAL LIFE APPLICATIONS OF CALCULUS:

- **Motion Analysis**

    - Helps in analyzing velocities and accelerations of moving objects.
    - Helps in understanding phenomenon like Projectile Motion.
- **Economics**
    - Used to analyze change in demand, supply, and profit.
    - Helps in decision making.
- **Medicine**
    - Helps in determining transmission/infection rate of a disease and rate of drug concentration in bloodstream.
    - Used to study growth of tumors.
- **Engineering**
    - Used to design structures, analyze circuits, and optimize designs for safety and efficiency.
- **Physics**
    - Essential for studying physical phenomenon.
- **Computer Graphics**
    - Used to design flawless animations.
- **Astronomy**
    - Used to predict celestial events.
- **Environmental Sciences**
    - Used to study changes in environmental factors over time.
- **Chemistry**
    - Used to determine reaction rates and equilibrium concentrations.

SINGLE VARIABLE CALCULUS:

- This branch deals with functions of a single variable.
- It includes topics like limits, derivatives, and integrals of functions with one independent variable.
- Single Variable Calculus is foundational and covers fundamental concepts such as rates of change, optimization, and area under curves.

SETTING STAGE FOR MULTIVARIABLE CALCULUS:

- Have you ever wondered how we can calculate the volume of irregularly shaped objects?
- Picture a world where you could predict the future behavior of dynamic systems, like weather patterns.

MULTIVARIABLE CALCULUS

- This branch extends calculus to functions of multiple variables.
- It involves concepts like partial derivatives, multiple integrals, vector calculus, and the analysis of functions in two or more dimensions.
- It involves understanding how quantities change with respect to multiple inputs
- Multivariable Calculus is essential for understanding phenomena that involve more complex relationships and higher dimensions

SOME EXAMPLES OF FUNCTIONS OF SEVERAL VARIABLES:

I. **The Distance Function:** This function is used to find the distance of any point from the origin in plane.

$$d(x,y) = \sqrt{x^2 + y^2}$$

This is a direct consequence of the famous "Pythagorean Theorem".

II. **RGB Color Function:** This function is used to map components of red, green and blue color to a specific color in digital map.

$$C(r,g,b) = (r,g,b)$$

III. **Average Function:** This function is used to find the average of two inputs x and y as follows:

$$A(x,y) = \frac{x+y}{2}$$

IV. **Electric Potential Function:** This function is used to find electric potential in space due to a point charge located at origin.

$$V(x,y,z) = \frac{kQ}{\sqrt{x^2 + y^2 + z^2}}$$

COURSE DETAILS:

Course Name: Multivariable Calculus

Course Code: MATH3122

Pre-Requisite: Calculus and Analytical Geometry

Course Outlines: Function of several variables and Partial Differentiation, Multiple Integral, Line and Surface Integral, Green's and Stoke's Theorem, Fourier Series: Periodic Functions, Functions of any period P-2L, Even and Odd Functions, Half Range Expansions, Fourier Transform: Laplace Transform, Z-transform.

Recommended Books: Multivariable Calculus by James Stewart, Calculus and Analytical Geometry by Swokowski, And Multivariable Calculus by Howard, A. Albert.

LECTURE-2

Multivariable Functions

Definition

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A real-valued function f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables x to X,. We also call the x's the function's input variables and call w the function's output variable.

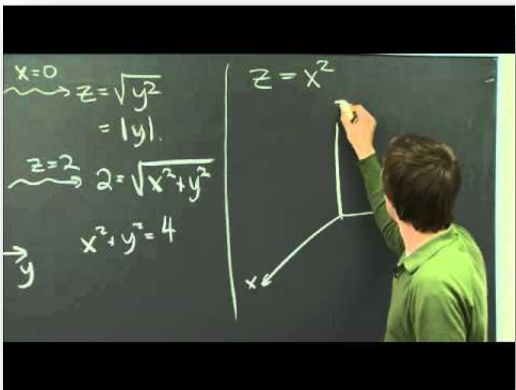
SOURCE: CALCULUS BY THOMAS (PAGE NO. 965/1144)

Examples

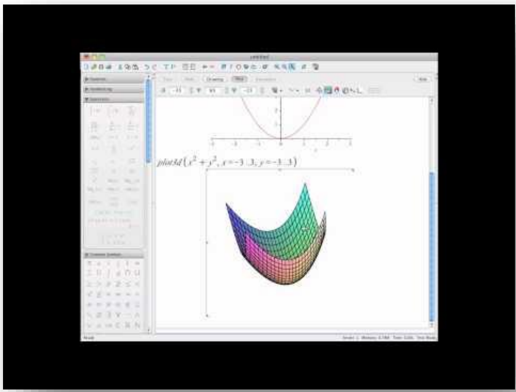
Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire Plane	$[-1,1]$
$w = \sqrt{x^2 + y^2 + z^2}$	Entire Space	$[0, \infty)$
$w = xy \ln z$	Half-Space i.e. $z > 0$	$(-\infty, \infty)$

SOURCE: CALCULUS BY THOMAS (PAGE NO. 966/1145)

The graph of a function of two variables is also known as **Surface**. Here is how you may plot it manually.



NOTE: You may use graphing utilities like Maple/Mathematica/MATLAB to visualize a function of two variables to further enhance your vision.



You may also plot your very own functions of two variables using following link:

https://www.geogebra.org/m/mCV2enZ2

Here is a real life application of a function of two variables:

Graphical Example: A Weather Map

Figure 12.1 shows a weather map from a newspaper. What information does it convey? It displays the predicted high temperature, T, in degrees Fahrenheit (°F), throughout the US on that day. The curves on the map, called *isotherms*, separate the country into zones, according to whether T is in the 60s, 70s, 80s, 90s, or 100s. (*Iso* means same and *therm* means heat.) Notice that the isotherm separating the 80s and 90s zones connects all the points where the temperature is exactly 90°F.

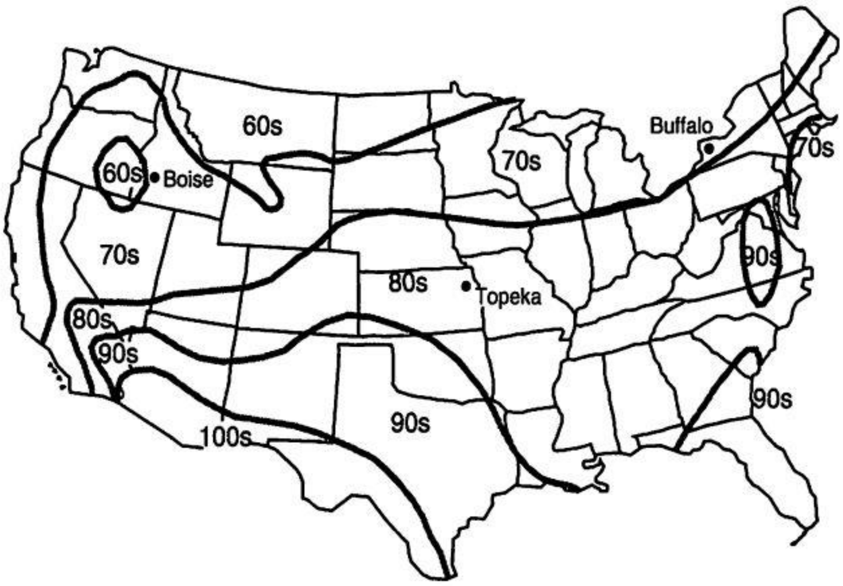


Figure 12.1 Weather map showing predicted high temperatures, T, on a summer day

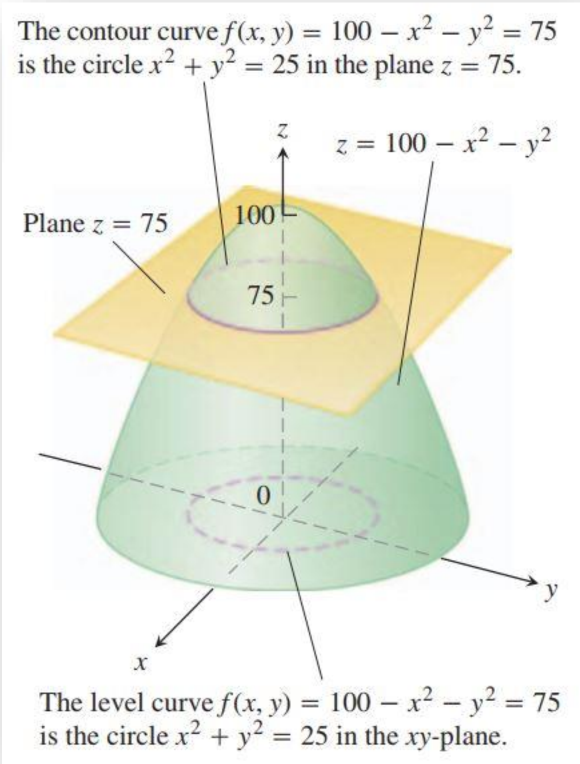
SOURCE: CALCULUS BY HUGHES (PAGE NO. 638/15)

Level Curves OR Contour Curves

Definition

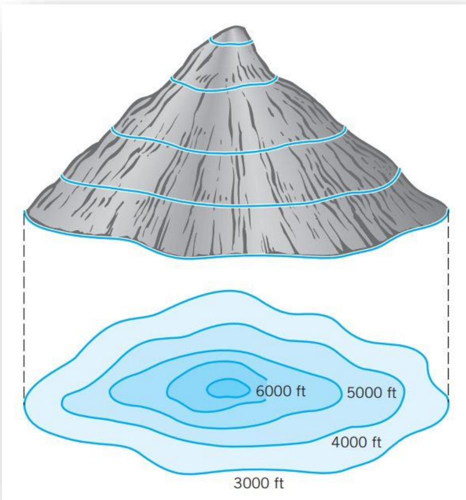
The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a Level Curve of $f(x, y)$.

SOURCE: CALCULUS BY THOMAS (PAGE NO. 968/1147)



Level curves helps in guessing the steepness of the graph. The more distant level curves are to each other, the less steep is the graph. Topographic maps of mountains are evident of their usefulness.

SOURCE: CALCULUS BY STEWART (PAGE NO. 907/265)



NOTE: You may observe level curve in real time on following link:

https://www.geogebra.org/m/M2P4KsRe

Limit of A Multivariable Function

Definition

We say that a function $f(x, y)$ approaches the limit L as (x, y) approaches (x_0, y_0) and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

If, for every number $\epsilon > 0$, there exist a number $\delta > 0$ such that for all (x, y) in the domain of $f(x, y)$,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

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Limit of A Multivariable Function

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Definition 2

The limit of a function of two variables at a specific point is the value that the function approaches as you get arbitrarily close to that point from all possible directions in the plane. In other words, it's the value the function "settles down" to as you approach the point from different paths.

This concept captures the idea of how the function behaves as you get very close to a particular point on its domain. If the function's output remains consistently close to a specific value as you approach the point from various directions, then that value is considered the limit of the function at that point.

Examples

1. $\lim_{(x,y)\to(1,2)} (x^3 + 3xy - 2y^2) = -1$ 2. $\lim_{(x,y)\to(5,5)} \frac{x^2-y^2}{x-y} = 10$ 3. $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ doesn't exist. 4. $\lim_{(x,y)\to(1,2)} \frac{x^2y^2}{x^4+y^4}$ doesn't exist. 5. $\lim_{(x,y)\to(0,0)} \frac{y^3}{(x+1)(y-1)} = 0$ 6. $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{3x^2+y^4}$ doesn't exist. 7. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+4}-2} = 4$ 8. $\lim_{(x,y,z)\to(0,0,0)} \frac{y^2+3xz^2}{x^2+2y^2+z^4}$ doesn't exist.	For regaining insights about these examples, visit: https://shorturl.at/EKL57
9. $\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$ doesn't exist.	**Source:** Calculus by Ron Larson (Page# 901/153)

10. Show that the function $g(x,y) = \frac{x^2y}{x^4+y^2}$ has same limiting value (i.e. 0) along any line through the origin and yet its limit as $(x,y) \to (0,0)$ does not exist.	**Source:** Calculus by Sallas-Etgen-Hille (Page# 778/802)

Continuity of a Multivariable Function

Definition

A function $f(x,y)$ is continuous at a point (x_0, y_0) if

- f is defined at (x_0, y_0).
- $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists.
- $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(x_0, y_0)$

A function is continuous if it is continuous at every point of its domain.

Examples

Discuss continuity at (0,0) of

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Source: Calculus by Thomas (Page# 979/1158)

Types of Discontinuity

Removable Discontinuity

A function which is discontinuous at a point due to the fact that the said function is not defined therein (But limit exists) can be made continuous by setting the value of function at that point equal to the limit of the function at the point. This makes this newly designed function a continuous one. This type of discontinuity is known as Removable Discontinuity.

Non-Removable Discontinuity

Suppose a function which is discontinuous at a point due to the fact that neither the said function is defined therein nor the limit exists and hence cannot be made continuous. This type of discontinuity is known as Non-Removable Discontinuity.

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Partial Derivatives

Definition

Let $z = f(x,y)$ be a function of two variables. If x is changed to $x + \delta x$ and y remains constant, then the change δz in z is given by

$$\delta z = f(x + \delta x, y) - f(x,y)$$

If the ratio

$$\frac{\delta z}{\delta x} = \frac{f(x + \delta x, y) - f(x,y)}{\delta x}$$

Approaches to a finite limit as $\delta x \to 0$, then this limit is called the **Partial Derivative** of z w.r.t x. It is denoted by any of the following:

$$\frac{\partial z}{\partial x} \quad OR \quad f_x \quad OR \quad \frac{\partial f}{\partial x}$$

Similarly, we can define partial derivative w.r.t 'y'.

Common Intuition

Partial derivatives are like measuring how one variable affects another, but keeping everything else constant. Imagine baking a cake with different ingredients like flour and sugar. How much will the sweetness change if you just slightly change the amount of flour without changing the amount of sugar? That's a partial derivative — it tells you how changing one ingredient affects the flavor, while the other remains constant.

Real Life Examples

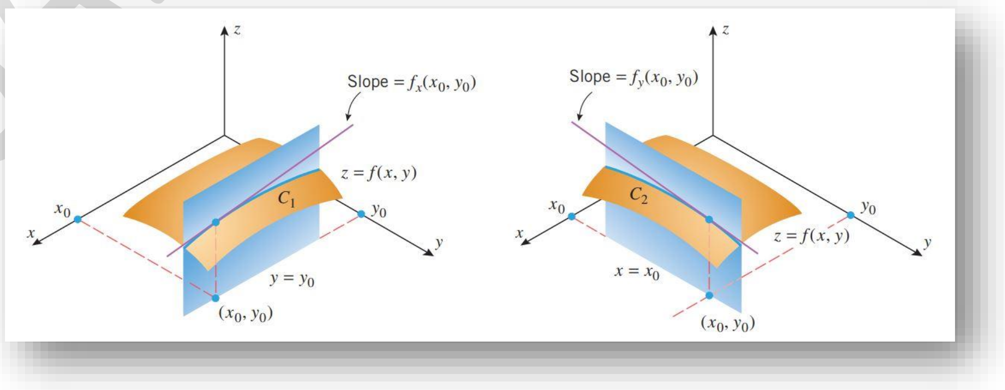
- **Heat Transfer Mechanics:** Engineers use part of the results to analyze heat transfer. For example, when designing a cooling system for electronic devices, varying degrees of temperature change are specified in the product to ensure efficient heat transfer.
- **Weather forecasting:** Meteorologists use partial results to model how changes in temperature, pressure, and humidity affect the weather. This helps them make accurate weather forecasts. **(See Details on Page No. 924(282) of Stewart's Book and Page No. 929(951) of Anton's Book)**
- **Rapid chemical reactions:** In pharmaceuticals, partial derivatives are used to study the rate of reactions. By understanding how reactant concentrations change over time, chemists can optimize production conditions and product yields.

- **Optics and Lens Design:** Optics uses partial derivatives to calculate the bending of light as it passes through lenses or other optical elements. This is important for optical systems such as cameras, microscopes and telescopes.
- **Fluid Development:** Engineers use partial results to analyze the flow of fluids (water or gas) in pipelines, sewers, and other systems. Understanding how pressure and velocity change dynamically helps to optimize water transport systems.
- **Agronomy:** Agricultural researchers use partial sources to study crop growth. By analyzing how crop yields change through factors such as sunlight, water and soil nutrients, farming practices can be improved.
- **System analysis in construction:** Designers and civil engineers use partial results to study the distribution of loads and forces in structures. This helps ensure the safety and stability of buildings, bridges and other structures.
- **Medical Imaging:** Partial derivatives in MRI and other medical images aid in image reconstruction

Partial Derivatives of Algebraic Functions

$\boldsymbol{f(x,y)}$	$\boldsymbol{f_x}$	$\boldsymbol{f_y}$
$3x^2y - 5x\cos\pi y$	$6xy - 5\cos\pi y$	$3x^2 + 5\pi x\sin\pi y$
$\tan^{-1}\frac{y}{x}$	$-\frac{y}{x^2+y^2}$	$\frac{x}{x^2+y^2}$
$\ln\sqrt{x^2+y^2}$	$\frac{x}{x^2+y^2}$	$\frac{y}{x^2+y^2}$

Geometrical Interpretation of Partial Derivatives



For a function of two variables, the partial derivative with respect to x represents the slope of the tangent line in the x-direction at a given point on the surface. Similarly, the partial derivative with respect to y represents the slope of the tangent line in the y-direction at that point.

Implicit Partial Differentiation

If the equation $f(x, y) = c$ defines $'y'$ as a differentiable function of $'x'$, then

$$\frac{dy}{dx} = -\frac{f_x}{f_y}, \qquad f_y \neq 0$$

Example

Given $y^3 + y^2 - 5y - x^2 + 4 = 0$. Find dy/dx.

Solution

As,

$$f_x = -2x$$
$$f_y = 3y^2 + 2y - 5$$

Thus,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{2x}{3y^2 + 2y - 5}$$

Higher Order Partial Derivatives

Suppose that f is a function of two variables x and y. Since the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are also functions of x and y, these functions may themselves have partial derivatives. This gives rise to four possible ***second-order*** partial derivatives of f, which are defined by

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$$

Differentiate twice with respect to x.

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}$$

Differentiate twice with respect to y.

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

Differentiate first with respect to x and then with respect to y.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Differentiate first with respect to y and then with respect to x.

The last two cases are called ***Mixed Second Order Partial Derivatives***.

Example

Find all second order partial derivatives of $f(x, y) = x^2y^3 + x^4y$

Solution

As,

$$f_x = 2xy^3 + 4x^3y$$

And

$$f_y = 3x^2y^2 + x^4$$

Thus,

$$f_{xx} = 2y^3 + 12x^2$$
$$f_{yy} = 6x^2y$$
$$f_{xy} = 6xy^2 + 4x^3$$
$$f_{yx} = 6xy^2 + 4x^3$$

In general, the two mixed second order partial derivatives are **NOT** necessarily equal. And this brings us to an important question: When do these mixed second order partial derivatives equate each other?

The answer to this question lies in Clairaut's famous theorem.

Clairaut's Theorem

If $f(x, y)$ is a function with continuous second order partial derivatives, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

i.e. the order of differentiation is immaterial.

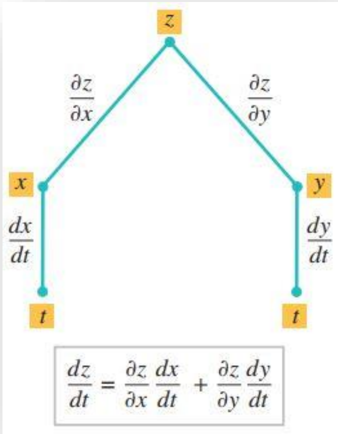
Note: In above example, mixed second order partial order derivatives are equal because both of them are simple polynomials which are always continuous.

Chain Rule for Partial Differentiation

Case-1

If $z = f(x, y)$ and both $'x'$ and $'y'$ are functions of another (single) variable $'t'$, then

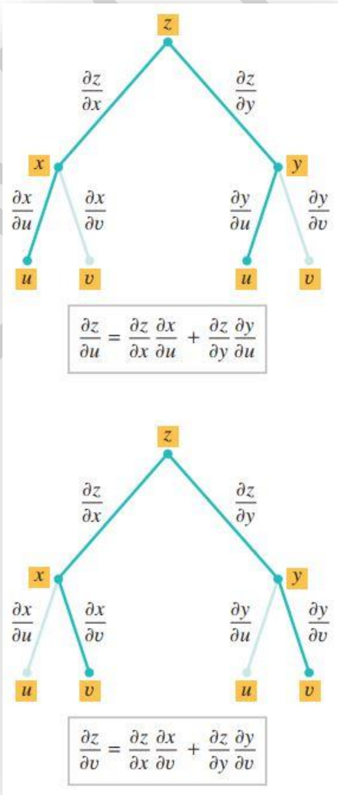
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



Case-2

If $z = f(x, y)$ and $x = x(u, v)$ and $y = y(u, v)$ i.e. both $'x'$ and $'y'$ are themselves functions of two variables $'u'$ and $'v'$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \qquad And \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$



ASSIGNMENT NO. 1

QUESTION NO. 1

Discuss the continuity of a function of two variables in relation to partial derivatives. Illustrate your point with diagrams to ensure good score.

QUESTION NO. 2

A model for heat index is given by

$$H = 0.885t - 22.4h + 1.20th - 0.544$$

Where $'H'$ is the heat index, 't' is the air temperature in degree Celsius, and 'h' is relative humidity in decimal form.

- Find H_t and H_h when $(t, h) = (30°\text{C}, 0.80)$
- Which has a greater impact on H, 't' or 'h'? Explain!

GOODLUCK!

Unveiling Fourier Series: A Mathematical Journey Through Even and Odd Functions

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A Lecture Series on Multivariable Calculus, 2023

Outline

What is a Periodic Function?

Definition

A function $f(x)$ is known as a **Periodic Function** if it satisfies the property:

$$f(x) = f(x + T)$$

where $T \in \mathbb{R}$ is the period, the smallest positive value for which the function repeats.

Examples of Periodic Functions

Examples

1. **Square Wave Function:** The square wave function is another periodic function that alternates between two constant values over a specified period.

$$f(x) = \begin{cases} 1, & 0 \leq x < \pi \\ -1, & \pi \leq x < 2\pi \end{cases}$$

2. **Triangle Wave Function:** The triangle wave function is another periodic function that has a sawtooth-like shape.

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ -x, & 1 \leq x < 2 \end{cases}$$

What is an Even Function?

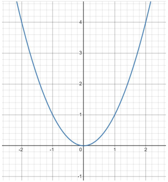
Definition

A function $f(x)$ is said to be an **Even Function** if $f(-x) = f(x)$.

Geomtrical Interpretation and Examples

Geometrically, one can see that an even function is symmetric about y-axis. Examples of even function include $f(x) = x^2$ and $f(x) = \cos x$.

Figure: Graph of an even function i.e. $f(x) = x^2$



What is an Odd Function?

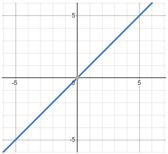
Definition

A function $f(x)$ is said to be an **Odd Function** if $f(-x) = -f(x)$.

Geomtrical Interpretation and Examples

Geometrically, one can see that an odd function is symmetric about origin. Examples of odd function include $f(x) = x^3$ and $f(x) = \sin x$.

Figure: Graph of an odd function i.e. $f(x) = x$



Some Key Results Related to Even and Odd Functions

Product of Even and Odd Functions

1. The product of an even functions with another function OR an odd function with another odd function will always result in an even function.
2. The product of an even and an odd functions will produce an odd function.

Some Key Results Related to Even and Odd Functions

Integral Formulas For Even/Odd Functions over $[-k, k]$

1. For any even function $f(x)$, we have

$$\int_{-k}^{k} f(x)dx = 2\int_{0}^{k} f(x)dx$$

2. For any odd function $g(x)$, we have

$$\int_{-k}^{k} g(x)dx = 0$$

Turning Noise into Music

Here is how it works . . .

1. **Every Sound is Made of Waves:** All sounds, like your voice or a musical instrument, are like a mix of waves. Think of waves in the ocean, going up and down.
2. **Breaking Down the Sound:** Fourier Series helps you break down the sound into different waves, just like taking apart a puzzle. You find the simple "building blocks" of the sound.
3. **Create Music:** Once you know these building blocks (the waves), you can use them to create beautiful music. You can decide how loud each building block should be and put them together in the right way.

What is Fourier Series?

Definition

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines over some interval $[-k, k]$ and is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{k}\right) + b_n \sin\left(\frac{n\pi x}{k}\right) \right]$$

$$a_0 = \frac{1}{2k} \int_{-k}^{k} f(x) dx, \quad a_n = \frac{1}{k} \int_{-k}^{k} f(x) \cos\left(\frac{n\pi x}{k}\right) dx$$

$$b_n = \frac{1}{k} \int_{-k}^{k} f(x) \sin\left(\frac{n\pi x}{k}\right) dx$$

Coefficients For Even and Odd Functions

Value of a_0, a_n, and b_n for Even Functions

The value of a_0, a_n, and b_n for an Even Function $f(x)$ is given as follows:

$$a_0 = \frac{1}{k}\int_0^k f(x)dx$$

$$a_n = \frac{2}{k}\int_0^k f(x)\cos\left(\frac{n\pi x}{k}\right)dx$$

$$b_n = 0$$

Coefficients For Even and Odd Functions

Value of a_0, a_n, and b_n for Odd Functions

The value of a_0, a_n, and b_n for an odd Function $f(x)$ is given as follows:

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{k} \int_0^k f(x) \sin\left(\frac{n\pi x}{k}\right) dx$$

A Reminder!

Even and odd functions are a special subset of periodic functions that have specific symmetry properties, and they simplify the calculations for their Fourier series. However, the general Fourier series representation can be used for any periodic function, whether it exhibits even, odd, or neither of these symmetries.

Problem No. 1

Find fourier series of $f(x) = x$ over $[-\pi, \pi]$

Solution: Since $f(x) = x$ is an odd function, So

$$a_0 = 0 = a_n$$

Also,

$$b_n = \frac{-2}{n}(-1)^n$$

Finally we have,

$$f(x) = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin(nx)$$

Problem No. 2

Find fourier series of $f(x) = x^2$ over $[-\pi, \pi]$

Solution: Since $f(x) = x$ is an even function, So

$$b_n = 0$$

Also,

$$a_0 = \frac{\pi^2}{3}, \;\; a_n = \frac{4}{n^2}(-1)^n$$

Finally we have,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx)$$

Problem No. 3

Find fourier series of $f(x) = x - x^2$ over $[-\pi, \pi]$

Solution: The calculated values of a_0, a_n, and b_n are as follows:

$$a_0 = -\frac{2\pi^2}{3}$$

$$a_n = \frac{-4\left(-1\right)^n}{n^2}$$

$$b_n = \frac{-2\left(-1\right)^n}{n}$$

Finally we have,

$$f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{\left(-1\right)^n}{n^2} \cos\left(nx\right) - 2\sum_{n=1}^{\infty} \frac{\left(-1\right)^n}{n} \sin\left(nx\right)$$

Beyond Full Periods: Understanding Half Range Expansions in Fourier Series

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A Lecture Series on Multivariable Calculus, 2023

Outline

Motivation

Welcome to our exploration of "Half Range Expansions" within the world of Fourier Series. In this presentation, we'll uncover the art of simplifying Fourier Series through the lens of symmetry. By embracing the inherent symmetries of functions, we can streamline our calculations, unveiling the elegance of mathematical simplicity. We'll focus on two vital components: the Half Cosine Series and the Half Sine Series. These techniques empower us to dissect periodic functions efficiently, making this mathematical journey an invaluable asset across various domains, from engineering to signal processing. So, let's embark on a journey of mathematical elegance and symmetry through the lens of Half Range Expansions.

What is Half Range Cosine Series

Definition

The Half Range Cosine Series of a periodic function $f(x)$ over an interval $[0, k]$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{k}\right)$$

where

$$a_0 = \frac{1}{k}\int_0^k f(x)dx \quad \text{And} \quad a_n = \frac{2}{k}\int_0^k f(x)\cos\left(\frac{n\pi x}{k}\right)$$

Problem No. 1

Express f(x) = x as a half range cosine series over $(0, 2)$

Solution: The calculated values of a_0 and a_n are as follows:

$$a_0 = 1 \quad \text{And} \quad a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

The half range cosine series of $f(x) = x$ is, therefore, given by:

$$f(x) = x = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \cos\left(\frac{n\pi x}{2}\right)$$

Problem No. 2

Find half range cosine series of $f(x) = x\sin x$ over $(0, \pi)$

Solution: The calculated values of a_0 and a_n are as follows:

$$a_0 = 1 \quad \text{And} \quad a_n = \frac{2(-1)^n}{1-n} \quad (n \neq -1)$$

The value of a_1, when calculated separately, is $\frac{-1}{2}$. Finally, the required series is determined to be:

$$f(x) = x\sin x = 1 - \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n} \cos(nx)$$

OR

$$f(x) = x\sin x = \frac{1}{2} + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n} \cos(nx)$$

The Assignment Awaits — Conquer It!

Problem

Find half range cosine series of following function:

$$f(x) = \begin{cases} kx, & 0 \leq x \leq \frac{p}{2} \\ k\left(p - x\right), & \frac{p}{2} \leq x \leq p \end{cases}$$

What is Half Range Sine Series?

Definition

The Half Range Sine Series of a periodic function $f(x)$ over an interval $[0, k]$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{k}\right)$$

where

$$b_n = \frac{2}{k} \int_0^k f(x) \sin\left(\frac{n\pi x}{k}\right)$$

Problem No. 1

Find half range sine series of $f(x) = x(\pi - x)$ over $[0, \pi]$

Solution: The calculated value of b_n is as follows:

$$b_n = \frac{4}{n^3 \pi} \left[1 - (-1)^n\right]$$

The half range sine series of $f(x) = x(\pi - x)$ is, therefore, given by:

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} \frac{4}{n^3 \pi} \left[1 - (-1)^n\right] \sin(nx)$$

Problem No. 2

Find half range sine series of $f(x) = (2x - 1)$ over $[0, 1]$

Solution: The calculated value of b_n is as follows:

$$b_n = \frac{-2}{n\pi}\left[1 + (-1)^n\right]$$

The half range sine series of $f(x) = (2x - 1)$ is, therefore, given by:

$$f(x) = (2x - 1) = \sum_{n=1}^{\infty} \frac{-2}{n\pi}\left[1 + (-1)^n\right] \sin(n\pi x)$$

Embrace the Challenge!

Problem

Obtain half range sine series of $f(x) = e^{2x}$ over $[0, \pi]$

Transforming Insights: Fourier and Inverse Transform Explained with Examples

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A Lecture Series on Multivariable Calculus, 2023

Outline

Motivation

Get ready to unlock the secrets of signals and waves! In this lecture, we'll journey into the world of Fourier Transform and Inverse Fourier Transform. Imagine them as tools to take a complex mix of sounds, like your favorite song, and break it down into individual notes. Then, we'll learn how to put those notes back together to recreate the song. These techniques are like magic spells for understanding and manipulating data, and we'll explore how they work with easy-to-follow examples. Let's dive in!

What is Fourier Transform

Definition

If $f(x)$ be a function defined on $(-\infty, \infty)$ uniformly continuous in finite interval and $\int_{-\infty}^{\infty} |f(x)| dx$ converges. Then, Fourier Transform of $f(x)$ is given by:

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

Is Fourier Transformation a useful tool?

Utilizations of the Fourier Transform

The Fourier Transform is a fundamental mathematical tool with versatile applications. It's used to analyze signals in fields like signal processing, audio, and image analysis. Additionally, it plays a crucial role in quantum mechanics, medical imaging, spectroscopy, data compression, electrical engineering, and seismology. Its applications extend to finance, speech processing, astronomy, and more, enabling insights into the frequency components of complex data.

Problem No. 1

Find Fourier Transform of $f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$

We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

Clearly,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx}\, dx$$

After using the results for even/odd functions, finally we have:

$$F(s) = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin as}{s} \right)$$

Problem No. 2

Find Fourier Transform of $f(x) = \begin{cases} 1 - x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

We know that

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$\Rightarrow \quad F(s) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx} \left(1 - x^2\right) dx$$

Using the Euler's formula for e^{isx} and results for even/odd functions, we finally have:

$$F(s) = \frac{4}{\sqrt{2\pi}} \left(\frac{\sin s - s \cos s}{s^3} \right)$$

What is Inverse Fourier Transform

Definition

The inverse fourier transform of a function $f(s)$, denoted by $f(x)$, is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} f(s)\, ds$$

Problem No. 1

Statement

The fourier transform of

$$f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

is given by

$$f(s) = \frac{2}{\sqrt{2\pi}} \frac{\sin(as)}{s}$$

Use inverse fourier transformation to evaluate

$$\int_{-\infty}^{\infty} \frac{\sin s}{s} ds$$

Problem No. 1

Solution

The inverse fourier transformation is given by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

The expression that we are going to evaluate does not involve e^{-isx}, so we will put $x = 0$ to get rid of the unwanted e^{-isx} term from integrand. Also, this restrict $f(x) = 1$. Therefore, we have:

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \frac{\sin(as)}{s} ds = 1$$

Continued ...

Problem No. 1

Solution

Choose $a = 1$ in

$$\frac{\sin(as)}{s}$$

to get the required expression i.e.

$$\frac{\sin(s)}{s}$$

. Finally, using the results for even/odd functions we have,

$$\int_0^\infty \frac{\sin s}{s} ds = \frac{\pi}{2}$$

Problem No. 2

Statement

The fourier transform of

$$f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

is given by

$$f(s) = \frac{2}{\sqrt{2\pi}} \frac{\sin(as)}{s}$$

Use inverse fourier transformation to evaluate

$$\int_0^\infty \frac{\sin(as)\cos(sx)}{s} ds$$

Problem No. 2

Solution

The inverse fourier transformation is given by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}\frac{2}{\sqrt{2\pi}}\frac{\sin(as)}{s}ds = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

Use Euler's Formula to get

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(as)\cos(sx)}{s}ds + \frac{\iota}{\pi}\int_{-\infty}^{\infty}\frac{\sin(as)\sin(sx)}{s} = f(x)$$

Continued ...

Problem No. 2

Solution

Use results for even odd functions to get

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(as)\cos(sx)}{s}ds = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

Finally we have,

$$\int_{-\infty}^{\infty}\frac{\sin(as)\cos(sx)}{s}ds = \begin{cases} \pi, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

Unlocking Symmetry: Fourier Cosine Transform and Practical Applications

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A Lecture Series on Multivariable Calculus, 2023

Outline

Motivation

The Fourier Cosine Transform is a mathematical technique closely related to the Fourier Transform, specifically designed for even functions. It allows you to represent a real-valued, even function as a sum of cosine functions of different frequencies, simplifying the analysis of symmetric data.

What is Fourier Cosine Transform

Definition

The fourier cosine transform of a function $f(x)$ is given by

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(sx)\,dx = \bar{f}(s)$$

Problem No. 1

Find Fourier Cosine Transform of

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x \geq 2 \end{cases}$$

The fourier cosine transform of $f(x)$ is given by

$$f_c(s) = \sqrt{\frac{2}{\pi}} \int_0^1 x \cos(sx)\, dx \; + \; \sqrt{\frac{2}{\pi}} \int_1^2 (2 - x) \cos(sx)\, dx$$

Finally, we have:

$$f_c(s) = \sqrt{\frac{2}{\pi}} \left(\frac{2(1 - \cos(s))(\cos(s))}{s^2} \right)$$

Problem No. 2

Find Fourier Cosine Transform of $f(x) = e^{-ax}$ where $a \geq 0$

The fourier cosine transform of $f(x)$ is given by

$$f_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos(sx)\, dx$$

Using the fact

$$\int e^{-ax} \cos(sx)\, dx = \frac{e^{-ax}}{a^2 + s^2} \left(-a \cos x + s \sin(sx)\right)$$

we finally have

$$f_c(s) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2 + s^2}\right)$$

What is Inverse Fourier Cosine Transform

Definition

The inverse fourier cosine transform of a function $f_c(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(s) \cos(sx)\, ds$$

Problem No. 1

The Fourier Cosine Tranform of $f(x) = e^{-ax} \quad (a > 0)$ is given by $f_c(s) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+b^2}\right)$. Use Inverse Fourier Cosine Transform to prove that $\int_0^\infty \frac{\cos(mx)}{x^2+a^2}dx = \frac{\pi}{2a}e^{-am}$

The inverse fourier cosine transform of $f_c(s)$ is given by

$$e^{-ax} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+b^2}\right)\cos(sx)\,ds$$

Put $x = m$ and $s = m$ to get

$$\int_0^\infty \frac{\cos(mx)}{x^2+a^2}dx = \frac{\pi}{2a}e^{-am}$$

What is Fourier Sine Transform

Definition

The fourier sine transform of a function $f(x)$ is given by

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x)\sin(sx)\,dx = \bar{f}(s)$$

Problem No. 1

Find Fourier Sine Transform of

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x \geq 2 \end{cases}$$

The fourier sine transform of $f(x)$ is given by

$$f_s(s) = \sqrt{\frac{2}{\pi}} \int_0^1 x \sin(sx)\, dx + \sqrt{\frac{2}{\pi}} \int_1^2 (2 - x) \sin(sx)\, dx$$

Finally, we have:

$$f_s(s) = 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin s (1 - \cos s)}{s^2} \right]$$

Problem No. 2

Find Fourier Sine Transform of $f(x) = e^{-ax}$ where $a \geq 0$

The fourier sine transform of $f(x)$ is given by

$$f_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin(sx)\, dx$$

Using the fact

$$\int e^{-ax} \sin(sx)\, dx = \frac{e^{-ax}}{a^2 + s^2} \left(-a \sin sx - s \cos(sx)\right)$$

we finally have

$$f_s(s) = \sqrt{\frac{2}{\pi}} \left(\frac{s}{a^2 + s^2} \right)$$

What is Inverse Fourier Sine Transform

Definition

The inverse fourier Sine transform of a function $f_c(s)$ is given by

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(s) \sin(sx)\, ds$$

Problem No. 1

The Fourier Sine Tranform of $f(x) = e^{-ax} \quad (a > 0)$ is given by $f_s(s) = \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$. Use Inverse Fourier Cosine Transform to prove that $\int_0^\infty x\frac{\sin(mx)}{x^2+a^2}dx = \frac{\pi}{2}e^{-am}$

DO YOURSELF!

Exploring Laplace Transforms: A Gentle Introduction and Practical Examples

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A Lecture Series on Multivariable Calculus, 2023

Outline

Deciphering the core idea

Definition

Let $f(t)$ be a real-valued piecewise continuous function defined on $[0, \infty)$. The Laplace Transform of $f(t)$ is given by

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided the improper integral converges.

Motivation

Why do we study Laplace Transform

The Laplace Transform is a crucial mathematical tool in engineering, simplifying the analysis of linear differential equations. By converting these equations from the time to the frequency domain, it aids in solving problems in control systems, signal processing, and circuit analysis. Its key strength lies in transforming complex differential equations into manageable algebraic forms, making it indispensable for stability analysis and system design in diverse applications.

Overview of Applications in Computer Science

In computer science, the Laplace Transform is applied extensively in signal processing for signal analysis and modification. It plays a crucial role in control systems for modeling and stability analysis. Moreover, in algorithm analysis, the Laplace Transform enhances understanding and optimization, making it a versatile tool with wide-ranging applications in computational processes and efficiency studies.

Linearity Property

Definition

Let $f_1(t)$, $f_2(t)$, $f_3(t)$, ..., $f_n(t)$ be any function of t and let a_1, a_2, a_3, ..., a_n be constants. Then

$$L\{a_1 f_1(t) + a_2 f_2(t) + \cdots + a_n f_n(t)\}$$

$$= a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\} + \cdots + a_n L\{f_n(t)\}$$

Linearity Property

Example

Let $f_1(t) = \sin t$ and $f_2(t) = t^2$. Then

$$L\{f_1(t) + f_2(t)\} = L\left\{\sin t + t^2\right\} = L\{\sin t\} + L\left\{t^2\right\}$$

Time Shifting Property

Definition

If $f(t)$ is a piecewise continuous function for $t \geq 0$, and $L\{f(t)\} = F(s)$, then the Laplace Transform of $f(t-a)\,u(t-a)$ is given by $e^{-as}F(s)$.

Time Shifting Propert

Example

Suppose we have a function $f(t) = e^{2t}$ for $t \geq 0$. We want to find the Laplace Transform of $f(t-3)\,u(t-3)$, where $u(t-3)$ is the unit step function defined as

$$u(t-3) = \begin{cases} 0, & t < 3 \\ 1, & t \geq 3 \end{cases}$$

then by definition of Laplace Transform, we have

$$L\{f(t-3)\,u(t-3)\} = \int_0^\infty e^{-st} e^{2(t-3)} u(t-3)\,dt$$

Continued . . .

Time Shifting Property

Example Continues ...

Breaking this integral from previous slide into two parts i.e. before and after $t = 3$, we finally have

$$L\left\{f\left(t-3\right)u\left(t-3\right)\right\} = \frac{e^{-3s}}{s-2}$$

which is exactly same as $e^{-as}F\left(s\right)$.

Scaling Property

Definition

If $f(t)$ is a piecewise continuous function for $t \geq 0$, and $L\{f(t)\} = F(s)$, then the Laplace Transform of $e^{at}f(t)$ (i.e. the function is scaled by the factor e^{at}) is given by $F(s-a)$.

Scaling Property

Example

Let $f(t) = \sin t$. Find Laplace Transform of its scaled version when it is scaled by e^{3t}.

DO YOURSELF!

Laplace Transform of Unit Step Function

General Formula

The most general formula for finding Laplace Transform of a unite step function is given by

$$L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\,dt = \frac{e^{-as}}{s}$$

This answer is obtained by dividing the improper integral into two parts i.e. before and after $t = a$.

Laplace Transform of Dirac Delta Function

General Formula

The most general formula for finding Laplace Transform of a Dirac Delta function is given by

$$L\{\delta(t)\} = \int_0^\infty e^{-st}\delta(t)\,dt = 1$$

Also, keep in mind following formulae as well for future use:

$$L\{\delta(t-a)\} = e^{-as}$$

and

$$L\{f(t)\,\delta(t-a)\} = e^{-as}f(a)$$

Laplace Transform of Dirac Delta Function

Example: Find Laplace transform of $t^2\delta(t-3)$

Using $L\{f(t)\delta(t-a)\} = e^{-as}f(a)$, we have

$$L\left\{t^2\delta(t-3)\right\} = 9e^{-3s}$$

The Assignment Awits – Conquer it!

Problem

Find Laplace Transform of:

1. $\sin(at)$
2. $\cos(at)$
3. e^{at}

Unveiling the Inverse Laplace Transform: Bridging Time and Frequency Domains

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A Lecture Series on Multivariable Calculus, 2023

Outline

Deciphering the Core Idea

Definition

if $F(s)$ is the Laplace transform of a function $f(t)$, then the inverse Laplace transform of $F(s)$ is the function $f(t)$ and is represented as:

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

In symbols, this is read as the Inverse Laplace Transform of $F(s)$ equals $f(t)$. Note that, we are now emphasizing on intuition rather than complex calculations i.e. we will use a table of Laplace transform **(See at the end of this lecture)**.

Deciphering the Core Idea

Purpose

The Inverse Laplace Transform seamlessly transitions mathematical models from the frequency to the time domain, unlocking insights into dynamic systems crucial for engineering, physics, and scientific exploration.

Problem No. 1

Find Inverse Laplace Transform of $F(s) = \frac{s-2}{s^2-2}$

$$\mathcal{L}^{-1}\left\{\frac{s-2}{s^2-2}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-2} - \frac{2}{s^2-2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2-\left(\sqrt{2}\right)^2} - \frac{\sqrt{2}\sqrt{2}}{s^2-\left(\sqrt{2}\right)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2-\left(\sqrt{2}\right)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{\sqrt{2}\sqrt{2}}{s^2-\left(\sqrt{2}\right)^2}\right\}$$

$$= \cosh\sqrt{2}t - \sqrt{2}\sinh\sqrt{2}t$$

Practice Yourself

Problems

Find Inverse Laplace Transform of following:

1. $F(s) = \frac{3s+1}{s^2-6s+18}$
2. $F(s) = \frac{9s-67}{s^2-16s+49}$
3. $F(s) = \frac{as+b}{s^2+2cs+d}$

Exploring Z-Transform: Unveiling the Magic of Discrete Signal Transformation

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A Lecture Series on Multivariable Calculus, 2023

Outline

What is Z-Transform?

Definition

The Z-Transform of a sequence u_n denoted by $Z(u_n)$ is defined as

$$Z(u_n) = \sum_{n=-\infty}^{\infty} u_n z^{-n} = \bar{u}(z)$$

where $\bar{u}(z)$ is the Z-Transform of u_n and z is a complex number.

Why do we need it?

Raison d'être

The Z-Transform is vital for discrete-time signal analysis, simplifying signal representation and enabling system response, stability, and transfer function analysis. Its applications extend to Digital Signal Processing (DSP) and control system design, making it a crucial tool in diverse engineering fields.

Difference between Laplace Transform and Z-Transform

Aspect	**Laplace Transform**	**Z-Transform**
Time Domain	Continuous-time signals	Discrete-time signals
Variable of Transformation	Complex variable s	Complex variable z
Domain of Convergence	Complex s plane	Complex z plane
Applications	Control systems, circuit analysis	Digital signal processing, control systems
Integral/Summation	Integral over time	Summation over discrete time steps
Transform Pairs	e^{-st} in the kernel	z^{-n} in the kernel

Understanding Discrete Time Signals

Definition

A discrete-time signal is a sequence of values defined at distinct, separate time instances.

Examples

- **Unit Impulse Signal ($\delta[n]$):** Represents a single, unit amplitude at time $n = 0$.
- **Unit Step Signal ($u[n]$):** Takes the value 1 for $n \geq 0$ and 0 for $n < 0$.
- **Exponential Decay (a^n):** Decreases exponentially as n increases, where a is a constant.
- **Sinusoidal Signal ($\sin(\omega_0 n)$):** Oscillates with frequency ω_0 over discrete time steps.

Simple Properties of Z-Transform

Linearity

For any constants a and b and sequences u_n and v_n, the Z-Transform satisfies:

$$Z\{au_n + bv_n\} = aZ\{u_n\} + bZ\{v_n\}$$

Time Shifting

Shifting a sequence u_n by k units in time results in a multiplication by z^{-k} in the Z-Transform:

$$Z\{u_{n-k}\} = z^{-k}Z\{u_n\}$$

Simple Properties of Z-Transform

Scaling

Scaling a sequence u_n by a constant a results in a multiplication by a in the Z-Transform:

$$Z\{au_n\} = aZ\{u_n\}$$

Z-Transform of a^n for $n \geq 0$

Solution

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots$$

$$= \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}$$

Show that $Z\left((n)^p\right) = z\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right]$, $n \geq 0$

Solution

$$Z\left(u_n\right) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$Z\left((n)^{p-1}\right) = \sum_{n=0}^{\infty} (n)^{p-1} z^{-n}$$

$$\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right] = \sum_{n=0}^{\infty} (n)^{p-1}\left(-nz^{-n-1}\right)$$

$$\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right] = \sum_{n=0}^{\infty} -(n)^{p} z^{-n-1}$$

Continued . . .

Show that $Z\left((n)^p\right) = z\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right]$, $n \geq 0$

Solution

$$\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right] = \sum_{n=0}^{\infty} -(n)^p \frac{z^{-n}}{z}$$

$$\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right] = -\frac{1}{z}\sum_{n=0}^{\infty}(n)^p z^{-n}$$

$$\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right] = -\frac{1}{z}Z\left((n)^p\right)$$

$$-z\frac{d}{dz}\left[Z\left((n)^{p-1}\right)\right] = Z\left((n)^p\right)$$

Time for Action

Problem

Find Z-Transform of Sinusoidal Sequences (e.g. $\sin(w_0 n)$).