GENERAL TOPOLOGY

BY

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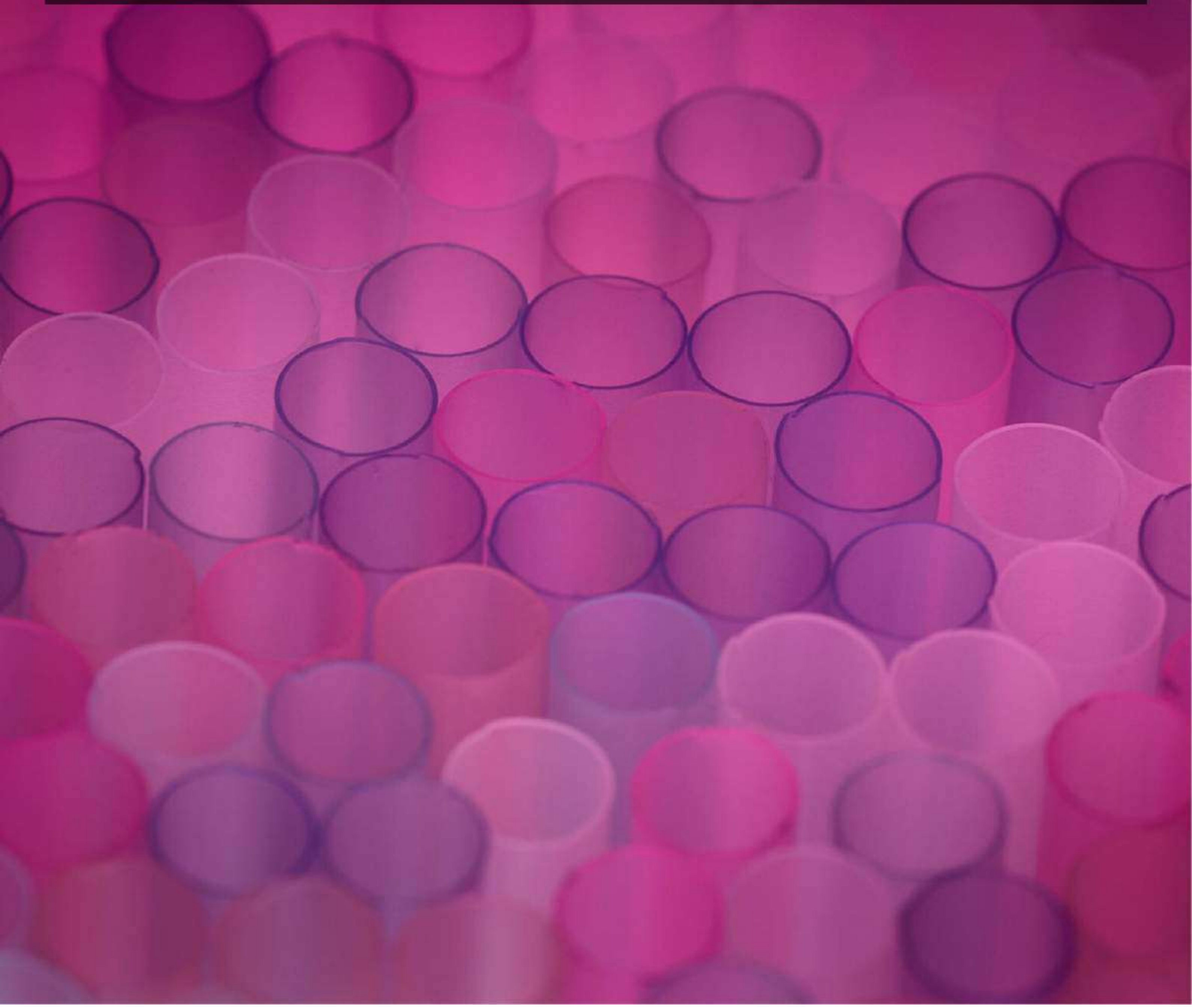

Recommended Books:

i) Elements of Topology and Functional Analysis by Dr. Abdul Majeed.

ii) Topology by James Munkres

iii) General Topology by Lipschutz

iv) Functional Analysis by Kreyszig

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Topology:

$X = \{1, 2, 3, 4\}$

$$P(X) = \begin{Bmatrix} \phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \\ \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\} \end{Bmatrix}$$

Definition:

Let X be a non-empty set. A collection of subsets of X is said to be the topology (τ) on X if the following properties are satisfied.

i) $\phi, X \in \tau$

ii) Arbitrary union of open sets is open.

iii) Finite intersection of open sets is open.

Note:

The members of τ are open sets and their complements are closed sets.

The pair (X, τ) or X is said to be topological space.

$X = \{1, 2, 3, 4\}$

$\tau_1 = \{\phi, X, \{1\}\}$ ✓

$\tau_2 = \{\phi, X, \{2\}\}$ ✓

$\tau_3 = \{\phi, X, \{1\}, \{2\}\}$ ✗

$\tau_4 = \{\phi, \{1\}, \{3\}, \{1,2,3,4\}\}$ ✗

$\tau_5 = \{\phi, X, \{1\}, \{4\}, \{1,4\}\}$ ✓

Discrete Topology:

Let $X \neq \phi$

$\tau = P(X)$

then τ is called discrete topology and X with this topology is called discrete space.

Note:-

If $\tau_1 \subseteq \tau_2$

then we call τ_2 is stronger (finer) than τ_1

Then discrete topology is the strongest topology.

Indiscrete Topology: (Trivial Topology)

Let $X \neq \phi$

$\tau = \{\phi, X\}$

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then τ is called indiscrete (trivial) topology.

Note:

The trivial topology is the weakest topology.

Sierprinski Topology:

Let $X = \{a, b\}$

$$\tau = \{\phi, X, \{a\}\}$$

then τ is called sierprinski topology and X together with this τ is called sierprinski space.

Usual Topology:

A collection of subsets of $\mathbb{R}$ which can be expressed as a union of open intervals, forms a topology on $\mathbb{R}$ and is called I usual topology on $\mathbb{R}$.

Cofinite Topology:

Let X be an infinite set.

$$\tau = \{\phi, A_\alpha : A_\alpha \subseteq X, \ A_\alpha^c \text{ is finite}\}$$

then τ is a topology on X, known as cofinite topology.

Note:

In cofinite topological space, open sets are infinite and closed sets are finite.

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$X = \{1,2,3\}$

$\tau_1 = \{\phi, X, \{1\}, \{1,2\}, \{1,3\}\}$

open sets: $\phi, X, \{1\}, \{1,2\}, \{1,3\}$

closed sets: $X, \phi, \{2,3\}, \{3\}, \{2\}$

$\tau_2 = \{\phi, X, \{2\}, \{2,3\}\}$

open sets: $\phi, X, \{2\}, \{2,3\}$

closed sets: $X, \phi, \{1,3\}, \{1\}$

$\{3\}, \{1,2\}$ are neither open nor closed.

closed set:
Let (X,τ) be a top. space and $A \subseteq X$ then A is said to be closed if A^c is open

Th:
Let (X,τ) be a topological space. Then
i) ϕ, X are closed
ii) union of finite no. of closed sets of τ is closed.
iii) Arbitrary intersection of closed sets of τ is closed.

Proof:- (i)

Let $\phi \in \tau$.

i.e. ϕ is open

$\Rightarrow \phi^c$ is closed

$\Rightarrow X$ is closed $\qquad \because \phi^c = X$

Let $X \in \tau$

i.e. X is open

$\Rightarrow X^c$ is closed

$\Rightarrow \phi$ is closed $\qquad \because X^c = \phi$

(ii)

Let $\{O_i : i = 1,2,\dots,n\}$ be finite collection of open sets of τ.

then $\bigcap_{i=1}^{n} O_i \in \tau$ is open

$\Rightarrow \left(\bigcap_{i=1}^{n} O_i\right)^c$ is closed

by De-Morgan's law

$\bigcup_{i=1}^{n} O_i^c$ is closed.

Hence, finite union of closed sets of τ is closed.

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(iii)

Let $\{O_\alpha : \alpha \in \Omega\}$ be an arbitrary collection of members of τ.

then $\bigcup_{\alpha\in\Omega} O_\alpha \in \tau$ is open

$\Rightarrow \left(\bigcup_{\alpha\in\Omega} O_\alpha\right)^c$ is closed

So, by De-Morgan's law

$\bigcap_{\alpha\in\Omega} O_\alpha^c$ is closed.

Results:

i) Φ, X are at a time open and closed.

ii) Arbitrary union of open sets of τ is open.

iii) Finite intersection of open sets of τ is open.

iii) Finite union of closed sets of τ is closed.

iv) Arbitrary union of closed sets of τ is closed.

Usual Topology:

A collection of subsets of $\mathbb{R}$ which can be expressed as a union of open intervals, forms a topology on $\mathbb{R}$ and is called usual topology on $\mathbb{R}$.

closure of a set:

Let (X,τ) be a topological space and $A \subseteq X$.

Then, the smallest closed superset of A is called closure of A.

or the intersection of all closed superset of A is called closure of A. It is denoted by $\bar{A}$ or $cl(A)$

~~$X = \{1,2,3,4,5\}$~~ i.e. $A \subseteq \bar{A}$, $\bar{A} = A \cup A^d$

~~$\tau = \{\Phi, X, \{1\}, \{3\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4,5\}, \{1,2,4,5\}\}$~~

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$X = \{4, 5, 6, 7\}$

$\tau = \{\phi, X, \{4\}, \{5\}, \{4,5\}\}$

$A = \{5, 6\}$

closed sets of τ : $X, \phi, \{5,6,7\}, \{4,6,7\}, \{6,7\}$

closed supersets of A : $X, \{5,6,7\}$

then

$\bar{A} = X \cap \{5,6,7\}$

$\bar{A} = \{5,6,7\}$

$B = \{6\}$

closed supersets of B : $X, \{5,6,7\}, \{4,6,7\}, \{6,7\}$

$\bar{B} = \{6,7\}$

$C = \{4,7\}$

closed supersets of C : $X, \{4,6,7\}$

$\bar{C} = \{4,6,7\}$

$\bar{A} = A \cup A^d$

<u>Th</u>:

Let (X, τ) be a topological space and $A \subseteq X$. Then,

i) $\bar{\phi} = \phi$, $\bar{X} = X$, $A \subseteq \bar{A}$

ii) A is closed if and only if $\bar{A} = A$

iii) $\bar{\bar{A}} = \bar{A}$, $\bar{\bar{\bar{A}}} = \bar{A}$

iv) For any subsets A, B of X

a) $A \subset B \Rightarrow \bar{A} \subseteq \bar{B}$

b) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

c) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof :- (i)

By definition,

$\bar{\phi}$ is the intersection of all closed supersets of ϕ

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(ii)

suppose that A is closed

$\because$ $A \subseteq A$

then A is ~~smallest~~ closed ~~subset~~ superset of A

but $\bar{A}$ is the smallest closed superset of A.

$$A \subseteq \bar{A} \subseteq A$$

$\Rightarrow$ $\bar{A} \subseteq A$ ——— (i)

also

$A \subseteq \bar{A}$ ——— (ii) $\quad\because\ \bar{A} = A \cup A^d$

(i), (ii) $\Rightarrow$

$$A = \bar{A}$$

<u>conversely:</u>

let $A = \bar{A}$

$\Rightarrow$ A is closed $\quad\because\ \bar{A}$ is closed

In particular, $\bar{\Phi} = \Phi$, $\bar{X} = X$

(ii)

$\because$ A is closed iff $\bar{A} = A$

$(\bar{A}) = \bar{A}$ $\Rightarrow$ $\bar{A}$ is closed

$\bar{\bar{A}} = \bar{A}$

$\Rightarrow$ $\bar{\bar{A}}$ is closed

$\Rightarrow$ $(\bar{\bar{\bar{A}}}) = \bar{\bar{A}} = \bar{A}$

$\bar{\bar{\bar{A}}} = \bar{A}$

(iii)

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Given that

$A \subseteq B$

$A \subseteq B \subseteq \bar{B}$ $\quad\because B \subseteq \bar{B}$

i.e. $\bar{B}$ is closed superset of A

but

$\bar{A}$ is the smallest closed superset of A

$A \subseteq \bar{A} \subseteq B \subseteq \bar{B}$

$\Rightarrow \quad \bar{A} \subseteq \bar{B}$

$$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$$

$\because$

$A \subseteq \bar{A}$ —(i) & $B \subseteq \bar{B}$ —(ii)

$A \cup B \subseteq \bar{A} \cup \bar{B}$

i.e. $\bar{A} \cup \bar{B}$ is closed superset of $A \cup B$

but

$\overline{A \cup B}$ is smallest closed superset of $A \cup B$

$\Rightarrow$

$A \cup B \subseteq \overline{A \cup B} \subseteq \cancel{\overline{A \cup B}}$ $\Rightarrow$ $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ —(i)

also

$A \subseteq A \cup B$ & $B \subseteq A \cup B$

$\Rightarrow$ $\bar{A} \subseteq \overline{A \cup B}$ & $\bar{B} \subseteq \overline{A \cup B}$

$\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ —(ii)

(i), (ii) $\Rightarrow$

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

$\because$ $A \subseteq \bar{A}$ & $B \subseteq \bar{B}$

$A \cap B \subseteq \bar{A} \cap \bar{B}$

i.e. $\bar{A} \cap \bar{B}$ is closed super set of $A \cap B$

but

$\overline{A \cap B}$ is smallest closed superset of $A \cap B$

$\Rightarrow$ $A \cap B \subseteq \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

$\Rightarrow$ $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

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Interior point of a set

Let (X, τ) be a topological space and $A \subseteq X$

Then, a point $x \in X$ is said to be an interior point of A if there exists atleast one open set U containing x such that $U \subseteq A$

$X = \{1, 2, 3, 4, 5\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

$A = \{3, 4\}$

$A^\circ = \{3\}$

Note: interior of A is the largest open set contained in A. i.e. $A^\circ \subseteq A$

Interior of ~~a~~ set:

Let (X, τ) be a topological space and $A \subseteq X$

Then, the set of all ~~limit~~ interior points of A is called interior of A. It is denoted by A°.

Closed set:

i) A is closed iff A^c is open

ii) ~~if~~ ~~A is~~ closed iff every limit point of A belongs to A. i.e. $A^d \subseteq A$

iii) A is closed iff $\bar{A} = A$

iv) A is closed iff $Fr(A) \subseteq A$ (i.e every frontier pt. of A belongs to A)

Open set:

i) A is open if $A \in \tau$

ii) A is open iff $A^\circ = A$; $\phi^\circ = \phi$, $X^\circ = X$

iii) If every interior point of A belongs to A then A is open.

alternate definition:

the ~~of~~ union of all open subsets of A. It is denoted by A° or $Int(A)$. $\phi^\circ = \phi$, $X^\circ = X$

$X = \{1, 2, 3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$

$A = \{1, 3\}$, $B = \{2\}$ = $C = \{2, 3\}$, $D = \{3\}$

$A^\circ = \{1\}$

$B^\circ = \{2\} = C^\circ$

$D^\circ = \phi$

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Limit point of

Let (X, τ) be a topological space and $A \subseteq X$.
A Point $x \in X$ is called limit point of A if every open set containing x has non-empty intersection with $A - \{x\}$.

Note:

Limit Point is also called cluster point, derived Point, accumulation point.

Example:

$$X = \{1,2,3,4,5\}$$

$$\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

~~all open sets of τ containing~~

$A = \{3,4\}$

Is 1 limit point of A?
all open sets containing 1
$X, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}$
$\{3,4\} \cap \{1\} = \phi$
so, 1 is not

for 2 ?
$X, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}$
$\{3,4\} \cap \{2\} = \phi$
so, 2 is not

for 3 ?
$X, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}$
$\{3\} \cap \{4\} = \phi$

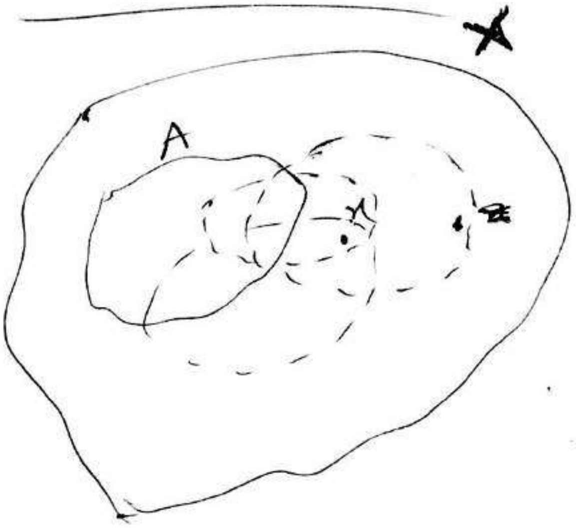
for 4 ?
$X \cap \{3\} = \{3\}$
so, 4 is

for 5 ?

Derived set:

Let (X, τ) be a topological space and $A \subseteq X$
Then, the set of all limit points of A is called derived set of A.
It is denoted by A^d.

$A^d = \{4,5\}$



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Th: Let (X,τ) be a topological space and $A \subseteq X$. Then, ⑥

i) $A^\circ \subseteq A$

by definition, it is proved.

another definitions of interior

$A^\circ = A - cl(A^c)$

$A^\circ = A - Fr(A)$

ii) $(A^\circ)^\circ = A^\circ$

$\because$ A° is open set by definition

by theorem:

A is open if and only if $A^\circ = A$

$(A^\circ)^\circ = A^\circ$

~~A~~ the required

(iii)

if $A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$

Proof:-

suppose

$A \subseteq B$

$A^\circ \subseteq A \subseteq B \qquad \because A^\circ \subseteq A$

i.e. A° is an open subset of B

but B° is largest open subset of B

$\Rightarrow A^\circ \subseteq A \subseteq B^\circ \subseteq B$

$\Rightarrow A^\circ \subseteq B^\circ$

(iv)

$(A \cap B)^\circ = A^\circ \cap B^\circ$

$\because A^\circ \subseteq A \quad \& \quad B^\circ \subseteq B$

$A^\circ \cap B^\circ \subseteq A \cap B$

ie. $A^\circ \cap B^\circ$ is an open subset of $A \cap B$

but $(A \cap B)^\circ$ is the largest open subset of $A \cap B$

$\Rightarrow A^\circ \cap B^\circ \subseteq (A \cap B)^\circ \subseteq A \cap B$

$\Rightarrow (A \cap B)^\circ \subseteq A \cap B$ ——— (i)

$\because A \cap B \subseteq A \quad ; \quad A \cap B \subseteq B$

$\Rightarrow (A \cap B)^\circ \subseteq A^\circ \quad ; \quad (A \cap B)^\circ \subseteq B^\circ$

$\because A \subseteq B \Rightarrow A^\circ \subseteq B^\circ$

$\Rightarrow (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ ——— (ii)

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(i), (ii) $\Rightarrow$

$$(A \cap B)^\circ = A^\circ \cap B^\circ$$

(v)

$$(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$$

Proof:

$\because$ $A \subseteq A \cup B$, $B \subseteq A \cup B$

$\Rightarrow$ $A^\circ \subseteq (A \cup B)^\circ$ — (i) ; $B^\circ \subseteq (A \cup B)^\circ$ — (ii)

$\Rightarrow$ $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$

$\Rightarrow$ $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$

Exterior point of a set:

Let (X, τ) be a topological space and $A \subseteq X$. Then, a point $x \in X$ is said to be an exterior point of A if there exists atleast one open set 'U' containing 'x' such that $U \subseteq A^c$

i.e. 'x' is exterior point of A if it is an interior point of A^c.

~~Exterior o Set of exterior points:~~ Exterior of a set:

Let A be a subset of a topological space (X, τ). Then, exterior of A is the set of all exterior points of A. It is denoted by ~~A~~ Ext(A).

Exp:

$X = \{1, 2, 3, 4, 5\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$

$A = \{2, 3, 4\} \Rightarrow A^c = \{1, 5\}$

$\Rightarrow$ $Ext(A) = \{1\}$

Note:-

$Ext(A) = Int(A^c)$

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Th:

Let (X, τ) be a topological space and $A \subseteq X$

Then

i) $(A^\circ)^c = \overline{A^c}$ i.e. $[Int(A)]^c = cl(A^c)$

ii) $A^\circ = A - \overline{A^c}$

iii) $Ext(X) = \phi$; $Ext(\phi) =$

iv) $Ext(A) = (\overline{A})^c$

v) $Ext(A \cup B) = Ext(A) \cap Ext(B)$

vi) $Ext(A \cap B) \supseteq Ext(A) \cup Ext(B)$

Proof :-

(i)

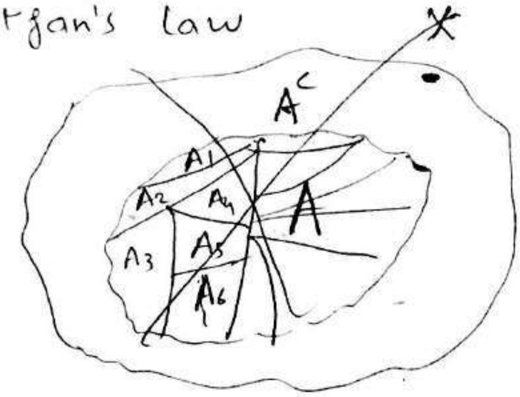
Let $\{A_\alpha : \alpha \in I\}$ be the collection of all open subsets of A. Then,

$$A^\circ = \bigcup_{\alpha \in I} A_\alpha$$

$$\Rightarrow (A^\circ)^c = \left(\bigcup_{\alpha \in I} A_\alpha\right)^c$$

~~by De-Morg~~

$$(A^\circ)^c = \bigcap_{\alpha \in I} A_\alpha^c \quad \text{by De-Morgan's law} \quad \underline{\quad\quad} (i)$$

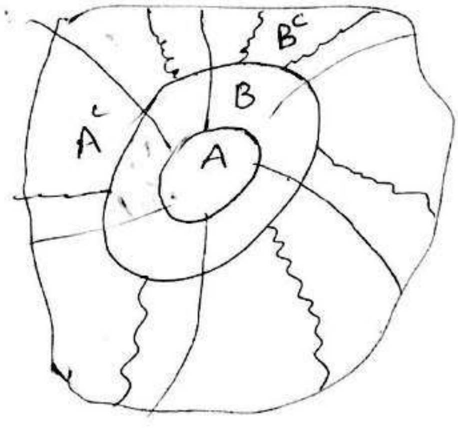


Now

$$A_\alpha \subseteq A \quad \forall \ \alpha \in I$$

$$\Rightarrow A^c \subseteq A_\alpha^c \quad \forall \ \alpha \in I$$

$\Rightarrow$
$\{A_\alpha^c : \alpha \in I\}$ is the collection of all closed supersets of A^c.



$$\overline{A^c} = \bigcap_{\alpha \in I} A_\alpha^c \quad \underline{\quad\quad} (ii)$$

(i), (ii) $\Rightarrow$

$$(A^\circ)^c = \overline{A^c}$$

(ii)

$A^\circ = A - \overline{A^c}$ i.e. $A^\circ = A - cl(A^c)$

Now

$$A - \overline{A^c}$$

$$= A \cap (\overline{A^c})^c \qquad \because A - B = A \cap B^c$$

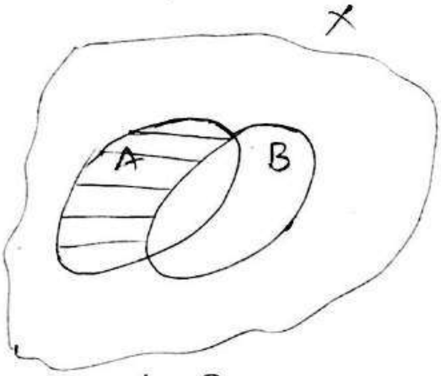
$$= A \cap [(A^\circ)^c]^c \qquad \because \overline{(A^c)} = (A^\circ)^c$$

$$= A \cap A^\circ$$

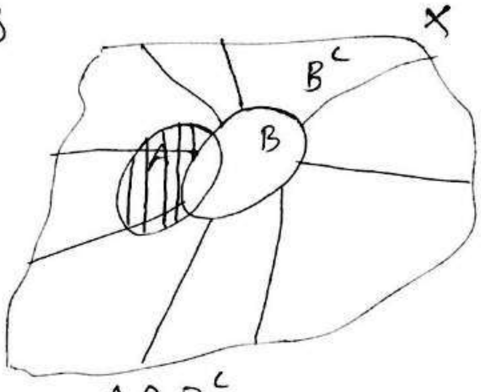
$$= A^\circ \qquad \because A^\circ \subseteq A$$

Hence, proved.

overlapping

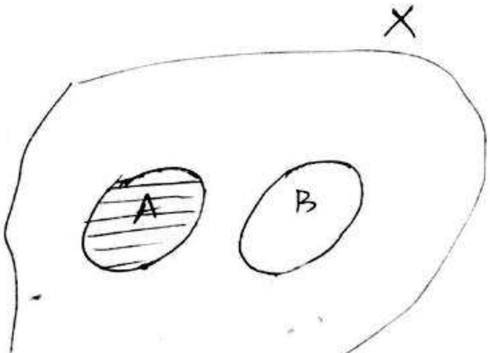


$A - B$

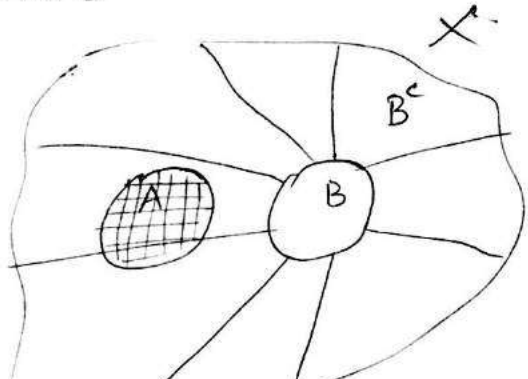


$A \cap B^c$

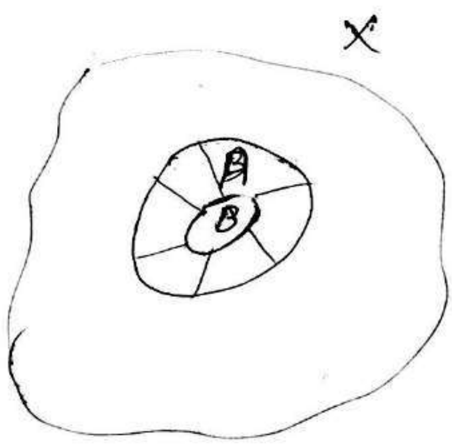
Disjoint



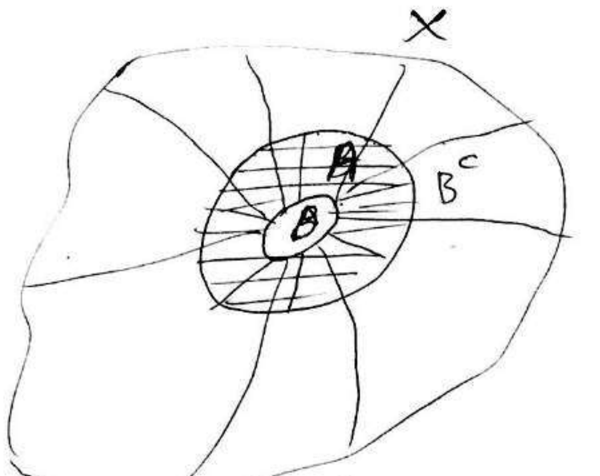
$A - B$



$A \cap B^c$



$A - B$



$A \cap B^c$

(iii)

$Ext(X) = \Phi$

Take

$Ext(X)$ ~~$= Int$~~

$= Int(X^c)$ $\quad\because Int(A) = Ext(A^c)$

$= Int(\Phi)$

$= \Phi$ $\quad\because \Phi$ is open and A is open iff $A^\circ = A$

also

$Ext(\Phi) = X$

take

$Ext(\Phi) = Int(\Phi^c)$

$= Int(X)$ $\quad\because \Phi^c = X$

$Ext(\Phi) = X$ $\quad\because X$ is open and A is open iff $A^\circ = A$

(iv)

$Ext(A) = (\overline{A})^c$

Take

$Ext(A)$

$\because Ext(A) = (A^c)^\circ$

Let $\{A_\alpha : \alpha \in I\}$ be the collection of all closed supersets of A. Then, by definition

$$\overline{A} = \bigcap_{\alpha \in I} A_\alpha$$

$$(\overline{A})^c = \left(\bigcap_{\alpha \in I} A_\alpha\right)^c$$

$$(\overline{A})^c = \bigcup_{\alpha \in I} A_\alpha^c \quad \text{——— (i)} \quad \text{by De-Morgan's law}$$

Now

$A \subseteq A_\alpha \quad \forall \alpha \in I$

$\Rightarrow A_\alpha^c \subseteq A^c \quad \forall \alpha \in I$

then $\{A_\alpha^c : \alpha \in I\}$ is the collection of all open subsets of A^c

$$(A^c)^\circ = \bigcup_{\alpha\in I} A_\alpha^c \quad \text{———} \quad (ii)$$

(i), (ii) $\Rightarrow$

$$(A^c)^\circ = (\overline{A})^c$$

$$Ext(A) = (\overline{A})^c$$

(v)

$$Ext(A\cup B) = Ext(A)\cap Ext(B)$$

Proof:-

Take

$Ext(A\cup B)$

$= Int[(A\cup B)^c]$ $\quad \because Ext(A) = Int(A^c)$

$= [(A\cup B)^c]^\circ$

by using De-Morgan's law

$= (A^c\cap B^c)^\circ$

$= (A^c)^\circ \cap (B^c)^\circ$ $\quad \because (A\cap B)^\circ = A^\circ \cap B^\circ$

$= Ext(A)\cap Ext(B)$ $\quad \because Ext(A) = Int(A^c)$

(vi)

$$Ext(A\cap B) \supseteq Ext(A)\cup Ext(B)$$

Take

$Ext(A\cap B)$

$= Int[(A\cap B)^c]$

$= [(A\cap B)^c]^\circ$

by De-Morgan's law

$= (A^c\cup B^c)^\circ$

$\supseteq (A^c)^\circ \cup (B^c)^\circ$ $\quad \because (A\cup B)^\circ \supseteq A^\circ\cup B^\circ$

$\supseteq Ext(A)\cup Ext(B)$

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Boundary (Frontier) Point:

Let (X, τ) be a topological space and $A \subseteq X$. Then,
a point $x \in X$ is called boundary (frontier) point of A if every open set containing 'x' has non-empty intersection with A and A^c.
and the set of all boundary (frontier) points of A is called boundary (frontier) of A.
It is denoted by $b(A)$ or $F_r(A)$
or

$$F_r(A) = \overline{A} \cap \overline{(A^c)}$$

Example:-

- $X = \{1, 2, 3, 4\}$

  $\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$

  $A = \{1, 2, 3\}$ $\qquad A^c = \{4\}$

  $F_r(A) = \{3, 4\}$

Th: Let (X, τ) be a topological space and $A \subseteq X$

i) $F_r(A) = F_r(A^c)$

ii) $A^\circ = A - F_r(A)$

iii) $\overline{A} = A \cup F_r(A)$

iv) $F_r(A)$ is closed subset of X.

v) A is both open and closed iff $F_r(A) = \phi$

vi) $F_r(A) \subseteq A$ if and only if A is closed.

Proof:-

by definition

$$F_r(A) = \overline{A} \cap \overline{(A^c)} \longrightarrow (i)$$

replace A by A^c

$$F_r(A^c) = \overline{A^c} \cap \left(\overline{A^c}\right)^c$$

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$= \overline{A^c} \cap \overline{A}$

$Fr(A^c) = \overline{A} \cap \overline{A^c}$ ______ (ii)

(i), (ii) $\Rightarrow$

$Fr(A) = Fr(A^c)$

(ii)

$A^o = A - Fr(A)$

Take

$A - Fr(A)$

$= A \cap [Fr(A)]^c$ $\quad\because A - B = A \cap B^c$

$= A \cap [\overline{A} \cap \overline{A^c}]^c$ $\quad$ by definition

$= A \cap [(\overline{A})^c \cup (\overline{A^c})^c]$ $\quad$ by De-Morgan's Law

$= [A \cap (\overline{A})^c] \cup [A \cap (\overline{A^c})^c]$

$= \phi \cup (A - \overline{A^c})$ $\quad\because A \subseteq B,\ A \cap B^c = \phi$

$A - B = A \cap B^c$

$= A - \overline{A^c}$

$= A^o$

the required

(iii)

$\overline{A} = A \cup Fr(A)$

Take

$A \cup Fr(A)$

$= A \cup (\overline{A} \cap \overline{A^c})$ $\quad\because$ (by definition)

$= (A \cup \overline{A}) \cap (A \cup \overline{A^c})$

$= \overline{A} \cap X$ $\quad\because A \subseteq \overline{A}$

$A \cup \overline{A^c} = X$

$= \overline{A}$

$\because A \cup \overline{A^c}$

$A \cup [A^c \cup (A^c)^d]$

$(A \cup A^c) \cup (A^c)^d \Rightarrow X \cup (A^c)^d \Rightarrow X$

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Corollary:

i) A is closed if and only if $Fr(A) \subseteq A$
i.e. A is closed if and only if every frontier point of A belongs to A.

~~corollary~~

ii) $Fr(A)$ is closed subset of X

iii) A is both open and closed if and only if $Fr(A) = \phi$

Proof:

suppose a subset A of a topological space X is both open and closed.
then by theorem
A is closed iff $\bar{A} = A$
A is open iff $A^{\circ} = A$

$\Rightarrow \quad A^{\circ} = \bar{A}$

$\Rightarrow A - Fr(A) = A \cup Fr(A)$
this relation is accepted if $Fr(A) = \phi$

Conversely:

Let $Fr(A) = \phi$

$\because \quad A^{\circ} = A - Fr(A) \quad ; \quad \bar{A} = A \cup Fr(A)$
$A^{\circ} = A - \phi \quad ; \quad \bar{A} = A \cup \phi$
$A^{\circ} = A \quad ; \quad \bar{A} = A$

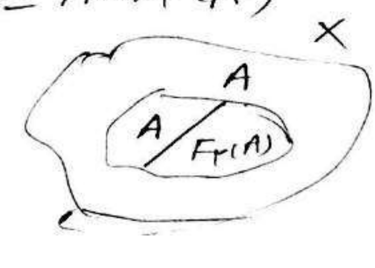
$\Rightarrow$ A is both open and closed.

Quiz Let (X, τ) —— and $A \subseteq X$

i) A is closed iff $Fr(A) \subseteq A$

proof:-
Let A is closed. Then
$A = \bar{A}$
$A = A \cup Fr(A) \quad \because \quad \bar{A} = A \cup Fr(A)$

$Fr(A) \subseteq A$



Conversely:

Let $Fr(A) \subseteq A$

$\because \quad \bar{A} = A \cup Fr(A)$

$\bar{A} = A \qquad\qquad \because Fr(A) \subseteq A$

$\Rightarrow A$ is closed $\qquad \because A$ is closed iff $\bar{A} = A$

Usual Topology $\qquad$ Pg # 2

Co-finite Topology

Let $X \neq \phi$

$\tau = \{A \subseteq X : A = \phi \text{ or } A^c \text{ is finite}\}$

Then τ is a topology on X known as co-finite topology or finite complement topology or Zariski topology.

Neighbour of a point:

Let (X,τ) be a topological space and $x \in X$. A subset N of 'X' is said to be neighbourhood of point 'x' if there exists atleast one open set 'U' such that

$$x \in U \subseteq N$$

In other words, if 'x' is an interior point of N. Then, N is called neighbourhood of x.

Note:
i) if N is open, then it is called open neighbourhood
ii) if N is closed, it is called closed neighbourhood

Neighbourhood system:

Let (X,τ) be a topological space and $x \in X$. Then, the collection of all neighbourhoods of x is called neighbour system for point x.

i.e. ~~all~~ set of all nbhds of 1 is called neighbourhood system for 1.

set of all nbhds of 2 is called neighbourhood system for 2.

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Sub-space:

Let (X, τ_x) be a topological space and $Y \subseteq X$

we define

$$\tau_y = \{ U : U = V \cap Y \ ; \ V \in \tau_x \}$$

then τ_y is a topology on Y known as relative topology. and (Y, τ_y) is called subspace of (X, τ_x).

$$X = \{1, 2, 3, 4\}$$

$$\tau = \{ \phi, X, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,2,3\}, \{1,3\}, \{2,3\} \}$$

$$Y = \{2, 3, 4\}$$

$$\tau_Y = \{ \phi, Y, \{2\}, \{3\}, \{2,3\} \}$$

Note:

It is not necessary for an open set of subspace to be open in the parent space.

$$X = \{1, 2, 3\}$$

$$\tau_x = \{ \phi, X, \{1\}, \{3\}, \{1,3\} \}$$

$Y = \{2, 3\}$; $Y = \{1, 3\}$

$\tau_y = \{ \phi, Y, \{3\} \}$ $\qquad$ $\tau_y = \{ \phi, Y, \{1\}, \{3\}, \{1,3\} \}$

Th: Let (X, τ_x) be ~~to~~ a topological space and (Y, τ_y) be its subspace. Then, every open subset of (Y, τ_y) is open in (X, τ_x) if and only if Y is open in (X, τ_x).

Proof:

we suppose that every open subset of (Y, τ_y) is also open in (X, τ_x).

$\because Y \in \tau_Y$

i.e. Y is open in (Y, τ_Y)

$\Rightarrow$ Y is open in (X, τ_X) $\because$ by above assumption

<u>Conversely:</u>

Let Y is open in (X, τ_X).

i.e. $Y \in \tau_X$

Let 'U' be an open subset of (Y, τ_Y)

i.e. $U \in \tau_Y$.

$\Rightarrow$ $V \cap Y = U$ for some $V \in \tau_X$

but

$V \cap Y = U \in \tau_X$ $\because$ $V \in \tau_X$; $Y \in \tau_X$

$\Rightarrow$ U is open in X, τ_X

---

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Base for a Topology:

Let (X, τ) be a topological space. A sub-collection $\mathcal{B}$ of τ is said to be base for the topology τ of X if every member of τ can be written as union of some members of $\mathcal{B}$.

If $\mathcal{B}$ is a base for τ. Then, members of ~~τ~~ $\mathcal{B}$ are called basic open sets.

we can say that the base generates the topology

Example:

$X \neq \phi$

$\tau = P(X)$

~~then~~ $\mathcal{B} = \{ \{x\} : x \in X \}$

then $\mathcal{B}$ is always a base for discrete topology.

$X = \{1, 2, 3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$\mathcal{B} = \{\{1\}, \{2\}, \{3\}\}$

Note:

Ground set is a base of indiscrete topology.

i.e. $X \neq \phi$

$\tau = \{\phi, X\}$

$\mathcal{B} = \{X\}$

Th: Let (X, τ) be a topological space. A collection $\mathcal{B} = \{B_\alpha : \alpha \in I\}$ of sets in τ is a base for τ if and only if, for any open set U and any point $x \in U$, there is B_α such that

$$x \in B_\alpha \subseteq U$$

(top. space) کا ہر (openset) اور ہر (open set) میں (pt) کو (contain) کرتا ہے
ان دونوں کے (corresponding) (Bases) کا ایک ایسا (open set) ہو جو (pt)
کو (contain) کرے اور اس 'U' کا (subset) ہو۔

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Proof:

we suppose that

$B = \{ B_\alpha : \alpha \in I \}$

is a base for τ.

Let 'U' be an open set with $x \in U$

$\because$ B is a base for τ

$\Rightarrow$

$$U = \bigcup_{\alpha \in I'} B_\alpha \qquad \because I' \subseteq I$$

$$\Rightarrow \quad x \in \bigcup_{\alpha \in I'} B_\alpha \qquad \because x \in U \ \cancel{\text{for some } \alpha \in I'}$$

$$\Rightarrow \quad x \in B_\alpha \qquad \because \text{for some } \alpha \in I'$$

$$\Rightarrow \quad x \in B_\alpha \subseteq U \qquad \because \text{for some } \alpha \in I$$

Conversely:

Let $B = \{ B_\alpha : \alpha \in I \}$ be sub-collection of members of τ.

we suppose that

$U \in \tau$ with $x \in U$

and there exists $B_x \in B$ such that

$$x \in B_x \subseteq U$$

$$\Rightarrow \quad \{x\} \subseteq B_x \subseteq U$$

$$\Rightarrow \quad \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} B_x \subseteq \bigcup_{x \in U} U$$

$$\Rightarrow \quad U \subseteq \bigcup_{x \in U} B_x \subseteq U$$

$$U = \bigcup_{x \in U} B_x$$

$\Rightarrow$ every $U \in \tau$ can be written as union of some members of ~~τ~~ B.

$\Rightarrow$ B is a base for τ.

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Assignment # 1

Th: A family $\mathbb{B}$ of subsets of τ is a base for τ if and only if

i) $X = \bigcup B_\alpha$ where $B_\alpha \in \mathbb{B}$

ii) For $B_1, B_2 \in \mathbb{B}$ and $x \in B_1 \cap B_2$ there is $B \in \mathbb{B}$ such that

$$x \in B \subseteq B_1 \cap B_2$$

Sub-Base :

A collection S of subsets of X is said to be sub-base for some topology τ on X if all finite intersection of ~~of~~ members of S forms a base for topology.

Note:

Any collection of subsets of X whose union is X forms some topology on X.

Examples:

$X = \{a, b, c, d\}$

$S = \{\{a\}, \{b, c\}, \{b, d\}\}$

all finite intersection

$\phi, \{b\}$

then

$\mathbb{B} = \{\{a\}, \{b\}, \{b, c\}, \{b, d\}\}$

is a base for topology

$$\tau = \{\phi, X, \{a\}, \{b\}, \{b, c\}, \{b, d\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$$

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$X = \mathbb{R}$

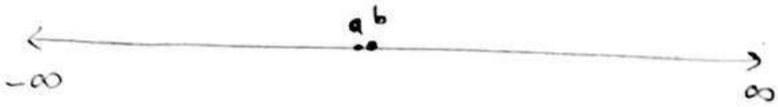
$S = \{ (-\infty, b), (a, \infty) : a, b \in \mathbb{R} \}$

forms a base for usual topology.

$b > a \; ; \; b = a \; ; \; b < a$

$\downarrow$

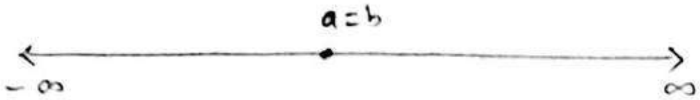
$b > a$



- finite intersection

  (a, b)

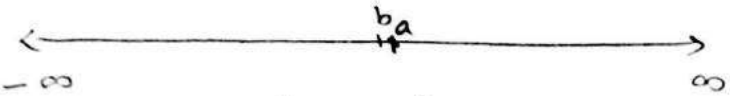
$b = a$



- finite intersection

  ϕ

$b < a$



- finite intersection

  ϕ

$\mathbb{B} = \{ (a, b) : a, b \in \mathbb{R} \}$

$\tau = \{ \phi, X \} \cup \{ (a, b) : a, b \in \mathbb{R} \}$

$X = \{1, 2, 3\}$

$S = \{ \{1\}, \{2\}, \{3\} \}$

all finite intersection of members of S.

ϕ

$\mathbb{B} = \{ \phi, \{1\}, \{2\}, \{3\} \}$

$\tau = \{ \phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\} \}$

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Th: Let S be a non-empty collection of subsets of X, such that

$$X = \bigcup_{S \in \mathcal{S}} S$$

Then, S is a sub-base for some topology on X.

i.e.
any collection of subsets of X whose union is X, forms some topology on X.

Neighbourhood Base (Local base) (Base at a pt.) **at a point:**

Let (X, τ) be a topological space and $x \in X$. A sub-collection $\mathbb{B}_x$ of τ is said to be neighbourhood base or simply a base at x if for any $U \in \tau$ with $x \in U$, there is a $B \in \mathbb{B}_x$ such that

$$x \in B \subseteq U$$

اُس (pt) کو (contain) کرنے والے کسی بھی (openset) 'U' کے (corresponding)، (sub-collection)
کا ایسا (open set) B ہو جو کہ اُس (pt.) کو (contain) کرے اور 'U' کا بھی (subset) ہو

Examples:

$X = \{1, 2, 3\}$
$\tau = \{\phi, X, \{1\}, \{2\}, \{1, 2\}\}$
$B = \{X, \{1\}, \{2\}\}$

$X = \{1, 2, 3\}$
$\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$
$B_1 = \{\{1\}\}$; $B_2 = \{\{2\}\}$; $B_3 = \{\{3\}\}$

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Th:
A collection $\mathcal{B}$ of open sets in a topological space (X, τ) is a base for τ if and only if $\mathcal{B}$ contains base at each Point.

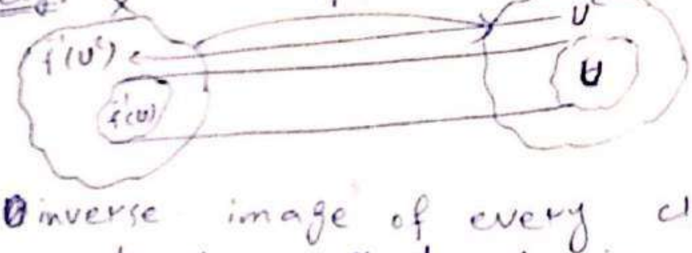
$$\mathcal{B} = \{\{1\}, \{2\}, \{3\}\}$$

Th:
A function $f: X \to Y$ is continuous on X if and only if inverse image of every closed is closed.

Proof:

Let $f: X \to Y$ is continuous on X.
and 'V' be a closed subset of Y.
then we have to show that $f^{-1}(V)$ is closed.
as $V \subseteq Y$ is closed
$\Rightarrow V^c \subseteq Y$ is open
by theorem:
f is continuous if and only if inverse image of every open is open
$\Rightarrow f^{-1}(V^c)$ is open in X
as $f^{-1}(V^c) = X - f^{-1}(V)$
$\Rightarrow f^{-1}(V)$ is closed in X.

Conversely:



Let inverse image of every closed is closed. Then we have to show that f is continuous on X.
let 'U' be an open subset of Y.
then we have to show $f^{-1}(U)$ is open in X.
as $U \subseteq Y$ is open
$\Rightarrow U^c \subseteq Y$ is closed.
by supposition $f^{-1}(U^c)$ is closed in X.
i.e. $f^{-1}(U^c) = X - f^{-1}(U)$ is closed in X
$\Rightarrow f^{-1}(U)$ is open in X
$\Rightarrow f$ is continuous

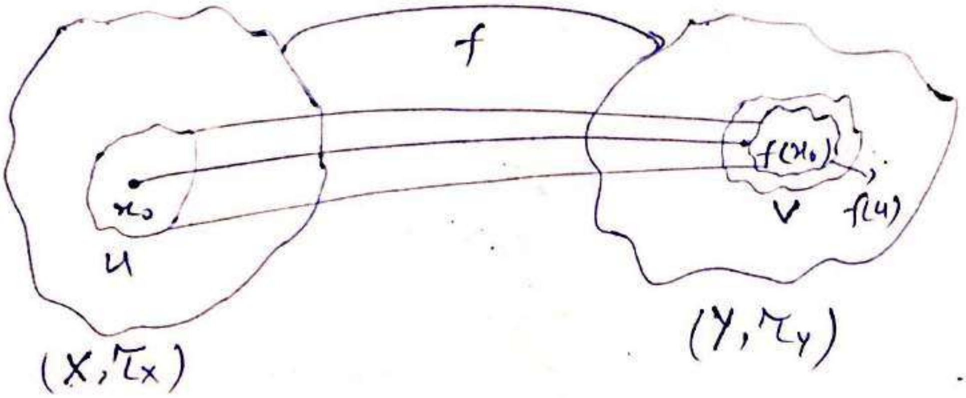
Continuity at a point:

Let (X, τ_X) and (Y, τ_Y) be two topological spaces, and $f: X \to Y$ is a function

let $x_0 \in X$

then 'f' is said to be continuous at x_0 if for each open set 'V' containing $f(x_0)$ there exists an open set 'U' in X such that

$$x_0 \in U \quad \text{and} \quad f(U) \subseteq V$$



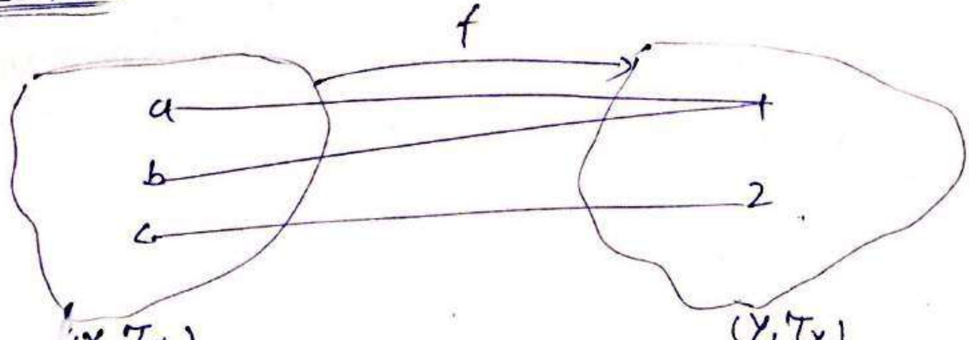
Example:

$X = \{a, b, c\}$ $\qquad$ $Y = \{1, 2\}$

$\tau_X = \{\Phi, X, \{a\}, \{a, b\}\}$ $\qquad$ $\tau_Y = \{\Phi, Y, \{1\}, \{2\}\}$

then $f: X \to Y$ defined by $f(a) = f(b) = 1$, $f(c) = 2$ is continuous at ~~x=~~ a and b but not at c.

Explanation:



at a

$x_0 = a$

$f(x_0) = f(a)$

$f(x_0) = 1$

let $V = \{1\} \in \tau_Y$ containing '1'

$\exists$ $U = \{a, b\} \in \tau_X$ s.t.

$f(U) = f(\{a, b\})$

$f(U) = \{1\} \subseteq V$

$\Rightarrow$ $f(U) \subseteq V$

$\Rightarrow$ f is continuous at a & b

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at c:

$x_0 = c$

$f(x_0) = f(c)$

$f(x_0) = 2$

Let $V = \{2\} \in \tau_Y$ containing '2'

$\exists$ $U = X \in \tau_X$ s.t.

$f(U) = f(X)$

$f(U) = Y \nsubseteq V$

$\Rightarrow$ $f(U) \nsubseteq V$

$\Rightarrow$ f is not continuous at $x = c$

Continuous function:

A function $f: X \to Y$ is continuous on X if f is continuous at every point of X.

Examples:

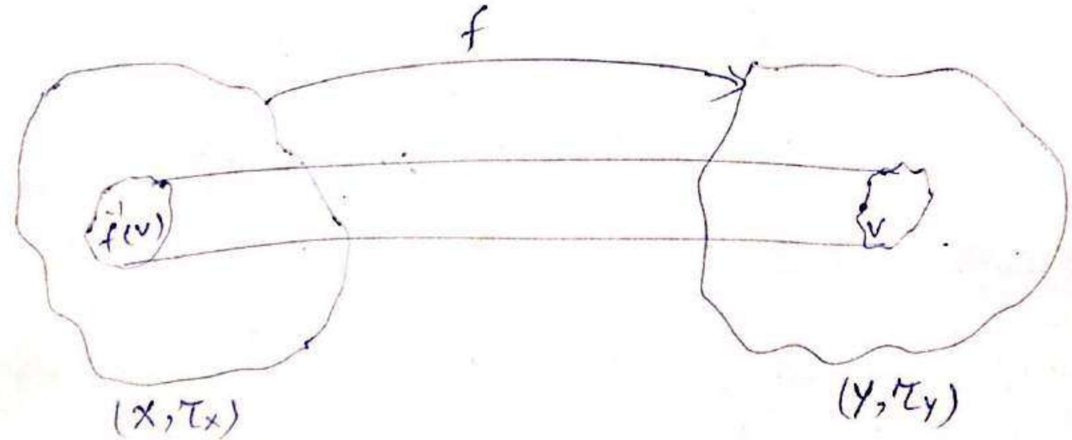
1) Let 'X' be an arbitrary topological space and 'Y' be indiscrete topological space. then $f: X \to Y$ is continuous.

i.e. Any function from arbitrary topological space to indiscrete topological space is continuous.

2) Every function from discrete topological space to arbitrary topological space is continuous.

Definition:

A function $f: X \to Y$ is continuous on X if $f^{-1}(V)$ is open in X, for every open set V of Y.



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$X = \{1,2,3\}$; $Y = \{1,2,3\}$

$\tau = P(X)$ $\tau =$ arbitrary

Th:

Let X and Y be topological spaces. A function $f: X \to Y$ is continuous on X if and only if for each subset V open in Y, $f^{-1}(V)$ is open in X.

Proof:-

suppose $f: X \to Y$ is continuous on X.

Let $V \in \tau_Y$

then we have to show $f^{-1}(V)$ is open

Let $x \in f^{-1}(V)$

$\Rightarrow f(x) \in V$

$\because f$ is continuous

then $\exists$ $U \in \tau_X$ s.t.

$x \in U$ & $f(U) \subseteq V$

$\Rightarrow U \subseteq f^{-1}(V)$

$\Rightarrow x \in U \subseteq f^{-1}(V)$

$\Rightarrow f^{-1}(V)$ is open

Conversely:

suppose inverse image of each open set in Y is open in X.

then we have to show that f is continuous on X.

Let $x \in X$, then to show f is continuous on X

let V be an open set in Y containing $f(x)$

i.e. $f(x) \in V$

$\Rightarrow x \in f^{-1}(V) = U$

by supposition U is open in X

$\Rightarrow f(U) \subseteq V$

$\Rightarrow f$ is continuous at $x \in X$

Since, x is arbitrary

So, f is continuous at each point of X.

Hence, $f: X \to Y$ is continuous on X.

Th:

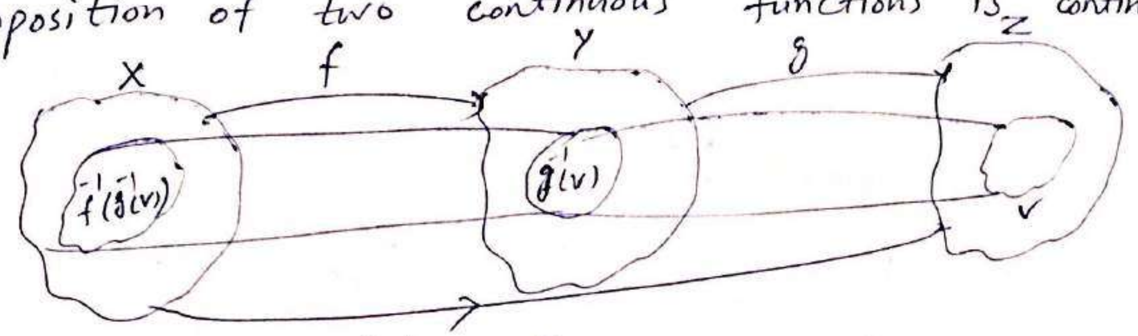
Let X, Y, Z be topological spaces,

and $f: X \to Y$, $g: Y \to Z$

be continuous functions. Thus $g \circ f: X \to Z$ is continuous.

i.e. composition of two continuous functions is continuous.

Proof:



$g \circ f: Y \to Z$

Let V be an open subset in Z.

Since, $g: Y \to Z$ is continuous.

$\Rightarrow g^{-1}(V)$ is open in Y

also $f: X \to Y$ is continuous

$\Rightarrow f^{-1}(g^{-1}(V))$ is open in X

$f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$

Hence, $g \circ f: X \to Z$ is continuous.

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Example:

Let τ_1 and τ_2 be two topologies on a set X.
A function $f:(X,\tau_1) \longrightarrow (X,\tau_2)$
is continuous if and only if τ_1 is stronger (finer) than τ_2. i.e. $\tau_2 \subseteq \tau_1$

~~Remarks~~ corollary:

A function $f: X \to Y$ is continuous on X if and only if for every subset C closed in Y $f^{-1}(C)$ is closed.
i.e inverse image of every closed is closed.

Corollary:

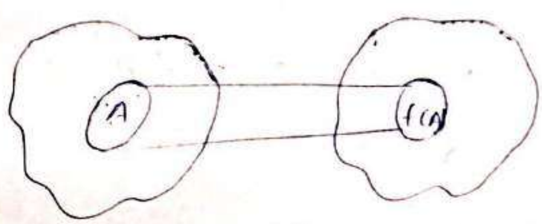
A function $f: X \longrightarrow Y$ is continuous on X if and only if for any subset A of X

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Proof:

suppose that $f: X \longrightarrow Y$ is continuous on X.
and $A \subseteq X$

$\because f(A) \subseteq \overline{f(A)}$ $\quad \because A \subseteq \overline{A}$



$\Rightarrow A \subseteq f^{-1}\,\overline{f(A)}$

$\because \overline{f(A)}$ is closed
and by theorem inverse image of every closed is closed.

$\Rightarrow f^{-1}\,\overline{f(A)}$ is closed in X.
and $f^{-1}\,\overline{f(A)}$ is closed superset of A.
but $\overline{A}$ is the smallest closed superset of A.
i.e. $A \subseteq \overline{A} \subseteq f^{-1}\,\overline{f(A)}$

$\Rightarrow \overline{A} \subseteq f^{-1}\,\overline{f(A)}$

$\Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$

Conversely:

Suppose that ~~for~~ for any $A \subseteq X$

$$f(\overline{A}) \subseteq \overline{f(A)}$$

Then, we have to show $f: X \longrightarrow Y$ is continuous

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Let $C \subseteq Y$ is closed

then we show $A = f^{-1}(C)$ is closed in X

$$\cancel{f(\bar{A}) \subseteq \overline{f(A)}}$$

$$\cancel{f(\overline{f^{-1}(C)}) \subseteq \overline{f(f^{-1}(C))}}$$

$$\cancel{f(\overline{f^{-1}(C)}) \subseteq \bar{C}}$$

$$\cancel{f(\overline{f^{-1}(C)}) \subseteq C} \qquad \because C \text{ is closed}$$

$\because$

$$f(\bar{A}) \subseteq \overline{f(A)}$$

$$\subseteq \overline{f(f^{-1}(C))} \qquad \because A = f^{-1}(C)$$

$$\subseteq \bar{C}$$

$$\subseteq C \qquad \because C \text{ is closed}$$

$$\Rightarrow \quad f(\bar{A}) \subseteq \overline{f(A)} \subset C$$

also

$$f(\bar{A}) \subseteq \overline{f(A)}$$

$$\Rightarrow \quad \bar{A} \subseteq f^{-1}\overline{f(A)}$$

$$\Rightarrow \quad \bar{A} \subset f^{-1}(C)$$

$$\Rightarrow \quad \bar{A} \subseteq A \text{ ---- (i)} \qquad \because A = f^{-1}(C)$$

since $A \subset \bar{A}$ ---- (ii)

(i), (ii) $\Rightarrow$

$$A = \bar{A}$$

$\Rightarrow$ $f^{-1}(C)$ is closed in X.

Hence, image of every closed is closed.

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Remark:

1- Let $\mathbb{B}$ be a base for some topology on Y. Then, a function $f: X \to Y$ is continuous if and only if, for each basic open set B in Y, $f^{-1}(B)$ is open in X.

Open mapping (function)

A function $f: X \to Y$ is said to be open if the image of every open is open.

closed mapping (function)

A function $f: X \to Y$ is said to be closed if the image of every closed is closed.

Example:

$$X = \{x, y, z\}$$

$$\tau_X = \{\phi, X, \{y\}, \{x, y\}, \{y, z\}\}$$

$$Y = \{1, 2, 3\}$$

$$\tau_Y = \{\phi, Y, \{1\}\}$$

Then $f: X \to Y$ defined as

$$f(x) = 2 \qquad f(y) = 1 \quad , \quad f(z) = 3$$

is continuous but not open.

$\because \quad f(\{x, y\}) = \{1, 2\} \notin \tau_Y$

Homeomorphism:

Let X and Y be topological spaces. A function $f: X \to Y$ is said to be homeomorphism if

1- f is bijective
2- f is continuous
3- f^{-1} is continuous (f is open).

and two spaces X and Y are said to be homeomorphic if there is a homeomorphism between them. we write $X \cong Y$

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Example:

$f:(a,b) \longrightarrow (c,d)$ defined by

$$f(x) = c + \frac{d-c}{(b-a)} \cdot (x-a)$$

is bijective and continuous.

$f^{-1}:(c,d) \longrightarrow (a,b)$

$$f^{-1}(x) = \frac{b-a}{d-c} \cdot (x-c) + a$$

Remark:

i) The identity mapping $I: X \rightarrow X$ is a homeomorphism. i.e. $X \cong X$

ii) if $f: X \longrightarrow Y$ is homeomorphism.

then $f^{-1}: Y \longrightarrow X$ is a homeomorphism.

i.e. $X \cong Y$ then $Y \cong X$

iii) If $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ are homeomorphisms then $g \circ f: X \longrightarrow Z$ is homeomorphism.

i.e. $X \cong Y$, $Y \cong Z$

then $X \cong Z$

Equal sets:

same and equal no. of elements

Equivalent sets:

- ~~same~~ equal no. of elements
- two sets are equivalent iff they have the same cardinality (is the no. of elements.)
- two sets are equivalent iff they have one-to-one correspondence between them.
- iff there exists a bijection between them.

Denumerable set:

A set S is said to be denumerable (or countably infinite) if there exists a bijection of $\mathbb{N}$ onto S.

i.e. there exists a bijection with $\mathbb{N}$.

$f : \mathbb{N} \longrightarrow S$

Examples:

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

$f : \mathbb{N} \longrightarrow \mathbb{Z}$

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & -4 & 5 & -5 \end{array}$$

$\mathbb{Q}$ is denumerable. , $\mathbb{N}$,

Countable set:

A set S is countable if it is finite or denumerable.

Finite Set:

A set S is finite if it is either empty or it has n elements for some $n \in \mathbb{N}$

George Cantor

First countable space:

A topological space (X, τ) is said to be first countable if its each neighbourhood base (or local base) at a point $x \in X$ is countable.

i.e. every local base is countable.

Second countable space:

A topological space (X, τ) is said to be second countable if it has a countable base.

Dense set in topological space:

Let (X, τ) be a topological space and $A \subseteq X$

Then, A is dense in X if

$$\overline{A} = X$$

Separable space:

A topological space (X, τ) is said to be separable if it has a countable dense subset.

i.e.

$A \subseteq X$

i) A is countable

ii) $\overline{A} = X$

Examples:

i) $\mathbb{R}$ is separable.

because $\mathbb{Q}$ is countable and dense in $\mathbb{R}$

i.e. $\overline{\mathbb{Q}} = \mathbb{R}$

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iii) Indiscrete space is always separable.

$X = \{1,2,3,4\}$

$\tau = \{\phi, X\}$

$A = X$

$\Rightarrow A$ is countable

and $\overline{A} = X$ (check)

(iii)

~~$X = [0,1]$~~. $X = \{1,2,3,4\}$

~~$\tau =$~~ $\tau = \{\phi, X, \{1\}, \{2,3\}, \{1,2,3\}, \{4\}, \{1,4\}, \{2,3,4\}\}$

$A = \{1,3,4\}$, $B = \{1,4\}$

then A is dense in X but B is not.

i.e. $\overline{A} = X$; $\overline{B} \neq X$

George contor 1807

Two sets (finite or infinite) have the same cardinality if there exists a bijection between them.

- $f: \mathbb{N} \to \mathbb{N}$
  $f(x) = x$. $\forall x \in \mathbb{N}$

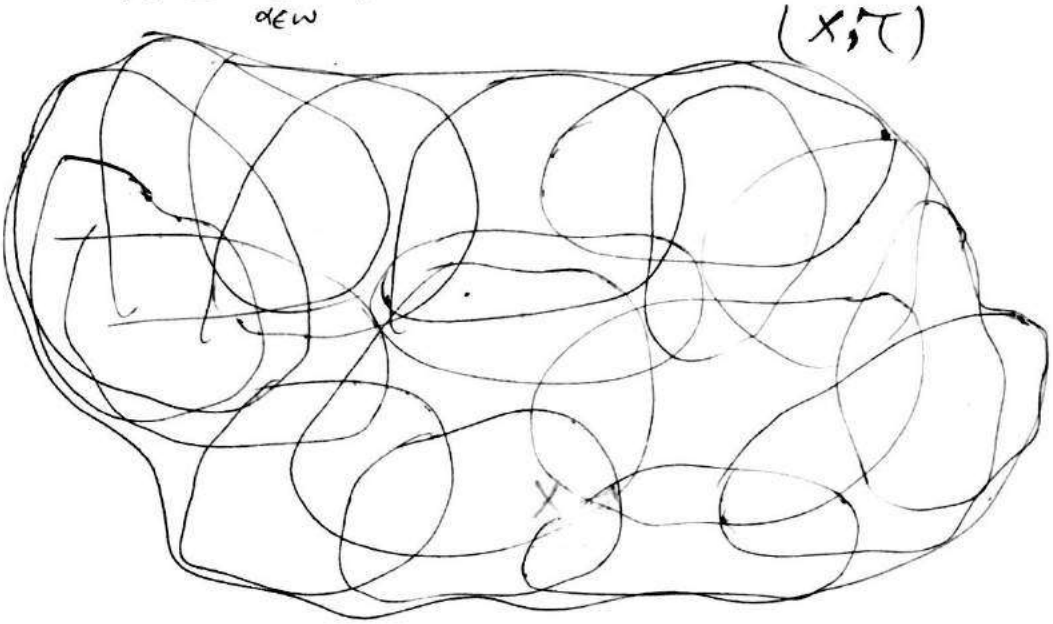
- $f: \mathbb{N} \to \mathbb{Z}$

$$f(x) = \begin{cases} \frac{1-n}{2} & n \text{ is odd} \\ \frac{n}{2} & n \text{ is even} \end{cases}$$

Open Cover: Cantor (1845-1918)

Let (X, τ) be a topological space. Then, a collection of open subsets $\{A_\alpha : \alpha \in \omega\}$ is said to be an open cover for X if

$$X = \bigcup_{\alpha \in \omega} A_\alpha$$

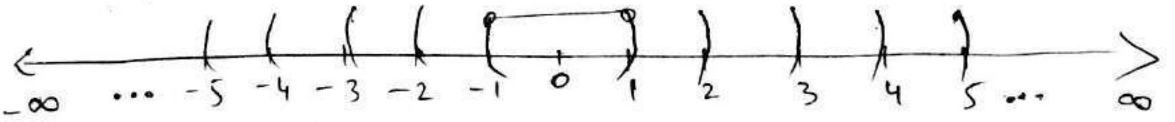


Example:

$X = \mathbb{R}$ with usual topology

$C = \{(-n, n) : n \in \mathbb{N}\}$

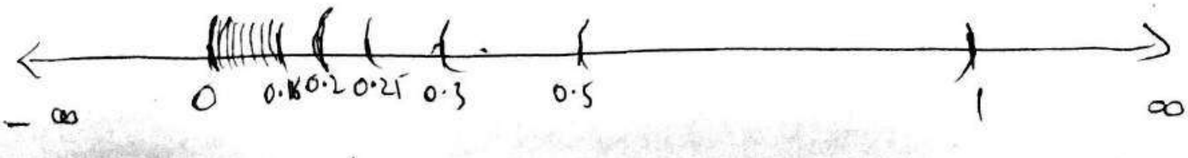
is an open cover for X.



ii) $X = (0,1)$ with usual topology

$C = \{(\frac{1}{n}, 1) : n \in \mathbb{N} - \{0\}\}$

is an open cover for X



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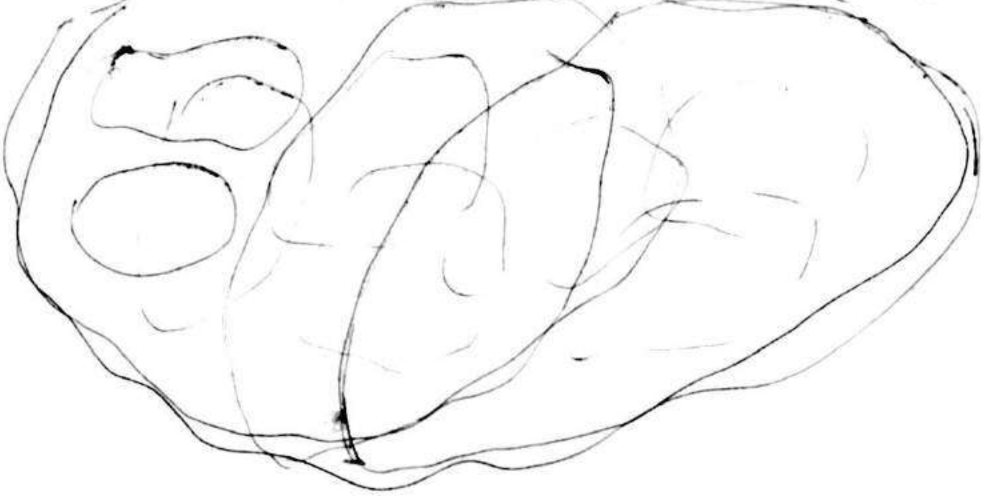
Sub-cover of an open cover:

A subset of an open cover that still X.

(X, τ)



i.e. $C = \{A_\alpha : \alpha \in w\}$ is an open cover of X.

then $\{A_{\alpha'} : \alpha' \in w' \subseteq w\}$ is a subcover of C

if $X = \bigcup_{\alpha' \in w'} A_{\alpha'}$

(Li-Lof)

<u>Lindelöf space</u>:

A topological space (X, τ) is said to be Lindelöf space if every open cover has a countable sub-cover.

<u>Th</u>:

i) Every second countable space is first countable but the converse is not true.

ii) Every second countable space is separable.

iii) Every second countable space is Lindelöf space.

iv) Every closed subspace of Lindelöf is Lindelöf.

Proof:

Let (X,τ) be a second countable space i.e. it has a countable base.

$$\mathbb{B} = \{B_\alpha : \alpha \in w = \{1,2,3,\dots\}\}$$

Let $x \in X$

then

$$B_x = \{B_\alpha : x \in B_\alpha \in \mathbb{B},\ \alpha \in w' \subseteq w\}$$

$\because$ w is countable.

$\Rightarrow$ w' is countable

$\Rightarrow$ B_x is countable

Hence, X is first countable.

coverse is not true in general.

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Separation Axioms:

$T_0, T_1, T_2, T_3, T_{\frac{3}{2}}, T_4$

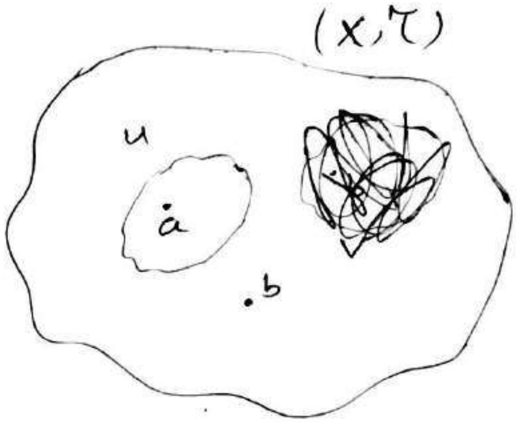
T_0-space:

A topological space (X, τ) is said to be T_0-space, for any two distinct points a, b of X, there is atleast one open set which contains one of the points but not the other.

i.e. $\forall\ a, b \in X \quad , a \neq b$

$\exists\ u \in \tau_x$ s.t.

$a \in u$ but $b \notin u$



Examples:

i) $\mathbb{R}$ with usual topology.

ii) Sierprinski space.

$X = \{a, b\}$

$\tau = \{\phi, X, \{a\}\}$

Note: Indiscrete space is not T_0-space.

$X \neq \phi$

$\tau = \{\phi, X\}$

Th: Every subspace of T_0 is T_0.

Proof: Let (X, τ_x) be T_0-space and (Y, τ_y) be its subspace. Then, we have to show Y is T_0.

Let $a, b \in Y \quad , a \neq b$

$\Rightarrow a, b \in X \qquad \because Y \subseteq X$

Since, X is T_0-space.

then there exists atleast one open set u such that

$a \in u$ but $b \notin u$

$\Rightarrow a \in u \cap Y = u_1$ but $b \notin u \cap Y = u_1$

$\Rightarrow u_1$ is an open set in Y which contains 'a' but not 'b' $\Rightarrow Y$ is T_0-space.

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Th: A space X is T_0 if and only if, for any $a, b \in X$, $a \neq b \Rightarrow \overline{\{a\}} \neq \overline{\{b\}}$

Proof:

Suppose X is T_0- space.

then for $a, b \in X$, $a \neq b$ there is atleast one open set 'u' s.t.

$a \in u$ but $b \notin u$

$\because a \neq b$

$\Rightarrow a \notin \{b\}$

~~and a is not~~

and $a \notin \{b\}^d$

i.e. a is not limit pt. of $\{b\}$.

because 'u' is an open set which contains 'a' but $u \cap \{b\} = \phi$

$\Rightarrow a \notin \overline{\{b\}}$ ——— (i) $\quad \because \overline{\{b\}} = \{b\} \cup \{b\}^d$

but

$a \in \{a\}$ and $a \in \overline{\{a\}}$

$\Rightarrow a \in \overline{\{a\}}$ ——— (ii)

(i), (ii) $\Rightarrow$

$\overline{\{a\}} \neq \overline{\{b\}}$

Conversely:

suppose for any $a, b \in X$, $a \neq b$

$\overline{\{a\}} \neq \overline{\{b\}}$

Then, we have to show that X is T_0- space.

we suppose on contrary that X is not T_0- space.

$\Rightarrow$ every open set which contains 'a' ~~but not 'b'~~ also contains 'b'.

Let 'u' be an open set such that

$a \in u$, $b \in u$

$\Rightarrow a \in u \cap \{b\} \neq \phi$

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$\Rightarrow \quad a \in \{b\}^d$

$\Rightarrow \quad a \in \overline{\{b\}} \qquad \because \overline{\{b\}} = \{b\} \cup \{b\}^d$

$\Rightarrow \quad \overline{\{a\}} \subseteq \overline{\{b\}}$ ____ (i)

similarly,

$\overline{\{b\}} \subseteq \overline{\{a\}}$ ____ (ii)

$\Rightarrow \quad \overline{\{a\}} = \overline{\{b\}}$

which is contradiction against the fact that $\{a\} \neq \{b\}$

then we cannot suppose X is not T_0-space.

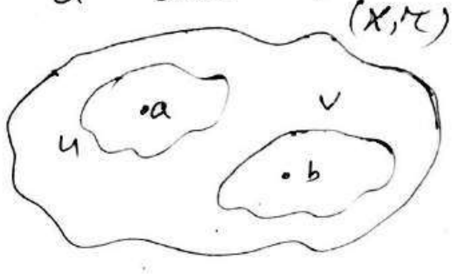
Then, X is T_0-space.

__T_1- space:__

A topological space (X, τ) is said to be T_1 if for any $a, b \in X$, $a \neq b$ there exists two open sets 'u' and 'v' such that

$a \in u \quad , \quad a \notin V$

$b \in V \qquad b \notin u$



__Note:__

Every T_1 is T_0 but the converse is not true in general.

__Counter example:__ (sierprinski space)

$X = \{a, b\}$

$\tau = \{\phi, X, \{a\}\}$

is T_0 but not T_1.

__Th:__

Every sub-space of T_1 is T_1

Examples:

i) Discrete space is T_1

$\Rightarrow$ also T_0

ii) $\mathbb{R}$ with usual topology is T_1

$\Rightarrow$ also T_0

$\forall\ a, b \in \mathbb{R}$, $a \neq b$

$(a, \frac{r}{2})$, $(b, \frac{r}{2})$ where $|a-b| = r$

Th: Every T_1 is T_0 but the converse is not true.

Proof: Let (X, τ) be a T_1-space. then for $a, b \in X$, $a \neq b$ there exists open set 'u' and 'v' such that

$a \in u$ $b \notin u$

$b \in v$ $a \notin v$

$\Rightarrow$ u is an open set of X which contains 'a' but not b.

$\Rightarrow$ X is T_0-space.

Conversely: Sierprinski space is T_0 but not T_1.

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Th: Every subspace of T_1 is T_1.

Proof:

Let (X, τ_x) be a T_1-space and (Y, τ_Y) be its sub-space.

Then, we have to show Y is T_1.

Let $a, b \in Y$, $a \neq b$

$\Rightarrow$ $a, b \in X$ $\because Y \subseteq X$

Since, X is T_1.

then there exists two open sets 'u' and 'v' such that

$a \in u$, $b \in v$

$a \notin v$ $b \notin u$

$\Rightarrow$ $a \in u \cap Y = u_1$ $b \in v \cap Y = v_1$

$a \notin v \cap Y = v_1$ $b \notin u \cap Y = u_1$

$\Rightarrow$

$a \in u_1$, $b \in v_1$

$a \notin v_1$ $b \notin u_1$

$\Rightarrow$ u_1 and v_1 are two open sets of Y which contain one but not the other.

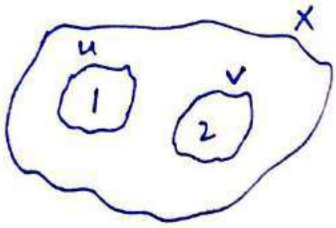
$\Rightarrow$ Y is T_1-space.

• Every discrete space is T_1.

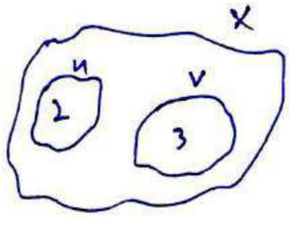
$X = \{1, 2, 3\}$

$\tau = \{\phi, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

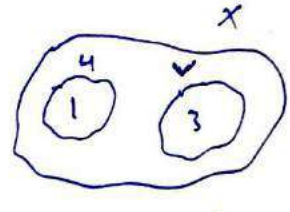
for $1, 2 \in X$ — $1 \in \{1\}$ & $2 \in \{2\}$



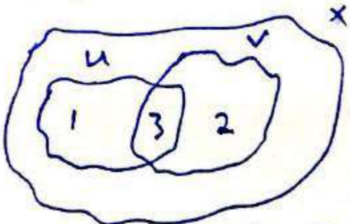
$2, 3 \in X$ — $2 \in \{2\}$ & $3 \in \{3\}$



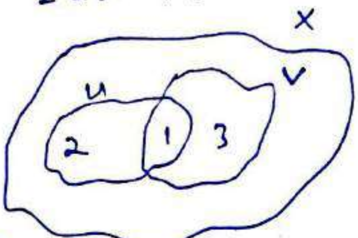
$1, 3 \in X$ — $1 \in \{1\}$ & $3 \in \{3\}$



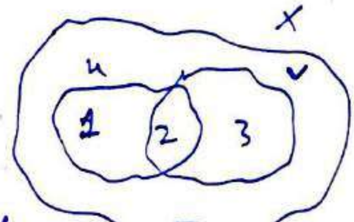
for $1, 2 \in X$ — $1 \in \{1,3\}$, $2 \in \{2,3\}$



$2, 3 \in X$ — $2 \in \{1,2\}$ & $3 \in \{1,3\}$



$1, 3 \in X$ — $1 \in \{1,2\}$ & $3 \in \{2,3\}$



Corollary: Every finite T_1-space is discrete.

Th: Let (X, τ) be a topological space. Then, the followings statements are equivalent.

i) X is T_1-space.

ii) Each singleton subset of X is closed.

iii) Each subset A of X is the intersection of its open supersets.

(iii)

$\{1\} = \{1,2\} \cap \{1,3\}$

$\{2\} = \{1,2\} \cap \{2,3\}$

$\{3\} = \{1,3\} \cap \{2,3\}$

Proof: (i) — (ii)

Let X is T_1-space and $x \in X$,

then we have to show $\{x\}$ is closed.

i.e. $\{x\}^c$ is open

Let $y \in \{x\}^c$

$\Rightarrow y \neq x$

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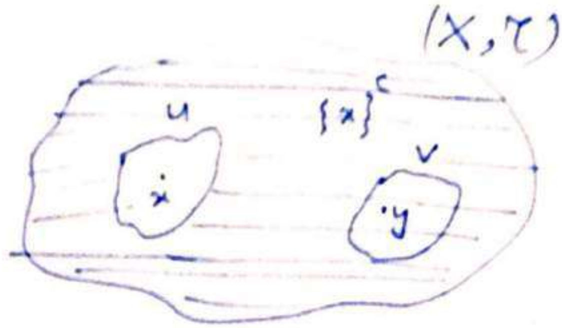
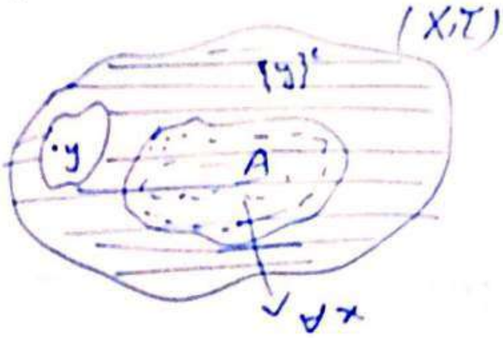
$\because$ X is T_1-space.
then there exists two open sets U & V s.t.
$x \in U$ & $y \in V$



also $V \subseteq \{x\}^c$
$\Rightarrow y \in V \subseteq \{x\}^c$
$\because$ ~~{x}ᶜ~~ V is open
$\Rightarrow \{x\}^c$ is open
i.e. $\{x\}$ is closed.

(ii) —— (iii)
Suppose each singleton subset of X is closed.
and $A \subseteq X$
we can write

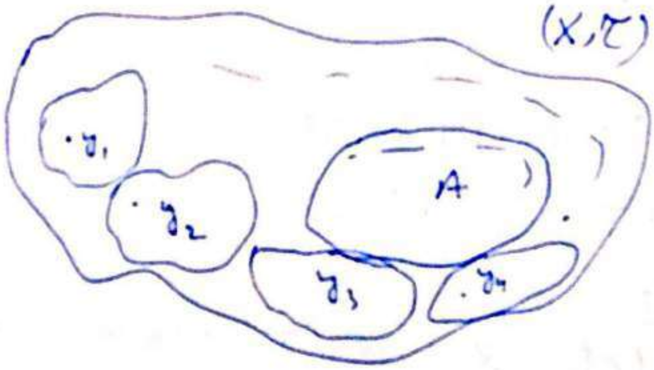


$$A = \bigcup_{x \in A} \{x\}$$

Let $y \in X$ s.t. $y \notin A$
$\Rightarrow y \neq x \quad \because \forall x \in A$
$\because$ $\{y\}$ is closed by supposition
$\Rightarrow \{y\}^c$ is open
also $A \subseteq \{y\}^c \quad \forall y \notin A$

$$\Rightarrow A = \bigcap_{y \notin A} \{y\}^c.$$

i.e. A is the intersection of its open supersets.



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(iii) — (i)

Suppose each subset A of X is the intersection of its open supersets.

Then, we have to show that X is T_1.

Let $x, y \in X$, $x \neq y$

$\Rightarrow \{x\}$ & $\{y\}$ is the intersection of its open supersets.

then there must exists an open superset U of $\{x\}$ which does not contain y.

i.e. U is an open superset s.t.

$x \in U$ but $y \notin U$

similarly

V is an open set s.t.

$y \in V$ but $x \notin V$

$\Rightarrow$ X is T_1.

Corollary:

i) Every finite T_1 space is discrete.

ii) In a T_1-space, no finite subset has a limit point.

T_2- space: (Hausdorff space)

A topological space (X, τ) is said to be T_2 if for any $a, b \in X$, $a \neq b$ there exists two open sets U & V s.t.

$a \in U$ $b \in V$

and $U \cap V = \phi$

Examples:

i) Every discrete space is T_2.

ii) Indiscrete space is not T_2.

iii) Sierprinski space is not T_2.

Th: Every T_2 is T_1 but the converse is not true.

Let (X,τ) be a T_2-space.
then for any $x,y \in X$; $x \neq y$
there exists two open sets u & v s.t.

$x \in u$, $y \in v$

and $u \cap v = \phi$

$\Rightarrow$ X is T_1-space.
because it also satisfy the T_1-axioms.

Conversely:

Converse is not true in general.
An infinite set with co-finite topology is T_1 but not T_2.

proof:-
Let X is an infinite set. Then, we have to show that X with cofinite topology is not T_2.
we suppose on contrary X is T_2.
then for any $a, b \in X$, $a \neq b$
there exists open sets u & v s.t.

$a \in u$ $b \in v$

and $u \cap v = \phi$

$(u \cap v)^c = (\phi)^c$

$u^c \cup v^c \neq X$

L.H.S is the union of two finite sets
but R.H.S is infinite.
which is impossible.
then we cannot suppose X is T_2.
$\Rightarrow$ X is not T_2.

Th: Every subspace of T_2 is T_2.

~~$u \cap v = \phi$~~

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Th: Every subspace of T_2 is T_2.

Proof:-

Let (X, τ_X) be a T_2-space and (Y, τ_Y) be its subspace. Then, we have to show Y is T_2.

Let $x, y \in Y$, $x \neq y$

$\Rightarrow$ $x, y \in X$ $\because Y \subseteq X$

Since, X is T_2.

then there exists two open sets U & V s.t.

$x \in U$ $y \in V$

and $U \cap V = \phi$

$\Rightarrow$ $x \in U \cap Y = U_1$ $y \in V \cap Y = V_1$

i.e. U_1 & V_1 are two open sets of Y which contains one of the point.

now

$$U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$$

$$= (U \cap V) \cap Y$$

$$= \phi \cap Y$$

$$U_1 \cap V_1 = \phi$$

$\Rightarrow$ Y is T_2.

Th: Let X be a T_1-space and $A \subseteq X$, if $x \in X$ is a Limit point of A then every open set containing 'x' contains infinite no. of distinct points of A.

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Regular space:-

A topological space (X,τ) is said to be regular if for any closed set A and a point not in A, there exists two open sets U and V such that

$x \in U$, $A \subseteq V$ and

$U \cap V = \phi$

Example:

$$X = \{a, b\}$$

$$\tau = \{\phi, X, \{a\}, \{b\}\}$$

Th:

The following statements are equivalent

i) X is regular.

ii) For any open set U in X and $x \in U$, there is an open set V containing x such that

$$x \in \overline{V} \subseteq U$$

iii) Each element of X has a local base containing closed sets.

T_3-space:

A regular T_1-space is called T_3.

Th:

Every T_3-space is T_1.

Proof:

Let X is regular space and U be an open set with $x \in U$.

then we have to show there exists an open set V in X containing x s.t.

$$x \in \overline{V} \subseteq U$$

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as u is open and $x \in u$

$\Rightarrow$ u^c is closed and $x \notin u^c$

Since, X is regular

then there exists open sets V and V_1 s.t.

$x \in V$, $u^c \subseteq V_1$

and $V \cap V_1 = \phi$

now

$u^c \subseteq V_1$

$\Rightarrow$ $V_1^c \subseteq u$

also

$V \cap V_1 = \phi$

$V \subseteq V_1^c$

$\Rightarrow$ $x \in V \subseteq V_1^c \subseteq u$

Since V_1 is open

$\Rightarrow$ V_1^c is closed.

i.e. V_1^c is closed superset of V

but $\bar{V}$ is the smallest closed superset of V

$\Rightarrow$ $x \in V \subseteq \bar{V} \subseteq V_1^c \subseteq u$

$\Rightarrow$ $x \in \bar{V} \subseteq u$

(ii) —— (iii)

Let u be an open set with $x \in u$

there exists V be an open set containing x s.t.

$x \in \bar{V} \subseteq u$

this shows that local base at x contains sets of the form $\bar{V}$ which is of course closed set.

(iii) —— (i)

Let $x \in X$ and A be a closed subset of X such that $x \notin A$

$\Rightarrow x \in A^c$ and A^c is open

i.e. A^c is open nbhd of x.

by supposition, there is a closed set B in the local base at x such that

$x \in B \subseteq A^c$

now

$B \subseteq A^c$

$\Rightarrow$

$A \subseteq B^c$

Let $U = B$ and $V = B^c$

then U is open as U is in local base and V is open because ~~$V = B^c$~~ B is closed.

and $x \in U$, $A \subseteq V$

and $U \cap V = \phi$

$\Rightarrow X$ is regular.

Th: Every subspace of regular is regular.

Proof:

Let (X, τ_X) be a regular space and (Y, τ_Y) be its subspace. Then, we have to show Y is regular.

Let A be a closed set in Y and $x \in Y$ such that $x \notin A$

Now as A is closed in Y and Y is subspace of X, so then there exists a closed set B in X, such that

$$A = B \cap Y$$

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Further

$x \notin A \Rightarrow x \notin B \cap Y$

$\Rightarrow x \notin B \quad \because x \in Y$

since, X is regular
then for a closed set B in X and $x \in X$ such that $x \notin B$, there exists two open sets U and V in X such that

$x \in U \quad , \quad B \subseteq V$

and $\quad U \cap V = \Phi$

$\Rightarrow x \in U \cap Y = U_1 \quad , \quad B \subseteq V \cap Y = V_1$

$B \subseteq V$
$B \cap Y \subseteq V \cap Y$
$A \subseteq V_1$
and A is closed is Y

as U and V are open in X
$\Rightarrow U_1$ and V_1 are open in Y.
also

$$U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$$
$$= (U \cap V) \cap Y$$
$$= \Phi \cap Y$$
$$U_1 \cap V_1 = \Phi$$

$\Rightarrow Y$ is regular

Completely regular space:

A topological space (X, τ) is said to be completely regular if for any closed set A in X and $x \in X$ such that $x \notin A$ there exists a continuous function

$f : X \longrightarrow [0,1]$ such that

~~$f(x)$~~ $\quad f(x) = 0$ and $f(A) = 1$

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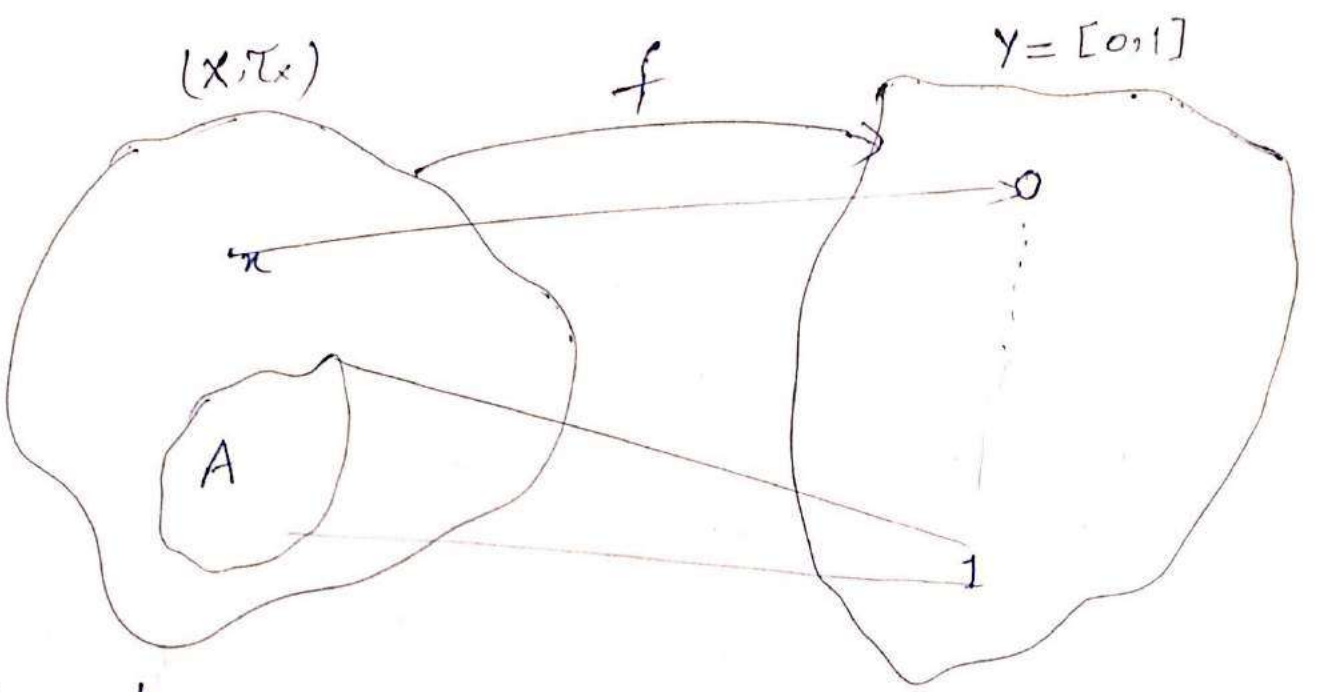

Example:

Every metric space is completely regular.

Th:

Every completely regular space is regular.

Proof:

Let (X, τ) be a completely regular space. Then, we have to show X is regular.
Let A be a closed subset of X and $x \in X$ such that $x \notin A$
As X is completely regular.
then there exists a continuous function $f : X \to [0,1]$ such that

$f(x) = 0 \qquad f(A) = 1$

Let $u = [0, \frac{1}{2})$ and $V = (\frac{1}{2}, 1]$
then u and v are open in $[0,1]$
As f is continuous
$\Rightarrow f^{-1}(u)$ and $f^{-1}(v)$ are open in X.

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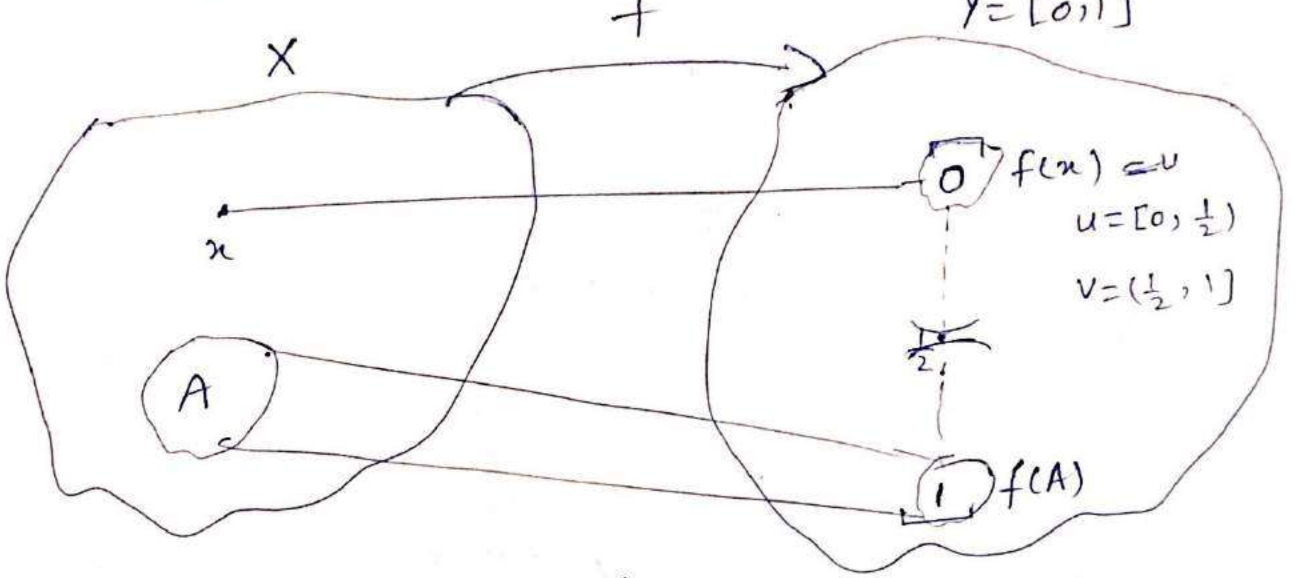
as $x \in f^{-1}(U)$, $A \subseteq f^{-1}(V)$

and
$$f^{-1}(U) \cap f^{-1}(V) = \phi$$

$\Rightarrow X$ is regular.



$f(x) \in U \Rightarrow x \in f^{-1}(U)$

$f(A) \subseteq V \quad A \subseteq f^{-1}(V)$

$f^{-1}(U) \cap f^{-1}(V) = \phi$

Th: Every subspace of completely regular is completely regular.

Proof:

Let (X, τ_X) be a completely regular space and (Y, τ_Y) be its subspace. Then, we have to show Y is completely regular.

Let A be a closed set of Y and $x \in Y$ such that $x \notin A$

$\Rightarrow x \in X$ ~~and~~ A $\because Y \subseteq X$

and A is closed in Y and Y is subspace

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of X, then there exists a closed subset B of X such that

$$A = B \cap Y$$

Since, X is completely regular. (closed set B and $x \notin B$)
then there exists a continuous function

$f : X \longrightarrow [0,1]$ such that

$f(x) = 0$ and $f(B) = 1$

now define

$g : Y \longrightarrow [0,1]$ by

$g(x) = f(x) \quad \forall \; x \in Y$

then $x \in Y$

$\Rightarrow \quad g(x) = f(x) = 0$

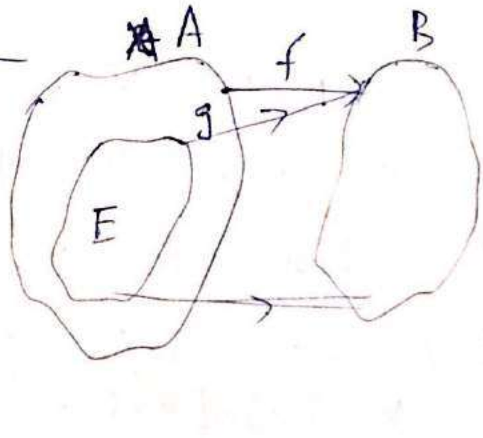
and $\quad g(A) = f(A) \quad \because A \subseteq Y$

$\quad = f(B \cap Y)$

$\quad = f(B) \cap f(Y)$

$g(A) = 1$

As g is restriction of f and f is continuous. So, g is also continuous.
Hence, Y is completely regular.



Restriction function:

$f : A \longrightarrow B$

$E \subseteq A$

$g : E \longrightarrow B$

$g(x) = f(x) \quad \forall \; x \in E$

then g is restriction of f

$T_{3\frac{1}{2}}$ space or Tychonoff space (تيكانوف)

A completely regular T_1-space is called $T_{3\frac{1}{2}}$.

Normal space:

A topological space (X,τ) is said to be normal if for every pair of disjoint closed sets A, B of X, there exists disjoint open sets U, V such that

$$A \subseteq U \quad , \quad B \subseteq V$$

Example: • Every discrete space.

$X = \{a,b,c\}$

$\tau = \{\Phi, X, \{a\}, \{b\}, \{a,b\}, \{c\}, \{b,c\}, \{a,c\}\}$

closed sets: $X, \Phi, \{b,c\}, \{a,c\}, \{c\}, \{a,b\}, \{a\}, \{b\}$

T_4- space:

A normal T_1-space is called T_4.

Note:

Normal may not be regular

but $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$

Th: A T_4-space is regular.

Proof:

Let (X,τ) be a T_4-space.

i.e. X is normal as well as T_1-space.

Then, we have to show that X is regular.

Let F be a closed subset of X and $x \notin F$

$\because$ X is T_1-space.

$\{x\}$ and F are disjoint closed sets.

because in T_1-space, every singleton set is closed.
also given X is normal
then for pair of disjoint closed sets $\{x\}$ and F
there exists disjoint open sets U and V s.t.

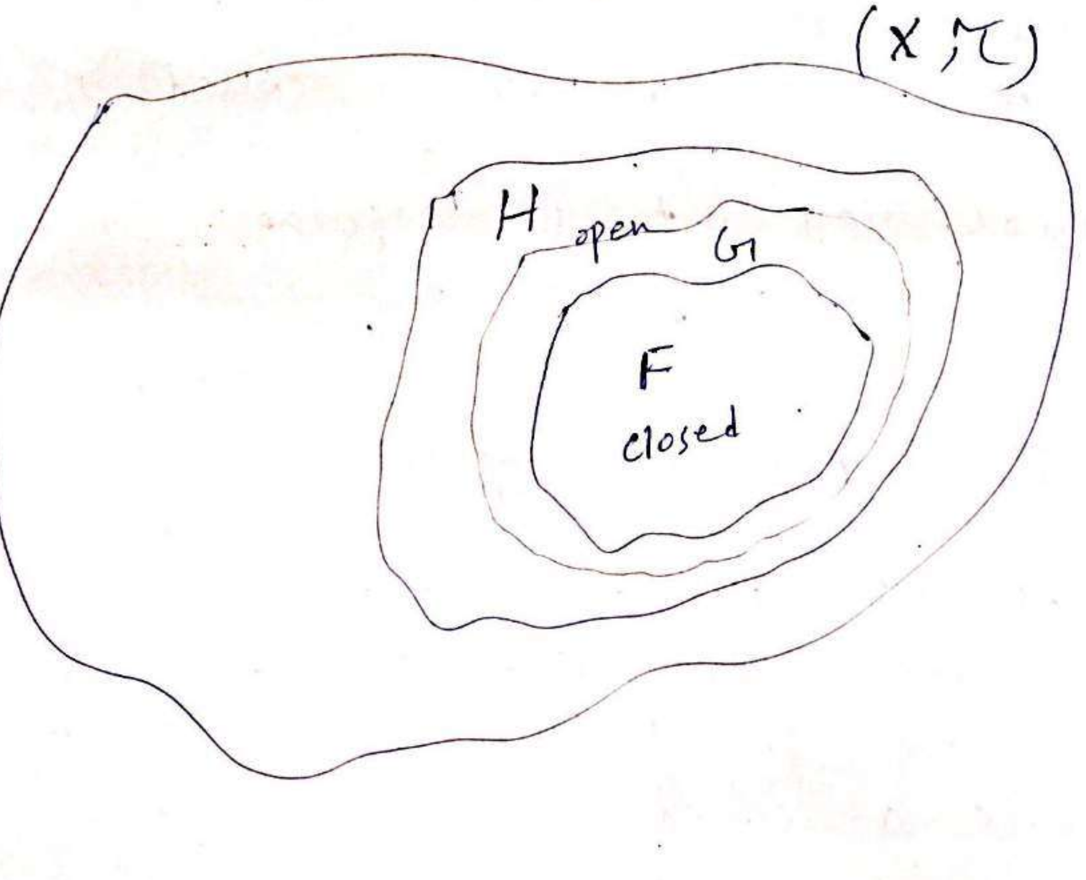
$$\{x\} \subseteq U \quad , \quad F \subseteq V$$
$$U \cap V = \Phi$$

$\Rightarrow \quad x \in U \quad , \quad F \subseteq V$

$\Rightarrow \quad X$ is regular.

Th: A topological space (X, τ) is normal iff for each closed set ~~H containing F~~ F and an open set H containing F, there exists an open set G s.t.

$$F \subseteq G \subseteq \overline{G} \subseteq H$$



Proof:-

Let (X,τ) is normal and

$F \subseteq H$

where F is closed and H is open

$\Rightarrow H^c$ is closed. and $F \cap H^c = \phi$

$\because X$ is normal.

there exists two open sets G and U such that

$F \subseteq G \quad , H^c \subseteq U$

and $G \cap U = \phi$

$\Rightarrow \quad G \subseteq U^c$

$\Rightarrow \quad F \subseteq G \subseteq U^c \qquad\qquad ? \; F \subseteq G$

$\Rightarrow F \subseteq G \subseteq \overline{G} \subseteq \overline{U^c} \in U^c \subseteq H \qquad ? \; G \subseteq \overline{G}$

$\Rightarrow F \subseteq G \subseteq \overline{G} \subseteq H \qquad\qquad G \subseteq U^c$

$\Rightarrow \overline{G} \subseteq \overline{U^c}$

Conversely:

Let F_1 and F_2 be two disjoint closed sets of X. Then

$F_1 \subseteq F_2^c$

where F_2^c is open

by hypothesis

there exists an open set G such that

$F_1 \subseteq G \subseteq \overline{G} \subseteq F_2^c$

$\Rightarrow \overline{G} \subseteq F_2^c$

$\Rightarrow F_2 \subseteq (\overline{G})^c$

$\Rightarrow F_1 \subseteq G \qquad\qquad F_2 \subseteq (\overline{G})^c$

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$$G \cap (\overline{G})^c = \phi$$

$\Rightarrow$ X is normal

Th: Every closed subspace of a normal is normal.

Proof: Let (X, τ_X) be a normal space and (Y, τ_Y) be its closed subspace.

Then, we have to show Y is normal.

Let A and B be disjoint closed subsets of Y

i.e.

$$A = U_1 \cap Y \quad , \quad B = U_2 \cap Y$$

where

U_1 and U_2 are closed in X.

$\Rightarrow$ A and B are closed in X.

$\because$ X is normal

there exists disjoint open sets V_1 and V_2 s.t

$$A \subseteq V_1 \quad , \quad B \subseteq V_2$$

$$A \subseteq V_1 \cap Y \quad , \quad B \subseteq V_2 \cap Y$$

where

$V_1 \cap Y$ and $V_2 \cap Y$ are open in Y.

now

$$(V_1 \cap Y) \cap (V_2 \cap Y)$$

$$= (V_1 \cap V_2) \cap Y$$

$$= \phi \cap Y$$

$$= \phi$$

$\Rightarrow$ Y is normal.

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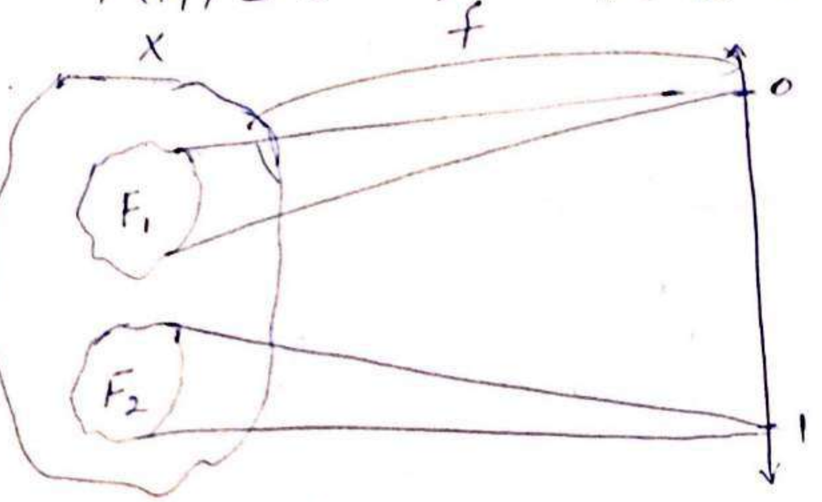
Urysohn's lemma:

Let (X, τ) be a normal space. If F_1, F_2 are any disjoint closed sets in X. Then, there exists a continuous function

$$f : X \longrightarrow [0,1] \quad \text{with}$$

$$f(F_1) = 0 \quad \& \quad f(F_2) = 1$$



Note: For finite sets

i) Let $\{X_1, X_2, \cdots, X_n\}$ be a finite family of sets. Then, the cartesian product

$$\prod_{\alpha=1}^{n} X_\alpha = X_1 \times X_2 \times X_3 \times \cdots \times X_n$$

$$= \{x = (x_1, x_2, \cdots, x_n) \mid x_\alpha \in X_\alpha \ \forall \ \alpha = 1, 2, \cdots, n\}$$

For arbitrary collection

Let $\{X_\alpha : \alpha \in \nabla\}$ be an arbitrary family of sets. Then, the cartesian product is given by

$$\prod_{\alpha \in \nabla} X_\alpha$$

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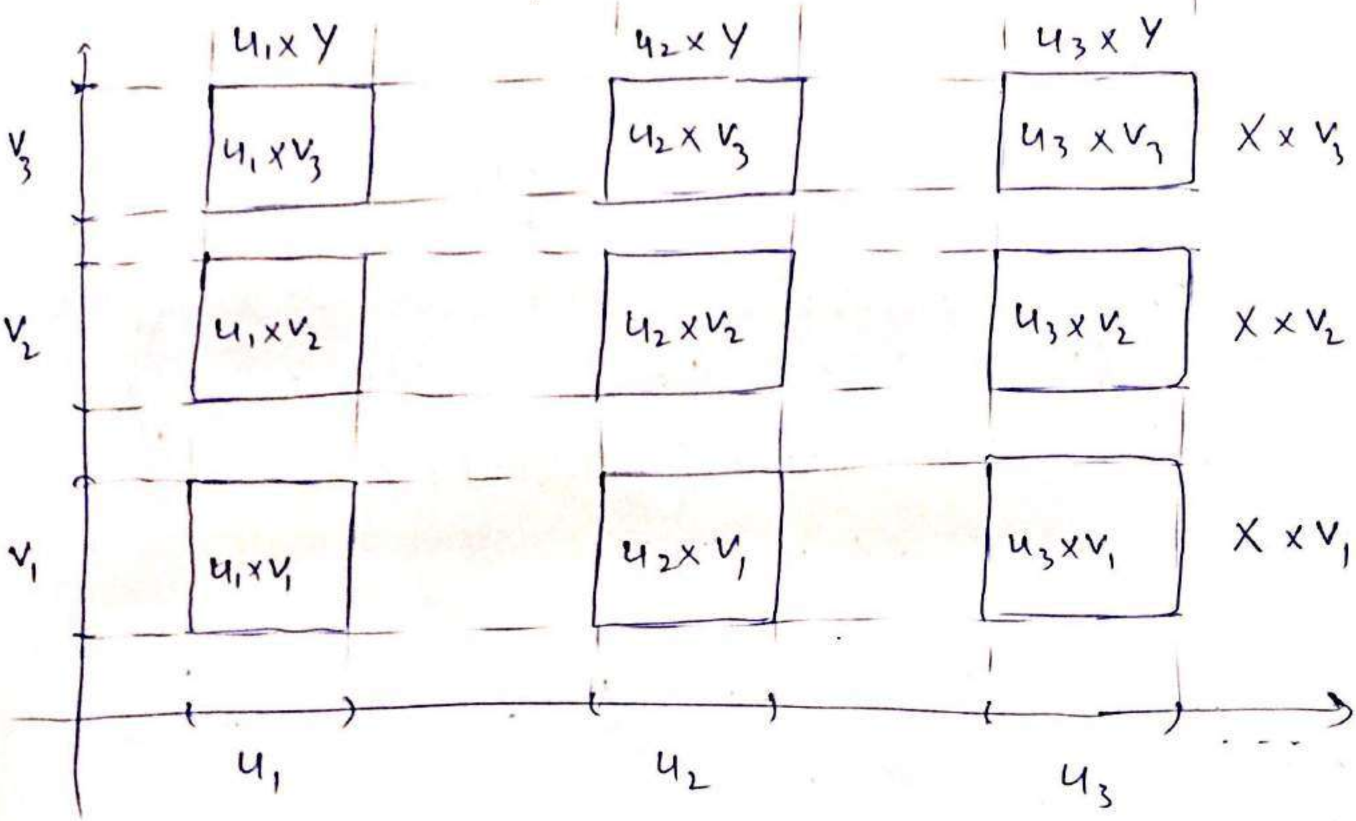
Product Topology:

Let X and Y be two topological spaces. Then, the product topology (box topology) on $X \times Y$ is the topology having as basis the collection β of all subsets of the form $U \times V$ where U is an open subset of X and V is an open subset of Y.



$$\beta = \left\{ \begin{array}{l} \phi, U_1 \times V_1, U_2 \times V_1, U_3 \times V_1, U_1 \times V_2, U_2 \times V_2, U_3 \times V_2, U_1 \times V_3 \\ U_2 \times V_3, U_3 \times V_3, X \times V_1, X \times V_2, X \times V_3, U_1 \times Y, U_2 \times Y, U_3 \times Y \end{array} \right\}$$

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$$X_1 = \{a, b, c\}$$

$$\tau_1 = \{\Phi, \{a\}, \{b\}, \{a,b\}, X_1\}$$

$$X_2 = \{1, 2\}$$

$$\tau_2 = \{\Phi, \{1\}, X_2\}$$

Solution

$$\beta_1 = \{\{a\}, \{b\}, X_1\}$$

$$\beta_2 = \{\{1\}, X_2\}$$

then basis for $X_1 \times X_2$ is given by

$$\beta = \left\{ \begin{array}{l} \{(a,1)\}, \{(a,1),(a,2)\}, \{(b,1)\}, \{(b,1),(b,2)\}, \{(a,1), \\ (b,1),(c,1)\}, \{(a,1),(a,2),(b,1),(b,2), \cancel{(c_1,c_2)}\, (c,1),(c,2)\} \end{array} \right\}$$

then

$$\tau_1 \times \tau_2 = \left\{ \begin{array}{l} \{(a,1)\}, \{(a,1),(a,2)\}, \{(b,1)\}, \{(b,1),(b,2)\}, \{(a,1),(b,1) \\ (c,1)\}, \{(a,1),(a,2),\overset{X\times Y}{(b,1)},(b,2),(c,1),(c,2)\}, \{(a,1),(b,1)\} \\ \{(a,1),(b,1),(b,2)\}, \{(a,1),(a,2),(b,1)\}, \{(a,1),(a,2),(b,1), \\ \qquad (b,2)\} \\ \{(a,1),(a,2),(b,1),(c,1), \{(a,1),(b,1),(b,2),(c,1)\}, \\ \{(a,1),(a,2),(b,1),(b,2),(c,1)\} \end{array} \right\}$$

$$X = \{a, b, c\} \Rightarrow \tau_x = \{\Phi, X, \{a\}, \{b\}, \{a,b\}\}$$

$$Y = \{1,2,3,4\} \Rightarrow \tau_y = \{\Phi, Y, \{1,2\}, \{3,4\}\}$$

Sol.

$$\beta_{\tau_x} = \{\{a\}, \{b\}, X\}$$

$$\beta_{\tau_y} = \{\{1,2\}, \{3,4\}\}$$

Base for $X \times Y$

$$\tau_x \times \tau_y = \{ \quad\quad \}$$

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Separated sets:

In simple words, neither overlapping nor touching

Connected sets:

A set is connected if it is all in one piece.
i.e. which cannot be partitioned into two or more than non-empty subsets.

Connected space:

A topological space (X, τ) is said to be connected if there does not exists a pair A, B of non-empty disjoint open sets such that

$$A \cup B = X$$

ایسی (top. space) جس میں آپ کو ایک بھی (pair) نہ ملے دو
(non-empty disjoint open sets) کا کہ جن کا (union) 'X' بنائے۔

Dis-connected space:

A space which is not connected is called dis-connected.

Examples:

i) Every indiscrete space $\tau = \{\phi, X\}$ is connected.

ii) Every discrete space $\tau = P(X)$ is disconnected.

iii) An infinite set with co-finite topology is connected.

iv)

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An infinite set with co-finite topology is connected.

Proof:-
we suppose on contrary that an infinite set with co-finite topology is disconnected.
Then, there exists a pair of non-empty disjoint open sets A, B s.t.

$A \cup B = X$

$\because$ $A \cap B = \phi$

by De-Morgan's law

$A^c \cup B^c = X$

which is not possible.

$\Rightarrow$ X is ~~disco~~ connected.

(iv)

$\mathbb{R}$ is connected.

$(-\infty, 1) \cup (1, \infty) \neq \mathbb{R}$

(v)

On the real line, an interval is connected.

(vi)

$A = (0,1) \cup (2,3)$

then A is dis-connected.

(vii)

Each point on $\mathbb{R}$ is in one piece, hence each pt. set $\{x\}$ is connected.

(viii)

Sierprinski space is connected.

$X = \{0,1\} \xrightarrow{\text{which defines}}$ only two points

$\tau = \{\phi, X, \{0\}\}$

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(ix)

$\mathbb{Q} \subseteq \mathbb{R}$ is disconnected.

$A = \mathbb{Q} \cap (-\infty, r)$ $\quad$ r is irrational

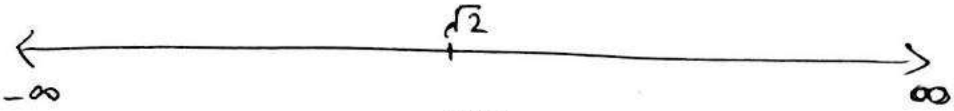
$B = \mathbb{Q} \cap (r, \infty)$

i.e.

if $r = \sqrt{2}$

$A = \mathbb{Q} \cap (-\infty, \sqrt{2})$, $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$

$A = \{x \in \mathbb{Q} : x < \sqrt{2}\}$, $B = \{x \in \mathbb{Q} : x > \sqrt{2}\}$



(x)

$X = \{a, b, c\}$

$\tau = \{\Phi, X, \{a, b\}, \{c\}\}$

then X is dis-connected.

Th: A topological space (X, τ) is dis-connected iff 'X' contains a non-empty set A which is both open and closed.

Proof: Let X is dis-connected. then there exists two non-empty disjoint open sets A, B s.t.

$A \cup B = X$

$\because B$ is open

$\Rightarrow B^c$ is closed

but $B^c = A$ $\quad \because A \cap B = \Phi$ & $A \cup B = X$

$\Rightarrow A$ is closed

$\Rightarrow A$ is both open and closed.

Conversely:- suppose A is a non-empty subset of X which is both open and closed.

$\because A$ is closed $\Rightarrow A^c$ is open $\Rightarrow \{A, A^c\}$ is a disconnection for X.

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Th: The continuous surjective image of a connected space is connected.

Proof:

Let $f: X \longrightarrow Y$

be a continuous surjective mapping.

Let 'X' be a connected space. Then, we have to show that

$f(X) = Y$ is connected.

we suppose on contrary that Y is dis-connected. then there exists two non-empty disjoint open sets A, B such that

$A \cup B = Y$

$\because f: X \longrightarrow Y$ is continuous

$\Rightarrow f^{-1}(A)$ and $f^{-1}(B)$ are open in X.

now

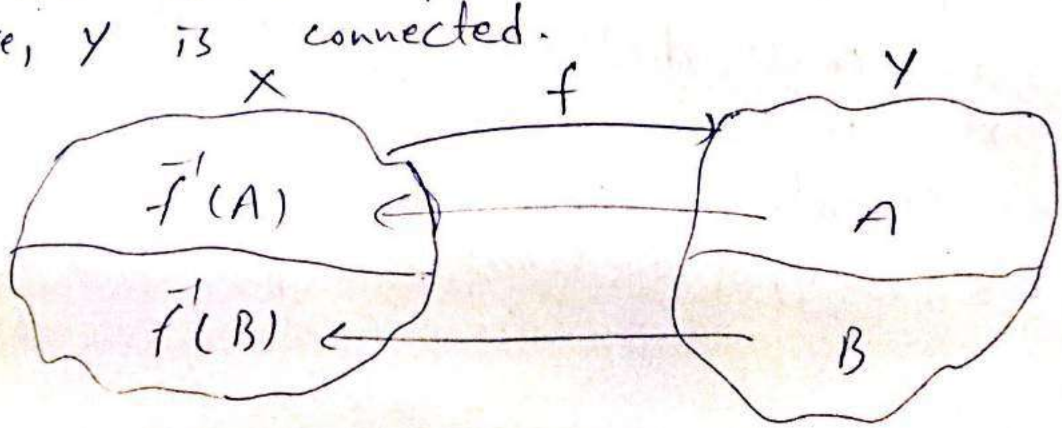
$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X$$

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$$

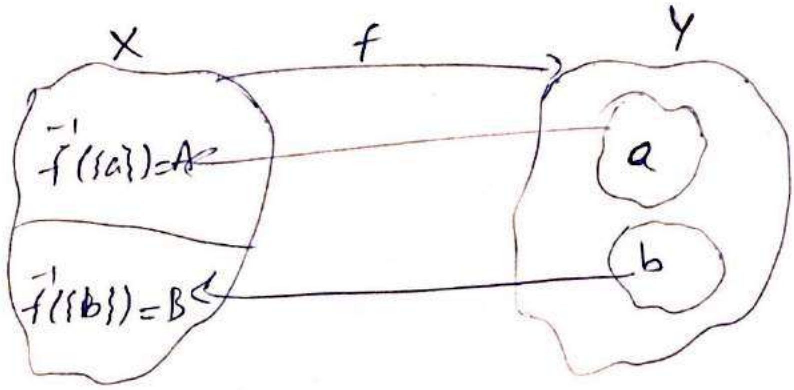
$\Rightarrow \{f^{-1}(A), f^{-1}(B)\}$ is a disconnection for X which is contradiction against the fact that X is connected.

Hence, we cannot suppose Y is disconnected.

Therefore, Y is connected.



Th: A space X is connected if and only if there does not exists a surjective continuous function f from X onto the two point discrete space.



<u>proof:</u>

Let X is connected space.
we suppose on contrary that there exists a continuous surjective function $f : X \longrightarrow \cancel{X} = \{a, b\}$
where $\cancel{X}$ is a two point discrete space
since $a, b \in Y$
$\Rightarrow$ $\{a\}$ & $\{b\}$ are open in Y
$\because$ f is continuous
$\Rightarrow$ $f^{-1}(\{a\}) = A$ & $f^{-1}(\{b\}) = B$ are open in X.

now
$$f^{-1}(\{a\}) \cup f^{-1}(\{b\}) = f^{-1}(\{a, b\}) = f^{-1}(Y) = X$$

$$f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = \cancel{f^{-1}(\{a,b\})}\ \cancel{f^{-1}(\{a,b\})} = \cancel{f^{-1}(Y)}\ f^{-1}(\{a\} \cap \{b\}) = f^{-1}(\phi) = \phi$$

$\Rightarrow$ $\{A, B\}$ is a disconnection for X.
which is contradiction against the fact that X is connected.
~~Hence, we cannot suppose~~
Hence, our supposition was wrong.

Conversely:

we have to show that X is connected.
we suppose on contrary that X is disconnected.
Then, there exists two non-empty disjoint open sets A, B s.t.

$$A \cup B = X$$

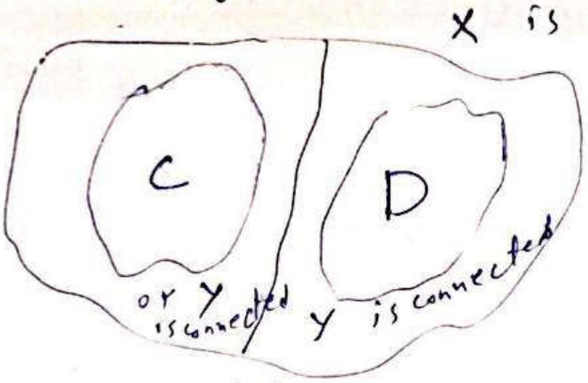
Then, the function

$$f : X \to Y \quad \text{as}$$

$$f(A) = a \qquad f(B) = b$$

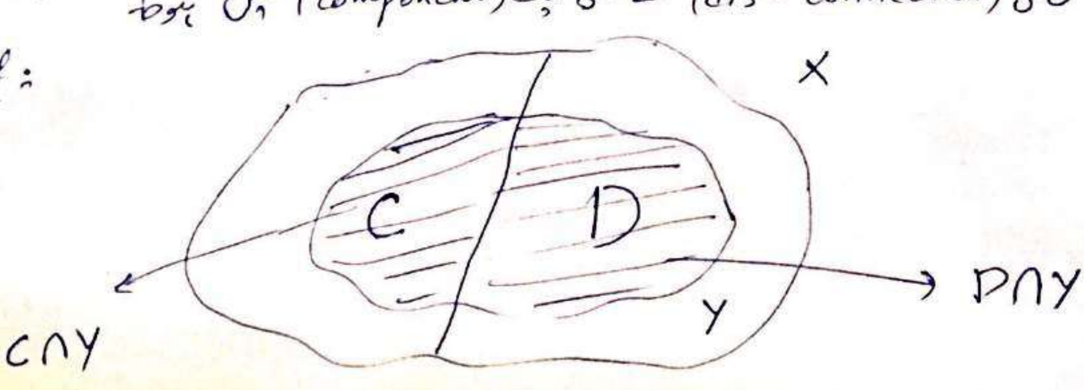
is a continuous surjective.
which is contraction to our supposition.

Th: Let (X, τ) be a dis-connected space, If C and D form a separation for X and if Y is connected subspace of X. Then, Y lies entirely with either C or D.



اگر کسی (dis-connected top. space) کے اندر کوئی (connected subset) پڑا ہو
تو وہ اس کی (dis-connection) کے کسی ایک (component) میں ہوگا

Proof:



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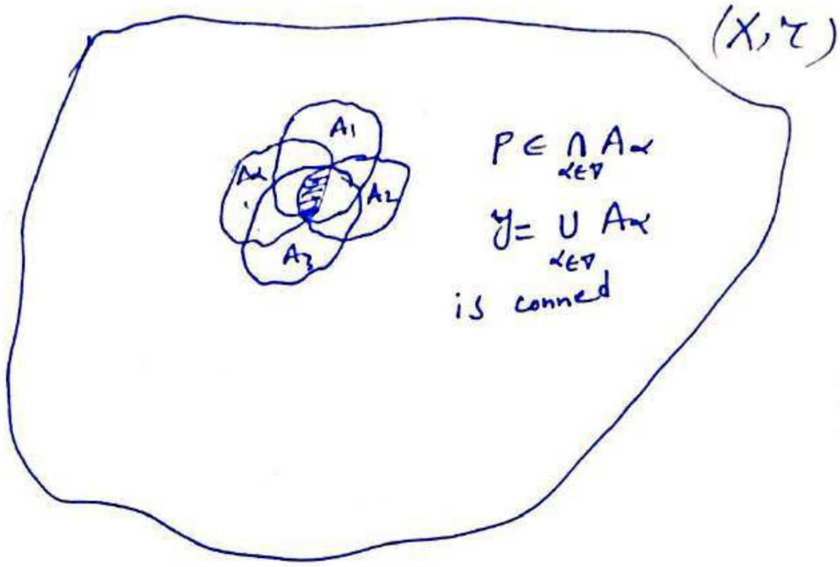
Since C and D are both open in X.
then the sets $C \cap Y$ and $D \cap Y$ are open in Y

Then.
$$(C \cap Y) \cap (D \cap Y) = \phi$$
and
$$(C \cap Y) \cup (D \cap Y) = Y$$
if they both are non-empty, then they will form separation of Y
but Y is connected
therefore
$$C \cap Y = \phi \quad \text{or} \quad D \cap Y = \phi$$
if $C \cap Y = \phi \Rightarrow Y \subseteq D$
if $D \cap Y = \phi \Rightarrow Y \subseteq C$
$\Rightarrow Y \subseteq C$ or $Y \subseteq D$
the required.

Th:
Prove that the union of a collection of connected subspaces of a topological space (X,τ) that have a point in common is connected.



Proof:
Let $\{A_\alpha : \alpha\in\nabla\}$ be a collection of connected subspaces of a topological space (X,τ).
Let $P \in \bigcap_{\alpha\in\nabla} A_\alpha$
then we have to show that
$Y = \bigcup_{\alpha\in\nabla} A_\alpha$ is connected.
we suppose on contrary that Y is dis-connected
Then, there exists two non-empty disjoint open sets C and D s.t.
$Y = C \cup D$
$\Rightarrow P \in C$ or $P \in D$
suppose
$P \in C$
Since A_α is connected
then by theorem, it must lie entirely in C.
i.e. $A_\alpha \subseteq C$ for every $\alpha\in\nabla$
$\Rightarrow Y = \bigcup_{\alpha\in\nabla} A_\alpha \subseteq C$
$\Rightarrow D$ is empty
which is contradiction against the fact that D is non-empty.

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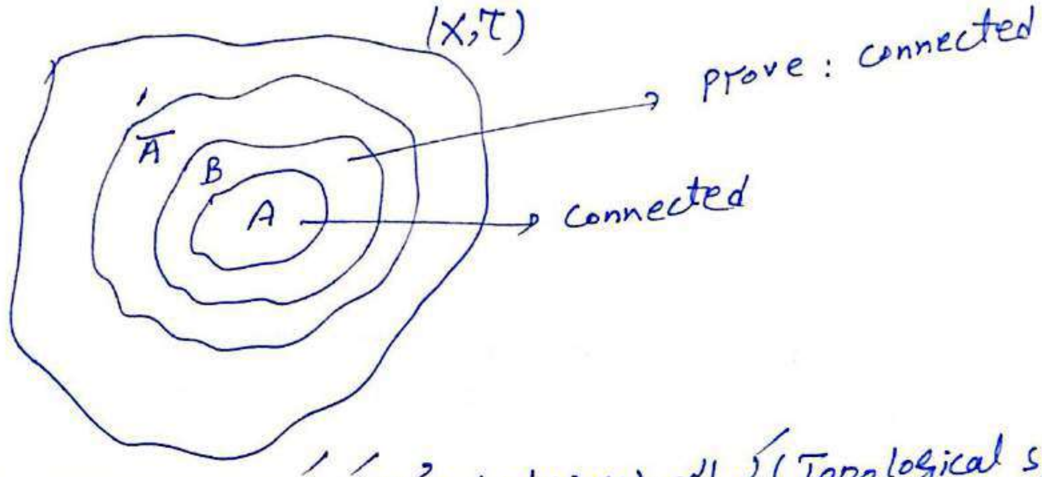
Hence, our supposition was wrong that y is dis-connected.

therefore, y is connected.

Th: Let A be a connected subspace of a topological space (X, τ), if

$$A \subseteq B \subseteq \overline{A}$$

then B is also connected subspace of X.



(Topological space) کی ایسی (subspace) جو اس کی کسی (connected subspace) کو
(contain) کرے اور اس کے (closure) میں (contained) ہو۔ (connected) کہلائے گی۔

<u>Proof:-</u> we have to show that B is connected.

we suppose on contrary that B is dis-connected.

then there exists two non-empty dis-joint open sets C and D s.t.

$$B = C \cup D$$

$\because$ $A \subseteq B = C \cup D$ and A is connected

then A must lie entirely in C or D.

suppose $A \subseteq C$

$$\Rightarrow \overline{A} \subseteq \overline{C}$$

$$\Rightarrow \overline{C} \cap D = \phi$$

$\Rightarrow$ B cannot intersect D.

which is contradiction against the fact that D is non-empty

Hence, B is connected.

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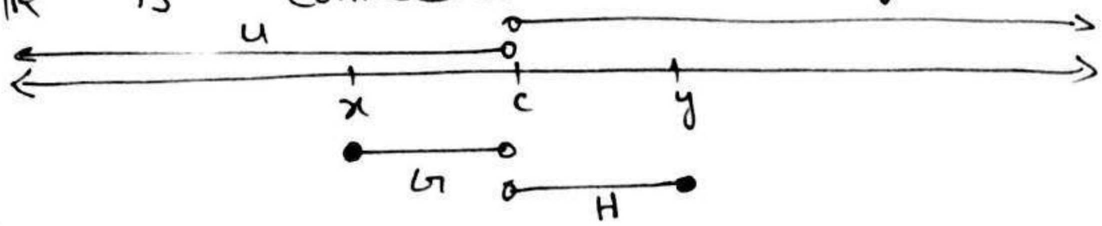
Th: Finite cartesian product of connected spaces is connected.

Th: $\mathbb{R}$ is connected.



Proof:

we suppose on contrary that $\mathbb{R}$ is dis-connected.
Then, there exists two non-empty dis-joint open sets u and v s.t.

$$u \cup v = \mathbb{R}$$

Let $x \in u$ and $y \in v$ with $x < y$
suppose

$$G = u \cap [x, y] \quad \& \quad H = v \cap [x, y]$$

now

$$G \cup H = \{u \cap [x, y]\} \cup \{v \cap [x, y]\}$$

$$= (u \cup v) \cap [x, y]$$

$$= \mathbb{R} \cap [x, y]$$

$$G \cup H = [x, y]$$

Observe, G is bounded above by 'y' and by the least upper bound property of $\mathbb{R}$ G has least upper bound say 'c' $\in \mathbb{R}$
then $c \in [x, y]$
we derive a contradiction by showing that
$c \notin G$ and $c \notin H$
To show that $c \notin H$
suppose $c \in H$
since $x \in H$ and H is open in $[x, y]$
then there exists $d \in \mathbb{R}$ s.t.

$$x < d < c \quad \text{and} \quad (d, c] \subset H$$

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$\Rightarrow$ ~~G~~ d is an upper bound of G

$\Rightarrow$ d is least upper bound of G

Which is contradiction

$\Rightarrow c \notin H$

Similarly,

$c \notin G$

but $c \in [x,y]$.

$\Rightarrow \mathbb{R}$ is connected.

Th: A subspace X of $\mathbb{R}$ is connected iff X is an interval

---

Locally closed set: A subset A of X is called locally closed if

$$A = B \cap C$$

where B is open and C is closed.

Examples:

i) Every closed set is locally closed.

i.e. $\underset{\text{closed}}{A} = \underset{\text{closed}}{A} \cap \underset{\text{open}}{X}$

ii) Every open set is locally closed.

$\underset{\text{open}}{A} = \underset{\text{open}}{A} \cap \underset{\text{closed}}{X}$

iii)

$A^\circ = A^\circ \cap \overline{A}$

iv) ~~A~~ Every interval of $\mathbb{R}$

$[1,2) = (0,2) \cap [1,2]$

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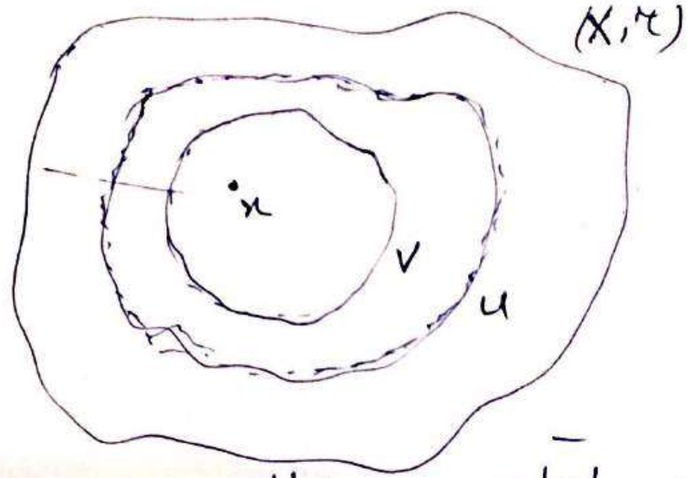
Locally connected space:

ایسی (top. space) جس کے ہر (pt.) کا (neighbourhood) کسی نہ کسی (connected sets) کو (contain) کرے۔

A topological space (X,τ) is said to be locally connected at $x \in X$ if for every nbhd u of x there is a connected nbhd V of x which is contained in u.



If X is locally connected at each of its points, then it is simply called locally connected space.

Example:
Every interval of $\mathbb{R}$ is both connected and locally connected.

Component of a topological space:

A maximal (largest) connected subset of a topological space X is called component of X.

Note: If X is itself is connected, then the only component of X is X itself

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Th: A topological space (X, τ) is locally connected if and only if each component of each open set is open.

Proof: Let (X, τ) be a locally connected space an 'u' be an open set of X.
Let 'C' be a component of U.
Then we have to show C is open.
Let $p \in C$
$\because$ X is locally connected.
$\therefore$ there is a connected nbhd 'V' of p s.t
$V \subseteq U$
if $V \nsubseteq C$
then C is a proper subset of the connected set $V \cup C$.
Therefor $V \subseteq C$
Hence, C is open

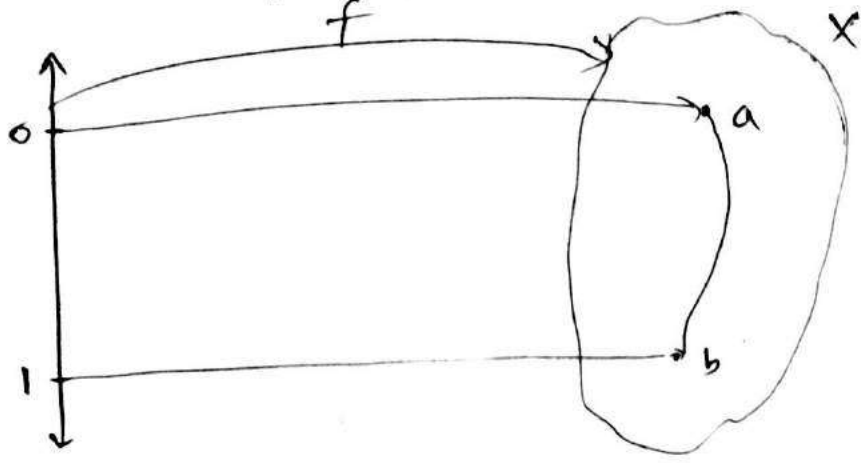
Conversely:
suppose each component of each open set is open.
Then, we have to show that X is locally connected.
Let $p \in X$ and u be a nbhd of p.
then, the component 'V' of U that contains 'p' is a connected nbhd of p s.t
$V \subseteq U$
$\Rightarrow$ X is locally connected.

Path:

A path in a topological space X is a continuous function $f:[0,1] \longrightarrow X$ s.t.

$$f(0) = a \quad \text{and} \quad f(1) = b \quad \forall\ a, b \in X$$

then we say f is a path from a to b.



Path-connected space:

A topological space (X, τ) is said to be path-connected if for every $x, y \in X$ there exists a path from x to y.

Th: Every path connected is connected but the converse is not true.

Proof:

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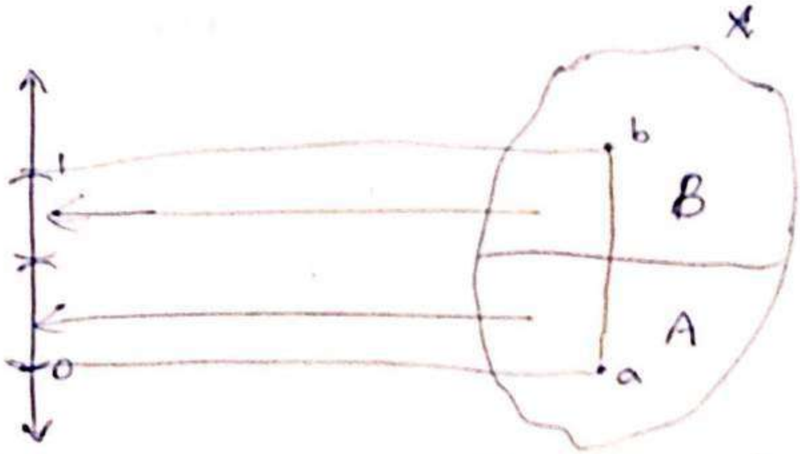
Th: Every path connected is connected

Proof:



Let (X,τ) be a path connected. Then, we have to show that X is connected.
we suppose on contrary that X is disconnected
Then, there exists two non-empty disjoint open sets A and B s.t.

$$A \cup B = X$$

$\because$ X is path connected.
then there exists a path in X.
i.e. $f : [0,1] \longrightarrow X$ s.t.

$$f(0) = a, \quad f(1) = b \quad \forall\ a, b \in X$$

also f is continuous
then $f^{-1}(A)$ & $f^{-1}(B)$ are two non-empty disjoint open sets of $[0,1]$ s.t.

$$[0,1] = f^{-1}(A) \cup f^{-1}(B)$$

which is contradiction against the fact that $[0,1]$ is ~~dis~~ connected.
Hence, we cannot suppose X is dis-connected.
Therefore, X is connected.

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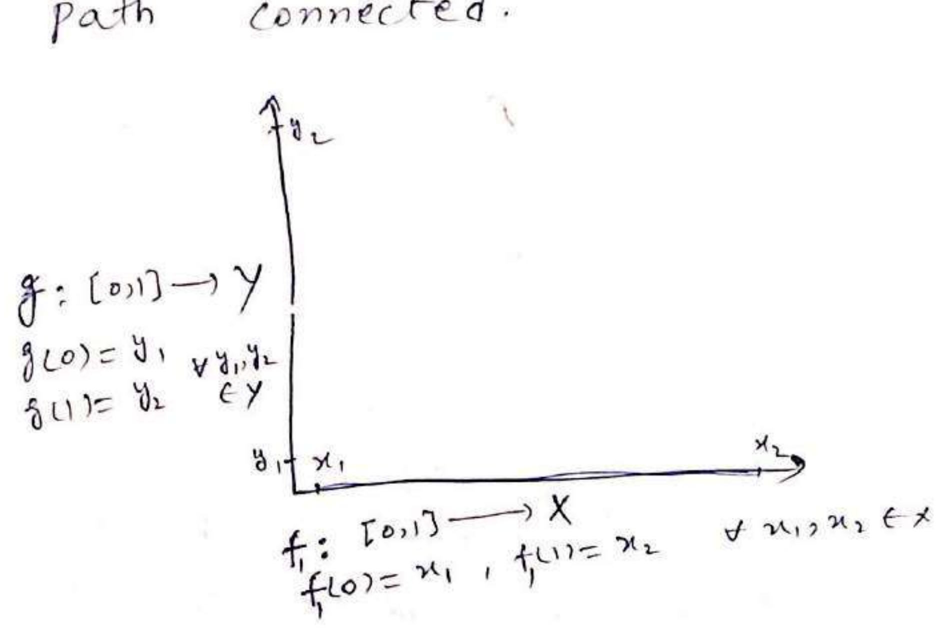
Th:
Let (X, τ_X) and (Y, τ_Y) be path connected spaces. Prove that $(X \times Y, \tau)$ is path connected. i.e. product space of path connected spaces is path connected.



Proof:
Let (x_1, y_1) & (x_2, y_2) be two points of $X \times Y$.

$\because$ X is path connected.
then there exists a path in X.
i.e. $f: [0,1] \longrightarrow X$ s.t.
$f(0) = x_1$, $f(1) = 1$ $\forall x_1, x_2 \in X$

also Y is path connected
then there exists a path in Y.
i.e. $g: [0,1] \longrightarrow Y$ s.t.
$g(0) = y_1$, $g(1) = y_2$ $\forall y_1, y_2 \in Y$

define
$h: [0,1] \longrightarrow X \times Y$ as

$h(t) = (f \times g)t$

$h(t) = (f(t), g(t))$

$h(0) = (f(0), g(0))$

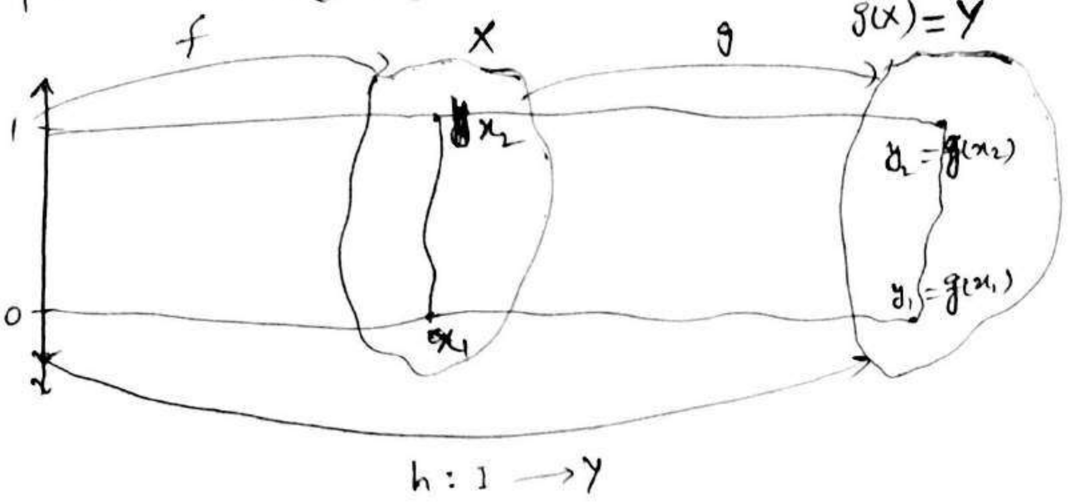
$h(0) = (x_1, y_1)$

similarly

$h(1) = (x_2, y_2)$

$\Rightarrow$ $X \times Y$ is path-connected.

Th: Continuous image of path connected space is path-connected



Proof:-

let $y_1, y_2 \in g(X) = Y$ s.t.

$g(x_1) = y_1$, $g(x_2) = y_2$ $\forall\ x_1, x_2 \in X$

$\because$ X is path connected.

then there exists a path in X from x_1 to x_2.

i.e. $f : I \longrightarrow X$ s.t.

$f(0) = x_1$, $f(1) = x_2$ $\forall\ x_1, x_2 \in X$

define

$h : I \longrightarrow Y$ by

$h(t) = g \circ f(t)$

$h(t) = g[f(t)]$

$h(0) = g[f(0)]$

$h(0) = g(x_1) = y_1$

$h(1) = g[f(1)]$

$h(1) = g(x_2) = y_2$

also h is continuous.

$\Rightarrow$ there exists a path from y_1 to y_2

$\Rightarrow$ $g(X) = Y$ is path connected.

Th: If $\{A_i : i \in N\}$ is a collection of path connected subsets of a space (X, τ) and

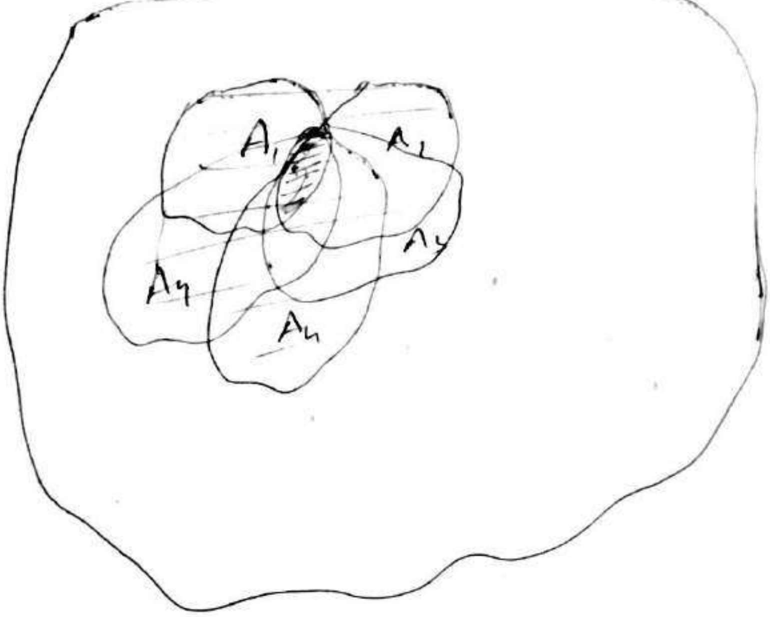
$\bigcap_{i \in N} A_i \neq \phi$ (they have a pt. common)

then $\bigcup_{i \in N} A_i$ is path connected

i.e. Countable union of path connected sets is path connected.

Proof:-

(X, τ)



Proof:- let $x, y \in \bigcup_{i \in N} A_i$

where $x \in A_{i1}$, $y \in A_{i2}$

Let $z \in \bigcap_{i \in N} A_i \neq \phi$

$\Rightarrow$ $z \in A_{i1}$ and $z \in A_{i2}$

$\because$ A_{i1} is path connected

then there exists a path in A_{i1} from x to z.

i.e. $f : [0,1] \longrightarrow A_{i1}$

$f(0) = x$, $f(1) = z$ $\forall$ $x, z \in A_{i1}$

also A_{i2} is path connected

then there exists a path in A_{i2} from z to y

i.e. $g : [0,1] \longrightarrow A_{i2}$

$g(0) = z$, $g(1) = y$ $\forall$ $z, y \in A_{i2}$

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define

~~$h : [0,1] \longrightarrow A_{i_2} \times A_i$~~

$$h(t) = \begin{cases} f(2t) & 0 \le t < \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

also h is continuous

$\Rightarrow$ there exists a path in $\bigcup_{i\in N} A_i$

$\Rightarrow$ $\bigcup_{i\in N} A_i$ is path connected.

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