Functional Analysis

MUHAMMAD USMAN HAMID
ZEESHAN AHMAD

Objectives of course:

This course extends methods of linear algebra and analysis to spaces of functions, in which the interaction between algebra and analysis allows powerful methods to be developed. The course will be mathematically sophisticated and will use ideas both from linear algebra and analysis.

Muhammad Usman Hamid

University of Sargodha

Course Contents:

- **Metric Spaces:** A quick review, completeness and convergence, completion,
- **Normed Spaces:** linear spaces, normed spaces, difference between a metric and a normed space, Banach spaces, bounded and continuous linear operators and functionals, dual spaces, finite dimensional spaces, F. Riesz Lemma, the Hahn-Banach theorem, the HB theorem for complex spaces, The HB theorem for normed spaces, the open mapping theorem, the closed graph theorem, uniform boundedness principle and its applications.
- **Inner-Product Spaces:** Inner-Product space, Hilbert Space, orthogonal and orthonormal sets, orthogonal complements, Gram-Schmidt orthogonalization process, representation of functionals, Reiz-representation theorem, and weak convergence.
- **Banach-Fixed-Point Theorem:** Applications in differential and integral equations.

Recommended Books:

- Kreyszig E. Introductory Functional Analysis with Applications.
- Dunford N. and Schwartz J.T. Linear Operators, Interscience publishers.
- Curtain R.F., Pritchard A.J. Functional Analysis in Modern Applied Mathematics.
- Friedman A. Foundations of Modern Analysis.
- Rudin W. Functional Analysis.
- Functional Analysis by Dr. Abdul Majeed.
- Functional Analysis by Z. R. Bhatti.

METRIC SPACES

Functional analysis is an abstract branch of mathematics that originated from classical analysis. It deals with analysis of functional (functions of functions). It concerned with infinite dimensional vector spaces (mainly function space) and mappings between them. It deals with abstract spaces and different operators define on these spaces. Its development started about eighty years ago, and nowadays functional analytic methods and results are important in various fields of mathematics and its applications. The impetus came from linear algebra, linear ordinary and partial differential equations, calculus of variations, approximation theory and, in particular, linear integral equations, whose theory had the greatest effect on the development and promotion of the modern ideas.

Metric space, Metric

A metric space is a pair (X, d), where X is a non-empty set and d is a metric on X (or distance function on X), that is, A function $d: X \times X \to \mathbb{R}^+$ is said to be metric on $X \neq \varphi$ such that for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and nonnegative. i.e. $d(x, y) \geq 0$; $\forall x, y \in X$

(M2) $d(x, y) = 0$ if and only if $x = y$

(M3) $d(x, y) = d(y, x)$; $\forall x, y \in X$ (Symmetry)

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle Inequality).

If (X, d) is metric space then X is called underlying set. Its elements are called points.

Induced Metric

A subspace $(Y, \tilde{d})$ of (X, d) is obtained if we take a subset $Y \subset X$ and restrict d to $Y \times Y$; thus the metric on Y is the restriction $\tilde{d} = d\,|_{Y \times Y}$ is called the **metric induced** on Y by d.

What is the use of Metric Space in daily life?

In mathematics a metric space is a set where a distance (called metric) is defined between elements of the set. Metric Space methods have been employed for decades in various applications, for example in internet search engines, image classification, or protein classification.

Can a Metric Space be empty?

A metric space is formally defined as a pair. The empty set is not such a pair, so it is not a metric space in itself.

What is the difference between normed space and a metric space?

A metric provides us a notion of the distance between points in a space, a norm gives us a notion of the length of an individual vector. A norm can only be defined on a vector space, while a metric can be defined on any set.

Examples

- **Real line** $\mathbb{R}$**:** This is the set of all real numbers, taken with the usual metric defined by $d(x,y) = |x-y|$ this is also called usual metric space.
- **Euclidean plane** $\mathbb{R}^2$**:** The metric space $\mathbb{R}^2$, called the Euclidean plane, is obtained if we take the set of ordered pairs of real numbers, written $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2)$, etc., and the Euclidean metric defined by
  $d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$
  Or $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$ this is also called the **taxicab metric**.
- **Three-dimensional Euclidean space** $\mathbb{R}^3$
  This metric space consists of the set of ordered triples of real numbers $x = (\xi_1, \xi_2, \xi_3), y = (\eta_1, \eta_2, \eta_3)$, etc., and the Euclidean metric defined by $d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2}$
- **Euclidean space** $\mathbb{R}^n$
  This space is obtained if we take the set of all ordered n-tuples of real numbers, written $x = (\xi_1, \xi_2, \dots, \xi_n), y = (\eta_1, \eta_2, \dots, \eta_n)$ etc., and the Euclidean metric defined by
  $d(x,y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots + (\xi_n - \eta_n)^2}$
- **Unitary Space C^n, complex plane C**
  This space is obtained if we take the set of all ordered n-tuples of complex numbers, written $x = (\xi_1, \xi_2, \dots, \xi_n), y = (\eta_1, \eta_2, \dots, \eta_n)$ etc., and the Euclidean metric defined by
  $d(x,y) = \sqrt{|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2 + \cdots + |\xi_n - \eta_n|^2}$
  (C^n is sometimes called complex Euclidean n-space.)
  When n = 1 this is the complex plane C with the usual metric defined by $d(x,y) = |x-y|$
- **Sequence space** $\boldsymbol{l^\infty}$**:** This space X (say) is a set of all bounded sequences of complex numbers; that is, every element of X is a complex sequence $(\xi_1, \xi_2, \dots)$ briefly $x = (\xi_j)$ such that for all $j = 1, 2, \dots$ we have $|\xi_j| \le c_x$ where c_x is a real number which may depend on x, but does not depend on j. This space is a metric space with defined metric $d(x,y) = Sup_{j \in N} |\xi_j - \eta_j|$

- **Function space C[a, b]:** This space X (say) is a set of all real-valued functions x, y, which are functions of an independent real variable t and are defined and **continuous** on a given closed interval J = [a, b]. This space is a metric space with defined metric $d(x,y) = max_{t\in J}|x(t) - y(t)|$
- **Discrete metric space or Trivial Metric space:** A space (X, d) is called discrete metric space, if we define the following metric on it
  $d(x,x) = 0,\ d(x,y) = 1\ (x \neq y)$.
  Or $d(x,y) = \begin{cases} 1 & ; x \neq y \\ 0 & ; x = y \end{cases}$

Problem: Show that the real line is a metric space.

Or Show that the set of all real numbers, taken with the usual metric defined by $d(x,y) = |x - y|$ is a metric space.

Solution Define a metric $d(x,y) = |x - y|$ for X = ℝ

(M1) Obviously $d(x,y) \geq 0\ ; \forall x, y \in X$

(M2) Let $x = y$ then $d(x,y) = |x - y| = |x - x| = 0 \Rightarrow d(x,y) = 0$

If $d(x,y) = 0 \Rightarrow |x - y| = 0 \Rightarrow x - y = 0 \Rightarrow x = y$

Thus $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = |x - y| = |y - x| = d(y,x)$

Thus $d(x,y) = d(y,x)\ ; \forall x, y \in X$

(M4) $d(x,z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z|$

Thus $d(x,z) \leq d(x,\ y) + d(y,z)$

Hence d is a metric on ℝ and is called usual metric on ℝ and $(\mathbb{R}, d)$ is called usual metric space on real line.

Problem: Show that the Euclidean Plane is a metric space.

Or Show that the set of ordered pairs of real numbers, written

$x = (\xi_1, \xi_2), y = (\eta_1, \eta_2)$, etc., defined $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by the following metric is a metric space.

$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$

Solution Define a metric $d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2}$ for X = $\mathbb{R}^2$

(M1) Obviously $d(x, y) \geq 0 \,; \forall x, y \in X$

(M2) Let $x = y$ then $(\xi_1, \xi_2) = (\eta_1, \eta_2)$ and

$\Leftrightarrow d(x, y) = \sqrt{(\xi_1 - \xi_1)^2 + (\xi_2 - \xi_2)^2} = 0 \Leftrightarrow d(x, y) = 0$

Thus $d(x, y) = 0$ if and only if $x = y$

(M3) $d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} = \sqrt{(\eta_1 - \xi_1)^2 + (\eta_2 - \xi_2)^2} = d(y, x)$

Thus $d(x, y) = d(y, x) \;; \forall x, y \in X$

(M4) Let $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2), z = (\gamma_1, \gamma_2)$ then
$d(x, z) = \sqrt{(\xi_1 - \gamma_1)^2 + (\xi_2 - \gamma_2)^2}$

If points x,y,z are non collinear then they form vertices of triangle therefore

Thus $d(x, y) < d(x, z) + d(z, y)$

But if points x,y,z are collinear then $d(x, y) = d(x, z) + d(z, y)$

Thus $d(x, y) \leq d(x, z) + d(z, y)$

Hence d is a usual metric on $\mathbb{R}^2$. **Euclidean Metric** also called **Usual metric**.

This is also called the **taxicab metric**. Taxicab Geometry gets its name from the fact that **taxis** can only drive along streets, rather than moving as the crow flies. Euclidian Distance between two sets as the taxi driving. It is known as Taxicab metric as it measures the distance a taxi would travel from a point to some other point if there were no one way streets.

Problem

Consider $X = \mathbb{R}^2$ with $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$ With $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathbb{R}^2$

Show that d is a metric on $\mathbb{R}^2$.

Solution Define a metric $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$ for X = $\mathbb{R}^2$

(M1) Obviously $|\xi_1 - \eta_1| + |\xi_2 - \eta_2| \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall x, y \in X$

(M2) Let $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2| = 0 \Leftrightarrow \xi_1 = \eta_1, \xi_2 = \eta_2$

$\Leftrightarrow (\xi_1, \xi_2) = (\eta_1, \eta_2) \Leftrightarrow x = y$

Thus $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2| = |\eta_1 - \xi_1| + |\eta_2 - \xi_2| = d(y,x)$

Thus $d(x,y) = d(y,x)\ ; \forall x, y \in X$

(M4) Let $d(x,y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$

$d(x,y) = |\xi_1 - \gamma_1 + \gamma_1 - \eta_1| + |\xi_2 - \gamma_2 + \gamma_2 - \eta_2|$

$d(x,y) \leq |\xi_1 - \gamma_1| + |\gamma_1 - \eta_1| + |\xi_2 - \gamma_2| + |\gamma_2 - \eta_2|$

$d(x,y) \leq (|\xi_1 - \gamma_1| + |\xi_2 - \gamma_2|) + (|\gamma_1 - \eta_1| + |\gamma_2 - \eta_2|)$

Thus $d(x,y) \leq d(x, z) + d(z,y)$ Hence d is a metric on $\mathbb{R}^2$

Question Show that non – negativity of a metric using (M2) and (M4).

Solution

(M4) $d(x,z) \leq d(x, y) + d(y,z)$

z = x $\Rightarrow d(x,x) \leq d(x,y) + d(y,x) \Rightarrow d(x,x) \leq 2d(x,y)$

using (M2) we have $d(x,x) = 0 \Rightarrow 0 \leq 2d(x,y)$

$\Rightarrow d(x,y) \geq 0$

Example

Let X be any non-empty set and d is a metric defined over X. Let m be any natural number so that we define $d_m(x,y) = md(x,y)$ for any x, y ∈ X. We are to show that $d_m(.,.)$ is also a metric. The new metric spaces $\{(X, d_m): m = 1, 2, \dots\}$ are thus obtained from (X, d).

Solution

(i) $d_m(x,y) = md(x,y) \geq 0 \quad \forall$ x, y ∈ X

(ii) $d_m(x,y) = 0 \Leftrightarrow md(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

(iii) $d_m(x,y) = d_m(y,x)$ since $d(x,y) = d(y,x)$

(iv) For any x, y, z ∈ X we have

$$d_m(x,y) = md(x,y) \leq m(d(x,z) + d(z,y))$$

$$d_m(x,y) \leq md(x,z) + md(z,y)$$

$$d_m(x,y) \leq d_m(x,z) + d_m(z,y)$$

Hence $d_m(.,.)$ is a metric space. This metric is called **dilation metric**.

Remark

The choice of m being a natural number has no specific advantage. However for $m > 1$, a ‘dilation’ and for $0 < m < 1$, a ‘contraction’ of distance occurs.

Example-19: *Show that* $d(x,y)=\left|\frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}\right|$ *is a metric on real line R.*

Solution: M_1): Since the modulus of the difference of any real numbers is always nonnegative, so

$$d(x,y)=\left|\frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}\right|\geq 0$$

M_2): $d(x,y)=0 \Leftrightarrow \left|\frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}\right|=0$

$$\Leftrightarrow \frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}=0$$

$$\Leftrightarrow \frac{x}{1+\sqrt{1+x^2}}=\frac{y}{1+\sqrt{1+y^2}}$$

Since the left hand side and the right hand side of last expression are identical which is only possible if $x=y$, so

$$d(x,y)=0$$

$$\Leftrightarrow x=y$$

M_3): $d(x,y)=\left|\frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}\right|=\left|-\left(\frac{y}{1+\sqrt{1+y^2}}-\frac{x}{1+\sqrt{1+x^2}}\right)\right|$

$$=\left|\frac{y}{1+\sqrt{1+y^2}}-\frac{x}{1+\sqrt{1+x^2}}\right|=d(y,x)$$

M_4): $d(x,z)=\left|\frac{x}{1+\sqrt{1+x^2}}-\frac{z}{1+\sqrt{1+z^2}}\right|$

$$=\left|\frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}+\frac{y}{1+\sqrt{1+y^2}}-\frac{z}{1+\sqrt{1+z^2}}\right|$$

$$\leq\left|\frac{x}{1+\sqrt{1+x^2}}-\frac{y}{1+\sqrt{1+y^2}}\right|+\left|\frac{y}{1+\sqrt{1+y^2}}-\frac{z}{1+\sqrt{1+z^2}}\right|$$

$\Rightarrow d(x,z)\leq d(x,y)+d(y,z)$

Hence d is a metric on real line R.

Problem

Does $d\ (x, y)\ = (x - y)^2$ define a metric on the set of all real numbers?

Or

Show that $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is not a metric, where $\mathbb{R}$ is the set of real numbers and d is defined by $d\ (x, y)\ = (x - y)^2$

Solution Define a metric $d\ (x, y)\ = (x - y)^2$ for X = $\mathbb{R}$

(M1) Obviously $(x - y)^2 = d(x, y) \geq 0\ ; \forall x, y \in X$

(M2) Let $d(x, y) = (x - y)^2 = 0 \Leftrightarrow x - y = 0 \Leftrightarrow x = y$

Thus $d(x, y) = 0$ if and only if $x = y$

(M3) $d(x, y) = (x - y)^2 = (y - x)^2 = d(y, x)$

Thus $d(x, y) = d(y, x)\ ; \forall x, y \in X$

(M4) Suppose triangular inequality holds in $\forall x, y, z \in X = \mathbb{R}$ then

$d(x, z) \leq d(x, y) + d(y, z) \Rightarrow (x - z)^2 \leq (x - y)^2 + (y - z)^2$

$(0 - 2)^2 \leq (0 - 1)^2 + (1 - 2)^2$ $\because x, y, z \in R \therefore 0,1,2 \in \mathbb{R}$

$4 \leq 2$

This is not true, so triangular inequality not holds. Hence d is not a metric on $\mathbb{R}$

Problem Prove General Triangular inequality.

Solution We will prove it by using mathematical inducation.

For $n = 3$ $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ induction is true. …….(1)

Suppose for $n = k$ $d(x_1, x_k) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k)$

Suppose for $n = k + 1$ $d(x_1, x_{k+1}) \leq d(x_1, x_k) + d(x_k, x_{k+1})$

$\Rightarrow d(x_1, x_{k+1}) \leq d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k) + d(x_k, x_{k+1})$ using 1

$\Rightarrow d(x_0, x_n) \leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)$ proved

Problem **Discrete Space is infact a metric space.**

Let X be a non – empty set. Define a function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$d(x,y) = \begin{cases} 1 & ; x \neq y \\ 0 & ; x = y \end{cases}$ $\forall x, y \in X$ then show that d is a metric on X.

Solution Define a metric $d(x,y) = \begin{cases} 1 & ; x \neq y \\ 0 & ; x = y \end{cases}$ $\forall x, y \in X$

(M1) Obviously $d(x,y) \geq 0\, ; \forall x, y \in X$

(M2) $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = d(y,x); \forall x, y \in X$

(M4) If $x = y, y = z, x = z\ \forall x, y, z \in X$ then $d(x,y) = d(x,\ z) = d(z,y) = 0$ then $d(x,z) = 0 = 0 + 0 = d(x,y) + d(y,z)$

$d(x,z) = d(x,y) + d(y,z)$

If $x \neq y, y \neq z, x \neq z\ \forall x, y, z \in X$ then $d(x,y) = d(x,\ z) = d(z,y) = 1$ then $d(x,z) = 1 < 1 + 1 = d(x,y) + d(y,z)$ implies $d(x,z) < d(x,y) + d(y,z)$

Thus $d(x,z) \leq d(x,y) + d(y,z)$

Hence d is a metric on X and (X,d) is called **discrete or trivial** metric space.

Question

If (X,d) is a metric space then show that $|d(x,a) - d(y,b)| \leq d(x,y) + d(a,b)$.

Solution Suppose that $d(x,a) \leq d(x,y) + d(y,b) + d(b,a)$

$\Rightarrow d(x,a) - d(y,b) \leq d(x,y) + d(a,b)$ ………(1) since d metric space

Also $d(y,b) \leq d(y,x) + d(x,a) + d(a,b)$

$\Rightarrow d(y,b) - d(x,a) \leq d(x,y) + d(a,b)$

$\Rightarrow -[d(x,a) - d(y,b)] \leq d(x,y) + d(a,b)$ ………(2) since d metric space

Combining (1) and (2) $|d(x,a) - d(y,b)| \leq d(x,y) + d(a,b)$.

Question

If (X, d) is a metric space then show that $|d(x,z) - d(y,z)| \leqq d(x,y)$.

Solution Since d metric space therefore by triangular inequality

$d(x,z) \leq d(x,y) + d(y,z) \Rightarrow d(x,z) - d(y,z) \leq d(x,y)$ ………(1)

Also $d(y,z) \leq d(y,x) + d(x,z) \Rightarrow -d(x,y) \leq d(x,z) - d(y,z)$ ………(2)

Combining (1) and (2) $\Rightarrow -d(x,y) \leq d(x,z) - d(y,z) \leq d(x,z)$

$\Rightarrow |d(x,z) - d(y,z)| \leqq d(x,y)$

Question If (X, d) is a metric space then show that $d(x,y) = \frac{2\, d(x,y)}{1+2\, d(x,y)}$ is also a metric on X.

Solution Define a metric $d(x,y) = \frac{2\, d(x,y)}{1+2\, d(x,y)}$

(M1) Obviously $\frac{2\, d(x,y)}{1+2\, d(x,y)} = d(x,y) \geq 0\ ; \forall x, y \in X$

(M2) Let $d(x,y) = \frac{2\, d(x,y)}{1+2\, d(x,y)} = 0$

$\Leftrightarrow 2d(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

Thus $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = \frac{2\, d(x,y)}{1+2\, d(x,y)} = \frac{2\, d(y,x)}{1+2\, d(y,x)} = d(y,x)$

Thus $d(x,y) = d(y,x)\ ; \forall x, y \in X$

(M4) Let $d(x,y) = \frac{2\, d(x,y)}{1+2\, d(x,y)} \leq \frac{2\,[d(x,z)+d(z,y)]}{1+2\,[d(x,z)+d(z,y)]}$

$d(x,y) \leq \frac{2\, d(x,z)}{1+2\,[d(x,z)+d(z,y)]} + \frac{2\, d(z,y)}{1+2\,[d(x,z)+d(z,y)]}$

$d(x,y) \leq \frac{2\, d(x,z)}{1+2\, d(x,z)} + \frac{2\, d(y,z)}{1+2d(y,z)}$

Thus $d(x,y) \leq d(x,\ z) + d(z,y)$ Hence d is a usual metric on X.

Question Show that $d(x,y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in \mathbb{R}$ is a metric on $\mathbb{R}$.

Solution Define a metric $d(x,y) = \frac{|x-y|}{1+|x-y|}$

(M1) Obviously $|x-y| \geq 0 \Rightarrow \frac{|x-y|}{1+|x-y|} = d(x,y) \geq 0$

(M2) Let $d(x,y) = \frac{|x-y|}{1+|x-y|} = 0 \Leftrightarrow |x-y| = 0$

$\Leftrightarrow x - y = 0 \Leftrightarrow x = y$ Thus $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d(y,x)$ Thus $d(x,y) = d(y,x)$

(M4) Let $d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|x-z+z-y|}{1+|x-z+z-y|} \leq \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|}$

$d(x,y) = \frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|}$

$d(x,y) = \frac{|x-y|}{1+|x-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|}$

Thus $d(x,y) \leq d(x, z) + d(z,y)$ Hence d is a usual metric on $\mathbb{R}$.

Question

If (X,d) is a metric space then show that $d(x,y) = \frac{1-d(x,y)}{1+d(x,y)}$ is not a metric on X.

Solution Define a metric $d(x,y) = \frac{1-d(x,y)}{1+d(x,y)}$

(M1) Obviously $\frac{1-d(x,y)}{1+d(x,y)} = d(x,y) \geq 0$

(M2) Let $d(x,y) = \frac{1-d(x,y)}{1+d(x,y)} = 0 \Rightarrow 1 - d(x,y) = 0 \Rightarrow d(x,y) = 1$

But $d(x,y) = 0$

Hence $d(x,y) = \frac{1-d(x,y)}{1+d(x,y)}$ is not a metric on X.

Question If $x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ then show that

$d(x,y) = \sum_{i=1}^{n} |\xi_i - \eta_i| = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + \cdots + (\xi_n - \eta_n)^2}$ is a metric on $\mathbb{R}^n$.

Solution Define a metric $d(x,y) = \sum_{i=1}^{n} |\xi_i - \eta_i|$ for X = R^n

(M1) Obviously $|\xi_i - \eta_i| \geq 0 \Rightarrow \sum_{i=1}^{n} |\xi_i - \eta_i| = d(x,y) \geq 0\ ; \forall x, y \in X$

(M2) Let $d(x,y) = \sum_{i=1}^{n} |\xi_i - \eta_i| = 0 \Leftrightarrow \xi_i = \eta_i \Leftrightarrow x = y$

Thus $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = \sum_{i=1}^{n} |\xi_i - \eta_i| = \sum_{i=1}^{n} |\eta_i - \xi_i| = d(y,x)$

Thus $d(x,y) = d(y,x)\ ; \forall x, y \in X$

(M4) Let $d(x,y) = \sum_{i=1}^{n} |\xi_i - \eta_i| = \sum_{i=1}^{n} |\xi_i - \gamma_i + \gamma_i - \eta_i|$

$d(x,y) = \sum_{i=1}^{n} |\xi_i - \eta_i| \leq \sum_{i=1}^{n} |\xi_i - \gamma_i| + \sum_{i=1}^{n} |\gamma_i - \eta_i|$

Thus $d(x,y) \leq d(x, z) + d(z,y)$ Hence d is a metric on R^n

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Question

Let $d(x,y) = max(|x|, |y|)$ for all $x, y \in \mathbb{R}$ then show that it is not a metric on $\mathbb{R}$.

Solution Define a metric $d(x,y) = max(|x|, |y|)$ for X = $\mathbb{R}$

(M1) Obviously $max(|x|, |y|) = d(x,y) \geq 0$

(M2) Let $d(x,y) = max(|x|, |y|) = 0 \Leftrightarrow x = 0, y = 0 \Leftrightarrow x = y$

But if $x = 1, y = 1$ then $max(|x|, |y|) = 1$

A contradiction to $d(x,y) = max(|x|, |y|) = 0$

Thus d is not a metric on $\mathbb{R}$.

Problem:

Let X be a non empty set and $d: X \times X \to \mathbb{R}^+$ be a metric space on X. then $d': X \times X \to \mathbb{R}^+$ defined by $d'(x,y) = min\big(1, d(x,y)\big)$ is a metric space.

Solution

Define a metric $d'(x,y) = min\big(1, d(x,y)\big)$ for X.

(M1) Obviously $d'(x,y) = min\big(1, d(x,y)\big) \geq 0 \quad \because 1 \geq 0$ and $d(x,y) \geq 0$

(M2) Let $d'(x,y) = 0 \Leftrightarrow min\big(1, d(x,y)\big) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

(M3) $d'(x,y) = min\big(1, d(x,y)\big) = min\big(1, d(y,x)\big) = d'(y,x)$

Thus $d(x,y) = d(y,x) \ ; \forall x, y \in X$

(M4) Let $d'(x,z) = min\big(1, d(x,z)\big)$ then

$d'(x,z) \leq 1$ or $d'(x,z) \leq d(x,z)$

We want to show that $d'(x,z) \leq d'(x,y) + d'(y,z)$

For this consider $d(x,z) \geq 1, d(x,y) \geq 1, d(y,z) \geq 1$

Then $d'(x,z) = 1, d'(x,y) = 1, d'(y,z) = 1$

$\Rightarrow d'(x,z) \leq d'(x,y) + d'(y,z) \qquad \because 1 \leq 1 + 1$

Again consider $d(x,z) < 1, d(x,y) < 1, d(y,z) < 1$

Then $d'(x,z) = d(x,z), d'(x,y) = d(x,y), d'(y,z) = d(y,z)$

Then $d(x,z) \leq d(x,y) + d(y,z) \qquad \because d$ is metric on X

$\Rightarrow d'(x,z) \leq d'(x,y) + d'(y,z)$

Hence proved $d'(x,y) = min\big(1, d(x,y)\big)$ is a metric on X.

Question

If (X, d) is a metric space then show that $d'(x,y) = \frac{d(x,y)}{1+\ d(x,y)}$ is also a metric on X.

Solution Define a metric $d'(x,y) = \frac{d(x,y)}{1+\ d(x,y)}$

(M1) Obviously $\frac{d(x,y)}{1+d(x,y)} = d'(x,y) \geq 0\ ; \forall x, y \in X$

(M2) Let $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = 0$

$\Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

Thus $d'(x,y) = 0$ if and only if $x = y$

(M3) $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$

Thus $d'(x,y) = d'(y,x)\ ; \forall x, y \in X$

(M4) Since d is metric on X therefore $d(x,y) \leq d(x,\ z) + d(z,y)$

Let $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+\ [d(x,z)+d(z,y)]}$

$d'(x,y) \leq \frac{d(x,z)}{1+\ [d(x,z)+d(z,y)]} + \frac{d(z,y)}{1+\ [d(x,z)+d(z,y)]}$

$d'(x,y) \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(y,z)}{1+d(y,z)}$

Thus $d'(x,y) \leq d'(x,z) + d'(z,y)$ Hence d' is a usual metric on X.

Example Show that $d(x,y) = \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2}$ is a metric.

Solution Define a metric $d(x,y) = \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2}$

(M1) The sum of two real valued, finite and non negative functions is non negative and real. $\Rightarrow d(x,y) \geq 0$

(M2) $d(x,y) = 0 \Leftrightarrow \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2} = 0$

$\Leftrightarrow d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2 = 0 \Leftrightarrow d_1(x_1,y_1)^2 = d_2(x_2,y_2)^2 = 0$

$\Leftrightarrow d_1(x_1,y_1) = 0 \Leftrightarrow x_1 = y_1$ and $\Leftrightarrow d_2(x_2,y_2) = 0 \Leftrightarrow x_2 = y_2 \Leftrightarrow x = y$

(M3) $d(x,y) = \sqrt{d_1(x_1,y_1)^2 + d_2(x_2,y_2)^2}$

$d(x,y) = \sqrt{d_1(y_1,x_1)^2 + d_2(y_2,x_2)^2} = d(y,x)$

(M4) Let $x = (x_1,y_1), y = (x_2,y_2), z = (x_3,y_3)$

$d(x_1,y_1) \leq d(x_1,z_1) + d(z_1,y_1)$

$d(x_2,y_2) \leq d(x_2,z_2) + d(z_2,y_2)$

Squaring above both

$d_1(x_1,y_1)^2 \leq d_1(x_1,z_1)^2 + d_1(z_1,y_1)^2 + 2d_1(x_1,z_1)d_1(z_1,y_1)$

$d_2(x_2,y_2)^2 \leq d_2(x_2,z_2)^2 + d_2(z_2,y_2)^2 + 2d_2(x_2,z_2)d_2(z_2,y_2)$

Adding above both

$d(x,y)^2 \leq d(x,z)^2 + d(z,y)^2 + 2\sum_{j=1}^{2} d_j(x_j,z_j)d_j(z_j,y_j)$

$d(x,y)^2 \leq d(x,z)^2 + d(z,y)^2 + 2\sqrt{\sum_{j=1}^{2} d_j(x_j,z_j)^2}\sqrt{\sum_{j=1}^{2} d_j(z_j,y_j)^2}$ **Schawarz**

$d(x,y)^2 \leq d(x,z)^2 + d(z,y)^2 + 2d(x,z)d(z,y) = [d(x,z)^2 + d(z,y)^2]^2$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Example

Show that $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$ is a metric.

Solution

Define a metric $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$

(M1) The sum of two real valued, finite and non negative functions is non negative and real. $\Rightarrow d(x,y) \geq 0$

(M2) $d(x,y) = 0 \Leftrightarrow d_1(x_1,y_1) + d_2(x_2,y_2) = 0$

$\Leftrightarrow d_1(x_1,y_1) = d_2(x_2,y_2) = 0$

$\Leftrightarrow d_1(x_1,y_1) = 0 \Leftrightarrow x_1 = y_1$ and $\Leftrightarrow d_2(x_2,y_2) = 0 \Leftrightarrow x_2 = y_2$

$\Leftrightarrow x = y$

(M3) $d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$

$d(x,y) = d_1(y_1,x_1) + d_2(y_2,x_2) = d(y,x)$

(M4) Let $x = (x_1,y_1), y = (x_2,y_2), z = (x_3,y_3)$

$d(x_1,y_1) \leq d(x_1,z_1) + d(z_1,y_1)$

$d(x_2,y_2) \leq d(x_2,z_2) + d(z_2,y_2)$

$d(x,y) = d_1(x_1,y_1) + d_2(x_2,y_2)$

$d(x,y) \leq d(x_1,z_1) + d(z_1,y_1) + d_1(x_1,y_1) + d_2(x_2,y_2)$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Example

Show that $d(x,y) = max[d_1(x_1,y_1), d_2(x_2,y_2)]$ is a metric.

Solution

Define a metric $d(x,y) = max[d_1(x_1,y_1), d_2(x_2,y_2)]$

(M1) The maximum of two real valued, finite and non negative functions is non negative and real. $\Rightarrow d(x,y) \geq 0$

(M2) $d(x,y) = 0 \Leftrightarrow max[d_1(x_1,y_1), d_2(x_2,y_2)] = 0$

$\Leftrightarrow d_1(x_1,y_1) = d_2(x_2,y_2) = 0$

$\Leftrightarrow d_1(x_1,y_1) = 0 \Leftrightarrow x_1 = y_1$ and $\Leftrightarrow d_2(x_2,y_2) = 0 \Leftrightarrow x_2 = y_2$

$\Leftrightarrow x = y$

(M3) $d(x,y) = max[d_1(x_1,y_1), d_2(x_2,y_2)]$

$d(x,y) = max[d_1(y_1,x_1), d_2(y_2,x_2)] = d(y,x)$

(M4) Let $x = (x_1,y_1), y = (x_2,y_2), z = (x_3,y_3)$

$d(x_1,y_1) \leq d(x_1,z_1) + d(z_1,y_1)$

$d(x_2,y_2) \leq d(x_2,z_2) + d(z_2,y_2)$

$max[d_1(x_1,y_1), d_2(x_2,y_2)]$

$$\leq max[\{d(x_1,z_1) + d(z_1,y_1)\}, \{d(x_2,z_2) + d(z_2,y_2)\}]$$

$max[d_1(x_1,y_1), d_2(x_2,y_2)]$

$$\leq max[d(x_1,z_1), d(x_2,z_2)] + max[d(z_1,y_1), d(z_2,y_2)]$$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Sequence Space s.

This space consists of the set of all (bounded or unbounded) sequences of complex numbers and the metric d defined by

$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$ Where $x = (\xi_j)$ and $y = (\eta_j)$.

Solution Define a metric $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$

(M1) $|\xi_j - \eta_j| \geq 0 \Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall x, y \in X$

(M2) $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} = 0 \Leftrightarrow |\xi_j - \eta_j| = 0$

$\Leftrightarrow (\xi_j) = (\eta_j) \Leftrightarrow x = y$

(M3) $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\eta_j - \xi_j|}{1+|\eta_j - \xi_j|} = d(y,x)$

(M4) Define $f(t) = \frac{t}{t+1}$ then $f'(t) = \frac{1}{(t+1)^2} > 0$

$\Rightarrow f$ is monotonically increasing $\because t_1 < t_2 \Rightarrow f(t_1) < f(t_2)$

Hence using the result $|a+b| \leq |a| + |b|$

$\Rightarrow f(|a+b|) \leq f(|a|+|b|) \Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}$

$\Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

Using $a = \xi_j - e_j$ and $b = e_j - \eta_j \Rightarrow \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq \frac{|\xi_j - e_j|}{1+|\xi_j - e_j|} + \frac{|e_j - \eta_j|}{1+|e_j - \eta_j|}$

$\Rightarrow \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq \frac{1}{2^j} \frac{|\xi_j - e_j|}{1+|\xi_j - e_j|} + \frac{1}{2^j} \frac{|e_j - \eta_j|}{1+|e_j - \eta_j|}$

$\Rightarrow \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - e_j|}{1+|\xi_j - e_j|} + \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|e_j - \eta_j|}{1+|e_j - \eta_j|}$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \Rightarrow d$ is metric on given Space.

Question

For a sequence space ‘s’ $d(x,y) \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$ defines a metric on the sequence s. show tha we can obtain another metric by replacing $\frac{1}{2^j}$ with $u^j > 0$ such that $\sum_{j=1}^{\infty} u^j$ converges.

Solution Define $d(x,y) = \sum_{j=1}^{\infty} u^j \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$

(M1) $|\xi_j - \eta_j| \geq 0 \Rightarrow \sum_{j=1}^{\infty} u^j \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall x, y \in X$

(M2) $d(x,y) = \sum_{j=1}^{\infty} u^j \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} = 0 \Leftrightarrow |\xi_j - \eta_j| = 0$

$\Leftrightarrow (\xi_j) = (\eta_j) \Leftrightarrow x = y$

(M3) $d(x,y) = \sum_{j=1}^{\infty} u^j \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} = \sum_{j=1}^{\infty} u^j \frac{|\eta_j - \xi_j|}{1+|\eta_j - \xi_j|} = d(y,x)$

(M4) Define $f(t) = \frac{t}{t+1}$ then $f'(t) = \frac{1}{(t+1)^2} > 0$

$\Rightarrow f$ is monotonically increasing $\because t_1 < t_2 \Rightarrow f(t_1) < f(t_2)$

Hence using the result $|a+b| \leq |a| + |b|$

$\Rightarrow f(|a+b|) \leq f(|a|+|b|) \Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}$

$\Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \Rightarrow \frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

Using $a = \xi_j - e_j$ and $b = e_j - \eta_j \Rightarrow \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq \frac{|\xi_j - e_j|}{1+|\xi_j - e_j|} + \frac{|e_j - \eta_j|}{1+|e_j - \eta_j|}$

$\Rightarrow u^j \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq u^j \frac{|\xi_j - e_j|}{1+|\xi_j - e_j|} + u^j \frac{|e_j - \eta_j|}{1+|e_j - \eta_j|}$

$\Rightarrow \sum_{j=1}^{\infty} u^j \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|} \leq \sum_{j=1}^{\infty} u^j \frac{|\xi_j - e_j|}{1+|\xi_j - e_j|} + \sum_{j=1}^{\infty} u^j \frac{|e_j - \eta_j|}{1+|e_j - \eta_j|}$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \Rightarrow d$ is metric on given Space.

Space B(A) of bounded functions

Consider the space B(A) of all function defined and bounded on a given set A, and the metric d is defined by $d(x,y) = sup_{t\in A}\,|x(t) - y(t)|$ then show that given space is a metric space.

Solution Define a metric $d(x,y) = sup_{t\in A}\,|x(t) - y(t)|$

(M1) $|x(t) - y(t)| \geq 0 \Rightarrow sup_{t\in A}\,|x(t) - y(t)| \geq 0 \Rightarrow d(x,y) \geq 0$

(M2) $d(x,y) = sup_{t\in A}\,|x(t) - y(t)| = 0 \Leftrightarrow |x(t) - y(t)| = 0$

$\Leftrightarrow x(t) - y(t) = 0 \Leftrightarrow x(t) = y(t) \Leftrightarrow x = y$

(M3) $d(x,y) = sup_{t\in A}\,|x(t) - y(t)| = sup_{t\in A}\,|y(t) - x(t)| = d(y,x)$

(M4) Let $x, y, z \in B(A)$ then

$|x(t) - y(t)| = |x(t) - z(t) + z(t) - y(t)|$

$\Rightarrow |x(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)|$

$\Rightarrow sup_{t\in A}|x(t) - y(t)| \leq sup_{t\in A}|x(t) - z(t)| + sup_{t\in A}|z(t) - y(t)|$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

$\Rightarrow d$ is metric on given Space.

Function Space C[a, b]

This space is a set of all real – valued continuous functions x, y, ... which are functions of an independent real variable t and are defined and **continuous** on a given closed interval J = [a, b]. Show that this space is a metric space with defined metric $d(x, y) = max_{t\in J}|x(t) - y(t)|$

Solution Define a metric $d(x, y) = max_{t\in J}|x(t) - y(t)|$

(M1) $|x(t) - y(t)| \geq 0 \Rightarrow max_{t\in J}|x(t) - y(t)| \geq 0 \Rightarrow d(x, y) \geq 0$

(M2) $d(x, y) = max_{t\in J}|x(t) - y(t)| = 0 \Leftrightarrow |x(t) - y(t)| = 0$

$\Leftrightarrow x(t) - y(t) = 0 \Leftrightarrow x(t) = y(t) \Leftrightarrow x = y$

(M3) $d(x, y) = max_{t\in J}|x(t) - y(t)| = max_{t\in J}|y(t) - x(t)| = d(y, x)$

(M4) Let $x, y, z \in C[a, b]$ then

$|x(t) - y(t)| = |x(t) - z(t) + z(t) - y(t)|$

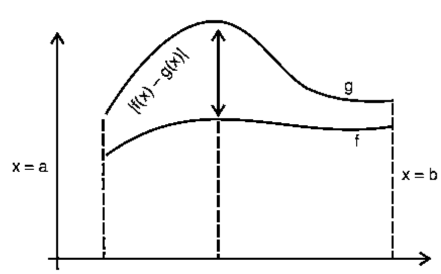
$\Rightarrow |x(t) - y(t)| \leq |x(t) - z(t)| + |z(t) - y(t)|$

$\Rightarrow max_{t\in J}|x(t) - y(t)| \leq max_{t\in J}|x(t) - z(t)| + max_{t\in J}|z(t) - y(t)|$

$\Rightarrow d(x, y) \leq d(x, z) + d(z, y) \Rightarrow d$ is metric on given Space.

This metric d is called the **sup metric/ max metirc** on C[a, b], where C[a, b] be the set of all real-valued continuous functions over [a, b].

In example, the so called supmetric or uniform metric geometrically presents maximum pointwise separation between two continuous functions f and g defined over [a, b].



Integral Metric Define d on C[a,b] by $d(x,y)=\int_a^b |x(t)-y(t)|\,dt$ then show that d is metric on given Space.

Solution Define a metric $d(x,y)=\int_a^b |x(t)-y(t)|\,dt \quad \forall t\in[a,b]\,, a<b$

(M1) $|x(t)-y(t)|\geq 0 \Rightarrow \int_a^b |x(t)-y(t)|\,dt \geq 0 \Rightarrow d(x,y)\geq 0$

(M2) $d(x,y)=\int_a^b |x(t)-y(t)|\,dt = 0 \Leftrightarrow |x(t)-y(t)|=0$

$\Leftrightarrow x(t)-y(t)=0 \Leftrightarrow x(t)=y(t) \Leftrightarrow x=y \quad \forall t\in[a,b]$

(M3) $d(x,y)=\int_a^b |x(t)-y(t)|\,dt = \int_a^b |y(t)-x(t)|\,dt = d(y,x)$

(M4) Let $x,y,z\in C[a,b]$ then $|x(t)-y(t)|=|x(t)-z(t)+z(t)-y(t)|$

$\Rightarrow |x(t)-y(t)|\leq |x(t)-z(t)|+|z(t)-y(t)|$

$$\Rightarrow \int_a^b |x(t)-y(t)|\,dt \leq \int_a^b |x(t)-z(t)|\,dt + \int_a^b |z(t)-y(t)|\,dt$$

$\Rightarrow d(x,y)\leq d(x,z)+d(z,y) \Rightarrow d$ is metric on given Space.

Pseudometric Let X be a non-empty set. A function $d: X\times X\to \mathbb{R}^+$ is said to be pseudometric if and only if

i. $d(x,x)=0\,;\forall\ x\in X$
ii. $d(x,y)=d(y,x)\ ;\forall x,y\in X$ (Symmetry)
iii. $d(x,y)\leq d(x,\ z)+d(z,y)$ (Triangle Inequality).

Or

A pseudometric satisfies all axioms of a metric except $d(x,x)=0$ may not imply x = y but x = y implies $d(x,x)=0$.

Example: $d(x,y)=|x_1-y_1|$ is a pseudometric on $\mathbb{R}^2$.

With $x=(2,3), y=(2,5) \Rightarrow d(x,y)=|2-2|=0$ but $x\neq y$

Note: Every metric is a Pseudometric, but pseudometric is not metric.

Question

Find all metrices on a set X consisting of two points, consisting of one point.

Solution

For two points (say) $x, y \in X$

(M1) Obviously $d(x,y) \geq 0 \,; \forall x, y \in X$

(M2) $d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = d(y,x); \forall x, y \in X$

(M4) This point is the consequence of (M1) to (M3).

Thus any two points of X, satisfying (M1) to (M3) is a metric on X.

If X has only one point then (M3) to (M4) are trivial. i.e.

$d(x,y) = d(x,x) = 0$. Thus any non – negative function is a metric on X.

Question

Let d be a metric on X, determine all constants ‘k’ such that

i. kd ii. $k + d$

Metric on X.

Solution

If X has more than one point then zero function is not metric, this implies $k \neq 0$. Hence generally show that for any positive real number k, lead to kd being a metric on X. i.e.

Let d be a metric on X then

i. For $\boldsymbol{k > 0}$
(M1) Obviously $kd(x,y) \geq 0 \,; \forall x, y \in X$ since $d(x,y) \geq 0$
(M2) $kd(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$
(M3) $kd(x,y) = kd(y,x); \forall x, y \in X$ since d is metric.
(M4) Since d is metric for all $x, y, z \in X$ then

$d(x, z) \leq d(x, y) + d(y, z)$
$kd(x, z) \leq kd(x, y) + kd(y, z)$

Hence kd is a metric for all $k > 0$ on X.

For $\boldsymbol{k < 0}$
Obviously $kd(x, y) \leq 0$; $\forall x, y \in X$ since $d(x, y) \geq 0$

Hence kd is not a metric for all $k < 0$ on X.

ii. To find all constant k such that $k + d$ is a metric.
For $\boldsymbol{k > 0}$
(M1) Obviously $k + d(x, y) \geq 0$; $\forall x, y \in X$ since $d(x, y) \geq 0$
(M2) if $x = y \Leftrightarrow d(x, y) = 0$ but $k + d(x, y) \neq 0$ for $k > 0$

Hence $k + d$ is not a metric for all $k > 0$ on X.

For $\boldsymbol{k < 0}$

Since $d(x, y) \geq 0$ but we may not sure about $k + d(x, y)$ be non – negative for $k < 0$. Hence $k + d$ is not a metric for all $k < 0$ on X.

For $\boldsymbol{k = 0}$

$k + d$ is a metric because actually $k + d = d$ and d is a metric on X.

$\boldsymbol{l^p}$ Space

Let $p \geq 1$ be a fixed real number. Then the space of all sequences $x = (\xi_j)_1^\infty = (\xi_1, \xi_2, \dots)$ of numbers such that $\sum_1^\infty |\xi_j|^p = |\xi_1|^p + |\xi_2|^p + \cdots$ converges is called the l^p space. i.e. $l^p = \left\{x = (\xi_j)_1^\infty : \sum_{j=1}^\infty |\xi_j|^p < \infty\right\}$

The metric d defined on l^p space is $d(x, y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p}$

Where $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty$ and $\sum_{j=1}^\infty |\xi_j|^p < \infty, \sum_{j=1}^\infty |\eta_j|^p < \infty.$

Remember: In Mathematics l^p spaces are function spaces define using a natural generalization of the *p – norm* for finite dimensional vector spaces. They are sometime called Lebesgue Space named after Henry Lebesgue.

Real and Complex $\boldsymbol{l^p}$ Space

If we take only real sequences $\xi_{j's}$ (satisfying $\sum_{j=1}^\infty |\xi_j|^p < \infty$), we get the **real $\boldsymbol{l^p}$ space**, and if we take complex sequences $\xi_{j's}$ (satisfying $\sum_{j=1}^\infty |\xi_j|^p < \infty$), we get the **complex $\boldsymbol{l^p}$ space**.

$\boldsymbol{l^2}$ Space/ Hilbert Sequence Space

Let $p = 2$ be a fixed real number. Then the space of all sequences such that $x = (\xi_j)_1^\infty = (\xi_1, \xi_2, \dots)$ of numbers such that $\sum_1^\infty |\xi_j|^2$ converges is called the l^2 space. i.e. $l^2 = \left\{x = (\xi_j)_1^\infty : \sum_{j=1}^\infty |\xi_j|^2 < \infty\right\}$

The metric d defined on l^2 space is $d(x, y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^2\right)^{1/2}$

Where $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty$ and $\sum_{j=1}^\infty |\xi_j|^2 < \infty, \sum_{j=1}^\infty |\eta_j|^2 < \infty.$

This space was introduced and studied by D. Hilbert (1912) in connection with integral equations and is the earliest example of what is now called a Hilbert space.

Example

Show that $l^p = \left\{x = (\xi_j)_1^\infty : \sum_{j=1}^\infty |\xi_j|^p < \infty\right\}$ with $d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p}$ is a metric space.

Where $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty$ and $\sum_{j=1}^\infty |\xi_j|^p < \infty, \sum_{j=1}^\infty |\eta_j|^p < \infty$.

Solution Define a metric $d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p}$

(M1) $|\xi_j - \eta_j| \geq 0 \Rightarrow \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p} \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall j$

(M2) $d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p} = 0 \Leftrightarrow |\xi_j - \eta_j| = 0$

$\Leftrightarrow (\xi_j) = (\eta_j)\ \forall j \Leftrightarrow x = y$

(M3) $d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p} = \left(\sum_{j=1}^\infty |\eta_j - \xi_j|^p\right)^{1/p} = d(y,x)$

(M4) Let $x, y, z \in l^p$ then

$$d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p} = \left(\sum_{j=1}^\infty |\xi_j - e_j + e_j - \eta_j|^p\right)^{1/p}$$

$$d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^\infty |\xi_j - e_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^\infty |e_j - \eta_j|^p\right)^{1/p}$$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \Rightarrow d$ is metric on given Space.

Example

Show that $l^2 = \left\{x = \left(\xi_j\right)_1^\infty : \sum_{j=1}^\infty \left|\xi_j\right|^2 < \infty\right\}$ with $d(x, y) = \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2}$ is a metric space on $X = \mathbb{R}^n$.

Where $x = \left(\xi_j\right)_1^\infty, y = \left(\eta_j\right)_1^\infty$ and $\sum_{j=1}^\infty \left|\xi_j\right|^2 < \infty, \sum_{j=1}^\infty \left|\eta_j\right|^2 < \infty$.

Solution Define a metric $d(x, y) = \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2}$

(M1) $\left|\xi_j - \eta_j\right| \geq 0 \Rightarrow \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2} \geq 0 \Rightarrow d(x, y) \geq 0 \; ; \forall j$

(M2) $d(x, y) = \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2} = 0 \Leftrightarrow \left|\xi_j - \eta_j\right| = 0$

$\Leftrightarrow \left(\xi_j\right) = \left(\eta_j\right) \; \forall j \Leftrightarrow x = y$

(M3) $d(x, y) = \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2} = \left(\sum_{j=1}^\infty \left|\eta_j - \xi_j\right|^2\right)^{1/2} = d(y, x)$

(M4) Let $x, y, z \in l^2$ then

$$d(x, y) = \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2} = \left(\sum_{j=1}^\infty \left|\xi_j - e_j + e_j - \eta_j\right|^2\right)^{1/2}$$

$$d(x, y) = \left(\sum_{j=1}^\infty \left|\xi_j - \eta_j\right|^2\right)^{1/2} \leq \left(\sum_{j=1}^\infty \left|\xi_j - e_j\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^\infty \left|e_j - \eta_j\right|^2\right)^{1/2}$$

$\Rightarrow d(x, y) \leq d(x, z) + d(z, y) \Rightarrow d$ is metric on given Space.

Example

Let l^1 denote the set of all sequences $x = (\xi_j)_1^\infty$ in $\mathbb{R}$ where $\sum_{j=1}^{\infty}|\xi_j|$ converges. Define $d: l^1 \times l^1 \to \mathbb{R}$ by $d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j|$ for all $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty$ then show that d is a metric space on l^1 .

Where $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty$ and $\sum_{j=1}^{\infty}|\xi_j|^2 < \infty, \sum_{j=1}^{\infty}|\eta_j|^2 < \infty$.

Solution Define a metric $d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j|$

(M1) $|\xi_j - \eta_j| \geq 0 \Rightarrow \sum_{j=1}^{\infty}|\xi_j - \eta_j| \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall j$

(M2) $d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j| = 0 \Leftrightarrow |\xi_j - \eta_j| = 0$

$\Leftrightarrow (\xi_j) = (\eta_j)\ \forall j \Leftrightarrow x = y$

(M3) $d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j| = \sum_{j=1}^{\infty}|\eta_j - \xi_j| = d(y,x)$

(M4) Let $x, y, z \in l^1$ then

$d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j| = \sum_{j=1}^{\infty}|\xi_j - e_j + e_j\eta_j|$

$d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j| \leq \sum_{j=1}^{\infty}|\xi_j - e_j| + \sum_{j=1}^{\infty}|e_j - \eta_j|$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y) \Rightarrow d$ is metric on given Space.

Example

Let $l^\infty = \{x = (\xi_j) : Sup_j|\xi_j| < \infty\}$. Define $d(x,y) = Sup_j|\xi_j - \eta_j|$ for all $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty$ then show that d is a metric space on l^∞ .

Solution Define a metric $d(x,y) = \sum_{j=1}^{\infty}|\xi_j - \eta_j|$

(M1) $|\xi_j - \eta_j| \geq 0 \Rightarrow Sup_j|\xi_j - \eta_j| \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall j$

(M2) $d(x,y) = Sup_j|\xi_j - \eta_j| = 0 \Leftrightarrow |\xi_j - \eta_j| = 0$

$\Leftrightarrow (\xi_j) = (\eta_j)\ \forall j \Leftrightarrow x = y$

(M3) $d(x,y) = Sup_j|\xi_j - \eta_j| = Sup_j|\eta_j - \xi_j| = d(y,x)$

(M4) Let $x = (\xi_j), y = (\eta_j), z = (\lambda_j)$

$|\xi_j - \eta_j| = |\xi_j - \lambda_j + \lambda_j - \eta_j| \leq |\xi_j - \lambda_j| + |\lambda_j - \eta_j|$

$\Rightarrow Sup_j|\xi_j - \eta_j| \leq Sup_j|\xi_j - \lambda_j| + Sup_j|\eta_j - \lambda_j|$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

$\Rightarrow Sup_j|\xi_j - \eta_j|$ satisfy (M4) for l^∞.

Question

If A is a subspace of l^∞ consisting of all sequences of zeros and ones. What is induced metric on A?

Answer

For any distinct $x, y \in A$ we have $d(x,y) = 1$ because there are sequences of zeros and ones, also $d(x,y) = 0$ which are not distinct. Thus induced metric on A is discrete metric.

Conjugate Index

Let p be a real number $(p > 1)$. A real number q is said to be conjugate index of p if $\frac{1}{p} + \frac{1}{q} = 1$. i.e. if $p = 2$ then $\frac{1}{2} + \frac{1}{q} = 1$ gives $q = 2$.

Auxiliary Inequality

Let $\propto, \beta$ be any positive real numbers. And let $p > 1$ define q such that $\frac{1}{p} + \frac{1}{q} = 1$ p and q are then called **conjugate exponents** then prove that $\propto \beta \leq \frac{\propto^p}{p} + \frac{\beta^q}{q}$.

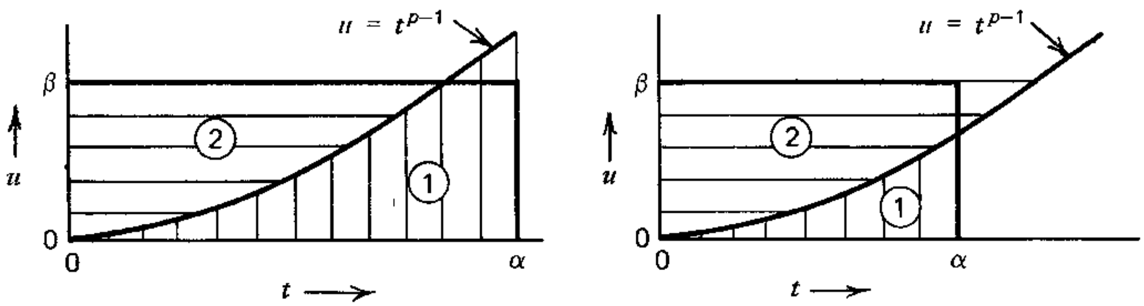
Proof: We have $\frac{1}{p} + \frac{1}{q} = 1$ then $p + q = pq$ so that $1 + \frac{q}{p} = q$ then

$(1 - p)(1 - q) = 1 \Rightarrow (p - 1)(q - 1) = 1 \Rightarrow \frac{1}{p-1} = q - 1$

Consider the function $u = t^{p-1}$; $0 \leq t \leq \infty$

$\Rightarrow t = u^{\frac{1}{p-1}} \Rightarrow t = u^{q-1}$

Consider the following figure



Let $\propto$ and β be any positive numbers. Since $\propto \beta$ is the area of the rectangle in Figure, we thus obtain by integration the inequality

$$\propto \beta \leq \int_0^{\propto} t^{p-1} dt + \int_0^{\beta} u^{q-1} du = \left|\frac{t^p}{p}\right|_0^{\propto} + \left|\frac{u^q}{q}\right|_0^{\beta} = \left(\frac{\propto^p}{p} - \frac{0^p}{p}\right) + \left(\frac{\beta^q}{q} - \frac{0^q}{q}\right)$$

$$\propto \beta \leq \frac{\propto^p}{p} + \frac{\beta^q}{q}$$

We may use another form as $\qquad \propto^{\frac{1}{p}} \beta^{\frac{1}{q}} \leq \frac{\propto}{p} + \frac{\beta}{q}$

Question

Using $\propto\beta \le \frac{\propto^p}{p} + \frac{\beta^q}{q}$ (Auxiliary Inequality) show that geometric mean does not exceed arithmetic mean.

Solution: We have $\propto\beta \le \frac{\propto^p}{p} + \frac{\beta^q}{q}$ then choose $p = q = 2$ so $\frac{1}{p} = \frac{1}{q} = \frac{1}{2}$ then

$$\Rightarrow \propto\beta \le \frac{\propto^2}{2} + \frac{\beta^2}{2} \Rightarrow 2\propto\beta \le \propto^2 + \beta^2 \Rightarrow 2\propto\beta + 2\propto\beta \le \propto^2 + \beta^2 + 2\propto\beta$$

$$\Rightarrow 4\propto\beta \le (\propto + \beta)^2 \Rightarrow \propto\beta \le \left(\frac{\propto+\beta}{2}\right)^2 \Rightarrow \sqrt{\propto\beta} \le \frac{\propto+\beta}{2}$$

$$\Rightarrow \boldsymbol{G.M \le A.M}$$

Some people are afraid of heights.

Not me,

I'm afraid of widths.

Steven Wright

The Holder Inequality

Let $x = (\xi_j)_1^\infty \in l^p, y = (\eta_j)_1^\infty \in l^q$ where $p > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$ then prove that

$\sum_{j=1}^{\infty}|\xi_j\eta_j| \leq (\sum_{k=1}^{\infty}|\xi_k|^p)^{1/p}(\sum_{m=1}^{\infty}|\eta_m|^q)^{1/q}$

This inequality was given by O. Holder (1889).

Proof:

Let $x = (\bar{\xi}_j)_1^\infty \in l^p, y = (\bar{\eta}_j)_1^\infty \in l^q$ such that $\sum_{j=1}^{\infty}|\bar{\xi}_j|^p = 1$ and $\sum_{j=1}^{\infty}|\bar{\eta}_j|^q = 1$, and if we let $|\bar{\xi}_j| = \propto$ and $|\bar{\eta}_j| = \beta$ then by auxiliary inequality

$\propto \beta \leq \frac{\propto^p}{p}+\frac{\beta^q}{q} \Rightarrow |\bar{\xi}_j||\bar{\eta}_j| \leq \frac{|\bar{\xi}_j|^p}{p}+\frac{|\bar{\eta}_j|^q}{q} \Rightarrow \sum_{j=1}^{\infty}|\xi_j\eta_j| \leq \frac{\sum_{j=1}^{\infty}|\bar{\xi}_j|^p}{p}+\frac{\sum_{j=1}^{\infty}|\bar{\eta}_j|^q}{q}$

$\Rightarrow \sum_{j=1}^{\infty}|\bar{\xi}_j\bar{\eta}_j| \leq \frac{1}{p}+\frac{1}{q} = 1 \Rightarrow \sum_{j=1}^{\infty}|\bar{\xi}_j\bar{\eta}_j| \leq 1$ …………..(1)

Let $x = (\bar{\xi}_j)_1^\infty \in l^p, y = (\bar{\eta}_j)_1^\infty \in l^q$ then define $x = (\hat{\xi}_j)_1^\infty \in l^p, y = (\hat{\eta}_j)_1^\infty \in l^q$ such that $\hat{\xi}_j = \frac{\xi_j}{(\sum_{k=1}^{\infty}|\xi_k|^p)^{1/p}}$ and $\hat{\eta}_j = \frac{\eta_j}{(\sum_{m=1}^{\infty}|\eta_m|^q)^{1/q}}$ then $\sum_{j=1}^{\infty}|\hat{\xi}_j|^p = 1$ and $\sum_{j=1}^{\infty}|\hat{\eta}_j|^q = 1$ then applying (1) to both $(\hat{\xi}_j)_1^\infty$ and $(\hat{\eta}_j)_1^\infty$ we get

$\sum_{j=1}^{\infty}|\hat{\xi}_j\hat{\eta}_j| \leq 1$

$\Rightarrow \sum_{j=1}^{\infty}\left|\frac{\xi_j}{(\sum_{k=1}^{\infty}|\xi_k|^p)^{1/p}} \times \frac{\eta_j}{(\sum_{m=1}^{\infty}|\eta_m|^q)^{1/q}}\right| \leq 1$

$$\Rightarrow \sum_{j=1}^{\infty}|\boldsymbol{\xi_j\eta_j}| \leq \left(\sum_{k=1}^{\infty}|\boldsymbol{\xi_k}|^{\boldsymbol{p}}\right)^{\boldsymbol{1/p}}\left(\sum_{m=1}^{\infty}|\boldsymbol{\eta_m}|^{\boldsymbol{q}}\right)^{\boldsymbol{1/q}}$$

The Minkowski Inequality

Let $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty \in l^p$ where $p \geq 1$ then prove that

$$\left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{\infty} |\eta_m|^p\right)^{\frac{1}{p}}$$

For finite sums this inequality was given by H. Minkowski (1896).

Proof:

Let $p = 1$ then given inequality is trivially true, since $|\xi_j + \eta_j| \leq |\xi_j| + |\eta_j|$ therefore $\quad \sum_{j=1}^{\infty} |\xi_j + \eta_j| \leq \sum_{k=1}^{\infty} |\xi_k| + \sum_{m=1}^{\infty} |\eta_m|$

If $p > 1$ then suppose that $w_j = \xi_j + \eta_j$ then

$$|w_j|^p = |w_j|^{1+p-1} = |w_j||w_j|^{p-1} = |\xi_j + \eta_j||w_j|^{p-1} \leq |\xi_j||w_j|^{p-1} + |\eta_j||w_j|^{p-1}$$

$$|w_j|^p \leq |\xi_j||w_j|^{p-1} + |\eta_j||w_j|^{p-1}$$

$$\sum_{j=1}^{n} |w_j|^p \leq \sum_{j=1}^{n} |\xi_j||w_j|^{p-1} + \sum_{j=1}^{n} |\eta_j||w_j|^{p-1} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

Where ‘n’ is fixed arbitrarily

Now consider $I = \sum_{j=1}^{n} |\xi_j||w_j|^{p-1}$ then obviously $(|\xi_j|)_1^\infty \in l^p$

We claim that $\left(|w_j|^{p-1}\right)_1^\infty \in l^q$, to see this we consider

$$\left(\sum_{j=1}^{n} |w_j|^{p-1}\right)^q = \sum_{j=1}^{n} |w_j|^{(p-1)q} = \sum_{j=1}^{n} |w_j|^p < \infty \qquad \because \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \left(|w_j|^{p-1}\right)_1^\infty \in l^q$$

Then by Holder’s Inequality we get

$$\sum_{j=1}^{n} |\xi_j||w_j|^{p-1} \leq \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |w_j|^{(p-1)q}\right)^{\frac{1}{q}} \qquad \ldots\ldots\ldots\ldots\ldots(2)$$

Similarly

$\sum_{j=1}^{n}|\eta_j||w_j|^{p-1} \le (\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|w_j|^{(p-1)q}\right)^{\frac{1}{q}}$ ……………(3)

Putting (2),(3) in (1)

$$\sum_{j=1}^{n}|w_j|^p \le (\sum_{k=1}^{n}|\xi_k|^p)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|w_j|^{(p-1)q}\right)^{\frac{1}{q}} + (\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}\left(\sum_{j=1}^{n}|w_j|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$\sum_{j=1}^{n}|w_j|^p \le \left[(\sum_{k=1}^{n}|\xi_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}\right]\left(\sum_{j=1}^{n}|w_j|^{(p-1)q}\right)^{\frac{1}{q}}$$

$$\sum_{j=1}^{n}|w_j|^p \le \left[(\sum_{k=1}^{n}|\xi_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}\right]\left(\sum_{j=1}^{n}|w_j|^{p}\right)^{\frac{1}{q}} \qquad \because \frac{1}{p}+\frac{1}{q}=1$$

$$\left(\sum_{j=1}^{n}|w_j|^p\right)^{1-\frac{1}{q}} \le (\sum_{k=1}^{n}|\xi_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}$$

$$\left(\sum_{j=1}^{n}|w_j|^p\right)^{\frac{1}{p}} \le (\sum_{k=1}^{n}|\xi_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}$$

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n}|w_j|^p\right)^{\frac{1}{p}} \le \lim_{n\to\infty}(\sum_{k=1}^{n}|\xi_k|^p)^{\frac{1}{p}} + \lim_{n\to\infty}(\sum_{m=1}^{n}|\eta_m|^p)^{\frac{1}{p}}$$

$$\left(\sum_{j=1}^{\infty}|w_j|^p\right)^{\frac{1}{p}} \le (\sum_{k=1}^{\infty}|\xi_k|^p)^{\frac{1}{p}} + (\sum_{m=1}^{\infty}|\eta_m|^p)^{\frac{1}{p}}$$

$$\boldsymbol{\left(\sum_{j=1}^{\infty}|\xi_j+\eta_j|^p\right)^{1/p} \le \left(\sum_{k=1}^{\infty}|\xi_k|^p\right)^{\frac{1}{p}} + \left(\sum_{m=1}^{\infty}|\eta_m|^p\right)^{\frac{1}{p}}}$$

Cauchy – Schwarz Inequality from the Holder Inequality

Let $x=(\xi_j)_1^{\infty}\in l^p, y=(\eta_j)_1^{\infty}\in l^q$ where $p=2, q=2$ then

$$\sum_{j=1}^{\infty}|\xi_j\eta_j| \le (\sum_{k=1}^{\infty}|\xi_k|^2)^{1/2}(\sum_{m=1}^{\infty}|\eta_m|^2)^{1/2}$$

Or $\left(\sum_{j=1}^{\infty}|\xi_j\eta_j|\right)^2 \le \sum_{k=1}^{\infty}|\xi_k|^2\sum_{m=1}^{\infty}|\eta_m|^2$

Question

By using Cauchy – Schwarz Inequality show that

$(|\xi_1 + \xi_2 + \cdots + \xi_n|)^2 \leq n(|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)$

Solution

By Cauchy – Schwarz Inequality we have

$\sum_{j=1}^{\infty}|\xi_j\eta_j| \leq (\sum_{k=1}^{\infty}|\xi_k|^2)^{1/2}(\sum_{m=1}^{\infty}|\eta_m|^2)^{1/2}$

For $\eta_j = 1$

$\sum_{j=1}^{\infty}|\xi_j.1| \leq (\sum_{k=1}^{\infty}|\xi_k|^2)^{1/2}(\sum_{m=1}^{\infty}|1|^2)^{1/2}$

$\sum_{j=1}^{\infty}|\xi_j| \leq (\sum_{k=1}^{\infty}|\xi_k|^2)^{1/2}(\sum_{m=1}^{\infty} 1)^{1/2}$

$\left(\sum_{j=1}^{\infty}|\xi_j|\right)^2 \leq (\sum_{k=1}^{\infty}|\xi_k|^2)(1 + 1 + \cdots + 1)$

$\left(\sum_{j=1}^{\infty}|\xi_j|\right)^2 \leq (\sum_{k=1}^{\infty}|\xi_k|^2)\big(n(1)\big)$

$\left(\sum_{j=1}^{\infty}|\xi_j|\right)^2 \leq (\sum_{k=1}^{\infty}|\xi_k|^2)(n)$

$\left(\sum_{j=1}^{\infty}|\xi_j|\right)^2 \leq n(\sum_{k=1}^{\infty}|\xi_k|^2)$

$(|\xi_1 + \xi_2 + \cdots + \xi_n|)^2 \leq n(|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2)$

Hence proved.

Example-23: *Let X be a set of all 2×2 matrices. If for any $A=\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$, $B=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\in X$, $d(A,B)=\sqrt{\sum_{n=1}^{4}(a_n-b_n)^2}$, then show that d is a metric on X.*

Solution: M_1): $d(A,B)=\sqrt{\sum_{n=1}^{4}(a_n-b_n)^2}\geq 0$

M_2): $d(A,B)=0 \Leftrightarrow \sqrt{\sum_{n=1}^{4}(a_n-b_n)^2}=0$

$$\Leftrightarrow \sqrt{(a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2+(a_4-b_4)^2}=0$$

$$\Leftrightarrow (a_1-b_1)^2+(a_2-b_2)^2+(a_3-b_3)^2+(a_4-b_4)^2=0$$

$$\Leftrightarrow (a_1-b_1)^2=0,\ (a_2-b_2)^2=0,\ (a_3-b_3)^2=0,\ (a_4-b_4)^2=0$$

$$\Leftrightarrow a_1-b_1=0,\ a_2-b_2=0,\ a_3-b_3=0,\ a_4-b_4=0$$

$$\Leftrightarrow a_1=b_1,\ a_2=b_2,\ a_3=b_3,\ a_4=b_4$$

$$\Leftrightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}=\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$$

$$\Leftrightarrow A=B$$

M_3): $d(A,B)=\sqrt{\sum_{n=1}^{4}(a_n-b_n)^2}=\sqrt{\sum_{n=1}^{4}(b_n-a_n)^2}=d(B,A)$

M_4): $d(A,C)=\sqrt{\sum_{n=1}^{4}(a_n-c_n)^2}=\sqrt{\sum_{n=1}^{4}(a_n-b_n+b_n-c_n)^2}$

Let $x_n=a_n-b_n, y_n=b_n-c_n$, then the last equation takes the form

$$d(A,C)=\left|\sqrt{\sum_{n=1}^{4}(x_n+y_n)^2}\right| \qquad \ldots(1)$$

Using Minkoski's inequality, we have

$$\sqrt{\sum_{n=1}^{4}(x_n+y_n)^2}\leq\sqrt{\sum_{n=1}^{4}x_n^2}+\sqrt{\sum_{n=1}^{4}y_n^2}$$

$$\Rightarrow \left|\sqrt{\sum_{n=1}^{4}(x_n+y_n)^2}\right| \le \left|\sqrt{\sum_{n=1}^{4}x_n^2}+\sqrt{\sum_{n=1}^{4}y_n^2}\right| \le \left|\sqrt{\sum_{n=1}^{4}x_n^2}\right| + \left|\sqrt{\sum_{n=1}^{4}y_n^2}\right|$$

Using this in (1), we have

$$d(A,C) = \left|\sqrt{\sum_{n=1}^{4}(x_n+y_n)^2}\right| \le \left|\sqrt{\sum_{n=1}^{4}x_n^2}\right| + \left|\sqrt{\sum_{n=1}^{4}y_n^2}\right|$$

$$\Rightarrow d(A,C) \le \left|\sqrt{\sum_{n=1}^{4}(a_n-b_n)^2}\right| + \left|\sqrt{\sum_{n=1}^{4}(b_n-c_n)^2}\right|$$

$$\Rightarrow d(A,C) \le d(A,B)+d(B,C)$$

This shows that d is a metric on X.

Bounded Metric: A metric d on X is said to be *bounded metric* on X if there exists a positive number M such that $d(x,y) \le M$ for every pair of points x and y of X. In this case, the metric space X is said to be a *bounded space*.

If the metric d is not bounded, it is called an *unbounded metric* and the metric space is said to be an *unbounded space*.

The discrete metric is an example of a bounded metric, because $d(x,y) \le 1$ for all pairs of points x and y of discrete metric space.

Example-20: *If (X,d) is a metric space, and $k>0$ then show that $d_1(x,y) = \dfrac{kd(x,y)}{1+kd(x,y)}$ is a bounded metric on X.*

Solution: M_1): $d_1(x,y) = \dfrac{kd(x,y)}{1+kd(x,y)} \ge 0 \qquad \because k>0, d(x,y) \ge 0$

$$M_2\text{)}: d_1(x,y) = 0 \Leftrightarrow \frac{kd(x,y)}{1+kd(x,y)} = 0$$

$$\Leftrightarrow kd(x,y) = 0$$

$$\Leftrightarrow d(x,y) = 0 \qquad \because k \neq 0$$

$$\Leftrightarrow x = y \qquad \because d \text{ is a metric}$$

$$M_3\text{)}: d_1(x,y) = \frac{kd(x,y)}{1+kd(x,y)} = \frac{kd(y,x)}{1+kd(y,x)} \qquad \because d \text{ is a metric}$$

M_4): Since d is a metric on X, so for all $x,y,z \in X$,

$$d(x,z) \le d(x,y) + d(y,z)$$

$$\Rightarrow kd(x,z) \le kd(x,y) + kd(y,z) \because k>0$$

$$\Rightarrow 1 + kd(x,z) \le 1 + kd(x,y) + kd(y,z)$$

$$\Rightarrow \frac{1}{1+kd(x,y)+kd(y,z)} \le \frac{1}{1+kd(x,z)}$$

$$\Rightarrow -\frac{1}{1+kd(x,z)} \le -\frac{1}{1+kd(x,y)+kd(y,z)}$$

$$\Rightarrow 1-\frac{1}{1+kd(x,z)} \leq 1-\frac{1}{1+kd(x,y)+kd(y,z)}$$

$$\Rightarrow \frac{kd(x,z)}{1+kd(x,z)} \leq \frac{kd(x,y)+kd(y,z)}{1+kd(x,y)+kd(y,z)}$$

$$\Rightarrow \frac{kd(x,z)}{1+kd(x,z)} \leq \frac{kd(x,y)}{1+kd(x,y)+kd(y,z)} + \frac{kd(y,z)}{1+kd(x,y)+kd(y,z)}$$

$$\leq \frac{kd(x,y)}{1+kd(x,y)} + \frac{kd(y,z)}{1+kd(y,z)}$$

$$\Rightarrow d_1(x,z) \leq d_1(x,y)+d_1(y,z), \quad \forall x,y,z \in X$$

This shows that d_1 is also a metric on X.

$$d_1(x,y) = \frac{kd(x,y)}{1+kd(x,y)} < \frac{1+kd(x,y)}{1+kd(x,y)} = 1, \ \forall x,y \in X$$

$$\Rightarrow d_1(x,y) < 1, \ \forall x,y \in X$$

So d_1 is bounded on X.

Distance between sets

Let (X, d) be a metric space and $, A, B \subset X$. The distance between A and B denoted $d(A,B)$ is defined as $d(A,B) = inf\{d(a,b): a \in A, b \in B\}$
If $A = \{x\}$ is a singleton subset of X, then $d(A,B)$ is written as $d(x,B)$ and is called distance of point x from the set B.

Theorem

If (X,d) is a metric space then show that $|d(x,A) - d(y,A)| \leq d(x,y)$.

Solution Let $z \in A$ then $d(x,z) \leq d(x,y) + d(y,z)$

$\Rightarrow inf_{z\in A} d(x,z) \leq d(x,y) + inf_{z\in A} d(y,z)$

$\Rightarrow d(x,A) \leq d(x,y) + d(y,A) \Rightarrow d(x,A) - d(y,A) \leq d(x,y)$ ………(1)

$d(y,z) \leq d(y,x) + d(x,z)$

$\Rightarrow inf_{z\in A} d(y,z) \leq d(y,x) + inf_{z\in A} d(x,z)$

$\Rightarrow d(y,A) \leq d(x,y) + d(x,A) \Rightarrow -\big(d(x,A) - d(y,A)\big) \leq d(x,y)$ ………(2)

Combining (1) and (2) $|d(x,A) - d(y,A)| \leq d(x,y)$.

Diameter of a set

The diameter $d(A)$ of a nonempty set $A \subset X$ in a metric space (X,d) is defined to be $d(A) = Sup_{x,y\in A}\, d(x,y)$.

Note: For an empty set φ , following convention are adopted

(i) $d(\varphi) = -\infty$ Some authors take $d(\varphi)$ also as 0.

(ii) $d(p,\varphi) = \infty$ i.e. distance of a point p from empty set is ∞.

(iii) $d(A,\varphi) = \infty$, where A is any non-empty set.

Bounded Set

Let (X,d) be a metric space and $A \subset X$ be a non-empty set. 'A' is said to be bounded if diameter of A is finite. i.e. $d(A) < \infty$

Theorem Show that the union of two bounded sets A and B in a metric space is a bounded set.

Proof Let (X, d) be a metric space and $A, B \subset X$ be a non-empty set. We wish to prove $A \cup B$ is bounded.

Let $x, y \in A \cup B$ then clearly $x, y \in A$ or $x, y \in B$

If $x, y \in A$ then since A is bounded therefore $d(x, y) < \infty$

And hence $d(A \cup B) = Sup_{x,y \in A \cup B}\, d(x, y) < \infty$ Implies $A \cup B$ is bounded.

Similarly if $x, y \in B$ then $A \cup B$ is bounded.

Now if $x \in A$ and $y \in B$ then

$d(x, y) \leq d(x, a) + d(a, b) + d(b, y)$ where $a \in A$ and $b \in B$

Since $d(x, a), d(a, b), d(b, y)$ are finite therefore $d(x, y) < \infty$. And hence $A \cup B$ is bounded.

Theorem Show that $d(A) = 0$ iff A consists of single point.

Proof For this consider

$d(A) = 0 \Leftrightarrow Sup_{x,y \in A}\, d(x, y) = 0 \Leftrightarrow d(x, y) = 0 \Leftrightarrow x = y \in A \Leftrightarrow \{x\} = A$

Hence $d(A) = 0$ iff A consists of single point.

Question If $A \cap B \neq \varphi$ then $d(A, B) = 0$. What about converse?

Proof For this consider $A \cap B \neq \varphi$ and

Let $x \in A \cap B$ then clearly $x \in A$ and $x \in B$

If $x \in A$ and $x \in B$ then $d(x, x) = 0$ since $d(x, y) = 0 \Leftrightarrow x = y$

Then $Sup_{\substack{x \in A \\ x \in B}}\, d(x, x) = 0 \Rightarrow d(A, B) = 0$

Conversly suppose that $d(A, B) = 0 \Rightarrow Sup_{\substack{x \in A \\ y \in B}}\, d(x, y) = 0$

$\Leftrightarrow d(x, y) = 0$ may or may not exists. So converse may or may not exists.

Example-24: *Show with the help of an example that the distance between two nonempty disjoint sets may be zero.*

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Solution: Let (R,d) be the metric space.

Let $A=\{n: n\in N\}$, the set of natural numbers

$$B=\left\{n-\frac{1}{n}: n\geq 2, n\in N\right\}$$

Then B has no natural numbers, so $A\cap B=\phi$. Now

$$d(A,B)=\inf\{d(a,b): a\in A, b\in B\}$$

$$=\inf\left\{d\left(n,n-\frac{1}{n}\right)\right\}$$

$$=\inf\left\{\left|n-\left(n-\frac{1}{n}\right)\right|\right\}=\inf\left\{\left|n-n+\frac{1}{n}\right|\right\}$$

$$=\inf\left\{\frac{1}{n}\right\}$$

$$=0 \qquad \because \frac{1}{n}\to 0 \text{ as } n\to\infty$$

Question Show that distance does not define a metric on power set of X.

Proof For this consider $d(A,B) = inf_{\substack{a\in A \\ b\in B}} d(a,b)$

And $X = \{1,2,3\}$ also let $A = \{1\}, B = \{1,2\}$ be subsets of X.

Let $d(A,B) = inf_{\substack{a\in A \\ b\in B}} d(a,b) = inf_{\substack{a\in A \\ b\in B}} \{d(1,1), d(1,2)\} = inf\ \{0,1\} = 0$

But $A \neq B$, M2 fails here. Thys (X,d) is not a metric on X.

Question

Show that $x \in \bar{A}$ iff $d(x,A) = 0$ here A is non – empty subset of X.

Proof Suppose $x \in \bar{A}$ then $x \in A$ or $x \in A^d$

If $x \in A$ then $inf_{y\in A}\, d(x,y)$ is obtained at $y = x$ implies $d(x,A) = 0$

If $x \in A^d$ then each neighbourhood $B_0(x)$ of x contains at least one point $y \in A$ distinct from x with $d(x,y) < \varepsilon$.

Suppose $\varepsilon \to 0$ then $d(x,y) = 0$ implies $d(x,A) = 0$.

Conversely suppose that $d(x,A) = 0$ then $inf_{x\in A}\, d(x,y) = 0$

If infimum is obtained then $x \in A \subseteq \bar{A}$

Hence $x \in \bar{A}$.

Theorem

The diameter of a closed ball $\bar{B}(a;r)$ in a metric space (X,d) is $\leq 2r$.

Proof

Let $x, y \in \bar{B}(a;r)$. Then $d(x,a) \leq r$, $d(y,a) \leq r$.

Hence $d(x,y) \leq d(x,a) + d(a,y)$

$\leq 2r.$

Thus $\delta(B(a;r)) = \sup_{x,y\in B(a;r)} d(x,y) \leq 2r.$

Open Ball Let (X, d) be a metric. An open ball in (X, d) is denoted by

$B(x_0; r) = \{x \in X: d(x, x_0) < r\}$

Where x_0 is called center of ball and r is called radius of ball and $r \geq 0$.

Closed Ball Let (X, d) be a metric. A closed ball in (X, d) is denoted by

$\tilde{B}(x_0; r) = \{x \in X: d(x, x_0) \leq r\}$

Where x_0 is called center of ball and r is called radius of ball and $r \geq 0$.

Sphere Let (X, d) be a metric. A sphere in (X, d) is denoted by

$S(x_0; r) = \{x \in X: d(x, x_0) = r\}$

Where x_0 is called center of ball and r is called radius of ball and $r \geq 0$.

Keep in mind: $S(x_0; r) = \tilde{B}(x_0; r) - B(x_0; r)$

Examples Consider the set of real numbers with usual metric

$d(x, y) = |x - y|\ \forall x, y \in \mathbb{R}$ then

$B(x_0; r) = \{x \in \mathbb{R}: d(x, x_0) < r\} = \{x \in \mathbb{R}: |x - x_0| < r\}$

i.e. $x_0 - r < x < x + r = (x_0 - r, x_0 + r)$

i.e. **open ball** is the real line with usual metric is an **open interval**.

$\tilde{B}(x_0; r) = \{x \in \mathbb{R}: d(x, x_0) \leq r\} = \{x \in \mathbb{R}: |x - x_0| \leq r\}$

i.e. $x_0 - r \leq x \leq x + r = [x_0 - r, x_0 + r]$

i.e. **closed ball** in a real line is a **closed interval**.

$S(x_0; r) = \{x \in \mathbb{R}: d(x, x_0) = r\} = \{x_0 - r, x_0 + r\}$

i.e. two points $x_0 - r$ and $x_0 + r$ are only.

Examples

Consider a discrete metric space (X,d) where $d(x,y)=\begin{cases}1 & ; x\neq y\\ 0 & ; x=y\end{cases}$ then

$B(x_0;1)=\{x\in X: d(x,x_0)<1\}=\{x_0\}$

$\tilde{B}(x_0;1)=\{x\in X: d(x,x_0)\leq 1\}=X-\{x_0\}$

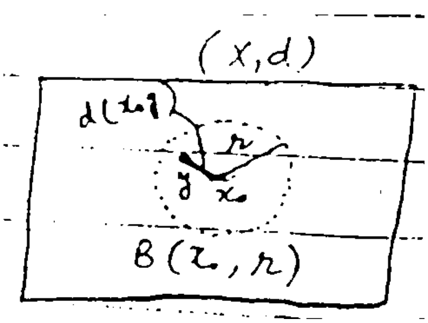
$S(x_0;r)=\{x\in X: d(x,x_0)=r\}=\varphi$ if $r\neq 1$

Open set

Let (X,d) be a metric. A subset 'A' of a metric space X is said to be open if it contains a ball about each of its points. i.e. for every $x\in A$ there exists an open ball $B(x;r)\subset A \Rightarrow x\in B(x;r)\subset A$

Theorem An open ball in metric space X is open set.

Proof Let $B(x_0;r)$ be an open ball with center x_0 and radius r in a metric space (X,d). And let $y\in B(x_0;r)$ and define $r_1=r-d(x_0,y)$



We claim that $B(y;r_1)\subseteq B(x_0;r)$

To see this let $z\in B(y;r_1)$ then

$d(z,x_0)\leq d(z,y)+d(y,x_0)<r_1+(r-r_1)=r$

$d(z,x_0)<r \Rightarrow z\in B(x_0;r)\Rightarrow B(y;r_1)\subseteq B(x_0;r)$

Hence $B(x_0;r)$ is an open set.

Question

What is an open ball in R and in C? Also in C[a,b].

Answer

- An open ball in R is an open interval $(x_0 - r, x_0 + r)$.
- An open ball in C is an open disk $D = \{z \in C: |z - x_0| < 1\}$
- An open ball in C[a,b] is any continuous function satisfying $Sup|x(t) - x_0(t)| < 1$

Question

Consider $C[0,2\pi]$ and determine the smallest r such that $y \in B(x;r)$ where $x(t) = Sint$ and $y(t) = Cost$.

Solution

Metric defined on $C[a,b]$ is $\quad d(x,y) = max_{t\in J}|x(t) - y(t)|$

Let $z(t) = Cost - Sint \quad$ then $\quad z'(t) = -Sint - Cost$

Put $z'(t) = 0$ then $-Sint - Cost = 0$

$\Rightarrow Sint = -Cost \Rightarrow \frac{Sint}{Cost} = -1 \Rightarrow Tant = -1 \Rightarrow t = \frac{3\pi}{4}, \frac{7\pi}{4}$ put in $z(t)$

$$z(t) = Cost - Sint = Cos\left(\frac{3\pi}{4}\right) - Sin\left(\frac{3\pi}{4}\right) = Cos\left(\frac{\pi}{2} + \frac{\pi}{4}\right) - Sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$z(t) = Cost - Sint = -Sin\left(\frac{\pi}{4}\right) - Cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$$

Also

$$z(t) = Cost - Sint = Cos\left(\frac{7\pi}{4}\right) - Sin\left(\frac{7\pi}{4}\right) = Cos\left(\frac{3\pi}{2} + \frac{\pi}{4}\right) - Sin\left(\frac{3\pi}{2} + \frac{\pi}{4}\right)$$

$$z(t) = Cost - Sint = Sin\left(\frac{\pi}{4}\right) + Cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

Thus he smallest r such that $y \in B(x;r)$ is $\sqrt{2}$

Question If $A \subseteq B$ then $\delta(A) \leq \delta(B)$.

Solution: Given that $A \subseteq B$. Then we know that for $x, a \in A$ and $y, b \in B$ we have

$d(x,a) \leq d(y,b) \Rightarrow Sup_{x,a\in A} d(x,a) \leq Sup_{y,b\in B} d(y,b) \Rightarrow \delta(A) \leq \delta(B)$

Theorem (a) Let (X,d) be a metric space then X and φ are open sets.

Proof It follows by noting that φ is open since φ has no elements and, obviously, X is open.

Theorem (b)

Let (X,d) be a metric space then Union of any number of open sets is open.

Proof Any point 'x' of the union $\cup$ of open sets belongs to (at least) one of these sets, call it M, and M contains a ball B about x since M is open. Then $B \subset U$, by the definition of a union. This proves the result.

Theorem (c)

Let (X,d) be a metric space then Intersection of a finite number of open sets is open.

Proof If 'y' is any point of the intersection of open sets $M_1, \dots, M_n$ then each M_j contains a ball about 'y' and a smallest of these balls is contained in that intersection. This proves the result.

Limit point of a set

Let (X,d) be a metric space and $A \subset X$, then $x \in X$ is called a limit point or accumulation point of A if for every open ball $B(x;r)$ with centre x,

$B(x;r) \cap \{A - \{x\}\} \neq \varphi$ i.e. every open ball contain a point of A other than x.

Or Let (X,d) be a metric space and $A \subset X$, then $x \in X$ is called a limit point or accumulation point of A if every neighbourhood B_x of x contains at least one element of A other than 'x' i.e. $B_x \cap \{A/\{x\}\} \neq \varphi$

Theorem

Let (X, d) be a metric space and x be a limit point of a subset A but not in A. Then every open ball $B(x; r)$ contains an infinite number of points of A.

Proof

Let x be a limit point of A but not in A.

Suppose that, for an open ball $B(x; r)$,

$$B(x; r) \cap A = \{x_1, x_2, \ldots, x_n\}$$

Put $d(x, x_i) = r_i$ and $r^* = \min(r_1, r_2, \ldots, r_n)$.

Then $B(x; r^*) = \{y : d(y, x) < r^*\}$ has no point common with A. That is, $B(x; r^*) \cap A = \varnothing$. So x is not a limit point of A, a contradiction. Hence $B(x; r) \cap A$ is infinite..

Theorem Let (X,d)be a metric space and $A \subset X$. If $x \in X$ is a limit point of A. then every open ball $B(x;r)$ with centre x contain an infinite numbers of point of A.

Proof Suppose $B(x;r)$ contain only a finite number of points of A. Let $a_1, a_2, a_3, \dots a_n$ be those points. And let $d(x, a_i) = r_i$; $i = 1,2, \dots, n$.

Also consider $r' = min(r_1, r_2, \dots, r_n)$. Then the open ball $B(x;r')$contain no point of A other than x. then x is not limit point of A. This is a contradiction therefore $B(x;r)$ must contain infinite numbers of point of A.

Closed Set

A subset A of metric space X is closed if it contains every limit point of itself. The set of all limit points of A is called the derived set of A and denoted by A^c, A' .

Or A subset A of a metric space X is said to be open if it contains a ball about each of its points. A subset A of X is said to be closed if its complement (in X) is open, that is, $A^c = X - A$ is open.

Theorem

A subset A of a metric space is closed if and only if its complement A^c is open

Proof

Suppose A is closed, we prove A^c is open. For this let $x \in A^c$ then $x \notin A$. Then it means 'x' is not a limit point of A. Then by definition of a limit there exists an open ball $B(x;r)$ such that $B(x;r) \cap A = \varphi$.

This implies that $B(x;r) \subset A^c$. Since x is an arbitrary point of A^c. So A^c is open.

Conversely, assume that A^c is an open then we prove A is closed. i.e. A contain all of its limit points.

Let x be an accumulation point of A. and suppose $x \in A^c$ then there exists an open ball $B(x;r) \subset A^c$ implies $B(x;r) \cap A = \varphi$

This shows that x is not a limit point of A. this is a contradiction to our assumption.

Hence $x \in A$. Accordingly A is closed. The proof is complete.

Theorem: A closed ball is a closed set.

Proof:

Let $\bar{B}(x;r)$ be a closed ball. We have to prove $\bar{B}^c(x;r) = C$ is an open ball.

Let $y \in C$ then $d(x,y) > r$. Let $d(x,y) = r_1 > r$ and take $r_2 = r_1 - r$

Consider the open ball $B\left(y;\frac{r_2}{2}\right)$ and we are to prove that $B\left(y;\frac{r_2}{2}\right) \subset C$

Let $z \in B\left(y;\frac{r_2}{2}\right)$ then $d(z,y) < \frac{r_2}{2}$

By using Triangular Inequality $d(x,y) \leq d(x,z) + d(z,y)$

$\Rightarrow d(x,y) \leq d(z,x) + d(z,y) \Rightarrow d(x,y) - d(z,y) \leq d(z,x)$

$\Rightarrow r_1 - \frac{r_2}{2} < d(z,x) \Rightarrow \frac{2r_1 - r_2}{2} < d(z,x) \Rightarrow \frac{2r_1 - r_1 + r}{2} < d(z,x) \Rightarrow \frac{r_1 + r}{2} < d(z,x)$

$\Rightarrow \frac{r+r}{2} < d(z,x) \Rightarrow r < d(z,x) \qquad \because r_2 = r_1 - r > 0 \Rightarrow r_1 > r$

$\Rightarrow d(z,x) > r \Rightarrow z \notin \bar{B}(x;r) \Rightarrow z \in B\left(y;\frac{r_2}{2}\right) \subset C \Rightarrow \bar{B}^c(x;r) = C$ is open set.

Consequently $\bar{B}(x;r)$ is closed set.

Closure of a Set

Let (X,d) be a metric space and $M \subset X$. Then closure of M is denoted by $\bar{M} = M \cup M'$ where M' is the set of all limit points of M. It is the smallest closed superset of M.

Dense Set

Let (X,d)be a metric space the a set $M \subset X$ is called dense in X if $\bar{M} = X$.

Countable Set

A set A is countable if it is finite or there exists a function $f: A \to \mathbb{N}$which is one-one and onto, where $\mathbb{N}$ is the set of natural numbers. e.g. $\mathbb{N}, \mathbb{Q}$ and $\mathbb{Z}$ are countable sets . The set of real numbers, the set of irrational numbers and any interval are not countable sets.

Remark

If M is dense in X, then every open ball in X, no matter how small, will contains point of M

In other words, there does not point $x \in X$ which has a neighborhood that does not contain point of M.

Theorem

Let (X, d) be a metric space $M \subset X$ is dense if and only if M has non-empty intersection with any open subset of X.

Proof

Assume that M is dense in X. Then $\bar{M} = X$. Suppose there is an open set $G \subset X$ such that $M \cap G = \varphi$. Then if $x \in G$ then $M \cap (G/\{x\}) = \varphi$ which show that x is not a limit point of M.

This implies $x \notin M'$ but $x \in X$ thus $\bar{M} \neq X$. This is a contradiction.

Consequently $M \cap G \neq \varphi$ for any open $G \subset X$.

Conversely suppose that $M \cap G \neq \varphi$ for any open $G \subset X$. We prove $\bar{M} = X$.

For this let $x \in X$. If $x \in M$ then $x \in M \cup M' = \bar{M}$ then $\bar{M} = X$.

If $x \notin M$ then let $\{G_i\}$ be the family of all the open subset of X such that $x \in G_i$for every i. Then by hypothesis $M \cap G_i \neq \varphi$ for any i. i.e. G_i contain point of M other than x.

This implies that x is an accumulation point of M. i.e. $x \in M'$

Accordingly $x \in M \cup M' = \bar{M}$ then $\bar{M} = X$. The proof is complete.

Separable Space

A metric space (X, d) is said to be separable if it contains a countable dense subsets.

Examples: The real line $\mathbb{R}$ is separable.

Proof: The set $\mathbb{Q}$ of all rational numbers is countable and is dense in $\mathbb{R}$.

Examples: The complex plane $\mathbb{C} = \{x + iy: x, y \in \mathbb{R}\}$ is separable.

Proof: Consider $M = \{p + iq: p, q \in \mathbb{Q}\} \subset \mathbb{C}$. As $\mathbb{Q}$ is countable then M is countable. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ therefore $\overline{M} = \mathbb{C}$. So $\mathbb{C}$ is separable.

Examples: A discrete metric space X is separable if and only if X is countable.

Proof: Let (X, d) be a discrete metric space then $d(x,y) = \begin{cases} 1 & ; x \neq y \\ 0 & ; x = y \end{cases}$

Suppose that X is separable. Let $M \subset X$ and $x \in X/M$ then for any $m \in M$ we have $d(x,m) = 1$ because $x \neq m$. If we draw an open ball with centre x and radius $\frac{1}{3}$, then it will not intersect with M. This means that <u>no proper subset</u> of X is dense in X. Since X is separable, the only dense set in X itself. So X is countable.

Conversely suppose that X is countable. Then $\bar{X} = X$ and hence X is separable.

Examples: The space l^∞ is not separable.

Proof: Since we know that $l^\infty = \left\{x = \left(\xi_j\right)_1^\infty : Sup_j\left|\xi_j\right| < \infty\right\}$

With $d(x,y) = Sup_i\left|\xi_j - \eta_j\right|$ Where $x = \left(\xi_j\right)_1^\infty, y = \left(\eta_j\right)_1^\infty \in l^\infty$

Let $y = (\eta_1, \eta_2, \eta_3, \ldots)$ be a sequence of zeros and ones. i.e. $\eta_j = 0$ or $\eta_j = 1$. Then since $Sup_j\left|\eta_j\right| < \infty$ therefore $y \in l^\infty$. With this y construct a real number $\hat{y}$ whose binary representation is $\hat{y} = \frac{\eta_1}{2} + \frac{\eta_2}{2} + \frac{\eta_3}{2} + \cdots \in [0,1]$

We now use the facts that there is one – to – one correspondence between [0,1] and the sequences of 0's and 1's in l^∞ and the set of points in the interval [0,1] is uncountable, each $\hat{y} \in [0,1]$ has a binary representation, and different $\hat{y}'s$ have different binary representations. Hence there are uncountably many sequences of zeros and ones. The metric on l^∞ shows that any two of them which are not equal must be of distance 1 apart. If we let each of these sequences be the center of a small ball, say, of radius 1/3, these balls do not intersect and we have uncountably many of them. If M is any dense set in l^∞, each of these nonintersecting balls must contain an element of M. Hence M cannot be countable. Since M was an arbitrary dense set, this shows that l^∞ cannot have dense subsets which are countable. Consequently, l^∞ is not separable.

Examples: The space l^p with $1 \le p < \infty$ is separable.

Proof: Since we know that $l^p = \left\{x = (\xi_j)_1^\infty : \sum_{j=1}^\infty |\xi_j|^p < \infty\right\}$

With $d(x,y) = \left(\sum_{j=1}^\infty |\xi_j - \eta_j|^p\right)^{1/p}$ Where $x = (\xi_j)_1^\infty, y = (\eta_j)_1^\infty \in l^p$

To prove l^p is separable, we have to establish the existence of a set in l^p which is countable and dense in l^p.

Let M be the set of all sequences y of the form $y = (\eta_1, \eta_2, \dots, \eta_n, 0,0, \dots)$

where n is any positive integer and the $\eta_{j's}$ are rational. M is countable. Because $\mathbb{Q}$ is countable. We show that M is dense in l^p.

i.e. $M = \{y = (\eta_1, \eta_2, \dots, \eta_n, 0,0, \dots) : \eta_j \in \mathbb{Q}\} \subseteq l^p$

Let $x = (\xi_j)_1^\infty \in l^p$ be arbitrary. Then for every $\epsilon > 0$ there is an n (depending on ϵ) such that $\sum_{j=n+1}^\infty |\xi_j|^p < \frac{\epsilon^p}{2}$

Since the rationals are dense in $\mathbb{R}$, for each ξ_j there is a rational η_j close to it. Hence we can find $y \in M$ satisfying $\sum_{j=1}^n |\xi_j - \eta_j|^p < \frac{\epsilon^p}{2}$

It follows that $[d(x,y)]^p = \sum_{j=1}^n |\xi_j - \eta_j|^p + \sum_{j=n+1}^\infty |\xi_j - 0|^p < \frac{\epsilon^p}{2} + \frac{\epsilon^p}{2}$

$[d(x,y)]^p < \epsilon^p \Rightarrow d(x,y) < \epsilon$

M is dense in l^p. This proves l^p is separable.

Boundary: A boundary point x of a set $A \subset (X,d)$ is a point of X (which may or may not belong to A) such that every neighborhood of x contains points of A as well as points not belonging to A; and the boundary (or frontier) of A is the set of all boundary points of A.

Convergence of a sequence, Limit of Sequence:

A sequence (x_n) in a metric space (X,d) is said to converge or to be convergent if there is an $x \in X$ such that $\lim_{n\to\infty}(x_n, x) = 0$

i.e. $d(x_n, x) \to 0$ as $n \to \infty$ or $x_n \to x$ as $n \to \infty$

'x' is called the limit of (x_n) and we write $\lim_{n\to\infty}(x_n) = x$

This means that for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \in \; ; \forall n > n_0$

i.e. $x_{n's}$ must be closed to 'x' for sufficiently larger 'n'.

We say that (x_n) converges to x or has the limit x. If (x_n) is not convergent, it is said to be divergent.

Remember:

The limit 'x' of a convergent sequence (x_n) is a metric space (X,d) must be a point of X. For example consider $X = (0,1]$ with $d(x,y) = |x-y|$ if $(x_n) = \left(\frac{1}{n}\right)$ then although $\lim_{n\to\infty} d(x_n, 0) = \lim_{n\to\infty}|x_n - 0|$ but $x_n \nrightarrow 0$ as $0 \notin X$

Theorem

If (x_n) is converges then limit of (x_n) is unique.

Proof:

Suppose $x_n \to a, x_n \to b$ as $n \to \infty$

Then $0 \le d(a,b) \le d(a, x_n) + d(x_n, b) \to 0 + 0$ as $n \to \infty$

$\Rightarrow d(a,b) = 0 \Rightarrow a = b$

Hence the limit is unique.

Question

Find a sequence which converges to 0 but is not in any space l^p where $1 \le p < \infty$.

Solution

Consider $a_n = \frac{1}{ln(n+1)}$

Since $ln(n+1) < n^{\frac{1}{p}}$ then $\frac{1}{ln(n+1)} > \frac{1}{n^{\frac{1}{p}}} \Rightarrow \left(\frac{1}{ln(n+1)}\right)^p > \frac{1}{n} \Rightarrow \frac{1}{n} < \left(\frac{1}{ln(n+1)}\right)^p$

Sequence $\frac{1}{n} \to 0$ as $n \to \infty$

But $\sum_{n=0}^{\infty} \frac{1}{n} > 0$ so $\frac{1}{n} \notin l^p$

Question

Find a sequence $x \in l^p$ with $p > 1$ but $x \notin l^1$.

Solution

We know that $x = \left(\frac{1}{n^2}\right)$ is a convergent sequence with $p > 1$.

And $\sum_{n=0}^{\infty} \frac{1}{n^2} < \infty \Rightarrow x \in l^p$.

If $p = 1$ then $x \notin l^1$.

Question

Show that $Sup_j\left|\xi_j - \eta_j\right|$ satisfy (M4) for l^∞.

Solution Let $x = (\xi_j), x = (\eta_j), x = (\lambda_j)$

$$\left|\xi_j - \eta_j\right| = \left|\xi_j - \lambda_j + \lambda_j - \eta_j\right| \le \left|\xi_j - \lambda_j\right| + \left|\lambda_j - \eta_j\right|$$

$$\Rightarrow Sup_j\left|\xi_j - \eta_j\right| \le Sup_j\left|\xi_j - \lambda_j\right| + Sup_j\left|\eta_j - \lambda_j\right|$$

$$\Rightarrow d(x,y) \le d(x,z) + d(z,y)$$

$\Rightarrow Sup_j\left|\xi_j - \eta_j\right|$ satisfy (M4) for l^∞.

Bounded Set

Let (X, d) be a metric space then a nonempty subset $M \subset X$ a bounded set if its diameter $d(M) = Sup_{x,y \in M}\, d(x, y)$ is finite.

If M is bounded, then $M \subset B(x_0, r)$, where $x_0 \in X$ is any point and r is a (sufficiently large) real number, and vice versa.

Bounded Sequence

Let (X, d) be a metric space then a sequence (x_n) in X is bounded if there exists M and a point $x \in X$ such that $d(x_n, x) < M \quad ; \forall n > n_0$

Lemma

Let (X, d) be a metric space then a convergent sequence in X is bounded and its limit is unique.

Proof:

Suppose that (x_n) is convergent sequence in X and let $x_n \to x$ as $n \to \infty$ where $x \in X$. Then for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \in \; ; \forall n > n_0$.

In particular for $\in = 1$ there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x) < 1 \; ; \forall n > n_1$.

Define $\lambda = max\left\{1, d(x_1, x), d(x_2, x), \dots, d\left(x_{n_1}, x\right)\right\}$ and $K = max\{\in, \lambda\}$

Then by using triangular inequality for arbitrary $x_i, x_j \in (x_n)$ we have

$0 \leq d\left(x_i, x_j\right) \leq d(x_i, x) + d\left(x, x_j\right) \leq K + K = M$ (say)

Then obviously $d(x_n, x) < M \; ; \forall n > n_0$ and hence (x_n) is bounded.

To prove uniqueness of limit; Suppose $x_n \to a, x_n \to b$ as $n \to \infty$

Then $0 \leq d(a, b) \leq d(a, x_n) + d(x_n, b) \to 0 + 0$ as $n \to \infty$

i.e. $d(a, b) \leq 0$ but $d(a, b) \geq 0$

$\Rightarrow d(a, b) = 0 \Rightarrow a = b$

Hence the limit is unique.

Remember:

Converse of above theorem is not true. i.e. a bounded sequence needs not to be convergent. For example $(x_n) = (-1)^n = -1,1,-1,1, \ldots$ is bounded but not convergent.

Lemma

Let (X, d) be a metric space then if $x_n \to x$ and $y_n \to y$ in X,

Then $d(x_n, y_n) \to d(x, y)$

Proof:

By using triangular inequality $d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$

$\Rightarrow d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$ ……………..(i)

Similarly using triangular inequality $d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$

$\Rightarrow d(x, y) - d(x_n, y_n) \leq d(x_n, x) + d(y_n, y)$

$\Rightarrow -\big(d(x_n, y_n) - d(x, y)\big) \leq d(x_n, x) + d(y_n, y)$ ……………..(ii)

From (i) and (ii) $|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y)$ ……………..(iii)

Since $x_n \to x$ as $n \to \infty$ then for all $\in > 0$ there exists $n_1 \in \mathbb{N}$ such that $d(x_n, x) < \frac{\in}{2}$; $\forall n > n_1$.

Also since $y_n \to y$ as $n \to \infty$ then for all $\in > 0$ there exists $n_2 \in \mathbb{N}$ such that $d(y_n, y) < \frac{\in}{2}$; $\forall n > n_2$.

Let $n_0 = max\{n_1, n_2\}$ then from (iii) we obtain

(3) $\Rightarrow |d(x_n, y_n) - d(x, y)| < \frac{\in}{2} + \frac{\in}{2} = \in$; $\forall n > n_0$

$\Rightarrow d(x_n, y_n) \to d(x, y)$ as $n \to \infty$

i.e. $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$

Cauchy Sequence

A sequence (x_n) in a metric space (X,d) is said to Cauchy sequence if for any $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \in$; $\forall m,n > n_0$

i.e. $d(x_m, x_n) \to 0$ as $m,n \to \infty$

Remember Let (x_n) be a sequence in the discrete space (X,d). If (x_n) be a Cauchy sequence, then for $\in = \frac{1}{2}$, there is a natural number n_0 depending on $\in$ such that $d(x_m, x_n) < \frac{1}{2}$; $\forall m,n > n_0$

Since in discrete space d is either 0 or 1 therefore $d(x_m, x_n) = 0$ implies $x_m = x_n$ (say). Thus a Cauchy sequence in (X,d) become constant after a finite number of terms, i.e. $(x_n) = (x_1, x_2, x_3, \dots, x_{n_0}, x, x, x, \dots)$

Theorem

Let (X,d) be a metric space then every convergent sequence in (X,d) is a Cauchy sequence but converse not holds (a Cauchy sequence may not be convergent).

Proof

Suppose that (x_n) is convergent sequence in X and let $x_n \to x$ as $n \to \infty$ where $x \in X$. Then for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \frac{\in}{2}$; $\forall n > n_0$.

And using triangular inequality for all $m,n > n_0$ we have

$$d(x_m, x_n) \le d(x_n, x) + d(x_m, x) < \frac{\in}{2} + \frac{\in}{2}$$

$$\Rightarrow d(x_m, x_n) < \in \ ; \forall m,n > n_0$$

Hence (x_n) is a Cauchy sequence.

Conversely A Cauchy sequence may not be convergent. For this consider $X = (0,1]$ with $d(x,y) = |x - y|$ then $(x_n) = \left(\frac{1}{n}\right)$ is a Cauchy sequence because

$$d(x_m, x_n) = |x_m - x_n| = \left|\frac{1}{m} - \frac{1}{n}\right| \le \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right| = \frac{1}{m} + \frac{1}{n} \to 0 \text{ as } m,n \to \infty$$

Obviously $x_n \to 0$ but $0 \notin X$. So that (x_n) is not a convergent sequence.

Subsequence

Let $(a_1, a_2, a_3, \dots)$ be a sequence in (X, d) and let $(i_1, i_2, i_3, \dots)$ be a sequence of positive integers such that $i_1 < i_2 < i_3 < \cdots$ then $(a_{i_1}, a_{i_2}, a_{i_3}, \dots)$ is called subsequence of $(a_n : n \in \mathbb{N})$.

Theorem

Let (x_n) be a Cauchy sequence in(X, d), then (x_n) converges to a point $x \in X$ if and only if (x_n) has a convergent subsequence $\left(x_{n_k}\right)$ which converges to $x \in X$.

Proof

Suppose $x_n \to x \in X$ then (x_n) itself is a subsequence which converges to $x \in X$.

Conversely, assume that $\left(x_{n_k}\right)$ is a subsequence of (x_n) which converges to x. Then for any $\in > 0$ there is $n_0 \in \mathbb{N}$ such that $d\left(x_{n_k}, x\right) < \frac{\in}{2}$; $\forall n_k > n_0$.

Furthermore (x_n) is Cauchy sequence Then for the $\in > 0$ there is $n_1 \in \mathbb{N}$ such that $d(x_m, x_n) < \frac{\in}{2}$; $\forall m, n > n_1$

Suppose $n_2 = max(n_0, n_1)$ then by using the triangular inequality we have

$$d(x_n, x) = d\left(x_n, x_{n_k}\right) + d\left(x_{n_k}, x\right) < \frac{\in}{2} + \frac{\in}{2} \quad ; \forall n_k, n > n_2$$

$d(x_n, x) < \in$ and this shows that $x_n \to x$

Theorem Let (x_n) be a Cauchy sequence in(X, d), and If (x_n) converges to $x \in X$, then every subsequence $\left(x_{n_k}\right)$ also converges to $x \in X$.

Proof Suppose $x_n \to x$ then for any $\in > 0$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \in$. Then in particular $d\left(x_{n_k}, x\right) < \in$; $\forall n_k > n_0$.

Hence $x_{n_k} \to x \in X$

Example Let $X = (0,1)$ then $(x_n) = (x_1, x_2, x_3, \dots) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ is a sequence in X. Then $x_n \to 0$ but 0 is not a point of X.

Theorem

Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure. Then $x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$.

Proof Let $x \in \bar{M}$ then $x \in M \cup M^d$

Case – I: If $x \in M$ then the sequence $(x, x, x, \dots)$ lies in M and converges to x.

Case – II: If $x \notin M$ then $x \in M^d$ that is 'x' is a limit point of M and hence for each $n = 1,2,3, \dots$ the open ball $B\left(x, \frac{1}{n}\right)$ contain infinite number of point of M.

We choose $x_n \in M$ for each $B\left(x, \frac{1}{n}\right)$. Thus we obtain a sequence (x_n) of points of M and since $\frac{1}{n} \to 0$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$.

Conversely suppose that there exists a sequence (x_n) in M such that $x_n \to x$. Then we are to prove that $x \in \bar{M}$.

Case – I: If $x \in M$ then $x \in M \cup M^d$ and $x \in \bar{M}$ as $\bar{M} = M \cup M^d$

Case – II: If $x \notin M$ then every neighbourhood of x contain infinite number of terms of (x_n). Then x is a limit point of M. i.e. $x \in M^d$ and $x \in M \cup M^d = \bar{M}$

Hence from both cases $x \in \bar{M}$.

Theorem

Let M be a nonempty subset of a metric space (X, d) and $\bar{M}$ its closure. Then M is closed if and only if the situation $x_n \in M$, $x_n \to x$ implies that $x \in M$.

Proof Suppose that M is closed. i.e. $M = \bar{M}$ and let $x_n \in M$ also $x_n \to x$ then by theorem

"$x \in \bar{M}$ if and only if there is a sequence (x_n) in M such that $x_n \to x$"

We have $x \in \bar{M} = M$ implies $x \in M$.

Conversely If (x_n) is in M and $x_n \to x$, then $x \in \bar{M}$ then by hypothesis $M = \bar{M}$, then M is closed.

Remember

A sequence (x_n) of real or complex numbers converges on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$, respectively, if and only if it satisfies the Cauchy convergence criterion, that is, if and only if for every given $\in > 0$ there is an $n_0 = n_0(\in)$ such that $|x_m - x_n| < \in$; $\forall m, n > n_0$

Here $|x_m - x_n|$ is the distance $d(x_m, x_n)$ from x_m to x_n on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$. Hence we can write the inequality of the Cauchy criterion in the form $d(x_m, x_n) < \in$; $\forall m, n > n_0$.

And if a sequence (x_n) satisfies the condition of the Cauchy criterion, we may call it a Cauchy sequence. Then the Cauchy criterion simply says that a sequence of real or complex numbers converges on $\mathbb{R}$ or in $\mathbb{C}$ if and only if it is a Cauchy sequence. This refers to the situation in $\mathbb{R}$ or $\mathbb{C}$. Unfortunately, in more general spaces the situation may be more complicated, and there may be Cauchy sequences which do not converge. Such a space is then lacking a property which is so important that it deserves a name, namely, *completeness*. This consideration motivates the following definition, which was first given by M. Frechet (1906).

Complete Metric Space, Completeness

A metric space (X, d) is said to be complete if every Cauchy sequence in X converges in X (that is, has a limit which is an element of X).

Remember

The real line and the complex plane are complete metric spaces by Cauchy Convergence Criterian.

Complete Space (Examples)

i. The discrete space is complete.
Since in discrete space a Cauchy sequence becomes constant after finite terms i.e. (x_n) is Cauchy in discrete space if it is of the form $(x_1, x_2, x_3, \dots x_n = b, b, b, \dots)$
ii. The set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ of integers with usual metric is complete.
iii. The set of rational numbers with usual metric is not complete.

Subspace

Let (X,d) be a metric space and $M \subset X$ then M is called subspace if M is itself a metric space under the metric d.

Theorem

A subspace M of a complete metric space (X,d) is itself complete if and only if the set M is closed in X.

Proof

Assume that M is complete we prove M is closed. i.e. $M = \bar{M}$

Obviously $M \subseteq \bar{M}$

Now let $x \in \bar{M}$ then there is a sequence (x_n) in M such that $x_n \to x$

Since convergent sequence is a Cauchy and M is complete then $x_n \to x \in M$

Since x was arbitrary point of M this implies $\bar{M} \subseteq M$

Therefore $M = \bar{M}$ and Consequently M is closed.

Conversely

Suppose M is closed. i.e. $M = \bar{M}$ and (x_n) is a Cauchy sequence in M.

Then (x_n) is Cauchy in X and since X is complete so $x_n \to x \in X$

Also $x \in \bar{M}$ and $M \subset X$.

Since M is closed i.e. $M = \bar{M}$. Then $x_n \to x \in M$

Therefore $x \in M$.

Hence M is complete.

Construction of Completeness Proofs

In various applications a set X is given (for instance, a set of sequences or a set of functions), and X is made into a metric space. This we do by choosing a metric d on X. The remaining task is then to find out whether (X, d) has the desirable property of being complete. To prove completeness, we take an arbitrary Cauchy sequence (x_n) in X and show that it converges in X. For different spaces, such proofs may vary in complexity, but they have approximately the same general pattern:

i. Construct an element 'x' (to be used as a limit).
ii. Prove that 'x' is in the space considered.
iii. Prove convergence $x_n \to x$ (in the sense of the metric).

We will use the facts ℝ and ℂ are complete.

Theorem The Real Line is Complete.

Proof Let (x_n) be any Cauchy sequence of real numbers.

We first prove that (x_n) is bounded. For this let $\in = 1 > 0$ then there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) = |x_m - x_n| < 1 \ ; \forall m, n > n_0$.

In particular for $m \geq n_0$ we have $\left|x_{n_0} - x_n\right| < 1 \Rightarrow x_{n_0} - 1 \leq x_n \leq x_{n_0} + 1$

Let $\alpha = max\{x_1, x_2, x_3, \dots, x_{n_0} + 1\}$ and $\beta = min\{x_1, x_2, x_3, \dots, x_{n_0} - 1\}$ then

$\beta \leq x_n \leq \alpha$ for all n. This shows that (x_n) is bounded with β as lower bound and α as upper bound.

Secondly we prove (x_n) has convergent subsequence$\left(x_{n_i}\right)$.

If the range of the sequence is $\{x_n\} = \{x_1, x_2, x_3, \dots\}$ is finite, then one of the term is the sequence say 'b' will repeat infinitely i.e. $b, b, b, \dots$. Then $(b, b, b, \dots)$ is a convergent subsequence which converges to 'b'.

If the range is infinite then by the Bolzano Weirestrass theorem, the bounded infinite set $\{x_n\}$ has a limit point, say 'b'.

Then each of the open interval $S_1 = (b-1, b+1), S_2 = \left(b - \frac{1}{2}, b + \frac{1}{2}\right)$,

$S_3 = \left(b - \frac{1}{3}, b + \frac{1}{3}\right)$, ... has an infinite numbers of points of the set $\{x_n\}$. i.e. there are infinite numbers of terms of the sequence (x_n) in every open interval S_n. We choose a point x_{i_1} from S_1, then we choose a point x_{i_2} from S_2 such that $i_1 < i_2$. i.e. the terms x_{i_2} comes after x_{i_1} in the original sequence (x_n). Then we choose a term x_{i_3} such that $i_2 < i_3$, continuing in this manner we obtain a subsequence $\left(x_{i_n}\right) = \left(x_{i_1}, x_{i_2}, x_{i_3}, \dots\right)$.

It is always possible to choose a term because every interval contain an infinite numbers of terms of the sequence (x_n).

Since $b - \frac{1}{n} \to b$ and $b + \frac{1}{n} \to b$ as $n \to \infty$. Hence we have convergent subsequence $\left(x_{i_n}\right)$ whose limit is 'b'.

Lastly we prove that $x_n \to b \in \mathbb{R}$

Since (x_n) is a Cauchy therefore for any $\in > 0$ there is an $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \frac{\in}{2}$; $\forall m, n > n_0$.

Also since $x_{i_n} \to b$ there is a natural number i_m such that $i_m > n_0$ then for all $m, n, i_m > n_0$ we have

$$d(x_n, b) = |x_n - b| = \left|x_n - x_{i_m} + x_{i_m} - b\right| \le \left|x_n - x_{i_m}\right| + \left|x_{i_m} - b\right|$$

$$d(x_n, b) < \frac{\in}{2} + \frac{\in}{2} = \in \Rightarrow d(x_n, b) = |x_n - b| < \in$$

Hence $x_n \to b \in \mathbb{R}$

Thus Real Line is complete.

Example Euclidean space $\mathbb{R}^n$ is complete.

Proof

Let (x_m) be a Cauchy sequence in $\mathbb{R}^n$ where $(x_m) = \left(\xi_1^{(m)}, \xi_2^{(m)}, \xi_3^{(m)}, \dots, \xi_n^{(m)}\right)$ therefore for any $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_r) = \left(\sum_{j=1}^{n} \left|\xi_j^{(m)} - \xi_j^{(r)}\right|^2\right)^{\frac{1}{2}} < \epsilon \;\; ; \forall m, r > n_0. \quad \dots\dots\dots\dots(1)$$

$$\Rightarrow \sum_{j=1}^{n} \left|\xi_j^{(m)} - \xi_j^{(r)}\right|^2 < \epsilon^2 \;\; ; \forall m, r > n_0 \Rightarrow \sum_{j=1}^{n} \left|\xi_j^{(m)} - \xi_j^{(r)}\right| < \epsilon \;\; ; \forall j = 1,2,3, ..$$

$\Rightarrow \xi_j^{(m)}$ is a Cauchy sequence of real numbers for every $j = 1,2,3, ..$ in $\mathbb{R}$.

Since $\mathbb{R}$ is complete therefore $\xi_j^{(m)} \to \xi_j \in \mathbb{R}$.

Using the n limits we define $(x) = (\xi_1, \xi_2, \dots, \xi_n)$ then clearly $x \in \mathbb{R}^n$

As $r \to \infty$ in (1) we have $d(x_m, x) = \left(\sum_{j=1}^{n} \left|\xi_j^{(m)} - \xi_j\right|^2\right)^{\frac{1}{2}} < \epsilon \;\; ; \forall m > n_0$

$\Rightarrow d(x_m, x) < \epsilon \;\; ; \forall m > n_0 \Rightarrow x_m \to x \in \mathbb{R}^n$

This proves that Euclidean space $\mathbb{R}^n$ is complete.

Example Unitary space $\mathbb{C}^n$ is complete.

Proof In the above theorem if we take $n = 2$ then we see complex plane $\mathbb{C} = \mathbb{R}^2$ is complete. Moreover the unitary space $\mathbb{C}^n$ is complete.

Example The space $l^\infty = \left\{x = (\xi_j) : Sup_j \left|\xi_j\right| < \infty\right\}$ is complete.

Proof Let (x_m) be a Cauchy sequence in l^∞ where $x_m = \left(\xi_j^{(m)}\right)_1^\infty$ then for any $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_r) = Sup_j \left|\xi_j^{(m)} - \xi_j^{(r)}\right| < \epsilon \;\; ; \forall m, r > n_0. \dots\dots\dots\dots(1)$$

$$\Rightarrow \left|\xi_j^{(m)} - \xi_j^{(r)}\right| < \epsilon \;\; ; \forall m, r > n_0, \forall j = 1,2,3, ..$$

$\Rightarrow \xi_j^{(m)}$ is a Cauchy sequence of real/complex numbers for every $j = 1,2,3,..$ in $\mathbb{R}$ or $\mathbb{C}$. And since $\mathbb{R}$ is complete therefore $\xi_j^{(m)} \to \xi_j \in \mathbb{R}$.

Using the n limits we define $x = (\xi_j)_1^\infty = (\xi_1, \xi_2, \dots)$.

As $r \to \infty$ in (1) we have $d(x_m, x) = Sup_j \left|\xi_j^{(m)} - \xi_j\right| <\in \ ; \forall m > n_0$

$\Rightarrow d(x_m, x) <\in \ ; \forall m > n_0 \Rightarrow x_m \to x$

Now we prove that ‘x’ is bounded and is from l^∞.

Since $x_m = \left(\xi_j^{(m)}\right)_1^\infty \in l^\infty$ then there exists a real number k_m such that for all ‘j’ $\left|\xi_j^{(m)}\right| \le k_m$, then using triangular inequality

$\left|\xi_j\right| = \left|\xi_j - \xi_j^{(m)} + \xi_j^{(m)}\right| \le \left|\xi_j - \xi_j^{(m)}\right| + \left|\xi_j^{(m)}\right| \le \in + k_m \Rightarrow Sup_j\left|\xi_j\right| < \infty$

This proves that $x_m \to x \in l^\infty$.

And hence $l^\infty = \left\{x = (\xi_j) : Sup_j\left|\xi_j\right| < \infty\right\}$ is complete.

Example The space ‘c’ consists of all convergent sequences $x = (\xi_j)_1^\infty$ of complex numbers, with the metric induced from the space l^∞ then the space ‘c’ is complete.

Or Show that $c = \left\{x = (\xi_j)_1^\infty : x \text{ is convergent}\right\}$ is a complete metric space.

Proof Since every convergent sequence is bounded therefore $c \subseteq l^\infty$. Since ‘c’ is subspace of complete metric space l^∞, to show that ‘c’ is complete it is sufficient to prove that ‘c’ is closed. i.e. $c = \bar{c}$.

Obviously $c \subseteq \bar{c}$ ………….(1)

Let $x = (\xi_j)_1^\infty \in \bar{c}$ then there exists a sequence $x_n = \left(\xi_j^{(n)}\right)_1^\infty$ in ‘c’ such that $x_n \to x$ as $n \to \infty$. Then for any $\in > 0$ there is an $n_0 \in \mathbb{N}$ such that

$d(x_n, x) = Sup_j \left|\xi_j^{(n)} - \xi_j\right| < \frac{\epsilon}{3}$; $\forall n > n_0$

$\left|\xi_j^{(n)} - \xi_j\right| < \frac{\epsilon}{3}$; $\forall n > n_0$. ....(2)

Then in particular for $n = n_0$ and $\forall j$ we have

$d(x_{n_0}, x) = Sup_j \left|\xi_j^{(n_0)} - \xi_j\right| < \frac{\epsilon}{3}$; $\forall n_0$

$\left|\xi_j^{(n_0)} - \xi_j\right| < \frac{\epsilon}{3}$; $\forall n_0$. ....(3)

Now as $x_{n_0} = \left(\xi_j^{(n_0)}\right)_1^\infty$ is a convergent sequence and since every convergent sequence is Cauchy sequence therefore there exists $n_1 \in \mathbb{N}$ such that

$\left|\xi_j^{(n_0)} - \xi_k^{(n_0)}\right| < \frac{\epsilon}{3}$; $\forall j, k > n_1$. ....(4)

Then by using triangular inequality we have

$\left|\xi_j - \xi_k\right| = \left|\xi_j - \xi_j^{(n_0)} + \xi_j^{(n_0)} - \xi_k^{(n_0)} + \xi_k^{(n_0)} - \xi_k\right|$

$\left|\xi_j - \xi_k\right| \leq \left|\xi_j - \xi_j^{(n_0)}\right| + \left|\xi_j^{(n_0)} - \xi_k^{(n_0)}\right| + \left|\xi_k^{(n_0)} - \xi_k\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$

$\left|\xi_j - \xi_k\right| < \epsilon$; $\forall j, k > n_1$

Hence x is Cauchy in l^∞ and x is convergent.

Therefore $x \in c$ and $\bar{c} \subseteq c$ ....(5)

Hence from (1) and (5) $c = \bar{c}$

i.e. C is closed in l^∞ and l^∞ is complete.

Since we know that a subspace of complete space is complete if and only if it is closed in the space. Consequently c is complete.

Remark: A metric space X may be complete under one metric but by changing the metric on X, it may happen that the metric space is no longer a complete a complete metric space. This is illustrated in following example.

Example

Consider $C[a,b] = \{x : x \text{ } is \text{ } continuous \text{ } on \text{ } [a,b]\}$ and introduce two metric on it by $d_1(x,y) = max_{t\in[a,b]}|x(t) - y(t)|$ and $d_2(x,y) = \int_a^b |x(t) - y(t)|\, dt$

Then show that $(C[a,b], d_1)$ is complete but $(C[a,b], d_2)$ is not complete.

Example (a)

Consider $C[a,b] = \{x : x \text{ } is \text{ } continuous \text{ } on \text{ } [a,b]\}$ and introduce metric on it by $d_1(x,y) = max_{t\in[a,b]}|x(t) - y(t)|$ then show that $(C[a,b], d_1)$ is complete.

Proof To show that $(C[a,b], d_1)$ is complete let (x_m) be a Cauchy sequence in $C[a,b]$ then for any $\in > 0$ there is an $n_0 \in \mathbb{N}$ such that

$d_1(x_m, x_r) = max_{t\in[a,b]}|x_m(t) - x_r(t)| < \in \ ; \forall m, r > n_0$ …………(1)

Then there exists $t_0 \in [a,b]$ at which maximum attained so that we have

$max_{t_0\in[a,b]}|x_m(t_0) - x_r(t_0)| < \in \ ; \forall m, r > n_0$

$|x_m(t_0) - x_r(t_0)| < \in \ ; \forall m, r > n_0$ …………(2)

$\Rightarrow x_m(t_0)$ is a Cauchy sequence of real/complex numbers for every $m = 1,2,3, ..$ in $\mathbb{R}$ or $\mathbb{C}$. And since $\mathbb{R}$ is complete therefore $x_m(t_0) \to x(t_0) \in \mathbb{R}$.

In this way we can associate with each $t \in [a,b]$,a unique real number $x(t)$. This define a function $x: [a,b] \to \mathbb{R}$.

Using $r \to \infty$ in (1) get $max_{t\in[a,b]}|x_m(t) - x(t)| < \in \ ; \forall m > n_0$ …………(3)

And for all $t \in [a,b]$ we have $|x_m(t) - x(t)| < \in \ ; \forall m > n_0$

$\Rightarrow x_m \to x$ uniformly on $[a,b]$ (Since it does not depends upon ‘t’)

Since (x_m) is a sequence of continuous functions on $[a,b]$ and the convergence is uniform, therefore the limit function ‘x’ is continuous .i.e. $x \in C[a,b]$

(With reference: If a sequence of continuous function on a closed interval converges uniformly, then the limit function is continuous)

From (3) we have $d_1(x_m, x) = max_{t\in[a,b]}|x_m(t) - x(t)| <\in\ ; \forall m > n_0$

i.e. $\Rightarrow x_m \to x \in C[a,b]$ under metric d_1 as $m \to \infty$

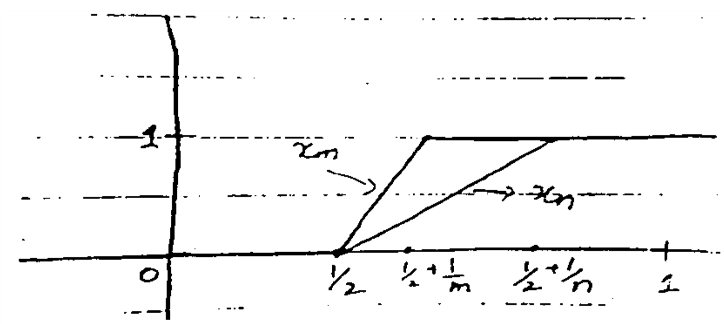
Hence $(C[a,b], d_1)$ is complete

Example (b)

Consider $C[a,b] = \{x\colon x\ is\ continuous\ on\ [a,b]\}$ and introduce metric on it by $d_2(x,y) = \int_a^b |x(t) - y(t)|\, dt$ then show that $(C[a,b], d_2)$ is not complete.

Proof

Let $[a,b] = [0,1]$ and consider the function x_m as show in figure below;



$m \neq n$

$m > n$

$m < n$

$x_m \to \frac{1}{2} + \frac{1}{m}$

$x_5 \to \frac{1}{2} + \frac{1}{5}$

Let $m > n$ so that $\frac{1}{m} < \frac{1}{n}$ then consider

$$d_2(x_m, x_n) = \int_0^1 |x_m(t) - x_n(t)|\, dt$$

$$d_2(x_m, x_n) = \int_0^{\frac{1}{2}} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x_n(t)|\, dt$$

$$+ \int_{\frac{1}{2}+\frac{1}{m}}^{\frac{1}{2}+\frac{1}{n}} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{n}}^{1} |x_m(t) - x_n(t)|\, dt$$

$$d_2(x_m, x_n) = \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{m}}^{\frac{1}{2}+\frac{1}{n}} |x_m(t) - x_n(t)|\, dt$$

Here $\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x_n(t)|\, dt + \int_{\frac{1}{2}+\frac{1}{m}}^{\frac{1}{2}+\frac{1}{n}} |x_m(t) - x_n(t)|\, dt \to 0$ as $m, n \to \infty$

$\Rightarrow (x_m)$ is a Cauchy sequence in $C[a,b]$

Suppose there exists $x \in C[a,b]$ such that $x_m \to x$ under metric d_2 as $m \to \infty$

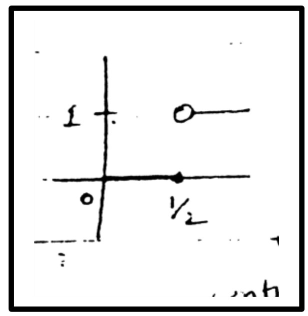
i.e. $d_2(x_m, x) \to 0$ as $m \to \infty$ then

$$d_2(x_m, x) = \int_0^1 |x_m(t) - x(t)|\, dt$$

$$d_2(x_m, x) = \int_0^{\frac{1}{2}} |x_m(t) - x(t)|\, dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x(t)|\, dt$$

$$+ \int_{\frac{1}{2}+\frac{1}{m}}^{1} |x_m(t) - x(t)|\, dt$$

$$d_2(x_m, x) = 0 + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x(t)|\, dt + 1$$



Here $\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{m}} |x_m(t) - x(t)|\, dt \to 0$ as $m, n \to \infty$

This happens if $x(t) = \begin{cases} 0 & ; 0 \le t \le \frac{1}{2} \\ 1 & ; \frac{1}{2} \le t \le 1 \end{cases}$

Implies $x \notin C[a,b]$ a contradiction. Hence $(C[a,b], d_2)$ is not complete.

Example The space l^p is complete; here p is fixed and $1 \le p < \infty$.

Proof Let (x_m) be a Cauchy sequence in l^p where $x_m = \left(\xi_j^{(m)}\right)_1^\infty$ then for any $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that

$$d(x_m, x_r) = \left(\sum_{j=1}^{\infty} \left|\xi_j^{(m)} - \xi_j^{(r)}\right|^p\right)^{\frac{1}{p}} < \epsilon \;\; ; \forall m, r > n_0. \quad \ldots\ldots\ldots\ldots(1)$$

$$\Rightarrow \left|\xi_j^{(m)} - \xi_j^{(r)}\right| < \epsilon \;\; ; \forall m, r > n_0, \forall j = 1,2,3, ..$$

$\Rightarrow \xi_j^{(m)}$ is a Cauchy sequence of real/complex numbers for every $j = 1,2,3, ..$ in $\mathbb{R}$ or $\mathbb{C}$. And since $\mathbb{R}$ is complete therefore $\xi_j^{(m)} \to \xi_j \in \mathbb{R}$.

Using the n limits we define $x = (\xi_j)_1^\infty = (\xi_1, \xi_2, \ldots)$.

As $r \to \infty$ in (1) we have $d(x_m, x) = \left(\sum_{j=1}^{\infty} \left|\xi_j^{(m)} - \xi_j\right|^p\right)^{\frac{1}{p}} <\in \ ; \forall m > n_0$ …..(2)

$\Rightarrow \sum_{j=1}^{\infty} \left|\xi_j^{(m)} - \xi_j\right|^p <\in^p \ ; \forall m > n_0$ this shows that $x_m - x \in l^p$ but $x_m \in l^p$

$\Rightarrow x_m - (x_m - x) \in l^p = x \in l^p$

From (2) we have $d(x_m, x) <\in \ ; \forall m > n_0$ i.e. $x_m \to x$

This proves that $x_m \to x \in l^p$.

And hence $l^p = \left\{x = (\xi_j)_1^{\infty} : \sum_{j=1}^{\infty} |\xi_j|^p < \infty\right\}$ is complete

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Example The function space $\mathbb{C}[a, b]$ is complete; here [a, b] is any given closed interval on $\mathbb{R}$.

Proof Let (x_m) be a Cauchy sequence in $\mathbb{C}[a, b]$ then for any $\in > 0$ there is an $n_0 \in \mathbb{N}$ such that

$d(x_m, x_n) = max_{t \in J} |x_m(t) - x_n(t)| <\in \ ; \forall m, n > n_0$. ……(1) where $J = [a, b]$

Then for any $t = t_0 \in J$ we have $\Rightarrow |x_m(t_0) - x_n(t_0)| <\in \ ; \forall m, n > n_0$

$\Rightarrow x_m(t_0)$ is a Cauchy sequence of real/complex numbers for every in $\mathbb{R}$ or $\mathbb{C}$. And since $\mathbb{R}$ is complete therefore $x_m(t_0) \to x(t_0) \in \mathbb{R}$.

In this way for every $t \in J$, we can associate a unique real number $x(t)$ with $x_n(t)$. This defines a function $x(t)$ on J.

We prove that $x(t) \in \mathbb{C}[a, b]$ and $x_m(t) \to x(t)$ as $m \to \infty$

From (1) we see that $|x_m(t) - x_n(t)| <\in \ ; \forall m, n > n_0$ for every $t \in J$

As $n \to \infty$ we have $|x_m(t) - x(t)| <\in \ ; \forall m > n_0$

Since the convergence is uniform and the $x_{n's}$ are continuous, the limit function $x(t)$ is continuous, as it is well known from the calculus. Then $x(t)$ is continuous.

Hence $x(t) \in \mathbb{C}[a, b]$, also $|x_m(t) - x(t)| <\in$ as $m \to \infty$

Therefore $x_m(t) \to x(t) \in \mathbb{C}[a, b]$ and the proof is complete.

Example If (X, d_1) and (Y, d_2) are complete then $X \times Y$ is complete.

Where the metric defined is as $d(x, y) = max[d_1(\xi_1, \xi_2), d_2(\eta_1, \eta_2)]$ with $x = (\xi_1, \eta_1), y = (\xi_2, \eta_2)$ and $\xi_1, \xi_2 \in X,\ \eta_1, \eta_2 \in Y$.

Proof Let (x_m) be a Cauchy sequence in $X \times Y$ then for any $\in > 0$ there is an $n_0 \in \mathbb{N}$ such that

$d(x_m, x_n) = max\left[d_1\left(\xi^{(m)}, \xi^{(n)}\right), d_2\left(\eta^{(m)}, \eta^{(n)}\right)\right] < \in\ ; \forall m, n > n_0$

$\Rightarrow d_1\left(\xi^{(m)}, \xi^{(n)}\right) < \in$ and $d_2\left(\eta^{(m)}, \eta^{(n)}\right) < \in$

$\Rightarrow \xi^{(m)}$ is a Cauchy sequence in X and $\eta^{(m)}$ is a Cauchy sequence in Y . And since X and Y are complete therefore $\xi^{(m)} \to \xi \in X$ and $\eta^{(m)} \to \eta \in Y$.

Let $x = (\xi, \eta)$ then $x \in X \times Y$.

Also $d(x_m, x) = max\left[d_1\left(\xi^{(m)}, \xi\right), d_2\left(\eta^{(m)}, \eta\right)\right] < \in\ ; \forall m > n_0$

$\Rightarrow d(x_m, x) = max\left[d_1\left(\xi^{(m)}, \xi\right), d_2\left(\eta^{(m)}, \eta\right)\right] \to 0$ as $m \to \infty$

Hence $x_m(t) \to x(t) \in X \times Y$ and the proof is complete.

Theorem (Just Read): Convergence $x_m \to x$ in the space C[a, b] is uniform convergence, that is, (x_m) converges uniformly on [a, b] to x. Hence the metric on C[a, b] describes uniform convergence on [a, b] and, for this reason, is sometimes called the uniform metric.

Examples of Incomplete Metric Spaces

- **Space Q:** This is the set of all rational numbers with the usual metric given by $d(x, y) = |x - y|$, where $x, y \in \boldsymbol{Q}$, and is called the rational line. Q is not complete.
- **Polynomials:** Let X be the set of all polynomials considered as functions of t on some finite closed interval J = [a, b] and define a metric d on X by $d(x, y) = max_{t \in J}|x(t) - y(t)|$. This metric space (X, d) is not complete. In fact, an example of a Cauchy sequence without limit in X is given by any sequence of polynomials which converges uniformly on J to a continuous function, not a polynomial.

Nested Sequence

A sequence sets $A_1, A_2, A_3, \ldots$ is called nested if $A_1 \supset A_2 \supset A_3, \ldots$

Cantor's Intersection Theorem (Nested Interval Property)

A metric space (X, d) is complete if and only if every nested sequence of non-empty closed subset of X, whose diameter tends to zero, has a non-empty intersection.

Proof Suppose (X, d) is complete and let $A_1 \supset A_2 \supset A_3, \ldots$ be a nested sequence of closed subsets of X.

Since A_i is non-empty we choose a point x_n from each A_n. And then we will prove

$(x_1, x_2, x_3, \ldots)$ is Cauchy in X.

Let $\in > 0$ be given, since $\lim_{n\to\infty} d(A_n) = 0$ then there is $n_0 \in \mathbb{N}$ such that $d\left(A_{n_0}\right) < \in$. Then for $m, n > n_0$ we have $d(x_m, x_n) < \in$

This shows that (x_n) is Cauchy in X. Since X is complete so $x_n \to p \in X$ (Say)

We prove $p \in \cap_n A_n$. For this contrarily suppose that $p \notin \cap_n A_n$ then there exists $k \in \mathbb{N}$ such that $p \notin A_k$.

Since A_k is closed therefore $d(p, A_k) = \delta > 0$.

Consider the open ball $B\left(p; \frac{\delta}{2}\right)$ then A_k and $B\left(p; \frac{\delta}{2}\right)$ are disjoint.

Now $x_k, x_{k+1}, x_{k+2}, \ldots$ all belong to A_k then all these points do not belong to $B\left(p; \frac{\delta}{2}\right)$. This is a contradiction as p is the limit point of (x_n).

Hence $p \in \cap_n A_n$

Conversely, assume that every nested sequence of closed subset of X has a non-empty intersection. Let (x_n) be Cauchy in X, where $(x_n) = (x_1, x_2, x_3, \ldots)$. Then consider the sets

$A_1 = \{x_1, x_2, x_3, \ldots\}, A_2 = \{x_2, x_3, x_4, \ldots\}, \ldots, A_k = \{x_n : n \geq k\}$

Then we have $A_1 \supset A_2 \supset A_3, \ldots$

We prove $\lim_{n\to\infty} d(A_n) = 0$

Since (x_n) is a Cauchy Sequence then for any $\in > 0$ there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$ we have $d(x_m, x_n) < \in$. i.e. $\lim_{n\to\infty} d(A_n) = 0$

Now $d(\overline{A_n}) = d(A_n)$ then $\lim_{n\to\infty} d(A_n) = \lim_{n\to\infty} d(\overline{A_n}) = 0$

Also $\overline{A_1} \supset \overline{A_2} \supset \overline{A_3}, \ldots$ then by hypothesis $\cap_n \overline{A_n} \neq \varphi$. And let $p \in \cap_n \overline{A_n}$

We are to prove $x_n \to p \in X$.

Since $\lim_{n\to\infty} d(\overline{A_n}) = 0$ therefore $k_0 \in \mathbb{N}$ such that $d\left(\overline{A_{k_0}}\right) < \in$

Then for $n > k_0$ we have $x_n, p \in \overline{A_n}$ Implies $d(x_n, p) < \in$ for all $n > k_0$.

This proves that $x_n \to p \in X$ and we are done.

$\in -$Neighborhood of x_0:

An open ball $B(x_0; \in)$ of radius $\in > 0$ is often called an $\in -$Neighborhood of x_0.

Neighbourhood of a Point

Let (X, d) be a metric space and $x_0 \in X$ and a subset $N \subset X$ is called a neighbourhood of x_0 if there exists an open ball $B(x_0; \in)$ with centre x_0 such that $B(x_0; \in) \subset N$

Or By a neighborhood of x_0 we mean any subset of X which contains an $\in -$neighborhood of x_0.

Shortly “neighbourhood” is written as “nhood”.

Interior Point

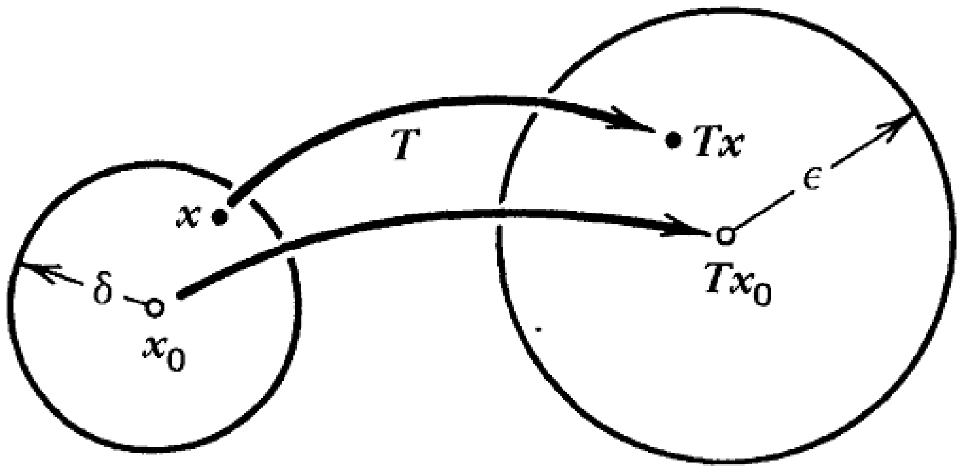
Let (X, d) be a metric space and $A \subset X$, a point $x_0 \in X$ is called an interior point of A if there is an open ball $B(x_0; r)$ with centre x_0 such that $B(x_0; r) \subset A$

Or We call x_0 an interior point of a set $A \subset X$ if A is a neighborhood of x_0. The set of all interior points of A is called interior of A and is denoted by $int(A)$ or A^0. It is the largest open set contain in A. i.e. $A^0 \subset A$.

Continuity

Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be metric spaces. A mapping (function) $T: X \rightarrow Y$ is said to be continuous at a point $x_0 \in X$ if for every $\in > 0$ there is a $\delta > 0$ such that $\tilde{d}(Tx, Tx_0) < \in$ for all x satisfying $d(x, x_0) < \delta$.

T is said to be continuous if it is continuous at every point of X.



Alternative:

$T: X \rightarrow Y$ is continuous at $x_0 \in X$ if for any $\in > 0$, there is a $\delta > 0$ such that

$x_0 \in B(x_0, \delta) \Rightarrow Tx_0 \in B(Tx_0, \in)$ or $T\big(B(x_0, \delta)\big) \subseteq B(Tx_0, \in)$

……………………………………………………………………

Remark

In calculus we usually write $y = f(x)$. A corresponding notation for the image of 'x' under T would be $T(x)$. However, to simplify formulas in functional analysis, it is customary to omit the parentheses and write Tx.

Theorem (Continuous mapping)

A mapping T of a metric space X into a metric space Y is continuous if and only if the inverse image of any open subset of Y is an open subset of X.

Or A mapping $T: X \to Y$ is continuous at $x_0 \in X$ if and only if $T^{-1}(O)$ is open in X. Where O is open in Y.

Proof

Suppose that T is continuous. Let O be an open subset of Y, we have to show that $T^{-1}(O)$ is open in X.

Let $x \in T^{-1}(O)$ then $Tx \in O$ and since O is open therefore there exists $\in > 0$ such that $B(Tx, \in) \subseteq O$. Also since T is continuous there exists $\delta > 0$ such that $T\big(B(x,\delta)\big) \subseteq B(Tx, \in) \subseteq O$. Then $B(x,\delta) \subseteq T^{-1}(O)$.

So that $T^{-1}(O)$ is open.

Conversely, assume that the inverse image of every open set in Y is an open set in X. i.e. $T^{-1}(O)$ is open where O is open subset of Y. We have to show that T is continuous. For this let $x \in X$ and let $\in > 0$ then $Tx \in Y$ and let $B(Tx, \in)$ be an open ball in Y, then by hypothesis $T^{-1}\big(B(Tx, \in)\big)$ is open in X.

Since $x \in X$ then there exists $\delta > 0$ such that $B(x,\delta) \subseteq T^{-1}\big(B(Tx, \in)\big)$

Implies $T\big(B(x,\delta)\big) \subseteq B(Tx, \in)$

Hence T is continuous at $x \in X$. Since $x \in X$ is an arbitrary number therefore T is continuous at every point of X.

Continuous mapping Theorem

A mapping $T: X \to Y$ of a metric space (X, d) into a metric space (Y, $\bar{d}$) is continuous at a point $x_0 \in X$ if and only if $x_n \to x_0$ implies $Tx_n \to Tx_0$

Proof Suppose that T is continuous at $x_0 \in X$. Then for all $\in > 0$, there is a $\delta > 0$ such that $d(x, x_0) < \delta \Rightarrow \bar{d}(Tx, Tx_0) < \in$ ....(1)

Suppose that $x_n \to x_0$ then there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \delta$; $\forall n > n_0$

By (1) $\bar{d}(Tx_n, Tx_0) < \in$; $\forall n > n_0$

Implies $Tx_n \to Tx_0$

Conversely Suppose that $x_n \to x_0$ implies $Tx_n \to Tx_0$

To prove T is continuous at x_0 suppose on contrary T is not continuous at x_0 then for all $\in > 0$, there is a $\delta > 0$ such that $d(x, x_0) < \delta \Rightarrow \bar{d}(Tx, Tx_0) \geq \in$

In particular $\delta = \frac{1}{n}$ there is an x_n such that $d(x_n, x_0) < \delta \Rightarrow \bar{d}(Tx_n, Tx_0) \geq \in$
$\Rightarrow x_n \to x_0 \Rightarrow Tx_n \nrightarrow Tx_0$ as $n \to \infty$ a contradiction.

Hence T is continuous at $x_0 \in X$

Isometry/Isometric mapping, Isometric Spaces

Let $X = (X, d)$ and $Y = (Y, d')$ be metric spaces. Then:

(a) A mapping T of X into Y is said to be **isometric** or an **isometry** if T preserves distances, that is, if for all $x, y \in X$, we have $d'(Tx, Ty) = d(x, y)$, where Tx and Ty are the images of x and y, respectively.

(b) The space X is said to be **isometric** with the space Y if there exists a bijective isometry of X onto Y. The spaces X and Y are then called isometric spaces.

Hence isometric spaces may differ at most by the nature of their points but are indistinguishable from the viewpoint of metric. And in any study in which the nature of the points does not matter, we may regard the two spaces as identical as two copies of the same "abstract" space.

Theorem (Completion)

For a metric space $X = (X, d)$ there exists a complete metric space $\hat{X} = (\hat{X}, \hat{d})$ which has a subspace W that is isometric with X and is dense in $\hat{X}$. This space $\hat{X}$ is unique except for isometries, that is, if $\tilde{X}$ is any complete metric space having a dense subspace $\widetilde{W}$ isometric with X, then $\tilde{X}$ and $\hat{X}$ are isometric.

Proof

The proof is somewhat lengthy, but straightforward. We subdivide it into four steps (a) to (d). We construct:

a) $\hat{X} = (\hat{X}, \hat{d})$
b) An isometry T of X onto W, where $\overline{W} = X$
c) Completeness of $\hat{X}$
d) Uniqueness of $\hat{X}$, except for isometries.

Construction of $\hat{X} = (\hat{X}, \hat{d})$

Let (x_n) and (x_n') be Cauchy sequences in X. Define (x_n) to be equivalent to (x_n'), written $(x_n) \sim (x_n')$, if $\lim_{n\to\infty} d(x_n, x_n') = 0$

Let $\hat{X}$ be the set of all equivalence classes $\hat{x}, \hat{y}, \ldots$ of Cauchy sequences thus obtained. We write $(x_n) \in \hat{x}$ to mean that (x_n) is a member of $\hat{x}$ (a representative of the class $\hat{x}$).

We now set $\hat{d}(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ where $(x_n) \in \hat{x}$ and $(y_n) \in \hat{y}$. We show that this limit exists. We have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

$$\Rightarrow d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n)$$

And a similar inequality with ‘m’ and ‘n’ interchanged. Together,

$$\Rightarrow -[d(x_n, x_m) + d(y_m, y_n)] \leq d(x_n, y_n) - d(x_m, y_m)$$

Hence $$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n)$$

Since (x_n) and (y_n) are Cauchy, we can make the right side as small as we please. This implies that the limit $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ exists because $\mathbb{R}$ is complete.

We must also show that the limit $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ is independent of the particular choice of representatives. In fact, if $(x_n)\sim(x_n{}')$ and $(y_n)\sim(y_n{}')$, then by $\lim_{n\to\infty} d(x_n, x_n{}') = 0$, we have

$|d(x_n, y_n) - d(x_n{}', y_n{}')| \leq d(x_n, x_n{}') + d(y_n, y_n') \to 0$ as $n \to \infty$,

Which implies the assertion $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n{}', y_n{}')$

We prove that $\hat{d}$ in $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ is a metric on $\hat{X}$. Obviously, $\hat{d}$ satisfies $\hat{d}(\hat{x},\hat{y}) \geq 0$ as well as $\hat{d}(\hat{x},\hat{x}) = 0$ and $\hat{d}(\hat{x},\hat{y}) = \hat{d}(\hat{y},\hat{x})$

Furthermore, $\hat{d}(\hat{x},\hat{y}) = 0 \Rightarrow (x_n)\sim(y_n) \Rightarrow \hat{x} = \hat{y}$

Gives $\hat{d}(\hat{x},\hat{y}) = 0$ if and only if $\hat{x} = \hat{y}$, and $\hat{d}(\hat{x},\hat{y}) \leq \hat{d}(\hat{x},\hat{z}) + \hat{d}(\hat{z},\hat{y})$ for $\hat{d}$ follows from $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$ by letting $n \to \infty$.

An isometry T of X onto W, where $\overline{W} = X$

Define a mapping $T: X \to W \subset \hat{X}$ by $Tb = \hat{b}$

We see that T is an isometry since $\hat{d}(\hat{x},\hat{y}) = \lim_{n\to\infty} d(x_n, y_n)$ becomes simply $\hat{d}(\hat{b},\hat{c}) = d(b,c)$. Here $\hat{c}$ is the class of (y_n) where $y_n = c$ for all 'n'.

Any isometry is injective, and $T: X \to W$ is surjective since $T(X) = W$. Hence W and X are isometric.

We show that W is dense in $\hat{X}$. We consider any $\hat{x} \in \hat{X}$. Let $(x_n) \in \hat{x}$. For every $\epsilon > 0$ there is an n_0 such that $d(x_n, x_{n_0}) < \frac{\epsilon}{2}$ $(n_0 > N)$.

Let $(x_{n_0}, x_{n_0}, \ldots) \in \hat{x}_{n_0}$ then $\hat{x}_{n_0} \in W$ then

$\hat{d}(\hat{x}, \hat{x}_{n_0}) = \lim_{n\to\infty} d(x_n, x_{n_0}) \leq \frac{\epsilon}{2} < \epsilon$

This shows that every ϵ - neighborhood of the arbitrary $\hat{x} \in \hat{X}$ contains an element of W. Hence W is dense in $\hat{X}$.

Completeness of $\widehat{X}$

Let $(\hat{x}_n)$ be any Cauchy sequence in $\hat{X}$. Since W is dense in $\hat{X}$, for every $\hat{x}_n$ there is a $\hat{z}_n \in W$ such that $\hat{d}(\hat{x}_n, \hat{z}_n) < \frac{1}{n}$

Hence by the triangle inequality,

$$\hat{d}(\hat{z}_m, \hat{z}_n) \leq \hat{d}(\hat{z}_m, \hat{x}_m) + \hat{d}(\hat{x}_m, \hat{x}_n) + \hat{d}(\hat{x}_n, \hat{z}_n) < \frac{1}{m} + \hat{d}(\hat{x}_m, \hat{x}_n) + \frac{1}{n}$$

and this is less than any given $\in > 0$ for sufficiently large m and n because $(\hat{x}_m)$ is Cauchy. Hence $(\hat{z}_m)$ is Cauchy. Since $T: X \rightarrow W$ is isometric and $\hat{z}_m \in W$, the sequence (z_m), where $z_m = T^{-1}\hat{z}_m$, is Cauchy in X.

Let $\hat{x} \in \hat{X}$ be the class to which (z_m) belongs. We show that '$\hat{x}$' is the limit of $(\hat{x})$. Now using triangular inequality

$$\hat{d}(\hat{x}_n, \hat{x}) \leq \hat{d}(\hat{x}_n, \hat{z}_n) + \hat{d}(\hat{z}_n, \hat{x}) < \frac{1}{n} + \hat{d}(\hat{z}_n, \hat{x}) \qquad \ldots\ldots\ldots\ldots(1)$$

Since $(z_m) \in \hat{x}$ (see right before) and $\hat{z}_n \in W$, so that $(z_n, z_n, \ldots) \in \hat{z}_n$ the inequality (1) becomes

$$\hat{d}(\hat{x}_n, \hat{x}) < \frac{1}{n} + \lim_{m\to\infty} d(z_n, z_m)$$

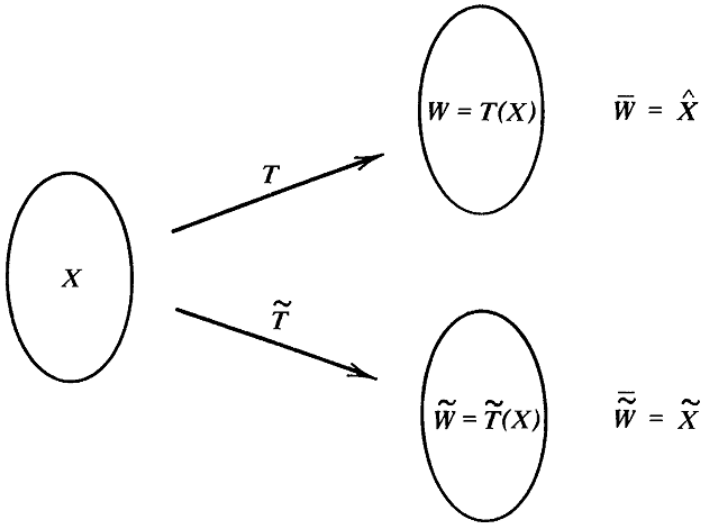
and the right side is smaller than any given $\in > 0$ for sufficiently large 'n'. Hence the arbitrary Cauchy sequence $(\hat{x}_n)$ in $\hat{X}$ has the limit $\hat{x} \in \hat{X}$, and $\hat{X}$ is complete.

Uniqueness of $\widehat{X}$ except for isometries

If $(\tilde{X}, \tilde{d})$ is another complete metric space with a subspace $\widetilde{W}$ dense in $\tilde{X}$ and isometric with X, then for any $\tilde{x}, \tilde{y} \in \tilde{X}$ we have sequences $(\tilde{x}_n)$, $(\tilde{y}_n)$ in $\widetilde{W}$ such that $\tilde{x}_n \rightarrow \tilde{x}$ and $\tilde{y}_n \rightarrow \tilde{y}$; hence $\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} \tilde{d}(\tilde{x}_n, \tilde{y}_n)$

Follows from $\left|\tilde{d}(\tilde{x}, \tilde{y}) - \tilde{d}(\tilde{x}_n, \tilde{y}_n)\right| \leq \tilde{d}(\tilde{x}, \tilde{x}_n) + \tilde{d}(\tilde{y}_n, \tilde{y}) \rightarrow 0$ as $n \rightarrow \infty$,

Since $\widetilde{W}$ is isometric with $W \subset \hat{X}$ and $\overline{W} = \hat{X}$, the distances on $\tilde{X}$ and $\hat{X}$ must be the same. Hence $\tilde{X}$ and $\hat{X}$ are isometric.



Rare (or nowhere dense in X)

Let X be a metric, a subset $M \subseteq X$ is called rare (or nowhere dense in X) if $\bar{M}$ has no interior point i.e. $int(\bar{M}) = \varphi$.

Meager (or of the first category)

Let X be a metric, a subset $M \subseteq X$ is called meager (or of the first category) if M can be expressed as a union of countably many rare subset of X.

Non-meager (or of the second category)

Let X be a metric, a subset $M \subseteq X$ is called non-meager (or of the second category) if it is not meager (of the first category) in X.

Or Let X be a metric, a subset $M \subseteq X$ is called non-meager (or of the second category) if M cannot be expressed as a union of countably many rare subset of X.

Example

Consider the set $\mathbb{Q}$ of rationales as a subset of a real line $\mathbb{R}$. Let $q \in \mathbb{Q}$, then $\{q\} = \overline{\{q\}}$ because $\mathbb{R} - \{q\} = (-\infty, q) \cup (q, \infty)$ is open. Clearly $\{q\}$ contain no open ball. Hence $\mathbb{Q}$ is nowhere dense in $\mathbb{R}$ as well as in $\mathbb{Q}$. Also since $\mathbb{Q}$ is countable, it is the countable union of subsets $\{q\}: q \in \mathbb{Q}$.

Thus $\mathbb{Q}$ is of the first category.

Bair's Category Theorem

If $X \neq \varphi$ is complete then it is non-meager in itself.

Or A complete metric space is of second category.

Proof Suppose that X is meager in itself then $X = \cup_1^\infty M_k$, where each M_k is rare in X. Since M_1 is rare then $int(\overline{M_1}) = \varphi$ i.e. $\overline{M_1}$ has no non-empty open subset but X has a non-empty open subset (i.e. X itself) then $\overline{M_1} \neq X$

This implies $\overline{M_1}^c = X - \overline{M_1}$ is a non-empty and open.

We choose a point $p_1 \in \overline{M_1}^c$ and an open ball $B_1 = B(p_1, \epsilon_1) \subseteq \overline{M_1}^c$ where $\epsilon_1 < \frac{1}{2}$

Now $\overline{M_2}^c$ is non-empty and open then there exists a point $p_2 \in \overline{M_2}^c$ and open ball $B_2 = B(p_2, \epsilon_2) \subseteq \overline{M_2}^c \cap B\left(p_1, \frac{\epsilon_1}{2}\right)$

($\overline{M_2}$ has no non-empty open subset then $\overline{M_2}^c \cap B\left(p_1, \frac{\epsilon_1}{2}\right)$ is non-empty and open.)

So we have chosen a point p_2 from the set $\overline{M_2}^c \cap B\left(p_1, \frac{\epsilon_1}{2}\right)$ and an open ball $B(p_2, \epsilon_2)$ around it, where $\epsilon_2 < \frac{\epsilon_1}{2} < \frac{1}{2} \cdot \frac{1}{2} < \frac{1}{2}$ then $\epsilon_2 < \frac{1}{2}$.

Proceeding in this way we obtain a sequence of balls B_k such that

$B_{k+1} \subseteq B\left(p_k, \frac{\epsilon_k}{2}\right) \subseteq B_k$ where $B_k = B(p_k, \epsilon_k)$ then the sequence of centers p_k is such that for $m > n$ then $d(p_k, p_{k-1}) < \frac{\epsilon_k}{2} < \frac{1}{2^{k+1}} \to 0$ as $k \to \infty$

Hence the sequence (p_k) is Cauchy. Since X is complete therefore $p_k \to p \in X$

Also $d(p_k, p) \leq d(p_k, p_{k-1}) + d(p_{k-1}, p)$

$d(p_k, p) < \frac{\epsilon_k}{2} + d(p_{k-1}, p) < \epsilon_k + d(p_{k-1}, p) \to \epsilon_k + 0$ as $k - 1 \to \infty$

$\Rightarrow p \in B_k \;; \forall k$ i.e. $p \in \overline{M_k}^c$ since $B_k = \overline{M_k}^c \cap B\left(p_{k-1}, \frac{\epsilon_{k-1}}{2}\right)$

$\Rightarrow B_k \subseteq \overline{M_k}^c \;; \forall k \Rightarrow B_k \cap \overline{M_k} = \varphi \;; \forall k \Rightarrow p \notin \overline{M_k} \;; \forall k \Rightarrow p \notin X \;; \forall k$

This is a contradiction. And hence the theorem.

NORMED SPACES, BANACH SPACES

Particularly useful and important metric spaces are obtained if we take a vector space and define on it a metric by means of a norm. The resulting space is called a normed space. If it is a complete metric space, it is called a Banach space. The theory of normed spaces, in particular Banach spaces, and the theory of linear operators defined on them are the most highly developed parts of functional analysis. The present chapter is devoted to the basic ideas of those theories.

Remark

- A **normed space** is a vector space with a metric defined by a norm; the latter generalizes the length of a vector in the plane or in three-dimensional space.
- A **Banach space** is a normed space which is a complete metric space. A normed space has a completion which is a Banach space. In a normed space we can also define and use infinite series.
- A mapping from a normed space X into a normed space Y is called an operator.
- A mapping from X into the scalar field **R** or **C** is called a functional. Of particular importance are so-called bounded linear operators and bounded linear functionals since they are continuous and take advantage of the vector space structure.
- It is basic that the set of all bounded linear operators from a given normed space X into a given normed space Y can be made into a normed space, which is denoted by **B(X, Y)**. Similarly, the set of all bounded linear functionals on X becomes a normed space, which is called the dual space **X'** of **X**.
- In analysis, infinite dimensional normed spaces are more important than finite dimensional ones. The latter are simpler and operators on them can be represented by matrices.
- We denote spaces by X and Y, operators by capital letters (preferably T), the image of an x under T by Tx (without parentheses), functionals by lowercase letters (preferably *f*) and the value of *f* at an 'x' by *f(x)* (with parentheses). This is a widely used practice.

We know that in many cases a vector space X may at the same time be a metric space because a metric d is defined on X. However, if there is no relation between the algebraic structure and the metric, we cannot expect a useful and applicable theory that combines algebraic and metric concepts. To guarantee such a relation between "algebraic" and "geometric" properties of X we define on X a metric d in a special way as follows. We first introduce an auxiliary concept, the norm (definition below), which uses the algebraic operations of vector space. Then we employ the norm to obtain a metric d that is of the desired kind. This idea leads to the concept of a normed space. It turns out that normed spaces are special enough to provide a basis for a rich and interesting theory, but general enough to include many concrete models of practical importance. In fact, a large number of metric spaces in analysis can be regarded as normed spaces, so that a normed space is probably the most important kind of space in functional analysis, at least from the viewpoint of present day applications.

Normed Space

A normed space X is a vector space with a norm defined on it, (Also called a normed vector space or normed linear space. The definition was given (independently) by S. Banach (1922), H. Hahn (1922) and N. Wiener (1922). The theory developed rapidly, as can be seen from the treatise by S. Banach (1932) published only ten years later.)

Let X be a vector space. A real valued function $\|.\|: X \to \mathbb{R}$ is called a norm on X if it satisfies the following axioms;

i. $\|x\| \geq 0 \quad ; \forall x \in X$
ii. $\|x\| = 0 \Leftrightarrow x = 0 \quad ; \forall x \in X$
iii. $\|\propto x\| = |\propto|\|x\| \quad ; \forall x \in X$ and all scalars $\propto \in \mathbb{R}$ or $\propto \in \mathbb{C}$
iv. $\|x + y\| \leq \|x\| + \|y\| \quad ; \forall x, y \in X$ (Triangle inequality);

Here 'x' and 'y' are arbitrary vectors in X and $\propto$ is any scalar.

The pair $(X, \|.\|)$ Is called normed space.

Example-1: *Show that every normed space X is a metric space with the metric $d: X\times X\to R$ defined by*

$$d(x,y)=\|x-y\|,\quad \forall\, x,y\in X$$

PU, 2016 (BS Math); PU, 2015 (M.Sc. Math)

Solution: In the following we check all the axioms of metric.

$M_1)$: $d(x,y)=\|x-y\|\geq 0, \qquad \forall\, x,y\in X$

$M_2)$: $d(x,y)=0 \Leftrightarrow \|x-y\|=0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$

$M_3)$: $d(x,y)=\|x-y\|=\|-(y-x)\|=|-1|\|y-x\|=\|y-x\|=d(y,x)$

$M_4)$: $d(x,z)=\|x-z\|=\|x-y+y-z\|\leq\|x-y\|+\|y-z\|$

$\Rightarrow d(x,z)\leq d(x,y)+d(y,z) \qquad \forall\, x,y,z\in X$

This shows that X is also a metric space.

Example-2: *Show that $\|P\|=|x|+|y|$ is a norm on R^2, where $P=(x,y)$, $x,y\in R$.*

Solution: $N_1)$: $\|P\|=|x|+|y|\geq 0, \qquad P(x,y)\in R^2$

$N_2)$: $\|P\|=0 \Leftrightarrow |x|+|y|=0$

$\Leftrightarrow |x|=0,\ |y|=0$

$\Leftrightarrow x=0,\ y=0$

$\Leftrightarrow (x,y)=(0,0)$

$\Leftrightarrow P=0$

$N_3)$: $\|\alpha P\|=\|\alpha(x,y)\|=\|(\alpha x,\alpha y)\|=|\alpha x|+|\alpha y|$

$=|\alpha||x|+|\alpha||y|=|\alpha|[|x|+|y|]$

$\Rightarrow \|\alpha P\|=|\alpha|\|P\|, \qquad \forall\, P\in R^2$

$N_4)$: $\|P_1+P_2\|=|x_1+x_2|+|y_1+y_2|\leq|x_1|+|x_2|+|y_1|+|y_2|$

$\Rightarrow \|P_1+P_2\|\leq|x_1|+|y_1|+|x_2|+|y_2|$

$\Rightarrow \|P_1+P_2\|\leq\|P_1\|+\|P_2\|, \qquad \forall\, P_1,P_2\in R^2$

This shows that $\|P\|=|x|+|y|$ is a norm on R^2.

Example-3: *Let $\underline{x}(x_1,x_2,..,x_n) \in R^n$, then show that $\|\underline{x}\| = \sum_{i=1}^{n}|x_i|$ is a norm on R^n.*

Solution: N_1): $\|\underline{x}\| = \sum_{i=1}^{n}|x_i| \geq 0 \quad \because |x_i| \geq 0,\ 1 \leq i \leq n$

N_2): $\|\underline{x}\| = 0 \Leftrightarrow \sum_{i=1}^{n}|x_i| = 0 \Leftrightarrow |x_i| = 0, \quad 1 \leq i \leq n$

$$\Leftrightarrow x_i = 0,\ 1 \leq i \leq n$$

$$\Leftrightarrow \underline{x} = 0$$

N_3): $\|\alpha\underline{x}\| = \sum_{i=1}^{n}|\alpha x_i| = |\alpha|\sum_{i=1}^{n}|x_i| = |\alpha|\|\underline{x}\|, \quad \forall \underline{x} \in R^n, \alpha \in F$

N_4): $\|\underline{x}+\underline{y}\| = \sum_{i=1}^{n}|x_i + y_i| = \sum_{i=1}^{n}|x_i + y_i|$

$$\leq \sum_{i=1}^{n}[|x_i| + |y_i|] = \sum_{i=1}^{n}|x_i| + \sum_{i=1}^{n}|y_i|$$

$\Rightarrow \|\underline{x}+\underline{y}\| \leq \|\underline{x}\| + \|\underline{y}\|, \qquad \forall \underline{x}, \underline{y} \in R^n$

Hence $\|\underline{x}\| = \sum_{i=1}^{n}|x_i|$ is a norm on R^n.

Example-4: *Let $P(x,y) \in R^2$, then show that $\|P\| = \max\{|x|,|y|\}$ is a norm on R^2.*

Solution: N_1): $\|P\| = \max\{|x|,|y|\} \geq 0$

N_2): $\|P\| = 0 \Leftrightarrow \max\{|x|,|y|\} = 0 \Leftrightarrow |x| = 0,\ |y| = 0$

$$\Leftrightarrow x = 0,\ y = 0$$

$$\Leftrightarrow (x,y) = (0,0)$$

$$\Leftrightarrow P = 0$$

N_3): $\|\alpha P\| = \max(|\alpha x|,|\alpha y|) = \max(|\alpha||x|,|\alpha||y|)$

$$= |\alpha|\max(|x|,|y|) = |\alpha|\|P\|$$

N_4): $\|P_1 + P_2\| = \max(|x_1 + x_2|,|y_1 + y_2|) \leq \max(\{|x_1| + |x_2|\},\{|y_1| + |y_2|\})$

$$\leq \max(|x_1|,|y_1|) + \max(|x_2|,|y_2|)$$

$\Rightarrow \|P_1 + P_2\| \leq \|P_1\| + \|P_2\|, \quad P_1, P_2 \in R^2$

Hence $\|P\| = \max\{|x|,|y|\}$ is a norm on R^2.

Example-5: *Show that the function $\|.\|: R^n \to R$ defined as*

$$\|\underline{x}\| = \max\{|x_i|, 1 \leq i \leq n\}$$

is a norm on R^n, where $\underline{x} = (x_1, x_2, ..., x_n) \in R^n$.

Solution: N_1): Since $\max\{|x_i|, 1 \leq i \leq n\}$, being the maximum of n nonnegative quantities, is nonnegative, therefore,

$$\|\underline{x}\| = \max\{|x_i|, 1 \leq i \leq n\} \geq 0, \quad \forall \underline{x} \in R^n$$

N_2): $\|\underline{x}\| = 0 \Leftrightarrow \max\{|x_i|, 1 \leq i \leq n\} = 0$

$$\Leftrightarrow |x_i| = 0, \qquad 1 \leq i \leq n$$

$$\Leftrightarrow x_i = 0, \qquad 1 \le i \le n.$$
$$\Leftrightarrow (x_1, x_2, \ldots, x_n) = 0$$
$$\Leftrightarrow \underline{x} = 0$$

N_3) $\|\alpha \underline{x}\| = \max\{|\alpha x_i|, 1 \le i \le n\} = \max\{|\alpha||x_i|, 1 \le i \le n\}$
$$= |\alpha| \max\{|x_i|, 1 \le i \le n\} = |\alpha| \|\underline{x}\|, \qquad \forall\, \underline{x} \in R^n, \alpha \in F$$

N_4) $\|\underline{x} + \underline{y}\| = \max\{|x_i + y_i|, 1 \le i \le n\} \le \max\{|x_i| + |y_i|, 1 \le i \le n\}$
$$\le \max\{|x_i|, 1 \le i \le n\} + \max\{|y_i|, 1 \le i \le n\}$$
$$\Rightarrow \|\underline{x} + \underline{y}\| \le \|\underline{x}\| + \|\underline{y}\|, \qquad \forall\, \underline{x}, \underline{y} \in R^n$$

This shows that $\|\underline{x}\| = \max\{|x_i|, 1 \le i \le n\}$ is a norm on R^n.

Example-6: *If* $(X, \|.\|)$ *is a normed space, then show that*
$$\big|\|x\| - \|y\|\big| \le \|x - y\|$$
for all $x, y \in X$.

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Solution: For any $x, y \in X$, we have
$$\|x\| = \|x - y + y\| \le \|x - y\| + \|y\|$$
$$\Rightarrow \|x\| - \|y\| \le \|x - y\| \qquad \ldots(1)$$
and
$$\|y\| = \|y - x + x\| \le \|y - x\| + \|x\|$$
$$\Rightarrow \|y\| \le \|x - y\| + \|x\| \because \|y - x\| = \|x - y\|$$
$$\Rightarrow -\|x - y\| \le \|x\| - \|y\| \qquad \ldots(2)$$

Combining (1) and (2), we have
$$-\|x - y\| \le \|x\| - \|y\| \le \|x - y\|$$
$$\Rightarrow \big|\|x\| - \|y\|\big| \le \|x - y\|$$

Example-7: *If $(X, \|.\|)$ is a normed space, then show that*

$$\big|\,\|x-z\|-\|y-z\|\,\big| \le \|x-y\|, \qquad \forall x, y, z \in X$$

Solution: Since $\|.\|$ is a norm on X, so by the triangle inequality

$$\|x-z\| \le \|x-y\| + \|y-z\|$$

$$\Rightarrow \|x-z\| - \|y-z\| \le \|x-y\| \qquad \ldots(1)$$

Similarly, $\|y-z\| \le \|y-x\| + \|x-z\|$

$$\Rightarrow \|y-z\| \le \|x-y\| + \|x-z\| \qquad \because \|y-x\| = \|x-y\|$$

$$\Rightarrow -\|x-y\| \le \|x-z\| - \|y-z\| \qquad \ldots(2)$$

Combining (1) and (2), we have

$$\Rightarrow -\|x-y\| \le \|x-z\| - \|y-z\| \le \|x-y\|$$

$$\Rightarrow \big|\,\|x-z\|-\|y-z\|\,\big| \le \|x-y\|, \ \forall x,y,z \in X$$

Example-8: *If $(X, \|.\|)$ is a normed space, then show that*

$$\big|\,\|x-a\|-\|y-b\|\,\big| \le \|x-y\| + \|a-b\|, \ \forall a,b,x,y,z \in X$$

Solution: Since $\|.\|$ is a norm on X, so by the triangle inequality

$$\|x-a\| \le \|x-y\| + \|y-b\| + \|b-a\|$$

$$\Rightarrow \|x-a\| \le \|x-y\| + \|y-b\| + \|a-b\| \qquad \because \|b-a\| = \|a-b\|$$

$$\Rightarrow \|x-a\| - \|y-b\| \le \|x-y\| + \|a-b\| \qquad \ldots(1)$$

Again using the triangle inequality, we have

$$\|y-b\| \le \|y-x\| + \|x-a\| + \|a-b\|$$

$$\Rightarrow \|y-b\| \le \|x-y\| + \|x-a\| + \|a-b\| \qquad \because \|y-x\| = \|x-y\|$$

$$\Rightarrow -\|x-y\| - \|a-b\| \le \|x-a\| - \|y-b\|$$

$$\Rightarrow -\big[\|x-y\| + \|a-b\|\big] \le \|x-a\| - \|y-b\| \qquad \ldots(2)$$

Combining (1) and (2), we have

$$-\big[\|x-y\| + \|a-b\|\big] \le \|x-a\| - \|y-b\| \le \|x-y\| + \|a-b\|$$

$$\Rightarrow \big|\,\|x-a\| - \|y-b\|\,\big| \le \|x-y\| + \|a-b\|$$

Example-9: *For all* x, y, z *in a normed space X, show that*

$$d(x+z, y+z) = d(x, y)$$

Solution: $d(x+z, y+z) = \|(x+z)-(y+z)\|$

$$= \|x+z-y-z\| = \|x-y\| = d(x, y)$$

Example-10: *For all* x, y, z *in a normed space X, show that*

$$d(x+y+a, y+a+z) = d(x, z)$$

Solution: $d(x+y+a, y+a+z) = \|(x+y+a)-(y+a+z)\|$

$$= \|x+y+a-y-a-z\| = \|x-z\| = d(x, z)$$

Example-11: *Let X be a normed linear space over a field F, then for all* $x, y \in X, \alpha \in F$, *show that* $d(\alpha x, \alpha y) = |\alpha| d(x, y)$.

Solution: $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = |\alpha|\|x-y\| = |\alpha| d(x, y)$

Example-14: *Let $(X,\|\cdot\|_X)$ and $(X,\|\cdot\|_Y)$ be two normed linear spa*

Let $P(x,y)\in X\times Y$, $x\in X, y\in Y$. Show that $\|P\|=\sqrt{\|x\|_X^2+\|y\|_Y^2}$

norm on $X\times Y$.

Solution: N_1): $\|P\|=\sqrt{\|x\|_X^2+\|y\|_Y^2}\geq 0$

N_2): $\|P\|=0\Leftrightarrow\sqrt{\|x\|_X^2+\|y\|_Y^2}=0\Leftrightarrow\|x\|_X^2+\|y\|_Y^2=0$

$$\Leftrightarrow\|x\|_X^2=0, \|y\|_Y^2=0\Leftrightarrow\|x\|_X=0, \|y\|_Y=0$$

$$\Leftrightarrow x=0, y=0 \qquad \because \|\cdot\|_X, \|\cdot\|_Y \text{ are norms}$$

$$\Leftrightarrow (x,y)=0$$

$$\Leftrightarrow P=0$$

N_3): $\|\alpha P\|=\sqrt{\|\alpha x\|_X^2+\|\alpha y\|_Y^2}=\sqrt{|\alpha|^2\|x\|_X^2+|\alpha|^2\|y\|_Y^2}$

$$=|\alpha|\sqrt{\|x\|_X^2+\|y\|_Y^2}=|\alpha|\|P\|$$

N_4): Let $P_3=(x_3,y_3)\in X\times Y$. Since $\|.\|_X, \|\cdot\|_Y$ are norms, so

$$\|x_1-x_3\|_X\leq\|x_1-x_2\|_X+\|x_2-x_3\|_X$$

and $\|y_1-y_3\|_Y\leq\|y_1-y_2\|_Y+\|y_2-y_3\|_Y$

therefore $\|P_1-P_3\|=\sqrt{\|x_1-x_3\|_X^2+\|y_1-y_3\|_Y^2}$

$$\Rightarrow\|P_1+P_2\|=\sqrt{\|x_1+x_2\|_X^2+\|y_1+y_2\|_Y^2}$$

$$\leq\sqrt{\left[\|x_1\|_X+\|x_2\|_X\right]^2+\left[\|y_1\|_Y+\|y_2\|_Y\right]^2} \quad ...(1)$$

$$\because \|\cdot\|_X, \|\cdot\|_Y \text{ are norms}$$

For $n=2$, Minkoski's inequality

$$\sqrt{\sum_{k=1}^{n}(a_k+b_k)^2} \le \sqrt{\sum_{k=1}^{n}a_k^2} + \sqrt{\sum_{k=1}^{n}b_k^2}$$

reduces to $\sqrt{(a_1+b_1)^2+(a_2+b_2)^2} \le \sqrt{a_1^2+a_2^2} + \sqrt{b_1^2+b_2^2}$ ...(2)

Putting $a_1 = \| x_1 \|_X, \quad b_1 = \| x_2 \|_X,$

$a_2 = \| y_1 \|_Y, \quad b_2 = \| y_2 \|_Y$

in (2), we have

$$\sqrt{[\| x_1 \|_X + \| x_2 \|_X]^2 + [\| y_1 \|_Y + \| y_2 \|_Y]^2} \le \sqrt{\| x_1 \|_X^2 + \| y_1 \|_Y^2} + \sqrt{\| x_2 \|_X^2 + \| y_2 \|_Y^2}$$

Combining this with (1), we have

$$\| P_1 + P_2 \| \le \sqrt{[\| x_1 \|_X + \| x_2 \|_X]^2 + [\| y_1 \|_Y + \| y_2 \|_Y]^2}$$

$$\le \sqrt{\| x_1 \|_X^2 + \| y_1 \|_Y^2} + \sqrt{\| x_2 \|_X^2 + \| y_2 \|_Y^2}$$

$$\Rightarrow \| P_1 + P_2 \| \le \| P_1 \| + \| P_2 \|$$

This shows that $\| P \| = \sqrt{\| x \|_X^2 + \| y \|_Y^2}$ is a norm on $X \times Y$.

Banach Space

A Banach space is a complete normed space (complete in the metric defined by the norm).

Metric Induced By the Norm

A metric d on a norm X can be defines by using the norm $\|.\|$ on X as

$d(x,y) = \|x-y\|$

Then this metric is called the matric induced by the norm. It has the following properties;

i. $d(x,y) = \|x-y\| \geq 0$
ii. $d(x,y) = \|x-y\| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$
iii. $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$
iv. $d(x,y) = \|x-y\| = \|x-z+z-y\| = \|x-z\| + \|z-y\|$
$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Remember that every normed space is a metric space but the converse is not true in general. i.e. Metric space needs not to be normed space.

Examples

- **Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$:** These are Banach spaces with norm defined by $\|x\| = \left(\sum_{j=1}^{n}\left|\xi_j\right|^2\right)^{\frac{1}{2}} = \sqrt{|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2}$
  in fact here norm induced the metric
  $d(x,y) = \|x-y\| = \sqrt{|\xi_1-\eta_1|^2 + |\xi_2-\eta_2|^2 + \cdots + |\xi_n-\eta_n|^2}$
- **Space l^p:** It is a Banach space with norm given by $\|x\| = \left(\sum_{j=1}^{\infty}\left|\xi_j\right|^p\right)^{\frac{1}{p}}$

  In fact, this norm induces the metric $d(x,y) = \|x-y\| = \left(\sum_{j=1}^{\infty}\left|\xi_j-\eta_j\right|^p\right)^{\frac{1}{p}}$

- **Space l^∞:** This is a Banach space obtained from the norm defined by $\|x\| = Sup_j\left|\xi_j\right|$
- **Space C[a, b]:** This is a Banach space with norm given by $\|x\| = max_{t\in J}|x(t)|$ where $J = [a,b]$.

2. The spaces $\mathbf{R}^n$ and $\mathbf{C}^n$ are Banach spaces. Here we prove that $\mathbf{R}^n$ is complete. The proof for completeness of $\mathbf{C}^n$ is similar.

Let $\{\mathbf{x}^{(p)}\}$ be a Cauchy sequence in $\mathbf{R}^n$,

$$x^{(p)} = (x_1^{(p)}, x_2^{(p)}, \cdots, x_n^{(p)}), \ p = 1, 2, \cdots$$

Then, given any $\varepsilon > 0$, there is a natural number n_0 such that

$$\forall p, q \quad p, q \geq n_0 \Rightarrow \|\mathbf{x}^{(p)} - \mathbf{x}^{(q)}\| = \sqrt{\sum_{i=1}^{n} |x_i^{(p)} - x_i^{(q)}|^2} < \varepsilon$$

Hence

$$\forall p, q; p, q \geq n_0 \Rightarrow |x_i^{(p)} - x_i^{(q)}| \leq \|\mathbf{x}^{(p)} - \mathbf{x}^{(q)}\| < \varepsilon$$

So, for each i, $\{x_i^{(p)}\}$ is a Cauchy sequence of real numbers. Since R is complete, $\{x_i^{(p)}\}$ converges to a real number x_i, say $i = 1, 2, \ldots, n$. But this implies that for the ε already chosen, there is a natural number p_i such that

$$\forall p; p \geq p_i \Rightarrow |x_i^{(p)} - x_i| < \varepsilon/\sqrt{n} \qquad (1)$$

Take $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, where $x_i = \lim_{p\to\infty} x_i^{(p)}$. Then $\mathbf{x} \in \mathbf{R}^n$. We show that $\lim_{p\to\infty} \mathbf{x}^{(p)} = \mathbf{x}$. Let $p_0 = \max(p_1, p_2, \ldots, p_n)$

$$\forall p; p \geq p_0 \geq n_0 \Rightarrow \|\mathbf{x}^{(p)} - \mathbf{x}\| = \sqrt{\sum_{i=1}^{n} |x_i^{(p)} - x_i|^2} < \varepsilon$$

by (1). Hence $\{\mathbf{x}^{(p)}\}$ converges to $\mathbf{x} \in \mathbf{R}^n$, as required. Thus $\mathbf{R}^n$ is complete and hence is a Banach space.

✓ **3. The Space l^∞.** Recall that This space consists of all bounded sequences $\mathbf{x} = \{x_i\}$ of real or complex numbers with addition and scalar multiplication defined by:

$$\mathbf{x} + \mathbf{y} = \{x_i + y_i\},$$
$$a\mathbf{x} = \{ax_i\}.$$

The norm in l^∞ is defined by:

$$\|\mathbf{x}\| = \operatorname*{Sup}_{i=1}^{\infty} |x_i|$$

We show that l^∞ is a Banach space.

Let $\{\mathbf{x}^{(p)}\}$ be any Cauchy sequence in l^∞, $\mathbf{x}^{(p)} = \{x_i^{(p)}\}$. Then, given any $\varepsilon > 0$, there is a natural number n_0 such that:

$$\forall p, q; p, q \geq n_0 \Rightarrow \|\mathbf{x}^{(p)} - \mathbf{x}^{(q)}\| = \operatorname*{Sup}_{i=1}^{\infty} |x_i^{(p)} - x_i^{(q)}| < \varepsilon$$

So, for each $i = 1, 2, \ldots,$

$$\forall p, q; p, q \geq n_0 \Rightarrow |x_i^{(p)} - x_i^{(q)}| \leq \|\mathbf{x}^{(p)} - \mathbf{x}^{(q)}\| < \varepsilon$$

Hence $\{x_i^{(p)}\}$ is a Cauchy sequence of real (or complex) numbers. Since $\mathbf{R}$ (or $\mathbf{C}$) is complete, $\{x_i^{(p)}\}$ converges to x_i, say, for each $i = 1, 2, \ldots$

Take $x = \{x_i\}$. We show that $x \in l^\infty$ and $\lim_{p\to\infty} x^{(p)} = x$.

Since $x_i^{(p)} \to x_i$, there is a natural number n_1 such that

$$\forall p; p \geq n_1 \Rightarrow |x_i^{(p)} - x_i| < \varepsilon/2, i = 1, 2, \ldots. \tag{1}$$

That is,

$$\forall p, p \geq n_1 \Rightarrow \|\mathbf{x}^{(p)} - \mathbf{x}\| = \operatorname*{Sup}_{i=1}^{\infty} |x_i^{(p)} - x_i| \leq \varepsilon/2 < \varepsilon$$

Hence $\mathbf{x}^{(p)} \to \mathbf{x}$. Also, from (1),

$$\begin{aligned} |x_i| &= |x_i - x_i^{(p)} + x_i^{(p)}| \\ &\leq |x_i - x_i^{(p)}| + |x_i^{(p)}| \\ &< \varepsilon/2 + k_p \end{aligned}$$

Now $\varepsilon/2 + k_p$ is a finite number, independent of i. Hence $\mathbf{x} = \{x_i\} \in l^\infty$. This proves the completeness of l^∞.

Sequence Space ‘s’ is a metric space but not a normed space

This space consists of the set of all (bounded or unbounded) sequences of complex numbers i.e. $s = \{x : x \text{ is bounded or unbounded}\}$ and the metric d defined by

$d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$ Where $x = (\xi_j)$ and $y = (\eta_j)$.

Solution Define a metric $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$

We have already shown that it is a metric space. We just show that it is normed space or not. For this suppose that d is induced by norm $\|.\|$ on s. Then

$d(x,y) = \|x - y\| \Rightarrow d(x,0) = \|x\| \geq 0 \Rightarrow \|x\| \geq 0$

$\Rightarrow \|x\| = d(x,0) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j|}{1+|\xi_j|}$

$\Rightarrow \|\propto x\| = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\propto \xi_j|}{1+|\propto \xi_j|} \neq \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\propto||\xi_j|}{1+|\propto||\xi_j|} \neq |\propto|\|x\| \Rightarrow \|\propto x\| \neq |\propto|\|x\|$

Hence s cannot be a normed space.

Euclidean space $\mathbb{R}^n$ and unitary space $\mathbb{C}^n$: These are Banach spaces with norm defined by $\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{\frac{1}{2}} = \sqrt{|\xi_1|^2 + |\xi_2|^2 + \cdots + |\xi_n|^2}$ in fact here norm induced the metric

$d(x,y) = \|x - y\| = \sqrt{|\xi_1 - \eta_1|^2 + |\xi_2 - \eta_2|^2 + \cdots + |\xi_n - \eta_n|^2}$

Proof

We have already shown that these are complete metric spaces. We just show that these are normed spaces. Let $\|.\|$ be a norm on $\mathbb{R}^n$ defined by

$\|x\| = \left(\sum_{j=1}^{n} |\xi_j|^2\right)^{\frac{1}{2}}$ where $x = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ then

i. $\|x\| \geq 0$; $\forall x \in \mathbb{R}^n$ since $\left(\sum_{j=1}^{n} |\xi_j|^2\right)^{\frac{1}{2}} \geq 0$

ii. $\|x\| = \left(\sum_{j=1}^{n}\left|\xi_j\right|^2\right)^{\frac{1}{2}} = 0 \Leftrightarrow \left|\xi_j\right| = 0 \Leftrightarrow x = 0 \quad ;\forall x \in \mathbb{R}^n, i = 1,2,\dots,n$

iii. $\|\propto x\| = \left(\sum_{j=1}^{n}\left|\propto \xi_j\right|^2\right)^{\frac{1}{2}} = \sqrt{\propto^2}\left(\sum_{j=1}^{n}\left|\xi_j\right|^2\right)^{\frac{1}{2}} = |\propto|\|x\| \quad ;\forall x \in \mathbb{R}^n$ and all scalars $\propto \in \mathbb{R}$ or $\propto \in \mathbb{C}$

iv. $\|x+y\| = \left(\sum_{j=1}^{n}\left|\xi_j + \eta_j\right|^2\right)^{\frac{1}{2}} \le \left(\sum_{j=1}^{n}\left|\xi_j\right|^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{n}\left|\eta_j\right|^2\right)^{\frac{1}{2}}$
$\|x+y\| \le \|x\| + \|y\| \quad ;\forall x, y \in \mathbb{R}^n$ (Triangle inequality)

Here 'x' and 'y' are arbitrary vectors in $\mathbb{R}^n$ and $\propto$ is any scalar. The pair $(\mathbb{R}^n, \|.\|)$ Is normed space. And hence $(\mathbb{R}^n, \|.\|)$ Is Banach (complete metric + normed) space.

Remark

By similar argument we can show that $(\mathbb{C}^n, \|.\|)$ Is Banach (complete metric + normed) space.

Space l^p: The space l^p with $p \ge 1$ is a Banach space with norm given by $\|x\| = \left(\sum_{j=1}^{\infty}\left|\xi_j\right|^p\right)^{\frac{1}{p}}$

Solution Let $\{x_n\}$ be a Cauchy Sequence in l^p where $\{x_n\} = \left\{x_j^{(n)}\right\}_1^\infty$ then for any $\epsilon > 0$ there exists $n_0 \epsilon \mathbb{N}$ such that

$$\|x_n - x_m\| = \sqrt[p]{\sum_1^\infty \left|x_j^{(n)} - x_j^{(m)}\right|^p} < \epsilon; \forall m, n \ge n_0$$

$\Rightarrow \left|x_j^{(n)} - x_j^{(m)}\right| < \epsilon; \forall m, n \ge n_0 \Rightarrow x_j^{(n)}$ is a Cacuchy sequence in **R** and since **R** is complete Therefore $x_j^{(n)} \to x_j \in \boldsymbol{R}$ then $\left|x_j^{(n)} - x_j\right| \to 0$ for each j

Next suppose that $x = \{x_j\}$ then $\|x_n - x\| = \left(\sum_1^\infty \left|x_j^{(n)} - x_j\right|^p\right)^{\frac{1}{p}} \to 0$ Since $\left|x_j^{(n)} - x_j\right| \to 0$ for each $j \Rightarrow x_n \to x$

Now $x = x_n - (x_n - x) \in l^p$ then $x_n \to x \in l^p$. This shows that $\{x_n\}$ converges in l^p. Hence l^p is a Hilbert space.

Space C[a, b]: This is a Banach space with norm given by $\|x\| = max_{t\in j}|x(t)|$ where $J = [a, b]$

Proof

We have already shown that it is complete metric space. We just show that this is normed spaces. Let $\|.\|$ be a norm on **C[a, b]** defined by $\|x\| = max_{t\in j}|x(t)|$ then

i. $\|x\| \geq 0$ since $max_{t\in j}|x(t)| \geq 0$
ii. $\|x\| = max_{t\in j}|x(t)| = 0 \Leftrightarrow |x(t)| = 0 \Leftrightarrow x = 0$
iii. $\|\propto x\| = max_{t\in j}|\propto x(t)| = |\propto| max_{t\in j}|x(t)| = |\propto|\|x\|$
iv. $\|x + y\| = max_{t\in j}|x(t) + y(t)| \leq max_{t\in j}|x(t)| + max_{t\in j}|y(t)|$
$\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)

Hence $(\mathbf{C[a, b]}, \|.\|)$ Is Banach (complete metric + normed) space.

Space C[a, b]:

This is not a Banach space with norm given by $\|x\| = \int_a^b |x(t)|\, dt$

Proof

We have already shown that it is not complete metric space. We just show that this is normed spaces. Let $\|.\|$ be a norm on **C[a, b]** defined by $\|x\| = \int_a^b |x(t)|\, dt$ then

i. $\|x\| \geq 0$ since $\int_a^b |x(t)|\, dt \geq 0$
ii. $\|x\| = \int_a^b |x(t)|\, dt = 0 \Leftrightarrow a = b \text{ or } |x(t)| = 0 \Leftrightarrow x = 0$
iii. $\|\propto x\| = \int_a^b |\propto x(t)|\, dt = |\propto| \int_a^b |x(t)|\, dt = |\propto|\|x\|$
iv. $\|x + y\| = \int_a^b |x(t) + y(t)|\, dt \leq \int_a^b |x(t)|\, dt + \int_a^b |y(t)|\, dt$
$\|x + y\| \leq \|x\| + \|y\|$ (Triangle inequality)
Hence $(\mathbf{C[a, b]}, \|.\|)$ Is not a Banach (metric + normed) space.

Remark: Since R is a Banach space and Q is subspace of R such that Q is not closed, so Q is not a Banach space.

11.6 QUOTIENT SPACES

Let N be a normed space and S a subspace of N. For any $x \in N$, the set

$$x + S = \{x + s : s \in S\}$$

is called a *coset of S determined by x or a translate of S by x.* The set

$$\{x + S : x \in N\}$$

of all cosets of S in N is a linear space under addition and scalar multiplication defined by:

$$x + S + y + S = x + y + S, x, y \in N$$

and $$\alpha(x + S) = \alpha x + S, x \in N, \alpha \in F,$$

This set of cosets of S in N is called the *quotient space* of N by S and is denoted by N/S.

For any subspace S of a linear space N, the dimension of N/S is called the *deficiency* of S.

As a special case, a subspace of N having deficiency 1 is called a *hyperplane* in N. Its cosets are also hyperplanes.

We can make N/S a normed linear space as follows:

Let $\|.\|$ be the norm in N. For an $x + S \in N/S$, put

$$\|x + S\|_1 = \operatorname*{Inf}_{s \in S} \|x + s\| = d(x, S) \qquad (1)$$

where d is the metric induced by the norm $\|.\|$ on N.

11.6.1 Theorem

If S is a closed subspace of a normed space $(N, \|.\|)$, then N/S is also a normed space under the norm defined by

$$\|x + S\|_1 = \operatorname*{Inf}_{s \in S} \|x + s\| = d(x, S) \qquad (1)$$

Proof

N_1: Clearly

$$\|x + S\|_1 \geq 0$$

Also $\|x + S\|_1 = 0$ if and only if $\underset{s \in S}{\text{Inf}} \|x + s\| = 0$, so that, by the property of infimum, there is a sequence $\{s_n\}$ in S such that

$$\|x + s_n\| \to 0 \text{ as } n \to \infty$$

But then $x + s_n \to 0$ that is $s_n \to -x$ as $n \to \infty$. Since S is a closed subspace, $x \in S$. Hence

$$x + S = S, \text{ the zero element of } N/S.$$

N_2. Let $x + S, y + S \in N/S$, $x, y \in N$. Then

$$x + S + y + S = x + y + S \in N/S$$

By definition of $\|.\|_1$ in N/S, there are sequences $\{x_n\}$ and $\{y_n\}$ in S such that

$$\lim_{n\to\infty} \|x + x_n\| = \|x + S\|_1, \lim_{n\to\infty} \|y + y_n\| = \|y + S\|_1$$

Hence, for any x, y in N and the definition of infimum,

$$\|x + S + y + S\|_1 = \|x + y + S\|_1 \le \|x + y + x_n + y_n\|$$
$$\le \|x + x_n\| + \|y + y_n\|$$

Taking limit as $n \to \infty$, we have:

$$\|x + S + y + S\|_1 = \|x + y + S\|_1$$
$$\le \lim_{n\to\infty} \|x + x_n\| + \lim_{n\to\infty} \|y+y_n\|$$
$$\le \|x + S\|_1 + \|y + S\|_1$$

so that N_2 is satisfied.

N_3. For any scalar α and $x + S \in N/S$, consider the element

$$\alpha(x + S) = \alpha x + S$$

If $\alpha = 0$, then

$$\|\alpha(x + S)\|_1 = \|0x + S\|_1 = \|S\|_1 = 0 = |\alpha| \|x + S\|_1$$

So let $\alpha \neq 0$. Then

$$\| ax + S \|_1 = \inf_{s \in S} \| ax + s \|$$
$$= \inf_{s' \in S} \|\alpha x + \alpha s'\|$$
$$= |\alpha| \inf_{s' \in S} \|x + s'\|$$
$$= |\alpha| \, \|x + S\|_1$$

Hence $(N/S, \|.\|_1)$ is a normed space.

We now discuss the structure of N/S, S a closed subspace of N, in relation to that of N. That is, we discuss the question of completeness of N/S if N is complete. To this end we prove:

✓ 11.6.2 Theorem

Let S be a closed subspace of a Banach space N. Then N / S, with the norm defined by (1), is also a Banach space.

Proof

To prove that N/S is a Banach space, we have to prove that every Cauchy sequence in N/S converges to a point of N/S. Since a Cauchy sequence is convergent if and only if it has a convergent subsequence, we shall show that every Cauchy sequence in N/S contains a convergent subsequence.

Let $\{x_n + S\}$, $x_n \in N$ be a Cauchy sequence in N/S. Then, given any $\varepsilon > 0$, there is a natural number n_1 such that:

$$\forall m, n; m, n \geq n_1 \Rightarrow \|x_m + S - (x_n + S)\|_1 = \|x_m - x_n + S\|_1 < \varepsilon$$

Take $\varepsilon = 1/2$ and $m = n_1$, $n = n_1 + 1$. Then

$$\|x_{n_1} + S - (x_{n_1+1} + S) = \|x_{n_1} - x_{n_1+1} + S\|_1 < 1/2$$

If we choose $\varepsilon = 1/4$, then there is a natural number n_2 such that

$$\|x_{n_2} + S - (x_{n_2+1} + S)\|_1 = \|x_{n_2} - x_{n_2+1} + S\|_1 < 1/4$$

Continuing in this way, we see that, in general, there is a natural number n_k such that

$$\| x_{n_k} + S - (x_{n_k+1} + S) \|_1 = \| x_{n_k} - x_{n_k+1} + S \|_1 < \frac{1}{2^k}$$

In each $x_{n_k} + S$ and $x_{n_k+1} + S$, select vectors y_k, y_{k+1} respectively such that

$$\|y_k - y_{k+1}\| < 1/2^k$$

Then, for any $k' > k$,

$$\begin{aligned} \|y_k - y_{k'}\| &= \|y_k - y_{k+1} + y_{k+1} - y_{k+2} + \ldots + y_{k'-1} - y_{k'}\| \\ &\le \|y_k - y_{k+1}\| + \|y_{k+1} - y_{k+2}\| + \ldots + \|y_{k'-1} - y_{k'}\| \\ &< \frac{1}{2^k} + \frac{1}{2^{k+1}} + \ldots. + \frac{1}{2^{k'}} \\ &< \frac{\frac{1}{2^k}}{1 - \frac{1}{2}} = \frac{1}{2^{k-1}} \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus $\{y_k\}$ is a Cauchy sequence in N. Since N is complete, $\{y_k\}$ converges to a point y of N. Hence

$$\|x_{n_k} + S - (y + S)\|_1 \le \|y_k - y\| \to 0 \text{ as } k \to \infty$$

so that the subsequence.

$$x_{n_k} + S \to y + S \in N/S$$

But then $x_n + S \to y + S$ by Theorem 5.1.4. Hence N/S is complete.

Reference

5.1.4 Theorem

In a metric space (X, d), every convergent sequence is a Cauchy sequence.

Can every metric on a vector space be obtained from a norm? The answer is no. A counterexample is the space s. In fact, s is a vector space, but its metric d defined by $d(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_j - \eta_j|}{1+|\xi_j - \eta_j|}$ cannot be obtained from a norm. This may immediately be seen from the following lemma which states two basic properties of a metric d obtained from a norm. The first property, as expressed by (i), is called *the translation invariance* of d.

Lemma (Translation invariance)

A metric d induced by a norm on a normed space X satisfies.

i. $d(x+a, y+a) = d(x,y)$
ii. $d(\propto x, \propto y) = |\propto|\, d(x,y)$

For all $x, y, a \in X$ and every scalar $\propto$.

Proof We have

$$d(x+a, y+a) = \|x+a-(y+a)\| = \|x-y\| = d(x,y)$$

$$d(\propto x, \propto y) = \|\propto x - \propto y\| = |\propto|\|x-y\| = |\propto|\, d(x,y)$$

- **Unit sphere** The sphere $S(0;1) = \{x \in X : \|x\| = 1\}$ in a normed space X is called the unit sphere.
- **Bounded set** A subset M in a normed space X is bounded if and only if there is a positive number c such that $\|x\| \le c$for every $x \in M$.
- **Closed subspace** a subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be induced by the norm on X. If Y is closed in X, then Y is called a closed subspace of X.
- **Subspace of a Banach space** a subspace Y of a Banach space X is a subspace of X considered as a normed space.
- **Theorem (Subspace of a Banach space)** A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Theorem (Subspace of a Banach space)

A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Proof

Suppose that Y is complete then we have to show that Y is closed. i.e. $Y = \bar{Y}$
Obviously $Y \subseteq \bar{Y}$ ………….(1)

Let $y \in \bar{Y}$ then there exists a sequence $(y_n)_1^\infty$ in Y such that $y_n \to y$, then $(y_n)_1^\infty$ being a convergent sequence is a Cauchy Sequence in Y. And since Y is complete therefore $y_n \to y \in Y$. i.e. $y \in Y$ then $\bar{Y} \subseteq Y$ ………….(2)

From (1) and (2) we have $Y = \bar{Y}$ Hence Y is closed.

Conversely suppose that Y is closed. i.e. $Y = \bar{Y}$ then we are to show that Y is complete. For this let $(y_n)_1^\infty$ be an arbitrary Cauchy Sequence in Y then $(y_n)_1^\infty$ will be a Cauchy Sequence in X. And as X is complete then $y_n \to y \in X$ (X is Banach) Then $y \in \bar{Y}$ ($\because$ y is limit point of Y i.e. $y \in Y^d$) Then $y \in Y$ $\because \bar{Y} = Y$

So that $y_n \to y \in Y$. Hence Y is complete.

Useful Definitions

- A sequence (x_n) in a normed space $(X, \|.\|)$ is **convergent** if X contains an x such that $\lim_{n\to\infty}\|x_n - x\| = 0$
  Then we write $x_n \to x$ and call 'x' the limit of (x_n).
- A sequence (x_n) in a normed space X is **Cauchy** if for every $\in > 0$ there is an n_0 such that $\|x_m - x_n\| <\in$ for all m, n > n_0
- If (x_k) is a sequence in a normed space X, then $\sum_1^\infty x_k$ is a series in X. we can associate with (x_k) the sequence (s_n) of partial sums $s_n = x_1 + x_2 + \cdots + x_n$ where $n = 1, 2, \ldots$ If (s_n) is convergent, say, $\lim_{n\to\infty} s_n = s$ then we say that the series $\sum_1^\infty x_k$ is convergent and we write $\sum_1^\infty x_k = s$ and if $\sum_1^\infty \|x_k\|$ converses then the series $\sum_1^\infty x_k$ is said to be absolutely convergent. However, we warn the reader that in a normed space X, absolute convergence implies convergence if and only if X is complete.

Note

- In case of $\mathbb{R}$ and $\mathbb{C}$ we have absolute convergence implies convergence. i.e. $\sum_1^\infty |x_k| < \infty \Rightarrow \sum_1^\infty x_k < \infty$
- But in normed space in general we have absolute convergence does not implies convergence. i.e. $\sum_1^\infty \|x_k\| < \infty \nRightarrow \sum_1^\infty x_k < \infty$

Example

Consider $l^\infty = \left\{x = \left(\xi_j\right)_1^\infty : \xi_j \text{ is Real (Complex)} \& Sup_j\left|\xi_j\right| < \infty\right\}$ with $\|x\| = Sup_j\left|\xi_j\right|$ and the induced metric on l^∞ is given by

$d(x,y) = \|x-y\| = Sup_j\left|\xi_j - \eta_j\right|$ and we clearly know that $(l^\infty, \|.\|)$ Is Banach Space. Let Y be the set of all sequences with only finitely many non – zero terms. i.e. $Y = \left\{y = \left(\eta_j\right)_1^\infty : \eta_j = 0 \ ; \forall j > n\right\}$ then $Y \subseteq l^\infty$ as $Sup_j\left|\eta_j\right| < \infty$.
$(y_n)_1^\infty = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right)$ lies in Y but
$\lim_{n\to\infty}(y_n)_1^\infty = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \dots\right) \notin Y$ then Y is not closed and therefore is not complete. Now we will show that absolute convergence does not implies convergence.

Define $(y_n)_1^\infty = \left(\eta_j^{(n)}\right)_1^\infty$ where $\eta_j^{(n)} = \begin{cases} \frac{1}{n^2} & ; j = n \\ 0 & ; j \neq n \end{cases}$ then

$y_1 = \eta_j^{(1)} = \left(\frac{1}{1^2}, 0, 0, \dots\right)$

$y_2 = \eta_j^{(2)} = \left(0, \frac{1}{2^2}, 0, 0, \dots\right)$

$\vdots \qquad\qquad\qquad \vdots$

$y_n = \eta_j^{(n)} = \left(0, 0, \dots, \frac{1}{n^2}, 0, 0, \dots\right)$

$\Rightarrow \|y_1\| = \frac{1}{1^2} \ , \ \|y_2\| = \frac{1}{2^2} \ , \dots, \ , \ \|y_n\| = \frac{1}{n^2}$

$\Rightarrow \sum_1^\infty \|y_n\| = \frac{1}{1^2} + \frac{1}{2^2} + \cdots = \sum_1^\infty \frac{1}{n^2} < \infty$

$\Rightarrow \sum_1^\infty \|y_n\| < \infty$ i.e. sequence is absolutely convergent.

($\sum_1^\infty \frac{1}{n^p}$ converges if $p > 1$)

But $\sum_1^\infty y_n = \left(\frac{1}{1^2}, \frac{1}{2^2}, \dots \frac{1}{n^2}, \frac{1}{(n+1)^2}, \frac{1}{(n+2)^2}, \dots\right) \notin Y \subseteq l^\infty \Rightarrow \sum_1^\infty y_n$ is not convergent. Hence in a normed space absolute convergence does not implies convergence.

Remember

- A subspace of a vector space X is a nonempty subset Y of X such that for all $y_1, y_2 \in Y$ and all scalars $\propto, \beta$ we have $\propto y_1 + \beta y_2 \in Y$. Hence Y is itself a vector space, the two algebraic operations being those induced from X. A special subspace of X is the *improper subspace* Y = X. Every other subspace of X $\{\neq \{0\}\}$ is called *proper*.
- Another special subspace of any vector space X is $Y = \{0\}$.
- A ***linear combination*** of vectors $x_1, x_2, \dots, x_m$ of a vector space X is an expression of the form $\propto_1 x_1 + \propto_2 x_2 + \cdots + \propto_m x_m$ where the coefficients $\propto_1, \propto_2, \dots, \propto_m$ are any scalars.
- For any nonempty subset $M \subseteq X$ the set of all linear combinations of vectors of M is called the span of M, written ***spanM***.
  Obviously, this is a subspace Yof X, and we say that Y is **spanned** or **generated** by M.
- **Linear independence, linear dependence:** Linear independence and dependence of a given set M of vectors $x_1, x_2, \dots, x_m$ $(m \geq 1)$ in a vector space X are defined by means of the equation
  $\propto_1 x_1 + \propto_2 x_2 + \cdots + \propto_m x_m = 0$
  where $\propto_1, \propto_2, \dots, \propto_m$ are scalars. Clearly, equation (above) holds for $\propto_1 = \propto_2 = \cdots = \propto_m = 0$. If this is the only $m - tuple$ of scalars for which equation holds, the set M is said to be linearly independent. M is said to be linearly dependent if M is not linearly independent, that is, if equation also holds for some $m - tuple$ of scalars, not all zero.
- An arbitrary subset M of X is said to be linearly independent if every nonempty finite subset of M is linearly independent. M is said to the linearly dependent if M is not linearly independent.

Schauder basis (or basis)

If a normed space $(X, \|.\|)$ contains a sequence (e_n) with the property that for every $x \in X$ there is a unique sequence of scalars $(\propto_n)$ such that

$\|x - (\propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n)\| \to 0$ as $n \to \infty$

Then (e_n) is called a Schauder basis (or basis) for X.

The series $\sum_1^\infty \propto_k e_k$ which has the sum X is then called the expansion of x with respect to (e_n), and we write $x = \sum_1^\infty \propto_k e_k$

Theorem

Show that if a normed space $(X, \|.\|)$ has a Schauder basis, it is separable.

Proof

Case – I: Suppose that $(X, \|.\|)$ Is a real normed space. Let $(e_n)_1^\infty$ be a Schauder basis for X. Let $M = \{x : x = \propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n \ ; \ \propto_j \in \mathbb{Q}\}$.

Since $\mathbb{Q}$ is countable therefore is countable M.

Now suppose $z \in X$, since $(e_n)_1^\infty$ is a Schauder basis for X, therefore there exists a sequence of scalars (real numbers) $(\propto_n)_1^\infty$ such that

$\|z - (\propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n)\| \to 0$ as $n \to \infty$

Then for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $\left\|z - \sum_1^n \propto_j e_j\right\| < \in \ ; \forall n > n_0$

$\Rightarrow z = \sum_1^n \propto_j e_j$ as $n \to \infty$

If $\propto_i \in \mathbb{Q}$ then $\sum_1^n \propto_j e_j \in M$

$\Rightarrow z \in M \Rightarrow z \in M \cup M^d \Rightarrow z \in \overline{M}$

If $\propto_j \notin \mathbb{Q}$ then we can approximate $\propto_j$ by rationals so that we can find corresponding combination $\propto_1' e_1 + \propto_2' e_2 + \cdots + \propto_n' e_n$, where $\propto_j' \in \mathbb{Q}$ which will approximate z. i.e. $\left\|z - \sum_1^n \propto_j e_j\right\| < \in$

Since z and $\in$ were arbitrary, so we conclude that every ball with center z and radius $\in$ contains an element of M implies $z \in M^d$

$\Rightarrow z \in M \cup M^d \Rightarrow z \in \overline{M}$

So M is dense and Hence X is separable.

Case – II: Suppose that $(X, \|.\|)$ Is a Complex normed space. Let $(e_n)_1^\infty$ be a Schauder basis for X. Let $M = \{x: x = \propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n\}$ where $Re \propto_j, Img \propto_j \in \mathbb{Q}$ then Since $\mathbb{Q}$ is countable therefore is countable M.

By similar argument as above we can show that $\overline{M} = X$

So M is dense and Hence X is separable.

Conversely, does every separable Banach space have a Schauder basis?

This is a famous question raised by Banach himself about forty years ago. Almost all known separable Banach spaces had been shown to possess a Schauder basis. Nevertheless, the surprising answer to the question is no. It was given only quite recently, by P. Enflo (1973) who was able to construct a separable Banach space which has no Schauder basis.

Theorem (No need to Prove)

Let $X = (X, \|.\|)$ be a normed space. Then there is a Banach space $\hat{X}$ and an isometry A from X onto a subspace W of $\hat{X}$ which is dense in $\hat{X}$. The space $\hat{X}$ is unique, except for isometries.

Theorem (Banach's Criterion)

A normed vector space X is complete if andonly if every absolutely convergent series in X is convergent.

Finite Dimensional Normed Spaces and Subspaces

Are finite dimensional normed spaces simpler than infinite dimensional ones? In what respect? These questions are rather natural. They are important since finite dimensional spaces and subspaces play a role in various considerations (for instance, in approximation theory and spectral theory). Quite a number of interesting things can be said in this connection. Hence it is worthwhile to collect some relevant facts, for their own sake and as tools for our further work.

A source for results of the desired type is the following lemma. Very roughly speaking it states that in the case of linear independence of vectors we cannot find a linear combination that involves large scalars but represents a small vector.

Lemma

Let $\{x_1, x_2, \dots, x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number $c > 0$ such that for every choice of scalars $\propto_1, \propto_2, \dots, \propto_n$ we have

$\|\propto_1 x_1 + \propto_1 x_2 + \dots + \propto_n x_n\| \geq c(|\propto_1| + |\propto_2| + \cdots + |\propto_n|)$ (c > 0). ……(1)

Proof

We write $s = |\propto_1| + |\propto_2| + \cdots + |\propto_n|$. If $s = 0$, all $\propto_j$ are zero, so that

$\|\propto_1 x_1 + \propto_1 x_2 + \dots + \propto_n x_n\| \geq c(|\propto_1| + |\propto_2| + \cdots + |\propto_n|)$ holds for any c.

Let $s > 0$. Then (1) is equivalent to the inequality which we obtain from (1) by dividing by s and writing $\beta_j = \frac{\propto_j}{s}$, that is,

$\|\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n\| \geq c$ ………………. (2) ; $\sum_1^n |\beta_j| = 1$

Hence it suffices to prove the existence of a c > 0 such that (2) holds for every *n-tuple* of scalars $\beta_1, \beta_2, \dots, \beta_n$ with $\sum |\beta_j| = 1$.

Suppose that this is false. Then there exists a sequence (y_m) of vectors

$y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + \dots + \beta_n^{(m)} x_n$ such that $\|y_m\| \to 0$ as $m \to \infty$

Now we reason as follows.

Since $\sum_1^n \left|\beta_j^{(m)}\right| = 1$, we have $\left|\beta_j^{(m)}\right| \leq 1$. Hence for each fixed 'j' the sequence $\left(\beta_j^{(m)}\right) = \left(\beta_1^{(m)}, \beta_2^{(m)}, \dots, \beta_n^{(m)}\right)$ is bounded. Consequently, by the Bolzano-Weierstrass theorem, $\left(\beta_1^{(m)}\right)$ has a convergent subsequence. Let β_1 denote the limit of that subsequence, and let $(y_{1,m})$ denote the corresponding subsequence of (y_m). By the same argument, $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence of scalars $\left(\beta_2^{(m)}\right)$ converges; let β_2 denote the limit. Continuing in this way, after *n* steps we obtain a subsequence

$(y_{n,m}) = (y_{1,m}, y_{2,m}, \dots)$ of (y_m) whose terms are of the form $y_{n,m} = \sum_1^n \gamma_j^{(m)} x_j$ and $\sum_1^n \left|\gamma_j^{(m)}\right| = 1$ with scalars $\gamma_j^{(m)}$ satisfying $\gamma_j^{(m)} \to \beta_j$ as $m \to \infty$. Hence, as $m \to \infty$ we have $y_{n,m} \to y = \sum_1^n \beta_j x_j$

where $\sum_1^n |\beta_j| = 1$, so that not all β_j can be zero. Since $\{x_1, x_2, \dots, x_n\}$ is a linearly independent set, we thus have $y \neq 0$.

On the other hand, $y_{n,m} \to y$ implies $\|y_{n,m}\| \to \|y\|$, by the continuity of the norm. Since $\|y_m\| \to 0$ by assumption and $(y_{n,m})$ is a subsequence of (y_m), we must have $\|y_{n,m}\| \to 0$. Hence $\|y\| = 0$, so that $y = 0$ by norm property.

This contradicts $y \neq 0$, and the lemma is proved.

Theorem-15: *Let $\{x_1, x_2, ..., x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number $c > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, ..., \alpha_n$, we have*

$$\| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq c(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$$

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Proof: If $|\alpha_1| + |\alpha_2| + ... + |\alpha_n| = 0$, then the inequality

$$\| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq c(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$$

is satisfied for every value of c and there is nothing to prove, so let

$$|\alpha_1| + |\alpha_2| + ... + |\alpha_n| \neq 0 \qquad ...(1)$$

Then $\alpha_j \neq 0$ for some $1 \leq j \leq n$.

Next suppose that $\beta = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$

Then $\dfrac{\alpha_j}{\beta} \neq 0$ for some $1 \leq j \leq n$.

Since $\{x_1, x_2, ..., x_n\}$ is linearly independent set, so

$$\frac{\alpha_1}{\beta} x_1 + \frac{\alpha_2}{\beta} x_2 + ... + \frac{\alpha_n}{\beta} x_n \neq 0$$

Since the norm of nonzero number is always positive, so we can find a number $c > 0$ such that

$$\left\| \frac{\alpha_1}{\beta} x_1 + \frac{\alpha_2}{\beta} x_2 + ... + \frac{\alpha_n}{\beta} x_n \right\| \geq c$$

$$\Rightarrow \left|\frac{1}{\beta}\right| \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq c$$

$$\Rightarrow \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq c|\beta|$$

$$\Rightarrow \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq c\left| |\alpha_1| + |\alpha_2| + ... + |\alpha_n| \right|$$

$$\Rightarrow \| \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \| \geq c(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$$

Theorem

Every finite dimensional subspace Y of a normed space $X = (X, \|.\|)$ is complete. In particular, every finite dimensional normed space is complete.

Proof

Let $(X, \|.\|)$ Be a norm space and Y is subspace of $(X, \|.\|)$ with $dim(Y) = n$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis for Y.

Let (y_m) be a Cauchy sequence in Y then for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|y_m - y_n\| < \in \quad ; \forall m, n > n_0$

Since $y_m \in Y$; $\forall m$ therefore $y_m = \propto_1^{(m)} e_1 + \propto_2^{(m)} e_2 + \dots + \propto_n^{(m)} e_n$

Now using $\|y_m - y_n\| < \in \quad ; \forall m, n > n_0$ we have

$$\left\|\left(\propto_1^{(m)} - \propto_1^{(n)}\right) e_1 + \left(\propto_2^{(m)} - \propto_2^{(n)}\right) e_2 + \dots + \left(\propto_n^{(m)} - \propto_n^{(n)}\right) e_n\right\| < \in \quad ; \forall m, n > n_0$$

Now using the result $\|\propto_1 x_1 + \propto_1 x_2 + \dots + \propto_n x_n\| \geq c(|\propto_1| + |\propto_2| + \cdots + |\propto_n|)$ we have $c(|\propto_1| + |\propto_2| + \cdots + |\propto_n|) \leq \|\propto_1 x_1 + \propto_1 x_2 + \dots + \propto_n x_n\|$ and hence

$$c\left(\sum_1^n \left|\propto_i^{(m)} - \propto_i^{(n)}\right|\right) \leq \left\|\left(\propto_1^{(m)} - \propto_1^{(n)}\right) e_1 + \left(\propto_2^{(m)} - \propto_2^{(n)}\right) e_2 + \dots + \left(\propto_n^{(m)} - \propto_n^{(n)}\right) e_n\right\| < \in \quad ; \forall m, n > n_0$$

$$c\left(\sum_1^n \left|\propto_i^{(m)} - \propto_i^{(n)}\right|\right) < \in \quad ; \forall m, n > n_0$$

$\sum_1^n \left|\propto_i^{(m)} - \propto_i^{(n)}\right| < \frac{\in}{c} \quad ; \forall m, n > n_0$ and for all i

Hence for each i; $\left(\propto_i^{(m)}\right)$ is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ and since $\mathbb{R}$ or $\mathbb{C}$ are complete therefore $\propto_i^{(m)} \to \propto_i \in \mathbb{R}$ (may take $\mathbb{C}$)

Implies since Y is linear combination of a basis of Y therefore

$$y_m = \propto_1^{(m)} e_1 + \propto_2^{(m)} e_2 + \dots + \propto_n^{(m)} e_n \to y = \propto_1 e_1 + \propto_2 e_2 + \dots + \propto_n e_n \in Y$$

Now we check whether this convergence is under the norm or not.

For this we consider

$\|y_m - y\| = \left\|\sum_1^n \left(\propto_i^{(m)} - \propto_i\right) e_i\right\| \le \sum_1^n \left|\propto_i^{(m)} - \propto_i\right| \|e_i\|$

$\|y_m - y\| \le k \sum_1^n \left|\propto_i^{(m)} - \propto_i\right|$ Where $k = max\|e_i\|$; $1 \le i \le n$

$\|y_m - y\| \le k \sum_1^n \left|\propto_i^{(m)} - \propto_i\right| \to 0$ as $m \to \infty$ $(\propto_i^{(m)} \to \propto_i$ as $m \to \infty)$

Implies $y_m \to y$ under $\|.\|$. And hence $(Y, \|.\|)$ Is complete.

We have already prove that "A subspace Y of a Banach Space $(X, \|.\|)$ Is complete if and only if it is closed"

From this result and previous result it is follows that

Theorem Every finite dimensional subspace Y of a normal space X is closed in X.

We shall need this theorem at several occasions in our further work.

Note that infinite dimensional subspaces need not be closed. (Above Theorem not Applicable)

Example: Let $X = C[0,1]$ be a normed space and $Y = span(x_0, x_1, \cdots)$, where $x_j(t) = t^j$, so that Y is the set of all polynomials. Y is not closed in X.

Example: Let $X = C[a, b]$ be a normed space with $\|x\| = max_{t\in[a,b]}|x(t)|$. We know that $(C[a, b], \|.\|)$ Is complete (i.e. a Banach Space) and $Y = span(1, t, t^2, \cdots)$, so that Y is the set of all polynomials.

Then $Y \subseteq C[a, b]$ (Since every polynomial is continuous function)

Obviously Y is infinite dimensional space.

Let (y_n) be a sequence in Y defined by $y_n = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$

Then $\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}\right) = e^t \notin Y$

Implies Y is not closed. Hence Y is not complete.

Another interesting property of a finite dimensional vector space X is that all norms on X lead to the same topology for X, that is, the open subsets of X are the same, regardless of the particular choice of a norm on X. The details are as follows.

Equivalent norms

A norm $\|.\|_1$ on a vector space X is said to be equivalent to a norm $\|.\|_2$ on X if there are positive numbers α and β such that for all $x \in X$ we have

$$\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2$$

This concept is motivated by the fact that Equivalent norms on X define the same topology for X.

Example Let $X = \mathbb{R}^2$ with norm $\|x\|_1 = |\xi_1| + |\xi_2|$; $x = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\|x\|_2 = (\sum_1^2 |\xi_i|^2)^{1/2} = \sqrt{|\xi_1|^2 + |\xi_2|^2}$ then show that $\|x\|_1$ and $\|x\|_2$ are equivalent norms.

Solution

$$\|x\|_1 = |\xi_1| + |\xi_2| = \sum_1^2 |\xi_i| = \sum_1^2 (1)|\xi_i| \leq (\sum_1^2 (1)^2)^{1/2} (\sum_1^2 |\xi_i|^2)^{1/2}$$

$$\|x\|_1 \leq \sqrt{2}\|x\|_2$$

$$\frac{1}{\sqrt{2}}\|x\|_1 \leq \|x\|_2 \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Now $\|x\|_2 = (\sum_1^2 |\xi_i|^2)^{1/2} = \sqrt{|\xi_1|^2 + |\xi_2|^2} \leq |\xi_1| + |\xi_2| = \|x\|_1$

$$\|x\|_2 \leq (1)\|x\|_1 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2) $\quad \frac{1}{\sqrt{2}}\|x\|_1 \leq \|x\|_2 \leq (1)\|x\|_1$

Implies $\|x\|_1$ and $\|x\|_2$ are equivalent norms.

Theorem

The relation of 'being equivalent to' among the norms that can be defined on a linear space $\mathbb{N}$ is an equivalence relation.

Proof

Reflexive: For any norm $\|.\|$ on $\mathbb{N}$ and for $x \in \mathbb{N}$

$\alpha\|x\| \leq \|x\| \leq \beta\|x\|$

Is satisfied for $\alpha = \beta = 1$. Hence $\|.\| \sim \|.\|$

Symmetric: If $\|.\|_1 \sim \|.\|_2$ then there are positive numbers α and β such that for all $x \in \mathbb{N}$ we have $\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2$

$\Rightarrow \frac{1}{\beta}\|x\|_1 \leq \|x\|_2 \leq \frac{1}{\alpha}\|x\|_1$

Hence $\|.\|_2 \sim \|.\|_1$

Transitive: If $\|.\|_1 \sim \|.\|_2$ and $\|.\|_2 \sim \|.\|_3$ then there are positive numbers α, β, α_1 and β_1 such that for all $x \in \mathbb{N}$ we have

$\alpha\|x\|_2 \leq \|x\|_1 \leq \beta\|x\|_2$ and $\alpha_1\|x\|_3 \leq \|x\|_2 \leq \beta_1\|x\|_3$

$\Rightarrow \alpha_1\|x\|_3 \leq \|x\|_2 \leq \frac{1}{\alpha}\|x\|_1 \leq \frac{\beta}{\alpha}\|x\|_2 \leq \frac{\beta}{\alpha}.\beta_1\|x\|_3$

$\Rightarrow \alpha_1\|x\|_3 \leq \frac{1}{\alpha}\|x\|_1 \leq \frac{\beta}{\alpha}.\beta_1\|x\|_3 \Rightarrow \alpha\alpha_1\|x\|_3 \leq \|x\|_1 \leq \beta\beta_1\|x\|_3$

Since $\alpha, \beta, \alpha_1, \beta_1 > 0$ therefore $\alpha\alpha_1, \beta\beta_1 > 0$

Hence $\|.\|_1 \sim \|.\|_3$.

Consequently the relation of 'being equivalent to' among the norms that can be defined on a linear space $\mathbb{N}$ is an equivalence relation.

Theorem (Every norm generate a topology)

Any two equivalent norms on a linear space $\mathbb{N}$ define (induced) the same topology on $\mathbb{N}$.

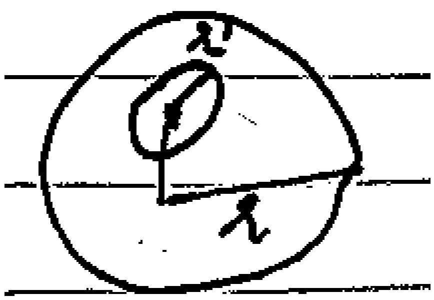
Proof: Let $\|.\|_1 \sim \|.\|_2$ then there are positive numbers a and b such that for all $x \in \mathbb{N}$ we have $a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2$

We show that every basic open ball in $(\mathbb{N}, \|.\|_1)$ is open in $(\mathbb{N}, \|.\|_2)$ and conversely.

For an $x \in \mathbb{N}$ let $B(x;r)$ be an open ball in $(\mathbb{N}, \|.\|_1)$, then we show that it is open ball in $(\mathbb{N}, \|.\|_2)$.

For this let $y \in B(x;r)$ then $\|x-y\|_1 = r_1 < r$

Consider $B_1(y;r')$ in $(\mathbb{N}, \|.\|_2)$ where $r' = \frac{r-r_1}{b}$



Then for any $z \in B_1(y;r')$ we have $\|z-y\|_2 < r'$ then

$\|z-x\|_1 = \|z-y+y-x\|_1 \leq \|z-y\|_1 + \|y-x\|_1$

$\|z-x\|_1 \leq b\|z-y\|_2 + r_1$ since $\|.\|_1 \sim \|.\|_2$ and $\|x-y\|_1 = r_1 < r$

$\|z-x\|_1 < br' + r_1 = b\left(\frac{r-r_1}{b}\right) + r_1 = r \Rightarrow \|z-x\|_1 < r$

Hence $z \in B(x;r)$ implies $z \in B_1(y;r') \subseteq B(x;r)$. Hence $B(x;r)$ is open ball in $(\mathbb{N}, \|.\|_2)$. Similarly we can conversely show that every basic open ball in $(\mathbb{N}, \|.\|_2)$ is open in $(\mathbb{N}, \|.\|_1)$. Hence Any two equivalent norms on a linear space $\mathbb{N}$ define (induced) the same topology on $\mathbb{N}$.

This theorem shows that equivalent norms preserve the Cauchy property of sequence.

Theorem

Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms on a linear space $\mathbb{N}$, then every Cauchy sequence in $(\mathbb{N}, \|.\|_1)$ is also Cauchy sequence in $(\mathbb{N}, \|.\|_2)$ and conversely.

Proof: Let (x_n) Cauchy sequence in $(\mathbb{N}, \|.\|_1)$ then for given any $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that

$\|x_m - x_n\|_1 < \in \quad ; \forall m, n > n_0$

$\|x_m - x_n\|_2 \leq \frac{1}{a}\|x_m - x_n\|_1 < \frac{\in}{a} \quad ; \forall m, n > n_0$ since $\|.\|_1 \sim \|.\|_2$

$\|x_m - x_n\|_2 < \in' \quad ; \forall m, n > n_0$

Hence (x_n) Cauchy sequence in $(\mathbb{N}, \|.\|_2)$. Converse is similar.

Theorem

Let $\|.\|_1$ and $\|.\|_2$ be equivalent norms on a linear space $\mathbb{N}$, then every Convergent sequence in $(\mathbb{N}, \|.\|_1)$ is also Convergent sequence in $(\mathbb{N}, \|.\|_2)$ and conversely.

Proof: Let (x_n) Convergent sequence in $(\mathbb{N}, \|.\|_1)$ then for given any $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that

$\|x_m - x\|_1 < \in \quad ; \forall n > n_0$

$\|x_m - x\|_2 \leq \frac{1}{a}\|x_m - x\|_1 < \frac{\in}{a} \quad ; \forall n > n_0$ since $\|.\|_1 \sim \|.\|_2$

$\|x_m - x\|_2 < \in' \quad ; \forall n > n_0$

Hence (x_n) Convergent sequence in $(\mathbb{N}, \|.\|_2)$. Converse is similar.

Zeroth Norm

The equation $\|x\|_0 = Sup_1^n|\propto_i| \quad ; \propto_i \in Field$ defines a norm on a normed space X and is said to be zeroth norm.

Theorem

Suppose $\|.\|_1$ and $\|.\|_2$ are equivalent norms defined on X. Let N be a finite dimensional subspace of $(X, \|.\|_1)$ then N is complete as subspace of $(X, \|.\|_1)$. In particular $(N, \|.\|_2)$ is complete.

Proof: Let $\|.\|_1 \sim \|.\|_2$ then there are positive numbers a and b such that for all $x \in X$ we have $a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2$ ……………..(1)

Let N be a finite dimensional subspace of $(X, \|.\|_1)$ and $\{e_1, e_2, \dots, e_n\}$ be a basis for N. Then each $y \in N$ has a unique representation $y = \sum_1^n \propto_i e_i$.

Suppose (y_m) be a Cauchy sequence in N then for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|y_m - y_n\|_1 < \in$; $\forall m, n > n_0$

$\Rightarrow \left\|\sum_1^n \propto_i^{(m)} e_i - \sum_1^n \propto_i^{(n)} e_i\right\|_1 < \in$; $\forall m, n > n_0$

$\Rightarrow \left\|\sum_1^n \left(\propto_i^{(m)} - \propto_i^{(n)}\right) e_i\right\|_1 < \in$; $\forall m, n > n_0$

$\Rightarrow \left\|\sum_1^n \left(\propto_i^{(m)} - \propto_i^{(n)}\right) e_i\right\|_2 \leq \frac{1}{a}\left\|\sum_1^n \left(\propto_i^{(m)} - \propto_i^{(n)}\right) e_i\right\|_1 < \frac{\in}{a}$; $\forall m, n > n_0$

$\Rightarrow \left\|\sum_1^n \left(\propto_i^{(m)} - \propto_i^{(n)}\right) e_i\right\|_2 < \frac{\in}{a}$; $\forall m, n > n_0$

$\Rightarrow Sup_1^n \left\|\propto_i^{(m)} - \propto_i^{(n)}\right\|_2 < \frac{\in}{a}$; $\forall m, n > n_0 \Rightarrow \left(\propto_i^{(m)}\right)$ is a Cauchy sequence in F.

since F being Real (Compex) is complete therefore for any $i = 1, 2, \dots, n$

$\left|\propto_i^{(m)} - a_i\right| \to 0$ as $m \to \infty$. Then for $y = \sum_1^n \propto_i e_i \in N$ we have

$\|y_m - y\|_1 = \left\|\sum_1^n \left(\propto_i^{(m)} - \propto_i\right) e_i\right\|_1 \leq \sum_1^n \left|\propto_i^{(m)} - a_i\right| \|e_i\|_1$

$\|y_m - y\|_1 \leq K \sum_1^n \left|\propto_i^{(m)} - a_i\right|$ where $K = Sup_1^n \|e_i\|_1$

$\Rightarrow \|y_m - y\|_1 \to 0$ as $m \to \infty$

Thus (y_m) converges to (y) in N. Hence N is complete as subspace of $(X, \|.\|_1)$. In particular $(N, \|.\|_2)$ is complete.

Theorem (Any two norms on a finite dimensional linear space are equivalent)

On a finite dimensional vector space X, any norm $\|.\|$ is equivalent to any other norm $\|.\|_0$

Proof

Let X be a finite dimensional normed space. And let $dim(X) = n$ also $\{e_1, e_2, \dots, e_n\}$ be a basis for X. Then each $x \in X$ has a unique representation $x = \propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n$ then

$\|x\| = \|\propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n\| \geq c(\sum_1^n |\propto_i|)$

$\Rightarrow \|x\| \geq c(\sum_1^n |\propto_i|) \Rightarrow (\sum_1^n |\propto_i|) \leq \frac{1}{c}\|x\|$ …………(1)

$\Rightarrow K(\sum_1^n |\propto_i|) \leq \frac{K}{c}\|x\|$ …………(2)

Now $\|x\|_0 = \|\propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n\|_0$

$\Rightarrow \|x\|_0 \leq |\propto_1|\|e_1\|_0 + |\propto_2|\|e_2\|_0 + \cdots + |\propto_n|\|e_n\|_0$

$\Rightarrow \|x\|_0 \leq K(\sum_1^n |\propto_i|)$ where $K = max\|e_i\|_0 \;\; ; 1 \leq i \leq n$

$\Rightarrow \|x\|_0 \leq K\frac{1}{c}\|x\|$ using (1)

$\Rightarrow \propto \|x\|_0 \leq \|x\|$ …………(3) with $\propto = \frac{c}{K}$

Similarly if $K' = max\|e_i\| \;\; ; 1 \leq i \leq n$ then $\|x\| \leq K'(\sum_1^n |\propto_i|)$ …………(4)

And $(\sum_1^n |\propto_i|) \leq \frac{1}{c\prime}\|x\|_0$ and with respect to this we get

$\Rightarrow K'(\sum_1^n |\propto_i|) \leq \frac{K\prime}{c\prime}\|x\|_0$ where $K' = max\|e_i\| \;\; ; 1 \leq i \leq n$

$\Rightarrow K'(\sum_1^n |\propto_i|) \leq \beta\|x\|_0$ …………(5) with $\beta = \frac{K\prime}{c\prime}$

Hence $\|x\| \leq K'(\sum_1^n |\propto_i|) \leq \beta\|x\|_0$ …………(6) using (4) and (5)

$\Rightarrow \propto \|x\|_0 \leq \|x\| \leq \beta\|x\|_0$ using (3) and (6)

Hence $\|.\|$ and $\|.\|_0$ are equivalent norms.

Compactness

A metric space X is said to be compact (or more precisely sequentially compact) if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X, that is, if every sequence in M has a convergent subsequence whose limit is an element of M.

Remark

There are three ways of defining the concept of compactness in a general Topological Space.

i. Every open cover has a finite subcover.
ii. Every countable cover has a finite subcover.
iii. Every sequence has a convergent subsequence.

Lemma

A compact subset M of a metric (X, d) space is closed and bounded.

Proof

Suppose that M is compact. We are to show that M is closed. i.e. $M = \bar{M}$

Obviously $M \subseteq \bar{M}$

Now let $x \in \bar{M}$ then there exists a sequence (x_n) in M such that $x_n \to x$

Since M is compact, (x_n) has a convergent subsequence which converges to x, so $x \in M$ (By definition of convergence) and we have $\bar{M} \subseteq M$

So $M = \bar{M}$ (i.e. M is closed)

Now we are to show that M is bounded. For this suppose M is compact but contrarily not bounded. Then there is an unbounded sequence (y_n) in M then $d(y_n, b) > n_0$ where b is fixed element of M. Then (y_n) cannot have convergent subsequence, which is a contradiction to the fact that M is compact. Our supposition was wrong and hence M is bounded.

Remark: The converse of this lemma is in general false.

Proof Let $X = l^\infty$ then $dim(X) = \infty$. If $x = \left(\xi_j\right)_1^\infty \in l^\infty$ then $\|x\| = Sup_j\left|\xi_j\right|$. Let $M = \{e_1 = (1,0,0, \dots), e_2 = (0,1,0, \dots), \dots\}$ then $\|e_1\| = \|e_2\| = 1$. So M is bounded. Since M is a point set, as no limit point is there, so we can suppose that all limit points lies in M, so M is closed. But M is not compact because the subsequence $e_1, e_5, e_{100}, e_{520}$ not converges to limit point.

Theorem

In a finite dimensional normed space X, any subset $M \subseteq X$ is compact (sequentially compact) if and only if M is closed and bounded.

Proof

Suppose that M is compact. We are to show that M is closed. i.e. $M = \bar{M}$

Obviously $M \subseteq \bar{M}$

Now let $x \in \bar{M}$ then there exists a sequence (x_n) in M such that $x_n \to x$

Since M is compact, (x_n) has a convergent subsequence which converges to x, so $x \in M$ (By definition of convergence) and we have $\bar{M} \subseteq M$

So $M = \bar{M}$ (i.e. M is closed)

Now we are to show that M is bounded. For this suppose M is compact but contrarily not bounded. Then there is an unbounded sequence (y_n) in M then $d(y_n, b) > n_0$ where b is fixed element of M. Then (y_n) cannot have convergent subsequence, which is a contradiction to the fact that M is compact. Our supposition was wrong and hence M is bounded.

Conversely

Let M be closed and bounded. We are to show that M is compact. Let $dim(X) = n$ and $\{e_1, e_2, \dots, e_n\}$ a basis for X. Let (x_m) be an arbitrary sequence in X then $x_m = \propto_1^{(m)} e_1 + \propto_2^{(m)} e_2 + \dots + \propto_n^{(m)} e_n$ for all $m \in \mathbb{N}$

Since M is bounded therefore (x_m) is bounded then there exists '$k > 0$' such that $\|x_m\| \leq k \quad ; \forall m \in \mathbb{N}$ then

$$k \geq \left\| \propto_1^{(m)} e_1 + \propto_2^{(m)} e_2 + \ldots + \propto_n^{(m)} e_n \right\| \geq c \left(\sum_1^n \left| \propto_i^{(m)} \right| \right) \quad ; \forall m \in \mathbb{N}$$

$$c \left(\sum_1^n \left| \propto_i^{(m)} \right| \right) \leq k \quad ; \forall m \in \mathbb{N}$$

$$\sum_1^n \left| \propto_i^{(m)} \right| \leq \frac{k}{c} \quad ; \forall m \in \mathbb{N} \qquad \ldots\ldots\ldots\ldots\ldots(1)$$

This means that for all $i = 1,2, \ldots, n$ the sequence $\left(\propto_i^{(m)}\right)$ is bounded. Then by Bolzano-Weierstrass Theorem, it has a convergent subsequence which converges to $\propto_i$, then (x_m) has a convergent subsequence (z_m) which converges to $z = \sum_1^n \xi_j e_j$. Since M is closed, therefore $z \in M$. This shows that the arbitrary sequence (x_m) in M has a subsequence which converges in M. Hence M is compact.

Theorem

Let N be a normed space in which every closed and bounded subset is compact. Then N is a Banach Space.

Proof

Let N be a normed space in which every closed and bounded subset is compact. Then every sequence in such a set has convergent subsequence. Let (x_n) be a Cauchy sequence in N and $A = (x_1, x_2, \ldots, x_n)$ as a Cauchy Sequence is bounded. Let $\bar{A}$ be the closure of A then by our assumption $\bar{A}$ is compact. So the sequence (x_n) in $\bar{A}$ has a convergent subsequence $\left(x_{n_i}\right)$.

Suppose that $x_{n_i} \to x$ then $x \in \bar{A}$.

Moreover (x_n) also convergent to x.

Hence $x \in \bar{A} \subseteq N$.

Thus N is a Banach Space.

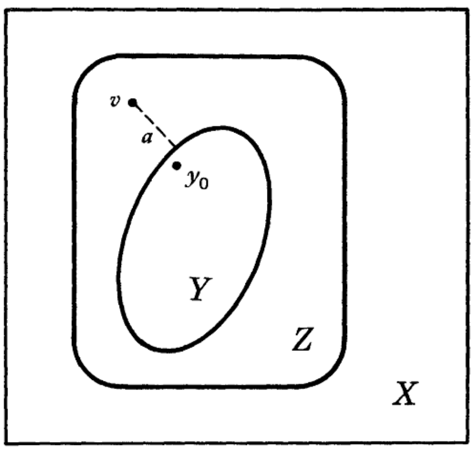
F. Riesz's Lemma

Let Y and Z be subspaces of a normed space X (of any dimension), and suppose that Y is closed and is a proper subset of Z. Then for every real number θ in the interval (0,1) there is a $z \in Z$ such that

$\|z\| = 1, \ \|z - y\| \geq \theta$ for all $y \in Y$.

Proof

We consider any $v \in Z - Y$ and denote its distance from Y by a, that is in the figure we have $a = inf_{y \in Y}\|v - y\| \geq 0$. Then clearly $a > 0$ since Y is closed and Y contains all of its limit points.



Choose $\theta \epsilon (0,1)$, then by definition of infimum, we can find a $y_0 \in Y$ such that
$a \leq \|v - y_0\| \leq \frac{a}{\theta}$ …………………..(1)

Note that $\frac{a}{\theta} > a$ since $0 < \theta < 1$

Let $z = c(v - y_0)$ where $c = \frac{1}{\|v - y_0\|}$ then

$\|z\| = \|c(v - y_0)\| = |c|\|v - y_0\| = \frac{1}{\|v - y_0\|}\|v - y_0\| \Rightarrow \|z\| = 1$

Now we show that $\|z - y\| \geq \theta$ for all $y \in Y$.

$\|z - y\| = \|c(v - y_0) - y\| = c \left\|(v - y_0) - \frac{1}{c}y\right\|$

$\|z\ - y\| = c\left\|v - \left(y_0 + \frac{1}{c}y\right)\right\| \geq c.a$ Since $\left(y_0 + \frac{1}{c}y\right) \in Y$ by definition

$\|z\ - y\| \geq \frac{1}{\|v-y_0\|}.a \geq \frac{\theta}{a}.a$ using (1) and $a > 0$

Hence $\|z\ - y\| \geq \theta$ for all $y \in Y$.

F. Riesz's Lemma (Another Form)

Let M be a proper closed subspace of a normed space N and a be any real number in the interval (0,1). Then there is an $x_a \in N$ such that $\|x_a\| = 1,\ \|x - x_a\| \geq a$ for all $x \in M$.

Proof

Since M is proper closed subspace of normed space N, then there exists $x_1 \in N/M$. Put $d = inf_{x\in M}\|x - x_1\| \geq 0$. Then clearly $d > 0$ for otherewise $x_1 \in \overline{M} = M$. A contradiction since M is closed and M contains all of its limit points.

Choose $a\epsilon(0,1)$, then by definition of infimum, we can find a $x_0 \in M$ such that $d \leq \|x_0 - x_1\| \leq \frac{d}{a} \Rightarrow \frac{1}{d} \geq \frac{1}{\|x_0-x_1\|} \geq \frac{a}{d}$

Note that $c \geq \frac{a}{d}$ Where $c = \frac{1}{\|x_0-x_1\|}$

Let $x_a = c(x_1 - x_0)$ then

$\|x_a\| = \|c(x_1 - x_0)\| = |c|\|x_1 - x_0\| = \frac{1}{\|x_0-x_1\|}\|x_1 - x_0\| \Rightarrow \|x_a\| = 1$

Now we show that $\|x - x_a\| \geq a$ for all $x \in M$.

$\|x - x_a\| = \|x - c(x_1 - x_0)\| = c\left\|\frac{x}{c} + x_0 - x_1\right\| = c\|z - x_1\|$ with $z = \frac{x}{c} + x_0$

$\|x - x_a\| = c\|z - x_1\| \geq cd$ Since $z \in M$ by definition

$\|x - x_a\| \geq d.\frac{a}{d} \geq a$

Hence $\|x - x_a\| \geq a$ for all $x \in M$.

In a finite dimensional normed space the closed unit ball is compact. Conversely, Riesz's lemma gives the following useful and remarkable theorem.

Theorem

If a normed space X has the property that the closed unit ball $M = \{x : \|x\| \leq 1\}$ is compact, then X is finite dimensional.

Proof

We assume that M is compact but $\dim X = \infty$, and show that this leads to a contradiction. We choose any x_1 of norm 1. This x_1 generates a one dimensional subspace X_1 of X, which is closed and is a proper subspace of X since $\dim X = \infty$. By Riesz's lemma there is an $x_2 \in X$ of norm 1 such that

$\|x_2 - x_1\| \geq \theta = \frac{1}{2}$

The elements x_1, x_2 generate a two dimensional proper closed sub-space X_2 of X. By Riesz's lemma there is an x_3 of norm 1 such that for all $x \in X_2$ we have

$\|x_3 - x\| \geq \frac{1}{2}$

In particular, $\|x_3 - x_1\| \geq \frac{1}{2}$ and $\|x_3 - x_2\| \geq \frac{1}{2}$

Proceeding by induction, we obtain a sequence (x_n) of elements $x_n \in M$ such that $\|x_m - x_n\| \geq \frac{1}{2}$ with $m \neq n$

Obviously, (x_n) cannot have a convergent subsequence. This contradicts the compactness of M. Hence our assumption $\dim X = \infty$ is false, and $\dim X < \infty$.

Compact sets are important since they are "well-behaved": they have several basic properties similar to those of finite sets and not shared by noncompact sets. In connection with continuous mappings a fundamental property is that compact sets have compact images, as follows.

Theorem

Let X and Y be metric spaces and T: $X \to Y$a continuous mapping. Then the image of a compact subset M of X under T is compact.

Proof

By the definition of compactness it suffices to show that every sequence (y_n) in the image $T(M) \subset Y$ contains a subsequence which converges in T(M).

Since $y_n \in T(M)$, we have $y_n = Tx_n$ For some $x_n \in M$. Since M is compact, (x_n) contains a subsequence (x_{n_k}) which converges in M. The image of (x_{n_k}) is a subsequence of (y_n) which converges in $T(M)$ by

"A mapping $\boldsymbol{T}$: $\boldsymbol{X} \to \boldsymbol{Y}$ of a metric space (X, d) into a metric space (Y, $\overline{\boldsymbol{d}}$) is continuous at a point $\boldsymbol{x_0} \in \boldsymbol{X}$ if and only if $\boldsymbol{x_n} \to \boldsymbol{x_0}$ implies $\boldsymbol{Tx_n} \to \boldsymbol{Tx_0}$" because T is continuous.

Hence $T(M)$ is compact.

From this theorem we conclude that the following property, well-known from calculus for continuous functions, carries over to metric spaces.

Corollary (Maximum and minimum)

A continuous mapping T of a compact subset M of a metric space X into **R** assumes a maximum and a minimum at some points of M.

Proof

$T(M) \subset R$ is compact by Theorem

"Let X and Y be metric spaces and $T\colon\ X \to Y$a continuous mapping. Then the image of a compact subset M of X under T is compact."

And closed and bounded by Lemma [applied to T(M)],

"A compact subset M of a metric space is closed and bounded."

So that $infT(M) \in T(M)$, $supT(M) \in T(M)$, and the inverse images of these two points consist of points of M at which Tx is minimum or maximum, respectively.

Local compactness: A metric space X is said to be locally compact if every point of X has a compact neighborhood. For example $\mathbb{R}$ and $\mathbb{C}$ and, more generally, $\mathbb{R}^n$ and $\mathbb{C}^n$ are locally compact.

Linear Operators

In calculus we consider the real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset of $\mathbb{R}$). Obviously, any such function is a mapping of its domain into $\mathbb{R}$. In functional analysis we consider more general spaces, such as metric spaces and normed spaces, and mappings of these spaces.

In the case of vector spaces and, in particular, normed spaces, a mapping is called an **operator**.

Of special interest are operators which "preserve" the two algebraic operations of vector space, in the sense of the following definition.

Linear operator

A linear operator T is an operator such that

(i) The domain $\mathfrak{D}(T)$ of T is a vector space and the range $\mathfrak{R}(T)$ lies in a vector space over the same field

(ii) for all $x, y \in \mathfrak{D}(T)$ and scalars $\propto$, we have
$T(x+y) = Tx + Ty$
$T(\propto x) = \propto Tx$

i.e. for all $x, y \in \mathfrak{D}(T)$ and scalars $\propto, \beta$, we have

$$T(\propto x + \beta y) = \propto Tx + \beta Ty$$

Observe the notation; we write Tx instead of $T(x)$; this simplification is standard in functional analysis. Furthermore, for the remainder of the book we shall use the following notations.

$\mathfrak{D}(T)$ denotes the domain of T.

$\mathfrak{R}(T)$ denotes the range of T.

$\mathcal{N}(T)$ denotes the null space of T.

By definition, the **null space** of a linear operator T is the set of all $x \in \mathfrak{D}(T)$such that $Tx = 0$. i.e. $\mathcal{N}(T) = \{x \in \mathfrak{D}(T) : Tx = 0\}$

(Another word for null space is "*kernel*" We shall not adopt this term since we must reserve the word "kernel" for another purpose in the theory of integral equations.)

Examples of different Operators

- **Identity operator:** The identity operator $I: X \to X$ is defined by $I(x) = x$ for all $x \in X$. We also write simply Ix for I; thus, $Ix = x$. This is linear operator, since $I(\propto x + \beta y) = \propto x + \beta y = \propto Ix + \beta Iy$
- **Zero operator:** The zero operator $0: X \to Y$ is defined by $0x = 0$ for all $x \in X$. This is linear operator, since $0(\propto x + \beta y) = 0 = \propto 0x + \beta\, 0y$
- **Differentiation:** Let X be the vector space of all polynomials on [a, b]. We may define a linear operator T on X by setting $Tx(t) = x'(t)$ for every

$x = x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in X$, where the prime denotes differentiation with respect to t. This operator T maps X onto itself. This is linear operator, since

$T\big(\propto x(t) + \beta y(t)\big) = \big(\propto x(t) + \beta y(t)\big)' = \propto x'(t) + \beta y'(t)$

$T\big(\propto x(t) + \beta y(t)\big) = \propto T\big(x(t)\big) + \beta T\big(y(t)\big)$

- **Integration:** A linear operator $T: [a, b] \to \mathbb{R}$ defined by $Tx = \int_a^b x dt$. Linear in the sense as follows;

  $T(\propto x + \beta y) = \int_a^b (\propto x + \beta y) dt = \propto \int_a^b x dt + \beta \int_a^b y dt = \propto Tx + \beta\, Ty$
- **Multiplication by t:** Another linear operator from C[a, b] into itself is defined by $Tx(t) = tx(t)$. T plays a role in physics (quantum theory)
- **Elementary vector algebra:** The cross product with one factor kept fixed defines a linear operator $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$. Similarly, the dot product with one fixed factor defines a linear operator $T_2 : \mathbb{R}^3 \to \mathbb{R}$, say, $T_2 x = x.a = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$ where $a = \big(\propto_j\big) \in \mathbb{R}^3$ is fixed.
- **Matrices:** A real matrix $A = \big(\propto_{jk}\big)$ with r rows and n columns defines an operator $T : \mathbb{R}^n \to \mathbb{R}^r$ by means of $y = Ax$ where $x = \big(\xi_j\big)$ has n components and $y = (\eta_j)$ has r components and both vectors are written as column vectors because of the usual convention of matrix multiplication; writing $y = Ax$ out, we have

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \cdot \\ \cdot \\ \cdot \\ \eta_r \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rn} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \cdot \\ \xi_n \end{bmatrix}.$$

T is linear because matrix multiplication is a linear operation. If A were complex, it would define a linear operator from $\mathbb{C}^n$ into $\mathbb{C}^r$.

In these examples we can easily verify that the ranges and null spaces of the linear operators are vector spaces. This fact is typical. Let us prove it, thereby observing how the linearity is used in simple proofs. The theorem itself will have various applications in our further work.

Theorem: Let T be a linear operator. Then:

i. The range $\Re(T)$ is a vector space.
ii. If $\dim \mathfrak{D}(T) = n < \infty$, then $\dim \Re(T) \leq n$.
iii. The null space $\mathcal{N}(T)$ is a vector space.

Proof

(a) Let $y_1, y_2 \in \Re(T)$ then there exists $x_1, x_2 \in \mathfrak{D}(T)$ such that
$y_1 = Tx_1, y_2 = Tx_2$
We know that $\mathfrak{D}(T)$ is a vector space therefore $\propto x_1 + \beta x_2 \in \mathfrak{D}(T)$
Then $T(\propto x_1 + \beta x_2) = \propto Tx_1 + \beta Tx_2 = \propto y_1 + \beta y_2 \in \Re(T)$
Hence $\Re(T)$ is a vector space.

(b) Consider $\dim \mathfrak{D}(T) = n < \infty$ and choose $y_1, y_2, \dots, y_n, y_{n+1}$ from $\Re(T)$ then there exists $x_1, x_2, \dots, x_n, x_{n+1}$ from $\mathfrak{D}(T)$ such that
$y_1 = Tx_1, y_2 = Tx_2, \dots, y_n = Tx_n, y_{n+1} = Tx_{n+1}$
Since $\dim \mathfrak{D}(T) = n$ therefore the set $\{x_1, x_2, \dots, x_n, x_{n+1}\}$ is linearly dependent. And hence
$\propto_1 x_1 + \propto_2 x_2 + \cdots + \propto_{n+1} x_{n+1} = 0 \quad ; \forall \propto_{i's} \neq 0$
$T(\propto_1 x_1 + \propto_2 x_2 + \cdots + \propto_{n+1} x_{n+1}) = T(0) = 0 \quad ; \forall \propto_{i's} \neq 0$
$\propto_1 Tx_1 + \propto_2 Tx_2 + \cdots + \propto_{n+1} Tx_{n+1} = 0 \quad ; \forall \propto_{i's} \neq 0$
$\propto_1 y_1 + \propto_2 y_2 + \cdots + \propto_{n+1} y_{n+1} = 0 \quad ; \forall \propto_{i's} \neq 0$
This shows that $\{y_1, y_2, \dots, y_n, y_{n+1}\}$ is a linearly dependent set because the $\propto_{i's}$ are not all zero. Then $\Re(T)$ has no linearly independent subset of $n + 1$ or more elements. By the definition this means that $\dim \Re(T) \leq n$.
This result tells that linear operators preserve linear dependence.

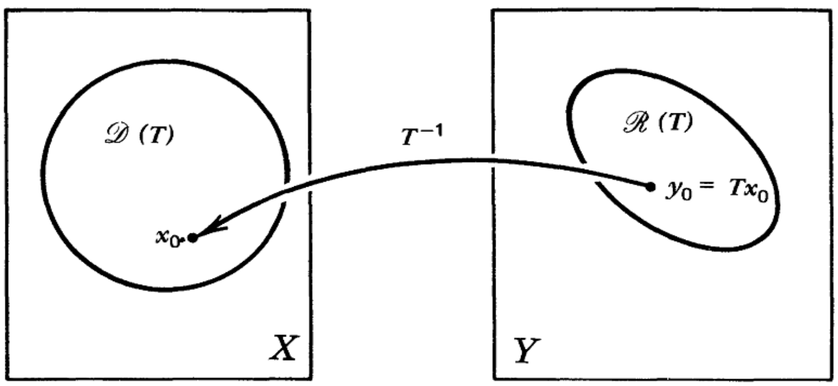
(c) Let $x_1, x_2 \in \mathcal{N}(T)$ then $Tx_1 = Tx_2 = 0$
Then $T(\propto x_1 + \beta x_2) = \propto Tx_1 + \beta Tx_2 = 0$
$\propto x_1 + \beta x_2 \in \mathcal{N}(T)$
Hence $\mathcal{N}(T)$ is a vector space.

Injective or one-to-one Mapping

A mapping $T: \mathfrak{D}(T) \to Y$ is said to be injective or one-to-one if different points in the domain have different images, that is, if for any $x_1, x_2 \in \mathfrak{D}(T)$

$x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$ Equivalently, $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$

In this case there exists the mapping $T^{-1}: \mathfrak{R}(T) \to \mathfrak{D}(T)$ which maps every $y_0 \in \mathfrak{R}(T)$ onto that $x_0 \in \mathfrak{D}(T)$ for which $Tx_0 = y_0$. The mapping is called the inverse of T.



Remark

- $T^{-1}Tx = x$ for all $x \in \mathfrak{D}(T)$
- $T^{-1}Ty = y$ for all $y \in \mathfrak{R}(T)$

In connection with linear operators on vector spaces the situation is as follows. The inverse of a linear operator exists if and only if the null space of the operator consists of the zero vector only. More precisely, we have the following useful criterion which we shall apply quite often.

Theorem

Let X, Y be vector spaces, both real or both complex. Let $T: \mathfrak{D}(T) \to Y$ be a linear operator with domain $\mathfrak{D}(T) \subset X$ and range $\mathfrak{R}(T) \subset Y$. Then:

a) The inverse $T^{-1}: \mathfrak{R}(T) \to \mathfrak{D}(T)$ exists if and only if $Tx = 0$ implies $x = 0$.
b) If T^{-1} exists, it is a linear operator.
c) If $\dim \mathfrak{D}(T) = n < \infty$ and T^{-1} exists, then $\dim \mathfrak{D}(T) = \dim \mathfrak{R}(T)$

Proof

a) Let $T^{-1}: \mathfrak{R}(T) \to \mathfrak{D}(T)$ exists then T is one – to – one and Suppose that $Tx = 0$.
Since T is one – to – one, therefore $Tx_1 = Tx_2$ implies $x_1 = x_2$
Choose $x_2 = 0$ and $x_1 = x$
So that $Tx = 0$ implies $x = 0$.

Conversly suppose that $Tx = 0$ implies $x = 0$. Then we have to prove that $T^{-1}: \mathfrak{R}(T) \to \mathfrak{D}(T)$ exists. For this we only need to prove T is one – to – one.

Let $x_1, x_2 \in \mathfrak{D}(T)$ such that $Tx_1 = Tx_2 \Rightarrow Tx_1 - Tx_2 = 0 \Rightarrow T(x_1 - x_2) = 0$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ Since T is linear.

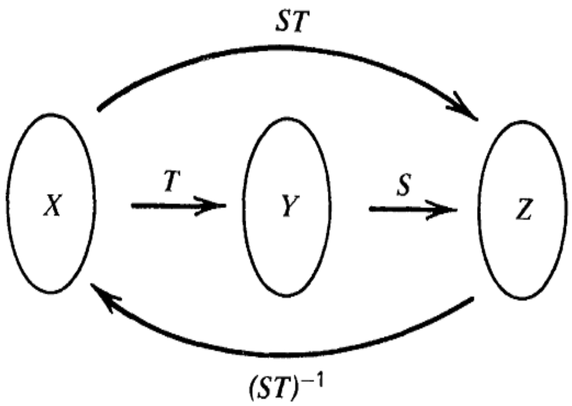
Hence T is one – to – one and $T^{-1}: \mathfrak{R}(T) \to \mathfrak{D}(T)$ exists

b) We assume that T^{-1} exists and show that T^{-1} is linear.
Let $y_1, y_2 \in \mathfrak{R}(T) = \mathfrak{D}(T^{-1})$ then there exists $x_1, x_2 \in \mathfrak{D}(T)$ such that
$y_1 = Tx_1, y_2 = Tx_2$
$\Rightarrow T^{-1}y_1 = x_1, T^{-1}y_2 = x_2$ then
$\propto y_1 + \beta y_2 = \propto Tx_1 + \beta Tx_2 = T(\propto x_1 + \beta x_2)$ as T is linear
$T^{-1}(\propto y_1 + \beta y_2) = \propto x_1 + \beta x_2 = \propto T^{-1}y_1 + T^{-1}y_2$
Hence T^{-1} is linear.

c) Suppose that $\dim \mathfrak{D}(T) = n < \infty$ and T^{-1} exists, also we know that if T is linear then $\dim \mathfrak{R}(T) \leq \dim \mathfrak{D}(T)$ …………..(1)
Since T^{-1} is linear therefore $\dim \mathfrak{R}(T^{-1}) \leq \dim \mathfrak{D}(T^{-1})$ then
$\dim \mathfrak{D}(T) \leq \dim \mathfrak{R}(T)$ …………..(2)
From (1) and (2) we have
$\dim \mathfrak{D}(T) = \dim \mathfrak{R}(T)$

Lemma (Inverse of product)

Let $T: X \rightarrow Y$ and $S: Y \rightarrow Z$ be bijective linear operators, where X, Y, Z are vector spaces (see Fig.). Then the inverse $(ST)^{-1}: Z \rightarrow X$ of the product (the composite) ST exists. And $(ST)^{-1} = T^{-1}S^{-1}$



Proof:

The operator $ST: X \rightarrow Z$ is bijective, so that $(ST)^{-1}$ exists. We thus have

$ST(ST)^{-1} = I_Z$

Where I_Z is the identity operator on Z.

Applying S^{-1} and using $S^{-1}S = I_Y$ (the identity operator on Y), we obtain

$S^{-1}ST(ST)^{-1} = T(ST)^{-1} = S^{-1}I_Z = S^{-1}$

Applying T^{-1} and using $T^{-1}T = I_X$, we obtain the desired result

$T^{-1}T(ST)^{-1} = (ST)^{-1} = T^{-1}S^{-1}$

Hence $(ST)^{-1} = T^{-1}S^{-1}$

Bounded and Continuous Linear Operators

The reader may have noticed that in the whole last section we did not make any use of norms. We shall now again take norms into account, in the following basic definition.

Bounded Linear Operator

Let X and Y be normed spaces and $T: \mathfrak{D}(T) \to Y$ a linear operator, where $\mathfrak{D}(T) \subset X$. The operator T is said to be **bounded** if there is a real number c such that for all $x \in \mathfrak{D}(T)$, $\|Tx\| \le c\|x\|$ or we can write $\frac{\|Tx\|}{\|x\|} \le c$

Formula $\|Tx\| \le c\|x\|$ shows that a bounded linear operator maps bounded sets in $\mathfrak{D}(T)$ onto bounded sets in Y. This motivates the term "bounded operator."

Warning: Note that our present use of the word "bounded" is different from that in calculus. Where a bounded function is one whose range is a bounded set. Unfortunately, both terms are standard. But there is little danger of confusion.

What is the smallest possible c (minimum of c) such that $\|Tx\| \le c\|x\|$ still holds for all nonzero $x \in \mathfrak{D}(T)$? [We can leave out x = 0 since Tx = 0 for x = 0]

By division, $\frac{\|Tx\|}{\|x\|} \le c$; $\|x\| \neq 0$ and this shows that c must be at least as big as the supremum of the expression on the left taken over $\mathfrak{D}(T) - \{0\}$.

Implies $Sup_{\substack{x \in \mathfrak{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \le c$. i.e. $c \ge Sup_{\substack{x \in \mathfrak{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$. Hence the answer to our question is that the smallest possible c in $\|Tx\| \le c\|x\|$ is that supremum. This quantity is denoted by $\|T\|$; thus

Norm of the Operator

The quantity $\|T\| = Sup_{\substack{x \in \mathfrak{D}(T) \\ x \neq 0}} \frac{\|Tx\|}{\|x\|}$ is called the norm of the operator.

And for a bounded linear operator T we can define it as $\|T\| = Sup_{\substack{x \in \mathfrak{D}(T) \\ \|x\| = 1}} \|Tx\|$

Also remember that if $c = \|T\|$ then $\|Tx\| \le c\|x\|$ becomes $\|Tx\| \le \|T\|\|x\|$

If $\mathfrak{D}(T) = \{0\}$ then $\|T\| = 0$ in this case $T = 0$ since $T0 = 0$

Lemma: Let T be a bounded linear operator. Then an alternative formula for the norm of T is $\|T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ \|x\|=1}}\|Tx\|$

Proof: Let $x \in \mathfrak{D}(T)$ be an arbitrary element and let $\|x\| = a$. Define $y = \frac{1}{a}x$ where $x \neq 0$. Then $\|y\| = 1$.

Consider $\|T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|Tx\|}{\|x\|}$

$\Rightarrow \|T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|Tx\|}{a} = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\left\|\frac{1}{a}Tx\right\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\left\|T\left(\frac{1}{a}x\right)\right\|$

$\Rightarrow \|T\| = Sup_{\substack{y\in\mathfrak{D}(T)\\ y\neq 0}}\|Ty\| \Rightarrow \boldsymbol{\|T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ \|x\|=1}}\|Tx\|}$

Lemma: Let T be a bounded linear operator. Then the norm defined by $\|T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|Tx\|}{\|x\|}$ satisfies all properties of Norm.

Proof:

N_1: Since $\frac{\|Tx\|}{\|x\|} \geq 0$ therefore $Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|Tx\|}{\|x\|} = \|T\| \geq 0$ for all $x \in \mathfrak{D}(T)$

N_2: $\|T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|Tx\|}{\|x\|} = 0 \Leftrightarrow Tx = 0 \ \ \forall x \in \mathfrak{D}(T) \Leftrightarrow x = 0 \ \ \forall x \in \mathfrak{D}(T)$

N_3: $\|\propto T\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|(\propto T)x\|}{\|x\|} = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|\propto Tx\|}{\|x\|} = |\propto| Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|Tx\|}{\|x\|}$

$\Rightarrow \|\propto T\| = |\propto|\|T\|$

N_4: $\|T_1 + T_2\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|(T_1+T_2)x\|}{\|x\|} = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|T_1x+T_2x\|}{\|x\|}$

$\Rightarrow \|T_1 + T_2\| \leq Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|T_1x\|}{\|x\|} + Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}}\frac{\|T_2x\|}{\|x\|}$

$\Rightarrow \|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$

Before we consider general properties of bounded linear operators, let us take a look at some typical examples, so that we get a better feeling for the concept of a bounded linear operator.

Examples

- **Identity operator:** The identity operator $I\colon X \to X$ on a normed space $X \neq \{0\}$ defined by $I(x) = x$ is linear and bounded and has norm $\|I\| = 1$.
  Proof: Since $\|Ix\| = \|x\| \leq 1\|x\|$
  $\Rightarrow \|I\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}} \frac{\|Ix\|}{\|x\|} = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}} \frac{\|x\|}{\|x\|} = 1$
- **Zero operator:** The zero operator $0\colon X \to Y$ defined by $0(x) = 0$ on a normed space X is linear and bounded and has norm $\|0\| = 0$.
  Proof: Since $\|0x\| = \|0\| = 0 \leq \|0\|\|x\|$
  $\Rightarrow \|0\| = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}} \frac{\|0x\|}{\|x\|} = Sup_{\substack{x\in\mathfrak{D}(T)\\ x\neq 0}} \frac{\|0\|}{\|x\|} = 0$
- **Differentiation operator:** Let X be the normed space of all polynomials on J = [0, 1] with norm given $\|x\| = max|x(t)|,\ t \in J$. A differentiation operator T is defined on X by $Tx(t) = x'(t)$. Where the prime denotes differentiation with respect to 't'. This operator is linear but not bounded.
  Proof: Consider the sequence (x_n) defined by $x_n = t^n$
  Then $\|x_n\| = max_{t\in[0,1]}|x_n(t)| = max_{t\in[0,1]}|t^n| = 1$
  $\Rightarrow \|Tx_n\| = \|x_n'\| = \|nt^{n-1}\| = max_{t\in[0,1]}|nt^{n-1}| = |n1^{n-1}| = n$
  $\Rightarrow \frac{\|Tx_n\|}{\|x_n\|} = \frac{n}{1} = n$
  Since $n \in \mathbb{N}$ is arbitrary, we are unable to find a fixed number 'c' such that $\|Tx_n\| \leq c\|x_n\|$
  Hence T is not bounded.
- **Integral operator:** We can define an integral operator $T\colon C[0,1] \to C[0,1]$ by $Tx = y = \int_0^1 k(t,\tau)x(\tau)d\tau$. Here k is a given function, which is called the kernel of T and is assumed to be continuous on the closed square G = J x J in the $t\tau$ −plane, where J = [0, 1]. This operator is linear. T is bounded.

Proof: This is linear operator as already proved

$T(\propto x + \beta y) = \int_0^1 k(t,\tau)(\propto x(\tau) + \beta y(\tau))d\tau = \propto Tx \ + \beta\ Ty$

Note that $k(t,\tau)$ being continuous on the closed square $[0,1] \times [0,1]$ is bounded then there exists k_0 such that $|k(t,\tau)| \leq k_0; \forall (t,\tau) \in [0,1] \times [0,1]$

We have $\|Tx\| = \|y\| = max_{t\in[0,1]}|y(t)| = max_{t\in[0,1]}\left|\int_0^1 k(t,\tau)x(\tau)d\tau\right|$

$\|Tx\| \leq max_{t\in[0,1]}\int_0^1 |k(t,\tau)||x(\tau)|d\tau \leq k_0 max_{t\in[0,1]}\int_0^1 |x(\tau)|d\tau$

$\|Tx\| \leq k_0 \int_0^1 max_{t\in[0,1]}|x(\tau)|d\tau = k_0\|x\|$

$\|Tx\| \leq k_0\|x\|$ implies that T is bounded.

- **Matrix:** A real matrix $A = (a_{ij})_{m\times n}$ defines an operator $T: \mathbb{R}^n \to \mathbb{R}^m$ defines by $y = Tx$ where $x = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$ and $y = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_m \end{pmatrix}$ are column vectors with *n* and *r* components, respectively.

  Now $y = Tx$ gives $\eta_i = \sum_{k=1}^{n} \propto_{ik} \xi_k$. Then T is linear and bounded.

  Proof: Consider

  $\|Tx\|^2 = \|y\|^2 = \sum_1^m \eta_i^2$

  $\|Tx\|^2 = \sum_1^m (\sum_{k=1}^n \propto_{ik} \xi_k)^2$

  $\|Tx\|^2 \leq \sum_{i=1}^m \left((\sum_{i=1}^n \propto_{ik}^2)^{\frac{1}{2}} (\sum_{k=1}^n \xi_k^2)^{\frac{1}{2}} \right)^2 = \left(\sum_{i=1}^m \sum_{k=1}^n \propto_{ik}^2\right)\|x\|^2$

  $\|Tx\|^2 \leq K^2\|x\|^2$

  $\|Tx\| \leq K\|x\|$

  Thus T is bounded.

Boundedness is typical; it is an essential simplification which we always have in the finite dimensional case, as follows.

Theorem: If a normed space X is finite dimensional, then every linear operator on X is bounded.

Or every linear operator on a finite dimensional norm space is bounded.

Proof: Let $dim\, X = n$ and $\{e_1, e_2, \dots, e_n\}$ a basis for X. Then every $x \in X$ can be expressed as $x = \sum_{i=1}^{n} \propto_i e_i$

$\Rightarrow Tx = T(\sum_{i=1}^{n} \propto_i e_i) = \sum_{i=1}^{n} \propto_i Te_i$ Since T is linear

$\Rightarrow \|Tx\| = \|\sum_{i=1}^{n} \propto_i Te_i\|$

$\Rightarrow \|Tx\| \leq \sum_{i=1}^{n} |\propto_i| \|Te_i\| \Rightarrow \|Tx\| \leq K \sum_{i=1}^{n} |\propto_i|$; $K = max_{1 \leq i \leq n} \|Te_i\|$

$\Rightarrow \|Tx\| \leq K \sum_{i=1}^{n} |\propto_i|$ ....(1)

Now since $x = \sum_{i=1}^{n} \propto_i e_i$ and $\{e_1, e_2, \dots, e_n\}$ is linearly independent therefore

$\|x\| = \|\sum_{i=1}^{n} \propto_i e_i\| \geq c \sum_{i=1}^{n} \propto_i$ $\Rightarrow \sum_{i=1}^{n} \propto_i \leq \frac{\|x\|}{c}$ ....(2)

Using (2) in (1) $\Rightarrow \|Tx\| \leq \frac{K}{c} \|x\|$

$\Rightarrow \|Tx\| \leq M\|x\|$ where $M = \frac{K}{c}$. Thus T is bounded.

We shall now consider important general properties of bounded linear operators.

- Operators are mappings, so that the definition of continuity applies to them. It is a fundamental fact that for a linear operator, continuity and boundedness become equivalent concepts.
- T is continuous if T is continuous at every point.

Theorem Let $T: \mathfrak{D}(T) \to Y$ be a linear operator, where $\mathfrak{D}(T) \subset X$ and X, Y are normed spaces. Then:

a) T is continuous if and only if T is bounded.
b) If T is continuous at a single point, it is continuous.

Theorem Let $T: \mathfrak{D}(T) \to Y$ be a linear operator, where $\mathfrak{D}(T) \subset X$ and X, Y are normed spaces. Then T is continuous if and only if T is bounded.

Proof: If $T = 0$ then the statement is trivially true.

Let $T \neq 0$ then $\|T\| \neq 0$. Assume that T is bounded, we are to prove that T is continuous.

For this let $x_0, x \in \mathfrak{D}(T)$ and $\epsilon > 0$, also define $\delta = \frac{\epsilon}{\|T\|}$ and let $\|x - x_0\| < \delta$

Then by linearity and boundedness of T we have $\|Tx - Tx_0\| = \|T(x - x_0)\|$

$\|Tx - Tx_0\| \leq \|T\|\|x - x_0\| < \|T\|\delta = \|T\|\frac{\epsilon}{\|T\|} = \epsilon$

$\|Tx - Tx_0\| < \epsilon$ implies that T is continuous.

Conversly suppose that T is continuous, then we are to show that T is bounded. Since T is continuous, then it is continuous at any arbitrary point $x_0 \in \mathfrak{D}(T)$ then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$\|Tx - Tx_0\| \leq \epsilon$ whenever $\|x - x_0\| \leq \delta$ …………..(1)

Now choose any $0 \neq y \in \mathfrak{D}(T)$ and set $x = x_0 + \frac{\delta}{\|y\|}y$ then $\|x - x_0\| = \delta$

$(1) \Rightarrow \|Tx - Tx_0\| \leq \epsilon \Rightarrow \|T(x - x_0)\| \leq \epsilon \Rightarrow \left\|T\left(\frac{\delta}{\|y\|}y\right)\right\| \leq \epsilon$

$\Rightarrow \frac{\delta}{\|y\|}\|Ty\| \leq \epsilon \Rightarrow \|Ty\| \leq \frac{\epsilon}{\delta}\|y\| \quad ; \forall y \neq 0$

$\Rightarrow \|Ty\| \leq c\|y\| \quad ; \forall y \neq 0, c = \frac{\epsilon}{\delta}$

Hence T is bounded.

Warning

Unfortunately, continuous linear operators are called "linear operators" by some authors. We shall not adopt this terminology; in fact, there are linear operators of practical importance which are not continuous.

Theorem Let $T: \mathfrak{D}(T) \to Y$ be a linear operator, where $\mathfrak{D}(T) \subset X$ and X, Y are normed spaces. Then if T is continuous at a single point, it is continuous.

Proof: Suppose that T is continuous at a single point in $\mathfrak{D}(T)$ then by theorem "T is continuous if and only if T is bounded" using second part of theorem we have T is bounded and hence continuous on $\mathfrak{D}(T)$ by using first part of this theorem.

Corollary: Let T be a bounded linear operator. Then:

a) $x_n \to x$ implies $Tx_n \to Tx$ where $x_n, x \in X$
b) The null space $\mathcal{N}(T)$ is closed.

Proof:

a) Consider $x_n \to x$ then $\|x_n - x\| \to 0$ as $n \to \infty$
$\Rightarrow \|Tx_n - Tx\| = \|T(x_n - x)\| \leq \|T\|\|x_n - x\| \to 0$
$\Rightarrow \|Tx_n - Tx\| \to 0$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$

b) We are to prove $\mathcal{N}(T) = \overline{\mathcal{N}(T)}$
Already know that $\mathcal{N}(T) \subseteq \overline{\mathcal{N}(T)}$ …………..(1)
Let $x \in \overline{\mathcal{N}(T)}$ then there is a sequence (x_n) in $\mathcal{N}(T)$ such that $x_n \to x$ then by theorem "for a bounded linear operator $x_n \to x$ implies $Tx_n \to Tx$" we have $Tx_n \to Tx$
Since $Tx_n = 0$; $\forall n \in \mathbb{N}$ (as $x_n \in \mathcal{N}(T)$; $\forall n \in \mathbb{N}$)
Therefore $Tx = 0$. i.e. $x \in \mathcal{N}(T)$
So that $\overline{\mathcal{N}(T)} \subseteq \mathcal{N}(T)$ …………..(2)
From (1) and (2) we have $\mathcal{N}(T) = \overline{\mathcal{N}(T)}$
Hence the null space $\mathcal{N}(T)$ is closed

It is worth noting that the range of a bounded linear operator may not be closed.

Theorem

Let $T: M \to N$ be a linear operator, Then T is continuous on N if and only if it is continuous at $0 \in N$.

Proof:

Suppose T is continuous on N then it is continuous at $0 \in N$.

Conversly

Suppose is continuous at $0 \in N$ then we have to prove T is continuous on N.

For this let $x_0 = 0 \in N$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in N$ we have $\|x - x_0\| = \|x\|$

$\Rightarrow \|Tx - Tx_0\| = \|Tx\| < \epsilon$

Hence for any $x_0 \in N$ we have $\|x - x_0\| < \delta$

$\Rightarrow \|Tx - Tx_0\| = \|T(x - x_0)\| < \epsilon$

Implies that T is continuous at x_0 and therefore also on N.

Theorem

Let $T: M \to N$ be a linear operator, and T is continuous on N then $\ker T$ is closed in N.

Proof:

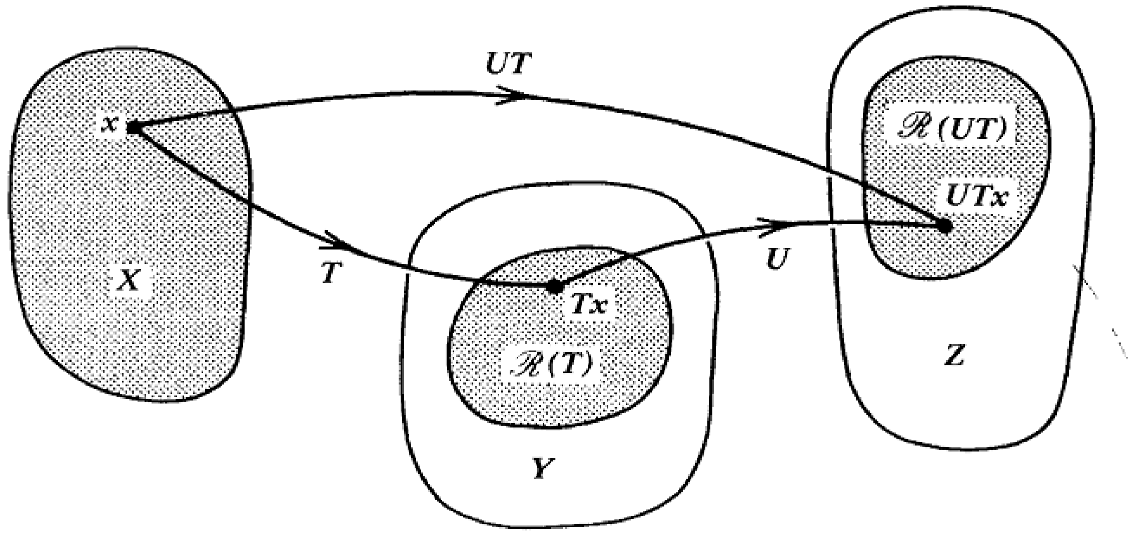
Suppose T is continuous on N and x be a limit point of $\ker T$. Then there is a sequence (x_n) in $\ker T$ such that $lim_{n\to\infty} x_n = x$

Then by continuity of T we have $0 = lim_{n\to\infty} Tx_n = Tx$

Hence $x \in \ker T$ and $\ker T$ is closed in N.

Composition (Product) of Mappings (Operators)

Let $T: X \to Y$ and $U: Y \to Z$ be two mappings (Operators) then their composition is defined as $UT: X \to Z$ by $UT(x) = U(Tx)$



Result

If $T: X \to Y$ and $U: Y \to Z$ are bounded linear operators then their composition $UT: X \to Z$ is also linear and bounded, moreover $\|UT\| \leq \|U\|\|T\|$

Or

If $T_1: X \to Y$ and $T_2: Y \to Z$ are bounded linear operators then their composition $T_2T_1: X \to Z$ is also linear and bounded, moreover $\|T_2T_1\| \leq \|T_2\|\|T_1\|$

Proof

Linearity: $T_2T_1(\propto x + \beta y) = T_2\big(T_1(\propto x + \beta y)\big)$

$T_2T_1(\propto x + \beta y) = T_2(\propto T_1x + \beta T_1 y)$ Since T_1 is linear

$T_2T_1(\propto x + \beta y) = \propto T_2(T_1x) + \beta T_2(T_1 y)$ Since T_2 is linear

$T_2T_1(\propto x + \beta y) = \propto T_2T_1x + \beta T_2T_1 y$ Hence T_2T_1 is linear

Boundedness: $\|T_2T_1(x)\| = \|T_2(T_1x)\| \leq c_1\|(T_1x)\|$ Since T_2 is bounded

$\|T_2T_1(x)\| \leq c_1c_2\|x\|$ Since T_1 is bounded

$\|T_2T_1(x)\| \leq k\|x\|$ Hence T_2T_1 is bounded

$\|\boldsymbol{T_2T_1}\| \leq \|\boldsymbol{T_2}\|\|\boldsymbol{T_1}\|$

$$\|T_2T_1\| = Sup_{\substack{x\in\mathfrak{D}(T_2T_1)\\ x\neq 0}} \frac{\|T_2T_1x\|}{\|x\|} = Sup_{\substack{x\in\mathfrak{D}(T_2T_1)\\ x\neq 0}} \frac{\|T_2(T_1x)\|}{\|x\|} \leq Sup_{\substack{x\in\mathfrak{D}(T_2T_1)\\ x\neq 0}} \frac{\|T_2\|\|T_1x\|}{\|x\|}$$

$$\|T_2T_1\| \leq \|T_2\| Sup_{\substack{x\in\mathfrak{D}(T_1)\\ x\neq 0}} \frac{\|T_1x\|}{\|x\|} \Rightarrow \|\boldsymbol{T_2T_1}\| \leq \|\boldsymbol{T_2}\|\|\boldsymbol{T_1}\|$$

Remember

- Two operators T_1 and T_2 are defined to be **equal**, written if they have the same domain $\mathfrak{D}(T_1) = \mathfrak{D}(T_2)$ and if $T_1x = T_2x$ for all $x \in \mathfrak{D}(T_1) = \mathfrak{D}(T_2)$
- The **restriction** of an operator $T: \mathfrak{D}(T) \to Y$ to a subset $B \subset \mathfrak{D}(T)$ is denoted by $T\mid_B$ and is the operator defined by $T\mid_B: B \to Y$, as $T\mid_B x = Tx$ for all $x \in B$.
- An **extension** of an operator $T: \mathfrak{D}(T) \to Y$ to a set $M \supset \mathfrak{D}(T)$ is an operator $\tilde{T}: M \to Y$ such that $\tilde{T}\mid_{\mathfrak{D}(T)} = T$, that is, $\tilde{T}x = Tx$ for all $x \in \mathfrak{D}(T)$. [Hence T is the restriction of $\tilde{T}$ to $\mathfrak{D}(T)$].

If $\mathfrak{D}(T)$ is a proper subset of M, then a given T has many extensions. Of practical interest are usually those extensions which preserve some basic property, for instance linearity (if T happens to be linear) or boundedness (if $\mathfrak{D}(T)$ lies in a normed space and T is bounded).

The following important theorem is typical in that respect. It concerns an extension of a bounded linear operator T to the closure $\overline{\mathfrak{D}(T)}$ of the domain such that the extended operator is again bounded and linear, and even has the same norm. This includes the case of an extension from a dense set in a normed space X to all of X. It also includes the case of an extension from a normed space X to its completion.

Theorem (Bounded linear extension, Principle of Extension by Continuity)

Let $T: \mathfrak{D}(T) \to Y$ be a bounded linear operator, where $\mathfrak{D}(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension $\tilde{T}: \overline{\mathfrak{D}(T)} \to Y$ where $\tilde{T}$ is a bounded linear operator of norm $\|\tilde{T}\| = \|T\|$.

Proof: First we show the existence of $\tilde{T}$. i.e. we justify that such a $\tilde{T}$ exists.

Let $x \in \overline{\mathfrak{D}(T)}$ then there exists a sequence (x_n) in $\mathfrak{D}(T)$ such that $x_n \to x$ then using the fact, T is linear we have

$\|Tx_n - Tx_m\| = \|T(x_n - x_m)\| \leq \|T\|\|x_n - x_m\| \to 0$ as $m, n \to \infty$

(Actually we use the fact; x_n being convergent sequence in Cauchy sequence.)

This implies that Tx_n is a Cauchy sequence in Y. Since Y is complete therefore $Tx_n \to y \in Y$. This means that $\lim_{n\to\infty} Tx_n = y$

We define $\tilde{T}$ by $\tilde{T}x = y = \lim_{n\to\infty} Tx_n$ then Clearly $\tilde{T}x = Tx \quad ; \forall x \in \mathfrak{D}(T)$

We now show that this definition of $\tilde{T}$ is independent of the particular choice of a sequence in $\mathfrak{D}(T)$ converging to x.

Suppose that $x_n \to x$ and $z_n \to x$. Then $v_m \to x$, where (v_m) is the sequence $(x_1, z_1, x_2, z_2, \dots)$

Hence (Tv_m) converges by ($x_n \to x$ implies $Tx_n \to Tx$), and the two subsequences (Tx_n) and (Tz_n) of (Tv_m) must have the same limit.

This proves that $\tilde{T}$ is uniquely defined at every $x \in \overline{\mathfrak{D}(T)}$.

$\tilde{T}$ is linear

$T(\propto x_n + \beta y_n) = \propto Tx_n + \beta T y_n$ such that $x_n \to x$ and $y_n \to y$

$\lim_{n\to\infty} T(\propto x_n + \beta y_n) = \propto \lim_{n\to\infty} Tx_n + \beta \lim_{n\to\infty} Ty_n$

$\tilde{T}(\propto x + \beta y) = \propto \tilde{T}x + \beta \tilde{T} y$

$\tilde{T}$ is Bounded

Let (x_n) be a sequence such that $x_n \to x$ then using the fact T is bounded we have

$\|Tx_n\| \leq \|T\|\|x_n\|$

$\Rightarrow \lim_{n\to\infty}\|Tx_n\| \leq \|T\| \lim_{n\to\infty}\|x_n\|$

$\Rightarrow \left\|\lim_{n\to\infty} Tx_n\right\| \leq \|T\| \left\|\lim_{n\to\infty} x_n\right\|$

$\Rightarrow \|\tilde{T}x\| \leq \|T\|\|x\|$

$\Rightarrow \tilde{T}$ is Bounded

$\|\tilde{\boldsymbol{T}}\| = \|\boldsymbol{T}\|$

Since $\|\tilde{T}x\| \leq \|T\|\|x\|$

$\Rightarrow \frac{\|\tilde{T}x\|}{\|x\|} \leq \|T\| \quad ; \forall x \neq 0$

$\Rightarrow Sup_{x\neq 0} \frac{\|\tilde{T}x\|}{\|x\|} \leq \|T\| \quad ; \forall x \neq 0$

$\Rightarrow \|\tilde{T}\| \leq \|T\|$ ……………(1)

Also $\|T\| \leq \|\tilde{T}\|$ ……………(2) Since $\mathfrak{D}(T) \subseteq \overline{\mathfrak{D}(T)}$

Combining (1) and (2) we have

$\|\tilde{T}\| = \|T\|$

LINEAR FUNCTIONALS

A functional is an operator whose range lies on the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$. And functional analysis was initially the analysis of functionals. The latter appear so frequently that special notations are used. We denote functionals by lowercase letters $f, g, h, \dots$, the domain of I by $\mathfrak{D}(f)$, the range by $\mathfrak{R}(f)$ and the value of f at an $x \in \mathfrak{D}(f)$ by $f(x)$, with parentheses.

Functionals are operators, so that previous definitions apply. We shall need in particular the following two definitions because most of the 'functionals' to be considered will be linear and bounded.

Functional (Function of Functions)

Let $M = \{f : f\ is\ vector\ space\}$ then $I : M \to \mathbb{R}$ is called functional.

Linear functional

A linear functional f is a linear operator with domain in a vector space X and range in the scalar field K of X; thus, $f : \mathfrak{D}(f) \to K$ where $K = \mathbb{R}$ if X is real and $K = \mathbb{C}$ if X is complex.

Bounded linear functional

A bounded linear functional f is a bounded linear operator with range in the scalar field of the normed space X in which the domain $\mathfrak{D}(f)$ lies.

Thus there exists a real number 'c' such that for all $x \in \mathfrak{D}(f)$, $|f(x)| \leq c\|x\|$

Furthermore, the norm of f is $\|f\| = Sup_{\substack{x \in \mathfrak{D}(f) \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$ Or $\|f\| = Sup_{\substack{x \in \mathfrak{D}(f) \\ x = 1}} |f(x)|$

Also remember that $|f(x)| \leq \|f\|\|x\|$

Remark: The result that we proved for bounded linear operator continue to hold true for bounded linear functional. i.e.

A linear functional f with domain $\mathfrak{D}(f)$ in a normed space is continuous if and only if f is bounded.

Examples

- **Norm:** The norm $\|.\|: X \to \mathbb{R}$ on a normed space (X, $\|.\|$) is a functional on X which is not linear (Since $\|x+y\| \le \|x\| + \|y\|$
- **Dot product:** The familiar dot product with one factor kept fixed defines a functional $f: \mathbb{R}^3 \to \mathbb{R}$ by means of $f(x) = x.a = \xi_1 \propto_1 + \xi_2 \propto_2 + \xi_3 \propto_3$ where $a = (\propto_1, \propto_2, \propto_3) = (\propto_i) \in \mathbb{R}^3$ is fixed. Then f is linear. f is bounded. Also $\|f\| = \|a\|$

  Solution

  - **Linearity:**

    $f(\propto x + \beta y) = (\propto x + \beta y).a = \propto x.a + \beta y.a = \propto f(x) + \beta f(y)$
  - **Boundedness:**

    $|f(x)| = |x.a| = |\|x\|\|a\|Cos\theta| \le \|x\|\|a\|$ $\quad\quad \therefore Cos\theta \le 1$

    $|f(x)| \le \|x\|\|a\|$
  - $\|\boldsymbol{f}\| = \|\boldsymbol{a}\|$

    Since $|f(x)| \le \|x\|\|a\|$ then $\frac{|f(x)|}{\|x\|} \le \|a\|$; $\forall x \ne 0$

    $Sup_{x\ne 0} \frac{|f(x)|}{\|x\|} \le \|a\|$; $\forall x \ne 0$

    $\|f\| \le \|a\|$ ……………(1)

    Now $\|f\| = Sup_{x\ne 0} \frac{|f(x)|}{\|x\|} \ge \frac{|f(a)|}{\|a\|} = \frac{|a.a|}{\|a\|} = \frac{\|a\|^2}{\|a\|} = \|a\|$

    $\|f\| \ge \|a\|$ ……………(2)

    From (1) and (2) $\quad\quad \|f\| = \|a\|$
- **Definite integral**

  The definite integral is a number if we consider it for a single function, as we do in calculus most of the time. However, the situation changes completely if we consider that integral for all functions in a certain function space. Then the integral becomes a functional on that space, call it f. As a space let us choose C[a, b];

  $f: \mathrm{C[a,b]} \to \mathbb{R}$ by $f(x) = \int_a^b x(t)dt$ where $x \in \mathrm{C[a,b]}$ then f is linear and bounded functional and $\|f\| = b - a$

  Solution

  - **Linearity:**

    $f(\propto x + \beta y) = \int_a^b (\propto x + \beta y)dt = \propto \int_a^b xdt + \beta \int_a^b ydt = \propto f(x) + \beta f(y)$

- **Boundedness:**

  We have $\|x\| = max_{t\in[a,b]}|x(t)|$ then

  $|f(x)| = \left|\int_a^b x(t)dt\right| \leq \int_a^b |x(t)|dt \leq (b-a)max_{t\in[a,b]}|x(t)|$

  $|f(x)| \leq (b-a)\|x\|$

- $\|\boldsymbol{f}\| = \|\boldsymbol{a}\|$

  Since $|f(x)| \leq (b-a)\|x\|$ then $\frac{|f(x)|}{\|x\|} \leq (b-a)$; $\forall x \neq 0$

  $Sup_{x\neq 0}\frac{|f(x)|}{\|x\|} \leq (b-a)$; $\forall x \neq 0$

  $\|f\| \leq (b-a)$ ……………(1)

  Now choose $x = x_0 = 1$

  $\|f\| = Sup_{x\neq 0}\frac{|f(x)|}{\|x\|} \geq \frac{|f(x_0)|}{\|x_0\|} = \frac{\left|\int_a^b x_0(t)dt\right|}{\|max_{t\in[a,b]}|x_0(t)|\|} = \frac{b-a}{1} = b-a$

  $\|f\| \geq b-a$ ……………(2)

  From (1) and (2) $\|f\| = b-a$

- **Space C[a, b]:** Another practically important functional on C[a, b] is obtained if we choose a fixed $t_0 \in J = [a,b]$ and set $f_1(x) = x(t_0)$; $x \in C[a,b]$ then f_1 is linear and bounded functional and $\|f_1\| = 1$
- **Space $\boldsymbol{l^2}$:** We can obtain a linear functional f on the Hilbert space l^2 by choosing a fixed $a = (\propto_j) \in l^2$ and setting $f(x) = \sum_{j=1}^{\infty} \xi_j \propto_j$ where $a = (\xi_j) \in l^2$. This series converges absolutely and f is bounded.

  Solution

- **Linearity:**

  $f(\propto x + \beta y) = \sum_{j=1}^{\infty}(\alpha\xi_j + \beta\eta_j)\propto_j = \alpha\sum_{j=1}^{\infty}\xi_j \propto_j + \beta\sum_{j=1}^{\infty}\eta_j \propto_j$

  $f(\propto x + \beta y) = \propto f(x) + \beta f(y)$

- **Boundedness:**

  $|f(x)| = \left|\sum_{j=1}^{\infty}\xi_j \propto_j\right| \leq \sum_{j=1}^{\infty}|\xi_j \propto_j| \leq \left(\sum_{j=1}^{\infty}|\xi_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{\infty}|\propto_j|^2\right)^{\frac{1}{2}}$

  $|f(x)| \leq c\|x\|$ where $\|x\| = \left(\sum_{j=1}^{\infty}|\xi_j|^2\right)^{\frac{1}{2}}$; $c = \left(\sum_{j=1}^{\infty}|\propto_j|^2\right)^{\frac{1}{2}}$

Algebraic dual space (Conjugate Space)

Let X be a vector space then the set of all linear functionals defined on X can itself be made into a vector space. This space is denoted by X* and is called the algebraic dual space of X

i.e. $X^* = \{f(x) : f \text{ } is \text{ } linear \quad ; \forall x \in X\}$

(Note that this definition does not involve a norm. The so-called dual space X' consisting of all bounded linear functionals on X). Its algebraic operations of vector space are defined in a natural way as follows (i.e. it satisfies the algebraic operation of a vector space).

- $S(x) = (f_1 + f_2)x = f_1(x) + f_2(x)$
- $P(x) = (\propto f)x = \propto f(x)$

Second Algebraic dual space

Let X* be a vector space then the set of all linear functionals defined on X* can itself be made into a vector space. This space is denoted by X** and is called the algebraic dual space of X^*.

i.e. $X^{**} = \{f(x) : f \text{ } is \text{ } linear \quad ; \forall x \in X^*\}$

Dual Space

The set of all continuous or bounded linear functionals on X becomes a normed space, which is called the dual space **X'** of **X**.

Remark: Dual space of a normed space also a dual space with defined norm.

Theorem (Dimension of X*)

For a finite dimensional normed space X show that $\dim X = \dim X^*$

Or Let X be n – dimensional norm space then its dual also n – dimensional.

Or Let X be an n – dimensional vector space and $E = \{e_1, e_2, \dots, e_n\}$ a basis for X. Then $E^* = \{f_1, f_2, \dots, f_n\}$ given by $f_j(e_i) = \delta_{ij} = \begin{cases} 0 & ; i \neq j \\ 1 & ; i = j \end{cases}$ is a basis for the algebraic dual X^* of X, and $\dim X = \dim X^* = n$.

Proof

Suppose $E = \{e_1, e_2, \dots, e_n\}$ be a basis for X then $x \in X$ could be written as $x = \sum_{j=1}^{n} a_j e_j$.

Define a linear functional $f_j : E \to F$ by $f_j(e_i) = \delta_{ij} = \begin{cases} 0 & ; i \neq j \\ 1 & ; i = j \end{cases}$

Also $f_j(x) = f_j\left(\sum_{j=1}^{n} a_j e_j\right) = \sum_{j=1}^{n} a_j f_j\left(e_j\right) = a_j$

We are to show that $E^* = \{f_1, f_2, \dots, f_n\}$ is a basis for X^*.

Step – I: Linear Independence

Let $\sum_{j=1}^{n} \beta_j f_j = 0$ then

$0 = O(x) = \left(\sum_{j=1}^{n} \beta_j f_j\right)x = \sum_{j=1}^{n} \beta_j f_j(x) = \sum_{j=1}^{n} \beta_j f_j(e_i) = \sum_{j=1}^{n} \beta_j \delta_{ij}$

$\Rightarrow \beta_j = 0$. This shows that $E^* = \{f_1, f_2, \dots, f_n\}$ is linearly independent.

Step – II: $\boldsymbol{E^* = \{f_1, f_2, \dots, f_n\}}$ generate $\boldsymbol{X^*}$

Let $f \in X^*$ then for any $x = \sum_{j=1}^{n} a_j e_j \in X$ we have

$f(x) = f\left(\sum_{j=1}^{n} a_j e_j\right) = \sum_{j=1}^{n} a_j f\left(e_j\right) = \sum_{j=1}^{n} a_j \gamma_j = \sum_{j=1}^{n} \gamma_j f_j(x) = \left(\sum_{j=1}^{n} \gamma_j f_j\right)x$

$\Rightarrow f = \sum_{j=1}^{n} \gamma_j f_j$. This shows that $E^* = \{f_1, f_2, \dots, f_n\}$ span X^*.

Hence $E^* = \{f_1, f_2, \dots, f_n\}$ is a basis of X^*.

Implies that $\dim X = \dim X^* = n$

Theorem (Dimension of X)**

A finite dimensional normed (linear) space X is isomorphic to its second dual. i.e. $X \cong X^{**}$

Proof Let X be finite dimensional normed (linear) space and X^{**} be its second dual. Then define $\varphi: X \to X^{**}$ as $\varphi(x) = g_x \ ; x \in X$ where $g_x: X^* \to F$ defined as $g_x(f) = f(x) \ ; f \in X^*$.

Step – I: $\boldsymbol{\varphi}$ is Linear

$\varphi(\propto x + \beta y) = g_{\propto x+\beta y} = g_{\propto x+\beta y}(f) = f(\propto x + \beta y) = \propto f(x) + \beta f(y)$

$\Rightarrow \varphi(\propto x + \beta y) = \propto g_x(f) + \beta g_y(f) \ ; f \in X^*$
$\Rightarrow \varphi(\propto x + \beta y) = \propto \varphi(x) + \beta\varphi(y)$

Step – II: $\boldsymbol{\varphi}$ is Injective

Let $x, y \in X \ and \ f \in X^*$ then take $\varphi(x) = \varphi(y)$

$\Rightarrow g_x = g_y \Rightarrow (g_x - g_y)f = 0 \Rightarrow g_x(f) - g_y(f) = 0 \Rightarrow f(x) - f(y) = 0$

$\Rightarrow f(x - y) = 0 \Rightarrow x - y = 0 \Rightarrow x = y \Rightarrow \varphi$ is Injective

Also φ is onto and hence φ is bijective. So $X \cong \varphi(X)$

Also $\varphi(X)$ is subspace of X^{**}.

Since X has finite dimension so $\dim X = \dim X^* = \dim X^{**}$ then $\varphi(X) \cong X^{**}$

Implies $X \cong X^{**}$

Lemma Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$.

Proof Let $\{e_1, e_2, \dots, e_n\}$ be a basis for X and $x_0 = \sum_{j=1}^{n} a_j e_j$. Then $f(x_0) = f\left(\sum_{j=1}^{n} a_j e_j\right) = \sum_{j=1}^{n} a_j f(e_j)$

By assumption this is zero for every $f \in X^*$. i.e. $f(x_0) = \sum_{j=1}^{n} a_j f(e_j) = 0$

$\Rightarrow f(x_0) = \sum_{j=1}^{n} a_j f(e_j) = a_j = 0 \Rightarrow x_0 = \sum_{j=1}^{n} a_j e_j = 0 \Rightarrow x_0 = 0$

Reflexive Space

A normed space X is said to be reflexive if there is an isometric isomorphism between X and X^{**} (second dual)

Canonical Mapping

To each $x \in X$ there corresponds a $g_x \in X^{**}$. This defines a mapping $C: X \to X^{**}$ by $C(x) = g_x$ this mapping is linear and is called the canonical mapping of X into X**. C is linear since its domain is a vector space. C is also called the **canonical embedding** of X into X**.

Theorem (Algebraic Reflexivity).

A finite dimensional vector space is algebraically reflexive.

Proof

Let the canonical mapping $C: X \to X^{**}$ is linear. $Cx_0 = 0$ means that for all $f \in X^*$ we have $(Cx_0)f = (g_{x_0})f = f(x_0)$ by the definition of C. This implies $x_0 = 0$ by Lemma "Let X be a finite dimensional vector space. If $x_0 \in X$ has the property that $f(x_0) = 0$ for all $f \in X^*$, then $x_0 = 0$.".

As we know that the mapping C has an inverse $C^{-1}: \Re(C) \to X$, where $\Re(C)$ is the range of C. We also have $\dim \Re(C) = \dim X$.

Now as we know that, $\dim X = \dim X^* = \dim X^{**}$.

Together, $\dim \Re(C) = \dim X^{**}$

Hence $\Re(C) = X^{**}$ because $\Re(C)$ is a vector space and a proper subspace of X^{**} has dimension less than $\dim X^{**}$.

By the definition, this proves algebraic reflexivity.

Space B(X,Y)

Let X and Y be normed spaces over the same field. The vector space of all bounded linear operators from X into Y is called B(X,Y) space. i.e.

$B(X,Y) = \{T: X \to Y: T \text{ is bounded and linear}\}$

Theorem

The vector space B(X,Y) of all bounded linear operators from a normed space X into a normed space Y is itself a normed space with norm defined by

$$\|T\| = Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = Sup_{\substack{x\in X\\ x=1}} \|Tx\|$$

Proof

$N_1)$ $\quad \|T\| = Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} \geq 0$

$N_2)$ $\quad \|T\| = 0 \Leftrightarrow Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = 0 \Leftrightarrow \|Tx\| = 0 \Leftrightarrow T = 0$

$N_3)$ $\quad \|\propto T\| = Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|\propto Tx\|}{\|x\|} = Sup_{\substack{x\in X\\ x\neq 0}} \frac{|\propto|\|Tx\|}{\|x\|} = |\propto| Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} = |\propto|\|T\|$

$N_4)$ $\quad \|T+S\| = Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|(T+S)x\|}{\|x\|} = Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx+Sx\|}{\|x\|} \leq Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|+\|Sx\|}{\|x\|}$

$\|T+S\| \leq Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Tx\|}{\|x\|} + Sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Sx\|}{\|x\|}$

$\|T+S\| \leq \|T\| + \|S\|$

Theorem (Completeness)

If Y is a Banach space, then B(X,Y) is a Banach space.

Proof

Let $\{T_n\}_1^\infty$ be a Cauchy sequence in B(X,Y) then for all $\in > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|T_n - T_m\| <\in_1 \quad ; \forall m, n \geq n_0$

Now consider for any $x \in X$ and $m, n \geq n_0$

$\|T_n x - T_m x\| = \|(T_n - T_m)x\| \leq \|T_n - T_m\|\|x\| <\in_1 \|x\| \Rightarrow \|T_n x - T_m x\| <\in$

Implies $\{T_n x\}_1^\infty$ is a Cauchy sequence in Y.

Since Y is a Banach space (Complete metric + norm) therefore $T_n x \to y \in Y$

Clearly the limit $y \in Y$ depends upon the choice of $x \in X$

This defines an operator $T: X \to Y$, where $y = Tx = \lim_{n\to\infty} T_n\, x$.

The operator T is Linear

$T(\propto x + \beta y) = \lim_{n\to\infty} T_n\,(\propto x + \beta y) = \lim_{n\to\infty}(\propto T_n x + \beta T_n y)$

$T(\propto x + \beta y) = \propto \lim\limits_{n\to\infty} T_n x + \beta \lim\limits_{n\to\infty} T_n y = \propto Tx + \beta Ty$

The operator T is Bounded

Since for any $x \in X$ and $m, n \geq n_0$ we have $\|T_n x - T_m x\| < \in$

$\|T_n x - Tx\| < \in$ letting $m \to \infty$

$\Rightarrow \|(T_n - T)x\| < \in \quad ; \forall n \geq n_0 \Rightarrow T_n - T$ is bounded.

$\Rightarrow T_n - T \in B(X,Y) \Rightarrow T = T_n - (T_n - T) \in B(X,Y) \Rightarrow T \in B(X,Y)$

To prove $T_n \to T$

Since we have $\|T_n x - Tx\| < \in_1 \|x\| \Rightarrow \|(T_n - T)x\| < \in_1 \|x\|$

$\Rightarrow \frac{\|(T_n - T)x\|}{\|x\|} < \in_1 \quad ; \forall n \geq n_0$

$\Rightarrow Sup_{x\neq 0} \frac{\|(T_n - T)x\|}{\|x\|} < \in_1 \quad ; \forall n \geq n_0 \Rightarrow \|T_n - T\| < \in_1 \quad ; \forall n \geq n_0$

$\Rightarrow T_n \to T$ with $\|.\|$

Hence B(X,Y) is a Banach space.

This theorem has an important consequence with respect to the dual space X' of X, which is defined as follows.

Definition (Dual space X')

Let X be a normed space. Then the set of all bounded linear functionals on X constitutes a normed space with norm defined by

$\|f\| = Sup_{\substack{x\in X \\ x\neq 0}} \frac{|f(x)|}{\|x\|} = Sup_{\substack{x\in X \\ x=1}} |f(x)|$

which is called the dual space of X and is denoted by X' .

Theorem

The dual space X' of a normed space X is a Banach space (whether or not X is).

Proof

Let $f: X \to Y$ be a linear functional then the set $X' = B(X, Y)$ consisting of all linear functionals $f: X \to Y$ such that $\|f\| = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|}$ is finite, is itself a normed space of linear functional. Because the following axioms are satisfied;

$N_1)$ $\quad \|f\| = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \geq 0$

$N_2)$ $\quad \|f\| = 0 \Leftrightarrow Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = 0 \Leftrightarrow |f(x)| = 0 \Leftrightarrow f = 0$

$N_3)$ $\quad \|\propto f\| = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|\propto f(x)|}{\|x\|} = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|\propto||f(x)|}{\|x\|} = |\propto| Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} = |\propto| \|f\|$

$N_4)$ $\quad \|f + g\| = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|(f+g)x|}{\|x\|} = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)+g(x)|}{\|x\|} \leq Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|+|g(x)|}{\|x\|}$

$$\|f + g\| \leq Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} + Sup_{\substack{x \in X \\ x \neq 0}} \frac{|g(x)|}{\|x\|}$$

$$\|f + g\| \leq \|f\| + \|g\|$$

Since X' itself is a normed space and hence a topological space, the topology on X' is called the strong topology in X'. Since Y is $\mathbb{R}$ or $\mathbb{C}$, which are complete. So X' as the space of all bounded linear functionals defined on X, is also complete and hence is a Banach Space. This is true even if X is not a Banach Space.

It is a fundamental principle of functional analysis that investigations of spaces are often combined with those of the dual spaces. For this reason it is worthwhile to consider some of the more frequently occurring spaces and find out what their duals look like. In this connection the concept of an isomorphism will be helpful in understanding the present discussion.

Isometric Isomorphism of a Normed Space

An isomorphism of a normed space X onto a normed space Y is a bijective linear operator $T: X \to Y$ which preserves the norm, that is, for all $x \in X$, $\|Tx\| = \|x\|$ (Hence T is isometric.) X is then called isomorphic with Y, and X and Y are called isomorphic normed spaces.

From an abstract point of view, X and Y are then identical, the isomorphism merely amounting to renaming of the elements (attaching a "tag" T to each point).

Or Let X and Y be normed spaces. A function $T: X \to Y$ is said to be an isometric isomorphism if

- T is bijective
- T is linear
- T preserves norm. i.e. for any $x \in X$, $\|Tx\| = \|x\|$

Example Show that The dual space (Conjugate Space) of $\mathbb{R}^n$ is $\mathbb{R}^n$.

Or Show that The dual space of $\mathbb{R}^n$ is isomorphic with $\mathbb{R}^n$.

Proof.

Let $\mathbb{R}^{n*}$ be the dual space of $\mathbb{R}^n$ then we have to show that $\mathbb{R}^{n*}$ is isomorphic to $\mathbb{R}^n$. For this we define a function $T: \mathbb{R}^{n*} \to \mathbb{R}^n$ as $T(f) = c_f$

Where $f \in \mathbb{R}^{n*}$ and $B = \{e_1, e_2, \dots, e_n\}$ be a basis of $\mathbb{R}^n$ and for each $x \in \mathbb{R}^n$ we have $x = \sum_1^n x_i e_i$ as well as $f(x) = f(\sum_1^n x_i e_i) = \sum_1^n x_i f(e_i) = \sum_1^n x_i c_i$ where we take $c_i = f(e_i)$ and $c_f = (c_1, c_2, \dots, c_n) \in \mathbb{R}^n$.

We are to show that T is an isomorphism.

***T* is bijective (Injective + Surjective)**

Let $f, f' \in \mathbb{R}^{n*}$ then $c_i = f(e_i), c_i' = f'(e_i)$ then consider for $i = 1,2, \dots, n$

$T(f) = T(f') \Rightarrow c_f = c_{f'} \Rightarrow (c_1, c_2, \dots, c_n) = (c_1', c_2', \dots, c_n') \Rightarrow c_i = c_i'$

$\Rightarrow f(e_i) = f'(e_i) \Rightarrow f(e_i) - f'(e_i) = 0 \Rightarrow (f - f')(e_i) = 0$

$\Rightarrow (f - f')x = \sum_1^n x_i(f - f')e_i = 0$ since $x = \sum_1^n x_i e_i$

$\Rightarrow f - f' = 0 \Rightarrow f = f' \Rightarrow$ ***T* is injective.**

Now let $b = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n$ then for any $x = \sum_1^n x_i e_i \in \mathbb{R}^n$ define a function $g: \mathbb{R}^n \to \mathbb{R}$ by $g(x) = \sum_1^n x_i b_i$. This g is linear and, because $\mathbb{R}^n$ is finite dimensional, g is bounded, so $g \in \mathbb{R}^{n*}$. We put then $T(g) = b \in \mathbb{R}^n$.

So that T is surjective.***T* is bijective.**

***T* is linear**

Let $f, f' \in \mathbb{R}^{n*}$ and $\propto, \propto' \in \mathbb{R}$ then

$T(\propto f + \propto' f') = c_{\propto f + \propto' f'} = (c_1'', c_2'', \dots, c_n'') = c_i'' = (\propto f + \propto' f')(e_i)$

$T(\propto f + \propto' f') = \propto f(e_i) + \propto' f'(e_i) = \propto c_i + \propto' c_i' = \propto c_f + \propto' c_{f'}$

$T(\propto f + \propto' f') = \propto T(f) + \propto' T(f')$

Hence T is linear.

***T* is norm preserving**

Let for any $x \in \mathbb{R}^n$ we have $\|f\| = Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \geq \frac{|f(x)|}{\|x\|} = \frac{|f(e_i)|}{\|e_i\|} = \|c_f\|$; $x \neq 0$

$\|f\| \geq \|c_f\|$ ………..(1)

Also for all $x \in \mathbb{R}^n$ we have $|f(x)| = |\sum_1^n x_i c_i| \leq \sqrt{\sum_1^n |x_i|^2} \sqrt{\sum_1^n |c_i|^2}$ Holder Inequality

$\Rightarrow |f(x)| \leq \|x\| \|c_f\| \Rightarrow \frac{|f(x)|}{\|x\|} \leq \|c_f\| \Rightarrow Sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{\|x\|} \leq \|c_f\|$

$\|f\| \leq \|c_f\|$ ………..(2)

Combining (1) and (2) $\|f\| = \|c_f\| = \|T(f)\|$

This show that $T: \mathbb{R}^{n*} \to \mathbb{R}^n$ is the isometric isomorphism between $\mathbb{R}^{n*}$ and $\mathbb{R}^n$.

Hence $\mathbb{R}^{n*} \cong \mathbb{R}^n$. i.e. dual space (Conjugate Space) of $\mathbb{R}^n$ is $\mathbb{R}^n$.

$\boldsymbol{l^1}$ **− space:** This space consists of all sequences $x = \{x_i\}$ with $\|x\| = \sum_1^\infty |x_i| < \infty$

$\boldsymbol{l^\infty}$ **− space:** This space consists all sequences $x = \{x_i\}$ with $\|x\| = Sup_1^\infty |x_i| < \infty$

Example Show that The dual space (Conjugate Space) of l^1 is l^∞.

Or Show that The dual space of l^1 is isomorphic with l^∞.

Proof

Let l^{1*} be the dual space of l^1 then we have to show that l^{1*} is isomorphic to l^∞. For this we define a function $T: l^{1*} \to l^\infty$ as $T(f) = c_f$

Where $f \in l^{1*}$ and $B = \{e_1, e_2, \dots, e_n\}$ be a basis of l^∞ and for each $x \in l^\infty$ we have $x = \sum_1^\infty x_i e_i$ as well as $f(x) = f(\sum_1^\infty x_i e_i) = \sum_1^\infty x_i f(e_i) = \sum_1^\infty x_i c_i$ where we take $\{c_i\} = f(e_i) = c_f$ and fistly we show that $c_f \in l^\infty$.

$|c_i| = |f(e_i)| \le \|f\|\|e_i\| \le \|f\| \quad ; \forall i$

$\Rightarrow \|c_f\| = Sup_1^\infty |c_i| \le \|f\| \quad ; \forall i \Rightarrow c_f \in l^\infty$

We are to show that T is an isomorphism.

***T* is bijective (Injective + Surjective)**

Let $f, f' \in l^{1*}$ then $c_i = f(e_i), c_i' = f'(e_i)$ then consider for $i = 1,2, \dots, n$

$T(f) = T(f') \Rightarrow c_f = c_{f'} \Rightarrow (c_1, c_2, \dots, c_n) = (c_1', c_2', \dots, c_n') \Rightarrow c_i = c_i'$

$\Rightarrow f(e_i) = f'(e_i) \Rightarrow f(e_i) - f'(e_i) = 0 \Rightarrow (f - f')(e_i) = 0$

$\Rightarrow (f - f')x = \sum_1^\infty x_i (f - f')e_i = 0$ since $x = \sum_1^\infty x_i e_i$

$\Rightarrow f - f' = 0 \Rightarrow f = f' \Rightarrow$ ***T* is injective.**

Now let $b=(b_1,b_2,\dots,b_n)\in l^\infty$ then for any $x=\sum_1^\infty x_ie_i\in l^\infty$ define a function $g: l^1\to F$ by $g(x)=\sum_1^\infty x_ib_i$. This g is linear. Also for any $x\in l^1$;

$$|g(x)|=|\sum_1^\infty x_ib_i|\le\sum_1^\infty|x_ib_i|\le Sup_1^\infty|b_i|\sum_1^\infty|x_i|\le\|b\|\|x\|$$

$$|g(x)|\le\|b\|\|x\|\Rightarrow\frac{|g(x)|}{\|x\|}\le\|b\|\Rightarrow\|g\|\le\|b\|$$

So $g\in l^{1*}$. We put then $T(g)=b\in l^\infty$.

So that T is surjective.***T* is bijective.**

***T* is linear**

Let $f,f'\in l^{1*}$ and $\propto,\propto'\in F$ then

$$T(\propto f+\propto'f')=c_{\propto f+\propto'f'}=(c_1'',c_2'',\dots,c_n'')=c_i''=(\propto f+\propto'f')(e_i)$$

$$T(\propto f+\propto'f')=\propto f(e_i)+\propto' f'(e_i)=\propto c_i+\propto'c_i'=\propto c_f+\propto'c_{f'}$$

$$T(\propto f+\propto'f')=\propto T(f)+\propto' T(f')$$

Hence T is linear.

***T* is norm preserving**

Let for any $x\in l^1$ we have

$$|f(x)|=|\sum_1^\infty x_ic_i|\le\sum_1^\infty|x_ic_i|\le Sup_1^\infty|c_i|\sum_1^\infty|x_i|\le\|c_f\|\|x\|\Rightarrow\frac{|f(x)|}{\|x\|}\le\|c_f\|$$

$$\|f\|\le\|c_f\| \qquad \dots\dots\dots\dots(1)$$

Also for all $x\in l^{1*}$ we have $|c_i|=|f(e_i)|\le\|f\|\|e_i\|\le\|f\|\,;\forall i$

$$\Rightarrow\|c_f\|=Sup_1^\infty|c_i|\le\|f\| \quad ;\forall i\Rightarrow\|f\|\ge\|c_f\| \qquad \dots\dots\dots\dots(2)$$

Combining (1) and (2) $\qquad \|f\|=\|c_f\|=\|T(f)\|$

This show that $T: l^{1*}\to l^\infty$ is the isometric isomorphism between l^{1*} and l^∞.

Hence $l^{1*}\cong l^\infty$. i.e. dual space (Conjugate Space) of l^1 is l^∞.

This shows that l^1 is not reflexive. As l^1 is separable but $l^{1*}\cong l^\infty$ is not.

l^p − space

This space consists of all sequences $x = \{x_i\}$ with $\|x\|_p = (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} < \infty$

Example For any $p > 1$, show that the dual space of l^p is l^q. Where q is conjugate exponent of p. Hence l^p is reflexive.

Or The dual space of l^p is l^q; here, $1 < P < +\infty$ and q is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof

Let l^{p*} be the dual space of l^p then we have to show that l^{p*} is isomorphic to l^q. For this we define a function $T: l^{p*} \to l^q$ as $T(f) = c_f$

Where $f \in l^{p*}$ and $B = \{e_1, e_2, \dots, e_n\}$ be a basis of l^p and for each $x \in l^q$ we have $x = \sum_1^\infty x_i e_i$ as well as $f(x) = f(\sum_1^\infty x_i e_i) = \sum_1^\infty x_i f(e_i) = \sum_1^\infty x_i c_i$ where we take $\{c_i\} = f(e_i) = c_f$ where $c_f \in l^q$.

We are to show that T is an isomorphism.

***T* is bijective (Injective + Surjective)**

Let $f, f' \in l^{p*}$ then $c_i = f(e_i), c_i' = f'(e_i)$ then consider for $i = 1,2, \dots, n$

$T(f) = T(f') \Rightarrow c_f = c_{f'} \Rightarrow (c_1, c_2, \dots, c_n) = (c_1', c_2', \dots, c_n') \Rightarrow c_i = c_i'$

$\Rightarrow f(e_i) = f'(e_i) \Rightarrow f(e_i) - f'(e_i) = 0 \Rightarrow (f - f')(e_i) = 0$

$\Rightarrow (f - f')x = \sum_1^\infty x_i (f - f')e_i = 0$ since $x = \sum_1^\infty x_i e_i$

$\Rightarrow f - f' = 0 \Rightarrow f = f' \Rightarrow$ ***T* is injective.**

Now let $b = (b_1, b_2, \dots, b_n) \in l^q$ then for any $x = \sum_1^\infty x_i e_i \in l^q$ define a function $g: l^p \to F$ by $g(x) = \sum_1^\infty x_i b_i$. This g is linear. Also using Holder Inequality;

$|g(x)| = |\sum_1^\infty x_i b_i| \leq (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} (\sum_1^\infty |b_i|^q)^{\frac{1}{q}} \leq \|x\|_p \|b\|_q \Rightarrow \|g\| \leq \|b\|_q$

So $g \in l^{p*}$. We put then $T(g) = b \in l^q$. So that T is surjective.***T* is bijective.**

***T* is linear**

Let $f, f' \in l^{p*}$ and $\propto, \propto' \in F$ then

$T(\propto f + \propto' f') = c_{\propto f + \propto' f'} = (c_1'', c_2'', \dots, c_n'') = c_i'' = (\propto f + \propto' f')(e_i)$

$T(\propto f + \propto' f') = \propto f(e_i) + \propto' f'(e_i) = \propto c_i + \propto' c_i' = \propto c_f + \propto' c_{f'}$

$T(\propto f + \propto' f') = \propto T(f) + \propto' T(f')$

Hence T is linear.

***T* is norm preserving**

Let for all $x \in l^{1*}$ we have

$|f(x)| = |\sum_1^\infty x_i c_i| \le \sum_1^\infty |x_i c_i| \le (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} (\sum_1^\infty |c_i|^q)^{\frac{1}{q}} \le \|x\|_p \|c_f\|_q$

$\Rightarrow \frac{|f(x)|}{\|x\|_p} \le \|c_f\|_q \quad ; \forall i \Rightarrow \|f\| \le \|c_f\|_q$ ....(1)

To show $\|f\| \ge \|c_f\|_q$

For any $\propto \in F$ define $\quad Signum\ function\ of \propto = sgn \propto = \begin{cases} \frac{|\propto|}{\propto} & ; \propto \neq 0 \\ 0 & ; \propto = 0 \end{cases}$

It is clear that $|sgn \propto| = 1$ and $\propto sgn \propto = |\propto|$ for all $\propto$.

Take a sequence $c^{(k)} = \left\{c_i^{(k)}\right\}$ in l^p such that $c_i^{(k)} = \begin{cases} |c_i|^{q-1} sgn c_i & ; i \le k \\ 0 & ; i > k \end{cases}$

Then for $i \le k$, using the definition of signum function $\left|c_i^{(k)}\right| = |c_i|^{q-1}$

Also from $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow (q-1)p = q$ we have $\left|c_i^{(k)}\right|^p = |c_i|^{(q-1)p} = |c_i|^q$

$\Rightarrow \left|c^{(k)}\right|_p = \left(\sum_1^k \left|c_i^{(k)}\right|^p\right)^{\frac{1}{p}} = \left(\sum_1^k |c_i|^q\right)^{\frac{1}{p}}$ ....(A)

$\Rightarrow f\left(c^{(k)}\right) = \sum_1^\infty c_i^{(k)} f(e_i) = \sum_1^\infty c_i^{(k)} c_i = \sum_1^k |c_i|^{q-1} |c_i| = \sum_1^k |c_i|^q$

$\Rightarrow f\left(c^{(k)}\right) = \sum_1^k |c_i|^q$ …………….(B)

Again using $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow (q-1)p = q$ we have

$$\left|f\left(c^{(k)}\right)\right| \leq \|f\| \left\|c^{(k)}\right\|_p \Rightarrow \left|f\left(c^{(k)}\right)\right| \leq \|f\| \left(\sum_1^k \left|c_i^{(k)}\right|^p\right)^{\frac{1}{p}}$$

$\Rightarrow \left|f\left(c^{(k)}\right)\right| \leq \|f\| \left(\sum_1^k |c_i|^q\right)^{\frac{1}{p}}$ using (A) …………….(C)

$\Rightarrow \left|f\left(c^{(k)}\right)\right| = \sum_1^k |c_i|^q \leq \|f\| \left(\sum_1^k |c_i|^q\right)^{\frac{1}{p}}$ using (B) and (C)

$$\Rightarrow \frac{\sum_1^k |c_i|^q}{\left(\sum_1^k |c_i|^q\right)^{\frac{1}{p}}} \leq \|f\| \Rightarrow \left(\sum_1^k |c_i|^q\right)^{1-\frac{1}{p}} \leq \|f\| \Rightarrow \left(\sum_1^k |c_i|^q\right)^{\frac{1}{q}} \leq \|f\|$$

$$\Rightarrow \left(\sum_1^\infty |c_i|^q\right)^{\frac{1}{q}} \leq \|f\| \Rightarrow \left(\sum_1^\infty |c_i|^q\right)^{\frac{1}{q}} \leq \|f\|$$

$\left\|c_f\right\|_q \leq \|f\|$ …………….(2) $\Rightarrow c_f \in l^q$

Combining (1) and (2) $\|f\| = \left\|c_f\right\|_q = \|T(f)\|$

This show that $T: l^{p*} \to l^q$ is the isometric isomorphism between l^{p*} and l^q.

Hence $l^{p*} \cong l^q$. i.e. dual space (Conjugate Space) of l^p is l^q. By similar method, the dual space of l^q is l^p. Hence the second dual of l^p is l^p. Therefore l^p is reflexive.

What is the significance of these and similar examples?

In applications it is frequently quite useful to know the general form of bounded linear functionals on spaces of practical importance, and many spaces have been investigated in that respect. Our examples give general representations of bounded linear functionals on $\mathbb{R}^n$, l^1 and l^p with $p > 1$.

INNER PRODUCT SPACES

In a normed space we can add vectors and multiply vectors by scalars, just as in elementary vector algebra. Furthermore, the norm on such a space generalizes the elementary concept of the length of a vector. However, what is still missing in a general normed space, and what we would like to have if possible, is an analogue of the familiar dot product $\vec{u}.\vec{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$ and resulting formulas, notably $|\vec{u}| = \sqrt{\vec{u}.\vec{u}}$ and the condition for orthogonality (perpendicularity) $\vec{u}.\vec{v} = 0$ which are important tools in many applications. Hence the question arises whether the dot product and orthogonality can be generalized to arbitrary vector spaces. In fact, this can be done and leads to inner product spaces and complete inner product spaces, called Hilbert spaces.

Inner product spaces are special normed spaces, as we shall see. Historically they are older than general normed spaces. Their theory is richer and retains many features of Euclidean space, a central concept being orthogonality. In fact, inner product spaces are probably the most natural generalization of Euclidean space, and the reader should note the great harmony and beauty of the concepts and proofs in this field. The whole theory was initiated by the work of D. Hilbert (1912) on integral equations. The currently used geometrical notation and terminology is analogous to that of Euclidean geometry and was coined by E. Schmidt (1908), who followed a suggestion of G. Kowalewski. These spaces have been, up to now, the most useful spaces in practical applications of functional analysis.

Important concepts, brief orientation about main content

An inner product space **X** is a vector space with an inner product $\langle x, y\rangle$ defined on it. The latter generalizes the dot product of vectors in three dimensional space and is used to define

i. a norm $\|.\|$ by $\|x\| = \sqrt{\langle x, x\rangle}$
ii. orthogonality by $\langle x, y\rangle = 0$.

A Hilbert space **H** is a complete inner product space. The theory of inner product ap.d Hilbert spaces is richer than that of general normed and Banach spaces. Distinguishing features are

(i) Representations of **H** as a direct sum of a closed subspace and its orthogonal complement.
(ii) Orthonormal sets and sequences and corresponding representations of elements of **H**.
(iii) the Riesz representation of bounded linear functionals by inner products.
(iv) The Hilbert-adjoint operator T* of a bounded linear operator T.

Orthonormal sets and sequences are truly interesting only if they are total. Hilbert-adjoint operators can be used to define classes of operators (self-adjoint, unitary, normal,which are of great imporance in applications.

Inner Product Space (Sesquilinear Space or $1\frac{1}{2}$ time linear Space)

An inner product space (or *pre-Hilbert space*) is a vector space X with an inner product defined on X. It is a mapping of $\langle .,. \rangle : X \times X \to F$ into the scalar field F of X; that is, with every pair of vectors x and y there is associated a scalar which is written $\langle x, y \rangle$ and is called the inner product of x and y, such that for all vectors x, y, z and scalars $\propto$ we have

- **(Additivity Axiom):** $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.
- **(Symmetry Axiom):** $\langle x, y \rangle = \langle y, x \rangle$ for real field.
  And $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for complex field
- **(Homogeneity Axiom):** $\langle \propto x, y \rangle = \propto \langle x, y \rangle$.
- **(Positivity Axiom):** $\langle x, x \rangle \geq 0$; and $\langle x, x \rangle = 0$ if and only if $x = 0$.

Norm on an Inner Product Space: An inner product on X defines a ***norm*** on X given by $\|x\| = \sqrt{\langle x, x \rangle}$. If $\|x\| = 1$ then x is a normalized or unit vector.

Metric on an Inner Product Space: An inner product on X defines a ***metric*** on X given by $d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}$
Hence inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.

Example: Let $(V, \langle ., . \rangle)$ be an inner product space over a field F then show that $\langle x, 0 \rangle = 0 = \langle 0, x \rangle \ ; x \in V$

Solution: $\langle 0, x \rangle = \langle 0. y, x \rangle = 0\langle y, x \rangle = 0 \ ; x \in V$

Similarly $\langle x, 0 \rangle = \overline{\langle 0, x \rangle} = \overline{\langle 0. y, x \rangle} = 0\overline{\langle y, x \rangle} = 0 \ ; x \in V$

Example: Let $(V, \langle ., . \rangle)$ be an inner product space over a field F then show that $\langle x, \propto y \rangle = \overline{\propto} \langle x, y \rangle \ ; x, y \in V, \propto \in F$

Solution: $\langle x, \propto y \rangle = \overline{\langle \propto y, x \rangle} = \overline{\propto \langle y, x \rangle} = \overline{\propto}\, \overline{\langle y, x \rangle} = \overline{\propto} \langle x, y \rangle \ ; x, y \in V, \propto \in F$

Example: Let $(V, \langle ., . \rangle)$ be an inner product space over a field F then show that $\langle x, z \rangle = \langle y, z \rangle \Rightarrow x = y \ ; x, y, z \in V$

Solution: $\langle x, z \rangle = \langle y, z \rangle \Rightarrow \langle x, z \rangle - \langle y, z \rangle = 0 \Rightarrow \langle x - y, z \rangle = 0 \Rightarrow x - y = 0$
$\Rightarrow x = y$

Example: Show that the Euclidean space $\mathbf{R}^2$ is an inner product defined by $\langle x, y \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2$

Solution $\boldsymbol{\langle x, x \rangle \geq 0}$

$\langle x, x \rangle = x_1x_1 - x_1x_2 - x_2x_1 + 2x_2x_2 = (x_1 - x_2)^2 + x_2^2 \geq 0$

- $\boldsymbol{\langle x, x \rangle = 0} \Leftrightarrow x_1x_1 - x_1x_2 - x_2x_1 + 2x_2x_2 = 0$
  $\Leftrightarrow (x_1 - x_2)^2 + x_2^2 = 0 \Leftrightarrow (x_1 - x_2)^2 = 0, x_2^2 = 0$
  $\Leftrightarrow x_1 - x_2 = 0, x_2 = 0 \Leftrightarrow x_1 = x_2 = 0 \Leftrightarrow x = 0$
- $\boldsymbol{\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle}$
  $\langle x + y, z \rangle = (x_1 + y_1)z_1 - (x_1 + y_2)z_2 - (x_2 + y_2)z_1 + 2(x_2 + y_2)z_2$
  $\langle x + y, z \rangle = x_1z_1 + y_1z_1 - x_1z_2 - y_2z_2 - x_2z_1 - y_2z_1 + 2x_2z_2 + 2y_2z_2 = x_1z_1 - x_1z_2 - x_2z_1 + 2x_2z_2 + y_1z_1 - y_2z_2 - y_2z_1 + 2y_2z_2$
  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- $\boldsymbol{\langle x, y \rangle = \langle y, x \rangle}$
  $\langle x, y \rangle = x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2 = y_1x_1 - y_1x_2 - y_2x_1 + 2y_2x_2 = \langle y, x \rangle$
- $\boldsymbol{\langle \propto x, y \rangle = \propto \langle y, x \rangle}$
  $\langle \propto x, y \rangle = \propto x_1y_1 - \propto x_1y_2 - \propto x_2y_1 + 2 \propto x_2y_2 = \propto (x_1y_1 - x_1y_2 - x_2y_1 + 2x_2y_2) \Rightarrow \langle \propto x, y \rangle = \propto \langle x, y \rangle$

Theorem

Let H be an inner product space then H is a normed space with defined norm $\|x\| = \sqrt{\langle x, x\rangle}$

Proof

$N_1)$ $\because \langle x, x\rangle \geq 0 \Rightarrow \sqrt{\langle x, x\rangle} \geq 0 \Rightarrow \|x\| \geq 0$

$N_2)$ $\|x\| = 0 \Leftrightarrow \sqrt{\langle x, x\rangle} = 0 \Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0$

$N_3)$ $\|\propto x\| = \sqrt{\langle \propto x, \propto x\rangle} = \sqrt{\propto \langle x, \propto x\rangle} = \sqrt{\propto\overline{\propto}\ \langle x, x\rangle}$

$\|\propto x\| = \sqrt{|\propto|^2 \langle x, x\rangle} = |\propto|\sqrt{\langle x, x\rangle} = |\propto|\|x\|$

$N_4)$ $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

$\|x + y\|^2 = \langle x + y, x + y\rangle = \langle x, x + y\rangle + \langle y, x + y\rangle$

$\|x + y\|^2 = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle x, y\rangle + \langle y, y\rangle$

$\|x + y\|^2 = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$

$\|x + y\|^2 \leq \langle x, x\rangle + 2|\langle x, y\rangle| + \langle y, y\rangle \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$

$\|x + y\|^2 \leq (\|x\| + \|y\|)^2 \Rightarrow \|x + y\| \leq \|x\| + \|y\|$

This shows that an inner product space is a normed space.

Theorem

An inner product is a metric space with metric defined by

$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$

Proof

(M1) $d(x,y) = \sqrt{\langle x-y, x-y\rangle} = \|x-y\| \geq 0 \Rightarrow d(x,y) \geq 0\ ; \forall x,y \in X$

(M2) $d(x,y) = \sqrt{\langle x-y, x-y\rangle} = \|x-y\| = 0 \Leftrightarrow x-y=0 \Leftrightarrow x=y$
$\Rightarrow d(x,y) = 0$ if and only if $x = y$

(M3) $d(x,y) = \sqrt{\langle x-y, x-y\rangle} = \|x-y\| = \|y-x\| = \sqrt{\langle y-x, y-x\rangle}$

$\Rightarrow d(x,y) = d(y,x)\ ; \forall x,y \in X$ (Symmetry)

(M4) $d(x,y) = \sqrt{\langle x-y, x-y\rangle} = \|x-y\| = \|x-z+z-y\|$

$d(x,y) \leq \|x-z\| + \|z-y\|$

$d(x,y) \leq d(x,\ z) + d(z,y)$ (Triangle Inequality).

Thus $d(x,y) = \sqrt{\langle x-y, x-y\rangle}$ satisfies all the properties of a metric and hence a metric space.

Inner Product Space Satisfies Parallelogram Equality

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

i.e. the sum of squares of the length of diagonals of parallelogram is equal to the sum of the squares of the lengths of sides of parallelogram.

Proof

$L.H.S = \|x+y\|^2 + \|x-y\|^2$

$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

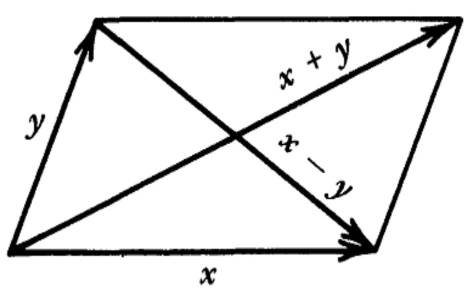
$= \langle x, x+y\rangle + \langle y, x+y\rangle + \langle x, x-y\rangle - \langle y, x-y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$

$= \langle x,x\rangle + \langle y,y\rangle + \langle x,x\rangle + \langle y,y\rangle$

$= 2\langle x,x\rangle + 2\langle y,y\rangle = 2(\langle x,x\rangle + \langle y,y\rangle) = 2(\|x\|^2 + \|y\|^2) = R.H.S$

This name is suggested by elementary geometry, as we see from Figure if we remember that the norm generalizes the elementary concept of the length of a vector. It is quite remarkable that such an equation continues to hold in our present much more general setting.



We conclude that if a norm does not satisfy Parallelogram Equality, it cannot be obtained from an inner product by the use of $\|x\| = \sqrt{\langle x,x\rangle}$. Such norms do exist. Without risking misunderstandings we may thus say:

Every inner product space is normed but Not all normed spaces are inner product spaces

Question

Every norm space is not an inner product space. Prove!

Proof

We will prove it by using an example.

Consider $C\left[0,\frac{\pi}{2}\right]$. i.e. A space of all continuous real valued functions on interval $\left[0,\frac{\pi}{2}\right]$. Then define the following norm;

$\|.\|: C\left[0,\frac{\pi}{2}\right] \to F$ with $\|f\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)|$

i. $\|f\| \geq 0$ Since $Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)| \geq 0$

ii. $\|f\| = 0 \Leftrightarrow Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)| = 0 \Leftrightarrow |f(x)| = 0 \Leftrightarrow f = 0$

iii. $\|\propto f\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|\propto f(x)| = |\propto| Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)| = |\propto|\|f\|$

iv. $\|f + g\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x) + g(x)| \leq Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)| + Sup_{x\in\left[0,\frac{\pi}{2}\right]}|g(x)|$

$\|f + g\| \leq \|f\| + \|g\|$

Hence $\left(C\left[0,\frac{\pi}{2}\right], \|.\|\right)$ is a normed space.

Now let $\|f\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x)|$ and choose $f, g \in C\left[0,\frac{\pi}{2}\right]$ such that;

$f(x) = Sinx$ and $g(x) = Cosx$ where $x \in \left[0,\frac{\pi}{2}\right]$ then

$\|f\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|Sinx| = 1$ as well as $\|g\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|Cosx| = 1$

$\|f + g\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x) + g(x)| = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$ using $x = \frac{\pi}{4}$

$\|f - g\| = Sup_{x\in\left[0,\frac{\pi}{2}\right]}|f(x) - g(x)| = 1 + 0 = 1$ using $x = \frac{\pi}{2}$ and $x = 0$

$\|f + g\|^2 + \|f - g\|^2 = 2(\|f\|^2 + \|g\|^2) \Rightarrow 3 \neq 4$

Hence $C\left[0,\frac{\pi}{2}\right]$ is not an inner product space.

Triangular Inequality for Vectors

$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Proof

$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle = \langle \vec{u}, \vec{u}+\vec{v} \rangle + \langle \vec{v}, \vec{u}+\vec{v} \rangle$

$\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle$

$\|\vec{u}+\vec{v}\|^2 \leq \langle \vec{u}, \vec{u} \rangle + 2|\langle \vec{u}, \vec{v} \rangle| + \langle \vec{v}, \vec{v} \rangle = \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$

$\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2 \Rightarrow \|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Remember: When does Equality Holds?

$\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$

$\Leftrightarrow \|\vec{u}+\vec{v}\|^2 = [\|\vec{u}\| + \|\vec{v}\|]^2$

$\Leftrightarrow \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$

$\langle \vec{u}, \vec{u} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{v} \rangle = \langle \vec{u}, \vec{u} \rangle + 2\|\vec{u}\|\|\vec{v}\| + \langle \vec{v}, \vec{v} \rangle$

$\Leftrightarrow \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle = 2\|\vec{u}\|\|\vec{v}\|$

$\Leftrightarrow \langle \vec{u}, \vec{v} \rangle + \overline{\langle \vec{u}, \vec{v} \rangle} = 2\|\vec{u}\|\|\vec{v}\|$

$\Leftrightarrow 2Re\langle \vec{u}, \vec{v} \rangle = 2\|\vec{u}\|\|\vec{v}\|$

$\Rightarrow Re\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\|\|\vec{v}\| \geq |\langle \vec{u}, \vec{v} \rangle|$ ………….(1)

But $Re\langle \vec{u}, \vec{v} \rangle \leq |\langle \vec{u}, \vec{v} \rangle|$ ………….(2)

From (1) and (2) $Re\langle \vec{u}, \vec{v} \rangle = |\langle \vec{u}, \vec{v} \rangle|$

$\Rightarrow \|\vec{u}\|\|\vec{v}\| = |\langle \vec{u}, \vec{v} \rangle|$

$\Leftrightarrow \vec{u} = 0, \vec{v} = 0$ or $\vec{u} = c\vec{v}$ for any $c \geq 0$

Cauchy–Schwarz Inequality

If $\vec{u}$ and $\vec{v}$ are vectors in a real inner product space **V**, then

$\langle \vec{u}, \vec{v} \rangle^2 \le \langle \vec{u}, \vec{u} \rangle^2 \langle \vec{v}, \vec{v} \rangle^2$ or $|\langle \vec{u}, \vec{v} \rangle| \le \|\vec{u}\| \|\vec{v}\|$

Proof

For any real number 't' consider $\langle t\vec{u} - \vec{v}, t\vec{u} - \vec{v} \rangle \ge 0$

$= t\langle \vec{u}, t\vec{u} - \vec{v} \rangle - \langle \vec{v}, t\vec{u} - \vec{v} \rangle = t\langle t\vec{u} - \vec{v}, \vec{u} \rangle - \langle t\vec{u} - \vec{v}, \vec{v} \rangle \ge 0$

$= t\{t\langle \vec{u}, \vec{u} \rangle - 1\langle \vec{v}, \vec{u} \rangle\} - \{t\langle \vec{u}, \vec{v} \rangle - 1\langle \vec{v}, \vec{v} \rangle\} \ge 0$

$= t^2\langle \vec{u}, \vec{u} \rangle - t\langle \vec{v}, \vec{u} \rangle - t\langle \vec{u}, \vec{v} \rangle + 1\langle \vec{v}, \vec{v} \rangle \ge 0$

$= t^2\|\vec{u}\|^2 - 2t\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \ge 0$

Let $t = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2}$ then $\langle t\vec{u} - \vec{v}, t\vec{u} - \vec{v} \rangle = \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{u}\|^4}\|\vec{u}\|^2 - 2\frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{u}\|^2}\langle \vec{u}, \vec{v} \rangle + \|\vec{v}\|^2 \ge 0$

$= \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{u}\|^2} - 2\frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{u}\|^2} + \|\vec{v}\|^2 \ge 0 \Rightarrow -\frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{u}\|^2} + \|\vec{v}\|^2 \ge 0 \Rightarrow \|\vec{v}\|^2 \ge \frac{\langle \vec{u}, \vec{v} \rangle^2}{\|\vec{u}\|^2}$

$\Rightarrow \|\vec{u}\|^2\|\vec{v}\|^2 \ge \langle \vec{u}, \vec{v} \rangle^2 \Rightarrow \langle \vec{u}, \vec{v} \rangle^2 \le \|\vec{u}\|^2\|\vec{v}\|^2$

$\Rightarrow |\langle \vec{u}, \vec{v} \rangle| \le \|\vec{u}\|\|\vec{v}\|$ or $\langle \vec{u}, \vec{v} \rangle^2 \le \langle \vec{u}, \vec{u} \rangle^2 \langle \vec{v}, \vec{v} \rangle^2$

Remember:

$|\langle \vec{u}, \vec{v} \rangle| = \|\vec{u}\|\|\vec{v}\|$

$\Leftrightarrow \|t\vec{u} - \vec{v}\| = 0$

$\Leftrightarrow t\vec{u} = \vec{v}$

$\Leftrightarrow \{\vec{u}, \vec{v}\}$ are L.Independent.

Appolonius Inequality

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2$$

Proof:

$$\|x+y\|^2=\langle x+y,x+y\rangle=\langle x,x+y\rangle+\langle y,x+y\rangle$$

$$\|x+y\|^2=\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle$$

$$\|x+y\|^2=\|x\|^2+2\langle x,y\rangle+\|y\|^2 \qquad \ldots\ldots\ldots\ldots\ldots..(1)$$

$$\|x-y\|^2=\langle x-y,x-y\rangle=\langle x,x-y\rangle-\langle y,x-y\rangle$$

$$\|x-y\|^2=\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle$$

$$\|x-y\|^2=\|x\|^2-2\langle x,y\rangle+\|y\|^2 \qquad \ldots\ldots\ldots\ldots\ldots..(2)$$

Adding (1) and (2) we have

$$\|x+y\|^2+\|x-y\|^2=\|x\|^2+2\langle x,y\rangle+\|y\|^2+\|x\|^2-2\langle x,y\rangle+\|y\|^2$$

$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \qquad \ldots\ldots\ldots\ldots\ldots..(3)$$

Put $x=z-x, y=z-y$ in (3)

$$\|z-x+z-y\|^2+\|z-x-z+y\|^2=2\|z-x\|^2+2\|z-y\|^2$$

$$\|2z-(x+y)\|^2+\|-x+y\|^2=2\|z-x\|^2+2\|z-y\|^2$$

$$4\left\|z-\frac{1}{2}(x+y)\right\|^2+\|-x+y\|^2=2\|z-x\|^2+2\|z-y\|^2$$

$$2\left\|z-\frac{1}{2}(x+y)\right\|^2+\frac{1}{2}\|x-y\|^2=\|z-x\|^2+\|z-y\|^2$$

Hence we get

$$\|z-x\|^2+\|z-y\|^2=\frac{1}{2}\|x-y\|^2+2\left\|z-\frac{1}{2}(x+y)\right\|^2$$

Polarization Inequality $\langle\vec{u},\vec{v}\rangle = \frac{1}{4}(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2)$

Proof: Consider $\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2$

$= \langle\vec{u}+\vec{v},\vec{u}+\vec{v}\rangle - \langle\vec{u}-\vec{v},\vec{u}-\vec{v}\rangle = \langle\vec{u},\vec{u}+\vec{v}\rangle + \langle\vec{v},\vec{u}+\vec{v}\rangle - \langle\vec{u},\vec{u}-\vec{v}\rangle + \langle\vec{v},\vec{u}-\vec{v}\rangle$

$= \langle\vec{u},\vec{u}\rangle + \langle\vec{u},\vec{v}\rangle + \langle\vec{v},\vec{u}\rangle - \langle\vec{u},\vec{u}\rangle + \langle\vec{u},\vec{v}\rangle + \langle\vec{v},\vec{u}\rangle - \langle\vec{v},\vec{v}\rangle$

$= \|\vec{u}\|^2 + 2\langle\vec{u},\vec{v}\rangle + \|\vec{v}\|^2 - \|\vec{u}\|^2 + 2\langle\vec{u},\vec{v}\rangle - \|\vec{v}\|^2 = 4\langle\vec{u},\vec{v}\rangle$

$\langle\vec{u},\vec{v}\rangle = \frac{1}{4}(\|\vec{u}+\vec{v}\|^2 - \|\vec{u}-\vec{v}\|^2)$

Theorem: Let V be an inner product space, V also a normed space if following axioms are true;

- $\|\vec{u}\| \geq 0 \ ; \ \|\vec{u}\| = 0 \Leftrightarrow \vec{u} = 0$
- $\|k\vec{u}\| = |k|\|\vec{u}\| \ ; \forall k \in F$
- $\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Proof

- $\boldsymbol{\|\vec{u}\| \geq 0 \ ; \ \|\vec{u}\| = 0 \Leftrightarrow \vec{u} = 0}$

Let $\vec{u} \neq 0$ then $\|\vec{u}\| = \sqrt{\langle\vec{u},\vec{u}\rangle} > 0 \Rightarrow \|\vec{u}\| > 0$

If $\vec{u} = 0$ then $\|\vec{u}\| = \sqrt{\langle\vec{u},\vec{u}\rangle} = 0 \Rightarrow \|\vec{u}\| = 0$

$\Rightarrow \|\vec{u}\| \geq 0 \ ; \ \|\vec{u}\| = 0 \Leftrightarrow \vec{u} = 0$

- $\boldsymbol{\|k\vec{u}\| = |k|\|\vec{u}\| \ ; \forall k \in F}$

Let $\|k\vec{u}\|^2 = \langle k\vec{u},k\vec{u}\rangle = kk\langle\vec{u},\vec{u}\rangle = k^2\|\vec{u}\|^2 \Rightarrow \|k\vec{u}\| = |k|\|\vec{u}\| \ ; \forall k \in F$

- $\boldsymbol{\|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|}$

$\|\vec{u}+\vec{v}\|^2 = \langle\vec{u}+\vec{v},\vec{u}+\vec{v}\rangle = \langle\vec{u},\vec{u}+\vec{v}\rangle + \langle\vec{v},\vec{u}+\vec{v}\rangle$

$\|\vec{u}+\vec{v}\|^2 = \langle\vec{u},\vec{u}\rangle + \langle\vec{u},\vec{v}\rangle + \langle\vec{v},\vec{u}\rangle + \langle\vec{v},\vec{v}\rangle = \langle\vec{u},\vec{u}\rangle + \langle\vec{u},\vec{v}\rangle + \langle\vec{u},\vec{v}\rangle + \langle\vec{v},\vec{v}\rangle$

$\|\vec{u}+\vec{v}\|^2 \leq \langle\vec{u},\vec{u}\rangle + 2|\langle\vec{u},\vec{v}\rangle| + \langle\vec{v},\vec{v}\rangle = \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2$

$\|\vec{u}+\vec{v}\|^2 \leq (\|\vec{u}\| + \|\vec{v}\|)^2 \Rightarrow \|\vec{u}+\vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$

Hilbert Space: A complete inner product spac is called Hilbert space. Or an inner product space in which every Cauchy sequence converges is said to be Hilbert Space.

Example: Show that the Euclidean space $\mathbf{R}^{\mathbf{n}}$ is a Hilbert space with inner product defined by $\langle x, x\rangle = \|x\|^2 = \sum_1^n |x_i|^2$.

Solution Let $\{x_n\}$ be a Cauchy Sequence in $\mathbf{R}^{\mathbf{n}}$ where $\{x_n\} = \left\{x_i^{(n)}\right\}_1^\infty$ then for any $\epsilon > 0$ there exists $n_0 \epsilon \mathbb{N}$ such that

$$\|x_n - x_m\| = \sqrt{\langle x_n - x_m, x_n - x_m\rangle} < \epsilon \quad ; \forall m, n \geq n_0$$

$$\Rightarrow \sqrt{\sum_1^n \left|x_i^{(n)} - x_i^{(m)}\right|^2} < \epsilon \quad ; \forall m, n \geq n_0$$

$$\Rightarrow \left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon \quad ; \forall m, n \geq n_0$$

$\Rightarrow x_i^{(n)}$ is a Cacuchy sequence in $\mathbf{R}$ and since $\mathbf{R}$ is complete

therefore $x_i^{(n)} \to x_i \in \boldsymbol{R}$

then there exists a natural number $n_i \epsilon \mathbb{N}$ such that $\left|x_i^{(n)} - x_i\right| < \frac{\epsilon}{\sqrt{p}} \quad ; \forall n \geq n_i$

$$\Rightarrow \left|x_1^{(n)} - x_1\right| < \frac{\epsilon}{\sqrt{p}} \quad ; \forall n \geq n_1$$

$$\left|x_2^{(n)} - x_2\right| < \frac{\epsilon}{\sqrt{p}} \quad ; \forall n \geq n_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\left|x_n^{(n)} - x_n\right| < \frac{\epsilon}{\sqrt{p}} \quad ; \forall n \geq n_n$$

If $x = (x_1, x_2, \ldots, x_n)$ then $x \epsilon R^n$.

Let $n' = max\{n_1, n_2, \ldots, n_n\}$ then for the above expression we have

$$\|x_n - x\| = \sqrt{\sum_1^n \left|x_i^{(n)} - x_i\right|^2}$$

$$\Rightarrow \|x_n - x\| = \sqrt{\sum_1^n \left|x_1^{(n)} - x_1\right|^2 + \left|x_2^{(n)} - x_2\right|^2 + \cdots + \left|x_n^{(n)} - x_n\right|^2}$$

$$\Rightarrow \|x_n - x\| < \sqrt{\frac{\epsilon^2}{n} + \frac{\epsilon^2}{n} + \cdots + \frac{\epsilon^2}{n}} = \sqrt{\frac{n\epsilon^2}{n}} \quad ; n \geq n'$$

$$\Rightarrow \|x_n - x\| < \epsilon \quad ; n \geq n' \Rightarrow \sqrt{\langle x_n - x, x_n - x\rangle} < \epsilon \quad ; n \geq n'$$

This shows that $\{x_n\}$ converges in R^n. Hence $\mathbf{R^n}$ is a Hilbert space.

Similary we can show that $\mathbf{C^n}$ is a Hilbert space with complex sequence.

Theorem: The space l^p with $p \geq 1$ of all sequecnes $x = \{x_i\}$ is a Hilbert space with inner product defined as $\sqrt{\langle x, x\rangle} = \|x\| = \left(\sum_1^\infty |x_i|^p\right)^{\frac{1}{p}}$

Solution Let $\{x_n\}$ be a Cauchy Sequence in l^p where $\{x_n\} = \left\{x_i^{(n)}\right\}_1^\infty$ then for any $\epsilon > 0$ there exists $n_0 \epsilon \mathbb{N}$ such that

$$\|x_n - x_m\| = \sqrt{\langle x_n - x_m, x_n - x_m\rangle} < \epsilon \quad ; \forall m, n \geq n_0$$

$$\Rightarrow \sqrt[p]{\sum_1^\infty \left|x_i^{(n)} - x_i^{(m)}\right|^p} < \epsilon \quad ; \forall m, n \geq n_0$$

$$\Rightarrow \left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon \quad ; \forall m, n \geq n_0$$

$\Rightarrow x_i^{(n)}$ is a Cacuchy sequence in **R** and since **R** is complete

Therefore $x_i^{(n)} \to x_i \in \boldsymbol{R}$ then $\left|x_i^{(n)} - x_i\right| \to 0$ for each *i*

Next suppose that $x = \{x_i\}$ then

$$\|x_n - x\| = \left(\sum_1^\infty \left|x_i^{(n)} - x_i\right|^p\right)^{\frac{1}{p}} \to 0 \text{ Since } \left|x_i^{(n)} - x_i\right| \to 0 \text{ for each } i$$

$$\Rightarrow x_n \to x$$

Now $x = x_n - (x_n - x) \in l^p$ then $x_n \to x \in l^p$

This shows that $\{x_n\}$ converges in l^p. Hence l^p is a Hilbert space.

Theorem

The space l^p with $p \neq 2$ is not an inner product space, hence not a Hilbert space.

Proof

We know that in an inner product space Parallelogram law holds. We check this law for give space l^p of all sequecnes $x = \{x_i\}$.

Let $x = (1,1,0,0,\dots), y = (1,-1,0,0,\dots) \in l^p$ then we are to prove

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\|x\| = (\sum_1^\infty |x_i|^p)^{\frac{1}{p}} = (|1|^p + |1|^p + |0|^p + |0|^p + \cdots)^{\frac{1}{p}} = 2^{\frac{1}{p}}$

$\|y\| = (\sum_1^\infty |y_i|^p)^{\frac{1}{p}} = (|1|^p + |-1|^p + |0|^p + |0|^p + \cdots)^{\frac{1}{p}} = 2^{\frac{1}{p}}$

$\|x+y\| = \|(1,1,0,0,\dots) + (1,-1,0,0,\dots)\| = \|(2,0,0,0,\dots)\| = 2$

$\|x-y\| = \|(1,1,0,0,\dots) - (1,-1,0,0,\dots)\| = \|(0,2,0,0,\dots)\| = 2$

$\|x+y\|^2 = \|x-y\|^2 = 4$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\Rightarrow 4 + 4 = 2\left(2^{\frac{1}{p}} + 2^{\frac{1}{p}}\right) \Rightarrow 8 = 4\left(2^{\frac{1}{p}}\right)$ this is only possible when $p \neq 2$.

$\Rightarrow$ l^p does not satisfy the parallelogram law with $p \neq 2$.

Hence The space l^p with $p \neq 2$ is not an inner product space, hence not a Hilbert space.

Theorem

The space l^{∞} of all b ounded sequecnes is a Hilbert space with inner product defined as $\sqrt{\langle x,x\rangle} = \|x\| = Sup_{i\in N}|x_i|$ where $x = \{x_i\} \in l^{\infty}$

Solution Let $\{x_n\}$ be a Cauchy Sequence in l^{∞} where $\{x_n\} = \left\{x_i^{(n)}\right\}_1^{\infty}$ then for any $\epsilon > 0$ there exists $n_0\epsilon\mathbb{N}$ such that

$$\|x_n - x_m\| = \sqrt{\langle x_n - x_m, x_n - x_m\rangle} < \epsilon \quad ;\forall m,n \geq n_0$$

$$\Rightarrow Sup_{i\in N}\left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon \quad ;\forall m,n \geq n_0$$

$$\Rightarrow \left|x_i^{(n)} - x_i^{(m)}\right| < \epsilon \quad ;\forall m,n \geq n_0$$

$\Rightarrow x_i^{(n)}$ is a Cacuchy sequence in **R** and since **R** is complete

Therefore $x_i^{(n)} \to x_i \in \boldsymbol{R}$

then there exists a natural number $n_i\epsilon\mathbb{N}$ such that $\left|x_i^{(n)} - x_i\right| < \frac{\epsilon}{2} \quad ;\forall n \geq n_i$

$$\Rightarrow \left|x_1^{(n)} - x_1\right| < \frac{\epsilon}{2} \quad ;\forall n \geq n_1$$

$$\left|x_2^{(n)} - x_2\right| < \frac{\epsilon}{2} \quad ;\forall n \geq n_2$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\left|x_n^{(n)} - x_n\right| < \frac{\epsilon}{2} \quad ;\forall n \geq n_n$$

If $x = (x_1, x_2, \dots, x_n)$ then $x\epsilon R^n$.

Let $n' = max\{n_1, n_2, \dots, n_n\}$ then for the above expression we have

$$|x_n - x| = \left|x_i^{(n)} - x_i\right| < \frac{\epsilon}{2} \quad ;\forall n \geq n'$$

$$\Rightarrow \|x_n - x\| = Sup_{i\in N}\left|x_i^{(n)} - x_i\right| < \frac{\epsilon}{2} \quad ;\forall n \geq n'$$

$$\Rightarrow \|x_n - x\| < \in \ ; n \geq n' \Rightarrow \sqrt{\langle x_n - x, x_n - x\rangle} < \in \ ; n \geq n'$$

This shows that $\{x_n\}$ converges in l^∞.

Now $|x| = |x_i| = \left|x_i^{(n)} - x_i - x_i^{(n)}\right| \leq \left|x_i^{(n)} - x_i\right| + \left|x_i^{(n)}\right|$

$|x| \leq \in + \left|x_i^{(n)}\right|$ ………………(1)

Since $\{x_n\} = \left\{x_i^{(n)}\right\}_1^\infty \in l^\infty$ being a bounded sequence could be written as for any real K; $\left|x_i^{(n)}\right| < K$

$(1) \Rightarrow |x| \leq \in +K$ this show that $x = \{x_i\} \in l^\infty$ is a bounded sequence.

Then $x_n \rightarrow x \in l^\infty$ and Hence l^∞ is a Hilbert space

Theorem

The space C[a, b] is not an inner product space, hence not a Hilbert space.

Proof

We know that in an inner product space Parallelogram law holds. We check this law for give space $C[a,b]$ with defined norm $\|x\| = max_{t\in J}|x(t)|$;$J = [a,b]$

Let $x = x(t) = 1, y = y(t) = \frac{t-a}{b-a} \in C[a,b]$ then we are to prove

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\|x\| = \|y\| = 1$

$x + y = x(t) + y(t) = 1 + \frac{t-a}{b-a}$ and $x - y = x(t) - y(t) = 1 - \frac{t-a}{b-a}$

$\|x+y\| = 2$ and $\|x-y\| = 1$

$\Rightarrow \|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

$\Rightarrow 5 = 2(1+1) \Rightarrow 8 \neq 5$

$\Rightarrow$ $C[a,b]$ does not satisfy the parallelogram law.

Hence the space $C[a,b]$ is not an inner product space, hence not a Hilbert space.

Theorem

The space $C(X,R)$ is a Hilbert space with inner product defined as $\sqrt{\langle f,f\rangle} = \|f\| = Sup_{x\in X}|f(x)|$

Solution Let $\{f_n(x)\}$ be a Cauchy Sequence in $C(X,R)$then for any $\epsilon > 0$ there exists $n_0 \epsilon \mathbb{N}$ such that

$\|f_n - f_m\| < \epsilon \quad ; \forall m,n \geq n_0 \Rightarrow Sup_{x\in X}|f_n(x) - f_m(x)| < \epsilon \quad ; \forall m,n \geq n_0$

$\Rightarrow |f_n(x) - f_m(x)| < \epsilon \quad ; \forall m,n \geq n_0$

$\Rightarrow f_n(x)$ is a Cacuchy sequence in **R** and since **R** is complete

Therefore $f_n(x) \to f(x) \in \boldsymbol{R}$ where $f(x)$ is function $f: X \to R$

Next we show that $f(x) \in C(X,R)$ for this it is enoght to show that f is continuous. Let $\epsilon > 0$ and $x_0 \in X$.

Since $f_n(x) \in C(X,R)$ then being continuous on X it will be continuous on $x_0 \in X$.

$\Rightarrow |f_n(x) - f_n(x_0)| < \frac{\epsilon}{3}$ whenever $\|x - x_0\|_X < \delta$

Since $f_n(x) \to f(x)$ so we have $n_1 \in N$ such that

$\Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{3} \quad ; \forall n \geq n_1$

Since $f_n(x_0) \to f(x_0)$ so we have $n_2 \in N$ such that

$\Rightarrow |f_n(x_0) - f(x_0)| < \frac{\epsilon}{3} \quad ; \forall n \geq n_2$

Now we are to show that $f(x) \to f(x_0)$ for this let

$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$

$|f(x) - f(x_0)| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$

$|f(x) - f(x_0)| < \frac{\epsilon}{3} + |f_n(x) - f_n(x_0)| + \frac{\epsilon}{3}$

$|f(x) - f(x_0)| < \frac{2}{3}\epsilon + |f_n(x) - f_n(x_0)|$

$|f(x) - f(x_0)| < \frac{2}{3}\epsilon + \frac{1}{3}\epsilon$ whenever $\|x - x_0\|_X < \delta$

$|f(x) - f(x_0)| < \epsilon$ whenever $\|x - x_0\|_X < \delta$

This shows that f is continuous at x_0, so f is continuous on X, so $f(x) \in C(X, R)$.

This shows that $f_n(x) \to f(x) \in C(X, R)$.

Hence $\in C(X, R)$is a Hilbert space

Other Examples

- **Space L^2[a,b]** is a Hilbert space with $\langle x, y \rangle = \int_a^b x(t)y(t)\,dt$
- **Hilbert sequence space $\boldsymbol{l^2}$** is a Hilbert space with $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ it is the prototype of a Hilbert space. It was introduced and investigated by D. Hilbert (1912) in his work on integral equations.

Theorem

Show that the space c consisting of all convergent sequecnes $x = \{x_n\}$ of real number (or complex numbers) with inner product defined as

$\sqrt{\langle x, x \rangle} = \|x\| = Sup_{n\in\mathbb{N}}|x_n|$ is a Hilbert Space.

Solution Since a convergent sequence is bounded and the space l^∞ consists of bounded sequences, so the space c is subspace of l^∞. Since l^∞ is Hilbert Space, so to show that c is Hilbert space it is enough to show that c is closed. For this we will have to show $c = \bar{c}$.

We already know that $c \subseteq \bar{c}$ ....(1)

Let $x = \{x_n\} \in \bar{c}$ then there must exists a sequence $\{x^{(p)}\} \in c$ such that $x^{(p)} \to x$

Hence then for any $\epsilon > 0$ there exists $n_0 \epsilon \mathbb{N}$ such that

$\|x^{(p)} - x\| < \frac{\epsilon}{3}$; $\forall p \geq n_0$

$\Rightarrow Sup_{n\in\mathbb{N}} \left|x_n^{(p)} - x_n\right| < \frac{\epsilon}{3}$; $\forall p \geq n_0$

$\Rightarrow \left|x_n^{(p)} - x_n\right| < \frac{\epsilon}{3} \quad ;\forall p \geq n_0$ ....(2)

Similarly for each fixed $n \in \mathbb{N}$ and $p = n_0$ we have

$\Rightarrow \left|x_n^{(n_0)} - x_n\right| < \frac{\epsilon}{3} \quad ;\forall p = n_0$ ....(3)

$\Rightarrow \left|x_m^{(n_0)} - x_m\right| < \frac{\epsilon}{3} \quad ;\forall p = n_0$ ....(4)

Since $x^{(n_0)} \in c$, so $x^{(n_0)} = \left\{x_n^{(n_0)}\right\}$ is the convergent sequence of real numbers.

Since every convergent sequence is Cauchy sequence, then for any $\epsilon > 0$ there exists $n_1 \epsilon \mathbb{N}$ such that

$\Rightarrow \left|x_m^{(n_0)} - x_n^{(n_0)}\right| < \frac{\epsilon}{3} \quad ;\forall m, n \geq n_1$ ....(5)

Now using (2), (3), and (4) we have

$|x_m - x_n| = \left|x_m + x_m^{(n_0)} - x_m^{(n_0)} + x_n^{(n_0)} - x_n^{(n_0)} - x_n\right|$

$|x_m - x_n| \leq \left|x_m^{(n_0)} - x_m\right| + \left|x_m^{(n_0)} - x_n^{(n_0)}\right| + \left|x_n^{(n_0)} - x_n\right|$

$|x_m - x_n| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$

$\Rightarrow |x_m - x_n| < \epsilon \quad ;\forall m, n \geq n_1$

This shows that $x = \{x_n\}$ is a Cauchy Sequence of real numbers. Since the set of real numbers is complete, so this Cauchy sequence converges. i.e. $x = \{x_n\}$ is convergent sequence. Then $x = \{x_n\} \in c$

Then $\quad \bar{c} \subseteq c$ ....(6)

Hence $c = \bar{c}$ and this shows that c is closed, so c is Hilbert Space.

Theorem

Show that the space c consisting of all sequecnes $x = \{x_n\}$ of real number (or complex numbers) converging to zero with inner product defined as

$\langle x, x\rangle = \|x\|^2 = (Sup_{n\in\mathbb{N}}|x_n|)^2$ is a Hilbert Space.

Solution Since a convergent sequence is bounded and the space l^{∞} consists of bounded sequences, so the space c is subspace of l^{∞}. Since l^{∞} is Hilbert Space, so to show that c is Hilbert space it is enough to show that c is closed. For this we will have to show $c = \bar{c}$.

We already know that $c \subseteq \bar{c}$

Let $x = \{x_n\} \in \bar{c}$ then there must exists a sequence $\{x^{(p)}\} \in c$ such that $x^{(p)} \to x$

Since the space c consists of those sequences which converges to zero so $x^{(p)} \to 0$

Since a sequence can converge at most one point so $x = 0$ i.e. $x = \{x_n\} = 0,0,0, \ldots$

This shows that $x_n \to 0$ i.e. $x \in c$

Then $\bar{c} \subseteq c$

Hence $c = \bar{c}$ and this shows that c is closed, so c is Hilbert Space.

Orthogonality

An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if $\langle x, y\rangle = 0$.

We also say that x and y are orthogonal, and we write $x \perp y$. Similarly, for subsets $A, B \subseteq X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$.

Orthogonal Set

A set $S = \{x_i : i \in I\}$ of vectors of an inner product space X is said to be an orthogonal set if distinct vectors of X are orthogonal, i.e.

$\langle x_i, y_i\rangle = 0$; $i \neq j$ and $x_i, y_i \in X$.

Theorem of Pythagoras

If x and y are orthogonal vectors in a real inner product space then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

Proof: Since x and y are orthogonal therefore $\langle x, y\rangle = 0$

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x+y\rangle + \langle y, x+y\rangle$$

$$\|x+y\|^2 = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle x, y\rangle + \langle y, y\rangle$$

$$\|x+y\|^2 = \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2 \qquad \because \langle x, y\rangle = 0$$

Generalized Pythagoras Theorem

If $x_1, x_2, x_3, \dots, x_n$ are piecewise orthogonal vectors in a real inner product space then $\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n}\|x_i\|^2$

Proof

$\|\sum_{i=1}^{n} x_i\|^2 = \langle \sum_{i=1}^{n} x_i , \sum_{j=1}^{n} x_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x_i, x_j \rangle$

$\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n} \langle x_i, x_i \rangle \quad \because \langle x_i, x_i \rangle = 0 \; ; i \neq j$

$\|\sum_{i=1}^{n} x_i\|^2 = \langle \sum_{i=1}^{n} x_i , \sum_{i=1}^{n} x_i \rangle$

$\|\sum_{i=1}^{n} x_i\|^2 = \sum_{i=1}^{n}\|x_i\|^2$

Lemma (Continuity of inner product)

If in an inner product space, $x_n \to x$ and $y_n \to y$, then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Proof:

Subtracting and adding a term, using the triangle inequality for numbers and, finally, the Schwarz inequality, we obtain

$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle|$

$|\langle x_n, y_n \rangle - \langle x, y \rangle| \leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle|$

$|\langle x_n, y_n \rangle - \langle x, y \rangle| \leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\| \to 0$ as $n \to \infty$

$|\langle x_n, y_n \rangle - \langle x, y \rangle| \to 0$ as $n \to \infty$

Hence $\quad \langle x_n, y_n \rangle \to \langle x, y \rangle$ as $n \to \infty$

Remember

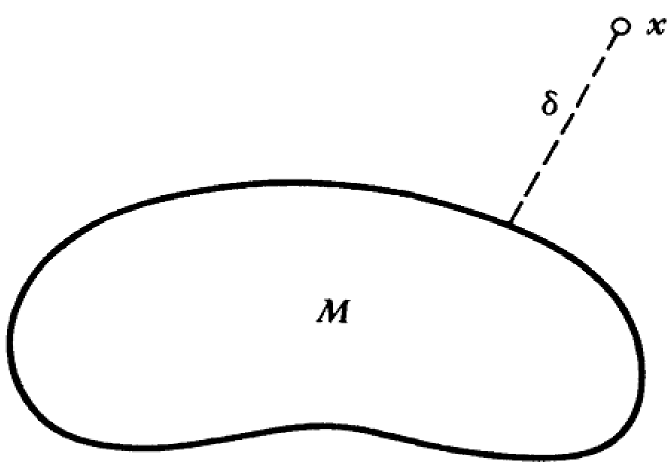
- **Theorem (Completion).** For any inner product space X there exists a Hilbert space H and an isomorphism A from X onto a dense subspace $W \subset H$. The space H is unique except for isomorphisms.
- A **subspace** Y of an inner product space X is defined to be a vector subspace of X taken with the inner product on X restricted to $Y \times Y$.
- **Theorem (Subspace):** Let Y be a subspace of a Hilbert space H, Then;
  a) Y is complete if and only if Y is closed in H.
  b) If Y is finite dimensional, then Y is complete.
  c) If H is separable, so is Y. More generally, every subset of a separable inner product space is separable.

Interesting to Remember

In a metric space X, **the distance** δ from an element $x \in X$ to a nonempty subset $M \subset X$ is defined to be $\delta = inf_{y \in M} d(x, y) \quad ; M \neq \varphi$

In a normed space this becomes $\delta = inf_{y \in M} \|x - y\| \quad ; M \neq \varphi$

A simple illustrative example is shown in the following figure.

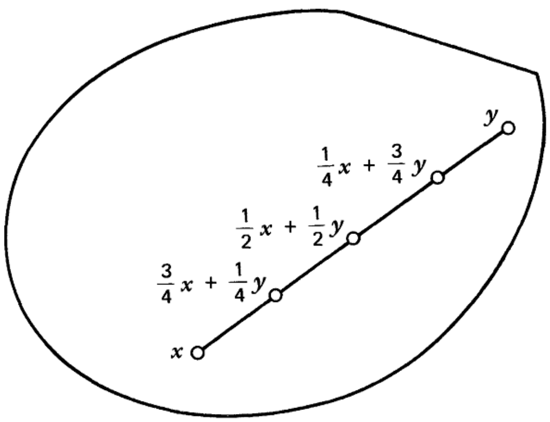


The **segment** joining two given elements x and y of a vector space X is defined to be the set of all $z \in X$ of the form

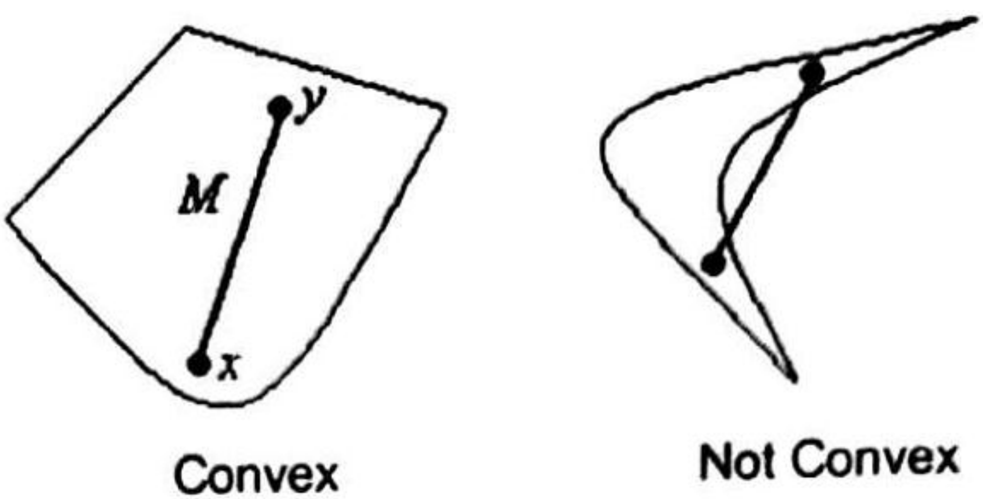
$z = \propto x + (1 - \propto) y \qquad ; (\propto \in R, 0 \leq \propto \leq 1).$

In fact for any $\propto \in [0,1]$, the point $z = \propto x + (1 - \propto) y$ is always the point of the segment with ends x and y.

A subset M of X is said to be **convex** if for every $x, y \in M$ the segment joining x and y is contained in M. In other words, a subset M of X is said to be **convex** if for any point $x, y \in M$, the closed segment with end points x and y is contained in M.



For instance, every subspace Y of X is convex, and the intersection of convex sets is a convex set. The empty set and the singleton sets are always convecx sets.



11.5.1 Examples

1. Any subspace of a linear space N is convex.

2. For any subspace S of N and $x \in N$, the set

$$x + S = \{x + s; s \in S\}$$

is convex.

Proof

Let $u, u' \in x + S$. Then $u = x + s$, $u' = x + s'$, for $s, s' \in S$. So, for any $\alpha \in [0, 1]$,

$$\begin{aligned} \alpha u + (1-\alpha) u' &= \alpha x + (1-\alpha) x + \alpha s + (1-\alpha) s' \\ &= x + \alpha s + (1-\alpha) s' \\ &= x + s'', s'' = \alpha s + (1-\alpha) s' \in S, \end{aligned}$$

is in $x + S$.

3. Intersection of any class of convex subsets of N is convex.

4. Let $T : N \to N'$ be a linear transformation and C a convex subset of N. Then $T(C)$ is also convex.

Proof

Here, for $u, u' \in T(C)$, there are $c, c' \in C$ such that

$$u = T(c), u' = T(c')$$

so that, for any $\alpha \in [0, 1]$,

$$\begin{aligned} \alpha u + (1-\alpha) u' &= \alpha T(c) + (1-\alpha) T(c') \\ &= T(\alpha c + (1-\alpha) c') \quad , \ T \text{ is linear} \\ &= T(c''), c'' \in C \end{aligned}$$

is an element of $T(C)$.

5. For any convex subsets K and L of a linear space N, the set

$$K + L = \{x + y : x \in K, y \in L\}$$

is convex.

Proof

Here for any $u, u' \in K + L$, there are $x, x' \in K, y, y' \in L$ such that

$$u = x + y, u' = x' + y'$$

so that, for any $\alpha \in [0, 1]$

$$\begin{aligned} \alpha u + (1-\alpha) u' &= \alpha x + (1-\alpha) x' + \alpha y + (1-\alpha) y' \\ &= x'' + y'', x'' \in K, y'' \in L \end{aligned}$$

is in $K + L$. Hence $K + L$ is convex.

6. Let C be a convex subset of N. Then for any scalar α, αC is also convex.

Proof

For if $u, u' \in \alpha C$ then there are c, c' in C such that

$$u = \alpha c, u' = \alpha c'.$$

Hence, for any $\gamma \in [0, 1]$,

$$\begin{aligned} \gamma u + (1-\gamma) u' &= \alpha(\gamma c + (1-\gamma) c') \\ &= \alpha c'', c'' \in C \end{aligned}$$

is in αC.

7. For any normed space N, the open and closed balls

$$B(x_0; r) = \{x \in N : \|x - x_0\| < r\}$$

$$\bar{B}(x_0; r) = \{x \in N : \|x - x_0\| \leq r\}$$

are convex. In particular the open and closed unit balls with centre at the origin are convex.

Proof

Let $x, x' \in B(x_0; r)$. Then $\|x - x_0\| < r$, $\|x' - x_0\| < r$ and for $\alpha \in [0, 1]$,

$$\begin{aligned}
\|\alpha x + (1-\alpha)x' - x_0\| &= \|\alpha x + (1-\alpha)x' - \alpha x_0 - (1-\alpha)x_0\| \\
&= \|\alpha(x - x_0) + (1-\alpha)(x' - x_0)\| \\
&\leq \alpha\|x - x_0\| + (1-\alpha)\|x' - x_0\| \\
&< (\alpha + 1 - \alpha)r \\
&< r
\end{aligned}$$

Hence $\alpha x + (1-\alpha)x' \in B(x_0; r)$ so that $B(x_0, r)$ is convex. Similarly $\bar{B}(x_0, r)$ is convex.

11.5.2 Theorem

For any convex set C in a linear space N and for any scalars α, β, $\alpha \geq 0$, $\beta \geq 0$,

$$\alpha C + \beta C = (\alpha + \beta)\, C$$

Proof

If $\alpha = 0$ or $\beta = 0$, then the equation holds trivially. Hence we suppose that $\alpha > 0$, $\beta > 0$. Let $z \in (\alpha + \beta)\, C$. Then there is a $c \in C$ such that

$$z = (\alpha + \beta)\, c = \alpha c + \beta c$$

so that $z \in \alpha C + \beta C$. Hence $(\alpha + \beta)\, C \subseteq \alpha C + \beta C$.

Conversely, let $u \in \alpha C + \beta C$. Then there are $c, d \in C$ such that

$$u = \alpha c + \beta d = (\alpha + \beta)\left(\frac{\alpha}{\alpha + \beta} c + \frac{\beta}{\alpha + \beta} d\right)$$

$$= (\alpha + \beta)\, w$$

where $$w = \frac{\alpha}{\alpha + \beta} c + \frac{\beta}{\alpha + \beta} d$$

Since $$\frac{\beta}{\alpha + \beta} = 1 - \frac{\alpha}{\alpha + \beta}$$

and C is convex, $w \in C$. Hence $u \in (\alpha + \beta)\, C$ so that $\alpha C + \beta C \subseteq (\alpha + \beta)\, C$. Consequently

$$\alpha C + \beta C = (\alpha + \beta) C,$$

11.5.4 Theorem

The closure of a convex subset of a normed space is a convex set.

Proof

Let C be a convex subset of a normed space N and $x, y \in \bar{C}$, the closure of C. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in C such that $x_n \to x$, $y_n \to y$. For any $\alpha \in [0, 1]$

$$\alpha x_n + (1-\alpha) y_n \in C.$$

Since addition and scalar multiplication in N are continuous,

$$\alpha x_n + (1-\alpha) y_n \to \alpha x + (1-\alpha) y$$

so that $\alpha x + (1-\alpha) y$, being a limit point of $\{\alpha x_n + (1-\alpha) y_n\}$, is in $\bar{C}$. Hence $\bar{C}$ is convex.

Minimizing Vector Theorem

Let X be an inner product space and $M \neq \varphi$ a convex subset which is complete (in the metric induced by the inner product).Then for every given $x \in X$ there exists a unique $y \in M$ such that $\delta = inf_{\bar{y} \in M} \|x - \bar{y}\| = \|x - y\|$

Proof

Existence: we have $\delta = inf_{\bar{y} \in M} \|x - \bar{y}\|$ then by the definition of an infimum there is a sequence (y_n) in M such that

$\delta_n \to \delta$ as $n \to \infty$ where $\delta_n = \|x - y_n\|$

We show that (y_n) is Cauchy.

Let $v_n = x - y_n$ then $\|v_n\| = \delta_n$ ……………(1)

Now $\|v_n + v_m\| = \|x - y_n + x - y_m\| = \|2x - (y_n + y_m)\|$

$\|v_n + v_m\| = 2\left\|x - \frac{1}{2}(y_n + y_m)\right\| = 2\left\|x - \left(\frac{1}{2}y_n + \left(1 - \frac{1}{2}\right)y_m\right)\right\|$

Since M is convex therefore $\frac{1}{2}(y_n + y_m) = \frac{1}{2}y_n + \left(1 - \frac{1}{2}\right)y_m \in M$

$\|v_n + v_m\| \geq 2\delta$ ……………(2) by definition of δ

Now by the parallelogram equality;

$\|y_n - y_m\|^2 = \|x - v_n - (x - v_m)\|^2 = \|v_m - v_n\|^2 = \|v_n - v_m\|^2$

$\|y_n - y_m\|^2 = -\|v_n + v_m\|^2 + 2(\|v_n\|^2 + \|v_m\|^2)$

$\|y_n - y_m\|^2 \leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2)$

$\|y_n - y_m\|^2 \leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \to 0$ as $m, n \to \infty$

implies that (y_n) is Cauchy. Since M is complete, therefore $\exists y \in M$ such that $y_n \to y \in M$. Since $y \in M$, we have $\|x - y\| \geq \delta$. Also,

$\|x - y\| = \|x - y_n + y_n - y\| \leq \|x - y_n\| + \|y_n - y\|$

$\|x - y\| \leq \delta_n + \|y_n - y\| \to \delta + 0$

$\Rightarrow \|x - y\| \leq \delta$

$\Rightarrow \|x - y\| = \delta$

Uniqueness:

We assume that $y \in M$ and $y_0 \in M$ both satisfy $\|x - y\| = \delta$ i.e.

$\delta = \|x - y\| = \|x - y_0\|$ and we will show that then $y = y_0$.

By using the parallelogram equality,

$\|y - y_0\|^2 = \|(y - x) - (y_0 - x)\|^2$

$\|y - y_0\|^2 = -\|(y - x) - (y_0 - x)\|^2 + 2(\|y - x\|^2 + \|y_0 - x\|^2)$

$\|y - y_0\|^2 = -\|-2x + (y + y_0)\|^2 + 2(\delta^2 + \delta^2)$

$\|y - y_0\|^2 = -4\left\|x - \left(\frac{1}{2}y + \left(1 - \frac{1}{2}\right)y_0\right)\right\|^2 + 4\delta^2$

Since M is convex therefore $\frac{1}{2}y + \left(1 - \frac{1}{2}\right)y_0 \in M$

$\|y - y_0\|^2 \leq -4\delta^2 + 4\delta^2 = 0$

$\Rightarrow \|y - y_0\| = 0$

$\Rightarrow y - y_0 = 0$

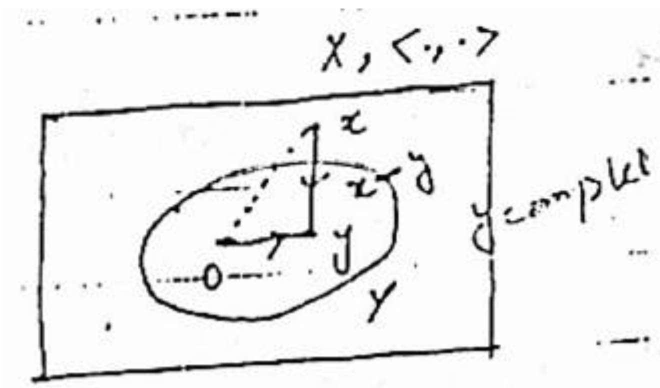
$\Rightarrow y = y_0$

Turning from arbitrary convex sets to subspaces, we obtain a lemma which generalizes the familiar idea of elementary geometry that the unique point y in a given subspace Y closest to a given x is found by "dropping a perpendicular from x to Y."

Lemma

Let $(X, \langle .,. \rangle)$ be an inner product space and let Y be a complete subspace of X and $x \in X$ fixed. Then $z = x - y$ is orthogonal to Y.

Proof



Suppose that $z \perp Y$ is not true then there exists $y_1 \in Y$ such that $\langle z, y_1 \rangle = \beta \neq 0$

Clearly, $y_1 \neq 0$ since otherwise $\langle z, y_1 \rangle = 0$.

Furthermore, for any scalar $\propto$ consider

$$\|z - \propto y_1\|^2 = \langle z - \propto y_1, z - \propto y_1 \rangle = \langle z, z - \propto y_1 \rangle - \propto \langle y_1, z - \propto y_1 \rangle$$

$$\|z - \propto y_1\|^2 = \langle z, z \rangle - \overline{\propto} \langle z, y_1 \rangle - \propto \langle y_1, z \rangle - \propto \overline{\propto} \langle y_1, y_1 \rangle$$

$$\|z - \propto y_1\|^2 = \langle z, z \rangle - \propto \langle y_1, z \rangle + \overline{\propto} \left[\propto \langle y_1, y_1 \rangle - \langle z, y_1 \rangle \right]$$

Consider $\propto = \frac{\langle z, y_1 \rangle}{\langle y_1, y_1 \rangle}$ then

$$\|z - \propto y_1\|^2 = \langle z, z \rangle - \frac{\langle z, y_1 \rangle}{\langle y_1, y_1 \rangle} \langle y_1, z \rangle + \overline{\propto} \left[\frac{\langle z, y_1 \rangle}{\langle y_1, y_1 \rangle} \langle y_1, y_1 \rangle - \langle z, y_1 \rangle \right]$$

$$\|z - \propto y_1\|^2 = \langle z, z \rangle - \frac{\langle z, y_1 \rangle}{\langle y_1, y_1 \rangle} \langle y_1, z \rangle = \langle z, z \rangle - \frac{\langle z, y_1 \rangle}{\langle y_1, y_1 \rangle} \langle z, y_1 \rangle = \langle z, z \rangle - \frac{\beta}{\langle y_1, y_1 \rangle} \beta$$

$\|z - \propto y_1\|^2 = \|z\|^2 - \frac{|\beta|^2}{\|y_1\|^2}$

Since $\|z\| = \|x - y\| = \delta$ therefore

$\|z - \propto y_1\|^2 < \delta^2$

But this is impossible because we have

$\|z - \propto y_1\| = \|x - y - \propto y_1\| = \|x - (y + \propto y_1)\| \geq \delta$

$\|z - \propto y_1\|^2 \geq \delta^2$

Hence $\langle z, y_1 \rangle = \beta \neq 0$ cannot hold, hence $z \perp Y$ and the lemma is proved.

Sum

The sum of two subspace Y and Z of a vector space X is denoted by Y + Z and is defined to be a set $Y + Z = \{y + z \quad ; y \in Y, z \in Z\}$.

Direct Sum

Let Y and Z be two subspace of a vector space X and if $Y \cap Z = \{0\}$, then Y + Z is called the direct sum of Y and Z and is denoted by $Y \oplus Z$.

Remember

A vector space X is said to be the direct sum of two subs paces Y and Z of X, written $X = Y \oplus Z$,

If each $x \in X$ has a unique representation $x = y + z \quad ; y \in Y, z \in Z$.

Then Z is called an **algebraic complement** of Y in X and vice versa, and Y, Z is called a **complementary pair of subspaces** in X

For example,

Y = R is a subspace of the Euclidean plane $\mathbf{R}^2$. Clearly, Y has infinitely many algebraic complements in R^2, each of which is a real line. But most convenient is a complement that is perpendicular. We make use of this fact when we choose a Cartesian coordinate system. In $\mathbf{R}^3$ the situation is the same in principle.

Remember

- **Theorem:** Let a linear space X be the sum of two subspace Y and Z, so that X = Y + Z then $X = Y \oplus Z$ if and only if $Y \cap Z = \{0\}$. i.e. Y and Z are disjoint.
- The condition in this theorem that the subspace Y and Z have only the origin in common, is often called disjointness of Y and Z.

Theorem

If M and N are linear closed subspace of a a Hilbert Space H, such that $M \perp N$, then the linear subspace $M + N$ is also closed.

Proof

For any subspace M and N their sum M + N is also a subspace. We have to show it is a closed subpsce. We need to show that all the limit points of $M + N$ are in $M + N$.

Let (z_n) be a sequence in $M + N$ converging to a limit point 'z' i.e. $z_n \to z$.

It is enough to show that $z \in M + N$. i.e. $z_n \to z \in M + N$.

Since $M \perp N$, so we see that $M \cap N = \{0\}$. i.e. M and N are disjoint. Then using the theorem "Let a linear space X be the sum of two subspace M and N, so that $X = M + N$ then $X = M \oplus N$ if and only if $M \cap N = \{0\}$. i.e. M and N are disjoint."

The sum $M + N$ can be strengthen to the direct sum $M \oplus N$ and thus each z_n can be expressed uniquely in the form $z_n = x_n + y_n$ where $x_n \in M, y_n \in N$.

Then we have;

$\|z_n - z_m\|^2 = \|(x_n + y_n) - (x_m + y_m)\|^2$

$\|z_n - z_m\|^2 = \|(x_n - x_m) + (y_n - y_m)\|^2$

Now using pythagorian theorem

"if $x \perp y$ in IPS then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$"

We have the following result

$\|z_n - z_m\|^2 = \|x_n - x_m\|^2 + \|y_n - y_m\|^2$

Now since (z_n) is a Cauchy sequence then

$\|z_n - z_m\| < \epsilon \quad ; m, n \geq n_0$

$\|z_n - z_m\|^2 < \epsilon^2 \quad ; m, n \geq n_0$

$\|x_n - x_m\|^2 + \|y_n - y_m\|^2 < \epsilon^2 \quad ; m, n \geq n_0$

$\|x_n - x_m\|^2 < \epsilon^2 \quad ; m, n \geq n_0$ and $\|y_n - y_m\|^2 < \epsilon^2 \quad ; m, n \geq n_0$

$\|x_n - x_m\| < \epsilon \quad ; m, n \geq n_0$ and $\|y_n - y_m\| < \epsilon \quad ; m, n \geq n_0$

So (x_n) and (y_n) are Cauchy sequences in M and N respectively. Also M and N are closed subspace of a Hilbert space H, therefore M and N are complete. So by completeness there exist vectors x and y in M and N such that;

$x_n \to x$ and $y_n \to y$

Since $x + y$ is in $M + N$, we have

$z = \lim_{n\to\infty} z_n = \lim_{n\to\infty}(x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$

$z = x + y$

This shows that z is also in $M + N$

Hence $M + N$ is closed.

Orthogonal Compliment

If Y is any subset of a Hilbert space H, then **orthogonal compliment** of Y is denoted by $Y^{\perp}$ and is defined by the formula

$Y^{\perp} = \{x \in H : \langle x, y \rangle = 0 \;\; \forall y \in Y\} = \{x \in H : x \perp Y\}$

Which is the set of all vectors orthogonal to Y.

Annihilator

Let Y be a subset of a Hilbert space H, then the set of all vectors of H which are orthogonal to Y is called the annihilator of Y and is denoted by $Y^{\perp}$. i.e.

$Y^{\perp} = \{x \in H : x \perp Y\}$

Remember

- The annihilator of $Y^{\perp}$ and is denoted by $Y^{\perp\perp}$. i.e. $Y^{\perp\perp} = \{x \in H : x \perp Y^{\perp}\}$
- $\{0\}^{\perp} = \left\{x \in H : x \perp \{0\}\right\} = H$ and $H^{\perp} = \{x \in H : x \perp H\} = \{0\}$

Theorem

Let Y be a subset of a Hilbert space H. Then $Y \subseteq Y^{\perp\perp}$

Proof

Let $x \in Y$ then $\langle x, y \rangle = 0$ for all $y \in Y^{\perp}$

$\Rightarrow x \perp Y^{\perp} \Rightarrow x \in Y^{\perp\perp} \Rightarrow Y \subseteq Y^{\perp\perp}$

Theorem

Let A and B be subset of a Hilbert space H. And $A \subseteq B$ Then $B^{\perp} \subseteq A^{\perp}$

Proof

Let $A \subseteq B$ and $x \in B^{\perp}$ then $\langle x, y \rangle = 0$ for all $y \in B$

$\Rightarrow \langle x, y \rangle = 0$ for all $y \in A \Rightarrow x \in A^{\perp} \Rightarrow B^{\perp} \subseteq A^{\perp}$

Theorem

Let A and B be subset of a Hilbert space H. Then $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$.

Proof

Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$ then $(A \cup B)^{\perp} \subseteq A^{\perp}$ and $(A \cup B)^{\perp} \subseteq B^{\perp}$

$\Rightarrow (A \cup B)^{\perp} \subseteq A^{\perp} \cap B^{\perp}$ ……………..(1)

Let $x \in A^{\perp} \cap B^{\perp}$ this means that $x \in A^{\perp}$ and $x \in B^{\perp}$ then by definition

$\langle x, u \rangle = 0$ for all $u \in A$ and $\langle x, v \rangle = 0$ for all $v \in B$

Hence $\langle x, v \rangle = 0$ for every $v \in A \cup B$ and so by definition $x \in (A \cup B)^{\perp}$

$\Rightarrow A^{\perp} \cap B^{\perp} \subseteq (A \cup B)^{\perp}$ ……………..(2)

From (1) and (2) $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

Theorem

Let A and B be subset of a Hilbert space H. Then $A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$.

Proof

Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$ then $A^{\perp} \subseteq (A \cap B)^{\perp}$ and $B^{\perp} \subseteq (A \cap B)^{\perp}$

$\Rightarrow A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$

Theorem

Let A be a subset of a Hilbert space H. Then $A^{\perp} = A^{\perp\perp\perp}$.

Proof

Since $A \subseteq A^{\perp\perp}$ then $(A^{\perp\perp})^{\perp} \subseteq A^{\perp} \Rightarrow A^{\perp\perp\perp} \subseteq A^{\perp}$

Also $A^{\perp} \subseteq (A^{\perp})^{\perp\perp} \Rightarrow A^{\perp} \subseteq A^{\perp\perp\perp}$

Hence $A^{\perp} = A^{\perp\perp\perp}$

Theorem

Let Y be a subset of a Hilbert space H. Then $Y \cap Y^{\perp} \subseteq \{0\}$.

Proof

If $Y \cap Y^{\perp} = \varphi$ then clearly $Y \cap Y^{\perp} = \varphi \subseteq \{0\}$ so the condition is true.

If $Y \cap Y^{\perp} \neq \varphi$ then let $x \in Y \cap Y^{\perp}$ implies $x \in Y$ and $x \in Y^{\perp}$ and so $\langle x, x\rangle = 0$ i.e. $\|x\|^2 = 0 \Rightarrow x = 0 \in \{0\} \Rightarrow x \in \{0\} \Rightarrow \boldsymbol{Y} \cap \boldsymbol{Y}^{\perp} \subseteq \{\mathbf{0}\}$

Theorem

Let Y be a subset of a Hilbert space H. Then $Y^{\perp}$ is closed linear subspace of H.

Proof

Let $x, y \in Y^{\perp}$ and $\propto, \beta \in F$ then we have to show that $\propto x + \beta y \in Y^{\perp}$

Since $x, y \in Y^{\perp}$ therefore $\langle x, u\rangle = 0$ and $\langle y, u\rangle = 0$ for every $u \in Y$ then

$\langle \propto x + \beta y, u\rangle = \propto \langle x, u\rangle + \beta\langle y, u\rangle = 0 \Rightarrow \propto x + \beta y \perp Y \Rightarrow \propto x + \beta y \in Y^{\perp}$.

This shows that $Y^{\perp}$ is linear subspace of H. Now we have to show that $Y^{\perp}$ is closed. For this we just show $Y^{\perp} = \overline{Y^{\perp}}$.

We already know that $Y^{\perp} \subseteq \overline{Y^{\perp}}$ …………..(1)

Now let $x \in \overline{Y^{\perp}}$ then there exists a sequence (x_n) in $Y^{\perp}$ such that $x_n \to x$

Now by using continuity of inner producsts for any $u \in Y$ we have

$\langle x, u\rangle = \langle \lim_{n\to\infty} x_n , u\rangle = \lim\limits_{n\to\infty} \langle x_n, u\rangle = 0 \Rightarrow x \perp Y \Rightarrow x \in Y^{\perp}$

Implies that $\overline{Y^{\perp}} \subseteq Y^{\perp}$ …………..(2)

Then $Y^{\perp} = \overline{Y^{\perp}}$ and $Y^{\perp}$ is closed subspace of H.

Theorem

Let Y be a closed linear subspace of a Hilbert space H. Then $Y \cap Y^{\perp} = \{0\}$.

Proof

Since we know that if Y be a subset of a Hilbert space H. Then

$Y \cap Y^{\perp} \subseteq \{0\}$ …………….(1)

Given that Y is closed linear subspace of H and we also know that $Y^{\perp}$ is closed linear subspace of H. Let $x \in Y \cap Y^{\perp}$ implies $x \in Y$ and $x \in Y^{\perp}$ and so $\langle x, x\rangle = 0$ i.e. $\|x\|^2 = 0 \Rightarrow x = 0 \Rightarrow 0 \in Y$ and $0 \in Y^{\perp} \Rightarrow 0 \in Y \cap Y^{\perp}$

$\Rightarrow \{0\} \subseteq Y \cap Y^{\perp}$ …………….(2)

Combining (1) and (2) we get $Y \cap Y^{\perp} = \{0\}$

Projection Theorem

Let Y be any closed subspace of a Hilbert space H. Then $H = Y \oplus Y^{\perp}$

Proof

Suppose $Y + Y^{\perp}$ is proper subspace of H then there is a non – zero vector $z \in H$ such that $z \perp (Y + Y^{\perp})$. i.e. $z \in (Y + Y^{\perp})^{\perp}$

Now $Y \subseteq (Y + Y^{\perp})$ implies $(Y + Y^{\perp})^{\perp} \subseteq Y^{\perp}$

Also we know $Y^{\perp} \subseteq (Y + Y^{\perp})$ implies $(Y + Y^{\perp})^{\perp} \subseteq Y^{\perp\perp}$

Then $z \in (Y + Y^{\perp})^{\perp} \subseteq Y^{\perp} \cap Y^{\perp\perp} = \{0\} \Rightarrow z = 0$ a contradiction.

Hence $Y + Y^{\perp}$ is the whole of H. i.e. $H = Y + Y^{\perp}$ since $Y \cap Y^{\perp} = \{0\}$

Thus $H = Y \oplus Y^{\perp}$

Theorem

Let Y be a closed subset of a Hilbert space H. Then $Y = Y^{\perp\perp}$

Proof

Let $x \in Y$ then $\langle x, y \rangle = 0$ for all $y \in Y^{\perp}$

$\Rightarrow x \perp Y^{\perp} \Rightarrow x \in Y^{\perp\perp} \Rightarrow Y \subseteq Y^{\perp\perp}$

Now let $x \in Y^{\perp\perp}$ and Y be a closed subset H also $H = Y \bigoplus Y^{\perp}$ so;

For each $x \in Y^{\perp\perp} \subseteq H$; $x = y + z$: $y \in Y, z \in Y^{\perp}$

But $Y \subseteq Y^{\perp\perp}$ therefore $y \in Y^{\perp\perp}$

$\Rightarrow z = x - y \in Y^{\perp\perp} \Rightarrow z \perp Y^{\perp}$

But $z \in Y^{\perp} \Rightarrow z \perp z \Rightarrow z = 0 \Rightarrow z = x - y = 0 \Rightarrow x = y \Rightarrow x \in Y \Rightarrow Y^{\perp\perp} \subseteq Y$

Hence from both cases $Y = Y^{\perp\perp}$.

Theorem

For any complete subspace Y of an inner product space V, Prove that $Y = Y^{\perp\perp}$.

Proof

Let $x \in Y$ then $\langle x, y \rangle = 0$ for all $y \in Y^{\perp}$

$\Rightarrow x \perp Y^{\perp} \Rightarrow x \in Y^{\perp\perp} \Rightarrow Y \subseteq Y^{\perp\perp}$

Now let $x \in Y^{\perp\perp}$ and Y be a complete subspace of V also $V = Y \bigoplus Y^{\perp}$ so;

For each $x \in Y^{\perp\perp} \subseteq V$; $x = y + z$: $y \in Y \subseteq Y^{\perp\perp}, z \in Y^{\perp}$

$\Rightarrow z = x - y \in Y^{\perp\perp} \Rightarrow z \perp Y^{\perp}$

But $z \in Y^{\perp} \Rightarrow z \perp z \Rightarrow z = 0 \Rightarrow z = x - y = 0 \Rightarrow x = y \Rightarrow x \in Y \Rightarrow Y^{\perp\perp} \subseteq Y$

Hence from both cases $Y = Y^{\perp\perp}$.

Theorem

Let A be proper complete subspace of an inner product space V, then Prove that $V = A \oplus A^{\perp}$

Proof

Let V be an inner product space and $A \neq \varphi$ and being a subspace is convex subset. Then for every given $x \in V$ there exists a unique $y \in A$ such that

$\|x - y\| = inf_{\bar{y} \in A}\|x - \bar{y}\|$

For each $x \in V/A = A'$ put $z = x - y$

Then $z \perp A \Rightarrow z \in A^{\perp} \subseteq V \Rightarrow z = x - y$

$\Rightarrow x = y + z \;\; ; y \in A, z \in A^{\perp}$

To see this expression is unique, suppose also $x = y_1 + z_1 \;\; ; y_1 \in A, z_1 \in A^{\perp}$

$\Rightarrow x = y + z = y_1 + z_1$

$\Rightarrow y - y_1 = z - z_1 \in A \cap A^{\perp} = \{0\}$

$\Rightarrow y - y_1 = z - z_1 \in \{0\} \Rightarrow y - y_1 = 0, z - z_1 = 0$

$\Rightarrow y = y_1, z = z_1$

Thus $V = A \oplus A^{\perp}$

Lemma (Keep in Mind)

The orthogonal complement $Y^{\perp}$ of a closed subspace Y of a Hilbert space H is the null space, $\mathcal{N}(P)$ of the orthogonal projection P of H onto Y.

Lemma

For any subset $M \neq \varphi$ of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$.

i.e. The set of all linear combiniations of vectors of M is dense in H iff $M^{\perp} = \{0\}$.

Proof

Let $x \in M^{\perp}$ and assume $V = span\ M$ to be dense in H.

Then $x \in \bar{V} = H$.

Then By Theorem ($x \in \bar{M} \Leftrightarrow (x_n)\ in\ M : x_n \to x$) there is a sequence (x_n) in V such that $x_n \to x$.

Since $x \in M^{\perp}$ and $M \perp V$, we have $\langle x_n, x \rangle = 0$.

The continuity of the inner product implies that $\langle x_n, x \rangle \to \langle x, x \rangle$ as $n \to \infty$.

Thus $\langle x, x \rangle = \|x\|^2 = 0$, so that $x = 0$.

Since $x \in M^{\perp}$ was arbitrary, this shows that $M^{\perp} = \{0\}$.

Conversely,

Suppose that $M^{\perp} = \{0\}$ and $V = span\ M$.

If $x \perp V$, then $x \perp M$, so that $x \in M^{\perp}$ and $x = 0$.

Hence $V^{\perp} = \{0\}$.

Since V is a subspace of H, then $H = \bar{V} \oplus \bar{V}^{\perp}$

We thus obtain $\bar{V} = H$.i.e. V is dense in H.

Orthonormal Vectors

Let X be an inner product space, the vectors $x, y \in X$ are said to be orthonoram if $\langle x, y\rangle = 0$ and $\|x\| = 1 = \|y\|$.

Orthonormal Sets and Sequences

An **orthogonal set** M in an inner product space X is a subset $M \subseteq X$ whose elements are pairwise orthogonal. An **orthonormal set** $M \subseteq X$ is an orthogonal set in X whose elements have norm 1, that is, for all $x, y \in M$,

$$\langle x, y\rangle = \begin{cases} 0 & ; x \neq y \\ 1 & ; x = y \end{cases}$$

If an orthogonal or orthonormal set M is countable, we can arrange it in a sequence (x_n) and call it an **orthogonal or orthonormal sequence**, respectively.

Orthonormal Basis

In an inner product space, a basis consisting of orthogonal vectors is called an orthogonal basis, and a basis consisting of orthonormal vectors is called an orthonormal basis. A familiar example of the orthonormal basis is the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$ for $\mathbf{R}^3$ with the Eucledian inner product.

Lemma

An orthonormal set is linearly independent.

Proof

Let $\{e_1, e_2, \ldots, e_n\}$ be orthonormal and consider the equation

$$\propto_1 e_1 + \propto_2 e_2 + \ldots + \propto_n e_n = 0$$

Consider $\langle \propto_1 e_1 + \propto_2 e_2 + \ldots + \propto_n e_n, e_j\rangle = 0$

$$\Rightarrow \langle \textstyle\sum_k \propto_k e_k, e_j\rangle = 0 \Rightarrow \sum_k \propto_1 \langle e_k, e_j\rangle = \propto_j \langle e_j, e_j\rangle = \propto_j = 0$$

This shows that orthonormal set $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

Examples

- **Euclidean space R^3:** In the space R^3, the three unit vectors (1, 0, 0), (0,1,0), (0,0,1) in the direction of the three axes of a rectangular coordinate system form an orthonormal set.
- **Space l^2:** In the space l^2, an orthonormal sequence is (e_n), where $e_n = (\delta_{nj})$ has the nth element 1 and all others zero.
- **Continuous functions:** Let X be the inner product space of all real-valued continuous functions on $[0, 2\pi]$ with inner product defined by
  $\langle x, y\rangle = \int_0^{2\pi} x(t)y(t)dt$
  An orthogonal sequence in X is (u_n), where $u_n(t) = \ cosnt \ ; n = 0,1, \ldots$
  Another orthogonal sequence in X is (v_n), where $v_n(t) = Sinnt\ ; n = 0,1, \ldots$

Advantage

A great advantage of orthonormal sequences over arbitrary linearly iadependent sequences is the following. If we know that a given x can be represented as a linear combination of some elements of an orthonormal sequence, then the orthonormality makes the actual determination of the coefficients very easy.

Fourier Coefficients

The inner products (x, ek) are called the Fourier coefficients of x with respect to the orthonormal sequence (e_k).

Geometrical Interpreatation of the Bessel Inequality اسکا حل آگے آرہا

A Geometrical Interpreatation of the Bessel Inequality is that the sum of the squares of the projections of a vector x onto a set of mutually perpendicular directions can not exceed the square of the length of the vector itself.

Bessel Inequality

Let (e_k) be anorthonormal sequence in an inner product space X. Then for every $x \in X$ we have $\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$

Proof

Let $Y_n = span\{e_1, e_2, \dots, e_n\}$ then for every $y \in Y_n$ we can express

$y = \sum_{k=1}^{n} \propto_k e_k \quad ; \propto_k = \langle y, e_k \rangle$

We claim that for a particular choice of $\propto_k$. i.e. $\propto_k = \langle x, e_k \rangle \ : x \in X$ but $x \notin Y_n$ then we can obtain $y \in Y_n$such that $z = (x - y) \perp y$ (will show this)

We first note that

$\|y\|^2 = \langle y, y \rangle = \langle \sum_{k=1}^{n} \propto_k e_k , \sum_{m=1}^{n} \propto_m e_m \rangle = \langle \sum_{k=1}^{n} \langle x, e_k \rangle e_k , \sum_{m=1}^{n} \langle x, e_m \rangle e_m \rangle$

$\|y\|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle \langle e_k, \sum_{m=1}^{n} \langle x, e_m \rangle e_m \rangle = \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\sum_{m=1}^{n} \langle x, e_m \rangle} \langle e_k, e_m \rangle$

$\|y\|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle \sum_{m=1}^{n} \overline{\langle x, e_m \rangle} \langle e_k, e_m \rangle$

$\|y\|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle \sum_{k=1}^{n} \overline{\langle x, e_k \rangle} \langle e_k, e_k \rangle$ taking $m = k$

$\|y\|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle \overline{\langle x, e_k \rangle} (1)$

$\|y\|^2 = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$ ………(1)

Now consider

$\langle z, y \rangle = \langle x - y, y \rangle = \langle x, y \rangle - \langle y, y \rangle = \langle x, \sum_{k=1}^{n} \propto_k e_k \rangle - \|y\|^2$

$\langle z, y \rangle = \langle x, \sum_{k=1}^{n} \langle x, e_k \rangle e_k \rangle - \|y\|^2 = \overline{\sum_{k=1}^{n} \langle x, e_k \rangle} \langle x, e_k \rangle - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$

$\langle z, y \rangle = \sum_{k=1}^{n} \overline{\langle x, e_k \rangle} \langle x, e_k \rangle - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$

$\langle z, y \rangle = 0$ Implies $z \perp y$

Now $z = x - y$ then using Pyhtagorian Theorem $\|z\|^2 = \|x\|^2 - \|y\|^2$

$0 \leq \|z\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \Rightarrow \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \leq \|x\|^2$

$\Rightarrow \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2$ if $n \to \infty$. Hence proved.

Bessel Inequality (Another Form)

Suppose $\{e_1, e_2, \ldots, e_k\}$ is an orthonormal set of vectors in an inner product space X. Let $x \in X$ be any arbitrary vector and c_k be the fourier coefficients of vector x with respect to e_k then $\sum_{k=1}^{r} c_k^2 \leq \|x\|^2$

Proof

Consider $\langle x - \sum_{k=1}^{r} c_k e_k\,, x - \sum_{k=1}^{r} c_k e_k\rangle \geq 0$

$\langle x, x\rangle - \langle x, \sum_{k=1}^{r} c_k e_k\rangle - \langle \sum_{k=1}^{r} c_k e_k\,, x\rangle + \langle \sum_{k=1}^{r} c_k e_k\,, \sum_{k=1}^{r} c_k e_k\rangle \geq 0$

$\|x\|^2 - 2\langle x, \sum_{k=1}^{r} c_k e_k\rangle + \sum_{k=1}^{r} c_k^2 \langle e_k, e_k\rangle \geq 0$

$\|x\|^2 - 2\sum_{k=1}^{r} c_k \langle x, e_k\rangle + \sum_{k=1}^{r} c_k^2 \langle e_k, e_k\rangle \geq 0$

$\|x\|^2 - 2\sum_{k=1}^{r} c_k \frac{\langle x, e_k\rangle}{\langle e_k, e_k\rangle} + \sum_{k=1}^{r} c_k^2 \geq 0$

$\|x\|^2 - 2\sum_{k=1}^{r} c_k^2 + \sum_{k=1}^{r} c_k^2 \geq 0$

$\|x\|^2 - \sum_{k=1}^{r} c_k^2 \geq 0$

$\sum_{k=1}^{r} c_k^2 \leq \|x\|^2$ Hence proved.

Series Related to Orthonormal Sequences and Sets

There are some facts and questions that arise in connection with the Bessel inequality. In this section we first motivate the term "Fourier coefficients," then consider infinite series related to orthonormal sequences, and finally take a first look at orthonormal sets which are uncountable.

Fourier series

A trigonometric series is a series of the form

$a_0 + \sum_{k=1}^{\infty}(a_k Coskt) + b_k Sinkt$

Where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} x(t)dt\,, a_k = \frac{1}{\pi}\int_0^{2\pi} x(t)Cosktdt\,, b_k = \frac{1}{\pi}\int_0^{2\pi} x(t)Sinktdt$
These coefficients are called the **Fourier coefficients** of x.

Theorem Let (e_k) be an orthonormal sequence in a Hilbert space H. Then:

a) The series $\sum_{k=1}^{\infty} \propto_k e_k$ converges (in the norm on H) if and only if the following series converges: $\sum_{k=1}^{\infty} |\propto_k|^2$
b) If $\sum_{k=1}^{\infty} \propto_k e_k$ converges, then the coefficients a_k are the Fourier coefficients $\langle x, e_k \rangle$, where x denotes the sum of $\sum_{k=1}^{\infty} \propto_k e_k$; hence in this case, $\sum_{k=1}^{\infty} \propto_k e_k$ can be written $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$
c) For any $x \in H$, the series $\sum_{k=1}^{\infty} \propto_k e_k$ with $a_k = \langle x, e_k \rangle$ converges (in the norm of H).

Proof

(a) Let $s_n = \propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n$ and $\sigma_n = |\propto_1|^2 + |\propto_2|^2 + \cdots + |\propto_n|^2$

Then, because of the orthonormality, for any m and $n > m$,

$$\|s_n - s_m\|^2 = \|(\propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_n e_n) - (\propto_1 e_1 + \propto_2 e_2 + \cdots + \propto_m e_m)\|^2$$

$$\|s_n - s_m\|^2 = \|\propto_{m+1} e_{m+1} + \propto_2 e_2 + \cdots + \propto_n e_n\|^2$$

$$\|s_n - s_m\|^2 = |\propto_{m+1}|^2 + |\propto_{m+2}|^2 + \cdots + |\propto_n|^2 = \sigma_n - \sigma_m$$

Hence (s_n) is Cauchy in H if and only if (σ_n) is Cauchy in R. Since H and R are complete, the first statement of the theorem follows. i.e.The series $\sum_{k=1}^{\infty} \propto_k e_k$ converges (in the norm on H) if and only if the following series converges: $\sum_{k=1}^{\infty} |\propto_k|^2$

(b) Let $\sum_{j=1}^{\infty} \propto_j e_j$ converges in H.Then $x = \sum_{j=1}^{\infty} \propto_j e_j$

So that $\langle x, e_k \rangle = \langle \sum_{j=1}^{\infty} \propto_j e_j, e_k \rangle$ for $k = 1,2, \ldots$

$\langle x, e_k \rangle = \sum_{j=1}^{\infty} \propto_j \langle e_j, e_k \rangle = \propto_k$ since $e_1, e_2, \ldots, e_n$ are orthonormal.

(c) Let there is an element $x \in H$ such that $c_k = \langle x, e_k \rangle$; $k = 1,2,3, \ldots$ then by Bessel inequality; $\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2 < \infty$

i.e. $\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 < \infty$ implies for any $x \in H$, the series $\sum_{k=1}^{\infty} \propto_k e_k$ with $\propto_k = \langle x, e_k \rangle$ converges (in the norm of H).

This complete the proof.

Riesz and Fischer Theorem

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal set in a Hilbert space H.Then for any sequence (c_k) of scalar the following statements are equivalent;

a) $(c_k) \in l^2$
b) $\sum_{k=1}^{\infty} c_k e_k$ converges in H.
c) There is an element $x \in H$ such that $c_k = \langle x, e_k \rangle$; $k = 1,2,3, \dots$

Proof

(a) Let $(c_k) \in l^2$ then $\sum_{k=1}^{\infty} |c_k|^2 < \infty$

For $n = 1,2,3, \dots$ let $s_n = \sum_{k=1}^{n} c_k e_k$

Then, because of the orthonormality, for any m and $n > m$,

$\|s_n - s_m\|^2 = \|(c_1 e_1 + c_2 e_2 + \cdots + c_n e_n) - (c_1 e_1 + c_2 e_2 + \cdots + c_m e_m)\|^2$

$\|s_n - s_m\|^2 = \|c_{n+1} e_{n+1} + c_2 e_2 + \cdots + c_n e_n\|^2$

$\|s_n - s_m\|^2 = \sum_{k=m+1}^{n} |c_k|^2 \to 0$ as $m, n \to \infty$ since $(c_k) \in l^2$ convergent

$\|s_n - s_m\|^2 \to 0$ as $m, n \to \infty$

Hence (s_n) is Cauchy in H. Since H is complete, therefore $s_n \to x \in H$.

Hence $\sum_{k=1}^{\infty} c_k e_k$ converges in H.

(b) Let $\sum_{j=1}^{\infty} c_j e_j$ converges in H.Then $x = \sum_{j=1}^{\infty} c_j e_j$

So that $\langle x, e_k \rangle = \langle \sum_{j=1}^{\infty} c_j e_j , e_k \rangle$ for $k = 1,2, \dots$

$\langle x, e_k \rangle = \sum_{j=1}^{\infty} c_j \langle e_j, e_k \rangle = c_k$ since $e_1, e_2, \dots, e_n$ are orthonormal.

(c) Let there is an element $x \in H$ such that $c_k = \langle x, e_k \rangle$; $k = 1,2,3, \dots$ then by Bessel inequality; $\sum_{k=1}^{\infty} |c_k|^2 = \sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \leq \|x\|^2 < \infty$

i.e. $\sum_{k=1}^{\infty} |c_k|^2 < \infty$ implies $(c_k) \in l^2$.

This complete the proof.

Representation of Functionals on Hilbert Spaces

It is of practical importance to know the general form of bounded linear functionals on various spaces. For general Banach spaces such formulas and their derivation can sometimes be complicated. However, for a Hilbert space the situation is surprisingly simple:

Riesz's Representation Theorem

Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely, $f(x) = \langle x, z\rangle$ where z depends on f, is uniquely determined by f and has norm $\|z\| = \|f\|$

Proof We prove that

(a) f has a representation $f(x) = \langle x, z\rangle$.
(b) z in $f(x) = \langle x, z\rangle$ is unique.
(c) formula $\|z\| = \|f\|$ holds.

(a) If $f = 0$, then $f(x) = \langle x, z\rangle$ hold if we take $z = 0$.

Let $f \neq 0$. To motivate the idea of the proof, lets investigate what properties z must have if a representation $f(x) = \langle x, z\rangle$ exists. First of all, $z \neq 0$ since otherwise $f = 0$. Second, if $f(x) = \langle x, z\rangle = 0$ for some $x \in H$ then $x \in \mathcal{N}(f)$ and $z \perp \mathcal{N}(f)$. Hence $z \in \mathcal{N}(f)^{\perp}$. This suggests that we consider $\mathcal{N}(f)$ and its orthogonal complement $\mathcal{N}(f)^{\perp}$.

Since $\mathcal{N}(f)$ is a vector space and closed therefore $H = \mathcal{N}(f) \oplus \mathcal{N}(f)^{\perp}$ Furthermore, $f \neq 0$ implies $\mathcal{N}(f) \neq H$, so that $\mathcal{N}(f)^{\perp} \neq \{0\}$ by the projection theorem.

Choose $0 \neq z_0 \in \mathcal{N}(f)^{\perp}$ and let $v = f(x)z_0 - f(z_0)x$ where $x \in H$ is arbitrary. Applying f, we obtain $f(v) = f(x)f(z_0) - f(z_0)f(x) = 0$ since f is linear. This show that $v \in \mathcal{N}(f)$.

Since $z_0 \in \mathcal{N}(f)^{\perp}$, we have $\langle v, z_0\rangle = 0$

$\langle f(x)z_0 - f(z_0)x, z_0\rangle = 0$

$f(x)\langle z_0, z_0\rangle - f(z_0)\langle x, z_0\rangle = 0$

$\Rightarrow f(x) = \frac{f(z_0)}{\langle z_0, z_0\rangle}\langle x, z_0\rangle = \frac{f(z_0)}{\|z_0\|^2}\langle x, z_0\rangle = \langle x, \frac{\overline{f(z_0)}}{\|z_0\|^2} z_0\rangle$

$\Rightarrow f(x) = \langle x, z\rangle$ where $z = \frac{\overline{f(z_0)}}{\|z_0\|^2} z_0$

Since $x \in H$ was arbitrary, f has a representation $f(x) = \langle x, z\rangle$.

(b) We prove that z in $f(x) = \langle x, z\rangle$ is unique. Suppose that for all $x \in H$ we have

$f(x) = \langle x, z_1\rangle = \langle x, z_2\rangle$

$\Rightarrow \langle x, z_1\rangle - \langle x, z_2\rangle = 0 \Rightarrow \langle x, z_1 - z_2\rangle = 0 \quad ; \forall x \in H$

Choosing $x = z_1 - z_2$ we get

$\Rightarrow \langle z_1 - z_2, z_1 - z_2\rangle = 0 \Rightarrow \|z_1 - z_2\|^2 = 0 \Rightarrow \|z_1 - z_2\| = 0$

Hence $z_1 - z_2 = 0$, so that $z_1 = z_2$, the uniqueness.

(c) We finally prove $\|z\| = \|f\|$.

If $f = 0$, then $z = 0$ and $\|z\| = \|f\|$ holds.

Let $f \neq 0$. Then $z \neq 0$. From $f(x) = \langle x, z\rangle$ with $x = z$ we obtain

$f(z) = \langle z, z\rangle = \|z\|^2$

$\Rightarrow \|z\|^2 = f(z) \leq |f(z)| \leq \|f\|\|z\|$

$\Rightarrow \|z\| \leq \|f\|$ ……………(1)

Now $|f(x)| = |\langle x, z\rangle| \leq \|x\|\|z\|$ by Schawarz Inequality

$\Rightarrow \frac{|f(x)|}{\|x\|} \leq \|z\| \Rightarrow Sup_{x\neq 0} \frac{|f(x)|}{\|x\|} \leq \|z\|$

$\Rightarrow \|f\| \leq \|z\|$ ……………(2)

From (1) and (2) $\|f\| = \|z\|$

Lemma

If $\langle v_1, w\rangle = \langle v_2, w\rangle$ for all w in an inner product space X, then $v_1 = v_2$. In particular, $\langle v, w\rangle = 0$ for all $w \in X$ implies $v = 0$.

Proof Let, for all w,

$$\langle v_1, w\rangle = \langle v_2, w\rangle \Rightarrow \langle v_1, w\rangle - \langle v_2, w\rangle = 0 \Rightarrow \langle v_1 - v_2, w\rangle = 0$$

Choosing $w = v_1 - v_2$ we get

$$\Rightarrow \langle v_1 - v_2, v_1 - v_2\rangle = 0 \Rightarrow \|v_1 - v_2\|^2 = 0 \Rightarrow \|v_1 - v_2\| = 0$$

Hence $v_1 - v_2 = 0$, so that $v_1 = v_2$.

Let, for all w, we have $\langle v, w\rangle = 0$ and Choosing $w = v$ we get

$$\Rightarrow \langle v, v\rangle = 0 \Rightarrow \|v\|^2 = 0 \Rightarrow \|v\| = 0 \Rightarrow v = 0$$

The practical usefulness of bounded linear functionals on Hilbert spaces results to a large extent from the simplicity of the Riesz representation $f(x) = \langle x, z\rangle$. Furthermore, $f(x) = \langle x, z\rangle$ is quite important in the theory of operators on Hilbert spaces. In particular, this refers to the Hilbert-adjoint operator T* of a bounded linear operator T which we shall define in the next section. For this purpose we need a preparation which is of general interest, too. We begin with the following definition.

SesquiIinear Form Let X and Y be vector spaces over the same field **K (=R or C)**.Then a sesquilinear form (or sesquilinear functional) h on $X \times Y$ is a mapping $h: X \times Y \to K$ such that for all $x,, x_1, x_2 \in X$ and $y,, y_1, y_2 \in Y$ and all scalars $\propto, \beta$,

(a) $h(x_1 + x_2, y) = h(x_1, y) + h(x_2, y)$
(b) $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$
(c) $h(\propto x, y) =\propto h(x, y)$
(d) $h(x, \beta y) = \bar{\beta} h(x, y)$

Hence h is linear in the first argument and conjugate **linear** in the second one. If X and Yare real (K = R), then $h(x, \beta y) = \beta h(x, y)$ is simply and h is called **bilinear** since it is linear in both arguments.

Bounded Sesquilinear Function

If X and Yare normed spaces and if there is a real number c such that for all $x \in X, y \in Y$ we have $|h(x,y)| \leq c\|x\|\|y\|$, then h is said to be **bounded sesquilinear**, and the number $\|h\| = Sup_{\substack{x \in X-\{0\} \\ y \in Y-\{0\}}} \frac{|h(x,y)|}{\|x\|\|y\|} = Sup_{\substack{\|x\|=1 \\ \|y\|=1}} |h(x,y)|$ is called the **norm** of h. Also we have $|h(x,y)| \leq \|h\|\|x\|\|y\|$.

Hilbert Adjoint Operator T*

Let $T: H_1 \to H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert-adjoint operator T* of T is the operator $T^*: H_2 \to H_1$ such that for all $x \in H_1$ and $y \in H_2$, we have $\langle Tx, y \rangle = \langle x, T^* y \rangle$

Theorem (Riesz Representation)

Let H_1 and H_2 be Hilbert spaces and $h: H_2 \times H_1 \to K$ is a bounded sesquilinear form. Then h has a representation $h(x,y) = \langle Sx, y \rangle$ where $S: H_1 \to H_2$ is a bourrded linear operator. S is uniquely determined by h and has norm $\|S\| = \|h\|$

Proof

We consider $\overline{h(x,y)}$. This is linear in y, because of the bar. And x is fixed. Then

$\overline{h(x,y)} = \langle y, z \rangle$ By Riesz Theorem

$\Rightarrow h(x,y) = \langle z, y \rangle$ …………..(1)

Here let $z \in H_2$ is unique but, of course, depends on our fixed $x \in H_1$. It follows that (1) with variable x defines an operator $S: H_1 \to H_2$ by $z = Sx$.

$(1) \Rightarrow h(x,y) = \langle Sx, y \rangle$ which is required. using $z = Sx$

Now S is linear. In fact, its domain is the vector space H_1. And we have from $h(x,y) = \langle Sx, y \rangle$ the following result

$\langle S(\alpha x_1 + \beta x_2), y \rangle = h(\alpha x_1 + \beta x_2, y)$

$\langle S(\alpha x_1 + \beta x_2), y \rangle = \alpha h(x_1, y) + \beta h(x_2, y)$

$\langle S(\alpha x_1 + \beta x_2), y\rangle = \alpha\langle Sx_1, y\rangle + \beta\langle Sx_2, y\rangle$

$\langle S(\alpha x_1 + \beta x_2), y\rangle = \langle \alpha Sx_1 + \beta Sx_2, y\rangle$

$\Rightarrow S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2$

Now S is bounded linear operator, then

$\|h\| = Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} = Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx,y\rangle|}{\|x\|\|y\|}$

Choose $y \geq Sx$

$\|h\| \geq Sup_{\substack{x\neq 0\\ Sx\neq 0}} \frac{|\langle Sx,Sx\rangle|}{\|x\|\|Sx\|} = Sup_{\substack{x\neq 0\\ Sx\neq 0}} \frac{\|Sx\|^2}{\|x\|\|Sx\|} = Sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|$

$\|h\| \geq \|S\|$ …………..(2)

Now by definition

$\|h\| = Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h(x,y)|}{\|x\|\|y\|} = Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx,y\rangle|}{\|x\|\|y\|} \leq Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{\|Sx\|\|y\|}{\|x\|\|y\|} = Sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|$

$\|h\| \leq \|S\|$ …………..(3)

Hence from (2) and (3) $\|h\| = \|S\|$

For Uniqueness

Let another operator $S: H_1 \to H_2$ for which $h(x,y) = \langle Tx, y\rangle$ but $h(x,y) = \langle Sx, y\rangle$ for $S: H_1 \to H_2$ then in this case

$\langle Tx, y\rangle = \langle Sx, y\rangle$

$Tx = Sx$

$T = S$ and hence S is unique.

Theorem (Existence)

The Hilbert-adjoint operator T* of $T: H_1 \to H_2$ where H_1 and H_2 are Hilbert spaces exists, is unique and is a bounded linear operator with norm $\|T^*\| = \|T\|$

Proof

Consider $h: H_2 \times H_1 \to K$ define by the formula $h\langle y, x\rangle = \langle y, Tx\rangle$ ………..(1)

and this defines a sesquilinear form on $H_2 \times H_1$ because the inner product is sesquilinear (linear in first argument and conjugate linear in second argument) and T is linear.

We claim that h is boubnded by Schwarz inequality. To see this consider;

$|h\langle y, x\rangle| = |\langle y, Tx\rangle| \leq \|y\|\|Tx\| \leq \|T\|\|x\|\|y\|$

$|h\langle y, x\rangle| \leq \|T\|\|x\|\|y\|$ ………..(2) implies h is boubnded.

By Riesz representation theorem for h; writing T* for S, replace by T* we have

$h\langle y, x\rangle = \langle T^* y, x\rangle$ ………..(3)

where $T^*: H_2 \to H_1$ is a uniquely determined bounded linear operator with norm $\|h\| = \|T^*\|$ ………..(4)

comparing (1) and (3) $\langle y, Tx\rangle = \langle T^* y, x\rangle$

$\Rightarrow \overline{\langle y, Tx\rangle} = \overline{\langle T^* y, x\rangle} \Rightarrow \langle Tx, y\rangle = \langle x, T^* y\rangle \Rightarrow T^*$ is Hilbert Adjoint Operator.

Now we prove $\|T^*\| = \|T\|$

From (2) we have $\frac{|h\langle y,x\rangle|}{\|x\|\|y\|} \leq \|T\|$

$\Rightarrow Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h\langle y,x\rangle|}{\|x\|\|y\|} \leq \|T\| \Rightarrow \|h\| \leq \|T\|$ ………..(5)

Now $\|h\| = Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|h\langle y,x\rangle|}{\|x\|\|y\|}$

$\Rightarrow \|h\| = Sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle y,Tx\rangle|}{\|x\|\|y\|} \geq Sup_{x\neq 0} \frac{|\langle Tx,Tx\rangle|}{\|x\|\|Tx\|} = Sup_{x\neq 0} \frac{\|Tx\|^2}{\|x\|\|Tx\|} = Sup_{x\neq 0} \frac{\|Tx\|}{\|x\|}$

$\Rightarrow \|h\| \geq \|T\|$ ....(6)

From (5) and (6) $\|h\| = \|T\|$ ....(7)

From (4) and (7) $\|T^*\| = \|T\|$

Lemma (Zero operator)

Let X and Y be inner product spaces and $Q: X \to Y$ a bounded linear operator. Then:

(a) $Q = 0$ if and only if $\langle Qx, y\rangle = 0$ for all $x \in X$ and $y \in Y$.
(b) If $Q: X \to X$, where X is complex, and $\langle Qx, x\rangle = 0$ for all $x \in X$, then $Q = 0$.

Proof. (a) $Q = 0 \Rightarrow Qx = 0 \Rightarrow \langle Qx, y\rangle = \langle 0, y\rangle = 0\langle w, y\rangle = 0$

Conversely, $\langle Qx, y\rangle = 0 \Rightarrow Qx = 0 \Rightarrow Q = 0$ for all x and y.

(b) By assumption, $\langle Qv, v\rangle = 0$ for every $v = ax + y \in X$, that is,

$0 = \langle Q(ax + y), ax + y\rangle = |a|^2\langle Qx, x\rangle + \langle Qy, y\rangle + a\langle Qx, y\rangle + \bar{a}\langle Qy, x\rangle$

The first two terms on the right are zero by assumption. a = 1 gives

$\langle Qx, y\rangle + \langle Qy, x\rangle = 0$ and $a = i$ gives $a = -i$ and $\langle Qx, y\rangle - \langle Qy, x\rangle = 0$

By addition, $\langle Qx, y\rangle = 0$, and $Q = 0$ follows from (a).

In part (b) of this lemma, it is essential that X be complex. Indeed, the conclusion may not hold if X is real. A counterexample is a rotation Q of the plane R^2 through a right angle. Q is linear, and $Qx \perp x$, hence $\langle Qx, x\rangle = 0$ for all $x \in R^2$, but $Q \neq 0$.

Self-Adjoint Operator

A bounded linear operator T: $H \rightarrow H$ on a Hilbert space H is said to be Self – adjoint or Hermitian if the T*=T,

Where the T* is the Hilber adjoint operator of T.

Unitary Operator

A bounded linear operator T: $H \rightarrow H$ on a Hilbert space H is said to be Unitary if T is bijective and $T^* = T^{-1}$

Where the T* is the Hilber adjoint operator of T.

Normal Operator

A bounded linear operator T: $H \rightarrow H$ on a Hilbert space H is said to be Normal if $TT^* = T^*T$

Where the T* is the Hilber adjoint operator of T.

Theorem The Gram-Schmidt Orthogonalisation Process

If **H** is a Hilbert space over **K** and $\{x_n\}_1^\infty$ is a linearly independent set in H, then we can find an orthonormal set $\{y_n\}_1^\infty$ in H so that

$span\{x_1, x_2, \dots, x_k\} = span\{y_1, y_2, \dots, y_k\}$ for all $k \geq 1$.

This procedure is used to find orthogonal basis and orthonormal basis set.

The Gram-Schmidt Orthogonalisation Process

Suppose $\{x_1, x_2, \dots, x_n\}$ forms a basis set for inner product space V.One can use to construct an orthogonal basis $\{y_1, y_2, \dots, y_n\}$ as follows;

$$y_1 = x_1$$

$$y_2 = x_2 - \frac{\langle x_2, y_1\rangle}{\langle y_1, y_1\rangle} y_1$$

$$y_3 = x_3 - \frac{\langle x_3, y_1\rangle}{\langle y_1, y_1\rangle} y_1 - \frac{\langle x_3, y_2\rangle}{\langle y_2, y_2\rangle} y_2$$

$$y_4 = x_4 - \frac{\langle x_4, y_1\rangle}{\langle y_1, y_1\rangle} y_1 - \frac{\langle x_4, y_2\rangle}{\langle y_2, y_2\rangle} y_2 - \frac{\langle x_3, y_3\rangle}{\langle y_3, y_3\rangle} y_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$y_n = x_n - \frac{\langle x_n, y_1\rangle}{\langle y_1, y_1\rangle} y_1 - \frac{\langle x_n, y_2\rangle}{\langle y_2, y_2\rangle} y_2 - \cdots - \frac{\langle x_n, y_{n-1}\rangle}{\langle y_{n-1}, y_{n-1}\rangle} y_{n-1}$$

In need for orthonormal set find the norms of $\{y_1, y_2, \dots, y_k\}$.

Example

$x_1 = (1,1,1,1), x_2 = (1,2,4,5)$

$y_1 = x_1 = (1,1,1,1)$

$$y_2 = x_2 - \frac{\langle x_2, y_1\rangle}{\langle y_1, y_1\rangle} y_1 = (1,2,4,5) - \frac{12}{4}(1,1,1,1) = (-2,-1,1,2)$$

For orthonormal

$$u_1 = \frac{y_1}{\|y_1\|} = \frac{1}{2}(1,1,1,1)$$

FUNDAMENTAL THEOREMS FOR NORMED AND BANACH SPACES

This chapter contains, roughly speaking, the basis of the more advanced theory of normed and Banach spaces without which the usefulness of these spaces and their applications would be rather limited. The four important theorems in the chapter are the Hahn-Banach theorem, the uniform bounded ness theorem, the open mapping theorem, and the closed graph theorem. These are the cornerstones of the theory of Banach spaces. (The first theorem holds for any normed space.)

Brief orientation about main content

- **Hahn-Banach theorem**
  This is an extension theorem for linear functionals on vector spaces. It guarantees that a normed space is richly supplied with linear functionals, so that one obtains an adequate theory of dual spaces as well as a satisfactory theory of adjoint operators.
- **Uniform boundedness theorem** by Banach and Steinhaus.
  This theorem gives conditions sufficient for $(\|T_n\|)$ to be bounded, where the $T_{n's}$ are bounded linear operators from a Banach into a normed space. It has various (simple and deeper) applications in analysis, for instance in connection with Fourier series, weak convergence, summability of sequences, numerical integration, etc.
- **Open mapping theorem**
  This theorem states that a bounded linear operator T from a Banach space onto a Banach space is an open mapping, that is, maps open sets onto open sets. Hence if T is bijective, T^{-1} is continuous ("bounded inverse theorem ").
- **Closed graph theorem**
  This theorem gives conditions under which a closed linear operator is bounded. Closed linear operators are of importance in physical and other applications.

Definitions

- A **partially ordered set** is a set M on which there is defined a partial ordering, that is, a binary relation which is written $\leq$ and satisfies the conditions
  a $\leq$ a for every $a \in M$. (Reflexivity)
  If a $\leq$ band b $\leq$ a, then a = b. (Antisymmetry)
  If a $\leq$ band b $\leq$ c, then a $\leq$ c. (Transitivity)

"Partially"emphasizes that M may contain elements a and b for which neither $a \leq b$ nor $b \leq a$ holds. Then a and b are called ***incomparable elements***. In contrast, two elements a and b are called ***comparable elements*** if they satisfy a $\leq$ b or b $\leq$ a (or both).

- A **totally ordered set** or chain is a partially ordered set such that every two elements of the set are comparable. In other words, a chain is a partially ordered set that has no incomparable elements.
- An **upper bound** of a subset W of a partially ordered set M is an element $u \in M$ such that $x \leq u$ for every $x \in W$.

Finite Functional

Let V be a linear space, a functional $p: V \to F$ is said to be finite if $p(x)$ is finite for all $x \in V$.

Convex Functional

Let V be a linear space, a functional $p: V \to F$ is said to be convex (or Semi norm) if

- $p(x) \geq 0$ for all $x \in V$
- $p(ax) = ap(x)$ for all $x \in V$ and real $a \geq 0$
- $p(x + y) \leq p(x) + p(y)$ for all $x, y \in V$

Zorn's Lemma

Let $M \neq \varphi$ be a partially ordered set. Suppose that every chain $C \subset M$ has an upper bound. Then M has at least one maximal element.

Hahn-Banach Theorem

The Hahn-Banach theorem is an extension theorem for linear functionals. It guarantees that a normed space is richly supplied with bounded linear functionals and makes possible an adequate theory of dual spaces, which is an essential part of the general theory of normed spaces. In this way the Hahn-Banach theorem becomes one of the most important theorems in connection with bounded linear operators. Furthermore, our discussion will show that the theorem also characterizes the extent to which values of a linear functional can be preassigned. The theorem was discovered by H. Hahn (1927), rediscovered in its present more general form (Theorem 4.2-1) by S. Banach (1929) and generalized to complex vector spaces by H. F. Bohnenblust and A.

Hahn – Banach Theorem for Real Spaces (Extension of linear functionals)

Let X be a real vector space and p a sub – linear (convex) functional on X. Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies $f(x) \leq p(x)$ for all $x \in Z$. Then f has a linear extension $\tilde{f}$ from Z to X satisfying $\tilde{f}(x) \leq p(x)$ for all $x \in X$.

That is, f is a linear functional on X, satisfies $\tilde{f}(x) \leq p(x)$ on X

And $f(x) = \tilde{f}(x)$ for every $x \in Z$.

Proof

We can suppose that $Z \neq X$ otherwise the theorem is trivial.

Step – I

In this part we shall prove that f can be extended onto a larger subspace without violating condition $f(x) \leq p(x)$ for all $x \in Z$.

Let $z \in X/Z$ and put $V = \{x + \propto z : x \in Z, \propto \in R\}$ then V is subspace of X and contains Z properly.

Define a function $f' : V \to R$ by $f'(x + \propto z) = f(x) + \propto f'(z) = f(x) + \propto c$ where $c = f'(z)$.

Clearly f' is a linear functional on V.

We show that it is possible to choose a real number c such that the 'majorisation' condition $f'(x+\propto z) \leq p(x+\propto z)$ is satisfied.

That is there exists a real number c such that

$f(x)+\propto c \leq p(x+\propto z)$

$\Rightarrow f\left(\frac{x}{\propto}\right) + c \leq p\left(\frac{x}{\propto} + z\right)$

$\Rightarrow c \leq p\left(\frac{x}{\propto} + z\right) - f\left(\frac{x}{\propto}\right)$ if $\propto > 0$

And

$\Rightarrow f\left(\frac{x}{\propto}\right) + c \geq -\left(-\frac{1}{\propto}\right) p(x+\propto z)$

$\Rightarrow f\left(\frac{x}{\propto}\right) + c \geq -p\left(-\frac{x}{\propto} - z\right)$

$\Rightarrow c \geq -p\left(-\frac{x}{\propto} - z\right) - f\left(\frac{x}{\propto}\right)$ if $\propto < 0$

Now for any arbitrary points y', y'' of Z we have

$f(y'') - f(y') = f(y'' - y') \leq p(y'' - y')$

$f(y'') - f(y') \leq p\big(y'' + z - (y' + z)\big)$

$f(y'') - f(y') \leq p\big(y'' + z + (-y' - z)\big)$

$f(y'') - f(y') \leq p(y'' + z) + p(-y' - z)$

Hence $-f(y') - p(-y' - z) \leq p(y'' + z) - f(y'')$

$Sup_{y' \in Z}\{-f(y') - p(-y' - z)\} \leq inf_{y'' \in Z}\{p(y'' + z) - f(y'')\}$ (Say)

$c' \leq c''$ Say for any arbitrary y', y''

Now choose c such that $c' \leq c \leq c''$

Then for this value of c the linear functional f' define on V by

$f'(x+\propto z) = f(x)+\propto c$

Satisfies the condition that $f'(x) \leq p(x)$ for all $x \in V$.

Because conditions $c \leq p\left(\frac{x}{\propto} + z\right) - f\left(\frac{x}{\propto}\right)$ and $c \geq -p\left(-\frac{x}{\propto} - z\right) - f\left(\frac{x}{\propto}\right)$ are satisfied.

Hence f' is an extension of f to a subspace V containing Z properly and satisfying condition $f(x) \leq p(x)$ for all $x \in Z$.

Step – II

Now suppose that X as a linear space, is generated by a countable set of elements $x_1, x_2, x_3, \dots, x_n, \dots$ in X. Then we construct a linear functional on X by induction on n. that is we construct a sequence of subspaces

$V_1 = \langle x_1, Z\rangle \subseteq V_2 = \langle x_2, V_2\rangle \subseteq \cdots \subseteq V_n = \langle x_n, V_{n-1}\rangle \dots$

This process extend the functional f onto the whole space X, since every x in X is in some subspace V_n.

Step – III

For the general case, that is, when no countable set generates X, the theorem is proved by applying Zorn's Lemma as follows;

Let F be the class of all possible extensions $\tilde{f}$ of f satisfying the condition

$\tilde{f}(x) \leq p(x)$ for all $x \in \mathcal{D}(\tilde{f})$ and $\tilde{f}(x) = f(x)$ for every $x \in \mathcal{D}(f)$.

Then F is non – empty because f' constructed above is in F.

We partially order F as follows;

For $f^*, g \in F$ we say that $f^* \leq g$ if and only if g is an extension of f^*.

That is $\mathcal{D}(f^*) \subseteq \mathcal{D}(g)$ and $g(x) = f^*(x)$ for every $x \in \mathcal{D}(f^*)$.

Now let C be a chain in F. Define a linear functional $\bar{f}$ as follows;

i. Domain of $\bar{f} = \cup_{g \in C} \mathcal{D}(g)$
ii. $\bar{f}(x) = g(x)$ for every $x \in \mathcal{D}(\bar{f}), x \in \mathcal{D}(g);\ g \in C$

It is clear that $\bar{f}$ is a linear extension of f and $\bar{f}(x) \leq p(x)$ for all $x \in \mathcal{D}(\bar{f})$.

So $\bar{f} \in F$ and is an upper bound for C.

By Zorn's Lemma, F has a maximal element f^* which is an extension of f and $f^*(x) \leq p(x)$ for all $x \in \mathcal{D}(f^*)$.

We claim that $\mathcal{D}(f^*) = X$. For otherwise, let $z \in X/\mathcal{D}(f^*)$ then as in step I, there is an extension f' of f^* to $\langle \mathcal{D}(f^*), z \rangle$, contradicting the maximality of f^*.

Hence f^* is the required extension of f and this prove the theorem completely.

Hahn – Banach Theorem for Real Spaces (Dr. AbdulMajeed)

Let p be a finite convex functional defined on a real linear space V and let U be a subspace of V. Let $f_0 : U \to \mathbf{R}$ be a linear functional such that

$$f_0(x) \leq p(x) \text{ for all } x \in U \tag{1}$$

Then f_0 can be extended to a linear functional f defined on V such that

$$f(x) \leq p(x) \text{ for all } x \in V$$

Proof

We can suppose that $U \neq V$, for otherwise the theorem is trivial.

Step I:

We shall first prove that f_0 can be extended onto a larger subspace without violating condition (1).

Let $z \in V \setminus U$ and put

$$V_1 = \{x + \alpha z : x \in U, \alpha \in \mathbf{R}\}.$$

Then V_1 is a subspace of V and contains U properly.

Define a function $f' : V_1 \to R$ by;

$$\begin{aligned} f'(x + \alpha z) &= f_0(x) + \alpha f'(z). \\ &= f_0(x) + \alpha c,\ c = f'(z). \end{aligned} \tag{2}$$

Then f' is a linear functional on V_1.

We show that it is possible to choose a real number c such that the 'majorisation' condition

$$f'(x + \alpha z) \leq p(x + \alpha z)$$

is satisfied. That is, there exists a real number c such that

$$f_0(x) + \alpha c \leq p(x + \alpha z)$$

i.e. $$f_0\left(\frac{x}{\alpha}\right) + c \leq p(x/\alpha + z)$$

i.e. $$c \leq p(x/\alpha + z) - f_0(x/\alpha) \tag{3}$$

if $\alpha > 0$, and

$$f_0(x/\alpha) + \quad c \geq -\left(\frac{1}{-\alpha}\right)p(x + \alpha z) = -p(-x/\alpha - z)$$

$$c \geq -p(-x/\alpha - z) - f_0(x/\alpha) \tag{4}$$

if $\alpha < 0$.

Now, for any two arbitrary points y', y'' of U, we have:

$$\begin{aligned} f_0(y'') - f_0(y') = f_0(y'' - y') &\leq p(y'' - y') \\ &\leq p(y'' + z - (y' + z)) \\ &\leq p(y'' + z + (-y' - z)) \\ &\leq p(y'' + z) + p(-y' - z) \end{aligned}$$

Hence

$$-f_0(y') - p(-y' - z) \leq p(y'' + z) - f_0(y'') \tag{5}$$

Put $$c' = \operatorname*{Sup}_{y' \in U} \{-f_0(y') - p(-y' - z)\}$$

$$c'' = \operatorname*{Inf}_{y'' \in U} \{p(y'' + z) - f_0(y'')\}$$

Then $\qquad c' \leq c''$

by (5) and the fact that y', y'' are arbitrary.

Now choose a c such that

$$c' \leq c \leq c''$$

Then, for this value of c, the linear functional f' defined on V_1 by (2) satisfies the condition that

$$f'(x) \leq p(x) \text{ for all } x \in V_1 \qquad (6)$$

because conditions (3) and (4) are satisfied. Hence f' is an extension of f_0 to a subspace V_1 containing U properly and satisfying condition (1).

Step II

Now suppose that V, as a linear space, is generated by a countable set of elements $x_1, x_2, \ldots, x_n, \ldots$, in V. Then we construct a linear functional on V by induction on n. That is, we construct a sequence of subspaces

$$V_1 = \langle x_1, U \rangle, V_2 = \langle x_2, V_1 \rangle, \ldots,$$
$$V_n = \langle x_n, V_{n-1} \rangle, \ldots$$

each contained in the next. This process extends the functional f_0 onto the whole space V, since every x in V is in some subspace V_n.

Step III

For the general case, that is, when no countable set generates V, the theorem is proved by applying Zorn's lemma as follows:

Let F be the class of all possible extensions f^* of f_0 satisfying the condition.

$$f^*(x) \leq p(x) \text{ for all } x \in D(f^*)$$

and $\qquad f^*(x) = f_0(x) \text{ for all } x \in D(f_0).$

Here $D(f_0)$ denotes the domain of f_0. Then F is non-empty because f' constructed above is in F. We partially order F as follows:

Let F be the class of all possible extensions f^* of f_0 satisfying the condition.

$$f^*(x) \leq p(x) \text{ for all } x \in D(f^*)$$

and $$f^*(x) = f_0(x) \text{ for all } x \in D(f_0).$$

Here $D(f_0)$ denotes the domain of f_0. Then F is non-empty because f' constructed above is in F. We partially order F as follows:

For $f, g \in F$, we say that

$$f \leq g$$

if and only if g is an extension of f, that is

$$D(g) \supseteq D(f)$$

and $$g(x) = f(x) \text{ for all } x \in D(f).$$

Now let C be a chain in F. Define a linear functional $\bar{f}$ as follows:

(i) Domain of $\bar{f} = \bigcup\limits_{g \in C} D(g)$,

(ii) for $x \in D(\bar{f})$,

$$\bar{f}(x) = g(x) \text{ for all } x \in D(g), g \in C$$

It is clear that $\bar{f}$ is a linear extension of f_0 and

$$\bar{f}(x) \leq p(x)$$

for all $x \in D(\bar{f})$. So $\bar{f} \in F$ and is an upper bound for C. By Zorn's lemma, F has a maximal element f which is an extension of f_0 and

$$f(x) \leq p(x)$$

for all $x \in D(f)$. We claim that $D(f) = V$, for otherwise, let $z \in V \backslash D(f)$. Then, as in step I, there is an extension f' of f to $<D(f), z>$, contradicting the maximality of f. Hence f is the required extension of f_0. This proves the theorem completely.

Hahn-Banach Theorem (Generalized) for Real Vector Spaces

The Hahn-Banach theorem concerns real vector spaces. A generalization that includes complex vector spaces was obtained by H.F.Bohnenblust and A.Sobczyk (1938):

Statement

Let X be a real vector space and p a real-valued functional on X which is subadditive, that is,

for all $x, y \in X,\ p(x+y) \leq p(x) + p(y)$

and for every scalar a satisfies $p(ax) = |a|p(x)$.

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies $|f(x)| \leq p(x)$ for all $x \in Z$. Then f has a linear extension $\tilde{f}$ from Z to X satisfying $\left|\tilde{f}(x)\right| \leq p(x)$ for all $x \in X$.

Proof

If X is real, the situation is simple.

Then $|f(x)| \leq p(x)$ for all $x \in Z$ implies $f(x) \leq p(x)$ for all $x \in Z$.

Hence by the Hahn-Banach theorem there is a linear extension $\tilde{f}$ from Z to X such that

$\tilde{f}(x) \leq p(x)$ for all $x \in X$. …………..(1)

From this and $p(ax) = |a|p(x)$ we obtain

$-\tilde{f}(x) = \tilde{f}(-x) \leq p(-x) = |-1|p(x) = p(x)$

$\Rightarrow \tilde{f}(x) \geq -p(x)$ …………..(2)

From (1) and (2) $\quad -p(x) \leq \tilde{f}(x) \leq p(x)$

Hence $\left|\tilde{f}(x)\right| \leq p(x)$ for all $x \in X$.

Hahn-Banach Theorem (Generalized) for Complex Vector Spaces

The Hahn-Banach theorem concerns real vector spaces. A generalization that includes complex vector spaces was obtained by H.F.Bohnenblust and A.Sobczyk (1938):

Statement

Let X be a complex vector space and p a real-valued functional on X which is subadditive, that is,

for all $x, y \in X,\ p(x+y) \leq p(x) + p(y)$

and for every scalar a satisfies $p(ax) = |a|p(x)$.

Furthermore, let f be a linear functional which is defined on a subspace Z of X and satisfies $|f(x)| \leq p(x)$ for all $x \in Z$. Then f has a linear extension $\tilde{f}$ from Z to X satisfying $\left|\tilde{f}(x)\right| \leq p(x)$ for all $x \in X$.

Proof

Let X be complex. Then Z is a complex vector space, too. Hence f is complex-valued, and we can write $f(x) = f_1(x) + if_2(x)$ for $x \in X$ where f_1 and f_2 are real-valued. For a moment we regard X and Z as real vector spaces and denote them by X_r and Z_r, respectively; this simply means that we restrict multiplication by scalars to real numbers (instead of complex numbers). Since f is linear on Z and f_1 and f_2 are real-valued, linear functionals on Z_r, Also $f_1(x) \leq |f(x)|$ because the real part of a complex number cannot exceed the absolute value.Hence by $|f(x)| \leq p(x)$ for all $x \in Z$, we have

$f_1(x) \leq p(x)$ for all $x \in Z_r$

By the Hahn-Banach theorem there is a linear extension $\tilde{f}_1$ of f_1 from Z_r to X_r such that $\tilde{f}_1(x) \leq p(x)$ for all $x \in X_r$

This takes care of f_1 and we now turn to f_2. Returning to Z and using $f = f_1 + if_2$ we have for every $x \in Z$

$i[f_1(x) + if_2(x)] = if(x) = f_1(ix) + if_2(ix)$

The real parts on both sides must be equal: $f_2(x) = -f_1(ix)$ for $x \in Z$

Hence if for all $x \in X$ we set $\tilde{f}(x) = \tilde{f}_1(x) - i\tilde{f}_1(ix)$ for $x \in X$

We see from $f_2(x) = -f_1(ix)$ that $\tilde{f}(x) = f(x)$ on Z.

This shows that $\tilde{f}$ is an extension of f from Z to X. Our remaining task is to prove that

i. $\tilde{f}$ is a linear functional on the complex vector space X
ii. $\tilde{f}$ satisfies $|\tilde{f}(x)| \leq p(x)$ on X.

That (i) holds can be seen from the following calculation which uses

$$\tilde{f}(x) = \tilde{f}_1(x) - i\tilde{f}_1(ix)$$

and the linearity of $\tilde{f}_1$ on the real vector space X_r; here $a + ib$ with real a and b is any complex scalar:

$$\tilde{f}\big((a+ib)x\big) = \tilde{f}_1\big((a+ib)x\big) - i\tilde{f}_1(i(a+ib)x)$$

$$\tilde{f}\big((a+ib)x\big) = a\tilde{f}_1(x) + b\tilde{f}_1(ix) - i\big[a\tilde{f}_1(ix) - b\tilde{f}_1(x)\big]$$

$$\tilde{f}\big((a+ib)x\big) = (a+ib)\big[\tilde{f}_1(x) - i\tilde{f}_1(ix)\big]$$

$$\tilde{f}\big((a+ib)x\big) = (a+ib)\tilde{f}(x)$$

We prove (ii). For any x such that $\tilde{f}(x) = 0$ this holds since $p(x) \geq 0$ by $p(x+y) \leq p(x) + p(y)$ and $p(ax) = |a|p(x)$.

Let x be such that $\tilde{f}(x) \neq 0$, Then we can write, using the polar form of complex quantities, $\tilde{f}(x) = |\tilde{f}(x)|e^{i\theta}$ thus $|\tilde{f}(x)| = \tilde{f}(x)e^{-i\theta} = \tilde{f}\big(e^{-i\theta}x\big)$. Since $|\tilde{f}(x)|$ is real, the last expression is real and thus equal to its real part. Hence by $p(ax) = |a|p(x)$ we have

$$|\tilde{f}(x)| = \tilde{f}\big(e^{-i\theta}x\big) = \tilde{f}_1\big(e^{-i\theta}x\big) \leq p\big(e^{-i\theta}x\big) = |e^{-i\theta}|p(x) = p(x)$$

This completes the proof.

Hahn-Banach Theorem for Complex Vector Spaces (Dr.AbdulMajeed)

Let p be a finite convex functional defined on a complex linear space V and let U be a subspace of V. Let f_0 be a linear functional defined on U satisfying the condition:

$$|f_0(x)| \leq p(x) \text{ for all } x \in U \qquad (1)$$

Then f_0 can be extended to a linear functional f on V such that

$$|f(x)| \leq p(x) \text{ for all } x \in V$$

Proof

Since V is a complex linear space, for each $v \in V$ and $\alpha = \alpha_1 + i\alpha_2 \in \mathbf{C}$, $\alpha v \in V$. If we restrict the scalars to real numbers only then V is a real linear space. Denote this space by $V_R(=V)$ and the corresponding subspace by $U_R(=U)$. Clearly p is a finite convex functional defined on $V_{\mathbf{R}}$ while f'_0 given by:

$$f_0'(x) = \text{real part of } f_0(x), x \in U_{\mathbf{R}}$$

is a real linear functional on $U_{\mathbf{R}}$. Hence, by the Hahn-Banach theorem for real spaces, there is a linear extension f_1 defined on all of $V_{\mathbf{R}}$ satisfying the conditions:

$$f_1(x) \leq p(x) \text{ for all } x \in V_{\mathbf{R}} \ (=V) \qquad (2)$$

and $\quad f_1(x) = f_0'(x)$ for all $x \in U_{\mathbf{R}}$ $(=U)$

Also, $\quad -f_1(x) = f_1(-x) \leq p(-x) = |-1|\, p(x) = p(x)$

Thus $\quad f_1(x) \geq -p(x)$ for all $x \in V_{\mathbf{R}}$ $\qquad (3)$

so that, from (2) and (3) one gets:

$$|f_1(x)| \leq p(x) \text{ for all } x \in V_{\mathbf{R}} \qquad (4)$$

Now we consider f_0 as a linear functional on the complex space U. So

$$f_0(x) = f_0'(x) + i\, f_0''(x) \qquad (5)$$

Since f_0 is linear on U,

$$i f_0(x) = f_0(ix) = f_0'(ix) + i f_0''(ix) \tag{6}$$

and also, multiplying (5) by i,

$$i f_0(x) = -f_0''(x) + i f_0'(x) \tag{7}$$

so that, comparing (6) and (7), we have

$$f_0''(x) = -f_0'(ix)$$

Hence $$f_0(x) = f_0'(x) - i f_0'(ix) \tag{8}$$

If f_1 denotes the linear extension of f_0' to the whole of V, as a real linear space, then put

$$f(x) = f_1(x) - i f_1(ix) \tag{9}$$

We show that the function f defined by (9) is the required linear extension of f_0 to V and satisfies the given condition.

Obviously f is an extension of f_0 to the whole of V. Also

$$\begin{aligned} f(x+y) &= f_1(x+y) - i f_1(i(x+y)) \\ &= f_1(x) + f_1(y) - i f_1(ix+iy) \\ &= f_1(x) + f_1(y) - i f_1(ix) - i f_1(iy) \\ &= f(x) + f(y) \end{aligned} \tag{10}$$

Finally we show that

$$|f(x)| \leq p(x) \text{ for all } x \in V.$$

Suppose, on the contrary, that $|f(x_0)| > p(x_0)$ for some $x_0 \in V$. Then, writing

$$f(x_0) = \rho e^{i\varphi}, \ \rho > 0,$$

if we put

$$y_0 = e^{-i\varphi} x_0$$

then $y_0 \in V$ and, using $|f(x_0)| = \rho$, we have

$$f_1(y_0) = Re\, f(y_0) = Re\,(e^{-i\varphi} f(x_0))$$

$$= \rho > p(x_0) = p(y_0)$$

which contradicts (4). Hence

$$|f(x)| \leq p(x) \text{ for all } x \in V$$

This complete the proof of theorem.

Hahn-Banach Theorem (Normed Spaces).

Let f be a bounded linear functional on a subspace Z of a normed space X. Then there exists a bounded linear functional $\tilde{f}$ on X which is an extension of f to X and has the same norm, i.e. $\|\tilde{f}\|_X = \|f\|_Z$

Where $\|\tilde{f}\|_X = Sup_{\substack{x\in X \\ \|x\|=1}} |\tilde{f}(x)|$ and $\|f\|_Z = Sup_{\substack{x\in Z \\ \|x\|=1}} |f(x)|$

(and $\|f\|_Z = 0$ in the trivial case $Z = \{0\}$).

Proof: If $Z = \{0\}$, then $f = 0$, and the extension is $\tilde{f} = 0$.

Let $Z \neq \{0\}$. We want to use **Hahn-Banach Theorem (Generalized)**. Hence we must first discover a suitable p.

For all $x \in Z$ we have $|f(x)| \leq \|f\|_Z\|x\|$

Comparing with $|f(x)| \leq p(x)$ we have $p(x) = \|f\|_Z\|x\|$

We see that p is defined on all of X.

Furthermore, p satisfies $p(x+y) \leq p(x) + p(y)$ on X since by the triangle inequality,

$p(x+y) = \|f\|_Z\|x+y\| \leq \|f\|_Z(\|x+y\|) \leq p(x) + p(y)$

Also $p(ax) = |a|p(x)$ because $p(ax) = \|f\|_Z\|ax\| = |a|\|f\|_Z\|x\| = |a|p(x)$

Hence we can now apply **Hahn-Banach Theorem (Generalized)** and conclude that there exists a linear functional $\tilde{f}$ on X which is an extension of f and satisfies

$|\tilde{f}(x)| \leq p(x) = \|f\|_Z\|x\|$

Taking the supremum over all $x \in X$ of norm 1, we obtain the inequality

$\|\tilde{f}\|_X = Sup_{\substack{x\in X \\ \|x\|=1}} |\tilde{f}(x)| \leq \|f\|_Z$

Since under an extension the norm cannot decrease, we also have $\|\tilde{f}\|_X \geq \|f\|_Z$. Together we obtain $\|\tilde{f}\|_X = \|f\|_Z$ and the theorem is proved.

Hahn-Banach Theorem for Normed Spaces (Dr. AbdulMajeed).

Let V be a normed space and U be a subspace of V. Let f_0 be a bounded linear functional on U with norm $\|f_0\|$. Then f_0 has a continuous linear extension f defined on V such that

$$\|f\| = \|f_0\|$$

Proof

Since f_0 is a bounded linear functional, $\|f_0\|$ is finite. Put

$$p(x) = \|f_0\| \; \|x\| \quad \text{for all } x \in V.$$

We show that p is a convex functional defined on V.

Clearly $p(x) \geq 0$. Also for any $\alpha \in F$,

$$p(\alpha x) = \|f_0\| \; \|\alpha x\| = |\alpha| \; \|f_0\| \; \|x\| = |\alpha| \, p(x), x \in V.$$

Moreover, for $x, y \in V$,

$$\begin{aligned} p(x + y) &= \|f_0\| \; \|x + y\| \\ &\leq \|f_0\| \; (\|x\| + \|y\|) \\ &\leq \|f_0\| \; \|x\| + \|f_0\| \; \|y\| \\ &\leq p(x) + p(y). \end{aligned}$$

Also $\quad |f_0(x)| \leq \|f_0\| \; \|x\|$

$$\leq p(x).$$

Thus, by the complex version of the Hahn-Banach Theorem, there is a linear functional f defined on V such that

$$|f(x)| \leq p(x) = \|f_0\|\,\|x\| \text{ for all } x \in V$$

$$\text{and} \qquad f(x) = f_0(x) \text{ for all } x \in U. \qquad (1)$$

From (1) we have:

$$\|f\| \leq \|f_0\| \qquad (2)$$

$$\text{Also} \qquad \|f\| = \sup_{\substack{x \neq 0 \\ x \in V}} \frac{|f(x)|}{\|x\|} \geq \sup_{\substack{x \neq 0 \\ x \in U}} \frac{|f_0(x)|}{\|x\|} = \|f_0\| \qquad (3)$$

Hence, from (2) and (3)

$$\|f\| = \|f_0\|$$

This completes the proof of the theorem for normed spaces.

Corrollary

Let N be a non-trivial normed space and $x_0 \neq 0$ be any point of N. Then there is a continuous (and so bounded) linear functional f defined on N such that

$$\|f\| = 1 \text{ and } f(x_0) = \|x_0\|$$

Proof

Let $0 \neq x_0 \in N$. Consider the subspace M generated by x_0. An arbitrary element of M is of the form ax_0, $a \in F$. Define a functional $f_0 : M \to F$ by:

$$f_0(y) = f_0(ax_0) = a\,\|x_0\|,\ y = ax_0 \in M, a \in F \qquad (1)$$

Then f_0 is linear because for y and $y' = a'x_0$ in M and α, α' in F, we have:

$$\begin{aligned} f_0(\alpha y + \alpha' y') &= f_0((\alpha a + \alpha' a')\,x_0) \\ &= (\alpha a + \alpha' a')\,\|x_0\| \ \text{ by (1)} \\ &= \alpha a\,\|x_0\| + \alpha' a'\,\|x_0\| \\ &= \alpha f_0(y) + \alpha' f_0(y') \end{aligned}$$

Also $\|f_0\| = \sup\limits_{\substack{y \neq 0 \\ y \in M}} \dfrac{|f_0(y)|}{\|y\|} = \sup\limits_{a \in F} \dfrac{|a|\,\|x_0\|}{|a|\,\|x_0\|} = 1$, $\because\ y \neq 0$ so that $a \neq 0$.

So f_0 is a bounded linear functional defined on M. By the Hahn-Banach Theorem for normed spaces, there is a linear extension f of f_0 to N such that

$$\|f\| = \|f_0\| = 1,\ f(y) = f_0(y) = a\|x_0\|, y = ax_0 \in M$$

Thus $\quad \|f\| = 1$ and $f(x_0) = \|x_0\|$

as required.

Theorem (Bounded linear functionals)

Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then there exists a bounded linear functional $\tilde{f}$ on X such that $\|\tilde{f}\| = 1\,; \tilde{f}(x_0) = \|x_0\|$

Proof

We consider the subspace Z of X consisting of all elements $x = ax_0$ where a is a scalar.

On Z we define a linear functional f by $f(x) = f(ax_0) = a\|x_0\|$

f is bounded and has norm $\|f\| = 1$ because

$|f(x)| = |f(ax_0)| = |a|\|x_0\| = \|ax_0\| = \|x\|$

Then **Hahn-Banach Theorem (Normed Spaces)** implies that f has a linear extension $\tilde{f}$ from Z to X, of norm $\|f\| = \|\tilde{f}\| = 1$.

Then From $f(x) = f(ax_0) = a\|x_0\|$ we see that; $\tilde{f}(x_0) = f(x_0) = \|x_0\|$

Corollary (Norm, zero vector)

For every x in a normed space X we have $\|x\| = Sup_{\substack{f\in X' \\ f\neq 0}} \frac{|f(x)|}{\|f\|}$.

Hence if x_0 is such that $f(x_0) = 0$ for all $f \in X'$, then $x_0 = 0$.

Proof

From **Theorem (Bounded linear functionals)** we have, writing x for x_0,

$$Sup_{\substack{f\in X' \\ f\neq 0}} \frac{|f(x)|}{\|f\|} \geq \frac{|\tilde{f}(x)|}{\|\tilde{f}\|} = \frac{\|x\|}{1} = \|x\|$$

and from $|f(x)| \leq \|f\|\|x\|$ we obtain $Sup_{\substack{f\in X' \\ f\neq 0}} \frac{|f(x)|}{\|f\|} \leq \|x\|$

Hence $\|x\| = Sup_{\substack{f\in X' \\ f\neq 0}} \frac{|f(x)|}{\|f\|}$

Uniform Boundedness Theorem/ Uniform Boundedness Principle

The uniform boundedness theorem (or uniform boundedness principle) by S. Banach and H. Steinhaus (1927) is of great importance. In fact, throughout analysis there are many instances of results related to this theorem, the earliest being an investigation by H. Lebesgue (1909).

The uniform boundedness theorem is often regarded as one of the corner stones of functional analysis in normed spaces, the others being the Hahn-Banach theorem, the open mapping theorem and the closed graph theorem. Unlike the Hahn-Banach theorem, the other three of these four theorems require completeness. Indeed, they characterize some of tl1e most important properties of Banach spaces which normed spaces in general may not have.

It is quite interesting to note that we shall obtain all three theorems from a common source. More precisely, we shall prove the so-called Baire's category theorem and derive from it the uniform boundedness theorem as well as the open mapping theorem. The latter will then readily entail the closed graph theorem.

Baire's category theorem has various other applications in functional- analysis and is the main reason why category enters into numerous proofs; for instance, the more advanced books by R. E. Edwards (1965) and J. L. Kelley and I. Namioka (1963).

Definition (Category). A subset M of a metric space X is said to be

(a) **Rare (or nowhere dense)** in X if its closure $\bar{M}$ has no interior points
(b) **Meager (or of the first category)** in X if M is the union of countably many sets each of which is rare in X,
(c) **Nonmeager (or of the second category)** in X if M is not meager in X.

Bair's Category Theorem

If $X \neq \varphi$ is complete then it is non-meager in itself.

Or A complete metric space is of second category.

From Baire's theorem we shall now readily obtain the desired uniform boundedness theorem. This theorem states that if X is a Banach space and a sequence of operators $T_n \in (X, Y)$ is bounded at every point $x \in X$, then the sequence is uniformly bounded. In other words, pointwise boundedness implies boundedness in some stronger sense, namely, uniform boundedness.

Uniform Boundedness Theorem (Banach Steinhaus Theorem)

Let (T_n) be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y such that $(\|T_n x\|)$ is bounded for every $x \in X$, say,

$$\|T_n x\| \leq c_x \qquad\qquad n = 1,2,\cdots$$

where c_x is a real number. Then the sequence of the norms $\|T_n\|$ is bounded, that is, there is a c such that

$$\|T_n\| \leq c \qquad\qquad n = 1,2,\cdots$$

Proof

For every $k \in N$, let $A_k \subset X$ be the set of all x such that

$$\|T_n x\| \leq k \qquad\qquad \text{for all n.}$$

A_k is closed.

Indeed, for any $x \in \bar{A}_k$ there is a sequence (x_j) in A_k converging to x.

This means that for every fixed n we have $\|T_n x_j\| \leq k$ and obtain $\|T_n x\| \leq k$ because T_n is continuous and so is the norm.

Hence $x \in A_k$, and A_k is closed.

By $\|T_n x\| \leq c_x$, each $x \in X$ belongs to some A_k.

Hence $\qquad X = \cup_{k=1}^{\infty} A_k$

Since X is complete, Baire's theorem implies that some A_k contains an open ball, say,

$B_o = B(x_o;\, r) \subset A_{k_o}$

Let $0 \neq x \in X$ be arbitrary. We set

$z = x_o + \gamma x$ with $\gamma = \frac{r}{2\|x\|}$

Then $\|z - x_o\| < r$, so that $z \in B_o$. By ($B_o = B(x_o;\, r) \subset A_{k_o}$) and from the definition of A_{k_o} we thus have $\|T_n z\| \leq k_o$ for all n.

Also $\|T_n x_o\| \leq k_o$ since $x_o \in B_o$.

From ($z = x_o + \gamma x$) we obtain $x = \frac{1}{\gamma}(z - x_o)$

$\Rightarrow \|T_n x\| = \frac{1}{\gamma}\|T_n(z - x_o)\|$

$\Rightarrow \|T_n x\| \leq \frac{1}{\gamma}(\|T_n z\| + \|T_n x_o\|) \leq \frac{4}{r}\|x\| k_o$ with $\gamma = \frac{r}{2\|x\|}$

Hence for all n, $\|T_n\| = Sup_{\|x\|=1}\|T_n x\| \leq \frac{4}{r} k_o$

$\Rightarrow \|T_n\| \leq c$ $n = 1,2,\cdots$

Where $c = \frac{4}{r} k_o$

Applications of Uniform Boundedness Theorem

- **Space of polynomials.** The normed space X of all polynomials with norm defined by $\|x\| = max_j|\propto_j|$ is not complete.
  ($\propto_0, \propto_1, \ldots$ the coefficients of x)

Proof: We construct a sequence of bounded linear operators on X which satisfies ($\|T_n x\| \le c_x$) but not ($\|T_n\| \le c$), so that X cannot be complete.

We may write a polynomial $x \neq 0$ of degree N_x in the form

$x(t) = \sum_{j=0}^{\infty} \propto_j t^j$ ($\propto_j = 0$ for $j > N_x$)

As a sequence of operators on X we take the sequence of functionals $T_n = f_n$ defined by $T_n 0 = f_n(0) = 0$ and $T_n x = f_n(x) = \propto_0 + \propto_1 + \cdots + \propto_{n-1}$

f_n is linear. f_n is bounded since $|\propto_j| \le \|x\|$ by ($\|x\| = max_j|\propto_j|$),

So that $|f_n(x)| \le n\|x\|$.

Furthermore, for each fixed $x \in X$ the sequence ($|f_n(x)|$) satisfies ($\|T_n x\| \le c_x$) because a polynomial x of degree N_x has N_{x+1} coefficients, so that by

$T_n 0 = f_n(0) = 0$ and $T_n x = f_n(x) = \propto_0 + \propto_1 + \cdots + \propto_{n-1}$

we have $|f_n(x)| \le N_{x+1}\|x\|$

Which is of the form ($\|T_n x\| \le c_x$).

We now show that (f_n) does not satisfy ($\|T_n\| \le c$), that is, there is no c such that $\|T_n\| = \|f_n\| \le c$ for all n. This we do by choosing particularly disadvantageous polynomials. For f_n we choose x defined by

$x(t) = 1 + t + t^2 + \cdots + t^n$

Then $\|x\| = 1$ by $\|x\| = max_j|\propto_j|$ and $f_n(x) = 1 + 1 + \cdots + 1 = n = n\|x\|$

Hence $\|f_n\| \ge \frac{|f_n(x)|}{\|x\|} = n$, so that ($\|f_n\|$) is unbounded.

- **Fourier series:** There exist real-valued continuous functions whose Fourier series diverge at a given point t_0.

Proof: Let X be the normed space of all real-valued continuous functions of period 2π with norm defined by $\|x\| = max|x(t)|$.

X is a Banach space with a = 0 and $b = 2\pi$. We may take $t_0 = 0$, without restricting generality. To prove our statement, we shall apply the uniform boundedness theorem to $T_n = f_n$ where $f_n(x)$ is the value at $t = 0$ of the nth partial sum of the Fourier series of x. Since for t = 0 the sine terms are zero and the cosine is one,

We see from the followings that

the Fourier series of a given periodic function x of period 2π is of the form

$$f_n(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}(a_m \cos mt + b_m \sin mt)$$

With

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \;\;,\;\; a_m = \frac{1}{\pi}\int_0^{2\pi} x(t)Cosmtdt \;\;,\;\; b_m = \frac{1}{\pi}\int_0^{2\pi} x(t)Sinmtdt$$

That $f_n(x) = \frac{1}{2}a_0 + \sum_{m=1}^{n} a_m$

$$f_n(x) = \frac{1}{\pi}\int_0^{2\pi} x(t)\left[\frac{1}{2} + \sum_{m=1}^{n} Cosmt\right]dt$$

We want to determine the function represented by the sum under the integral sign. For this purpose we calculate

$$2Sin\frac{1}{2}t\sum_{m=1}^{n} Cosmt = \sum_{m=1}^{n} 2Sin\frac{1}{2}tCosmt$$

$$2Sin\frac{1}{2}t\sum_{m=1}^{n} Cosmt = \sum_{m=1}^{n}\left[-Sin\left(m-\frac{1}{2}\right)t + Sin\left(m+\frac{1}{2}\right)t\right]$$

$$2Sin\frac{1}{2}t\sum_{m=1}^{n} Cosmt = -Sin\frac{1}{2}t + Sin\left(n+\frac{1}{2}\right)t$$

where the last expression follows by noting that most of the terms drop out in pairs. Dividing this by $Sin\frac{1}{2}t$ and adding 1 on both sides, we have

$$1 + 2\sum_{m=1}^{n} Cosmt = \frac{Sin\left(n+\frac{1}{2}\right)t}{Sin\frac{1}{2}t}$$

Consequently, the formula for $f_n(x)$ can be written in the simple form

$$f_n(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)\frac{Sin\left(n+\frac{1}{2}\right)t}{Sin\frac{1}{2}t}dt$$

$$f_n(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)q_n(t)dt \quad \text{with } q_n(t) = \frac{Sin\left(n+\frac{1}{2}\right)t}{Sin\frac{1}{2}t}$$

Using this, we can show that the linear functional f_n is bounded. In fact, by ($\|x\| = max|x(t)|$) and ($f_n(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)q_n(t)dt$),

From this we see that f_n is bounded. Furthermore, by taking the supremum over all x of norm one we obtain

$$\|f_n\| \leq \frac{1}{2\pi}\int_0^{2\pi}|q_n(t)|dt$$

The equality sign holds, when $|q_n(t)| = y(t)q_n(t)$

where $y(t) = +1$ at every t at which $q_n(t) \geq 0$ and $y(t) = -1$ elsewhere. y is not continuous, but for any given $\epsilon > 0$ it may be modified to a continuous x of norm 1 and

$$\|f_n\| = \frac{1}{2\pi}\int_0^{2\pi}|q_n(t)|dt$$

This sequence is unbounded, so that ($\|T_n\| \leq c$) does not hold. Since X is complete, this implies that ($\|T_n x\| \leq c_x$) cannot hold for all x. Hence there must be an $x \in X$ such that ($|f_n(x)|$) is unbounded. But by the definition of the $f_{n's}$ this means that the Fourier series of that x diverges at $t = 0$.

Open Mapping Theorem

We have discussed the Hahn-Banach theorem and the uniform boundedness theorem and shall now approach the third "big" theorem in this chapter, ***the open mapping theorem.*** It will be concerned with open mappings. These are mappings such that the image of every open set is an open set (definition below). Remembering our discussion of the importance of open sets, we understand that open mappings are of general interest.

More specifically, the open mapping theorem states conditions under which a bounded linear operator is an open mapping. As in the uniform boundedness theorem we again need completeness, and the present theorem exhibits another reason why Banach spaces are more satisfactory than incomplete normed spaces.

The theorem also gives conditions under which the inverse of a hounded linear operator is bounded. The proof of the open mapping theorem will be based on Baire's category theorem.

Let us begin by introducing the concept of an open mapping.

Definition (Open mapping)

Let X and Y be metric spaces. Then $T: \mathfrak{D}(T) \rightarrow Y$ with domain $\mathfrak{D}(T) \subset X$ is called an ***open mapping*** if for every open set in $\mathfrak{D}(T)$ the image is an open set in Y.

Lemma (Open unit ball)

A bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image T(B_0) of the open unit ball $B_0 = B(0;1) \subset X$ contains an open ball about $0 \in Y$.

The Open Mapping Theorem (Dr.AbdulMajeed)

Let N and M be Banach spaces and $T : N \to M$ be a surjective continuous linear operator. Then T is an open mapping.

Proof

To show that $T : N \to M$ is an open mapping, we have to prove that, for each open set U, $T(U)$ is open in M. For this, let $y \in T(U)$. Then there is an $x \in U$ such that

$$y = Tx$$

Since U is open and $x \in U$, there is an open ball $B(x; r) \subseteq U$. If now $B_1 = B(0; 1)$,

$$B(x; r) = x + rB_1 \subseteq U$$

by lemma 13.2.2. For the open ball B_1 in N, there is an open ball $B_\varepsilon^{(1)}$ with center at the origin in M such that

$$B_\varepsilon^{(1)} \subseteq T(B_1) \subset r\, T(B_1) = T(B_r)$$

Hence $$B^{(1)}(y; r) = y + B_\varepsilon^{(1)} \subseteq y + T(B_r) = Tx + T(B_r)$$

$$= T(x + B_r) \subseteq T(U).$$

Hence $T(U)$ is open.

13.2.2 Lemma

Let T be a surjective continuous linear operator from a Banach space N to a Banach space M. Then, for each open ball $B(0; 1)$, the image $T(B(0; 1))$ contains an open ball in M with center at the origin.

Open Mapping Theorem, Bounded Inverse Theorem.

A bounded linear operator T from a Banach space X onto a Banach space Y is an open mapping. Hence if T is bijective, T^{-1} is continuous and thus bounded.

Proof:

We prove that for every open set $A \subset X$ the image $T(A)$ is open in Y. This we do by showing that for every $y = Tx \in T(A)$ the set T(A) contains an open ball about $y = Tx$.

Let $y = Tx \in T(A)$.

Since A is open, it contains an open ball with center x. Hence $A - X$ contains an open ball with center 0.

Let the radius of the ball be r and set $k = \frac{1}{r}$, so that $r = \frac{1}{k}$.

Then $k(A - x)$ contains the open unit ball $B(0; 1)$.

Using Lemma

"A bounded linear operator T from a Banach space X onto a Banach space Y has the property that the image T(B_0) of the open unit ball $B_0 = B(0; 1) \subset X$ contains an open ball about $0 \in Y$. "

Now implies that $T[k(A - x)] = k[T(A) - Tx]$ contains an open ball about 0, and so does $T(A) - Tx$.

Hence T(A) contains an open ball about $Tx = y$. Since $y = Tx \in T(A)$ was arbitrary, $T(A)$ is open.

Finally, if $T^{-1}: Y \to X$ exists, it is continuous because T is open. Also since T^{-1} is linear, it is bounded.

Hence the result.

Closed Graph Theorem

Not all linear operators of practical importance are bounded. For instance, the differential operator in 2.7-5 is unbounded, and in quantum mechanics and other applications one needs unbounded operators quite frequently. However, practically all of the linear operators which the analyst is likely to use are so-called closed linear operators. This makes it worthwhile to give an introduction to these operators. In this section we define closed linear operators on normed spaces and consider some of their properties, in particular in connection with the important closed graph theorem which states sufficient conditions under which a closed linear operator on a Banach space is bounded.

Definition (Closed Linear operator).

Let X and Y be normed spaces and $T: \mathfrak{D}(T) \to Y$ is a linear operator with domain $\mathfrak{D}(T) \subset X$.Then T is called a closed linear operator if its graph

$$\mathcal{G}(T) = \{(x, y): x \in \mathfrak{D}(T), y = Tx\}$$

is closed in the normed space $X \times Y$, where the two algebraic operations of a vector space in $X \times Y$ are defined as usual, that is

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\propto (x, y) = (\propto x, \propto y)$$

($\propto$ a scalar) and the norm on $X \times Y$ is defined by

$$\|(x, y)\| = \|x\| + \|y\|$$

Closed Graph Theorem (Dr.AbdulMajeed)

Let N and M be Banach spaces and $T : N \to M$ be a linear operator. Then T is continuous if and only if the graph of T is a closed subspace of $N \times M$.

Proof

Suppose that $T : N \to M$ is a continuous linear operator. We show that the graph

$$G_T = \{(x, Tx) : x \in N\}$$

is closed in $N \times M$. For this, let $(x, y) \in \overline{G}_T$. Then there are sequences $\{x_n\}$ and $\{y_n\}$ in N and M respectively such that

$$x_n \to x,\ y_n \to y$$

Since T is continuous and $y_n = Tx_n$,

$$x_n \to x \Rightarrow Tx_n \to Tx = y.$$

Hence $(x, y) = (x, Tx) \in G_T$. Thus G_T is closed.

Conversely suppose that, for a linear operator $T : N \to M$, G_T is closed. Then G_T is a subspace of $N \times M$. Since N and M are Banach spaces and G_T is a closed subspace of the Banach space $N \times M$, G_T itself is complete and hence is a Banach space.

Consider the mapping $f : G_T \to N$ defined by:

$$f(x, Tx) = x \quad \text{for all } x \in N.$$

Then f is injective and linear. Also, since

$$\|f(x, Tx)\| = \|x\| \le \|(x, Tx)\|$$

by definition of the product norm, f is continuous. By the open mapping theorem, f^{-1} is continuous and so bounded. Moreover

$$\|Tx\| \le \|(x, Tx)\| = \|f^{-1}(x)\| \le \|f^{-1}\|\ \|x\|$$

Hence T is bounded and so is continuous as required.

Another Statement: A closed linear operator forms a banach space X into a Banach space Y is continuous.

Closed Graph Theorem

Let X and Y be Banach spaces and $T: \mathfrak{D}(T) \to Y$ a closed linear operator, where $\mathfrak{D}(T) \subset X$. Then if $\mathfrak{D}(T)$ is closed in X, the operator T is bounded.

Proof

We first show that $X \times Y$ with norm defined by $\|(x, y)\| = \|x\| + \|y\|$ is complete. Let (z_n) be Cauchy in $X \times Y$, where $z_n = (x_n, y_n)$.Then for every $\epsilon > 0$ there is an n_0 such that

$$\|z_n - z_m\| = \|x_n - x_m\| + \|y_n - y_m\| < \epsilon \qquad (m, n > n_0)$$

Hence (x_n) and (y_n) are Cauchy in X and Y, respectively, and converge, say, $x_n \to x$ and $y_n \to y$, because X and Y are complete.

Since $\|z_n - z_m\| < \epsilon \qquad (m, n > n_0)$

Therefore $\|z_n - z\| < \epsilon \qquad (n > n_0)$ as $m \to \infty$

This implies that $z_n \to z = (x, y)$

Since the Cauchy sequence (z_n) was arbitrary, $X \times Y$ is complete.

By assumption, $\mathcal{G}(T) = \{(x, y): x \in \mathfrak{D}(T), y = Tx\}$ is closed in $X \times Y$ and $\mathfrak{D}(T)$ is closed in X.

Hence $\mathcal{G}(T)$ and $\mathfrak{D}(T)$ are complete. By Theorem

"A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X."

We now consider the mapping $P: \mathcal{G}(T) \to \mathfrak{D}(T)$ defined by $(x, Tx) \mapsto x$ then P is linear. P is bounded because

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$$

P is bijective; in fact the inverse mapping is

$P^{-1}: \mathfrak{D}(T) \to \mathcal{G}(T)$ defined by $x \mapsto (x, Tx)$

Since $\mathcal{G}(T)$ and $\mathfrak{D}(T)$ are complete, we can apply the bounded inverse theorem and see that P^{-1} is bounded, say,

$\|(x, Tx)\| \leq b\|x\|$ for some b and all $x \in \mathfrak{D}(T)$.

Hence T is bounded because

$\|Tx\| \leq \|Tx\| + \|x\| = \|(x, Tx)\| \leq b\|x\|$ for all $x \in \mathfrak{D}(T)$.

Interesting to Know

- **Theorem (Closed linear operator)**
  Let $T: \mathfrak{D}(T) \to Y$ be a linear operator, where $\mathfrak{D}(T) \subset X$ and X and Yare normed spaces. Then T is closed if and only if it has the following property.
  If $x_n \to x$, where $x_n \in \mathfrak{D}(T)$, and $Tx_n \to y$, then $x \in \mathfrak{D}(T)$ and $Tx = y$.
- **Example (Ditlerential operator):** Let $X = C[O, 1]$ and $T: \mathfrak{D}(T) \to X$ defined by $x \mapsto x'$ where the prime denotes differentiation and $\mathfrak{D}(T)$ is the subspace of functions $x \in X$ which have a continuous derivative. Then T is not bounded, but is closed.
- Closed ness does not imply boundedness of a linear operator. Conversely, boundedness does not imply closedness.
- **Lemma(Closed operator):** Let $T: \mathfrak{D}(T) \to Y$ be a bounded linear operator with domain $\mathfrak{D}(T) \subset X$, where X and Y are normed spaces. Then:
  (a) If $\mathfrak{D}(T)$ is a closed subset of X, then T is closed.
  (b) If T is closed and Y is complete, then $\mathfrak{D}(T)$ is a closed subset of X.

BANACH FIXED POINT THEOREM

The Banach fixed point theorem is important as a source of existence and uniqueness theorems in different branches of analysis. In this way the theorem provides an impressive illustration of the unifying power of functional analytic methods and of the usefulness of fixed point theorems in analysis.

The Banach fixed point theorem or contraction theorem concerns certain mappings of a complete metric space into itself. It states conditions sufficient for the existence and uniqueness of a fixed point (point that is mapped onto itself). The theorem also gives an iterative process by which we can obtain approximations to the fixed point and error bounds. We consider three important fields of application of the theorem, namely, linear algebraic equations, ordinary differential equations, integral equations.

Fixed Point A ***fixed point*** of a mapping $T: X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself (is "kept fixed" by T), that is, $Tx = x$, the image Tx coincides with x.

For example, a translation has no fixed points, a rotation of the plane has a single fixed point (the center of rotation), the mapping $x \mapsto x^2$ of R into itself has two fixed points (0 and 1) and the projection $(\xi_1, \xi_2) \mapsto \xi_1$ of R^2 onto the ξ_1 −axis has infinitely many fixed points (all points of the ξ_1 −axis).

Contraction Let $X = (X, d)$ be a metric space. A mapping $T: X \to X$ is called a contraction on X if there is a positive real number $a < 1$ such that for all $x, y \in X$ we have $d(Tx, Ty) \leq ad(x, y) \quad ; a < 1$

Geometrically this means that any points x and y have images that are closer together than those points x and y; more precisely, the ratio $\frac{d(Tx,Ty)}{ad(x,y)}$ does not exceed a constant a which is strictly less than 1.

Iteration: By definition, this is a method such that we choose an arbitrary x_0 in a given set and calculate recursively a sequence $x_0, x_2, x_3, \ldots$ from a relation of the form $x_{n+1} = Tx_n$. that is, we choose an arbitrary x_0 and determine successively $x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \ldots$

Theorem 7.1 (Banach Contraction Theorem). *Let (X,d) be a complete metric space and $T : X \to X$ be a contraction mapping. Then T has a unique fixed point.*

Proof. We construct a sequence $\{x_n\}$ by the following iterative method.

Choose any arbitrary point $x_0 \in X$. Then $x_0 \neq T(x_0)$, otherwise x_0 is a fixed point of T and there is nothing to prove. So, we define

$$x_1 = T(x_0), x_2 = T(x_1), x_3 = T(x_2), \ldots, x_n = T(x_{n-1}) \quad \text{for all } n \in \mathbb{N}.$$

We claim that this sequence $\{x_n\}$ of points of X is a Cauchy sequence.

Since T is a contraction mapping with Lipschitz constant $0 < \alpha < 1$, for all $p = 1,2,\ldots$, we have

$$\begin{aligned} d(x_{p+1},x_p) &= d(T(x_p),T(x_{p-1})) \leq \alpha d(x_p,x_{p-1}) \\ &= \alpha d(T(x_{p-1}),T(x_{p-2})) \leq \alpha^2 d(x_{p-1},x_{p-2}) \\ &\quad \ldots\ldots\ldots\ldots\ldots \\ &\quad \ldots\ldots\ldots\ldots\ldots \\ &= \alpha^{p-1} d(T(x_1),T(x_0)) \leq \alpha^p d(x_1,x_0). \end{aligned}$$

Let m and n be any positive integers with $n < m$. Then by the triangle inequality, we have

$$\begin{aligned} d(x_m,x_n) &\leq d(x_m,x_{m-1}) + d(x_{m-1},x_{m-2}) + \cdots + d(x_{n+1},x_n) \\ &\leq \left(\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n\right) d(x_1,x_0) \\ &\leq \alpha^n \left(\alpha^{m-n-1} + \alpha^{m-n-2} + \cdots + 1\right) d(x_1,x_0) \\ &\leq \frac{\alpha^n}{1-\alpha} d(x_1,x_0). \end{aligned}$$

Since $\lim_{n\to\infty} \alpha^n = 0$ and $d(x_1,x_0)$ is fixed, the right hand side of the above inequality approaches to 0 as n tends to ∞. It follows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $\bar{x} \in X$ such that $x_n \to \bar{x}$. We show that this limit point $\bar{x}$ is a fixed point of T.

Since T is a contraction mapping, from the triangle inequality, we have

$$\begin{aligned} d(\bar{x},T(\bar{x})) &\leq d(\bar{x},x_n) + d(x_n,T(\bar{x})) \\ &= d(\bar{x},x_n) + d(T(x_{n-1}),T(\bar{x})) \\ &\leq d(\bar{x},x_n) + \alpha d(x_{n-1},\bar{x}) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $d(\bar{x},T(\bar{x})) = 0$ and so $T(\bar{x}) = \bar{x}$.

Now we show that the fixed point of T is unique. Suppose to the contrary that x and y are two distinct fixed points of T. Then $T(x) = x$ and $T(y) = y$. Since T is a contraction mapping, we have

$$d(x,y) = d(T(x),T(y)) \leq \alpha d(x,y) < d(x,y)$$

a contradiction. Hence $x = y$.

Banach Fixed Point Theorem (Contraction Theorem). Another proof

Consider a metric space $X = (X, d)$, where $X \neq \varphi$. Suppose that X is complete and let $T: X \to X$ be a contraction on X. Then T has precisely one fixed point.

Proof

We construct a sequence (x_n) and show that it is Cauchy, so that it converges in the complete space X, and then we prove that its limit x is a fixed point of T and T has no further fixed points. This is the idea of the proof.

We choose any $x_0 \in X$ and define the "iterative sequence" (x_n) by

$x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \dots, x_n = Tx_{n-1} = T^n x_0, \dots$

Clearly, this is the sequence of the images of x_0 under repeated application of T. We show that (x_n) is Cauchy. For this consider;

$d(x_{m+1}, x_m) = d(Tx_m, Tx_{m-1})$

$d(x_{m+1}, x_m) \leq a d(x_m, x_{m-1}) \qquad \because d(Tx, Ty) \leq a d(x, y)$

$d(x_{m+1}, x_m) \leq a d(Tx_{m-1}, Tx_{m-2})$

$d(x_{m+1}, x_m) \leq a^2 d(x_{m-1}, x_{m-2}) \qquad \because d(Tx, Ty) \leq a d(x, y)$

Continuing in this manner we get

$d(x_{m+1}, x_m) \leq a^m d(x_1, x_0)$

Hence by the triangle inequality and the formula for the sum of a geometric progression we obtain for $n > m$

$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$

$d(x_m, x_n) \leq (a^m + a^{m+1} + \cdots + a^{n-1}) d(x_0, x_1)$

$d(x_m, x_n) \leq a^m \frac{1-a^{n-m}}{1-a} d(x_0, x_1)$

Since $0 < a < 1$, in the numerator we have $1 - a^{n-m} < 1$. Consequently,

$$d(x_m, x_n) \leq \frac{a^m}{1-a} d(x_0, x_1) \qquad ; n > m$$

On the right, $0 < a < 1$ and $d(x_0, x_1)$ is fixed, so that we can make the right-hand side as small as we please by taking m sufficiently large (and $n > m$). This proves that (x_m) is Cauchy. Since X is complete, (x_m) converges, say, $x_m \to x$. We show that this limit X is a fixed point of the mapping T.

From the triangle inequality and $d(Tx, Ty) \leq ad(x, y)$ we have

$$d(x, Tx) \leq d(x, x_m) + d(x_m, Tx)$$

$$d(x, Tx) \leq d(x, x_m) + ad(x_{m-1}, x)$$

and can make the sum in the second line smaller than any preassigned $\in > 0$ because $x_m \to x$. We conclude that $d(x, Tx) = 0$, so that $x = Tx$ by (M2).

This shows that x is a fixed point of T.

x is the only fixed point of T because from $Tx = x$ and $T\tilde{x} = \tilde{x}$ we obtain by $d(Tx, Ty) \leq ad(x, y)$

$$d(x, \tilde{x}) = d(Tx, T\,\tilde{x}) \leq ad(x, \tilde{x})$$

which implies $d(x, \tilde{x}) = 0$ since $a < 1$.

Hence $x, \tilde{x}$ by (M2) and the theorem is proved.

Corollary (Iteration, error bounds).

Under the conditions of Banach Fixed Point Theorem (Contraction Theorem) the iterative sequence $x_0, x_1 = Tx_0, x_2 = T^2x_0, \dots, x_n = T^nx_0, \dots$ with arbitrary $x_0 \in X$ converges to the unique fixed point x of T. Error estimates are the ***prior estimate*** $d(x_m, x) \leq \frac{a^m}{1-a} d(x_0, x_1)$

And the ***posterior estimate*** $d(x_m, x) \leq \frac{a^m}{1-a} d(x_{m-1}, x_m)$

Proof

The first statement is obvious from the previous proof.

Since $d(x_m, x_n) \leq \frac{a^m}{1-a} d(x_0, x_1) \qquad ; n > m$

Then letting $n \to \infty$ we get $d(x_m, x) \leq \frac{a^m}{1-a} d(x_0, x_1)$

Now We derive $d(x_m, x) \leq \frac{a^m}{1-a} d(x_{m-1}, x_m)$

Since $d(x_m, x) \leq \frac{a^m}{1-a} d(x_0, x_1)$

Taking m = 1 and writing y_0 for x_0 and y_1 for x_1, we have;

$d(y_1, x) \leq \frac{a^m}{1-a} d(y_0, y_1)$

Setting $y_0 = x_{m-1}$ we have $y_1 = Ty_0 = x_m$ and obtain

$d(x_m, x) \leq \frac{a^m}{1-a} d(x_{m-1}, x_m)$

Theorem (Contraction on a ball).

Let T be a mapping of a complete metric space $X = (X, d)$ into itself. Suppose T is a contraction on a closed ball $Y = \{x : d(x, x_0) \leq r\}$, that is, T satisfies $d(Tx, Ty) \leq ad(x, y)$ for all $x, y \in Y$. Moreover, assume that

$d(x_0, T\,x_0) < (1 - a)r$

Then the iterative sequence $x_0, x_1 = Tx_0, x_2 = T^2x_0, \dots, x_n = T^nx_0, \dots$ converges to an $x \in Y$. This x is a fixed point of T and is the only fixed point of T in Y.

Proof

We merely have to show that all $x_m's$ as well as x lie in Y.

We put $m = 0$ in $d(x_m, x_n) \leq \frac{a^m}{1-a} d(x_0, x_1)$, change n to m and use $d(x_0, T\,x_0) < (1 - a)r$ to get $d(x_0, x_m) \leq \frac{1}{1-a} d(x_0, x_1) < r$

Hence all $x_m's$ are in Y. Also $x \in Y$ since $x_m \to x$ and Y is closed.

The assertion of the theorem now follows from the proof of Banach's theorem.

Lemma (Continuity)

A contraction T on a metric space X is a continuous mapping.

Application of Banach's Theorem to Linear Equations

Banach's fixed point theorem has important applications to iteration methods for solving systems of linear algebraic equations and yields sufficient conditions for convergence and error bounds.

To understand the situation, we first remember that for solving such a system there are various direct methods (methods that would yield the exact solution after finitely many arithmetical operations if the precision-the word length of our computer-were unlimited); a familiar example is Gauss' elimination method (roughly, a systematic version of the elimination taught in school). However, an iteration, or indirect method, may be more efficient if the system is special, for instance, if it is sparse, that is, if it consists of many equations but has only a small number of nonzero coefficients. (Vibrational problems, networks and difference approximations of partial differential equations often lead to sparse systems.) Moreover, the usual direct methods require about n3 /3 arithmetical operations (n = number of equations = number of unknowns), and for large n, rounding errors may become quite large, whereas in an iteration, errors due to roundoff (or even blunders) may be damped out eventually. In fact, iteration methods are frequently used to improve "solutions" obtained by direct methods.

Application of Banach's Theorem to Linear Equations

In this subsection, we present an application of Banach contraction theorem to find the solution of the following system of n linear equations with n unknowns:

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n. \end{array}\right\} \tag{7.2}$$

This system can be written as

$$\left.\begin{array}{l} x_1 = (1-a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1-a_{22})x_2 - a_{23}x_3 - \cdots - a_{2n}x_n + b_2 \\ x_3 = -a_{31}x_1 - a_{32}x_2 + (1-a_{33})x_3 - \cdots - a_{3n}x_n + b_3 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - a_{n3}x_3 - \cdots + (1-a_{nn})x_n + b_n. \end{array}\right\} \tag{7.3}$$

By letting $\alpha_{ij} = -a_{ij} + \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

system (7.3) is equivalent to the following system:

$$x_i = \sum_{j=1}^{n} \alpha_{ij}x_j + b_i \quad i = 1,2,3,\ldots,n. \tag{7.4}$$

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and

If $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

and $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, then the system (7.4) is equivalent to

$$x = Ax + b. \tag{7.5}$$

In other words, the problem is to find the fixed point of the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T(x) = Ax + b. \tag{7.6}$$

If T is a contraction mapping, then we can use Banach Contraction Theorem 7.1 and obtain the unique solution of $T(x) = x$ by the method of successive approximation.

The conditions under which T is a contraction mapping depend on the choice of the metric on $X = \mathbb{R}^n$. Here we discuss one case and left two others for exercise.

Theorem 7.5. *Let $X = \mathbb{R}^n$ be a metric space with the metric $d_\infty(x,y) = \max_{1\le i\le n} |x_i - y_i|$. If*

$$\sum_{j=1}^{n} |\alpha_{ij}| \le \alpha < 1 \quad \text{for all } i = 1,2,\dots,n, \tag{7.7}$$

then the linear system (7.2) *of n linear equations in n unknowns has a unique solution.*

Proof. Since $X = \mathbb{R}^n$ with respect to the metric d_∞ is complete, it is sufficient to prove that the mapping T defined by (7.6) is a contraction.

$$\begin{aligned}
d_\infty(T(x),T(y)) &= \max_{1\le i\le n} \left| \sum_{j=1}^{n} \alpha_{ij}(x_j - y_j) \right| \\
&\le \max_{1\le i\le n} \sum_{j=1}^{n} |\alpha_{ij}| \; |x_j - y_j| \\
&\le \max_{1\le i\le n} \left(\max_{1\le j\le n} |x_j - y_j| \right) \sum_{j=1}^{n} |\alpha_{ij}| \\
&= \max_{1\le i\le n} \sum_{j=1}^{n} |\alpha_{ij}| \, d_\infty(x,y) \\
&\le \alpha d_\infty(x,y).
\end{aligned}$$

Thus T is a contraction mapping. By Banach Contraction Theorem 7.1, the linear systems (7.2) has a unique solution.

Exercise 7.8. Let $X = \mathbb{R}^n$ be a metric space with the metric $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$. If

$$\sum_{i=1}^{n} |\alpha_{ij}| \leq \alpha < 1 \quad \text{for all } j = 1,2,\ldots,n, \tag{7.8}$$

then prove that the linear system (7.2) of n linear equations in n unknowns has a unique solution.
Hint:

$$d_1(T(x),T(y)) = \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \alpha_{ij}(x_j - y_j) \right|$$

$$\begin{aligned}
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |\alpha_{ij}| \; |x_j - y_j| \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} |\alpha_{ij}| \; |x_j - y_j| \\
&\leq \max_{1 \leq j \leq n} \sum_{i=1}^{n} |\alpha_{ij}| \, d_1(x,y) \\
&\leq \alpha d_1(x,y).
\end{aligned}$$

Exercise 7.9. Let $X = \mathbb{R}^n$ be a metric space with the metric $d_2(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/}$

If

$$\sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_{ij}|^2 \leq \alpha^2 < 1, \tag{7.9}$$

then prove that the linear system (7.2) of n linear equations in n unknowns has a unique solution.

Hint:

$$\begin{aligned}
[d_2(T(x),T(y))]^2 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \alpha_{ij}(x_j - y_j) \right|^2 \\
&\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{ij}| \; |x_j - y_j| \right)^2 \\
&\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\alpha_{ij}|^2 \sum_{j=1}^{n} |x_j - y_j|^2 \right).
\end{aligned}$$

So,

$$[d_2(T(x),T(y))]^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |\alpha_{ij}|^2 d_2(x,y) \leq \alpha^2 d_2(x,y).$$

Application of Banach's Theorem to Differential Equations

The most interesting applications of Banach's fixed point theorem arise in connection with function spaces. The theorem then yields existence and uniqueness theorems for differential and integral equations, as we shall see.

In fact, in this section let us consider an explicit ordinary differential equation of the first order $x' = f(t, x)$

An initial value problem for such an equation consists of the equation and an initial condition $x(t_0) = x_0$ where t_0 and x_0 are given real numbers.

We shall use Banach's theorem to prove the famous Picard's theorem which, while not the strongest of its type that is known, plays a vital role in the theory of ordinary differential equations. The idea of approach is quite simple: ODE (given above) with $x(t_0) = x_0$ will be converted to an integral equation, which defines a mapping T, and the conditions of the theorem will imply that T is a contraction such that its fixed point becomes the solution of our problem.

We give an application of Banach contraction theorem to prove the existence and uniqueness of the following ordinary differential equation with an initial condition:

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0. \tag{7.10}$$

Theorem 7.6 (Picard Theorem). *Let $f(x,y)$ be a continuous function of two variables defined on a rectangle $A = \{(x,y) : a \le x \le b, c \le y \le d\}$ and satisfy the following Lipschitz condition in the second variable:*

$$|f(x,y) - f(x,\hat{y})| \le \alpha |y - \hat{y}| \quad \textit{for all } y, \hat{y} \in [c,d]. \tag{7.11}$$

Further, let (x_0, y_0) be an interior point of A. Then the differential equation (7.10) with the given initial condition has a unique solution.

Proof. First of all, we show that the problem of determining the solution of differential equation (7.10) is equivalent to the problem of finding the solution of an integral equation. If $y = g(x)$ satisfies the differential equation (7.10) and has the property that $g(x_0) = y_0$, then integrating differential equation (7.10) from x_0 to x, we obtain

$$\left.\begin{aligned} g(x) - g(x_0) &= \int_{x_0}^{x} f(t, g(t))dt \\ g(x) &= y_0 + \int_{x_0}^{x} f(t, g(t))dt. \end{aligned}\right\} \tag{7.12}$$

Thus a unique solution of the differential equation (7.10) with the given initial condition is equivalent to a unique solution of (7.12). We apply Banach contraction theorem to determine the solution of (7.12).

By (7.11), there exists a constant $q > 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq q|y_1 - y_2|.$$

Since $f(x, y)$ is continuous on a compact subset A of $\mathbb{R}^2$, it is bounded and so there exists a positive constant m such that $|f(x, y)| \leq m$ for all $(x, y) \in A$.

Choose a positive constant p such that $pq < 1$ and the rectangle $B = \{(x, y) : -p + x_0 \leq x \leq p + x_0, \ -pm + y_0 \leq y \leq pm + y_0\}$ is contained in A.

Let X be the set of all real-valued continuous functions $y = g(x)$ defined on $[-p + x_0, p + x_0]$ such that $d(g(x), y_0) \leq mp$. The set X is a closed subset of the metric space $C[x_0 - p, x_0 + p]$ with sup metric. Since $C[x_0 - p, x_0 + p]$ is complete, X is complete.

Consider the transformation $T : X \to X$ defined by

$$T(g) = h \quad \text{where} \quad h(x) = y_0 + \int_{x_0}^{x} f(t, g(t))dt.$$

Since

$$d(h(x), y_0) = \sup \left| \int_{x_0}^{x} f(t, g(t))dt \right| \leq m(x - x_0) \leq mp,$$

$h(x) \in X$, **and so,** T **is well defined. For all** $g, g_1 \in X$, **we have**

$$\begin{aligned} d(T(g, T(g_1)) = d(h, h_1) &= \sup \left| \int_{x_0}^{x} [f(t, g(t)) - f(t, g_1(t))]\, dt \right| \\ &\leq \int_{x_0}^{x} \sup |f(t, g(t)) - f(t, g_1(t))|\, dt \\ &\leq q \int_{x_0}^{x} |g(t) - g_1(t)|\, dt \\ &\leq qp\, d(g, g_1), \end{aligned}$$

that is,

$$d(T(g), T(g_1)) \leq \alpha d(g, g_1)$$

where $0 \leq \alpha = qp < 1$.

Hence T is a contraction mapping on X into itself. By Banach Contraction Theorem 7.1, T has a unique fixed point $g^* \in X$. This unique fixed point g^* is the unique solution of the differential equation (7.10) and satisfies the given initial condition.

Picard's Existence and Uniqueness Theorem (ODE) /another form

Let f be continuous on a rectangle (Fig.1)

$R = \{ \{(t, x) : |t - t_0| \leq a, |x - x_0| \leq b\}$ and thus bounded on R, say (Fig. 2) $|f(t, x)| \leq c \quad ; \forall (t, x) \in R$

Suppose that f satisfies a ***Lipschitz condition*** on R with respect to its second argument, that is, there is a constant $\boldsymbol{k}$ (Lipschitz constant) such that for $(t, x), (t, v) \in R$ we have $|f(t, x) - f(t, v)| \leq k|x - v|$ Then the initial value problem $x' = f(t, x)$ with $(x(t_0) = x_0)$ has a unique solution. This solution exists on an interval $[t_0 - \beta, t_0 + \beta]$, where $\beta < min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}$

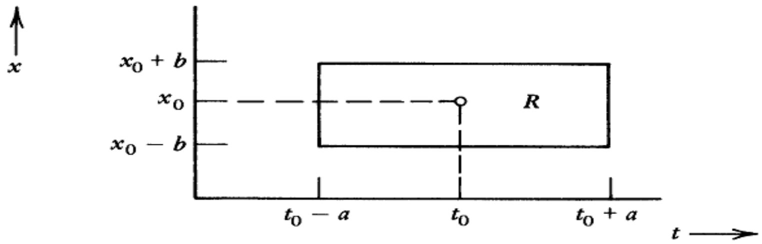


fig.1

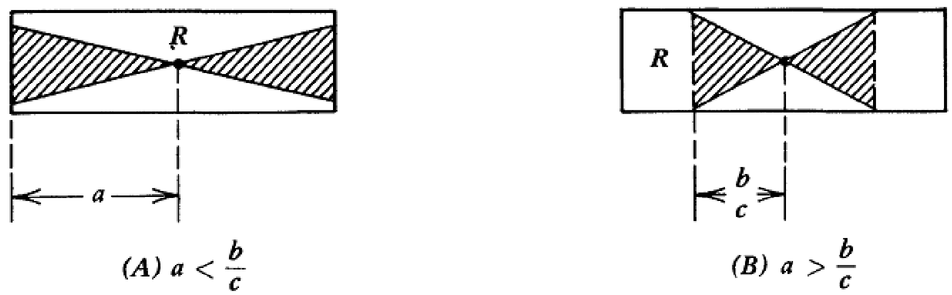


Fig. 54. Geometric illustration of inequality (2) for (A) relatively small c, (B) relatively large c. The solution curve must remain in the shaded region bounded by straight lines with slopes $\pm c$.

Fig 2

Proof

Let $C(J)$ be the metric space of all real-valued continuous functions on the interval $J = [t_0 - \beta, t_0 + \beta]$ with metric d defined by

$$d(x, y) = max_{t \in J}|x(t) - y(t)|$$

Where $C(J)$ is complete. Let $\tilde{C}$ be the subspace of $C(J)$ consisting of all those functions $x \in C(J)$ that satisfy $|x(t) - x_0| \leq c\beta$.

It is not difficult to see that $\tilde{C}$ is closed in $C(J)$, so that $\tilde{C}$ is complete. By integration we see that $x' = f(t, x)$ with $x(t_0) = x_0$ can be written $x = Tx$, where $T: \tilde{C} \to \tilde{C}$ is defined by $Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$

Indeed, T is defined for all $x \in \tilde{C}$, because $c\beta < b$ by $\beta < min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}$, so that if $x \in \tilde{C}$, then $\tau \in J$ and $(\tau, x(\tau)) \in R$, and the integral above exists since f is continuous on R. To see that T maps $\tilde{C}$ into itself, we can use

$Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$ and $|f(t,x)| \leq c$, obtaining

$$|Tx(t) - x_0| = \left|x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau - x_0\right| = \left|\int_{t_0}^{t} f(\tau, x(\tau))\, d\tau\right| \leq c|t - t_0|$$

$$|Tx(t) - x_0| \leq c\beta$$

We show that T is a contraction on $\tilde{C}$. By the Lipschitz condition,

$$|Tx(t) - Tv(t)| = \left|\int_{t_0}^{t} \left[f\big(\tau, x(\tau)\big) - f(\tau, v(\tau))\right] d\tau\right|$$

$$|Tx(t) - Tv(t)| \leq |t - t_0| max_{\tau \in J} K|x(t) - v(t)|$$

Since the last expression does not depend on t, we can take the maximum on the left and have

$d(Tx, Tv) \leq a d(x, v)$ where $a = k\beta$.

From $\beta < min\left\{a, \frac{b}{c}, \frac{1}{k}\right\}$ we see that $a = k\beta < 1$, so that T is indeed a contraction on $\tilde{C}$. Implies that T has a unique fixed point $x \in \tilde{C}$, that is, a continuous function x on J satisfying $x = Tx$. Writing $x = Tx$ out, we have by $Tx(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$$

Since $(\tau, x(\tau)) \in R$ where f is continuous, $x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$ may be differentiated.

Hence x is even differentiable and satisfies $x' = f(t,x)$ with $(x(t_0) = x_0)$. Conversely, every solution of $x' = f(t,x)$ with $(x(t_0) = x_0)$ must satisfy $x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))\, d\tau$.

This completes the proof.

Application of Banach's Theorem to Integral Equations

The most interesting applications of fixed point theorems arise when the underlying metric space is a function space. Here we discuss the existence and uniqueness of the Volterra integral equation by using Theorem 7.3.

Let K be a continuous function on $[a,b] \times [a,b]$ and let ϕ be a continuous function on $[a,b]$. Consider the equation

$$f(x) = \phi(x) + \lambda \int_a^x K(x,y) f(y)\,dy \quad \text{for all } x \in [a,b], \tag{7.13}$$

where λ is a parameter. It is called the *Volterra equation*.

Theorem 7.7. *For each $\lambda \in \mathbb{R}$, the Volterra equation* (7.13) *has a unique solution f that is continuous on $[a,b]$.*

Proof. Let $X = C[a,b]$ the set of all continuous real-valued functions defined on $[a,b]$ with the uniform metric. Since K is continuous, there exists a constant $k > 0$ such that $|K(x,y)| \leq k$ for all $x,y \in [a,b]$. Define the transformation $T : f \mapsto T(f)$ on X by

$$T(f(x)) = \phi(x) + \lambda \int_a^x K(x,y) f(y)\,dy.$$

For all $f, g \in X$, we have

$$|T(f(x)) - T(g(x))| = \left|\lambda \int_a^x K(x,y)\,|f(y) - g(y)|\,dy\right|$$
$$\leq |\lambda|\,k\,(x-a)\,d(f,g) \quad \text{for all } x \in [a,b].$$

Since $T^2(f) - T^2(g) = T\,(T(f) - T(g))$, we have

$$\left|T^2(f(x)) - T^2(g(x))\right| = \left|\lambda \int_a^x K(x,y)\,|T(f(y)) - T(g(y))|\,dy\right|$$
$$\leq |\lambda| \int_a^x |K(x,y)|\; |\lambda|\; k\;(y-a)\; d(f,g)dy$$
$$\leq |\lambda|^2 k^2 \int_a^x (y-a)dy\; d(f,g)$$
$$\leq \frac{|\lambda|^2 k^2\,(x-a)^2}{2} d(f,g).$$

Continuing this iterative process, we obtain

$$|T^n(f(x)) - T^n(g(x))| \leq \frac{|\lambda|^n k^n\,(x-a)^n}{n!} d(f,g) \quad \text{for all } x \in [a,b].$$

Hence,

$$|T^n(f) - T^n(g)| \leq \frac{[|\lambda|\,k\,(b-a)]^n}{n!} d(f,g).$$

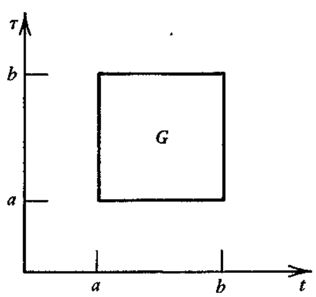
Recalling that $\frac{r^n}{n!} \to 0$ as $n \to \infty$ for any $r \in \mathbb{R}$, we conclude that there exists n such that T^n is a contraction mapping. Taking n sufficiently large to have $\frac{[|\lambda|\,k\,(b-a)]^n}{n!} < 1$. By Theorem 7.3, there exists a unique solution $f \in X$ satisfying $T(f) = f$. Obviously, if $T(f) = f$, then f solves (7.13).

Application of Banach's Theorem to Integral Equations

We finally consider the Banach fixed point theorem as a source of existence and uniqueness theorems for integral equations. An integral equation of the form

$$x(t) - \mu \int_a^b k(t,\tau)\, x(\tau) d\tau = v(t) \qquad \ldots\ldots\ldots\ldots\ldots..(1)$$

is called a ***Fredholm equation of the second kind***. Here, [a, b] is a given interval. x is a function on [a, b] which is unknown. μ is a parameter. The kernel k of the equation is a given function on the square $G = [a,b] \times [a,b]$ shown in Fig., and v is a given function on [a, b].



Integral equations can be considered on various function spaces. In this section we consider (1) on C[a, b], the space of all continuous functions defined on the interval J = [a, b] with metric d given by

$$d(x,y) = max_{\tau \in J} |x(t) - v(t)|.$$

For the proposed application of Banach's theorem it is important to note that C[a, b] is complete.

Theorem (Fredholm integral equation).

Suppose k and v in $x(t) - \mu\int_a^b k(t,\tau)\, x(\tau)d\tau = v(t)$ to be continuous on $J \times J$ and J = [a, b], respectively, and assume that μ satisfies $|\mu| < \frac{1}{c(b-a)}$ with c defined as $|k(t,\tau)| \leq c$. Then $x(t) - \mu\int_a^b k(t,\tau)\, x(\tau)d\tau = v(t)$ has a unique solution x on J. This function x is the limit of the iterative sequence $(x_0, x_1, \ldots)$, where x_0 is any continuous function on J and for $n = 0,1, \ldots$, and

$$x_{n+1}(t) = v(t) + \mu\int_a^b k(t,\tau)\, x_n(\tau)d\tau$$

The Volterra integral equation

An equation of the following form is called the Volterra integral equation

$$x(t) - \mu\int_a^t k(t,\tau)\, x(\tau)d\tau = v(t)$$

The difference between

$x(t) - \mu\int_a^b k(t,\tau)\, x(\tau)d\tau = v(t)$ and $x(t) - \mu\int_a^t k(t,\tau)\, x(\tau)d\tau = v(t)$

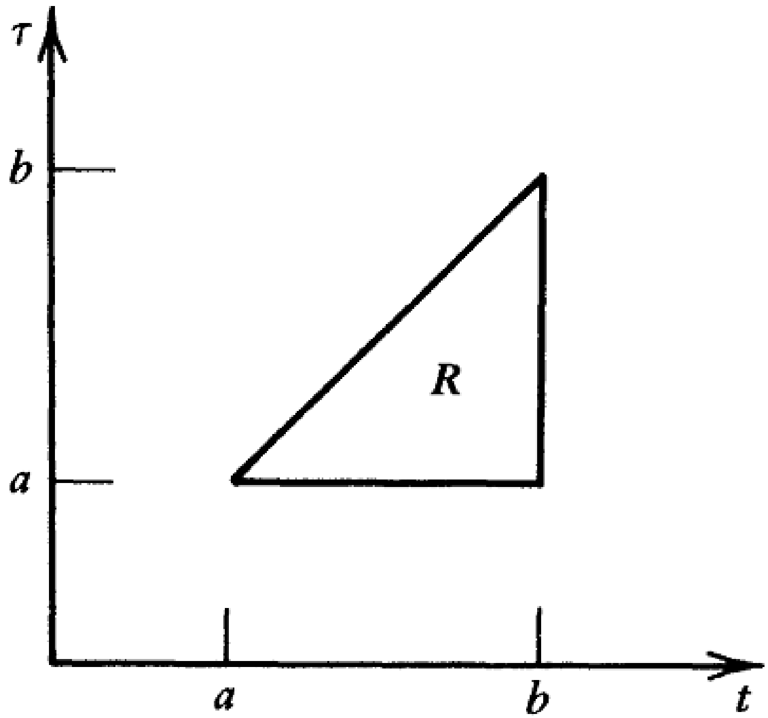
is that in first the upper limit of integration b is constant, whereas here in second it is variable. This is essential.

Theorem (Volterra integral equation).

Suppose that v in $x(t) - \mu\int_a^t k(t,\tau)\, x(\tau)d\tau = v(t)$ is continuous on [a, b] and the kernel k is continuous on the triangular region R in the $t\tau -$ plane given by $a \leq \tau \leq t,\ a \leq t \leq b$; see Fig. Then

$$x(t) - \mu\int_a^t k(t,\tau)\, x(\tau)d\tau = v(t)$$

has a unique solution x on [a, b] for every μ.



Lemma (Fixed point)

Let $T\colon X \to X$ be a continuous mapping on a complete metric space $X = (X, d)$, and suppose that T^m is a contraction on X for some positive integer m. Then T has a unique fixed point.

حرفِ آخر(08-02-2022)

خوش رہیں خوشیاں بانٹیں اور جہاں تک ہوسکے دوسروں کے لیے آسانیاں پیدا کریں۔

اللہ تعالیٰ آپ کو زندگی کے ہر موڑ پر کامیابیوں اور خوشیوں سے نوازے۔(اٰمین)

محمد عثمان حامد

چک نمبر 105 شمالی(گودھے والا) سرگودھا

UNIVERSITY OF SARGODHA

PUNJAB, PAKISTAN