FUNCTIONAL ANALYSIS

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Lecture # 1

Metric Space:

Let X be a non-empty set. Defined a function

f: $X \times X \rightarrow R$ s.t

M_1 : $d(x,y) \geq 0 \quad \forall \quad x,y \in X$

M_2 : $d(x,y) = 0 \Leftrightarrow x = y$

M_3 : $d(x,y) = d(y,x)$

M_4 : $d(x,z) \leq d(x,y) + d(y,z) \quad \forall \quad x,y,z$

Then d is called Metric in X and (X,d) is called Metric Space.

d: $l_2 \times l_2 \rightarrow R$ s.t $d(x,y) = \sqrt{\sum_{k=1}^{\infty}|x_k - y_k|^2}$

$x = \{x_k\}$, $y = \{y_k\} \in l_2$

s.t $\sum_{k=1}^{\infty}|x_k|^2 < \infty$ then (l_2, d) is Metric Space.

M_1 : $d(x,y) \geq 0 \quad \because \quad \sqrt{\sum_{k=1}^{\infty}|x_k - y_k|^2} \geq 0$

M_2 : $d(x,y) = 0 \quad \Rightarrow \quad \sqrt{\sum_{k=1}^{\infty}|x_k - y_k|^2} = 0$

$\sum_{k=1}^{\infty}|x_k - y_k|^2 = 0$

$x_k - y_k = 0 \qquad k = 1, 2, \ldots\ldots \infty$

$x_k = y_k \quad \Rightarrow \quad \{x_1, x_2, \ldots\ldots\} = \{y_1, y_2, \ldots\ldots\}$

$\underline{x = y}$

M_3 : $d(x,y) = \sqrt{\sum_{k=1}^{\infty}|x_k - y_k|^2} = \sqrt{\sum_{k=1}^{\infty}|y_k - x_k|^2} = d(y,x)$

M_4 : $d(x,z) \leq d(x,y) + d(y,z)$

$\sqrt{\sum_{k=1}^{\infty}|x_k - z_k|^2} = d(x,z) = \sqrt{\sum_{k=1}^{\infty}|x_k - y_k + y_k - z_k|^2}$

$\leq \sqrt{\sum_{k=1}^{\infty}|x_k - y_k|^2} + \sqrt{\sum_{k=1}^{\infty}|y_k - z_k|^2}$

$= d(x,y) + d(y,z)$

$|a + b| \leq |a| + |b|$ (Minkowski Inequality)

$\sqrt{\sum_{k=1}^{\infty}|a_k + b_k|^2} \leq \sqrt{\sum_{k=1}^{\infty}|a_k|^2} + \sqrt{\sum_{k=1}^{\infty}|b_k|^2}$

Question

$d': X \times X \to R$ s.t

$d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \quad \forall \quad x,y \in X$

Where d is metric on X then (X, d') is metric.

M_1 : $d'(x,y) \geq 0$ since $d(x,y) \geq 0$ by M_1

M_1 is satisfied or M_1 is True.

M_2 : $d'(x,y) = 0 \quad \Rightarrow \quad \frac{d(x,y)}{1+d(x,y)} = 0$

$d(x,y) = 0 \quad \Leftrightarrow \quad x = y \quad \because$ d is Metric on X

M_3 : $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)}$

$= d(y,x) \quad \because$ d is Metric

M_4 : $d'(x,z) = \frac{d(x,z)}{1+d(x,z)}$

$\leq \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$

$\leq \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$

$d'(x,z) \leq d'(x,y) + d'(y,z)$

Question

$d: X \times X \to R$ s.t

(i) $d(x,y) = 0$ iff $x=y$

(ii) $d(x,z) \leq d(x,y)+d(z,y)$

then (X,d) is Metric Space.

Solution:

M_1 : $d(x,z) \leq d(x,y) + d(z,y)$ by eq. (ii)

Put $z = x$

$d(x,x) \leq d(x,y) + d(x,y)$

$0 \leq 2d(x,y)$

$d(x,y) \geq 0 \qquad M_1$ is True.

M_2: d(x,y) = 0 iff x=y given in (i) So, M_2 is True.

M_3 : d(x,z) ≤ d(x,y) + d(z,y)

Put y = x

d(x,z) ≤ d(x,x) + d(z,x)

d(x,z) ≤ d(z,x) since d(x,x) = 0 _____ (1)

Replace x by z and z by x

d(z,x) ≤ d(x,z) _____(2)

d(x,z) = d(z,x) M_3 is True.

M_4 : d(x,z) ≤ d(x,y) + d(z,y) By (2)

d(x,z) ≤ d(x,y) + d(y,z)

M_4 is True.

Hence (X,d) is Metric Space.

Let (X,d) be a Metric Space then

d': X × X → R s.t

d'(x,y) = Min(d(x,y),1) then (X, d') is Metric Space.

M_1: d'(x,y) ≥ 0 Min(d(x,y),1) ≥ 0 since d is Metric on X

M_2: d'(x,y) = 0 ⇒ d(x,y) = 0 by M_2 x = y since d is Metric.

M_3 : d'(x,y) = Min(d(x,y),1)

= Min(d(y,x),1) = d(y,x)

Note: Min(d(x,y),1) ≥ 0 (Minimum is the answer. In this answer is 1)

M_3 : d'(x,z)Min(d(x,z),1) ≤ Min(d(x,y)+d(y,z)),1) ∵ d is Metric

≤ Min(d(x,y),1) +Min (d(y,z),1)

d'(x,z) ≤ d'(x,y) + d'(y,z)

(X, d') is Metric.

Example

Min (2+3,1) ≤ Min (2,1) + Min (3,1)

1 ≤ 1 + 1

Question:

(X_1, d_1) , (X_2, d_2) are Metric Spaces

X = $X_1 \times X_2$

d: X × X → R s.t

d(x,y) = max$(d_1(x_1, y_1), d_2(x_2, y_2))$

where x = $(x_1, x_2) \in$ X , y = $(y_1, y_2) \in$ X

then (X,d) is Metric Space.

Solution:

M_1 : Since d_1, d_2 is metric space

Then $d_i(x_i, y_i) \geq 0$ i = 1,2,……

$(x_1, y_1), d_2(x_2, y_2)$

max$[d_1(x_1, y_1), d_2(x_2, y_2)] \geq 0$

$d'[(x_1, x_2), (y_1, y_2)] \geq 0$

M_2 : $d'[(x_1, x_2), (y_1, y_2)] = 0 \Leftrightarrow$ max$[d_1(x_1, y_1), d_2(x_2, y_2)] = 0$

$d'[(x_1, x_2), (y_1, y_2)] = 0 \Leftrightarrow d_1(x_1, y_1), d_2(x_2, y_2) = 0$

$x_1 = y_1$, $x_2 = y_2 \Rightarrow (x_1, x_2) = (y_1, y_2)$

M_3 : $d'[(x_1, x_2), (y_1, y_2)] =$ max$[d_1(x_1, y_1), d_2(x_2, y_2)]$

$= d'[d_1(y_1, x_1), d_2(y_2, x_2)]$

$= d'[(y_1, y_2), (x_1, x_2)]$ d_1, d_2 are metric space.

M_4 : let max$[d_1(x_1, z_1), d_2(x_2, z_2)] = d_1(x_1, z_1)$ (i)

Since $d_1(x_1, y_1) \leq$ max$[d_1(x_1, y_1), d_2(x_2, y_2)]$ (ii)

$d_1(y_1, z_1) \leq$ max $[d_1(y_1, z_1), d_2(y_2, z_2)]$ (iii)

Adding (ii) and (iii)

$d_1(x_1, y_1) + d_1(y_1, z_1) \leq$ max$[d_1(x_1, y_1), d_2(x_2, y_2)]$+max $[d_1(y_1, z_1), d_2(y_2, z_2)]$

Since d_1 is metric

$d_1(x_1, z_1) \leq d_1(x_1, y_1) + d_1(y_1, z_1)$

$d_1(x_1, z_1) \leq \max[d_1(x_1, y_1), d_2(x_2, y_2)] + \max\ [d_1(y_1, z_1), d_2(y_2, z_2)]$

Put the value of $d_1(x_1, z_1)$ from (i)

$$\max[d_1(x_1, z_1)\ , d_2(x_2, z_2)] \leq \max[d_1(x_1, y_1), d_2(x_2, y_2)] + \max[d_1(y_1, z_1), d_2(y_2, z_2)]$$

$d'[(x_1, x_2), (z_1, z_2)] \leq d'[(x_1, x_2), (y_1, y_2)] + d'[(y_1, y_2), (z_1, z_2)]$

Hence d' is metric on $X_1 \times X_2$.

Lecture # 2

Question:

d: $X \times X \rightarrow$ R s.t

d(x,y) = $\sqrt{\sum_{i=1}^{\infty}|x_i - y_i|^2}$ is metric on X.

M_1 : d(x,y) ≥ 0 $\because$ $\sqrt{\sum_{i=1}^{\infty}|x_i - y_i|^2} \geq 0$

M_2 : d(x,y) = 0 $\Rightarrow$ $\sqrt{\sum_{i=1}^{\infty}|x_i - y_i|^2} = 0$

$\sum_{i=1}^{\infty}|x_i - y_i|^2 = 0$

$x_i - y_i = 0$ i =1, 2……… ∞

$x_i = y_i$ $\Rightarrow$ $\{x_1, x_2, \ldots\ldots\} = \{y_1, y_2, \ldots\ldots\}$

$\underline{x} = \underline{y}$

M_3 : d(x,y) = $\sqrt{\sum_{i=1}^{\infty}|x_i - y_i|^2}$ = $\sqrt{\sum_{i=1}^{\infty}|y_i - x_i|^2}$ = d(y,x)

M_4 : d(x,z) ≤ d(x,y) + d(y,z)

$\sqrt{\sum_{i=1}^{\infty}|x_i - z_i|^2}$ = d(x,z) = $\sqrt{\sum_{i=1}^{\infty}|x_i - y_i + y_i - z_i|^2}$

≤ $\sqrt{\sum_{i=1}^{\infty}|x_i - y_i|^2} + \sqrt{\sum_{i=1}^{\infty}|y_i - z_i|^2}$

= d(x,y) + d(y,z)

$|a + b| \leq |a| + |b|$ (Minkowski Inequality)

$\sqrt{\sum_{i=1}^{\infty}|a_i + b_i|^2} \leq \sqrt{\sum_{i=1}^{\infty}|a_i|^2} + \sqrt{\sum_{i=1}^{\infty}|b_i|^2}$

Question:

$|d(x,y) - d(x',y')| \leq d(x,x') + d(y,y')$

By M_4 : $d(x,y) \leq d(x,x') + d(x',y') + d(y,y')$

$d(x,y) - d(x',y') \leq d(x,x') + d(y,y')$ (1)

Interchanging x by x' & y by y'

$d(x',y') - d(x,y) \leq d(x',x) + d(y',y)$

Multiply by -1

-[$d(x,y) + d(x',y')$] ≤ $d(x,y)$- $d(x',y)$ (2)

$-[\ d(x,y) + d(x',y')] \le d(x,y) - d(x',y') \le d(x,x') + d(y,y')$

$$\left|d(x,y) - d(x',y')\right| \le d(x,x') + d(y,y')$$

Distance between two sets:

Suppose that A and B are subsets of metric space (X,d). Then

(i) $d(A,B) = \text{Inf}\ d(A,B) \qquad a \in A \quad , \quad b \in B$

(ii) If $A = \{x\}$

$d(A,B) = \text{Inf}\ d(x,B) \qquad x \in A$

Question:

Prove that $|d(x,A) - d(y,A)| \le d(x,y) \qquad$ when $A \subseteq X, \qquad x,y \in X$

Proof:

Def. of distance b/w point and a set $d(x,A) = \text{Inf}\ d(x,A)$

For any $z \in A$

$d(x,z) \le d(x,y) + d(y,z)$

So $d(x,A) = \text{Inf}\ d(x,z) \le d(x,y) + \text{Inf}\ d(y,z) \qquad z \in A$

$d(x,A) \le d(x,y) + d(y,A)$

$d(x,A) - d(y,A) \le d(x,y) \qquad (1)$

Interchanging x by y

$d(y,A) - d(x,A) \le d(y,x)$

$[d(x,A) - d(y,A)] \le d(y,x) \qquad |x| < \alpha \Rightarrow -\alpha < x < \alpha$

Or $-d(x,y) \le d(x,A) - d(y,A) \qquad (2)$

From (1) & (2)

$-d(x,y) \le d(x,A) - d(y,A) \le d(y,x) \Rightarrow |d(x,A) - d(y,A)| \le d(x,y)$

Diameter of a set:

Suppose that A is subset of metric space (X,d) then diameter of set is define as

(i) $\delta(A) = \text{Sup}\ d(x,y)$

(ii) If $A = \phi \quad , \quad \delta(\phi) = -\infty$

(iii) If $A = \{x\} \quad , \quad \delta(A) = 0$

Note: If diameter of set is finite then set is said to be bounded set.

Question:

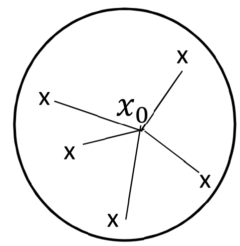
What is open ball, close ball and a sphere in a metric space.

Solution:

Open Ball:

$$B(x_0, r) = \{ x \in X : d(x, x_0) < r\}$$

For real line



$$B(x_0, r) = \{ x \in R : |x - x_0| < r\}$$

$|x - x_0| < r$

$-r < x - x_0 < r$

$x_0 - r < x < x_0 + r$

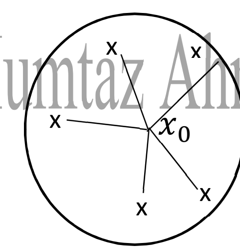
$]\, x_0 - r, x_0 + r\,[$ open interval

Close Ball:

$$B(x_0, r) = \{ x \in X : d(x, x_0) \leq r\}$$

For real line

$$\bar{B} = \{ x \in R : |x - x_0| \leq r\}$$



$|x - x_0| \leq r$

$-r \leq x - x_0 \leq r$

$x_0 - r \leq x \leq x_0 + r$

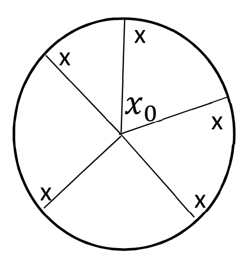
$[\, x_0 - r, x_0 + r\,]$

Sphere:

$$S(x_0) = \{ x \in X : d(x, x_0) = r\}$$

For real line

$$S(x_0) = \{ x \in R : |x - x_0| = r\}$$



$|x - x_0| = r$

$x - x_0 = \pm r$

$x = x_0 \pm r$

$\{ x_0 - r, x_0 + r \}$ Set

Question:

Show that every open ball is an open set in a metric space.

Solution:

Let B $(x_0, r) = \{$ x ∈ X : d(x,x_0) $< r\}$ be an open ball in metric space.

Let x ∈ B (x_0, r) , d(x,x_0) $< r$

put d(x,x_0) $= r_1$ then r_1 < r

Define $r_2 = r - r_1$ (1) ⇒ $r_2 > 0$

Now consider an open ball B(x, r_2) and let y ∈ B(x, r_2)

⇒ d(y,x) < r_2 (2)

By M_4 : d(y,x_0) ≤ d(y,x) + d(x,x_0)

d(y,x_0) < $r_2 + r_1$

$< r - r_1 + r_1$ $\because$ $r_2 = r - r_1$

d(y,x_0) < r

y ∈ B (x_0, r) , B(x, r_2) ≤ B (x_0, r)

⇒ B (x_0, r) is an open set.

Question:

Every close ball is a close set in metric space.

Solution: Let $\bar{B}$(a,r) = { x ∈ X : d(x,a) ≤ $r\}$ be a close ball.

To show $\bar{B}$(a,r) is an close set we shall prove that $\bar{B}'$(a,r) is open set.

Let x ∈ $\bar{B}'$(a,r) ⇒ d(x,a) > r

Take r_1 = d(x,a) – r then r_1> 0

Consider $B_1(x, r_1)$ be an open ball. We shall prove $B_1(x, r_1)$ ≤ $\bar{B}'$(a,r)

Let y ∈ $B_1(x, r_1)$ ⇒ d(y,x) < r_1

By M_4 d(x,a) ≤ d(x,y) + d(y,a) ⇒ d(y,a) ≥ d(x,a) - d(x, y)

d(y,a) ≥ d(x,a) - r_1 ⇒ d(y,a) ≥ d(x,a) - d(x, a)+r

d(y,a) > r , y ∈ $\bar{B}$(a,r) ⇒ $B_1(x, r_1)$ ≤ $\bar{B}'$(a,r)

$\bar{B}'$(a,r) is an open set. Hence $\bar{B}$(a,r) is a close set.

Lecture # 3

Sequence:

Suppose that (X,d) is a metric space a sequence in X is a function.

$$f : N \to X \qquad \forall \; n \in N$$

if f(x_n) = x_n then x_n will be nth term of seq$\{x_n\}$

e.g. $f(n) = \frac{n}{2} \qquad \forall \; n \in N$

$f(n) = 2n \qquad \forall \; n \in N$

$f(n) = 5n \qquad \forall \; n \in N$

$f(n) = \left\lfloor \frac{n}{2} \right\rfloor \qquad \forall \; n \in N$ → Floor Brackets

$f(n) = \lceil n \rceil \qquad \forall \; n \in N$ → Ceiling Brackets

Convergent Sequence:

A sequence $\{x_n\}$ in metric space (X,d) is said to be converges x ∈ X. If given any ε > 0 ∃ a natural number.

$$n_0 = n_0(\varepsilon)$$

s.t $\forall \;\; n \,;\, n \geq n_0 \Rightarrow d(x_n, x) < \varepsilon$

or $\forall \;\; n \,;\, n \geq n_0 \Rightarrow \lim_{n\to\infty} x_n = x$

or $\forall \;\; n \,;\, n \geq n_0 \Rightarrow x_n \to x$ when $n \to \infty$

Question:

Prove that a sequence in a (X,d) converges to one and only one limit.

Solution:

Let (X,d) be a metric space $\{x_n\}$ be a convergent sequence converges to two distinct points x and x' of X.

Let r = d(x, x') : r > 0

Since $x_n \to x$, for ε > 0 ∃ n_1 ∈ N s.t

$\forall \;\; n \,;\, n \geq n_1 \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$

Similarly, since $x_n \to x'$, for ε > 0 ∃ n_2 ∈ N s.t

$\forall \quad n\ ;\ n \geq n_2 \Rightarrow d(x_n, x') < \frac{\varepsilon}{2}$

Take $n_0 = \text{Max}(n_1, n_2)$

$\forall \quad n\ ;\ n \geq n_0 \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$

$d(x_n, x') < \frac{\varepsilon}{2}$

Now $r = d(x, x') \leq (x, x_n) + d(x_n, x')$

$r = d(x, x') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$r < r \qquad \because \varepsilon = r$

Which is contradiction.

So $\quad x = x'$

Note: Limit and limit points for a convergent sequence is same.

Cauchy Sequence:

Let (X,d) be a metric space a $\{x_n\}$ is said to a Cauchy sequence.

If $\quad \varepsilon > 0\ \exists\ n_0 = n_0(\varepsilon)$ Depending on

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$

Theorem:

Prove that every convergent sequence is a Cauchy sequence.

Proof: Suppose that $\{x_n\}$ is a convergent sequence in (X,d) metric space. And converges to a point $x \in X$.

Then for $\varepsilon > 0\ \exists\ n_1 \in N,\ n_1 = n_1(\varepsilon)$

s.t $\quad \forall \quad n \geq n_1 \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$

Also for same $n_2 \in N,\ n_2 = n_2(\varepsilon)$

$\forall \quad m\ ;\ m \geq n_2 \Rightarrow d(x_m, x) < \varepsilon$ Take $n_0 = \text{Max}\ (n_1, n_2)$

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow d(x_m, x_n) \leq d(x_m, x) + d(x_m, x)$

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow d(x_m, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$

Hence $\{x_n\}$ is a Cauchy Sequence in (X,d).

Theorem:

A Cauchy sequence in (X,d) metric space converges. Iff it has a convergent subsequence.

Proof:

Suppose that $\{x_n\}$ is a Cauchy sequence in (X,d) which converges to x $\in$ X.

Then $\{x_n\}$ itself is a convergent subsequence of it.

Conversely, let a Cauchy sequence $\{x_n\}$ has convergent subsequence $\{x_n\}$ converges to x $\in$ X.

Then for $\varepsilon > 0 \ \exists \ n_1, n_2 \in N$

$$n_1 = n_1(\varepsilon) \ , \qquad n_2 = n_2(\varepsilon)$$

s.t $\forall \quad n_k \ ; n_k \geq n_1 \Rightarrow d(x_n, x_{nk}) < \frac{\varepsilon}{2}$

$\forall \quad n_k \ ; n_k \geq n_2 \Rightarrow d(x_{nk}, x) < \frac{\varepsilon}{2}$

Take $n_0 = \text{Max}(n_1, n_2)$

$\forall$ n , n $\geq n_0 \Rightarrow d(x_n, x) \leq d(x_n, x_{nk}) + d(x_{nk}, x)$

$\forall$ n , n $\geq n_0 \Rightarrow d(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\forall$ n , n $\geq n_0 \Rightarrow d(x_n, x) < \varepsilon$

$\forall \ x_n \rightarrow x \in X$

Note: Every sequence itself is subsequence of it.

Complete Space:

Let $\{x_n\}$ be a Cauchy sequence in (X,d) if $x_n \rightarrow$ x $\in$ X then (X,d) is said to be complete space. e.g. R & C are complete spaces.

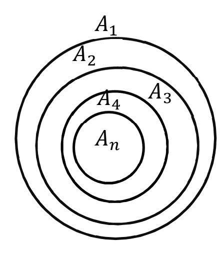
Dense Subset: If A $\subseteq$ X s.t $\overline{A}$ = X then A is dense in X.

Somewhere & Nowhere Dense Subset:

If A $\subseteq$ X , $(\overline{A}^{\,0}) \neq \phi$ then it is somewhere dense subset.

If $(\overline{A}^{\,0}) = \phi$ then it is nowhere dense subset.

Super Set



Nested Sequence:

Let $A_1, A_2 \dots\dots A_n, \dots\dots$ be a sequence of non-empty set in (X,d) s.t

(i) $A_n \supseteq A_{n+1}$, n = 1,2,…. (ii) $\delta(A_n) \rightarrow 0$ as n $\rightarrow \infty$

$\{A_n\}$ is nested sequence of set.

Lecture # 4

Normed Space:

Let N be a linear space over the field F (R or C) A norm on N is a function

$\|.\| : N \to R$ such that

N_1 : $\forall\ \underline{x} \in N$, $\|\underline{x}\| \geq 0$

N_2 : $\|\underline{x}\| = 0 \Leftrightarrow x = 0$

N_3 : $\|\alpha \underline{x}\| = |\alpha| \|x\|$, $\alpha \in F$

N_4 : $\|x + y\| \leq \|x\| + \|y\| \quad \forall\ x,y \in N$

$\|.\|$ is Norm and $(N, \|.\|)$ is Normed space.

Example:

Prove that l_p space consisting of all sequence $x = \{x_n\}$, $x_n \in F$ under

$\|.\| : l_p \to R$ such that

$\|x\| = \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p}$ where $\sum_{i=1}^{\infty} |x_i|^p < \infty$ is normed space then $(l_p, \|.\|)$ is normed space.

Solution:

N_1 : $\|x\| \geq 0 \quad \because \quad \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p} \geq 0$

N_2 : $\|x\| = 0 \Leftrightarrow \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p} = 0$

$\|x\| = 0 \Leftrightarrow \{x_1, x_2, \dots\dots x_n, \dots\} = 0$

$\|x\| = 0 \Leftrightarrow \underline{x} = 0$

N_3 : $\|\alpha x\| = \sqrt[p]{\sum_{i=1}^{\infty} |\alpha x_i|^p}$

$= |\alpha| \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p} = |\alpha| \|x\|$

N_4 : $\|x + y\| = \sqrt[p]{\sum_{i=1}^{\infty} |x_i + y_i|^p}$

By Minkoswki Inequality

$\|x + y\| \leq \sqrt[p]{\sum_{i=1}^{\infty} |x_i|^p} + \sqrt[p]{\sum_{i=1}^{\infty} |y_i|^p}$

$\|x + y\| \leq \|x\| + \|y\|$ Hence $(l_p, \|.\|)$ is normed space.

Question: Prove that l^n is a normed space under $\|.\| : l^n \to R$ such that

$$\|x\| = \sup_{i=1}^{n} |x_i|$$

Solution:

N_1 : $\|x\| \geq 0$ $\because$ $\sup_{i=1}^{n} |x_i| \geq 0$

N_2 : $\|x\| = 0$ $\Leftrightarrow$ $\sup_{i=1}^{n} |x_i| = 0$

$|x_i| = 0$ $\Leftrightarrow$ $\{x_1, x_2, \ldots \ldots . x_n\} = (0,0,0\ldots.0)$

$\|x\| = 0$ $\Leftrightarrow$ $\underline{x} = 0$

N_3 : $\|\alpha x\| = \sup_{i=1}^{n} |\alpha x_i| = |\alpha| \sup_{i=1}^{n} |x_i| = |\alpha| \|x\|$

N_4 : $\|x + y\| = \sup_{i=1}^{n} |x_i + y_i|$

$\|x + y\| \leq \sup_{i=1}^{n} |x_i| + \sup_{i=1}^{n} |y_i|$

$\|x + y\| \leq \|x\| + \|y\|$

Hence $(l^n, \|.\|)$ is normed space.

Question: Show that a normed space is a metric space.

Proof:

Let d: N × N → R such that

d(x,y) = $\|x - y\|$ $\forall$ x,y $\in$ N

M_1 : d(x,y) ≥ 0 $\because$ $\|x - y\| \geq 0$ By N_1

M_2 : d(x,y) = 0 $\Leftrightarrow$ $\|x - y\| = 0$ $\Leftrightarrow$ x = y by N_2

M_3 : d(x,y) = $\|x - y\|$

$= \|-1(y - x)\| = |-1| \|y - x\|$ $\because$ $\|\alpha x\| = |\alpha| \|x\|$

$= \|y - x\| \quad = d(y,x)$

M_4 : $d(x,z) = \|x - z\|$

$= \|x - y + y - z\|$

$\leq \|x - y\| + \|y - z\|$ By N_4

$d(x,z) \leq d(x,y) + d(y,z)$

Hence (N,d) is a metric space.

Question: If (N, ‖.‖) is a normed space then $|\|x\| - \|y\|| \leq \|x - y\|$

Solution:

Let $\|x\| = \|x - y + y\|$

$\leq \|x - y\| + \|y\|$ By N_4

$\|x\| - \|y\| \leq \|x - y\|$ (1)

$\|y\| = \|y - x + x\|$

$\leq \|y - x\| + \|y\| + \|x\|$ By N_4

$\|y\| \leq \|x - y\| + \|x\|$

$-\|x - y\| \leq \|x\| - \|y\|$ (2)

By (1) and (2)

$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$ Proved

$|\|x\| - \|y\|| \leq \|x - y\|$

Question:

∀ x, y, z ∈ N prove that

(i) d(x+y ,y+z) = d (x,z)
(ii) d(αx,αy) $= |\alpha|$d(x,y)

Solution:

We know that

d: N × N → R such that

$d(x,y) = \|x - y\|$

(i) $d(x+y, y+z) = \|x - y + y - z\|$

$= \|x - z\|$
$= d(x,z)$

(ii) $d(\alpha x,\alpha y) = \|\alpha x - \alpha y\|$

$= \|\alpha(x - y)\|$

$= |\alpha|\|x - y\|$ $\because \|\alpha x\| = |\alpha|\|x\|$

$= |\alpha| d(x,y)$

Lecture # 5

Question #1:

What is inner product space? State its axioms.

Solution:

Let V be a linear space over the field F (R or C) then an inner product space

$< . , . > : V \times V \to F$ such that

$I_1 : < x , x > \ \geq 0 \qquad \forall\, x \in V$

$I_2 : < x , x > \ = 0 \quad \Leftrightarrow \quad x = 0$

$I_3 : < x + y, z > \ = < x , z > + < y , z > \qquad \forall\, x,y,z \in V$

$I_4 : < \alpha x , y > \ = \alpha < x , z > \quad , \alpha \in F , \quad x,y \in V$

$I_5 : < \overline{x , y} > = < y, x > \quad$ or $\quad < \overline{y, x} > = < x, y >$

Then the pair (V, < . , . >) is called inner product.

Note:

$< x , \alpha y > = \overline{\alpha} < x , y >$

$< x, y+z > = < x,y > + <y,z>$

Question #2:

Show that every inner product space is a normed space.

Solution:

In an inner product space V then

$\|.\| : V \to R^+$ define

$\|x\| = \sqrt{< x, x >} \qquad \forall\, x \in V$

$N_1 : \|x\| \geq 0 \qquad$ since $< x , x > \ \geq 0$ By I_1

$N_2 : \|x\| = 0 \;\Leftrightarrow\; \sqrt{< x, x >} = 0$

$\Leftrightarrow < x , x > \ = 0$

$\Leftrightarrow \quad x = 0 \qquad$ By I_2

N_3 : $\|\alpha x\| = \sqrt{<\alpha x, \alpha x>}$ $\Rightarrow$ $\sqrt{\alpha\bar{\alpha} < x, x >}$

$= \sqrt{|\alpha|^2 < x, x >}$ $\Rightarrow$ $|\alpha|\sqrt{< x, x >}$

$= |\alpha|\|x\|$

N_4 : $\|x + y\|^2$ = < x+y , x+y>

= <x,x> + <x,y> + <y,x> + <y,y>

≤ <x,x> + 2Re<x,y> + <y,y>

= $\|x\|^2 +$ 2Re$\|x\|\|y\| + \|y\|^2$ since Re<x,y> ≤$\|x\|\|y\|$

$\|x + y\|^2 \leq (\|x\| + \|y\|)^2$

$\|x + y\| \leq \|x\| + \|y\|$

(V, $\|.\|$) is a normed space.

Question #3:

State and prove parallelogram law for inner product space.

Solution:

Statement:

Define a function

$\|.\| : V \to R^+$ such that

$\|x\| = \sqrt{< x, x >}$ $\forall\, x \in V$

$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ $\forall\, x,y \in V$

Proof:

$\|x + y\|^2$ = < x+y , x+y>

= <x,x> + <x,y> + <y,x> + <y,y>

= $\|x\|^2$ + <x, y> + <y,x> + $\|y\|^2$ (i)

And

$\|x - y\|^2$ = < x−y , x−y>

= <x,x> + <x, −y> + <−y,x> + <y,y>

= $\|x\|^2$ − <x, y>− <y,x> + $\|y\|^2$ (ii)

Adding (i) and (ii)

$\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y,x\rangle + \|y\|^2 + \|x\|^2 - \langle x, y\rangle - \langle y,x\rangle + \|y\|^2$

$= \quad 2\|x\|^2 + 2\|y\|^2$

Subtracting (ii) from (i)

$\|x+y\|^2 - \|x-y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y,x\rangle + \|y\|^2 - \|x\|^2 + \langle x, y\rangle + \langle y,x\rangle - \|y\|^2$

$= 2\langle x,y\rangle + 2\langle y,x\rangle$

$= 2[\langle x,y\rangle + \langle y,x\rangle]$

$= 2(2\mathrm{Re}\langle x,y\rangle)$

$= 4\mathrm{Re}\langle x,y\rangle$

Question #4:

Prove that $4\langle x,y\rangle = (\|x+y\|^2 - \|x-y\|^2) + i\,(\|x+iy\|^2 - \|x-iy\|^2)$

Solution:

Define a function

$\|.\| : V \to R^+$ such that

$\|x\| = \sqrt{\langle x, x\rangle} \qquad \forall\, x \in V$

$\|x+y\|^2 = \langle x+y\,,\, x+y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \langle x, y\rangle + \langle y,x\rangle + \|y\|^2 \qquad$ (i)

$\Rightarrow \qquad \|x+iy\|^2 = \langle x+iy\,,\, x+iy\rangle$

$= \langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle + \langle iy,iy\rangle$

$= \langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle + i\,\bar{\imath}\langle y,y\rangle$

$= \|x\|^2 + \langle x,iy\rangle + \langle iy,x\rangle - i\,i\,\|y\|^2$

$= \|x\|^2 + \langle x, iy\rangle + \langle iy,x\rangle + \|y\|^2 \qquad$ (ii)

$\Rightarrow \qquad \|x-y\|^2 = \langle x-y\,,\, x-y\rangle$

$= \langle x,x\rangle + \langle x, -y\rangle + \langle -y,x\rangle + \langle -y, -y\rangle$

$= \|x\|^2 - \langle x, y\rangle - \langle y,x\rangle + \langle y,y\rangle$

$$= \|x\|^2 - \langle x, y\rangle - \langle y,x\rangle + \|y\|^2 \qquad \text{(iii)}$$

$$\Rightarrow \qquad \|x - iy\|^2 = \langle x-iy\, , x-iy\rangle$$

$$= \langle x,x\rangle + \langle x, -iy\rangle + \langle -iy,x\rangle + \langle -iy, -iy\rangle$$

$$= \|x\|^2 - \langle x,iy\rangle - \langle iy,x\rangle + i\,\bar{i}\langle y,y\rangle$$

$$= \|x\|^2 - \langle x,iy\rangle - \langle iy,x\rangle - i\, i\, \|y\|^2$$

$$= \|x\|^2 - \langle x, iy\rangle - \langle iy,x\rangle + \|y\|^2 \qquad \text{(iv)}$$

Subtract (iii) from (i)

$$\|x + y\|^2 - \|x - y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y,x\rangle + \|y\|^2 - \|x\|^2 + \langle x, y\rangle + \langle y,x\rangle - \|y\|^2$$

$$= 2\langle x,y\rangle + 2\langle y,x\rangle$$

$$= 2[\langle x,y\rangle + \langle y,x\rangle]$$

$$= 2(2\text{Re}\langle x,y\rangle)$$

$$= 4\text{Re}\,\langle x,y\rangle \qquad \text{(v)}$$

Now Subtract (iv) from (ii)

$$\|x + iy\|^2 - \|x - iy\|^2 = \|x\|^2 + \langle x, iy\rangle + \langle iy,x\rangle + \|y\|^2 - \|x\|^2 + \langle x, iy\rangle + \langle iy,x\rangle - \|y\|^2$$

$$= 2\langle x+iy\rangle + 2\langle iy + x\rangle$$

$$= 2(\langle x+iy\rangle + \langle iy + x\rangle)$$

$$= 2(\text{Im}\langle x,y\rangle)$$

$$= 4\text{Im}\langle x,y\rangle \qquad \text{(vi)}$$

Add (v) and (vi)

$$(\|x + y\|^2 - \|x - y\|^2) + i(\|x + iy\|^2 - \|x - iy\|^2) = 4\text{Re}\langle x,y\rangle + i4\text{Im}\langle x,y\rangle$$

$$= 4(\text{Re}\langle x,y\rangle + i\text{Im}\langle x,y\rangle)$$

$$= 4\,\langle x,y\rangle \qquad \text{Proved}$$

Question #5:

Prove that every normed space is not an inner product space. Prove it by counter example.

Solution: Consider $c[0,\frac{\pi}{2}]$ i.e. A space of all continuous real valued function defined on $[0,\frac{\pi}{2}]$. Define $\|.\| : c[0,\frac{\pi}{2}] \to F$ such that $\|f\| = \sup_{x\in[0,\frac{\pi}{2}]} |f(x)|$

$N_1 : \|f\| \geq 0 \qquad \because \sup_{x\in[0,\frac{\pi}{2}]} |f(x)| \geq 0$

$N_2 : \|f\| = 0 \iff \sup_{x\in[0,\frac{\pi}{2}]} |f(x)| = 0$

$\iff |f(x)| = 0$

$\iff f = 0$

$N_3 : \|\alpha f\| = \sup_{x\in[0,\frac{\pi}{2}]} |\alpha f(x)| \qquad \Rightarrow \qquad |\alpha| \sup_{x\in[0,\frac{\pi}{2}]} |f(x)|$

$= |\alpha| \, \|f\|$

$N_4 : \quad \|f+g\| = \sup_{x\in[0,\frac{\pi}{2}]} |f(x)+g(x)|$

$\leq \sup_{x\in[0,\frac{\pi}{2}]} |f(x)| + \sup_{x\in[0,\frac{\pi}{2}]} |g(x)|$

$\|f+g\| \leq \|f\| + \|g\|$

$(c[0,\frac{\pi}{2}], \|.\|)$ is a normed space.

Now let $\|f\| = \sup_{t\in[0,\frac{\pi}{2}]} |f(t)|$

Let f,g $\in$ $c[0,\frac{\pi}{2}]$ such that

f(t) = sint

and g(t) = cost where $t \in [0,\frac{\pi}{2}]$

$\|f\| = \sup_{t\in[0,\frac{\pi}{2}]} |\text{sint}| = 1 \qquad , \qquad \|g\| = \sup_{t\in[0,\frac{\pi}{2}]} |\text{cost}| = 1$

$\|f+g\| = \sup_{t\in[0,\frac{\pi}{2}]} |\text{sint} + \text{cost}|$

دونوں فنکشن کی maximum value '0' اور '$\frac{\pi}{2}$' کے درمیان $\frac{\pi}{4}$ ہے۔

$\|f+g\| = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\|f-g\| = \sup_{t\in[0,\frac{\pi}{2}]} |\text{sint} - \text{cost}|$

جب دوسرا فنکشن نفی ہو رہا ہو تو پہلے کی maximum value اور دوسرے فنکشن کی zero value لیں گے۔

$\|f-g\| = 1$

By //gram identity

$\|f+g\|^2+\|f-g\|^2 \quad = \quad 2\|f\|^2+2\|g\|^2$

$(\sqrt{2})^2+(1)^2 \neq 2(1)+2(1)$

$3 \neq 4$

Hence c$[0,\frac{\pi}{2}]$ is not an inner product space.

Question #6:

How can you prove that inner product space is a normed space and hence a metric space also show that its converse may not be true [normed space is not an inner product space V]

Solution:

In an inner product space V

$\|.\|: V \to R$ such that

$\|x\| = \sqrt{<x,x>} \qquad \forall\, x \in V$

N_1 : $\|x\| \geq 0$ since $<x,x> \geq 0$ By I_1

N_2 : $\|x\| = 0 \Leftrightarrow \sqrt{<x,x>} = 0$

$\Leftrightarrow <x,x> = 0$

$\Leftrightarrow x = 0$ By I_2

N_3 : $\|\alpha x\| = \sqrt{<\alpha x, \alpha x>} = \sqrt{\alpha\bar{\alpha} <x,x>}$

$= \sqrt{|\alpha|^2 <x,x>} = |\alpha|\sqrt{<x,x>}$

$= |\alpha|\|x\|$

N_4 : $\|x+y\|^2 = <x+y, x+y>$

$= <x,x> + <x,y> + <y,x> + <y,y>$

$= <x,x> + 2\text{Re}<x,y> + <y,y>$

$\leq \|x\|^2 + 2\text{Re}\|x\|\|y\| + \|y\|^2$ since $\text{Re}<x,y> \leq \|x\|\|y\|$

$\|x+y\|^2 \leq (\|x\|+\|y\|)^2$

$\|x+y\| \leq \|x\|+\|y\|$

$(V, \|.\|)$ is a normed space.

Now to show normed space is metric space

Let $d: N \times N \rightarrow R$ s.t

$d(x,y) = \|x - y\|$

M_1 : $d(x,y) \geq 0$ since $\|x - y\| \geq 0$ By N_1

M_2 : $d(x,y) = 0 \Leftrightarrow \|x - y\| = 0$

$\Leftrightarrow x - y = 0$ By N_2

$\Leftrightarrow x = y$

M_3 : $d(x,y) = \|x - y\|$

$=\|-(y - x)\|$

$= |-1|\|y - x\|$ $\because \|\alpha x\| = |\alpha|\|x\|$

$= \|y - x\|$

$= d(y,x)$

M_4 : $d(x,z) = \|x - z\|$

$= \|x - y + y - z\|$

$\leq \|x - y\| + \|y - z\|$

$\leq d(x,y) + d(y,z)$

(N,d) is a metric space.

Now to show that its converse is not true or normal space is not an inner product space.

Let $\|f\| = \sup_{t\in[0,\frac{\pi}{2}]} |f(t)|$

Let $f,g \in c[0,\frac{\pi}{2}]$

Such that $f(t) = \sin t$ and $g(t) = \cos t$ where $t \in [0,\frac{\pi}{2}]$

$\|f\| = \sup_{t\in[0,\frac{\pi}{2}]} |\sin t| = 1$, $\|g\| = \sup_{t\in[0,\frac{\pi}{2}]} |\cos t| = 1$

$\|f + g\| = \sup_{t\in[0,\frac{\pi}{2}]} |\sin t + \cos t|$

$\|f+g\| = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\|f-g\| = \sup_{t\in[0,\frac{\pi}{2}]} |\sin t - \cos t|$

$\|f-g\| = 1$

By //gram identity

$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2$

$(\sqrt{2})^2 + (1)^2 \neq 2(1) + 2(1)$

$3 \neq 4$

Hence $c[0,\frac{\pi}{2}]$ is not an inner product space.

So, every normed is not an inner product space.

Conjugate Index (For MCQs)

Let P be a real number (P > 1). A real number q is said to be conjugate index of P if $\frac{1}{p} + \frac{1}{q} = 1$

i.e. if P = 2

$\Rightarrow \quad \frac{1}{2} + \frac{1}{q} = 1$

$\Rightarrow \quad q = 2$

Lecture # 6

★ Theorem:

Prove that R^n is a norm space and a Banach Space.

Proof:

The space R^n is a Euclidian space where $\underline{x} = (x_1, x_2, x_3, \ldots\ldots x_n) \in R^n$

$\underline{0} = (0,0,0,0,\ldots\ldots.0) \in R^n$

is a Linear space over **F (R or C).**

$\|.\| : R^n \to R$ Such that

$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$

$N_1 : \|x\| \geq 0 \qquad \therefore \sqrt{\sum_{i=1}^{n} |x_i|^2} \geq 0$

$N_2 : \|x\| = 0 \quad \Leftrightarrow \quad \sqrt{\sum_{i=1}^{n} |x_i|^2} = 0$

$\Leftrightarrow \quad x_i = 0$

$\Leftrightarrow (x_1, x_2, x_3, \ldots \ldots x_n) = (0,0,0,\ldots\ldots0)$

$\Leftrightarrow \quad \underline{x} = 0$

$N_3 : \|\alpha x\| = \sqrt{\sum_{i=1}^{n} |\alpha x_i|^2}$

$= |\alpha| \sqrt{\sum_{i=1}^{n} |x_i|^2}$

$= |\alpha| \|x\|$

$N_3 : \|x + y\| = \sqrt{\sum_{i=1}^{n} |x_i + y_i|^2}$

$\leq \sqrt{\sum_{i=1}^{n} |x_i|^2} + \sqrt{\sum_{i=1}^{n} |y_i|^2}$

$\|x + y\| \leq \|x\| + \|y\|$

Hence $(R^n, \|.\|)$ is a normed space.

(ii). R^n is a Banach Space.

Let $\{x^{(p)}\}$ be a cauchy in R^n

Then for $\varepsilon > 0 \; \exists \; n_0 \in N$, $n_0 = n_0(\varepsilon)$

s.t $\forall$ p,q ; p,q $\geq n_0 \Rightarrow \|x^{(p)} - x^{(q)}\| = \sqrt{\sum_{i=1}^{n} \left|x_i^{(p)} - x_i^{(q)}\right|^2} < \varepsilon$

$\Rightarrow \forall$ p,q ; p,q $\geq n_0 \Rightarrow \left|x^{(p)} - x^{(q)}\right| \leq \sqrt{\sum_{i=1}^{n} \left|x_i^{(p)} - x_i^{(q)}\right|} < \varepsilon$

$\Rightarrow \forall$ p,q ; p,q $\geq n_0 \Rightarrow \left|x^{(p)} - x^{(q)}\right| < \varepsilon$

$\{x^{(p)}\}$ is a cauchy sequence in R.

Since R is complete so

$x_i^{(p)} \rightarrow x_i \in$ R as P $\rightarrow \infty \quad \forall$ i = 1,2,3,…….,n

For ε > 0 (Already choosen) $\exists P_i \in$ N s.t

$\forall$ P; P $\geq P_i \Rightarrow \left|x_i^{(p)} - x_i\right| < \frac{\varepsilon}{\sqrt{n}}$

Take x = $\lim_{p\rightarrow\infty} x_i^{(p)}$ then x $\in R^n$

We shall prove $\lim\limits_{p\rightarrow\infty} x^p$ = x $\in R^n$

Let $P_0 = max(P_1, P_{2,\ldots\ldots} P_n)$

$\forall$ P ; P $\geq P_0 \Rightarrow \|x^{(p)} - x\| = \sqrt{\sum_{i=1}^{n} |x^{(p)} - x_i|^2}$

$\forall$ P ; P $\geq P_0 \Rightarrow \|x^{(p)} - x\| = \sqrt{\frac{\varepsilon^2}{n} + \frac{\varepsilon^2}{n} + \ldots\ldots\ldots\ldots + \frac{\varepsilon^2}{n}}$

$\forall$ P ; P $\geq P_0 \Rightarrow \|x^{(p)} - x\| < \varepsilon$

$\Rightarrow x^p \rightarrow$ x $\in R^n$ where P $\rightarrow \infty$

R^n is complete and Hence a Banach space.

★**Question: Prove that $C^n$is a normed space and hence a Banach Space.**

Proof:

The space C^n is Euclidian space where

$\underline{z} = (z_1, z_2, \ldots \ldots . z_n) \in C^n$

$\underline{0}$ = (0,0,0,……0) $\in C^n$

is a Linear space over F (R or C)

$\|.\| : C^n \rightarrow R$ Such that

$\|z\| = \sqrt{\sum_{i=1}^{n} |z_i|^2}$

$N_1 : \|z\| \geq 0 \qquad \therefore \sqrt{\sum_{i=1}^{n}|z_i|^2} \geq 0$

$N_2 : \|z\| = 0 \quad \Leftrightarrow \quad \sqrt{\sum_{i=1}^{n}|z_i|^2} = 0$

$\Leftrightarrow \quad z_i = 0$

$\Leftrightarrow (z_1, z_2, z_3, \ldots \ldots . . z_n) = (0,0,0,\ldots\ldots0)$

$\Leftrightarrow \quad \underline{z} = 0$

$N_3 : \|\alpha z\| = \sqrt{\sum_{i=1}^{n}|\alpha z_i|^2}$

$= |\alpha|\sqrt{\sum_{i=1}^{n}|z_i|^2}$

$= |\alpha|\|z\|$

$N_3 : \|z + z'\| = \sqrt{\sum_{i=1}^{n}|z + z'|^2}$

By Minkowski inequality

$\leq \sqrt{\sum_{i=1}^{n}|z_i|^2} + \sqrt{\sum_{i=1}^{n}|z'_i|^2}$

$\|z + z'\| \leq \|z\| + \|z'\|$

Hence $(C^n, \|.\|)$ is a normed space.

(ii). C^n is a Banach space

Let $\{z^{(p)}\}$ be a cauchy in C^n

Then for $\varepsilon > 0 \ \exists \ n_0 = n_0(\varepsilon)$

s.t $\forall$ p,q ; p,q $\geq n_0 \Rightarrow \left\|z^{(p)} - z^{(q)}\right\| = \sqrt{\sum_{i=1}^{n}\left|z_i^{(p)} - z_i^{(q)}\right|^2} < \varepsilon$

$\Rightarrow \forall$ p,q ; p,q $\geq n_0 \Rightarrow \left|z^{(p)} - z^{(q)}\right| \leq \sqrt{\sum_{i=1}^{n}\left|z_i^{(p)} - z_i^{(q)}\right|} < \varepsilon$

$\Rightarrow \forall$ p,q ; p,q $\geq n_0 \Rightarrow \left|z^{(p)} - z^{(q)}\right| < \varepsilon$

$\{z^{(p)}\}$ is a cauchy sequence in R.

Since R is complete so

$z_i^{(p)} \rightarrow z_i \in$ R as P $\rightarrow \infty \quad \forall$ i = 1,2,3,…….,n

For $\varepsilon > 0$ (Already choosen) $\exists \ P_i \in$ N s.t

$\forall$ P; $\quad$ P $\geq P_i \Rightarrow \left|z_i^{(p)} - z_i\right| < \frac{\varepsilon}{\sqrt{n}}$

Take z = $\lim_{p\to\infty} z_i^{(p)}$ then z $\in C^n$

We shall prove $\lim\limits_{p\to\infty} z^p$ = z $\in C^n$

Let $P_0 = max(P_1, P_{2\,,\ldots\ldots}P_n)$

$\forall$ P ; P $\geq P_0 \Rightarrow \|z^{(p)} - z\| = \sqrt{\sum_{i=1}^{n} |z^{(p)} - z_i\,|^{\,2}}$

$\forall$ P ; P $\geq P_0 \Rightarrow \|z^{(p)} - z\| = \sqrt{\frac{\varepsilon^2}{n} + \frac{\varepsilon^2}{n} + \ldots\ldots.\ldots.+\frac{\varepsilon^2}{n}}$

$\forall$ P ; P $\geq P_0 \Rightarrow \|z^{(p)} - z\| < \varepsilon$

$\Rightarrow z^p \to$ z $\in C^n$ where P $\to \infty$

C^n is complete and Hence a Banach space.

Lecture # 7

l^∞-Space:

A space of bounded sequence x = $\{x_i\}$ or real or complex numbers with addition and scalar multiplication defined by

$$x + y = \{x_i + y_i\}$$

$$\alpha x = \{\alpha x_i\}$$

Question: The norm in l^∞ can be defined as $\|\cdot\| : l^\infty \to R$ such that

$$\|x\| = Sup_{i=1}^{\infty}|x_i|$$

Prove l^∞ is normed space and hence a Banach space.

Solution:

l^∞ is normed space

$$N_1: \ \|x\| \geq 0 \quad \because \ \underset{i=1}{\overset{\infty}{Sup}}|x_i| \geq 0$$

$$N_2: \ \|x\| = 0 \quad \Leftrightarrow \ \underset{i=1}{\overset{\infty}{Sup}}|x_i| = 0$$

$$\Leftrightarrow \ |x_i| = 0$$

$$\Leftrightarrow \ (x_1, x_2, x_3, \ldots\ldots\ldots x_n, \ldots\ldots) = (0, 0, \ldots\ldots\ldots 0, \ldots\ldots)$$

$$\Leftrightarrow x = 0$$

$$N_3: \|\alpha x\| = \underset{i=1}{\overset{\infty}{Sup}}|\alpha x_i|$$

$$= |\alpha| \ \underset{i=1}{\overset{\infty}{Sup}}|x_i|$$

$$= |\alpha| \|x_i\|$$

$$N_4: \|x + y\| = \underset{i=1}{\overset{\infty}{Sup}}|x_i + y_i|$$

$$\leq \ \underset{i=1}{\overset{\infty}{Sup}}|x_i| + \underset{i=1}{\overset{\infty}{Sup}}|y_i|$$

$$\|x + y\| \leq \|x\| + \|y\| \qquad Hence\,(l^\infty, \|.\|)\, is\, a\, normed\, space.$$

l^∞ is Banach Space:

Let $\{x^{(p)}\}$ be a Cauchy sequence in l^∞

$x^{(p)} = \{x_i^{(p)}\}$. Then for ε > 0 ∃ $n_o \in N, n_o = n_o(\varepsilon)$

$$S.t \ \ \forall\, p,q\ ;\ p,q \geq n_o \Rightarrow \left\| x^{(p)} - x^{(q)} \right\| = \left\| x_i^{(p)} - x_i^{(q)} \right\| < \varepsilon$$

$$\forall\, p,q ; p,q \geq n_o \Rightarrow \left| x^{(p)} - x^{(q)} \right| \leq \left\| x^{(p)} - x^{(q)} \right\| < \varepsilon$$

$$\forall\, p,q ; p,q \geq n_o \Rightarrow \left| x^{(p)} - x^{(q)} \right| < \varepsilon$$

Hence $\{x_i^{(p)}\}$ is a Cauchy sequence in R or C. Since R or C is complete so

$x_i^{(p)} \to x_i$ $\quad\quad\quad$ ∀ i = 1,2,…….∞

Take x = $\{x_i\}$ we show that x $\in l^\infty$ and $\lim_{p\to\infty} x^{(p)}$ = x

Since $x_i^{(p)} \to x_i$ So, ∃ $n_1 \in$ N , $n_1 = n_1(\varepsilon)$ s.t

∀ p ; p ≥ $n_1 \Rightarrow \left| x_i^{(p)} - x_i \right| < \frac{\varepsilon}{2}$, i = 1,2,…..∞

That is p ; p ≥ $n_1 \Rightarrow \left\| x_i^{(p)} - x \right\| = Sup_{i=1}^{\infty} \left| x_i^{(p)} - x_i \right| \leq \frac{\varepsilon}{2} < \varepsilon$

$x^{(p)} \to x$ *Also from*

$$|x_i| = \left| x_i - x_i^{(p)} + x_i^{(p)} \right|$$

$$\leq \left| x_i - x_i^{(p)} \right| + \left| x_i^{(p)} \right|$$

$$|x_i| < \frac{\varepsilon}{2} + k_p \ (finite\, number)$$

Hence $x = \{x_i\} \in l^\infty$

$\Rightarrow l^\infty$ *is Banach space.*

$$\begin{array}{llll} x^p = x_1^{(1)} & x_2^{(1)}, \ldots\ldots\ldots & x_n^{(1)} \ldots\ldots \\ \quad\ x_1^{(2)} & x_2^{(2)}, \ldots\ldots\ldots & x_n^{(2)} \ldots\ldots \\ \quad\ . & . & . \ \ \ldots\ldots \\ \quad\ . & . & . \ \ \ldots\ldots \\ x = (x_1 & x_2 & x_n \ldots\ldots\ldots) \end{array}$$

$x^p \to x$ *is called Banach space*

Question:

What do you know about c-space show that it is a norm space and Hence a Banach Space.

Solution:

C-space:

Space of all convergent sequence in F (R or C) and it is a sub-space of l^∞ where

x = $\{x_i\}$ $\in$ C and

$\|\cdot\| : c \to F$ such that

$$\|x\| = \sup_{i=1}^{\infty} |x_i|$$

$$N_1: \ \|x\| \geq 0 \qquad \because \sup_{i=1}^{\infty} |x_i| \geq 0$$

$$N_2: \ \|x\| = 0 \quad \Leftrightarrow \quad \sup_{i=1}^{\infty} |x_i| = 0$$

$$\Leftrightarrow \quad |x_i| = 0$$

$$\Leftrightarrow \quad (x_1, x_2, x_3, \ldots\ldots\ldots x_n, \ldots.) = (0, 0, \ldots\ldots\ldots 0, \ldots\ldots)$$

$$\Leftrightarrow x = 0$$

$$N_3: \|\alpha x\| = \sup_{i=1}^{\infty} |\alpha x_i|$$

$$= |\alpha| \sup_{i=1}^{\infty} |x_i|$$

$$= |\alpha| \|x_i\|$$

$$N_4: \ \|x + y\| = \sup_{i=1}^{\infty} |x_i + y_i|$$

$$\leq \sup_{i=1}^{\infty} |x_i| + \sup_{i=1}^{\infty} |y_i|$$

$$\|x + y\| \leq \|x\| + \|y\| \qquad Hence (c, \|.\|) is\, a\, normed\, space.$$

Now to prove c is Banach space we shall prove that c is closed subspace of l^∞.

Let x be a limit point of c then x $\in$ l^∞ so that x = $\{x_i\}$.

By definition of limit point there is a sequence $\{x^{(p)}\}$ in C such that $\lim_{p\to\infty} x^{(p)} = x$

Hence for $\varepsilon > 0$ $\exists$ $n_o \in N$, $n_o = n_o(\varepsilon)$ s.t

$$S.t \quad \forall\, p\,;\; p \geq n_o \Rightarrow \left\| x^{(p)} - x \right\| = \sup_{i=1}^{\infty} \left\| x_i^{(p)} - x_i \right\|$$

$$\forall\, p\,; p \geq n_o \Rightarrow \left| x_i^{(p)} - x_i \right| \leq \left\| x^{(p)} - x \right\| < \varepsilon/3$$

$$\forall\, p\,; p \geq n_o \Rightarrow \left| x^{(p)} - x_i \right| < \varepsilon/3 \qquad \forall\; i = 1, 2, \ldots\ldots\ldots\infty$$

Now consider the sequence $x^{(n_o)} = \{x_i^{(n_o)}\}$

Then $x^{(n_o)} \in$ C and so is convergent.

But a convergent sequence will be a Cauchy sequence.

So, for $\varepsilon > 0$ $\exists$ $n_1 \in$ N s.t

$\forall$ i,j ; i,j $\geq n_1 \Rightarrow \left| x_i^{(n_o)} - x_j^{(n_o)} \right| < \varepsilon/3$

Take $n_2 = \max(n_o, n_1)$ then

$$\forall\, i,j\,;\; i,j \geq n_2 \Rightarrow \left| x_i - x_j \right| = \left| x_i - x_i^{(n_o)} + x_i^{(n_o)} - x_j^{(n_o)} + x_j^{(n_o)} - x_j \right|$$

$$\leq \left| x_i - x_i^{(n_o)} \right| + \left| x_i^{(n_o)} - x_j^{(n_o)} \right| + \left| x_j^{(n_o)} - x_j \right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

$\forall$ i,j ; i,j $\geq n_o \Rightarrow \left| x_i - x_j \right| < \varepsilon$

$\Rightarrow$ $\{x_i\}$ is a Cauchy sequence in R or C .

Since R or C is complete. So x = $\{x_i\}$ is convergent and hence x $\in$ C

Thus, c is closed subspace of l^∞

Thus, c is complete and hence Banach space.

" A subspace Y of complete metric space X is complete iff Y is close in X"

Question:

What is meant by C_o-space?? Show that it is a Banach Space.

Solution:

Space of all convergent sequence in F(R or C) and it is a subspace of C which is subspace of l^∞ where x = $\{x_i\} \in C_o$ and

$$\|\cdot\| : C_o \to F \quad \text{such that}$$

$$\|x\| = \sup_{i=1}^{\infty} |x_i|$$

To prove C_o is Banach space. We shall prove that C_o is closed subspace of l^∞.

Let x be limit point of C_o the x $\in l^\infty$

So that x = $\{x_i\}$

By definition of limit point there is a sequence $\{x^{(p)}\}$ in C_o such that $\lim_{p\to\infty} x^{(p)} = x$

Hence for $\varepsilon > 0$ $\exists$ $n_o \in N$, $n_o = n_o(\varepsilon)$ s.t

$\forall$ p ; $p \ge p_0 \Rightarrow \|x^{(p)} - x\| = \sup_{i=1}^{\infty} |x_i^{(p)} - x_i| < \varepsilon/3$

$\forall p; p \ge p_o \Rightarrow |x_i^{(p)} - x_i| \le \|x^{(p)} - x_i\| < \varepsilon/3$

$\forall p; p \ge p_o \Rightarrow |x^{(p)} - x_i| < \varepsilon/3 \qquad \forall\ i = 1, 2, \ldots\ldots\ldots\infty$

Now consider the sequence $x^{(n_o)} = \{x_i^{(n_o)}\}$

Then $x^{(n_o)} \in C_o$ and so is convergent.

But a convergent sequence will be a Cauchy sequence.

So, for $\varepsilon > 0$ $\exists$ $n_1 \in$ N s.t

$\forall$ i,j ; i,j $\ge n_1 \Rightarrow \left|x_i^{(n_o)} - x_j^{(n_o)}\right| < \varepsilon/3$

Take $n_2 = \max(n_o, n_1)$ then

$\forall$ i,j ; i,j $\ge n_2 \Rightarrow |x_i - x_j| = \left|x_i - x_i^{(n_o)} + x_i^{(n_o)} - x_j^{(n_o)} + x_j^{(n_o)} - x_j\right|$

$$\leq \left|x_i - x_i^{(n_o)}\right| + \left|x_i^{(n_o)} - x_j^{(n_o)}\right| + \left|x_j^{(n_o)} - x_j\right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$$

$\forall$ i,j ; i,j $\geq n_o \Rightarrow |x_i - x_j| < \varepsilon$

$\Rightarrow$ $\{x_i\}$ is a Cauchy sequence in R or C.

Since R or C is complete. So x = $\{x_i\}$ is convergent and hence x $\in C_o$

Thus, C_o is closed subspace of l^∞

Thus, C_o is complete and hence Banach space.

Question:

Give an example of space which is not a Banach space.

Solution:

The space Q of rational number is a subset of the Banach space R

We Know

$\overline{Q}$ = R $\neq$ Q

$\Rightarrow$ $\overline{Q} \neq$ Q is not closed in R

$\Rightarrow$ Q is not complete space

$\Rightarrow$ Q is not a Banach space

$A = \overline{A}$

Iff A is closed

Lecture # 8

Convex Set:

Let X be a normed space C is a subset of x then it is said to be convex set if

$\forall \quad x, y \in C \qquad \exists \ \alpha \in [0,1]$

s.t $\quad \alpha x + (1-\alpha)y \in C$

or $\quad \forall \ x, y \in C \qquad \exists \ \alpha,\beta \in [0,1]$

s.t $\alpha x+\beta y \in C \qquad$ when $\alpha+\beta = 1$

Note:

Every subspace of a Linear space is convex set but converse may not be true.

Theorem:

Prove that $x + S = \{x+s, s \in S\}$ where S is a subspace of N is convex.

Solution:

Let u, $u' \in x + S$

Then $\quad u = x + s$

$u' = x + s' , \qquad s, s' \in S$

Also $\quad \alpha \in [0,1]$

Consider

$\alpha u+(1-\alpha)u' = \alpha(x + s)+(1-\alpha)(x +s')$

$= \alpha x +\alpha s +(1-\alpha)x+(1-\alpha)s'$

$= \alpha x+(1-\alpha)x+\alpha s+(1-\alpha)s'$

$= \alpha x+(1-\alpha)x+ s'' \qquad \because s'' = \alpha s+(1-\alpha)s'$

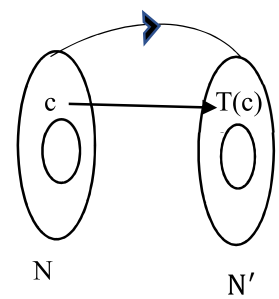
$= \alpha x+ x-\alpha x+s''$

$= x + s'' \quad \in \quad x + S$

$\Rightarrow \qquad$ x+S is convex set

Theorem:

Let $T: N \to N'$ be a linear transformation and C is convex subset of N. Then show that T(C) is also convex in N'.



Solution:

Let u, u' ∈ T(C) then ∃ c, c' ∈ C

s.t u = T(c)

u' = T(c')

For α ∈ [0,1]

αu + (1-α)u' = αT(c) +(1-α)T(c')

= T(αc) + T((1-α)c') ∵ T(αx) = αT(x)

= T(αc+(1-α)c') T(X) + T(y) = T(x+y)

= T(c'') ∈ T(C) ∵ c'' = αc+(1-α)c'

⇒ T(C) is convex in N'.

Question:

For any convex subset K and L of a Linear Space N. Prove that

K + L = { x+y; x ∈ K , y ∈ L} is convex.

Solution:

Let u, u' ∈ K+L then ∃ x, x' ∈ K and y, y' ∈ L

s.t

u = x+y

u' = x'+y' and α ∈ [0,1]

Consider

αu + (1-α)u' = α(x+y) +(1-α)T(x'+y')

= αx + αy +(1-α)x'+(1-α)y'

= αx+(1-α)x'+ αy +(1-α)y'

= (x''+y'') ∈ K+L

Where x'' = αx+(1-α)x' , y'' = αy+(1-α)y'

⇒ K+L is convex in N.

Question:

Let N be a norm space then define an open ball in norm space. Prove that open ball in norm space is convex.

Solution:

Open ball:

Let N be a normed space r > 0 the

$$B(x_0,r) = \{ x \in N : \|x - x_0\| < r \}$$

Let x, $x' \in B(x_0,r)$ then

$$\|x - x_0\| < r$$

And $\|x' - x_0\| < r$ and $\alpha \in [0,1]$

Consider

$$\|\alpha x + (1-\alpha)x' - x_0\| = \|\alpha x + (1-\alpha)x' + \alpha x_0 - \alpha x_0 - x_0\|$$

$$= \|\alpha(x - x_0) + (1-\alpha)x' - (1-\alpha)x_0\|$$

$$= \|\alpha(x - x_0) + (1-\alpha)(x' - x_0)\|$$

$$\leq \|\alpha(x - x_0)\| + \|(1-\alpha)(x' - x_0)\|$$

$$\leq |\alpha| \, \|(x - x_0)\| + |1-\alpha| \, \|(x' - x_0)\|$$

$$\leq \alpha r + (1-\alpha) r$$

$$\|\alpha x + (1-\alpha)x' - x_0\| < r$$

$\Rightarrow$ $\alpha x + (1-\alpha)x' \in B(x_0,r)$

$\Rightarrow$ $B(x_0,r)$ $\in$ is convex in N

Question:

Let N be a norm space then define a close ball in norm space. Prove that a close ball in a norm space is convex.

Solution:

Let N be a norm space r > 0

Then $\overline{B}(x_0,r) = \{ x \in N : \|x - x_0\| \leq r \}$

Let x, $x' \in \overline{B}(x_0,r)$ then

$\|x - x_0\| \leq$ r

And $\|x' - x_0\| \leq$ r and $\alpha \in [0,1]$

Consider

$\|\alpha x + (1-\alpha)x' - x_0\| = \|\alpha x + (1-\alpha)x' + \alpha x_0 - \alpha x_0 - x_0\|$

$= \|\alpha(x - x_0) + (1-\alpha)x' - (1-\alpha)x_0\|$

$= \|\alpha(x - x_0) + (1-\alpha)(x' - x_0)\|$

$\leq \|\alpha(x - x_0)\| + \|(1-\alpha)(x' - x_0)\|$

$\leq |\alpha| \, \|(x - x_0)\| + |1-\alpha| \, \|(x' - x_0)\|$

$\leq \alpha$ r + (1-α) r

$\leq$ r

$\Rightarrow$ $\alpha x + (1-\alpha)x' \in \overline{B}(x_0, r)$

$\Rightarrow$ $\overline{B}(x_0, r)$ $\in$ is convex in N

Theorem:

Let C be a convex set in Linear space N then for $\alpha \geq 0$, $\beta \geq 0$. Prove that

$(\alpha+\beta)C = \alpha C + \beta C$

Proof: Case-I

If $\alpha = 0$ or $\beta = 0$

Then $(\alpha+\beta)C = \alpha C + \beta C$

Holds trivially

Case-II

Let $\alpha > 0$, $\beta > 0$

And $z \in (\alpha+\beta)C$ then $\exists$ $c \in C$

s.t $z = (\alpha+\beta)c$

$z = \alpha c + \beta c \in \alpha C + \beta C$

$\Rightarrow$ $(\alpha+\beta)C \subseteq \alpha C + \beta C$ (1)

Conversely, Let $u \in \alpha C + \beta C$

Then $\exists$ c,d $\in$ C

s.t u = αc+βd = (α+β) $(\frac{\alpha c}{\alpha+\beta} + \frac{\beta d}{\alpha+\beta})$

= (α+β) $(\frac{\alpha}{\alpha+\beta} c + (1 - \frac{\alpha}{\alpha+\beta})d)$

= (α+β) $\omega \in$ (α+β)C

$\Rightarrow$ αC+βC $\subseteq$ (α+β) C …..(2)

where $\omega = (\frac{\alpha}{\alpha+\beta} c + (1 - \frac{\alpha}{\alpha+\beta})d) \in C$

By (1) and (2)

(α+β)C = αC +βC

Question:

Show that closure of a convex subset of a norm space is a convex set.

Solution:

Let C be convex subset of normal space N and x,y $\in \overline{C}$

Then there exists a sequence $\{x_n\}$ and $\{y_n\}$ in C s.t

$x_n \to$ x

$y_n \to$ y

For α $\in$ [0,1]

$\alpha x_n + (1-\alpha) y_n \in C$

$\because$ Addition and scalar multiplication is continuous so

$\alpha x_n + (1-\alpha) y_n \to \alpha x + (1-\alpha) y \in \overline{C}$

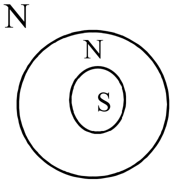
$\Rightarrow \overline{C}$ is convex in N

Lecture # 9

Quotient Space:

Suppose that N is a norm space S is a closed subspace of N then $\forall$ x $\in$ N

$x + S = \{x+s \; ; s \in S\}$ is called Co-set of S determined by S.



It should be noted that N/S (quotient space) is a linear space under addition and scalar multiplication defined by

$$x + S + y + S = x+y+ S \qquad ; x,y \in N$$

$$\alpha(x+S) = \alpha x + S \qquad ; x \in N , \alpha \in F$$

$$\|x+S\|_1 = \underset{s \in S}{Inf} \|x+s\|$$

Question:

Show that a quotient space N/S is a norm space under the norm

$$\|x+S\|_1 = \underset{s \in S}{Inf} \|x+s\|$$

Solution:

$$N_1 : \|x+S\|_1 \geq 0 \qquad \because \underset{s \in S}{Inf} \|x+s\| \geq 0$$

$$N_2 : \|x+S\| = 0 \qquad \Leftrightarrow \qquad \underset{s \in S}{Inf} \|x+s\| = 0$$

So by the property of infimum $\exists$ a Seq$\{S_n\}$ in S such that

$$\|x+s_n\| \to 0 \quad as \quad n \to \infty$$

$$But\,then \quad x + s_n \to 0 \;\; that\,is \quad s_n \to -x \quad as \quad n \to \infty$$

$$Since\; S\, is\; closed\; subspace\; of\; N$$

$$\Rightarrow x \in S \quad Hence$$

$$x + S = 0 \quad the\,zero\;element\,of\; N / S$$

$$\Rightarrow \|x+S\|_1 = 0 \quad \Leftrightarrow \quad x + S = 0$$

N_3: For any scalar α and x+s $\in$ N/S

Consider the elements

$$\alpha(x+S) = \alpha x+S$$

Case-I

If $\alpha \neq o$ then

$$\|\alpha(x+s)\|_1 = \|0.x+S\|_1 = \|S\|_1 = 0 = |\alpha|\|x+s\|_1$$

Case – II

If $\alpha \neq 0$ *then*

$$\|\alpha(x+s)\|_1 = \underset{s\in S}{Inf}\|\alpha(x+s)\|$$

$$= \underset{s\in S}{Inf}\|\alpha x+\alpha s\|$$

$$= |\alpha|\underset{s\in S}{Inf}\|x+s\|$$

$$= |\alpha|\|x+S\|$$

N_4: Let x+S and y+S $\in$ N/S

Then the sequence $\{x_n\}$ and $\{y_n\}$ in S

Such that

$$\underset{n\to\infty}{Lim}\|x+x_n\| = \|x+S\|_1, \quad \underset{n\to\infty}{Lim}\|y+y_n\| = \|y+S\|_1$$

Hence for any x,y in N and the def. of Infimum

$$\|x+s+y+s\|_1 = \|x+y+S\| \leq \|x+y+x_n+y_n\|$$

$$\|x+x_n\| \leq \|y+y_n\|$$

Taking limit as n $\to \infty$

$$\|x+s+y+S\|_1 = \|x+y+S\|_1 \leq \underset{n\to\infty}{Lim}\|x+x_n\| + \underset{n\to\infty}{Lim}\|y+y_n\|$$

$$\|x+s+y+S\| \leq \|x+S\| + \|y+S\|$$

Hence $(N/S, \|.\|)$ *is a normed space*

Question:

Show that quotient N/S is a Banach space under the norm

$$\|x+S\|_1 = \inf_{s\in S}\|x+s\|$$

Solution:

Let $\{x_n + S\}; x_n \in$ N Cauchy sequence in N/S

Then for ε > 0 ∃ $n_1 \in N, \quad n_1 = n_1(\varepsilon)$ s.t

$\forall m,n \,; m,n \geq n_1 \Rightarrow \|(x_m + S) - (x_n + S)\|_1 = \|x_m - x_n + S\| < \varepsilon$ ....(1)

$\because \|x+S\| = \inf_{s\in S}\|x+s\|$

Take $\varepsilon = \frac{1}{2}$, $m = n_1$ $and\, n = n_1 + 1$

$\forall m,n \,; m,n \geq n_1 \Rightarrow \|(x_{n_1} + S) - (x_{n_{1+1}} + S)\| = \|x_{n1} - x_{n1+1} + S\| < \frac{1}{2}$

If we choose ε = $\frac{1}{4}$ ∃ $n_2 \in N, \quad n_2 = n_2(\varepsilon)$ s.t

$\forall m,n \,; m,n \geq n_2 \Rightarrow \|(x_{n_2} + S) - (x_{n_{2+1}} + S)\| = \|x_{n2} - x_{n2+1} + S\| < \frac{1}{4}$

Continuing in this way

. . . .

. . . .

. . . .

. . . .

If we choose ε = $\frac{1}{2^k}$ ∃ $n_k \in N, \quad n_k = n_k(\varepsilon)$ $s.t$ s.t

$\forall m,n \,; m,n \geq n_k \Rightarrow \|(x_{nk} + S) - (x_{n_{k+1}} + S)\| = \|x_{nk} - x_{nk+1} + S\| < \frac{1}{2^k}$

In each $x_{n_k} + S$ *and* $x_{n_{k+1}} + S$ *select vectors* y_k *and* y_{k+1} *such that*

$\|y_k - y_{k+1}\| < \frac{1}{2^k} \quad by\ (1)$

Then for k > k'

$\left\|y_k - y_{k'}\right\| = \left\|y_k - y_{k+1} + y_{k+1} - y_{k+2} + y_{k+2} \ldots\ldots\ldots\ldots + y_{k-1} + y_{k'}\right\|$

$\leq \|y_k - y_{k+1}\| + \|y_{k+1} - y_{k+2}\| + \ldots\ldots\ldots\ldots\ldots\ldots + \left\|y_{k-1} + y_{k'}\right\|$

$\left\|y_k - y_{k'}\right\| < \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \ldots.. + \frac{1}{2^{k-1}}$

$< \frac{\frac{1}{2^k}}{1 - \frac{1}{2}} \qquad \because S_\infty = \frac{a}{1-r}$

$\left\|y_k - y_{k'}\right\| < \frac{1}{2^{k-1}} \qquad as \quad k \to \infty$

Thus $\{y_k\}$ is convergent sequence in N.

Since N is complete so $y_k \to$ y $\in$ N

Hence $\left\|(x_{nk} + S) - (x_{n_{k+1}} + S)\right\| \leq \|y_k - y\| \to 0$

$\Rightarrow x_{n_k} + S \to y + S \in N/S$

$N/S \ \ is \ \ complete$

We use this theorem in above proof

"A Cauchy sequence converges iff it has convergent subsequence"

Equivalent Norms:

Let N be a Norm space $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms define on N then $\|\cdot\|_1$ is said to be equivalent to $\|\cdot\|_2$ ($\|\cdot\|_1 \sim \|\cdot\|_2$) if $\exists$ a > 0 , b > 0 be real numbers such that

$$a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2 \qquad \forall \ x \in N$$

Question:

Show that the relation being equivalent to among the norms that can be defined on a linear space N is an equivalence relation.

Solution:

Step-I Reflexive relation

For any $\|\cdot\|$ on N the condition

$$a\|x\| \le \|x\| \le b\,\|x\| \qquad \forall\ x \in N$$

holds if a = 1 = b

$\Rightarrow\ \|\cdot\| \sim \|\cdot\|$

i.e. Relation is Reflexive

Step-II Symmetric relation:

Let $\|\cdot\|_1 \sim \|\cdot\|_2$ then $\exists$ a > 0 , b > 0 real numbers such that

$$a\|x\|_2 \le \|x\|_1 \le b\|x\|_2 \qquad \forall\ x \in N$$

$$a\|x\|_2 \le \|x\|_1 \quad \Rightarrow \quad \|x\|_2 \le \frac{1}{a}\|x\|_1$$

and $\quad b\|x\|_2 \le \|x\|_1 \quad \Rightarrow \quad \|x\|_2 \ge \frac{1}{b}\|x\|_1$

$\Rightarrow \frac{1}{b}\|x\|_1 \le \|x\|_2 \le \frac{1}{a}\|x\|_1$

$\Rightarrow\ \|\cdot\|_2 \sim \|\cdot\|_1$

i.e. Relation is symmetric

Step-III Transitive Relation:

Let $\|\cdot\| \sim \|\cdot\|_2$ and $\|\cdot\|_1 \sim \|\cdot\|_2$ then $\exists$ a > 0 , b > 0 , $a_1 > 0$, $b_1 > 0$ s.t

$$a\|x\|_1 \le \|x\| \le b\|x\|_1 \qquad (1)$$

$$a_1\|x\|_2 \le \|x\|_1 \le b_1\|x\|_2 \qquad (2) \qquad \forall\ x \in N$$

From (2)

$$a_1\|x\|_2 \le \|x\|_1 \quad \text{and} \quad \|x\|_1 \le b_1\|x\|_2$$

$$aa_1\|x\|_2 \le a\|x\|_1 \quad .....(3) \quad \text{and} \quad b\|x\|_1 \le bb_1\|x\|_2 \ .....(4)$$

Using (1) ,(3) and (4)

$$aa_1\|x\|_2 \le a\|x\|_1 \le \|x\| \le b\|x\|_1 \le bb_1\|x\|_2$$

$$aa_1\|x\|_2 \le \|x\| \le bb_1\|x\| \quad \Rightarrow \quad a_2\|x\|_2 \le \|x\| \le b_2\|x\| \ \because aa_1 = a_2\ ,\ bb_1 = b_2$$

$\Rightarrow\ \|\cdot\| \sim \|\cdot\|_2$ i.e. Relation is transitive

$\Rightarrow$ Hence Relation is Equivalence

Lecture # 10

Theorem:

Show that any two-equivalent norm on linear space N define the same topology on N.

Proof:

$\|.\| \sim \|.\|_1$ *on N then* $\exists\ a, b \in R, \quad a > 0, b > 0 \ s.t$

$a\|x\|_1 \leq \|x\| \leq b\|x\|_1 \qquad \ldots(1) \qquad \forall \quad x \in N$

We shall prove that a Basic open set in (N, $\|\cdot\|$) and conversely.

For this let x $\in$ N and B(x,r) be an open ball in (N, $\|\cdot\|$). We show that B(x,r) will open in (N, $\|\cdot\|$)

Let y $\in$ B(x,r) and $\|x-y\| = r_1 < r$

$\Rightarrow \quad \|x-y\| < r$ By def.

Conversely $B_1^*(y, r')$ is an open ball in (N, $\|\cdot\|$)

Where $r' = \left(\frac{r-r_1}{b}\right) > 0$

Then for any z $\in B_1^*(y, r')$

$\|z-y\|_1 < \ r'$

And $\|z-x\| = \|z-y+y-x\|$

$\leq \|z-y\| + \|y-x\|$

$\leq b\|z-y\|_1 + \|y-x\| \qquad \because$ by (1) $\|x\| = b\|x\|_1$

$< b\left(\frac{r-r_1}{b}\right) + r_1$

$\|z-x\| < r$

Hence z $\in$ B(x,r)

y $\in B_1^*(y, r') \subseteq$ B(x,r)

$\Rightarrow$ B(x,r) is open ball in (N, $\|.\|_1$)

Similarly, every open ball in (N, $\|.\|_1$) will be open in (N, $\|\cdot\|$)

Hence topologies induced by $\|\cdot\|$ and $\|.\|_1$ are same.

Theorem:

Let $\|\cdot\| \sim \|.\|_1$ then show that every Cauchy sequence in (N, $\|\cdot\|$) is also a Cauchy sequence in (N, $\|.\|_1$)

Proof:

$Given\ \|.\| \sim \|.\|_1\ on\ N\ then\ \exists\ a, b \in R, \quad a > 0, b > 0\ \ s.t$

$a\|x\|_1 \leq \|x\| \leq b\|x\|_1 \qquad \ldots(1) \qquad \forall \quad x \in N$

Let $\{x_n\}$ be a Cauchy sequence in (N, $\|\cdot\|$) Then for

Then for $\varepsilon > 0\ \exists\ n_0 \in N$, $n_0 = n_0(\varepsilon)$

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow \|x_m - x_n\| < \varepsilon$

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow \|x_m - x_n\|_1 \leq \frac{1}{a}\|x_m - x_n\| < \frac{\varepsilon}{a} \qquad \because$ by (1)

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow \|x_m - x_n\|_1 < \frac{\varepsilon}{a}$

$\forall \quad m,n\ ;\ m,n \geq n_0 \Rightarrow \|x_m - x_n\|_1 < \varepsilon' \qquad \because\ \varepsilon' = \frac{\varepsilon}{a}$

$\Rightarrow\ \{x_n\}$ is a Cauchy sequence in (N, $\|.\|_1$)

Question:

Let $\|\cdot\| \sim \|.\|_1$ then prove that a sequence $\{x_n\}$ in (N, $\|\cdot\|$) is converges to x $\in$(N, $\|\cdot\|$) iff $x_n \to x \in$ (N, $\|.\|_1$)

Proof: $Given\ \|.\| \sim \|.\|_1\ on\ N\ then\ \exists\ a, b \in R, \quad a > 0, b > 0\ \ s.t$

$a\|x\|_1 \leq \|x\| \leq b\|x\|_1 \qquad \ldots(1) \qquad \forall \quad x \in N$

Let $\{x_n\}$ be a convergent sequence in (N, $\|\cdot\|$)

Then for $\varepsilon > 0\ \exists\ n_0 \in N$, $n_0 = n_0(\varepsilon)$

$\forall \quad n\ ;\ n \geq n_0 \Rightarrow \|x_m - x\| < \varepsilon$

$\forall \quad n\ ;\ n \geq n_0 \Rightarrow \|x_m - x\|_1 \leq \frac{1}{a}\|x_m - x\| < \frac{\varepsilon}{a} \qquad \because$ by (1)

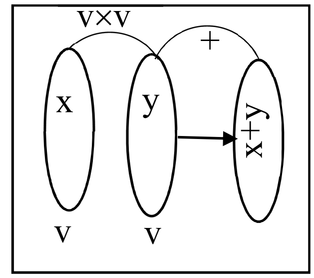
$\forall \quad n\ ;\ n \geq n_0 \Rightarrow \|x_m - x\|_1 < \frac{\varepsilon}{a}$

$\forall \quad n\ ;\ n \geq n_0 \Rightarrow \|x_m - x\|_1 < \varepsilon' \qquad \because\ \varepsilon' = \frac{\varepsilon}{a}$

$\Rightarrow\ \{x_n\}$ is a convergent sequence in (N, $\|.\|_1$)

Topological Linear Space:

A linear space V(F) where F is R or C is said to be topological linear space if



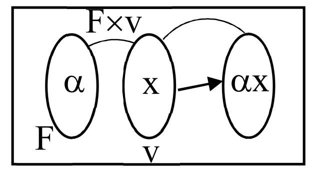
(i) V is linear space
(ii) V is topological space
(iii) The addition and scalar multiplication as function

$+ : V \times V \rightarrow V$ s.t

$(x,y) \rightarrow x+y$ $\forall\ x,y \in V$

And '.' : $F \times V \rightarrow V$ s.t

$(\alpha,x) \rightarrow \alpha x$ $\forall\ \alpha \in F,\ x \in V$



Linear Operator:

Let N and M are topological linear spaces then function $T : N \rightarrow M$ is said to be linear operator if for every

$a,b \in F,\ x,y \in N$

$T(ax+by) = aT(x) + bT(y)$ Or

(i) $T(x+y) = T(x) + T(y)$
(ii) $T(\alpha x) = \alpha T(x)$, $\alpha \in F$ and $x \in N$

Example:

$I: N \rightarrow N$ s.t

$T(x) = x$ $\forall\ x \in N$

For $a_1, a_2 \in F$, $x_1, x_2 \in N$

$I(a_1x_1 + a_2x_2) = I(a_1x_1) + I(a_2x_2)$

$= a_1 I(x_1) + a_2 I(x_2)$

Example:

O linear operator

$O : N \rightarrow N$ s.t

$a_1, a_2 \in F$, $x_1, x_2 \in N$

$O(a_1x_1 + a_2x_2) = O(a_1x_1) + O(a_2x_2)$

$= a_1.O(x_1) + a_2.O(x_2)$

Lecture # 11

Theorem:

Let T : N→M be surjective linear operation then

(i) T^{-1} exist iff T(x) = 0 ⇒ x = 0 , x ∈ N
(ii) T is bijective & dimN = n Then dim M = n

Proof:

(i) Suppose T^{-1} exist then T^{-1} is linear. Also for any

x ∈ N let T(x) = 0

⇒ T(x) = T(0) ∵ T(0) = 0

⇒ x = 0

Conversely,

Le T(x) = 0 to prove

T : N→M bijective , Suppose x_1, x_2 ∈ N

T(x_1) = T(x_2)

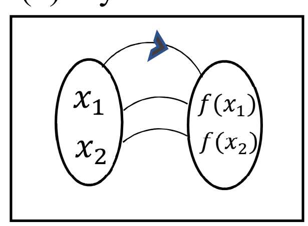
T(x_1) − T(x_2) = 0 ∵ T is linear

$x_1 - x_2 = 0$

$x_1 = x_2$

The function T^{-1} : N→M defined by T^{-1}(y) = x where T(x) = y

is then the inverse of T i.e.



T^{-1} : N→M exist

(∵ if a function is bijective then inverse exist)

Solution: (ii)

Let T : N→M is bijective function and dimN = n ∵ dim= dimension

Let B = { e_1, e_2,……. , e_n} be a basis of N we shall prove

B^* = {T(e_1),T(e_2),…..,T(e_n)} is a Basis of M

(i) B^* = {T(e_1),T(e_2),…..,T(e_n)} is L.I

Let $\sum_{i=1}^{n} a_i T(e_i) = 0$ $\quad$ *for some* $a_i \in \mathbb{F}; i = 1, 2, \ldots, n$

$T(\sum_{i=1}^{n} a_i e_i) = T(0)$ $\quad \because T(0) = 0$

$$\sum_{i=1}^{n} a_i e_i = 0$$

$\Rightarrow B^* \text{ is } L.I \quad \because B = \{e_1, e_2, \ldots\ldots, e_n\} \text{ is Basis of } N$

(ii) $\quad$ *Let* $y \in M$ $\quad \because T$ *is surjective so* $\exists$ *an* $x \in N$

s.t $\quad T(x) = y$

$\because \; B = \{e_1, e_2, \ldots\ldots, e_n\}$ *is Basis of* N $\exists\, a_1, a_2, \ldots. a_n \in \mathbb{F}$

s.t $x = \sum_{i=1}^{n} a_i e_i$

so $y = T(x)$

$$= T(\sum_{i=1}^{n} a_i e_i)$$

$$y = \sum_{i=1}^{n} a_i T(e_i)$$

$\Rightarrow B^* = \{T(e_1), T(e_2), \ldots., T(e_n)\}$ *span* M

Hence $\quad \dim N = n = \dim M$

Theorem:

Let T : N→M be a linear operator then prove that T is continuous on N iff T is bounded .

Proof:

Suppose that T is continuous on N then it is continuous ∀ x_o ∈ N so for ε > 0 ∃ δ > 0 s.t $\quad \forall \quad x \in N; \; \|x - x_0\| < \delta \;\; \Rightarrow \|T(x) - T(x_0)\| < \varepsilon \;\; \ldots(1)$

Let y ∈ N and put

$$x = x_0 + \frac{\delta}{2\|y\|} y$$

$$x - x_0 = \frac{\delta}{2\|y\|} y$$

$\because$ *T is linear and*

$$\|x - x_0\| = \left\| \frac{\delta}{2\|y\|} y \right\| = \frac{\delta}{2\|y\|} \|y\|$$

$$= \frac{\delta}{2} < \delta$$

We have

$$\|T(x) - T(x_0)\| = \|T(x - x_0)\|$$

$$= \left\| T\left(\frac{\delta}{2\|y\|} y \right) \right\|$$

$$= \frac{\delta}{2\|y\|} \|T(y)\| < \varepsilon \qquad by(1)$$

$$\|T(y)\| < \frac{2\varepsilon}{\delta} \|y\|$$

$$\|T(y)\| \leq K \|y\| \qquad \forall \quad y \in N$$

$$\Rightarrow \quad T : N \to M \quad is \ bounded$$

Conversely,

Let T : N→M is bounded linear operator then ∃ K > 0 s.t

$$\|T(x)\| \leq K\|x\| \qquad \forall \quad x \in N$$

So for any ε > 0 choose $\delta = \frac{\varepsilon}{K}$

$$\|x - x_0\| < \delta \Rightarrow \|T(x) - T(x_0)\| = \|T(x - x_0)\|$$

$$\leq K\|x - x_0\|$$

$$< K.\ \delta$$

$$< K \frac{\varepsilon}{K}$$

$$\|T(x) - T(x_0)\| < \varepsilon$$

⇒ T : N→M is continuous

Theorem:

Prove that every linear operator on a finite dimensional norm space is bounded.

Proof: Let N be a finite dimensional normed space and B = $\{e_1, e_2, \ldots\ldots, e_n\}$ be a basis of N

Let T : N→M be a linear operator

For any x ∈ N

$$x=\sum_{i=1}^{n}x_i e_i$$

Since T is linear

$$T(x)=T\left(\sum_{i=1}^{n}x_i e_i\right) \Rightarrow T(x)=\sum_{i=1}^{n}x_i T(e_i)$$

$$\|T(x)\|=\left\|\sum_{i=1}^{n}x_i\,T(e_i)\right\| \leq \sum_{i=1}^{n}|x_i|\|T(e_i)\|$$

$$\leq b\sum_{i=1}^{n}|x_i| \qquad ....(1) \qquad b=\sup_{i=1}^{n}\|T(e_i)\|$$

By a Lemma "Let $(N,\|.\|)$ be a normed space $B=\{x_1,x_2,\ldots,x_n\}$ be a basis of N

Then $\exists c>0$, s.t for $a_1,a_2,\ldots,a_n\in\mathbb{F}$

$$\left\|\sum_{i=1}^{n}a_i x_i\right\|\geq c\sum_{i=1}^{n}|a_i|"$$

In this case $\|x\|=\left\|\sum_{i=1}^{n}x_i e_i\right\|\geq c\sum_{i=1}^{n}|x_i|$ $\quad\because$ *by lemma*

$$\sum_{i=1}^{n}|x_i|\leq\frac{1}{c}\|x\| \qquad put\ in\ (1)$$

$$\sum_{i=1}^{n}|x_i|\leq\frac{b}{c}\|x\|$$

$$\leq K\|x\| \qquad \because K=\frac{b}{c}>0$$

⇒ T : N→M is bounded Linear operator

Lecture # 12

Finite Dimensional Normed Space:

Suppose that N be a normed space B $=\{x_1, x_2, ...x_n\}$ *is a basis of* N , $\forall\, x \in N$

(i) $\quad x = \sum_{i=i}^{n} a_i x_i \quad ; \quad a_i \in F\ ,\ i = 1,2,........n$

(ii) $\quad \{x_1, x_2, ...x_n\}$ *are L.I then* $\dim N = n$

Zero Norm:

$$\|x\|_0 = \underset{i=}{\overset{n}{Sup}} |a_i|$$

Question:

Suppose that $\|\cdot\| \sim \|.\|_0$

$\|x\|_0 = \underset{i=1}{\overset{n}{Sup}} \leq |a_i|$ *on a Norm space* N_1

Let N be a finite dimensional subspace on N_1 . Show that (N,$\|.\|_0$) is complete space or Banach space.

Solution:

Since $\|\cdot\| \sim \|.\|_0$ so $\exists$ a > 0 , b > 0 s.t

$$a\|x\|_0 \leq \|x\| \leq b\|x\| \quad \forall \ x \in N \qquad .....(1)$$

If N is finite dimensional subspace of N and $\{x_1, x_2, ...x_n\}$ is a basis of N

Then $\forall$ y $\in$ N is of the form

$$y = \sum_{i=i}^{n} a_i x_i$$

Let $\{y^p\}$ be a Cauchy sequence in N then for $\varepsilon > 0\ \exists\ n_0 \in$ N , $n_0 = n_0$ (ε) s.t

$$\forall\ p,q : p,q \geq n_0 \Rightarrow \left\|y^{(p)} - y^{(q)}\right\| = \left\|\sum_{i=1}^{n} a_i x_0^p - \sum_{i=1}^{n} a_i^q x_i\right\|$$

$$= \left\| \sum_{i=1}^{n} (a_i^{(p)} - a_i^{(q)}) x_i \right\| < \varepsilon$$

$$Since \ \ \|.\|_0 \sim \frac{1}{a}\|.\| \quad by \ (1)$$

$$\forall \ p,q : p,q \geq n_0 \Rightarrow \left\| y^{(p)} - y^{(q)} \right\|_0 = \frac{1}{a} \left\| y^{(p)} - y^{(q)} \right\| < \frac{\varepsilon}{a}$$

$$\forall \ p,q : p,q \geq n_0 \Rightarrow \underset{i=1}{\overset{n}{Sup}} \left| a_i^p - a_i^q \right| < \frac{\varepsilon}{a} \qquad \because \ \|x\|_0 = \underset{i=1}{\overset{n}{Sup}} |a_i|$$

$$\forall \ p,q : p,q \geq n_0 \Rightarrow \left| a_i^{(p)} - a_i^{(q)} \right| < \frac{\varepsilon}{a}$$

$\left\{ a_i^{(p)} \right\}$ is a cauchy sequence in F (R or C) Since F is complete so $a_i^{(p)} \to a_i$ as $p \to \infty$

i.e. $\left| a_i^{(p)} - a_i^{(q)} \right| \to 0 \ \ as \ \ p \to \infty$ ....(2)

put $y = \sum_{i=i}^{n} a_i x_i$

Then $y \in N$ and $\left\| y^p - y \right\| = \left\| \sum_{i=1}^{n} a_i^p x_i - a_i x_i \right\|$

$$= \left\| \sum_{i=1}^{n} (a_i^p - a_i) x_i \right\|$$

$$\leq \sum_{i=1}^{n} \left| a_i^p - a_i \right| \left\| x_i \right\|$$

$$\left\| y^p - y \right\| \leq k \sum_{i=1}^{n} \left| a_i^p - a_i \right| \to 0 \ \ as \ \ p \to \infty \quad by \ (2) \qquad \because k = \underset{i=n}{\overset{n}{Sup}} \|x_i\|$$

$y^p \to y \in N$

$\Rightarrow N \, is \ complete$

Theorem:

Any two norms on a finite dimensional linear space are equivalent.

Proof:

Suppose that $\|\cdot\|$ and $\|.\|_1$ be any two norms define on any norm space N and dimN = n

Let $\{x_1, x_2, ...x_n\}$ be a Basis of N. Then $\forall$ x $\in$ N

$$x = \sum_{i=i}^{n} a_i x_i$$

We know if $\{x_1, x_2, ...x_n\}$ is a Basis of N then $\exists$ c > 0 s.t

$$\|x\| = \left\| \sum_{i=1}^{n} a_i x_i \right\| \geq c \sum_{i=1}^{n} |a_i| \quad .....(1) \qquad \forall\ x \in N$$

$$\Rightarrow \quad \|x\| \geq c\ s \qquad \because S = \sum_{i=1}^{n} |a_i|$$

$$\Rightarrow \quad s \leq \frac{1}{c}\|x\| \quad ....(2)$$

Also $\|x\|_1 = \left\| \sum_{i=1}^{n} a_i x_i \right\|_1$

$$\leq \sum_{i=1}^{n} |a_i| \|x_i\|_1$$

$$\|x\|_1 \leq kS \qquad \because \quad k = \sup_{i=1}^{n} \|x_i\|_1$$

$$\|x\|_1 \leq \frac{k}{c}\|x\|$$

$$\frac{c}{k}\|x\|_1 \leq \|x\|$$

$$a\|x\|_1 \leq \|x\| \qquad ...(3) \qquad \because \quad \frac{c}{k} = a > 0$$

Similarly, $\|x\|_1 = \left\|\sum_{i=1}^{n} a_i x_i\right\|_1 \geq c' \sum_{i=1}^{n} |a_i| \quad \forall \quad x \in N$

$$\|x\|_1 \geq c' S$$

$$S \leq \frac{1}{c'}\|x\|_1 \qquad \ldots(4)$$

So, $\|x\| = \left\|\sum_{i=1}^{n} a_i x_i\right\|$

$$\leq \sum_{i=1}^{n} |a_i| \|x_i\|$$

$$\leq K' \sum_{i=1}^{n} |a_i| \qquad \because K' = \sup_{i=1}^{n} \|x_i\|$$

$$\|x\| \leq K' S$$

$$\|x\| \leq \frac{K'}{c'}\|x\|_1$$

$$\|x\| \leq b\|x\|_1 \quad \ldots(5) \qquad \because \frac{K'}{c'} = b$$

By (3) and (5)

$$a\|x\|_1 \leq \|x\| \leq b\|x\|_1 \qquad \forall \quad x \in N$$

$$\Rightarrow \quad \|\cdot\|_1 \sim \|\cdot\|$$

Hence any two norms on a finite dimensional Linear space are equivalent

Lecture # 13

Question:

Show that $\|.\|_0 \sim \|.\|_1$ where $\|x\|_0 = \underset{i=1}{\overset{n}{Sup}}|x_i|$, $x = (x_1, x_2, \ldots\ldots, x_n) \in R^n$

$\|x\|_1 = \sum_{i=1}^{n}|x_i|$, $x \in R^n$

Solution:

Let $x = (x_1, x_2, \ldots\ldots, x_n)$

$y = (1, 1, \ldots\ldots, 1)$

Then $\|x\|_1 = \sum_{i=1}^{n}|x_i|$

$= \sum_{i=1}^{n}|x_i y_i|$

$\leq \sum_{i=1}^{n}|x_i||y_i|$

$\leq \underset{i=1}{\overset{n}{Sup}}|x_i| \sum_{i=1}^{n}|y_i|$

$\|x\|_1 \leq n\|x\|_0$ ….(1)

Also $\|x\|_0 = \underset{i=1}{\overset{n}{Sup}}|x_i| \leq \sum_{i=1}^{n}|x_i|$

$\|x\|_0 \leq \|x\|_1$ …(2)

From (1) and (2)

1. $\|x\|_0 \leq \|x\|_1 \leq n\|x\|_0$ $\forall$ $x \in R^n$

$\Rightarrow$ $\|.\|_0 \sim \|.\|_1$

Question:

Show that $\|.\|_1 \sim \|.\|_2$ where $\|x\|_1 = \sum_{i=1}^{n}|x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^{n}|x_i|^2}$, $x \in R^n$

Solution: We know $a^2 + b^2 \leq (a+b)^2$, $a \geq 0, b \geq 0$

$$\sum_{i=1}^{2}|x_i|^2 \leq \left(\sum_{i=1}^{2}|x_i|\right)^2$$

For n $$\sum_{i=1}^{n}|x_i|^2 \leq \left(\sum_{i=1}^{n}|x_i|\right)^2$$

$$\sqrt{\sum_{i=1}^{n}|x_i|^2} \leq \sqrt{\left(\sum_{i=1}^{n}|x_i|\right)^2}$$

$$\sqrt{\sum_{i=1}^{n}|x_i|^2} \leq \sum_{i=1}^{n}|x_i|$$

$$\|x\|_2 \leq \|x\|_1 \qquad \ldots(1)$$

Also $\|x\|_1 = \sum_{i=1}^{n}|x_i|$

$$= \sum_{i=1}^{n}|x_i y_i| \qquad \because y = (1,1,\ldots\ldots 1)$$

$$\leq \sum_{i=1}^{n}|x_i||y_i|$$

$$\|x\|_1 \leq \sqrt{\sum_{i=1}^{n}|x_i|^2} \cdot \sqrt{\sum_{i=1}^{n}|y_i|^2}$$

$$\|x\|_1 \leq \sqrt{n}\,\|x\|_2$$

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \qquad \ldots(2)$$

By (1) and (2)

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1$$

$\Rightarrow$ $\|.\|_1 \sim \|.\|_2$

Question:

Show that d(x,y) = $\sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}$ is metric space.

Solution:

M_1 : d(x,y) ≥ 0 $\quad \because \sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|} \geq 0$

M_2 : d(x,y) = 0 $\Leftrightarrow \sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|} = 0$

$$\Leftrightarrow \frac{1}{2^1}\frac{|x_1-y_1|}{1+|x_1+y_1|}+\frac{1}{2^2}\frac{|x_2-y_2|}{1+|x_2+y_2|}+.....+\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n+y_n|} = 0$$

$$\Leftrightarrow \frac{1}{2^1}\frac{|x_1-y_1|}{1+|x_1+y_1|}=0,\ \frac{1}{2^2}\frac{|x_2-y_2|}{1+|x_2+y_2|}=0,......,\frac{1}{2^n}\frac{|x_n-y_n|}{1+|x_n+y_n|} = 0$$

$$\Leftrightarrow |x_1-y_1|=0,\ |x_2-y_2|=0,.......,|x_n-y_n|=0$$

$$\Leftrightarrow x_1-y_1=0,\ x_2-y_2=0,.......,x_n-y_n=0$$

$$\Leftrightarrow x_1=y_1,\ x_2=y_2,.......,x_n=y_n$$

$$\Rightarrow (x_1,x_2,.....,x_n)=(y_1,y_2,.....,y_n)$$

d(x,y) = 0 ⇔ x = y

M_3 : $d(x,y) = \sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-y_i|}{1+|x_i-y_i|}$

$= \sum_{i=1}^{n}\frac{1}{2^i}\frac{|y_i-x_i|}{1+|y_i-x_i|} = d(y,x)$

M_4 : $d(x,z) = \sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-z_i|}{1+|x_i-z_i|} = \sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-y_i+y_i-z_i|}{1+|x_i-y_i+y_i-z_i|}$

$$\leq \sum_{i=1}^{n}\frac{1}{2^i}\frac{|x_i-y_i|+|y_i-z_i|}{1+|x_i-y_i|+|y_i-z_i|}$$

$$\leq \sum_{i=1}^{n} \frac{1}{2^i} \left(\frac{|x_i - y_i|}{1+|x_i - y_i| + |y_i - z_i|} + \frac{|y_i - z_i|}{1+|x_i - y_i| + |y_i - z_i|} \right)$$

$$d(x,z) \leq \sum_{i=1}^{n} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i - y_i|} + \sum_{i=1}^{n} \frac{1}{2^i} \frac{|y_i - z_i|}{1+|y_i - z_i|}$$

$d(x,z) \leq d(x,y) + d(y,z)$

$\Rightarrow$ d(x,y) is metric space

Question:

Show that $d(x,y) = \frac{|x_i - y_i|}{1+|x_i - y_i|}$ is metric space.

Solution:

M_1 : $d(x,y) \geq 0$ $\quad\quad \because \sum_{i=1}^{n} \frac{|x_i - y_i|}{1+|x_i - y_i|} \geq 0$

M_2 : $d(x,y) = 0 \quad \Leftrightarrow \sum_{i=1}^{n} \frac{|x_i - y_i|}{1+|x_i - y_i|} = 0$

$$\Leftrightarrow \frac{|x_1 - y_1|}{1+|x_1 + y_1|} + \frac{|x_2 - y_2|}{1+|x_2 + y_2|} + \ldots + \frac{|x_n - y_n|}{1+|x_n + y_n|} = 0$$

$$\Leftrightarrow \frac{|x_1 - y_1|}{1+|x_1 + y_1|} = 0, \frac{|x_2 - y_2|}{1+|x_2 + y_2|} = 0, \ldots, \frac{|x_n - y_n|}{1+|x_n + y_n|} = 0$$

$\Leftrightarrow |x_1 - y_1| = 0,\ |x_2 - y_2| = 0, \ldots, |x_n - y_n| = 0$

$\Leftrightarrow x_1 - y_1 = 0,\ x_2 - y_2 = 0, \ldots, x_n - y_n = 0$

$\Leftrightarrow x_1 = y_1,\ x_2 = y_2, \ldots, x_n = y_n$

$\Rightarrow (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$

$d(x,y) = 0 \Leftrightarrow x = y$

M_3 : $d(x,y) = \sum_{i=1}^{n} \frac{|x_i - y_i|}{1+|x_i - y_i|}$

$$= \sum_{i=1}^{n} \frac{|y_i - x_i|}{1+|x_i - y_i|} = d(y,x)$$

M_4 : $d(x,z) = \sum_{i=1}^{n} \frac{|x_i - z_i|}{1+|x_i - z_i|} = \sum_{i=1}^{n} \frac{|x_i - y_i + y_i - z_i|}{1+|x_i - y_i + y_i - z_i|}$

$$= \sum_{i=1}^{n} \frac{|x_i - y_i| + |y_i - z_i|}{1+|x_i - y_i| + |y_i - z_i|}$$

$$\le \sum_{i=1}^{n} \left(\frac{|x_i - y_i|}{1+|x_i - y_i| + |y_i - z_i|} + \frac{|y_i - z_i|}{1+|x_i - y_i| + |y_i - z_i|} \right)$$

$$d(x,z) \le \sum_{i=1}^{n} \frac{|x_i - y_i|}{1+|x_i - y_i|} + \sum_{i=1}^{n} \frac{|y_i - z_i|}{1+|y_i - z_i|}$$

$d(x,z) \le d(x,y) + d(y,z)$

$\Rightarrow$ d(x,y) is metric space

Question:

Show that $d(x,y) = \sqrt{|x|^2 + |y|^2}$ is metric space.

Solution:

M_1 : $d(x,y) \ge 0$ $\because \sqrt{|x|^2 + |y|^2} \ge 0$

M_2 : $d(x,y) = 0$ $\Leftrightarrow \sqrt{|x|^2 + |y|^2} = 0$

$\Leftrightarrow |x|^2 + |y|^2 = 0$

$\Leftrightarrow |x|^2 = 0 \quad , |y|^2 = 0$

$\Leftrightarrow x = 0 \quad , y = 0$

$d(x,y) = 0 \quad \Leftrightarrow x = y$

M_3 : $d(x,y) = \sqrt{|x|^2 + |y|^2}$

$= \sqrt{|y|^2 + |x|^2} \quad \Rightarrow \quad d(x,y) = d(y,x)$

M_4 : $d(x,z) = \sqrt{|x|^2 + |z|^2} = \sqrt{|x|^2 + |y-y|^2 + |z|^2}$

$\leq \sqrt{|x|^2 + |y|^2 + |-y|^2 + |z|^2} \leq \sqrt{|x|^2 + |y|^2} + \sqrt{|y|^2 + |z|^2}$

$d(x,z) \leq d(x,y) + d(x,z) \quad \Rightarrow$ d(x,y) is metric space

Lecture # 14

Cauchy Schwarz Inequality:

For any x,y $\in$ V , V is an inner product space then

$$|<x,y>| \le \|x\|\|y\|$$

Proof:

Case-I If x = 0 , y = 0 then

$|<x,y>| \le \|x\|\|y\|$ holds trivially

Case-II Suppose at least one of x and y say x $\neq$ 0 , $\lambda \in$ F then

$$0 \le <x+\lambda y, x+\lambda y> = <x,x> + <x,\lambda y> + <\lambda y,x> + <\lambda y,\lambda y>$$

$$= <x,x> + \bar{\lambda} <x,y> + \lambda <y,x> + \lambda\bar{\lambda} <y,y>$$

Take a = <x,x> , b = <x,y> , c = <y,y>

$$0 \le a + \bar{\lambda} b + \lambda \bar{b} + \lambda\bar{\lambda} c$$

Let c $\neq$ 0 then y $\neq$ 0 . Define $\lambda = \frac{-b}{c}$

$$= a + \frac{\bar{b}}{c} b - \frac{b}{c}\bar{b} + \frac{b}{c}.\frac{\bar{b}}{c} c$$

$$= a - \frac{|b|^2}{c}$$

$$= \frac{ac - |b|^2}{c}$$

$0 \le ac - |b|^2$

$|b|^2 \le ac$

$|<x,y>|^2 \le <x,x>.<y,y>$ $\qquad \because <x,x> = \|x\|^2, <y,y> = \|y\|^2$

$|<x,y>|^2 \le \|x\|^2.\|y\|^2$

$$|<x,y>| \le \|x\|\|y\|$$

Case-III

If y = 0 then

$<x,y> = <x,0>$

$= < x, 0.z >$ $\qquad\qquad \because 0 = 0.z$

$= 0 <x,z>$

$< x , y > = 0 = \|x\|\|y\|$

In all three cases

$|< x, y >| \leq \|x\|\|y\|$

Question:

(a) For any sequence $\{x_n\}$ and $\{y_n\}$ in inner product space V. $x_n \to x$, $y_n \to y$ then prove $< x_n, y_n > \to <x ,y>$

(b) If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in V then $< x_n, y_n >$ is convergent sequence in F (R or C)

Solution (a):

Consider

$$|< x_n, y_n > - < x, y >| = |< x_n, y_n > - < x, y_n > + < x, y_n > - < x, y >|$$

$$= |< x_n - x, y_n > + < x, y_n - y >|$$

$$\leq |< x_n - x, y_n >| + |< x, y_n - y >| \qquad \because |a+b| \leq |a||b|$$

$$\leq \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\| \qquad \because |< x, y >| \leq \|x\|\|y\|$$

Given $\left.\begin{array}{l} x_n \to x \Rightarrow \| x_n - x\| \to 0 \\ y_n \to y \Rightarrow \| y_n - y\| \to 0 \end{array}\right]$ ….(1)

$\Rightarrow \quad |< x_n, y_n > - < x, y >| \to 0$ by (1)

$\Rightarrow \quad < x_n, y_n > \to <x ,y>$

(b). If $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in V

So $\| x_n - x_m\| \to 0$ and $\| y_n - y_m\| \to 0$ as every Cauchy sequence is bounded

Consider

$$\left| < x_n, y_n > - < x_m, y_m > \right| = \left| < x_n, y_n > - < x_n, y_m > + < x_n, y_m > - < x_m, y_m > \right|$$

$$= \left| < x_n, y_n - y_m > + < x_n - x_m, y_m > \right|$$

$$\leq \left| < x_n, y_n - y_m > \right| + \left| < x_n - x_m, y_m > \right| \quad \because |a+b| \leq |a||b|$$

$$\leq \|x_n\| \, \|y_n - y_m\| + \|x_n - x_m\| \, \|y_m\| \quad \because |< x, y >| \leq \|x\|\|y\|$$

$$\left| < x_n, y_n > - < x_m, y_m > \right| \to 0 \quad as \quad m \to \infty \ , n \to \infty$$

$\Rightarrow$ $\{<x_n,y_n>\}$ is a Cauchy sequence in F(R or C).

Since F(Ror C) is complete so $<x_n,y_n>$ is convergent in F (R or C)

Question: What is meant by orthogonal system in an inner product space?

Solution:

Orthogonal System:

For any inner product space V ; x,y $\in$ V are said to be orthogonal (perpendicular) if <x,y> = 0 and can be written as x $\perp$ y

Question:

Define Pythagorean theorem in particular and general form.

Pythagorean Theorem:

In any inner product space V and x,y $\in$ V , x $\perp$ y. Then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

L.H.S = $\|x+y\|^2$

= < x+y , x+y>

= <x,x> + <x,y> + <y,x> + <y,y>

Since x $\perp$ y $\Rightarrow$ <x,y> = <y,x> = 0

= <x,x> + 0+ 0 + <y,y>

$\|x+y\|^2 = \|x\|^2 + \|y\|^2$ = R.H.S

Or $$\left\| \sum_{i=1}^{2} x_i \right\|^2 = \sum_{i=1}^{2} \|x_i\|^2$$

Generalized form of Pythagorean Theorem:

$$\left\|\sum_{i=1}^{n} x_i\right\|^2 = \sum_{i=1}^{n}\|x_i\|^2$$

L.H.S = $\left\|\sum_{i=1}^{n} x_i\right\|^2$

$= <\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j> \qquad \because \|x\|^2 = <x,x>$

$= \sum_{i=1}^{n}\sum_{j=1}^{n} <x_i, x_j>$

$= \sum_{i=1}^{n} <x_i, x_i> \qquad \because <x_i, x_j> = 0 \ \forall \ i \neq j$

$= \sum_{i=1}^{n}\|x_i\|^2 = \text{R.H.S}$

Question:

(a) Define orthonormal system in an inner product space.

(b) $\|x_\alpha - x_\beta\| = \sqrt{2}$

(c) Consider the set S = $\{S_n(t), c_0(t), c_n(t); n = 1, 2, \ldots\ldots\}$

Where $S_n(t) = \frac{1}{\sqrt{\pi}} sinnt$, n = 1,2,….. , $c_0(t) = \frac{1}{\sqrt{2\pi}}$

$c_n(t) = \frac{1}{\sqrt{\pi}}$ cosnt , n = 1,2,………

In the real space c[o, 2π] with inner product space V defined by

<x,y> = $\int_0^{2\pi} x(t)y(t)dt$. Then show that S is an orthonormal system.

Solution: (a)

A set A = $\{x_\alpha ; \alpha \in \Omega\}$ of non-zero vectors in an inner product space V is said to be orthonormal system if $<x_\alpha , x_\beta> = 0$, $\alpha \neq \beta$, $\alpha, \beta \in \Omega$ and $<x_\alpha , x_\alpha> = 1$

$\|x_\alpha\|^2 = 1 \Rightarrow \|x_\alpha\| = 1, \alpha \in \Omega$

Solution: (b) $L.H.S = \|x_\alpha - x_\beta\|^2$

$$= < x_\alpha - x_\beta , x_\alpha - x_\beta >$$

$$= < x_\alpha , x_\alpha > - < x_\beta , x_\alpha > - < x_\alpha , x_\beta > + < x_\beta , x_\beta >$$

$$\because < x_\alpha , x_\beta > = < x_\beta , x_\alpha > = 0 \ , \ \alpha \neq \beta$$

$$= < x_\alpha , x_\alpha > + < x_\beta , x_\beta >$$

$$= \|x_\alpha\|^2 + \|x_\beta\|^2$$

$$= 1 + 1 = 2$$

$$\|x_\alpha - x_\beta\|^2 = 2$$

$$\|x_\alpha - x_\beta\| = \sqrt{2}$$

Solution (c) :

Step-I : To prove $< S_n , C_m > = 0 \qquad \forall \ m,n = 1,2,....... \quad for \ m = n$

$$< S_n , C_m > = < \frac{1}{\sqrt{\pi}} \sin nt , \frac{1}{\sqrt{\pi}} \cos nt >$$

$$< S_n , C_m > = \frac{1}{\pi} < \sin nt , \cos nt > \ = \frac{1}{\pi} \int_0^{2\pi} \sin nt \cos nt dt$$

$$= \frac{1}{\pi} \left| \frac{\sin^2 nt}{2n} \right|_0^{2\pi} \ = \frac{1}{2\pi n}(0)$$

$$< S_n , C_m > = 0$$

Step-II For $n \neq m$

$$< S_n , C_m > = < \frac{1}{\sqrt{\pi}} \sin nt , \frac{1}{\sqrt{\pi}} \cos mt >$$

$$< S_n , C_m > = \frac{1}{\pi} < \sin nt , \cos mt > \ = \frac{1}{\pi} \int_0^{2\pi} \sin nt \cos mt dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} 2 \sin nt \cos mt dt \ = \frac{1}{2\pi} \int_0^{2\pi} (\sin(n+m) + \sin(n-m)) t dt$$

$$= \frac{1}{2\pi}\left|\frac{-\cos(n+m)t}{n+m}\right|_0^{2\pi} + \frac{1}{2\pi}\left|\frac{-\cos(n-m)t}{n+m}\right|_0^{2\pi}$$

$$= 0 + 0 = 0$$

Also $< S_n, S_n > = < \frac{1}{\sqrt{\pi}}\sin nt, \frac{1}{\sqrt{\pi}}\sin nt > = \frac{1}{\pi} < \sin nt, \sin nt >$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin^2 t dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left(\frac{1-\cos nt}{2}\right) t dt$$

$$= \frac{1}{2\pi}\left|t - \frac{\sin 2nt}{2n}\right|_0^{2\pi}$$

$$= \frac{1}{2\pi}(2\pi) = 1$$

Similarly, $< C_n, C_n > = 1$, $< C_0, C_0 > = 1$

$\Rightarrow$ S is orthonormal system

Lecture # 15

Dual Space (Conjugate Space)

Let N be a normed space. Let f : N $\rightarrow$ F be a Linear functionals.

Then (f+g) : N $\rightarrow$ F defined by

$$(f+g)(x) = f(x) + g(x), \qquad x \in N$$

And for any $\alpha \in F$,

$$\alpha f : N \rightarrow F \qquad \text{defined by}$$

$$(\alpha f)(x) = \alpha . f(x) \qquad , x \in N$$

are also Linear functionals. If N' be the set of all Linear functionals defined on N then N' itself a linear space called Algebraic dual space of N.

If we consider only the continuous or bounded Linear functional on N then corresponding space is called dual or conjugate and it is denoted by N^*.

Let N be a normed space and N^* be the dual space of N. Let N^{**} be the dual space of N^* then N^{**} is called the second dual space or second conjugate space of N.

A norm space N is said to be reflexive if there is an isometric isomorphism b/w N and N^{**}

Isometric Isomorphism:

Let N and M are normed space A function $\phi : N \rightarrow M$ is said to be a isometric isomorphism if

(i) ϕ is bijective
(ii) ϕ is linear i.e. for a,b $\in$ F , x,y $\in$ N
$\phi(ax+by) = a\phi(x) + b\phi(y)$
(iii) ϕ preserves norms i.e. for any $x \in N$
$\|\phi(x)\| = \|x\|$

Theorem:

A finite dimensional normed or Linear Space N is isomorphic to its second dual space N^{**} i.e. $N \cong N^{**}$

Proof: Let N be a finite dimensional normed or linear space of dim = n and N^{**} be its second dual space. Define

$\phi : N \to N^{**}$ s.t

For each $x \in N$ we put

$\phi(x) = g_x$

Where $g_x : N^* \to F$ is defined by $g_x(f) = f(x)$, $f \in N^*$

(i) ϕ is linear

$\phi(ax+a'x') = g_{ax+a'x'}$

And $g_{ax+a'x'}(f) = f(ax + a'x')$

$= a.\, f(x) + a'.\, f(x')$

$= a.\, g_x(f) + a'.\, g_{x'}(f)$, $f \in N^*$

$= a\phi(x) + a'\phi(x')$

$\Rightarrow \phi$ is linear

(ii) ϕ is injective (one-one)

For $x, x' \in N$

$\phi(x) = \phi(x')$

$g_x = g_{x'}$

$(g_x - g_{x'})(f) = g_x(f) - g_{x'}(f)$

$= f(x) - f(x')$

$= f(x - x') = 0$ $\forall x \in N^*$

Hence $x - x' = 0$

$x = x'$

We use "If N is finite dimensional normed space and $x_0 \in N$ s.t

$f(x_0) = 0 \Rightarrow x_0 = 0 \quad \forall f \in N^*$"

$\Rightarrow \phi$ is one-one

(iii) ϕ is onto

Also $\phi(N)$ is subspace of N^{**}

$\because$ N has finite dimension

$\dim N = \dim N^* = \dim N^{**}$

so that $\phi(N) = N^{**}$

we use "Let N be a n-dimensional normed space then its dual N^* is also n-dimensional"

$\Rightarrow$ Hence N and N^{**} are isomorphic to each other.

Annihilators:

Let H be Hilbert space and $A \subseteq H$ for $x \in H$ we say x is orthogonal to A written as $x \perp A$ iff $< x,y > = 0 \ \forall \ y \in A$. The set of all vectors which are orthogonal to A is called the Annihilators and denoted by $A^{\perp}$.

Thus $\qquad A^{\perp} = \{x \in H \, , \, x \perp A\}$

For MCQ

(i) $A \subseteq A^{\perp\perp}$

(ii) $A \subseteq B \ \Rightarrow \ A^{\perp} \subseteq B^{\perp}$

(iii) $(A \cup B)^{\perp} = A^{\perp} \cap B^{\perp}$

(iv) $A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$

(v) $A^{\perp} = A^{\perp\perp\perp}$

(vi) $A \cap A^{\perp} \subseteq \{0\}$

(vii) $A^{\perp}$ is closed subspace of H

(viii) $\{0\}^{\perp} = H$

(ix) $H^{\perp} = \{0\}$

Complete Space:

If $x_n \to x \in (X,d)$

Banach Space:

If $x_n \to x \in (N, \| \cdot \|)$

Hilbert Space:

If $x_n \to x \in (X, < . \, , . >)$