Differential Geometry (Notes)

by

Ms. Kaushef Salamat

Partial Contents

These are handwritten notes. We are very thankful to Ms. Kaushef Salamat for these notes.

Part I

Space Curves

Space curve :-

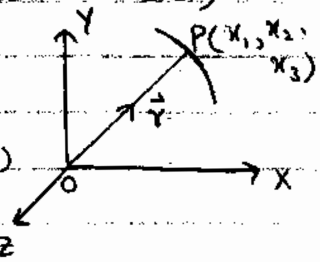
Space curve is the locus of a point whose position vector $\vec{r}$ with respect to some fixed origin can be expressed as a function of a single variable or parametre.

(i-e), $\vec{r} = \vec{r}(u)$ where $u \in R$ (real number)

$\vec{r} = (x_1, x_2, x_3)$

$\vec{r} = \vec{r}(u) = (x_1(u), x_2(u), x_3(u))$

where $x_1 = x_1(u)$, $x_2 = x_2(u)$, $x_3 = x_3(u)$



There are two types of curve.

(i) Twisted Curve :-

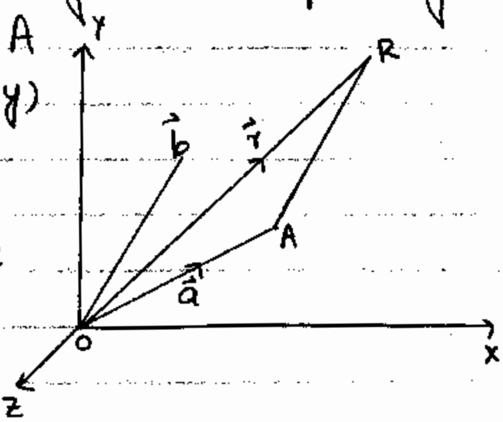
A curve which does not lie in a plane is known as twisted or skew curve.

(ii) Plane curve :-

A curve all of whose points lie in a plane is known as plane curve. (circle)

Equation of straight line (Plane curve) :-

Equation of straight line passing through a fixed point A and parallel to some (any) vector.



$\vec{OR} = \vec{OA} + \vec{AR}$

$\vec{r} = \vec{a} + u\vec{b} \rightarrow (1)$ where u is any real number.

$\vec{AR} = u\vec{b}$ because $\vec{AR} \parallel$ to $\vec{b}$

If $\vec{a} = (a_1, a_2, a_3)$,

$\vec{b} = (b_1, b_2, b_3)$

$\vec{r} = (x_1, x_2, x_3)$ put all values in (1)

$(x_1, x_2, x_3) = (a_1, a_2, a_3) + u(b_1, b_2, b_3)$

$(x_1, x_2, x_3) = (a_1 + ub_1, \; a_2 + ub_2, \; a_3 + ub_3)$

$\Rightarrow \; x_1 = a_1 + ub_1, \quad x_2 = a_2 + ub_2, \quad x_3 = a_3 + ub_3$

are Co-ordinates of a fixed point A.

$$\frac{x_1 - a_1}{b_1} = \frac{x_2 - a_2}{b_2} = \frac{x_3 - a_3}{b_3} = u$$

$\Rightarrow \; \vec{r} = \vec{a} + u\vec{b}$ is equation of st. line

A is fixed point and $\vec{b}$ is a vector parallel to straight line.

Example:-

A circle in xy-plane with radius "a" is given by $\vec{r} = (a\cos u, \; a\sin u, \; 0)$

$\vec{r} = (x, y, z) = (a\cos u, \; a\sin u, \; 0)$

$\Rightarrow \; x = a\cos u, \quad y = a\sin u, \quad z = 0$

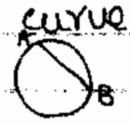
Tangent:-

A straight line touching the curve at a point is called tangent. 

Arc:-

Any portion of a curve is arc. 

Chord:-

A straight line cutting the curve at two points is called chord. 

Arc Length:-

Let $\vec{R} = \vec{R}(u)$ be a curve where $u \in I$, Let $[a, b] \subset I$ as u varies on a closed interval $[a, b]$ we obtain an arc of the curve $\vec{R} = \vec{R}(u)$.

Let $\Delta = \{a = u_0 < u_1 < u_2 - - - < u_n = b\}$

then $L(\Delta)(\text{Real}) = \sum_{i=1}^{n} |\vec{R}(u_i) - \vec{R}(u_{i-1})|$ is the length of arc w.r.t the subdivision Δ of $[a, b]$. Now, any addition of a point u_i to the subdivision Δ of $[a, b]$ gives another value of $L(\Delta)$.

Now, corresponding to all possible subdivision of $[a,b]$ we obtain a set $\{L(\Delta)\}$ of real numbers and the least upper bound of this set is known as arc length between "a" and "b"

(i-e). $\text{Arc-Length} = \sup_{i} \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})|$

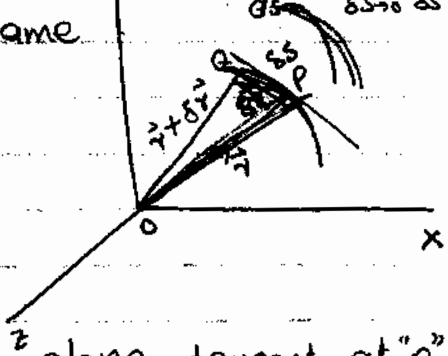
where sup is taken over all possible partitions of $[a,b]$

Equation of tangent :-

Let P be a point with position vector $\vec{r}$ on a curve. then when $\delta s \to 0$ (in limiting case) $r' = \frac{d\vec{r}}{ds} = \lim_{\delta s \to 0} \frac{\delta r}{\delta s}$

$|\delta\vec{r}|$ and δs have the same values. So, $|\delta\vec{r}| = |\delta s|$

So, $\left|\lim_{\delta s \to 0} \frac{\delta\vec{r}}{\delta s}\right| = 1$



$\Rightarrow |\vec{r}'| = \left|\frac{d\vec{r}}{ds}\right| = 1$

and the direction of $\vec{r}$ is along tangent at "P" and is parallel to the tangent at point 'P'. Hence $\vec{r}'$ is unit vector and this unit vector is known as unit tangent vector at point "P" and is denoted by "t".

So, $\vec{t} = \vec{r}' = \frac{d\vec{r}}{ds}$

Hence, the equation of tangent at point "P" with position vector $\vec{r}$ is given by

$$\vec{R} = \vec{r} + u\vec{t}$$

where $\vec{R}$ is ~~position vector of~~ r.v. of any point on tangent at point "P" and u is any real number.

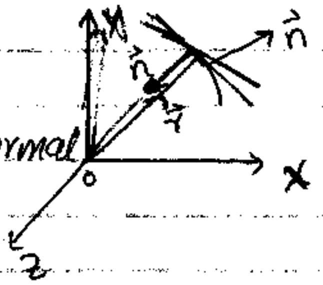
Oscillating plane :-

A plane through a point "p" having tangent at "p" and a consective point is known as oscillating plane.

In case of plane curves, the oscolating plane at any point "p" is the plane in which the curve lies.

Principal Normal :-

A unit vector lying in the oscolating plane at a point "p" and perpandicular to the tangent at point 'p' is known as unit principal normal at "p", and is denoted by "n".



Equation of principal normal at Point $\vec{R}$ is given by

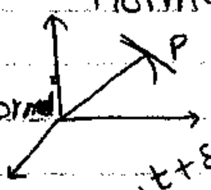
$$\vec{R} = \vec{r} + u\vec{n}$$

where $\vec{R}$ is any point on tangent at p and u is any real number.

Normal Plane :-

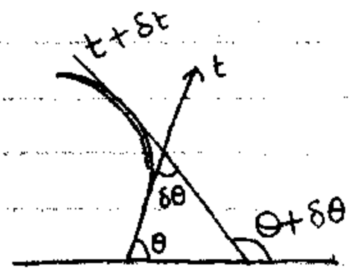
A plane through a point "p" perpandicular to the tangent at point "p" is known as normal plane at point "p"

"A line ⊥ to tangent is norm

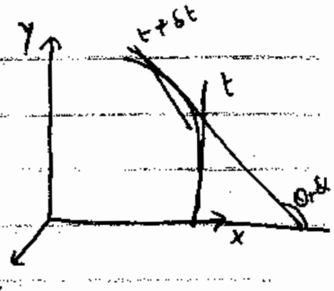


Curvature :-

The curvature at any point on a curve is defined as arc rate of rotation of tangent, i.s



denoted $k = \frac{d\theta}{ds}$ keppa "k"

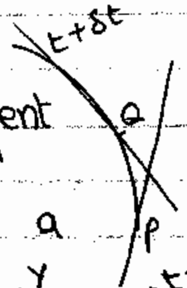


$$k = \frac{d\theta}{ds} = \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s}$$

Question :-

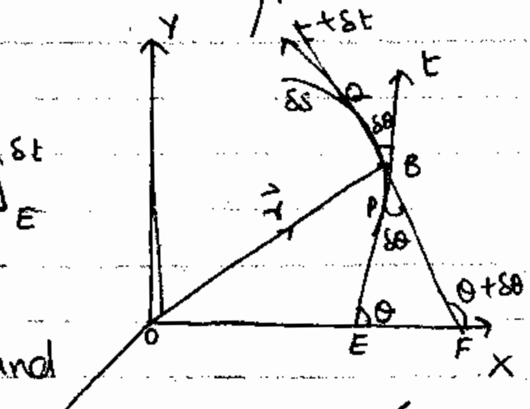
Prove that $\vec{r}'' = k\vec{n}$; where $\vec{r}$ is the position vector of any point "p" on a curve and $\vec{n}$ is unit principal normal vector at point "p".

Proof :-



Let $\vec{t}$ be unit tangent vector at point P and $\vec{t}+\vec{\delta t}$ be the unit tangent vector at a neighbouring point Q

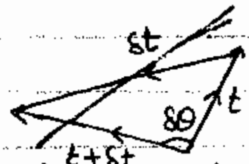
Now, we consider the triangle BEF



$|\vec{BE}| = |\vec{t}| = 1$

$|\vec{BF}| = |\vec{t}+\vec{\delta t}| = 1$

The angle between $\vec{t}$ and $\vec{t}+\vec{\delta t}$ is $\angle EBF = \delta\theta$



In the limiting case (when $\delta s \to 0$), $\vec{\delta t}$ is perpendicular to $\vec{t}$. Since δs is a scalar quantity. So $\frac{\vec{\delta t}}{\delta s}$ will be in the direction of $\vec{\delta t}$ and hence is $\perp$ to $\vec{t}$.

(i-e) $\frac{\vec{\delta t}}{\delta s} \perp \vec{t}$ (when $\delta s \to 0$)

So, $\frac{\vec{\delta t}}{\delta s}$ will be in the direction of normal at point P.

Also in limiting case (when $\delta s \to 0$), the value of $|\delta \vec{t}|$ will be the same as that of $\delta\theta$.

Now, $\vec{r}'' = t' = \dfrac{dt}{ds}$

$$= \lim_{\delta s \to 0} \frac{\delta t}{\delta s}$$

$$= \left| \lim_{\delta s \to 0} \frac{\delta t}{\delta s} \right| \cdot \vec{n}$$

where $\vec{n}$ is unit normal at point p.

$$= \lim_{\delta s \to 0} \frac{|\delta t|}{|\delta s|} \cdot \vec{n}$$

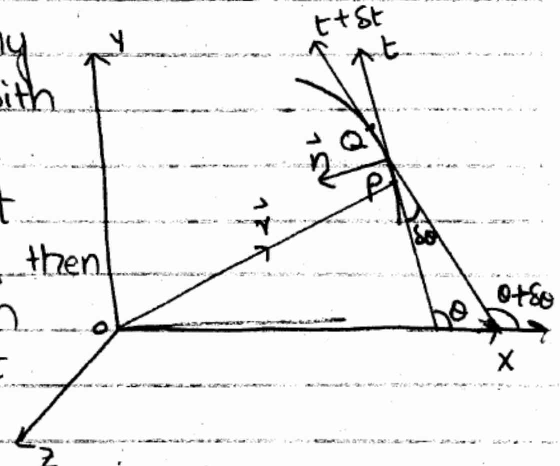
$$= \lim_{\delta s \to 0} \frac{\delta\theta}{\delta s} \cdot \vec{n}$$

$$= \frac{d\theta}{ds} \cdot \vec{n}$$

$$\vec{r}'' = k \cdot \vec{n}$$

Equation of oscolating plane :-

oscolating plane at a point "p" contains both tangent and normal at point "p", if p is any point on a curve with p.v $\vec{r}$, $\vec{t}$ and $\vec{n}$ are unit tangent and unit normal at point "p", then to derive the equation of oscolating plane at point "p".



Let Q be any point in the oscollating plane with p.v $\vec{R}$. Then, the vectors $\vec{R} - \vec{r}$, $\vec{t}$ and $\vec{n}$

are Co-planar. So, $[\vec{R}-\vec{r}, \vec{t}, \vec{n}] = 0$

$\Rightarrow [\vec{R}-\vec{r}, \vec{r}', \frac{\vec{r}''}{k}] = 0$

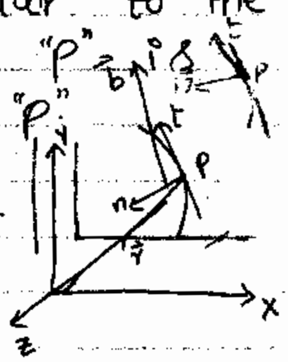
$\Rightarrow \frac{1}{k}[\vec{R}-\vec{r}, \vec{r}', \vec{r}''] = 0$

$\Rightarrow [\vec{R}-\vec{r}, \vec{r}', \vec{r}''] = 0$

is an equation of oscollating plane at point "p" with p.v $\vec{r}$ and here $\vec{R}$ is the p.v of any point lying in the oscollating plane.

Bi-normal vector:-

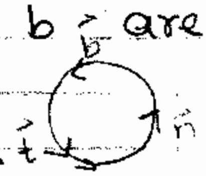
A vector perpandicular to the oscollating plane at a point "p" is known as bi-normal at point "p".



The unit vector perpandicular to oscollating plane at a point "p" is known as unit bi-normal vector and is denoted by $\vec{b}$.

Remark:-

The unit vectors $\vec{t}$, $\vec{n}$ and $\vec{b}$ are perpandicular to each other, So



$\vec{t}\cdot\vec{n} = \vec{n}\cdot\vec{b} = \vec{b}\cdot\vec{t} = 0$

and $\vec{t}\times\vec{n} = \vec{b}$, $\vec{n}\times\vec{b} = \vec{t}$ and $\vec{b}\times\vec{t} = \vec{n}$

Equation of Bi-normal:-

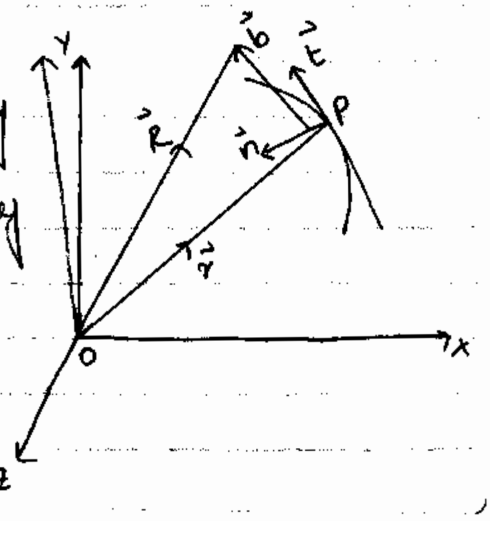
Equation of Bi-normal at point "p" is given by

$\vec{R} = \vec{r} + u\vec{b}$

where $\vec{R}$ is p.v of any point on bi-normal at point "p".

$\vec{R} = \vec{r} + u(\vec{t}\times\vec{n})$

$\vec{R} = \vec{r} + u(\vec{r}' \times \frac{\vec{r}''}{k})$



$$= \vec{r} + \frac{u}{k}(\vec{r}' \times \vec{r}'')$$

$$\vec{R} = \vec{r} + v(\vec{r}' \times \vec{r}'') \rightarrow (1)$$

where $v = \frac{u}{k}$, u is any real number.

(1) is eq of binormal at point "P" with position vector $\vec{R}$.

Torsion:-

The torsion of a curve at any point "P" is defined as the arc rate of rotation of Binormal, it is denoted by "τ"

$$\tau = \frac{d\phi}{ds} = \lim_{\delta s \to 0} \frac{\delta\phi}{\delta s}$$

where $\delta\phi$ is the angle between binormals $\vec{b}$ and $\vec{b} + \delta\vec{b}$ at two neighbouring points on the curve.

Question:-

Prove that $\vec{b}' = -\tau\vec{n}$

Sol:-

Available at
www.mathcity.org

Consider $\vec{t} \cdot \vec{b} = 0$

$$\vec{t}' \cdot \vec{b} + \vec{t} \cdot \vec{b}' = 0$$

put $\vec{t}' = \vec{r}'' = k\,\vec{n}$

$$k\vec{n} \cdot \vec{b} + \vec{t} \cdot \vec{b}' = 0$$

$$k \cdot 0 + \vec{t} \cdot \vec{b}' = 0$$

$$0 + \vec{t} \cdot \vec{b}' = 0$$

$$\Rightarrow \vec{t} \cdot \vec{b}' = 0$$

$\Rightarrow \vec{b}'$ is perpandicular to $\vec{t}$

Now consider

$$\vec{b} \cdot \vec{b} = 1$$

$$\vec{b}' \cdot \vec{b} + \vec{b} \cdot \vec{b}' = 0$$

$\Rightarrow \quad \vec{b}\cdot\vec{b}' + \vec{b}\cdot\vec{b}' = 0$

$\Rightarrow \quad 2\vec{b}\cdot\vec{b}' = 0$

$\Rightarrow \quad \vec{b}\cdot\vec{b}' = 0$

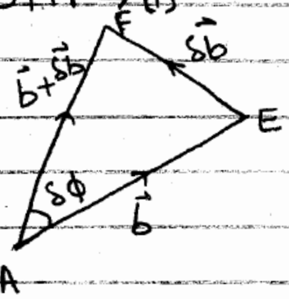
$\Rightarrow$ $\vec{b}'$ is perpandicular to $\vec{b}$

Hence, $\vec{b}'$ is a vector perpandicular to the vectors $\vec{t}$ and $\vec{b}$.

So, we can say that $\vec{b}'$ is $\parallel$ (parallel) to the vector $\vec{n}$ (normal).

Hence, we can write $\vec{b}' = \pm|\vec{b}'|\vec{n} \rightarrow (1)$

$\vec{AE} = \vec{b}, \quad \vec{AF} = \vec{b} + \delta\vec{b}$



$\angle EAF = \delta\phi$ angle between $\vec{b}$ and $\vec{b} + \delta\vec{b}$

$|\vec{AE}| = |\vec{AF}| = 1$

The limiting value of $|\delta\vec{b}|$ is the same as that of $\delta\phi$ when $\delta s \rightarrow 0$

So, $\left|\frac{db}{ds}\right| = \left|\lim_{\delta s \to 0} \frac{\delta b}{\delta s}\right|$

$$= \lim_{\delta s \to 0} \frac{|\delta b|}{\delta s}$$

$$= \lim_{\delta s \to 0} \frac{\delta\phi}{\delta s}$$

$$\left|\frac{db}{ds}\right| = \frac{d\phi}{ds}$$

$|\vec{b}'| = \left|\frac{db}{ds}\right| = \pm\tau \rightarrow (2)$

By using (1) and (2) we have

$\vec{b}' = -\tau\cdot\vec{n}$

where "$-$" signifies that whenever τ is +ve, $\vec{b}'$ and $\vec{n}$ are in opposite directions and parallel to each other.

Seret Frenet Formulas:-

The equations $\vec{t}' = k\vec{n}$ or $\vec{r}'' = k\vec{n} \rightarrow (1)$

$$\vec{b}' = -\tau\vec{n} \rightarrow (2)$$
$$\vec{n}' = \tau\vec{b} - k\vec{t} \rightarrow (3)$$

Eq (1), (2) and (3) are knows as Seret Frenet Formulas.

To derive equation (3)

Consider $\vec{n} = \vec{b} \times \vec{t}$

Differentiate it w.r.t s

$$\vec{n}' = \vec{b}' \times \vec{t} + \vec{b} \times \vec{t}'$$

put $\vec{b}' = -\tau\vec{n}$ and $\vec{t}' = k\cdot\vec{n}$

$$\vec{n}' = -\tau\vec{n} \times \vec{t} + \vec{b} \times k\vec{n}$$
$$= -\tau(-\vec{b}) + k(\vec{b} \times \vec{n}) \qquad \because \vec{b} \times \vec{n} = -\vec{t}$$
$$\vec{n}' = \tau\vec{b} - k\vec{t}$$

Question:-

Prove that $\tau = \frac{1}{k^2}[\vec{r}' \ \vec{r}'' \ \vec{r}''']$

Sol:-

we know that

$$\vec{r}' = \vec{t}$$
$$\vec{r}'' = k\cdot\vec{n}$$

To obtain $\vec{r}'''$

Differentiate $\vec{r}''$ w.r.t s.

$$\vec{r}''' = k'\cdot\vec{n} + k\cdot\vec{n}'$$
$$= k'\cdot\vec{n} + k\cdot(\tau\vec{b} - k\vec{t})$$
$$\vec{r}'' = k'\cdot\vec{n} + k\tau\vec{b} - (k\cdot k)\vec{t}$$

$$\vec{r}''' = k'\cdot\vec{n} + k\tau\vec{b} - k^2\vec{t}$$

Now Consider $[\vec{r}' \ \vec{r}'' \ \vec{r}'''] =$

$$\vec{r}''' = [\vec{t} \quad k\vec{n} \quad k'\cdot\vec{n} + k\tau\vec{b} - k^2\vec{t}]$$

$$\vec{r}''' = [\vec{t} \ \ k\vec{n} \ \ k'\vec{n}] + [\vec{t} \ \ k\vec{n} \ \ k\tau\vec{b}] + [\vec{t} \ \ k\vec{n} \ \ -k^2\vec{t}]$$

$\because [\vec{t} \ \ k\vec{n} \ \ k'\vec{n}] = kk'(\vec{t}\cdot\vec{n}\times\vec{n}) = kk'(\vec{t}\cdot 0) = 0$

$[\vec{t} \ \ k\vec{n} \ \ -k^2\vec{t}] = -kk^2[\vec{t}\cdot\vec{n}\times\vec{t}] = -kk^2[\vec{n}\times\vec{t}\cdot\vec{t}] = -kk^2(\vec{n}\cdot\vec{t}\times\vec{t})$
$= 0$

$$= 0 + [\vec{t} \quad k\vec{n} \quad k\tau\vec{b}] + 0$$

$$= kk\tau[\vec{t} \quad \vec{n} \quad \vec{b}]$$

$$= k^2\tau(1)$$

$$[\vec{r}' \quad \vec{r}'' \quad \vec{r}'''] = k^2\tau$$

$$\Rightarrow \quad \tau = \frac{1}{k^2}[\vec{r}' \quad \vec{r}'' \quad \vec{r}''']$$

Question:-

Find the curvature and torsion of the curve $\vec{r} = [a\cos\theta, a\sin\theta, a\theta\cot\beta]$

Sol:-

$$\vec{r} = (a\cos\theta \;,\; a\sin\theta \;,\; a\theta\cot\beta)$$

we know

$$\vec{r}'' = k\vec{n} \rightarrow (1)$$

$$\vec{b}' = -\tau\vec{n} \rightarrow (2)$$

$$\vec{r}' = \left(-a\sin\theta\frac{d\theta}{ds} \;,\; a\cos\theta\frac{d\theta}{ds} \;,\; a\cot\beta\frac{d\theta}{ds}\right)$$

$$\vec{t} = \vec{r}' = (-a\sin\theta \quad a\cos\theta \quad a\cot\beta)\frac{d\theta}{ds} \rightarrow (3)$$

$$\vec{t}\cdot\vec{t} = (a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2(\sin^2\theta + \cos^2\theta) + a^2\cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2 + a^2\cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = a^2(1 + \cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$$\frac{1}{a^2} = \operatorname{cosec}^2\beta\left(\frac{d\theta}{ds}\right)^2$$

$$\Rightarrow \left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2\operatorname{cosec}^2\beta}$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2}\sin^2\beta$$

$$\frac{d\theta}{ds} = \frac{1}{a}\sin\beta$$

put this in (3)

$$\vec{r}' = (-a\sin\theta \quad a\cos\theta \quad a\cot\beta)\frac{1}{a}\sin\beta$$

$$= (-\sin\theta \quad \cos\theta \quad \cot\beta)\sin\beta$$

$$\vec{r}'' = (-\cos\theta \quad -\sin\theta \quad 0)\frac{d\theta}{ds}\sin\beta$$

$$\vec{r}'' = (-\cos\theta \quad -\sin\theta \quad 0)\frac{1}{a}\sin^2\beta$$

Put $\vec{r}'' = k\vec{n}$

$$k\vec{n} = (-\cos\theta, -\sin\theta, 0)\frac{1}{a}\sin^2\beta$$

$$|k\vec{n}|^2 = (\cos^2\theta + \sin^2\theta)\frac{1}{a^2}(\sin^2\beta)^2 \qquad |\vec{r}''| = |k\vec{n}|$$

$$|\vec{r}''| = k|\hat{n}| = k$$

$$k = \frac{1}{a}\sin^2\beta = \frac{1}{a\operatorname{cosec}^2\beta}$$

is k curvature of the given curve.

To find torsion τ, we use the formula.

$$\tau = \frac{1}{k^2}[\vec{r}' \ \vec{r}'' \ \vec{r}'''] \rightarrow (1)$$

From above $\vec{r}'' = (-\cos\theta \quad -\sin\theta \quad 0)\frac{d\theta}{ds}\sin^2\beta$

Again differentiate it w.r.t "s"

$$\vec{r}''' = (+\sin\theta \quad -\cos\theta \quad 0)\frac{1}{a}\frac{d\theta}{ds}\sin^2\beta$$

Put $\frac{d\theta}{ds} = \frac{1}{a}\sin\beta$

$$\vec{r}''' = (\sin\theta \quad -\cos\theta \quad 0)\frac{1}{a}\left(\frac{1}{a}\sin\beta\right)\sin^2\beta$$

$$\vec{r}''' = (\sin\theta \quad -\cos\theta \quad 0)\frac{1}{a^2}\sin^3\beta$$

Putting all values in (1)

$$T = \frac{1}{k^2}\Big[(-\sin\theta \quad \cos\theta \quad \cot\beta)\sin\beta \quad (-\cos\theta \quad -\sin\theta \quad 0)\frac{1}{a^2}\sin\beta$$

$$(\sin\theta \quad -\cos\theta \quad 0)\frac{1}{a^2}\sin^3\beta\Big]$$

put $k = \frac{1}{a}\sin^2\beta$

$$T = \frac{1}{k^2}\begin{vmatrix} -\sin\theta\sin\beta & \cos\theta\sin\beta & \cot\beta\sin\beta \\ -\frac{1}{a}\cos\theta\sin^2\beta & -\frac{1}{a}\sin\theta\sin^2\beta & 0 \\ \frac{1}{a^2}\sin\theta\sin^3\beta & -\frac{1}{a^2}\cos\theta\sin^3\beta & 0 \end{vmatrix}$$

$$T = \frac{1}{K^2}\begin{vmatrix} -\sin\theta\sin\beta & \cos\theta\sin\beta & \frac{\cos\beta\sin\beta}{\sin\beta} \\ -\frac{1}{a}\cos\theta\sin^2\beta & -\frac{1}{a}\sin\theta\sin^2\beta & 0 \\ \frac{1}{a^2}\sin\theta\sin^3\beta & -\frac{1}{a^2}\cos\theta\sin^3\beta & 0 \end{vmatrix}$$

$$T = \frac{1}{K^2}\begin{vmatrix} -\sin\theta\sin\beta & \cos\theta\sin\beta & \cos\beta \\ -\frac{1}{a}\cos\theta\sin^2\beta & -\frac{1}{a}\sin\theta\sin^2\beta & 0 \\ \frac{1}{a^2}\sin\theta\sin^3\beta & -\frac{1}{a^2}\cos\theta\sin^3\beta & 0 \end{vmatrix}$$

$$T = \frac{1}{K^2}\cos\beta\left(\frac{1}{a^3}\cos^2\theta\sin^5\beta + \frac{1}{a^3}\sin^2\theta\sin^5\beta\right)$$

$$= \frac{1}{K^2}\frac{1}{a^3}\cos\beta\sin^5\beta\left(\cos^2\theta + \sin^2\theta\right)$$

$$T = \frac{1}{a^3}\frac{1}{K^2}\sin^5\beta\cos\beta \quad (2)$$

Put $k = \frac{1}{a}\sin^2\beta = \frac{1}{a\,\mathrm{cosec}^2\beta}$

$$T = \frac{1}{a^3}\frac{1}{\frac{1}{a^2}(\sin^2\beta)^2}\sin^5\beta\cos\beta$$

$$T = \frac{a^2\sin^5\beta\cos\beta}{a^3\sin^4\beta} = \frac{1}{a}\sin\beta\cos\beta$$

Equation of tangent in cartesian form:-

t Let $\vec{r} = \vec{r}(t)$ be a curve, then the equation of tangent at any point 'p' with position vector $\vec{r}$ is given by

$$\vec{R} = \vec{r} + u\vec{t} \rightarrow (1)$$

where u is real number

Now, $\vec{R} = (X, Y, Z)$

$\vec{r} = (x, y, z)$, $\vec{t} = \vec{r}'$

put these values in (1)

$$(X, Y, Z) = (x, y, z) + u\vec{r}'$$

$$(X, Y, Z) = (x, y, z) + u\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$\Rightarrow (X-x, Y-y, Z-z) = u\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$\Rightarrow X-x = u\frac{dx}{ds}, \quad Y-y = u\frac{dy}{ds}, \quad Z-z = u\frac{dz}{ds}$$

$$\Rightarrow \frac{X-x}{\frac{dx}{ds}} = \frac{Y-y}{u\frac{dy}{ds}} = \frac{Z-z}{\frac{dz}{ds}} = u$$

$$\Rightarrow \frac{X-x}{dx/ds} = \frac{Y-y}{dy/ds} = \frac{Z-z}{dz/ds} \rightarrow (2)$$

$\Rightarrow$

Now $\frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds}$, $\frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds}$, $\frac{dz}{ds} = \frac{dz}{dt}\frac{dt}{ds}$

Subsitute these values in (2)

$$\Rightarrow \frac{X-x}{\frac{dx}{dt}\frac{dt}{ds}} = \frac{Y-y}{\frac{dy}{dt}\frac{dt}{ds}} = \frac{Z-z}{\frac{dz}{dt}\frac{dt}{ds}}$$

$$\Rightarrow \frac{X-x}{dx/dt} = \frac{Y-y}{dy/dt} = \frac{Z-z}{dz/dt}$$

which is equation of tangent in cartesian form.

Equation of oscollating plane in cartesian form:-

The equation of oscollating plane at any point "p" on a curve is given by

$$[\vec{R}-\vec{r} \quad \vec{r}' \quad \vec{r}''] = 0 \rightarrow (1)$$

If $\vec{R} = (X, Y, Z)$

$\vec{r} = (x, y, z)$

$$\vec{r}' = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$\vec{r}'' = \left(\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}\right)$$

$\vec{R}-\vec{r} = (X-x, Y-y, Z-z)$

Substituting all values in (1)

$$\begin{vmatrix} X-x & Y-y & Z-z \\ dx/ds & dy/ds & dz/ds \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{vmatrix} = 0$$

which is equation of oscollating plane in cartesian form.

Example:-

Find the equation of oscollating plane at a point "p" on a circle with radius "$\vec{a}$" and centre at origin, where

$x = a\cos\theta$, $y = a\sin\theta$, $z = 0$

eq of tangent to the

$\vec{r} = (a\cos\theta, a\sin\theta, 0)$

Sol:-

$x = a\cos\theta$, $y = a\sin\theta$, $z = 0$

$$\frac{dx}{ds} = \frac{dx}{d\theta}\,\frac{d\theta}{ds}$$

$$\frac{d^2x}{ds^2} = \frac{d}{ds}\left(\frac{dx}{d\theta}\cdot\frac{d\theta}{ds}\right)$$

$$= \frac{d^2x}{ds\,d\theta}\cdot\frac{d\theta}{ds} + \frac{dx}{d\theta}\,\frac{d^2\theta}{ds^2} = \frac{d^2x}{d\theta\cdot d\theta}\,\frac{d\theta\cdot d\theta}{(ds)^2} + \frac{dx}{d\theta}\,\frac{d^2\theta}{ds^2}$$

$$\frac{d^2x}{ds^2} = \frac{d^2x}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 + \frac{dx}{d\theta}\,\frac{d^2\theta}{ds^2}$$

$$\frac{dy}{ds} = \frac{dy}{d\theta}\,\frac{d\theta}{ds}$$

$$\frac{d^2y}{ds^2} = \frac{d}{ds}\left(\frac{dy}{ds}\right) = \frac{d}{ds}\left(\frac{dy}{d\theta}\,\frac{d\theta}{ds}\right)$$

$$= \frac{d^2y}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 + \frac{d^2\theta}{ds^2}$$

Similarly,

$$\frac{dz}{ds} = \frac{dz}{d\theta}\,\frac{d\theta}{ds}$$

$$\frac{d^2z}{ds^2} = \frac{d}{ds}\left(\frac{dz}{ds}\right) = \frac{d}{ds}\left(\frac{dz}{d\theta}\,\frac{d\theta}{ds}\right)$$

$$= \frac{d^2z}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 + \frac{d^2\theta}{ds^2}$$

Now, the equation of oscullating plane in Cartesian form at any point $P(x, y, z)$ is given by

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \frac{dx}{ds} & \frac{dy}{ds} & \frac{dz}{ds} \\ \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \end{vmatrix} = 0$$

putting all values

$$\begin{vmatrix} X-x & Y-x & Z-z \\ \frac{dx}{d\theta}\frac{d\theta}{ds} & \frac{dy}{d\theta}\frac{d\theta}{ds} & \frac{dz}{d\theta}\frac{d\theta}{ds} \\ \frac{d^2x}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2+\frac{dx}{d\theta}\frac{d^2\theta}{ds^2} & \frac{d^2y}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2+\frac{dy}{d\theta}\frac{d^2\theta}{ds^2} & \frac{d^2z}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2+\frac{dz}{d\theta}\frac{d^2\theta}{ds^2} \end{vmatrix}$$

$$= \begin{vmatrix} X-x & Y-x & Z-z \\ \frac{dx}{d\theta}\frac{d\theta}{ds} & \frac{dy}{d\theta}\frac{d\theta}{ds} & \frac{dz}{d\theta}\frac{d\theta}{ds} \\ \frac{d^2x}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 & \frac{d^2y}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 & \frac{d^2z}{d\theta^2}\left(\frac{d\theta}{ds}\right)^2 \end{vmatrix} + \begin{vmatrix} X-x & Y-y & Z-z \\ \frac{dx}{d\theta}\frac{d\theta}{ds} & \frac{dy}{d\theta}\frac{d\theta}{ds} & \frac{dz}{d\theta}\frac{d\theta}{ds} \\ \frac{dx}{d\theta}\frac{d^2\theta}{ds^2} & \frac{dy}{d\theta}\frac{d^2\theta}{ds^2} & \frac{dz}{d\theta}\frac{d^2\theta}{ds^2} \end{vmatrix}$$

Taking common $\left(\frac{d\theta}{ds}\right)^2$ from I det and $\frac{d^2\theta}{ds^2}$ from II det

$$= \left(\frac{d\theta}{ds}\right)^3 \begin{vmatrix} X-x & Y-x & Z-z \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \\ \frac{d^2x}{d\theta^2} & \frac{d^2y}{d\theta^2} & \frac{d^2z}{d\theta^2} \end{vmatrix} + \frac{d\theta}{ds}\frac{d^2\theta}{ds^2} \begin{vmatrix} X-x & Y-y & Z-z \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \end{vmatrix}$$

$$= \left(\frac{d\theta}{ds}\right)^3 \begin{vmatrix} X-x & Y-y & Z-z \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \\ \frac{d^2x}{d\theta^2} & \frac{d^2y}{d\theta^2} & \frac{d^2z}{d\theta^2} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} X-x & Y-y & Z-z \\ dx/d\theta & dy/d\theta & dz/d\theta \\ d^2x/d\theta^2 & d^2y/d\theta^2 & d^2z/d\theta^2 \end{vmatrix} = 0$$

is equation of oscillating plane in polar form

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = 0$$

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta \quad \frac{dz}{d\theta} = 0$$

$$\frac{d^2x}{d\theta^2} = -a\cos\theta, \quad \frac{d^2y}{d\theta^2} = -a\sin\theta, \quad \frac{d^2z}{d\theta^2} = 0$$

Put all values in (3)

$$\begin{vmatrix} x - a\cos\theta & y - a\sin\theta & z - 0 \\ -a\sin\theta & a\cos\theta & 0 \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix} = 0$$

$$\Rightarrow z(a^2\sin^2\theta + a^2\cos^2\theta) = 0$$

$$\Rightarrow a^2 z(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow a^2 z = 0$$

$$\Rightarrow z = 0$$

Example:-

Find equation of tangent to the circle given by

$$\vec{r} = (a\cos\theta, a\sin\theta, 0)$$

Sol:-

$$\vec{r} = (a\cos\theta, a\sin\theta, 0)$$

Equation of tangent to the circle is given as

$$\frac{X - x}{\frac{dx}{d\theta}} = \frac{Y - y}{\frac{dy}{d\theta}} = \frac{Z - z}{\frac{dz}{d\theta}} \rightarrow (1)$$

$$x = a\cos\theta, \quad y = a\sin\theta, \quad z = 0$$

$$\frac{dx}{d\theta} = -a\sin\theta, \quad \frac{dy}{d\theta} = a\cos\theta, \quad \frac{dz}{d\theta} = 0$$

putting all values in (1)

$$\frac{X-x}{-a\sin\theta} = \frac{Y-y}{a\cos\theta} = \frac{Z-z}{0}$$

$$\Rightarrow \frac{X-x}{-a\sin\theta} = \frac{Y-y}{a\cos\theta} = \frac{Z-z}{0}$$

put $x = a\cos\theta$, $y = a\sin\theta$, $z=0$

$$\frac{X-a\cos\theta}{-a\sin\theta} = \frac{Y-a\sin\theta}{a\cos\theta} = \frac{Z-0}{0}$$

$$\Rightarrow \frac{X-a\cos\theta}{-a\sin\theta} = \frac{Y-a\sin\theta}{a\cos\theta} = \frac{Z}{0}$$

$$\Rightarrow \frac{X-a\cos\theta}{-a\sin\theta} = \frac{Z}{0} = , \quad \frac{Y-a\sin\theta}{a\cos\theta} = \frac{Z}{0}$$

$$\Rightarrow a\cos\theta(X-a\cos\theta) = -a\sin\theta(Y-a\sin\theta)$$

$$\Rightarrow a\cos\theta\, X - a^2\cos^2\theta = -a\sin\theta\, Y + a^2\sin^2\theta$$

$$a\cos\theta X + a\sin\theta Y = a^2\sin^2\theta + a^2\cos^2\theta$$

$$\Rightarrow X a\cos\theta + Y a\sin\theta = a^2(\sin^2\theta + \cos^2\theta)$$

$$\Rightarrow X(a\cos\theta) + Y(a\sin\theta) = a^2 (1)$$

$$\Rightarrow X(a\cos\theta) + Y(a\sin\theta) = a^2$$

is an equation of circle with radius "a"

Example:-

Let $\vec{\gamma} = \vec{\gamma}(t)$ be a curve then prove that

$$k = \frac{|\dot{\vec{\gamma}} \times \ddot{\vec{\gamma}}|}{|\dot{\vec{\gamma}}|^3}$$

where $\dot{\vec{\gamma}} = \dfrac{d\vec{\gamma}}{dt}$, $\ddot{\vec{\gamma}} = \dfrac{d^2\vec{\gamma}}{dt^2}$

Sol:-

To prove this result.

Consider $\dot{\vec{\gamma}} = \dfrac{dr}{dt}$

$$\dot{\vec{\gamma}} = \frac{dr}{ds} \cdot \frac{ds}{dt}$$

$$\dot{\vec{\gamma}} = \vec{\gamma}' \cdot \dot{s}$$

Again Differentiate w.r.t "t"

$$\ddot{\vec{\gamma}} = \frac{d}{dt}(\vec{\gamma}' \cdot \dot{s})$$

$$\ddot{\vec{\gamma}} = \frac{d}{dt}(\vec{\gamma}')\dot{s} + \frac{d}{dt}(\dot{s})\vec{\gamma}' \qquad \because \frac{d\vec{\gamma}'}{dt} = \frac{d\vec{\gamma}'}{ds}\frac{ds}{dt}$$

$$\ddot{\vec{\gamma}} = \vec{\gamma}''(\dot{s})^2 + \vec{\gamma}'\ddot{s} \qquad \because \vec{\gamma}''\dot{s}$$

$$\dot{\vec{\gamma}} \times \ddot{\vec{\gamma}} = \vec{\gamma}'\dot{s} \times (\vec{\gamma}''(\dot{s})^2 + \vec{\gamma}' \cdot \ddot{s})$$

$$= (\vec{\gamma}' \times \vec{\gamma}'')((\dot{s})^3) + (\vec{\gamma}' \times \vec{\gamma}')\dot{s}\ddot{s}$$

$$= (\vec{\gamma}' \times \vec{\gamma}'')(\dot{s})^3 + 0$$

$$= (\vec{\gamma}' \times \vec{\gamma}'')(\dot{s})^3$$

put $\vec{\gamma}'' = \vec{t}'$ and $\vec{\gamma}' = \vec{t}$

$$\vec{\gamma}'' = \vec{t}' = kn$$

$$= (\vec{t} \times \vec{t}')(\dot{s})^3$$

$$\dot{\vec{\gamma}} \times \ddot{\vec{\gamma}} = (\vec{t} \times k\hat{n})(\dot{s})^3$$

$$|\vec{\dot{r}} \times \vec{\ddot{r}}| = |k(\vec{t} \times \vec{n})(\dot{s})^3|$$

$$= k(\dot{s})^3 \, |\vec{t} \times \vec{n}| \qquad |\vec{t} \times \vec{n}| = 1$$

$$= k(\dot{s})^3$$

$$|\vec{\dot{r}} \times \vec{\ddot{r}}| \quad k = \frac{|\vec{\dot{r}} \times \vec{\ddot{r}}|}{(\dot{s})^3} \rightarrow (1)$$

Now Consider

$$\vec{\dot{r}} = \vec{r}' \dot{s}$$

$$\Rightarrow |\vec{\dot{r}}| = |\vec{r}' \dot{s}|$$

$$= \dot{s} \, |\vec{r}'| \qquad \vec{r}' = \vec{t}$$

$$= \dot{s} \, |\vec{t}| \qquad |\vec{t}| = 1$$

$$|\vec{\dot{r}}| = \dot{s}$$

So, $|\vec{\dot{r}}|^3 = (\dot{s})^3$

putting this in (1)

$$k = \frac{|\vec{\dot{r}} \times \vec{\ddot{r}}|}{|\vec{\dot{r}}|^3}$$

Example:-

Let $\vec{r} = \vec{r}(t)$ be a curve then prove that $[\vec{r}' \; \vec{r}'' \; \vec{r}'''] = \dfrac{[\vec{\dot{r}} \; \vec{\ddot{r}} \; \vec{\dddot{r}}]}{(\dot{s})^6}$

Sol:-

we know $\vec{\dot{r}} = \vec{r}' \cdot \dot{s}$

$$\vec{\ddot{r}} = \vec{r}''(\dot{s})^2 + \vec{r}' \ddot{s}$$

Now, Diff again it w.r.t "t"

$$\frac{d}{dt}(\vec{\ddot{r}}) = \frac{d}{dt}\left(\vec{r}''(\dot{s})^2 + \vec{r}' \ddot{s}\right)$$

$$= \frac{d}{dt}\left(\vec{r}''(\dot{s})^2\right) + \frac{d}{dt}\left(\vec{r}' \ddot{s}\right)$$

$$= \frac{d}{dt}(\vec{r}'')(\dot{s})^2 + \vec{r}'' \frac{d}{dt}((\dot{s})^2) + \frac{d}{dt}(\vec{r}') \ddot{s}$$

$$+\vec{r}'\frac{d}{dt}(\ddot{s})$$

$$\dddot{\vec{r}} = \frac{d\vec{r}''}{ds}\frac{ds}{dt}(\dot{s})^2 + \vec{r}''\left(2\dot{s}\frac{d\dot{s}}{dt}\right) + \frac{d\vec{r}'}{ds}\frac{ds}{dt}\ddot{s} + \vec{r}'\dddot{s}$$

$$\dddot{\vec{r}} = \vec{r}'''\dot{s}(\dot{s})^2 + \vec{r}''\,2\dot{s}\ddot{s} + d\vec{r}''\dot{s}\ddot{s} + \vec{r}'\dddot{s}$$

$$= \vec{r}'''(\dot{s})^3 + 3\vec{r}''(\dot{s}\ddot{s} + \vec{r}''\ddot{s} + \vec{r}'\dddot{s}$$

$$\dddot{\vec{r}} = \vec{r}'''(\dot{s})^3 + 3\vec{r}''(\dot{s})\ddot{s} + (\vec{r}''\,\vec{r}')\dddot{s}$$

$$\ddot{\vec{r}}\times\dddot{\vec{r}} = (\vec{r}''(\dot{s})^2 + \vec{r}'\ddot{s})\times(\vec{r}'''(\dot{s})^3 + 3\vec{r}''(\dot{s})\ddot{s} + (\vec{r}''\,\vec{r}')\dddot{s})$$

$$= (\vec{r}''\times\vec{r}''')(\dot{s})^5 + (\vec{r}'\times\vec{r}''')\ddot{s}(\dot{s})^3 + 3(\vec{r}''\times\vec{r}'')(\dot{s})^3\ddot{s} + \cancel{\ldots} + (\vec{r}'\times\vec{r}')\dot{s}\dddot{s}$$

$$= (\vec{r}''\times\vec{r}''')(\dot{s})^5 + (\vec{r}'\times\vec{r}''')(\dot{s})^3\ddot{s} + 3(\vec{r}''\times\vec{r}'')\dot{s}(\ddot{s})^2$$

$$+ (\vec{r}''\times\vec{r}')(\dot{s})^2\dddot{s}$$

$$+ (\vec{r}'\times\vec{r}')\dot{s}\dddot{s} + (\vec{r}'\times\vec{r}')\ddot{s}\dddot{s}]$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}}\times\dddot{\vec{r}} = (\vec{r}'\cdot\vec{r}''\times\vec{r}''')(\dot{s})^6 + (\vec{r}'\cdot\vec{r}'\times\vec{r}''')(\dot{s})^4\ddot{s}$$

$$= [(\vec{r}''\times\vec{r}''')(\dot{s})^5 + 3(\ldots)(\dot{s})^3\ddot{s}] +$$

$$[(\vec{r}'\ldots\vec{r}'')(\dot{s})^2\dddot{s} + (\vec{r}'\times\vec{r}'')\dot{s}\dddot{s}$$

$$\dot{\vec{r}}\cdot\ddot{\vec{r}}\times\dddot{\vec{r}} = (\vec{r}'\cdot\vec{r}''\times\vec{r}''')(\dot{s})^6 + (\vec{r}'\times\vec{r}'\cdot\vec{r}''')(\dot{s})^4\ddot{s}$$

$$[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}] + 3(\vec{r}'\times\vec{r}'\cdot\vec{r}'')(\dot{s})^2\ddot{s}^2 + (\vec{r}'\cdot\vec{r}'\times\vec{r}'')(\dot{s})^3\dddot{s}$$

"." and "x" can be interchanged and

$$\vec{r}'\times\vec{r}' = 0$$

So, $[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}] = [\vec{r}'\ \ \vec{r}''\ \ \vec{r}'''](\dot{s})^6 + 0 + 0$

$$+ (\vec{r}'\times\vec{r}'\cdot\vec{r}'')(\dot{s})^3\dddot{s}$$

$$= [\vec{r}'\ \ \vec{r}''\ \ \vec{r}'''](\dot{s})^6 + 0 + 0 + 0$$

$$[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}] = [\vec{r}'\ \ \vec{r}''\ \ \vec{r}'''](\dot{s})^6$$

$$\Rightarrow [\vec{r}'\ \ \vec{r}''\ \ \vec{r}'''] = \frac{[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}]}{(\dot{s})^6}$$

Question:-

Prove that for a curve $\vec{r} = \vec{r}(t)$

$$\vec{\tau} = \frac{[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}]}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2}$$

Sol:-

we know $\tau = \dfrac{1}{k^2}[\vec{r}'\ \ \vec{r}''\ \ \vec{r}'''] \to (1)$

and $k = \dfrac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$

put k-value in (1)

$$\tau = \frac{1}{\left[\dfrac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}\right]^2}[\vec{r}'\ \ \vec{r}''\ \ \vec{r}''']$$

$$\tau = \frac{1}{\dfrac{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2}{|\dot{\vec{r}}|^6}}[\vec{r}'\ \ \vec{r}''\ \ \vec{r}''']$$

put $[\vec{r}'\ \ \vec{r}''\ \ \vec{r}'''] = \dfrac{[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}]}{(\dot{s})^6}$

$$\tau = \frac{|\dot{\vec{r}}|^6}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2}\ \frac{[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}]}{(\dot{s})^6} \to (1)$$

we know

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}$$

$$\dot{\vec{r}} = \frac{d\vec{r}}{ds}\frac{ds}{dt}$$

$$\dot{\vec{r}} = \vec{r}'\ \dot{s}$$

we know $\vec{r}' = \vec{t}$

$$\dot{\vec{r}} = \vec{t}\ \dot{s}$$

$$|\dot{\vec{r}}| = |\vec{t}||\dot{s}|$$

$|\vec{t}| = 1$ because, $\vec{t}$ is unit vector

$$|\dot{\vec{r}}| = |\dot{s}|$$

$\Rightarrow\ |\dot{\vec{r}}|^6 = |\dot{s}|^6$

Putting this value in (1)

$$\tau = \frac{|s'|^6}{|\vec{r}' \times \vec{r}''|^2} \left[\frac{\vec{r}' \quad \vec{r}'' \quad \vec{r}'''}{(s')^6}\right]$$

$$\tau = \frac{[\vec{r}' \quad \vec{r}'' \quad \vec{r}''']}{|\vec{r}' \times \vec{r}''|^2}$$

Question:-

For a curve given by (i) $x = a(3u-u^3)$, $z = a(3u+u^3)$, $y = 3au^2$

(ii) $x = 3a(u-u^3)$, $y = 3au^2$, $z = 3a(u+u^3)$

Find curvature and Torsion.

Sol:-

(ii)

$$k = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} \rightarrow (1)$$

$$\tau = \frac{[\vec{r}' \quad \vec{r}'' \quad \vec{r}''']}{|\vec{r}' \times \vec{r}''|^2} \rightarrow (2)$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = 3a(u-u^3)\hat{i} + 3au^2\hat{j} + 3a(u+u^3)\hat{k}$$

$$\vec{r}' = 3a(1-3u^2)\hat{i} + 6au\hat{j} + 3a(1+3u^2)\hat{k}$$

$$\vec{r}'' = 3a(0-6u)\hat{i} + 6a\hat{j} + 3a(0+6u)\hat{k}$$

$$\vec{r}'' = -18au\hat{i} + 6a\hat{j} + 18au\hat{k}$$

$$\vec{r}''' = -18a\hat{i} + 0\hat{j} + 18a\hat{k}$$

$$|\vec{r}'|^2 = (3a(1-3u^2))^2 + (6au)^2 + (3a(1+3u^2))^2$$

$$= 9a^2(1-3u^2)^2 + 36a^2u^2 + 9a^2(1+3u^2)^2$$

$$= 9a^2(1+9u^4-6u^2) + 36a^2u^2 + 9a^2(1+9u^4+6u^2)$$

$$= 9a^2 + 81a^2u^4 - 54a^2u^2 + 36a^2u^2 + 9a^2 + 81a^2u^4 + 54a^2u^2$$

$$= 18a^2 + 198a^2u^4$$

$$= 2a^2(9+99u^4) = 18a^2(1+9u^4)$$

$$|\vec{\dot{r}}||\vec{\dot{r}}|^2 = \sqrt{18a^2(1+9u^4)\ (18a^2(1+9u^4))}$$

$$|\vec{\dot{r}}|^3 = 18a^2(1+9u^4)\sqrt{18a^2(1+9u^4)} = [18a^2(1+9u^4)]^{3/2}$$

$$\vec{\dot{r}} \times \vec{\ddot{r}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3a(1-3u^2) & 6au & 3a(1+3u^2) \\ -18au & 6a & 18au \end{vmatrix}$$

$$= \hat{i}(108a^2u^2 - 18a^2(1+3u^2)) - \hat{j}(54a^2u(1-3u^2) + 54a^2u(1+3u^2)) + \hat{k}(18a^2(1-3u^2) + 108a^2u^2)$$

$$= \hat{i}(108a^2u^2 - 18a^2 - 54a^2u^2) - \hat{j}(54a^2u[1-3u^2+1+3u^2]) + \hat{k}(18a^2 - 54a^2u^2 + 108a^2u^2)$$

$$= \hat{i}(54a^2u^2 - 18a^2) - \hat{j}(54a^2u(2)) + \hat{k}(54a^2u^2 + 18a^2)$$

$$\vec{\dot{r}} \times \vec{\ddot{r}} = \hat{i}(54a^2u^2 - 18a^2) - \hat{j}(108a^2u) + \hat{k}(54a^2u^2 + 18a^2)$$

$$|\vec{\dot{r}} \times \vec{\ddot{r}}|^2 = (54a^2u^2 - 18a^2)^2 + (108a^2u)^2 + (54a^2u^2 + 18a^2)^2$$

$$= 2916a^4u^4 - \cancel{1944a^4u^2} + 324a^4 + 11664a^4u^2 + 2916a^4u^4 + \cancel{1944a^4u^2} + 324a^4$$

$$= 2(2916a^4u^4) + 11664a^4u^2 + 2(324a^4)$$

$$= 5832a^4u^4 + 11664a^4u^2 + 648a^4$$

$$|\vec{\dot{r}} \times \vec{\ddot{r}}|^2 = 54(108a^4u^4 + 216a^4u^2 + 12a^4)$$

$$= 54a^4(108u^4 + 216u^2 + 12)$$

$$|\vec{\dot{r}} \times \vec{\ddot{r}}|^2 = 216a^4(27u^4 + 54u^2 + 3)$$

$$\vec{\dot{r}} \cdot \vec{\ddot{r}} \times \vec{\dddot{r}} = \begin{vmatrix} 3a(1-3u^2) & 6au & 3a(1+3u^2) \\ -18au & 6a & 18au \\ -18a & 0 & 18a \end{vmatrix}$$

$$= 3a(1-3u^2)(108a^2 - 0) - 6au(-324a^2u + 324a^2u) + 3a(1+3u^2)(0 + 108a^2)$$

$$= 324a^3 - \cancel{972a^3u^2} + 324a^3 + \cancel{972a^3u^2}$$

$$\vec{\dot{r}} \cdot \vec{\ddot{r}} \times \vec{\dddot{r}} = 648a^3$$

Putting all values in (1) and (2)

$$k = \frac{\sqrt{216a^4(27u^4 + 54u^2 + 3)}}{[18a^2(1+9a^4)]^{3/2}} = \frac{\sqrt{9u^4 + 18u^2 + 1}}{3a(9u^4 + 2u^2 + 1)^{3/2}}$$

$$[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}] = \begin{vmatrix} 3a(1-3u^2) & 6au & 3a(1+3u^2) \\ -18au & 6a & 18au \\ -18a & 0 & 18a \end{vmatrix}$$

$$[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = (3a)(6a)(18a)\begin{vmatrix} 1-3u^2 & 2u & 1+3u^2 \\ -3u & 1 & 3u \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 18a^3 \cdot 18\left[(1-3u^2)(1-0) - 2u(-3u+3u) + (1+3u^2)(0+1)\right]$$

$$= (18)^2 a^3 \left[1-3u^2-0+1+3u^2\right] = 648a^3$$

$$[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}] = 648a^3$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = 18a^2\sqrt{9u^2+1-6u^2+36u^2+1+9u^2+6u^2}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = 18a^2\sqrt{18u^4+36u^2+2}$$

$$\Rightarrow |\dot{\vec{r}} \times \ddot{\vec{r}}|^2 = (18a^2)^2(18u^4+36u^2+2) \text{ put in (2)}$$

$$\tau = \frac{648a^3}{(18a^2)^2\, 2(9u^4+8u^2+1)}$$

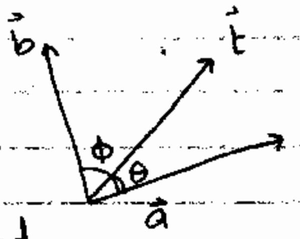
$$\tau = \frac{a^3}{a^4(9u^4+18u^2+1)} = \frac{1}{a(9u^4+18u^2+1)}$$

Questions:-

If the tangent and bi-normal at any point on a curve make angles θ and ϕ respectively with a fixed direction then prove that

$$\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = \frac{-k}{\tau}$$

Sol:-



Let $\vec{a}$ be a unit vector along the fixed direction with which tangent and binormal make angle θ and ϕ respectively

Now,

$$\vec{a}\cdot\vec{t} = |\vec{a}||\vec{t}|\cos\theta$$

$$\vec{a}\cdot\vec{t} = \cos\theta \rightarrow (1)$$

Also $\vec{b}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\phi$

$$\vec{a}\cdot\vec{b} = \cos\phi \rightarrow (2)$$

Differentiating (1) and (2) w.r.t s

$$\vec{a}\cdot\vec{t}' + 0\cdot\vec{t} = -\sin\theta\frac{d\theta}{ds} \rightarrow (3)$$

$$\vec{a}\cdot\vec{b}' + 0\cdot\vec{b} = -\sin\phi\frac{d\phi}{ds} \rightarrow (4)$$

Dividing (3) by (4)

$$\frac{\vec{a}\cdot\vec{t}'}{\vec{a}\cdot\vec{b}'} = \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}$$

Put $\vec{t}' = \vec{r}'' = k\vec{n}$ and $\vec{b}' = -\tau\vec{n}$

$$\frac{\vec{a}\cdot k\vec{n}}{\vec{a}\cdot(-\tau\vec{n})} = \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}$$

$$-\frac{k}{\tau}\left(\frac{\vec{a}\cdot\vec{n}}{\vec{a}\cdot\vec{n}}\right) = \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}$$

$$\Rightarrow \frac{-k}{\tau} = \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi}$$

$$\Rightarrow \frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{k}{\tau}$$

Rectifing plane :-

Rectifing plane at any point "p" on a curve is a plane through "p" and perpandicular to the normal at point "p". i.e.

Rectifing plane contain tangent and bi-normal so it is $\vec{t}\vec{b}$-plane.

Remark:-

(i) $\vec{t}\vec{b}$-plane is rectifing plane
(ii) $\vec{t}\vec{n}$-plane is oscolating plane
(iii) $\vec{n}\vec{b}$-plane is normal plane

Circle of Curvature :-

Circle of curvature at any point "p" on a curve is a circle passing through 3 consective points at point "p".

And its radius is known as radius of curvature, it is usual denoted by ρ and

$$\rho = \frac{1}{k}$$

The radius of curvature is always (along the) in the direction of normal at point "p".

The centre of circle of curvature is known as centre of curvature.

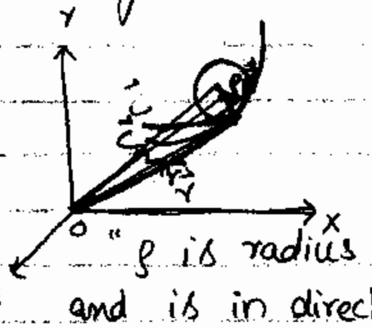
Locus of centre of curvature :-

(Locus means path)

Let $\vec{c}$ be the position vector of centre of curvature corresponding to a point "p" on a curve with position vector $\vec{r}$.

$$\vec{c} = \vec{r} + \rho\vec{n}$$

is know is locus of centre of curvature.



" ρ is radius and is in direction of normal so, we use $\rho\vec{n}$ ".

The tangent at any point on the locus of centre of curvature is along the vector

$$\frac{d\vec{c}}{ds} = \vec{c}\,'$$

$$\vec{c}' = \frac{d}{ds}(\vec{r} + \rho\vec{n})$$

$$\vec{c}' = \vec{r}' + \rho'\vec{n} + \rho\vec{n}'$$

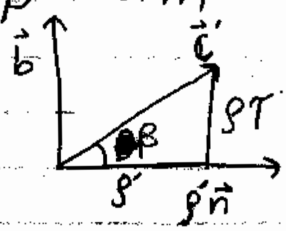
$$\vec{c}' = \vec{r}' + \rho'\vec{n} + \rho(\tau\vec{b} - k\vec{t})$$

$$\vec{c}' = \vec{r}' + \rho'\vec{n} + \rho\tau\vec{b} - \rho k\vec{t}$$

$$\vec{c}' = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \cancel{\rho}_{\frac{1}{k}} k\vec{t} \qquad \because \rho = \frac{1}{k}$$

$$\vec{c}' = \rho'\vec{n} + \rho\tau\vec{b}$$

is equation of tangent of locus of centre of...
Hence, the tangent at any point on the locus of centre of curvature lie in the normal plane; and if the tangent on the locus of centre of curvature makes an angle "β" with the normal $\vec{n}$ then



$$\tan\beta = \frac{\rho\tau}{\rho'\vec{n}}$$

$$\Rightarrow \quad \tan\beta = \frac{\rho\tau}{\rho'}$$

Question:-

Prove that for a curve

(i) If $k=0$ at all points then the curve is a straight line.

(ii) The curve is a plane iff $\tau = 0$

(iii) The necessary and sufficient condition for a curve to be a plane is

$$[\vec{r}' \ \vec{r}'' \ \vec{r}'''] = 0$$

Sol:-

(i) Let $\vec{r} = \vec{r}(s)$ be a curve

if $k = 0$

$k\vec{n} = 0$

$\Rightarrow \vec{t}' = \vec{r}'' = 0$

$\Rightarrow \dfrac{d^2\vec{r}}{ds^2} = 0$ Integrate both sides w.r.t "s".

we have

$$\frac{d\vec{r}}{ds} = \vec{a} \text{ (constant vector)}$$

$$\vec{r} = \vec{a}s + \vec{b}$$

is equation of a straight line.

Hence, the curve is a straight line.

(ii)

Suppose that the curve is a plane-curve, then $\vec{b}$ = constant

Differentiating both sides w.r.t "s"

$$\vec{b}' = 0$$

$$-\tau\vec{n} = 0$$

$\Rightarrow n \neq 0$ and $\tau = 0$

Conversly, suppose that $\tau = 0$

$\Rightarrow \tau \cdot \vec{n} = 0$

$\Rightarrow -\tau\vec{n} = 0$

$\Rightarrow \vec{b}' = 0$

Integrating both sides w.r.t "s"

$\vec{b}$ = constant (vector)

$\Rightarrow \vec{b} = \vec{b}_0$

Now, Consider

$$\frac{d}{ds}(\vec{r}\cdot\vec{b}_0) = \vec{r}'\cdot\vec{b}_0 + \vec{r}\cdot 0$$

$$= \vec{r}'\cdot\vec{b}_0$$

$$= \vec{t}\cdot\vec{b}_0$$

$$\frac{d}{ds}(\vec{r}\cdot\vec{b}_0) = \vec{t}\cdot\vec{b}$$

$\Rightarrow \frac{d}{ds}(\vec{r}\cdot\vec{b}_0) = 0$

Integrating both sides w.r.t "s"

$\Rightarrow \vec{r}\cdot\vec{b}_0 = \text{constant}$

(where $\vec{r}$ is moving point and $\vec{r} = (\vec{x}, \vec{y}, \vec{z})$ and $\vec{b}_0$ is constant and $\vec{b}_0 = (\vec{b}_1, b_2, \vec{b}_3)$

$\vec{r}\cdot\vec{b}_0 = (\vec{x}, \vec{y}, \vec{z})\cdot(\vec{b}_1, \vec{b}_2, \vec{b}_3)$

$\text{Constant} = \vec{x}\vec{b}_1 + \vec{y}\vec{b}_2 + z\vec{b}_3$

is a straight line)

$\Rightarrow$ Hence, the curve is a straight line and hence is a plane curve.

(iii)

Suppose that the curve is a plane curve Then $\tau = 0$

we know that

$[\vec{r}'\ \vec{r}''\ \vec{r}'''] = k^2\tau$

put $\tau = 0$

$[\vec{r}'\ \vec{r}''\ \vec{r}'''] = 0$

Conversly

$[\vec{r}'\ \vec{r}''\ \vec{r}'''] = 0$

Let $[\vec{r}'\ \vec{r}''\ \vec{r}'''] = 0$

we know

$\Rightarrow [\vec{r}'\ \vec{r}''\ \vec{r}'''] = k^2\tau$

$\Rightarrow k^2\tau = 0$

$\Rightarrow k = 0$ or $\tau = 0$

we shall prove that

$\tau = 0$ at all points of the curve.

Let $\tau \neq 0$ at some point of the curve

Then, $\tau \neq 0$ in some nbhd of that point. Since $k = 0$ in this nbhd, the arc of the curve in this nbhd

is a straight line.
This implies that $\tau = 0$ in this line in the nbhd of this point which is a contradiction to the hypothesis that $\tau \neq 0$.
This contradiction proves that $\tau = 0$ at all points of the curve.
So that the curve is a plane curve.
Thus, condition is sufficient also.

Note:-

In terms of the dash derivatives we have

$$[\vec{r}', \vec{r}'' \ \vec{r}'''] = u'^6 [\dot{\vec{r}} \ \ddot{\vec{r}} \ \dddot{\vec{r}}]$$

Since $u' = \dfrac{du}{ds} \neq 0$

$$[\dot{\vec{r}} \ \ddot{\vec{r}} \ \dddot{\vec{r}}] = 0$$

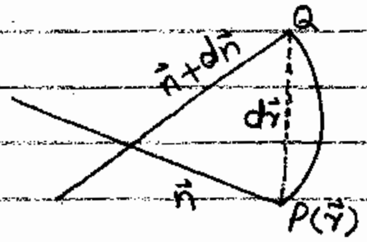
iff $[\vec{r}' \ \vec{r}'' \ \vec{r}'''] = 0$

which is necessary and sufficient condition for a space curve to be a plane curve.

Question:-

Prove that the principal normals at two consective points on a curve do not intersect unless $\tau = 0$

Sol:-



Let $P(\vec{r})$ and $Q(\vec{r}+d\vec{r})$ be two consective points on a curve.

Let $\vec{n}$ and $\vec{n}+d\vec{n}$ be unit principal vectors at points P and Q respectively. Now the vectors $d\vec{r}$, $\vec{n}$ and $\vec{n}+d\vec{n}$ are co-planar so

$$[d\vec{r} \quad \vec{n} \quad \vec{n}+d\vec{n}] = 0$$

$$\Rightarrow \left[\frac{d\vec{r}}{ds}ds \quad \vec{n} \quad \vec{n}+\frac{d\vec{n}}{ds}ds\right] = 0 \qquad \because \frac{d\vec{r}}{ds} = \vec{r}\,' = \vec{t}$$

$$\Rightarrow [\vec{t}ds \quad \vec{n} \quad \vec{n}+\vec{n}\,'ds] = 0$$

$$\Rightarrow [\vec{t}ds \quad \vec{n} \quad \vec{n}] + [\vec{t}ds \quad \vec{n} \quad \vec{n}\,'ds] = 0$$

$$\Rightarrow \quad 0 \quad + [\vec{t}ds \quad \vec{n} \quad (\tau\vec{b}-k\vec{t})ds] = 0$$

$$\Rightarrow [\vec{t}ds \quad \vec{n} \quad \tau\vec{b}ds] + [\vec{t}ds \quad \vec{n} \quad -k\vec{t}ds] = 0$$

$$\Rightarrow \tau(ds)^2[\vec{t} \quad \vec{n} \quad \vec{b}] - (ds)^2 k[\vec{t} \quad \vec{n} \quad \vec{t}] = 0$$

$$\Rightarrow \tau(ds)^2(1) - 0 = 0$$

$$\Rightarrow \tau(ds)^2 = 0$$

$$\Rightarrow \tau = 0$$

Remarks:-

$\vec{t}$, $\vec{n}$ and $\vec{b}$ are co-planar so, their scalar triple product is zero $[\vec{t} \quad \vec{n} \quad \vec{b}] = 0$

$[\vec{t} \quad \vec{n} \quad \vec{t}] = \vec{t}\cdot\vec{n}\times\vec{t} = \vec{n}\times\vec{t}\cdot\vec{t} = n\cdot\vec{t}\times\vec{t} = n\cdot 0 = 0$

Radius of Torsion :-

Radius of Torsion is the reciprocal of Torsion and it is denoted by 'σ'

$$\sigma = \frac{1}{\tau}$$

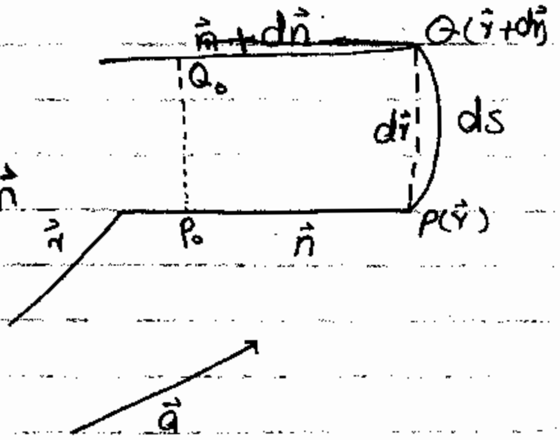
Question :-

Prove that the shortest distance between the principal normal at two consective points on a curve is

$$\frac{\rho\, ds}{\sqrt{\rho^2+\sigma^2}}$$

and the line of this shortest distance divides the radius of curvature in the ratio $\rho^2 : \sigma^2$

Sol:-



Let $P(\vec{r})$ and $Q(\vec{r}+d\vec{r})$ be two consective points on a curve and $\vec{n}$ and $\vec{n}+d\vec{n}$ be the unit principal normal at $\hat{P}$ and Q respectively.

Now, To find the shortest distance between the principal normals at $\hat{P}$ and Q.

we will first find a vector Perpandicular to both $\vec{n}$ and $\vec{n}+d\vec{n}$

The vector perpandicular to $\vec{n}$ and $\vec{n}+d\vec{n}$ is $[\vec{n}\times(\vec{n}+d\vec{n})] = [\vec{n}\times\vec{n} + \vec{n}\times d\vec{n}]$

$$= [0 + \vec{n}\times d\vec{n}]$$

$$= \vec{n}\times\frac{d\vec{n}}{ds}\,ds$$

$$= \vec{n} \times \vec{n}' \, ds$$
$$= \vec{n} \times (\tau \vec{b} - k\vec{t}) ds$$
$$= [\vec{n} \times \tau \vec{b} - \vec{n} \times k\vec{t}] ds$$

$$= [\tau(\vec{n} \times \vec{b}) - k(\vec{n} \times \vec{t})] ds$$
$$= [\tau \vec{t} - k(-\vec{b})] ds$$

$[\vec{n} \times (\vec{n} + d\vec{n})] = [\tau \vec{t} + k\vec{b}] ds$

Hence, the vector perpandicular to both $\vec{n}$ and $\vec{n} + d\vec{n}$ is $[\tau \vec{t} + k\vec{b}] ds$

Now the unit vector perpandicular to both $\vec{n}$ and $\vec{n} + d\vec{n}$ is

$$\frac{[\tau \vec{t} + k\vec{b}] ds}{\sqrt{[\tau^2 + k^2] ds^2}} = \frac{[\tau \vec{t} + k\vec{b}] ds}{\sqrt{\tau^2 + k^2} \, ds}$$

$$= \frac{\tau \vec{t} + k\vec{b}}{\sqrt{\tau^2 + k^2}}$$

Let $\hat{e} = \dfrac{\tau \vec{t} + k\vec{b}}{\sqrt{\tau^2 + k^2}}$

Now, To find the shortest distance between $\vec{n}$ and $\vec{n} + d\vec{n}$ is equal to the projection of $d\vec{r}$ upon $\hat{e}$

(i-e)

shortest distance $SD = d\vec{r} \cdot \hat{e}$

$$= d\vec{r} \cdot \left[\frac{\tau \vec{t} + k\vec{b}}{\sqrt{\tau^2 + k^2}} \right]$$

$$= \frac{1}{\sqrt{\tau^2 + k^2}} \left[d\vec{r} \cdot (\tau \vec{t} + k\vec{b}) \right]$$

$$= \frac{1}{\sqrt{\tau^2+k^2}}\left[\frac{d\vec{r}}{ds}ds\cdot(\tau\vec{t}+k\vec{b})\right]$$

$$SD = \frac{1}{\sqrt{\tau^2+k^2}}\left[\frac{d\vec{r}}{ds}\cdot(\tau\vec{t}+k\vec{b})\right]ds$$

$$SD = \frac{ds}{\sqrt{\tau^2+k^2}}\left[\vec{t}\cdot(\tau\vec{t}+k\vec{b})\right]$$

$$= \frac{ds}{\sqrt{\tau^2+k^2}}\left[\tau(\vec{t}\cdot\vec{t})+k(\vec{t}\cdot\hat{b})\right]$$

$$= \frac{ds}{\sqrt{\tau^2+k^2}}\left[\tau(1)+k(0)\right]$$

$$SD = \frac{\tau\, ds}{\sqrt{\tau^2+k^2}}$$

put $\tau = \frac{1}{\sigma}$ and $k = \frac{1}{\rho}$

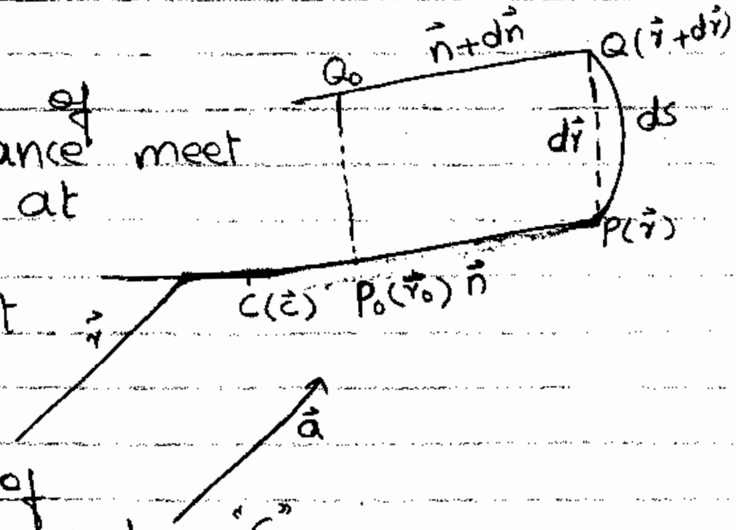
$$SD = \frac{\frac{1}{\sigma}ds}{\sqrt{\left(\frac{1}{\sigma}\right)^2+\left(\frac{1}{\rho}\right)^2}}$$

$$SD = \frac{\frac{1}{\sigma}ds}{\sqrt{\frac{1}{\sigma^2}+\frac{1}{\rho^2}}} = \frac{\frac{1}{\sigma}ds}{\sqrt{\frac{\rho^2+\sigma^2}{\rho^2\sigma^2}}}$$

$$= \frac{\frac{1}{\sigma}ds}{\sqrt{\rho^2+\sigma^2}\,\frac{1}{\rho\sigma}}$$

$$SD = \frac{\rho\, ds}{\sqrt{\rho^2+\sigma^2}}$$

is shortest distance between $\vec{n}$ and $\vec{n}+d\vec{n}$



Let the line of shortest distance meet the normal $\vec{n}$ at point P_0 and $\vec{n}+d\vec{n}$ at point Q_0.

Let $\vec{c}$ be the position vector of the centre of curvature "C" corresponding to the point P on the curve.

Here the vectors $\overrightarrow{Q_0Q}$, $\overrightarrow{QP_0}$ and $\overrightarrow{P_0Q_0}$ are co-planar.

The vector $\overrightarrow{Q_0Q}$ is along $\vec{n}+d\vec{n}$, the vector $\overrightarrow{P_0Q_0}$ is along the $(\tau\vec{t}+k\vec{b})ds$, and the vector $\overrightarrow{QP_0}$ is equal to

$\vec{r}_0-(\vec{r}+d\vec{r}) = \overrightarrow{QP_0}$

Now, since these vectors are co-planar so, the scalar triple product of

$$[\overrightarrow{QP_0} \quad \overrightarrow{Q_0Q} \quad \overrightarrow{P_0Q_0}] = 0$$

$$\Rightarrow [\vec{r}_0-(\vec{r}+d\vec{r}) \quad \vec{n}+d\vec{n} \quad (\tau\vec{t}+k\vec{b})ds] = 0 \rightarrow (1)$$

$\Rightarrow$ Now, the equation of normal at point "P" is

$$\vec{R} = \vec{r}+u\vec{n} \rightarrow (2)$$

Since, P_0 lies on the normal at point "P" so, it must satisfy eq (2)

$\vec{r}_0 = \vec{r} + u\vec{n}$

Substitute $\vec{r}_0$ - value in (1)

$$\left[\vec{r}+u\vec{n}-\vec{r}-d\vec{r} \quad \vec{n}+d\vec{n} \quad [\tau\vec{t}+k\vec{b}]ds\right]=0$$

$$\left[u\vec{n}-d\vec{r} \quad \vec{n}+d\vec{n} \quad [\tau\vec{t}+k\vec{b}]ds\right]=0$$

$$\left[u\vec{n}-\frac{d\vec{r}}{ds}ds \quad \vec{n}+\frac{d\vec{n}}{ds}ds \quad (\tau\vec{t}+k\vec{b})ds\right]=0$$

$$\Rightarrow \left[u\vec{n}-\vec{t}ds \quad \vec{n}+\vec{n}'ds \quad (\tau\vec{t}+k\vec{b})ds\right]=0$$

$$\Rightarrow \left[u\vec{n}-\vec{t}ds \quad \vec{n}+(\tau\vec{b}-k\vec{t})ds \quad (\tau\vec{t}+k\vec{b})ds\right]=0$$

$$\Rightarrow \left[u\vec{n}-\vec{t}ds \quad \vec{n}+\tau\vec{b}ds-k\vec{t}ds \quad \tau\vec{t}ds+k\vec{b}ds\right]=0$$

we write these in determinant, first Components of $\vec{t}$, then $\vec{n}$ and last $\vec{b}$.

$$\Rightarrow \begin{vmatrix} -ds & u & 0 \\ -kds & 1 & \tau ds \\ \tau ds & 0 & kds \end{vmatrix} = 0$$

$$-ds(kds-0) - u(-k^2(ds)^2 - \tau^2(ds)^2) - 0 = 0$$

$$-k(ds)^2 + uk^2(ds)^2 + u\tau^2(ds)^2 = 0$$

$$-k + uk^2 + u\tau^2 = 0$$

$$uk^2 + u\tau^2 = k$$

$$u(k^2+\tau^2) = k$$

$$u = \frac{k}{k^2+\tau^2}$$

Now $\overrightarrow{P_0P} = \vec{r}_0 - \vec{r}$

$$= \vec{r} + u\vec{n} - \vec{r}$$

$$\overrightarrow{P_0P} = u\vec{n}$$

$$|\overrightarrow{P_oP}| = |u\vec{n}|$$

$$|\overrightarrow{P_oP}| = u$$

$$|\overrightarrow{P_oP}| = \frac{k}{k^2+\tau^2}$$

$$|\overrightarrow{CP_o}| = |\overrightarrow{CP}| - |\overrightarrow{P_oP}|$$

$$= \rho - \frac{k}{k^2+\tau^2}$$

$$|\overrightarrow{CP_o}| = \frac{\rho(k^2+\tau^2) - k}{k^2+\tau^2} \qquad \text{put } \rho = \frac{1}{k}$$

$$= \frac{\frac{1}{k}(k^2+\tau^2) - k}{k^2+\tau^2}$$

$$= \frac{k + \tau^2/k - k}{k^2+\tau^2}$$

$$|\overrightarrow{CP_o}| = \frac{\frac{\tau^2}{k}}{k^2+\tau^2}$$

Now,

$$\frac{|\overrightarrow{CP_o}|}{|\overrightarrow{P_oP}|} = \frac{\frac{\tau^2/k}{k^2+\tau^2}}{\frac{k}{k^2+\tau^2}}$$

$$= \frac{\tau^2}{k} \cdot \frac{1}{k}$$

$$\frac{|\overrightarrow{CP_o}|}{|\overrightarrow{P_oP}|} = \frac{\tau^2}{k^2}$$

$$= \frac{\frac{1}{\sigma^2}}{\frac{1}{\rho^2}}$$

$$\frac{|\vec{CP_0}|}{|\vec{P_0P}|} = \frac{\rho^2}{\sigma^2}$$

Hence, the line of shortest distance between the principal normal at two consective points divides the radius of curvature in ratio $\rho^2 : \sigma^2$

Question :-

For any curve Prove that
$$\vec{t}' \cdot \vec{b}' = -k\tau$$

Sol :-

we know that $\vec{t}' = \vec{r}'' = k\vec{n}$

and $\vec{b}' = -\tau\vec{n}$

$$\vec{t}' \cdot \vec{b}' = k\vec{n} \cdot (-\tau\vec{n})$$
$$= -k\tau(\vec{n} \cdot \vec{n})$$
$$= -k\tau\,(1)$$
$$\vec{t}' \cdot \vec{b}' = -k\tau$$

Question :-

If m_1, m_2 and m_3 are moments about origin of the vectors $\vec{t}, \vec{n}$ and $\vec{b}$, then prove that

(i) $m_1' = k \cdot m_2$

(ii) $m_2' = \vec{b} - km_1 + \tau m_3$

(iii) $m_3' = -\vec{n} - \tau m_2$

where $(')$ denotes the derivative w.r.t "s"

Sol :-

m

$$m_1 = \vec{r} \times \vec{t}, \quad m_2 = \vec{r} \times \vec{n}, \quad m_3 = \vec{r} \times \vec{b}$$

(i) $m_1' = \frac{d}{ds}(\vec{r} \times \vec{t})$

$$m_1' = \frac{d\vec{r}}{ds} \times \vec{t} + \vec{r} \times \frac{d\vec{t}}{ds} \qquad \because \vec{r}' = \vec{t}$$

$$m_1' = t \times \vec{t} + \vec{r} \times \vec{t}'$$

$$= 0 + \vec{r} \times \vec{r}''$$

$$= \vec{r} \times k\vec{n}$$

$$= k(\vec{r} \times \vec{n})$$

$$m_1' = k\, m_2$$

(ii)

$$m_2 = \vec{r} \times \vec{n}$$

$$m_2' = \frac{d}{ds}(\vec{r} \times \vec{n})$$

$$= \frac{d\vec{r}}{ds} \times \vec{n} + \vec{r} \times \frac{d\vec{n}}{ds}$$

$$= \vec{t} \times \vec{n} + \vec{r} \times \vec{n}'$$

$$= \vec{b} + \vec{r} \times (\tau\vec{b} - k\vec{t})$$

$$= \vec{b} + \tau(\vec{r} \times \vec{b}) - k(\vec{r} \times \vec{t})$$

$$= \vec{b} - k m_1 + \tau m_3$$

(iii)

$$m_3 = \vec{r} \times \vec{b}$$

$$m_3' = \frac{d}{ds}(\vec{r} \times \vec{b})$$

$$= \vec{r}' \times \vec{b} + \vec{r} \times \vec{b}'$$

$$= \vec{t} \times \vec{b} + \vec{r} \times (-\tau\vec{n})$$

$$= -\vec{n} - \tau(\vec{r} \times \vec{n})$$

$$m_3' = -\vec{n} - \tau(m_2)$$

$$m_3' = -\vec{n} - \tau m_2$$

Question:-
If S_1 is the arc length of locus of centre of curvature of a curve then prove that

$$\frac{dS_1}{dS} = \frac{1}{k^2}\sqrt{k^2\tau^2 + k'^2} = \sqrt{\frac{\rho^2}{\sigma^2} + \rho'^2}$$

Sol:-
The locus of centre of curvature of the curve is given by

$$\vec{C} = \vec{r} + \rho\vec{n}$$

Differentiating both sides w.r.t s.

$$\frac{d\vec{c}}{ds} = \vec{r}' + \rho'\vec{n} + \rho\vec{n}'$$

$$\frac{d\vec{c}}{dS_1}\frac{dS_1}{dS} = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - k\vec{t})$$

$$= \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho k\vec{t}$$

$$= \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \vec{t} \qquad \rho = \frac{1}{k}$$

$$\frac{d\vec{c}}{dS_1}\frac{dS_1}{dS} = \rho'\vec{n} + \rho\tau\vec{b}$$

$$\vec{t}_1\frac{dS_1}{dS} = \rho'\vec{n} + \rho\tau\vec{b}$$

where $\vec{t}_1 = \dfrac{d\vec{c}}{dS_1}$

$$\Rightarrow \left|\vec{t}_1\frac{dS_1}{dS}\right| = \left|\rho'\vec{n} + \rho\tau\vec{b}\right|$$

$$|\vec{t}_1|\left|\frac{dS_1}{dS}\right| = \sqrt{(\rho')^2 + \rho^2\tau^2} \rightarrow (1) \qquad \because \tau = \frac{1}{\sigma}$$

$$\frac{dS_1}{dS} = \sqrt{(\rho')^2 + \rho^2\frac{1}{\sigma^2}}$$

$$\frac{ds_1}{ds} = \sqrt{\frac{\rho^2}{\sigma^2} + (\rho')^2}$$

Now $\rho = \frac{1}{k}$

$$\frac{d\rho}{ds} = \frac{d}{ds}\left(\frac{1}{k}\right)$$

$$\rho' = -\frac{1}{k^2}(k')$$

$$(\rho')^2 = \frac{k'^2}{k^4} \quad , \quad \rho^2 = \frac{1}{k^2}$$

putting this value in (1)

$$\frac{ds_1}{ds} = \sqrt{\frac{k'^2}{k^4} + \frac{1}{k^2}\tau^2}$$

$$= \sqrt{\frac{k'^2 + k^2\tau^2}{k^4}}$$

$$\frac{ds_1}{ds} = \frac{1}{k^2}\sqrt{k'^2 + k^2\tau^2}$$

Question:-

Find the curvature and torsion of the locus of centre of curvature for a curve with a constant curvature.

Sol:-

Let s_1 denote the arc length of locus of centre of curvature and the locus of centre of curvature is given by

$$\vec{c} = \vec{r} + \rho\vec{n}$$

Differentiate it w.r.t "s"

$$\frac{d\vec{c}}{ds} = \vec{r}' + \rho'\vec{n} + \rho\vec{n}'$$

$$= \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - k\vec{t})$$

$$\frac{d\vec{c}}{ds} = \vec{t} + 0 + \rho\tau\vec{b} - \rho k\vec{t} \qquad \because k = \frac{1}{\rho} \text{ is}$$

$$\frac{d\vec{c}}{ds} = \vec{t} + \rho\tau\vec{b} - \frac{1}{k}k\vec{t} \qquad \text{constant}$$

$$\text{so } \rho' = 0$$

$$= \vec{t} + \rho\tau\vec{b} - \vec{t}$$

$$\frac{d\vec{c}}{ds} = \rho\tau\vec{b}$$

$$\frac{d\vec{c}}{ds_1}\frac{ds_1}{ds} = \rho\tau\vec{b}$$

$$\vec{t}_1\frac{ds_1}{ds} = \rho\tau\vec{b} \qquad \because \frac{d\vec{c}}{ds_1} = \vec{t}_1$$

$$\Rightarrow \frac{ds_1}{ds} = \rho\tau \rightarrow \text{III and } \vec{t}_1 = \vec{b} \rightarrow \text{I}$$

Now $\vec{t}_1 = \vec{b}$

Differentiating w.r.t "s"

$$\vec{t}_1' = \vec{b}'$$

$$\frac{d\vec{t}_1}{ds} = -\tau\vec{n}$$

$$\frac{d\vec{t}_1}{ds_1}\frac{ds_1}{ds} = -\tau\vec{n}$$

$$k_1\vec{n}_1\frac{ds_1}{ds} = -\tau\vec{n} \qquad \text{From II}$$

$$k_1\vec{n}_1(\rho\tau) = -\tau\vec{n}$$

$$\tau(k_1\vec{n}_1) = -\tau\frac{1}{\rho}\vec{n}$$

$\vec{k}_1 \vec{n}_1 = -k\vec{n}$ $\qquad\qquad \because \frac{1}{\rho} = k$

$\Rightarrow$ $\vec{k}_1 = k$ and $\vec{n}_1 = -\vec{n} \to \text{III}$

Now

$$\vec{b}_1 = \vec{t}_1 \times \vec{n}_1$$

put $\vec{t}_1 = \vec{b}$, $\vec{n}_1 = -\vec{n}$

$$\vec{b}_1 = -(\vec{b} \times \vec{n})$$
$$= -(-\vec{t})$$
$$\vec{b}_1 = \vec{t}$$

Differentiating both sides w.r.t "s"

$$\frac{d\vec{b}_1}{ds} = \vec{t}'$$

$\because \vec{r}'' = \vec{t}' = k\vec{n}$

$$\frac{d\vec{b}_1}{ds_1}\frac{ds_1}{ds} = k\vec{n}$$

$$\vec{b}_1' \frac{ds_1}{ds} = k\vec{n}$$

$$-\tau_1 \vec{n}_1 \frac{ds_1}{ds} = k\vec{n}$$

put $\vec{n}_1 = -\vec{n}$, $\frac{ds_1}{ds} = \rho\tau$

$$-\tau_1(-\vec{n})\,\rho\tau = k\vec{n}$$

$$\tau\tau_1\vec{n} = \frac{1}{\rho}k\vec{n}$$

$$\tau\tau_1 = k^2$$

$$\tau_1 = \frac{k^2}{\tau}$$

Question:

Prove that the current point satisfies the differential equation

$$\frac{d}{ds}\left(\sigma\frac{d}{ds}\left(\rho\frac{d^2t}{ds^2}\right)\right) + \frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{dr}{ds}\right) + \frac{\rho}{\sigma}\frac{d^2r}{ds^2} = 0$$

Sol:-

$$\frac{d}{ds}\left(\sigma\frac{d}{ds}\left(\rho\frac{d^2r}{ds^2}\right)\right)+\frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{dr}{ds}\right)+\frac{\rho}{\sigma}\frac{d^2r}{ds^2} \rightarrow (1)$$

$$\frac{d\vec{r}}{ds} = \vec{r}' = \vec{t}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{r}'' = \vec{t}' = k\vec{n}$$

$$\sigma = \frac{1}{\tau} \text{ and } \rho = \frac{1}{k}$$

Putting all these values in (1)

$$\frac{d}{ds}\left(\sigma\frac{d}{ds}\left(\frac{1}{k}k\vec{n}\right)\right)+\frac{d}{ds}\left(\frac{\frac{1}{\tau}}{\frac{1}{k}}\vec{t}\right)+\frac{\frac{1}{k}}{\frac{1}{\tau}}k\vec{n}$$

$$=\frac{d}{ds}\left(\frac{1}{\tau}\frac{d}{ds}(\vec{n})\right)+\frac{d}{ds}\left(\frac{k}{\tau}\vec{t}\right)+\frac{\tau}{k}k\vec{n}$$

$$=\frac{d}{ds}\left(\frac{1}{\tau}\vec{n}'\right)+\frac{d\vec{t}}{ds}\left(\frac{k}{\tau}\right)+\tau\vec{n}$$

$$=\frac{d}{ds}\left(\frac{1}{\tau}(\tau\vec{b}-k\vec{t})\right)+\vec{t}'\frac{k}{\tau}+\tau\vec{n}$$

$$=\frac{d}{ds}\left(\vec{b}-\frac{k}{\tau}\vec{t}\right)+k\vec{n}\cdot\frac{k}{\tau}+\tau\vec{n}$$

$$=\vec{b}'-\frac{k}{\tau}\vec{t}'+k^2\frac{\vec{n}}{\tau}+\tau\vec{n}$$

$$=-\tau\vec{n}-\frac{k}{\tau}(k\vec{n})+\frac{k^2\vec{n}}{\tau}+\tau\vec{n}$$

$$=\frac{-k^2\vec{n}}{\tau}+\frac{k^2\vec{n}}{\tau}$$

$= 0$ R.H.S

$\Rightarrow$ L.H.S = R.H.S

$$\Rightarrow \frac{d}{ds}\left(\sigma\frac{d}{ds}\left(\rho\frac{d^2r}{ds^2}\right)\right)+\frac{d}{ds}\left(\frac{\sigma}{\rho}\frac{dr}{ds}\right)+\frac{\rho}{\sigma}\left(\frac{d^2r}{ds^2}\right)=0$$

Question:-

If the plane (oscolating plane) of curvature at every point on a curve passess through a fixed point then the curve is a plane (plane curve).

Sol:-

The equation of oscolating plane at any point $P(\vec{r})$ on a curve is given by

$$[\vec{R}-\vec{r} \,,\; \vec{r}' \;\; \vec{r}''] = 0$$

$$(\vec{R}-\vec{r})\cdot \vec{r}' \times \vec{r}'' = 0$$

$$(\vec{R}-\vec{r})\cdot \vec{t} \times (k\vec{n}) = 0 \qquad \because \vec{r}' = \vec{t} \,,\; \vec{r}'' = \vec{t}' = kn$$

$$k\,(\vec{R}-\vec{r})\cdot(\vec{t}\times\vec{n}) = 0$$

$$(\vec{R}-\vec{r})\cdot \vec{b} = 0$$

$\Rightarrow$ $(\vec{R}-\vec{r})$ and $\vec{b}$ are $\perp$ to each other.

Since, the plane of curvature at every point on the curve passes through a fixed point.

Let $\vec{R}_o$ be the position vector of the fixed point

$$\Rightarrow (\vec{R}_o - \vec{r})\cdot\vec{b} = 0$$

Differentiating w.r.t "s"

$$\frac{d}{ds}(\vec{R}_o - \vec{r})\cdot\vec{b} = 0$$

$$(0-\vec{r}')\cdot\vec{b} + (\vec{R}_o - \vec{r})\cdot\vec{b}' = 0 \qquad \because \vec{r}' = \vec{t}$$

$$-(\vec{t}\cdot\vec{b}) + (\vec{R}_o - \vec{r})\cdot\vec{b}' = 0$$

$$0 + (\vec{R}_o - \vec{r})\cdot(-\tau\vec{n}) = 0$$

$$\Rightarrow -\tau(\vec{R}_o - \vec{r})\cdot\vec{n} = 0$$

$$\Rightarrow \tau(\vec{R}_o - \vec{r})\cdot\vec{n} = 0$$

$$\Rightarrow \tau = 0 \quad \text{or} \quad (\vec{R}_o - \vec{r})\cdot\vec{n} = 0$$

If $\tau = 0$, then the curve is a plane-curve.

If $(\vec{R}_o - \vec{r})\cdot\vec{n} = 0$

$\Rightarrow$ $(\vec{R}_o - \vec{r})$ is perpandicular to $\vec{n}$

Also $(\vec{R}_0 - \vec{r}) \cdot \vec{b} = 0$

$\Rightarrow$ $(\vec{R}_0 - \vec{r})$ is perpandicular to $\vec{b}$

$\Rightarrow$ $(\vec{R}_0 - \vec{r})$ is in the direction of tangent at point "p". So, $(\vec{R}_0 - \vec{r}) = \lambda \vec{t}$ where λ is any real number

$\Rightarrow \quad (\vec{R}_0 - \vec{r}) = \lambda \vec{t}$

$\Rightarrow \quad \vec{R}_0 = \vec{r} + \lambda \vec{t}$

is equation of tangent at point $P(\vec{r})$ and passes through fixed point R_0

This shows that the tangent at every Point on the curve passess through a fixed point

Hence, the curve is straight line and hence is a plane-curve.

Question:-

Prove that for any curve

(i) $\vec{r}' \cdot \vec{r}'' = 0$

(ii) $\vec{r}' \cdot \vec{r}''' = -k^2$

(iii) $\vec{r}'' \cdot \vec{r}''' = kk'$

(iv) $\vec{r}' \cdot \vec{r}^{iv} = -3kk'$

(v) $\vec{r}'' \cdot \vec{r}^{iv} = k(k'' - k^3 - k\tau^2)$

(vi) $\vec{r}''' \cdot \vec{r}^{iv} = k'k'' + 2k^3k' + k^2\tau\tau' + kk'\tau^2$

Sol:-

(i) $\vec{r}' \cdot \vec{r}'' = 0$

put $\vec{r}' = \vec{t}$ and $\vec{r}'' = k\vec{n}$

$$\vec{r}' \cdot \vec{r}'' = \vec{t} \cdot (k\vec{n})$$
$$= k(\vec{t} \cdot \vec{n})$$
$$= k(0)$$
$$\vec{r}' \cdot \vec{r}'' = 0$$

(ii)

$\vec{r}' \cdot \vec{r}''' = -k^2$

put $\vec{\gamma}' = \vec{t}$ and $\vec{\gamma}'' = k\vec{n}$

$$\vec{\gamma}''' = k'\vec{n} + k\vec{n}'$$
$$= k'\vec{n} + k(\tau\vec{b} - k\vec{t})$$
$$\vec{\gamma}''' = k'\vec{n} + k\tau\vec{b} - k^2\vec{t}$$
$$\vec{\gamma}' \cdot \vec{\gamma}''' = \vec{t} \cdot (k'\vec{n} + k\tau\vec{b} - k^2\vec{t})$$
$$= k'(\vec{t}\cdot\vec{n}) + k\tau(\vec{t}\cdot\vec{b}) - k^2(\vec{t}\cdot\vec{t})$$
$$= k'(0) + k\tau(0) - k^2(1)$$
$$\vec{\gamma}' \cdot \vec{\gamma}''' = -k^2$$

(iii)

$$\vec{\gamma}'' \cdot \vec{\gamma}''' = kk'$$

we know $\vec{\gamma}'' = k\vec{n}$ and $\vec{\gamma}''' = k'\vec{n} + k\tau\vec{b} - k^2\vec{t}$

$$\vec{\gamma}'' \cdot \vec{\gamma}''' = k\vec{n} \cdot (k'\vec{n} + k\tau\vec{b} - k^2\vec{t})$$
$$= kk'(\vec{n}\cdot\vec{n}) + k^2\tau(\vec{n}\cdot\vec{b}) - k^3(\vec{n}\cdot\vec{t})$$
$$= kk'(1) + k^2\tau(0) + k^3(0)$$
$$\vec{\gamma}'' \cdot \vec{\gamma}''' = kk'$$

(iv)

$$\vec{\gamma}' \cdot \vec{\gamma}^{iv} = -3kk'$$

we know $\vec{\gamma}' = \vec{t}$; $\vec{b}' = -\tau\vec{n}$

$$\vec{\gamma}''' = k'\vec{n} \pm k\tau\vec{b} - k^2\vec{t}$$
$$\vec{\gamma}^{iv} = k''\vec{n} + k'\vec{n}' + k'\tau\vec{b} + k\tau'\vec{b} + k\tau\vec{b}' - k^2\vec{t}' - 2kk'\vec{t}$$
$$\vec{\gamma}^{iv} = k''\vec{n} + k'(\tau\vec{b} - k\vec{t}) + k'\tau\vec{b} + k\tau'\vec{b} + k\tau\vec{b}' - k^2\vec{t}' - 2kk'\vec{t}$$
$$= k''\vec{n} + k'\tau\vec{b} - kk'\vec{t} + k'\tau\vec{b} + k\tau'\vec{b} + k\tau\vec{b}' - k^2\vec{t}' - 2kk'\vec{t}$$
$$\vec{\gamma}^{iv} = k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} + k\tau(-\tau\vec{n}) - k^2\vec{t}' -$$
$$\vec{\gamma}' \cdot \vec{\gamma}^{iv} = \vec{t} \cdot (k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} - k\tau^2\vec{n} - k^2\vec{t}')$$
$$= -3kk'(\vec{t}\cdot\vec{t})$$
$$\vec{\gamma}' \cdot \vec{\gamma}^{iv} = -3kk'$$

(v)

$$\vec{\gamma}'' \cdot \vec{\gamma}^{iv} = k(k'' - k^3 - k\tau^2)$$

we know $\vec{\gamma}'' = k\vec{n}$

and $\vec{\gamma}^{iv} = k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} - k\tau^2\vec{n} - k^2\vec{t}'$

put $\vec{t}' = \vec{\gamma}'' = k\vec{n}$

$$\vec{\gamma}^{iv} = k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} - k\tau^2\vec{n} - k^2(k\vec{n})$$

$$\vec{r}^{iv} = k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} - k\tau^2\vec{n} - k^3\vec{n}$$

$$\vec{r}''\cdot\vec{r}^{iv} = k\vec{n}\cdot(k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} - k\tau^2\vec{n} - k^3\vec{n})$$
$$= kk''(\vec{n}\cdot\vec{n}) + 0 + 0 + 0 - k^2\tau^2(\vec{n}\cdot\vec{n}) - k^4(\vec{n}\cdot\vec{n})$$
$$= kk''(1) - k^2\tau^2(1) - k^4(1)$$
$$\vec{r}''\cdot\vec{r}^{iv} = kk'' - k^2\tau^2 - k^4$$
$$\vec{r}''\cdot\vec{r}^{iv} = k(k'' - k^3 - k\tau^2)$$

(vi)

$$\vec{r}'''\cdot\vec{r}^{iv} = k'k'' + 2k^3k' + k^2\tau\tau' + kk'\tau^2$$

we know

$$\vec{r}''' = k'\vec{n} + k\tau\vec{b} - k^2\vec{t}$$
$$\vec{r}^{iv} = k''\vec{n} + 2k'\tau\vec{b} - 3kk'\vec{t} + k\tau'\vec{b} - k\tau^2\vec{n} - k^3\vec{n}$$

$$\vec{r}'''\cdot\vec{r}^{iv} = (k'\vec{n} + k\tau\vec{b} - k^2\vec{t})\cdot((k'' - k\tau^2 - k^3)\vec{n} + (2k'\tau + k\tau')\vec{b} - 3kk'\vec{t})$$
$$\vec{r}'''\cdot\vec{r}^{iv} = k'(k'' - k\tau^2 - k^3)(\vec{n}\cdot\vec{n}) + k\tau(2k'\tau + k\tau')(\vec{b}\cdot\vec{b}) + 3k^3k'(\vec{t}\cdot\vec{t})$$

$$\vec{r}'''\cdot\vec{r}^{iv} = k'k'' - kk'\tau^2 - k^3k' + 2kk'\tau^2 + k^2\tau\tau' + 3k^3k'$$
$$\vec{r}'''\cdot\vec{r}^{iv} = k'k'' + 2k^3k' + k^2\tau\tau' + kk'\tau^2$$

Question:-

If nth derivative of $\vec{r}$ w.r.t s is given by $\vec{r}^n = a_n\vec{t} + b_n\vec{n} + c_n\vec{b}$ prove the reduction formula

$$a_{n+1} = a_n' - kb_n$$
$$b_{n+1} = b_n' + ka_n - \tau c_n$$
$$c_{n+1} = c_n' + \tau b_n$$

Sol:-

$$\vec{r}^n = a_n\vec{t} + b_n\vec{n} + c_n\vec{b}$$

Differentiate it w.r.t s.

$$\vec{r}^{n+1} = a_n'\vec{t} + a_n\vec{t}' + b_n'\vec{n} + b_n\vec{n}' + c_n'\vec{b} + c_n\vec{b}'$$

put $\vec{t}' = r'' = k\vec{n}$

$$\vec{n}' = \tau\vec{b} - k\vec{t} \quad \text{and} \quad \vec{b}' = -\tau\vec{n}$$

$$\vec{r}^{\,n+1} = a_n'\vec{t} + a_n(k\vec{n}) + b_n'\vec{n} + b_n(\tau\vec{b} - k\vec{t}) + c_n'\vec{b} + c_n(-\tau\vec{n})$$

$$\vec{r}^{\,n+1} = a_n'\vec{t} + ka_n\vec{n} + b_n'\vec{n} + b_n\tau\vec{b} - kb_n\vec{t} + c_n'\vec{b} - \tau c_n\vec{n}$$

$$\vec{r}^{\,n+1} = (a_n' - kb_n)\vec{t} + (ka_n + b_n' - \tau c_n)\vec{n} + (c_n' + \tau b_n)\vec{b} \to (1)$$

Given $\vec{r}^{\,n+1} = a_{n+1}\vec{t} + b_{n+1}\vec{n} + c_{n+1}\vec{b} \to (2)$

Comparing (1) and (2)

$$a_{n+1} = a_n' - kb_n$$
$$b_{n+1} = b_n' + ka_n - \tau c_n$$
$$c_{n+1} = c_n' + \tau b_n$$

Question:-

(i) $[\vec{t}' \;\; \vec{t}'' \;\; \vec{t}'''] = [\vec{r}'' \;\; \vec{r}''' \;\; \vec{r}'''']$

$$= k^3(k\tau' - k'\tau)$$
$$= k^5 \frac{d}{ds}\left(\tau/k\right)$$

Sol:-

$[\vec{t}' \;\; \vec{t}'' \;\; \vec{t}''']$

we know $\vec{t}' = \vec{r}'' = k\vec{n}$

$$\vec{t}' = k\vec{n}$$

$$\vec{t}'' = k'\vec{n} + k\vec{n}' \qquad \vec{n}' = \tau\vec{b} - k\vec{t}$$

$$= k'\vec{n} + k(\tau\vec{b} - k\vec{t}) \qquad \vec{b}' = -\tau\vec{n}$$

$$\vec{t}'' = k'\vec{n} + k\tau\vec{b} - k^2\vec{t}$$

$$\vec{t}''' = k''\vec{n} + k'\vec{n}' + k'\tau\vec{b} + k\tau'\vec{b} + k\tau\vec{b}' - k^2\vec{t}' - 2kk'\vec{t}$$

$$\vec{t}''' = k''\vec{n} + k'(\tau\vec{b} - k\vec{t}) + k'\tau\vec{b} + k\tau'\vec{b} + k\tau(-\tau\vec{n}) - k^2(\vec{r}'') - 2kk'\vec{t}$$

$$\vec{t}''' = k''\vec{n} + k'\tau\vec{b} - kk'\vec{t} + k'\tau\vec{b} + k\tau'\vec{b} - k\tau^2\vec{n} - k^2(k\vec{n}) - 2kk'\vec{t}$$

$$\vec{t}''' = k''\vec{n} - k\tau^2\vec{n} - k^3\vec{n} + 2k'\tau\vec{b} + k\tau'\vec{b} - 3kk'\vec{t}$$

$$\vec{t}''' = (k'' - k^3 - k\tau^2)\vec{n} + (2k'\tau + k\tau')\vec{b} - 3kk'\vec{t}$$

$$[\vec{t}' \ \vec{t}'' \ \vec{t}'''] = \begin{array}{c} \begin{array}{ccc} \vec{t} & \vec{n} & \vec{b} \end{array} \\ \begin{vmatrix} 0 & k & 0 \\ -k^2 & k' & k\tau \\ -3kk' & k''-k^3-k\tau^2 & 2k'\tau+k\tau' \end{vmatrix} \end{array}$$

$$= 0 \cancel{+} k\left(-k^2(2k'\tau+k\tau')+k\tau(3kk')\right)+0$$

$$= -k(-2k^2k'\tau - k^3\tau' + 3k^2k'\tau)$$

$$= -k(-k^3\tau' + k^2k'\tau)$$

$$= k \cdot k^2(k\tau' - k'\tau)$$

$$[\vec{t}' \ \vec{t}'' \ \vec{t}'''] = k^3(k\tau' - k'\tau)$$

$$= \frac{k^2 \cdot k^3(k\tau' - k'\tau)}{k^2}$$

$$= k^5\left(\frac{k\tau' - k'\tau}{k^2}\right)$$

$$[\vec{t}' \ \vec{t}'' \ \vec{t}'''] = k^5 \frac{d}{ds}\left(\frac{\tau}{k}\right)$$

$$[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = k^5 \frac{d}{ds}(\tau/k)$$

(ii)

$$[\vec{b}' \ \vec{b}'' \ \vec{b}'''] = \tau^3(k'\tau - k\tau')$$

$$= \tau^5 \frac{d}{ds}(k/\tau)$$

Sol:-

$[\vec{b}' \ \vec{b}'' \ \vec{b}''']$

we know $\vec{b}' = -\tau\vec{n}$

$\vec{b}'' = -\tau\vec{n}' - \tau'\vec{n}$

put $\vec{n}' = (\tau\vec{b} - k\vec{t})$

$$\vec{b}'' = -\tau(\tau\vec{b} - k\vec{t}) - \tau'\vec{n}$$

$$\vec{b}'' = -\tau^2\vec{b} + k\tau\vec{t} - \tau'\vec{n}$$

$$\vec{b}''' = -\tau^2\vec{b}' - 2\tau\tau'\vec{b} + k'\tau\vec{t} + k\tau'\vec{t} + k\tau\vec{t}' - \tau'\vec{n}' - \tau''\vec{n}$$

Put $\vec{b}' = -\tau\vec{n}$ and $\vec{n}' = (\tau\vec{b} - k\vec{t})$ and $\vec{t}' = k\vec{n}$

$$\vec{b}''' = -\tau^2(-\tau\vec{n}) + 2\tau\tau'\vec{b} + k'\tau\vec{t} + k\tau'\vec{t} + k\tau(k\vec{n}) - \tau''\vec{n} - \tau'(\tau\vec{b} - k\vec{t})$$

$$\vec{b}''' = +\tau^3\vec{n} + 2\tau\tau'\vec{b} + k'\tau\vec{t} + k\tau'\vec{t} + k^2\tau\vec{n} - \tau''\vec{n} - \tau\tau'\vec{b} + k\tau'\vec{t}$$

$$\vec{b}''' = (\tau^3 + k^2\tau - \tau'')\vec{n} + (-2\tau\tau' - \tau\tau')\vec{b} + (k'\tau + k\tau' + k\tau')\vec{t}$$

$$\vec{b}''' = (\tau^3 + k^2\tau - \tau'')\vec{n} + 3\tau\tau'\vec{b} + (k'\tau + 2k\tau')\vec{t}$$

$$[\vec{b}'\ \vec{b}''\ \vec{b}'''] = \begin{vmatrix} 0 & -\tau & 0 \\ k\tau & -\tau' & -\tau^2 \\ k'\tau + 2k\tau' & \tau^3 + k^2\tau - \tau'' & -3\tau\tau' \end{vmatrix}$$

$$= -(-\tau)(-3k'\tau^2\tau' + \tau^2(k'\tau + 2k\tau'))$$

$$= 0 + \tau(2k\tau\tau' + k'\tau^3 + 3k\tau^2\tau') + 0$$

$$= 2k\tau^3\tau' + k'\tau^4 + 3k\tau^3\tau'$$

$$= k'\tau^4 - k\tau^3\tau'$$

$$[\vec{b}'\ \vec{b}''\ \vec{b}'''] = \tau^3(k'\tau - k\tau')$$

$$= \frac{\tau^2 \cdot \tau^3 (k'\tau - k\tau')}{\tau^2}$$

$$= \tau^5 \frac{(k'\tau - k\tau')}{\tau^2}$$

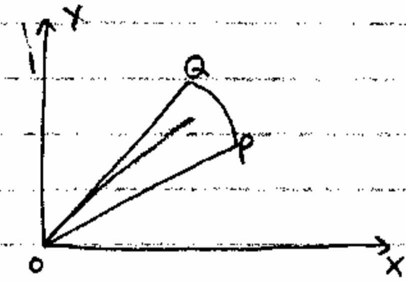
$$[\vec{b}'\ \vec{b}''\ \vec{b}'''] = \tau^5 \frac{d}{ds}\left(\frac{k}{\tau}\right)$$

Oscolating Sphere or Sphere of curvature or Sphere of closest contact with the curve:-

The sphere of closest contact at a point "p" on a curve is the sphere which passess through four points on the curve altimately co-incident with "p". It is also known as the oscolating sphere or sphere of curvature at point "p" on the curve its centre ξ (zetta) and radius "R" are called centre of spherical curvature and radius of spherical curvature.

The locus of centre of spherical curvature:-

The centre of spherical curvature of an oscolating sphere at a point "p" and an adjacent point Q on a curve lies on the plane which is perpandicular bisector of $\overline{PQ}$ and in the limiting position $(Q \to P)$ the plane is the normal plane at point "p".



Hence, in the limiting position the centre of spherical curvature is the intersection of three (3) normal planes at point "p".

Hence, the equation of the normal plane at a point 'p' with position vector $(\vec{r})$ on the curve is

$$(\vec{\xi} - \vec{r}) \cdot \vec{t} = 0 \rightarrow (1)$$

Differentiating both sides w.r.t 's'

$$0 - \vec{r}' \cdot \vec{t} + (\vec{\xi} - \vec{r}) \cdot \vec{t}' = 0$$

$$-\vec{t} \cdot \vec{t} + (\vec{\xi} - \vec{r}) \cdot k\vec{n} = 0$$

$$-1 + (\vec{\xi} - \vec{r}) \cdot k\vec{n} = 0$$

$$\Rightarrow (\vec{\xi} - \vec{r}) \cdot k\vec{n} = 1$$

$$(\vec{\xi} - \vec{r}) \cdot \vec{n} = \frac{1}{k} = \rho \rightarrow (2)$$

Differentiating again w.r.t "s"

$$0 - \vec{r}' \cdot \vec{n} + (\vec{\xi} - \vec{r}) \cdot \vec{n}' = \rho'$$

$$-\vec{t} \cdot \vec{n} + (\vec{\xi} - \vec{r}) \cdot (\tau\vec{b} - k\vec{t}) = \rho'$$

$$0 + (\vec{\xi} - \vec{r}) \cdot (\tau\vec{b} - k\vec{t}) = \rho'$$

$\Rightarrow \quad \tau(\vec{\xi}-\vec{r})\cdot\vec{b} - k(\vec{\xi}-\vec{r})\cdot\vec{t} = \rho'$

$(\vec{\xi}-\vec{r})\cdot\vec{t} = 0$ by (1)

$\Rightarrow \quad \tau(\vec{\xi}-\vec{r})\cdot\vec{b} - 0 = \rho'$

$\Rightarrow \quad \tau(\vec{\xi}-\vec{r})\cdot\vec{b} = \rho'$

$\Rightarrow \quad (\vec{\xi}-\vec{r})\cdot\vec{b} = \rho'\cdot\frac{1}{\tau}$

$\because \frac{1}{\tau} = \sigma$

$\Rightarrow \quad (\vec{\xi}-\vec{r})\cdot\vec{b} = \sigma\rho' \rightarrow (3)$

From eq (1), (2) and (3) we have

$$(\vec{\xi}-\vec{r}) = 0\cdot\vec{t} + \rho\cdot\vec{n} + \sigma\rho'\cdot\vec{b}$$

$$\vec{\xi}-\vec{r} = \rho\vec{n} + \sigma\rho'\vec{b}$$

$$\vec{\xi} = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$$

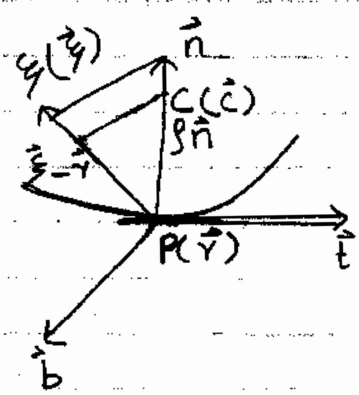
which is the equation of locus of centre of sphercal curvature at point "P" with position vector $\vec{r}$.

Now, the radius R of spherical curvature is $R = |\vec{\xi}-\vec{r}|$

$$= |\rho\vec{n} + \sigma\rho'\vec{b}|$$

$$R = \sqrt{\rho^2 + \sigma^2(\rho')^2}$$

Remark:-



$$C\xi = P\xi - PC$$

$$= (\vec{\xi}-\vec{r}) - \rho\vec{n}$$

Put $\vec{\xi}-\vec{r} = \rho\vec{n} + \sigma\rho'\vec{b}$

$$C\xi = \rho\vec{n} + \sigma\rho'\vec{b} - \rho\vec{n}$$

$$C\xi = \sigma\rho'\vec{b}$$

$$|C\xi| = \sigma\rho' \qquad \because |\vec{b}| = 1$$

Hence, the distance between the centre

of circular curvature and the centre of spherical curvature is "$\sigma\rho'$" and if the curve is of constant curvature then $\rho' = 0$.

Hence, in this case the centre of circular curvature and centre of spherical curvature are concide with each other

Question:-

If k and τ denotes the curvature and torsion of a curve $\vec{r} = \vec{r}(s)$ and k_1 and τ_1 be the curvature and torsion of locus of centre of spherical curvature. Then, prove that

$$k k_1 = \tau \tau_1$$

Sol:-

The equation of locus of centre of spherical curvature is given by

$$\vec{\xi} = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$$

Differentiating w.r.t "s"

$$\frac{d\vec{\xi}}{ds} = \vec{r}' + \rho'\vec{n} + \rho\vec{n}' + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'\vec{b}'$$

put $\vec{r}' = \vec{t}$, $\vec{n}' = (\tau\vec{b} - k\vec{t})$, $\vec{b}' = -\tau\vec{n}$

$$\frac{d\vec{\xi}}{ds} = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - k\vec{t}) + \rho'\sigma'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'(-\tau\vec{n})$$

$$= \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho k\vec{t} + \rho'\sigma'\vec{b} + \sigma\rho''\vec{b} - \sigma\tau\rho'\vec{n} \quad \because \tau = \frac{1}{\sigma}$$

$$= \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho k\vec{t} + \rho'\sigma'\vec{b} + \sigma\rho''\vec{b} - \rho'\vec{n}$$

$$= \vec{t} - \rho k \vec{t} + \rho \tau \vec{b} + \rho' \sigma' \vec{b} + \sigma \rho'' \vec{b}$$

$$= \vec{t} - \vec{t} + (\rho \tau + \rho' \sigma' + \sigma \rho'') \vec{b} \qquad k = \frac{1}{\rho}$$

$$\frac{d\vec{\xi}}{ds} = (\rho \tau + \rho' \sigma' + \rho'' \sigma) \vec{b}$$

$$\frac{d\vec{\xi}}{ds_1} \frac{ds_1}{ds} = (\rho \tau + \rho' \sigma' + \rho'' \sigma) \vec{b}$$

where S_1 is the arc of locus of centre of spherical curvature.

$$\Rightarrow \vec{t}_1 \frac{ds_1}{ds} = (\rho \tau + \rho' \sigma' + \rho'' \sigma) \vec{b}$$

where $\vec{t}_1 = \frac{d\vec{\xi}}{ds_1}$

$$\Rightarrow \vec{t}_1 = \vec{b} \rightarrow (1) \text{ and } \frac{ds_1}{ds} = \rho \tau + \rho' \sigma' + \rho'' \sigma \rightarrow (2)$$

Differentiating eq (1) w.r.t "S_1"

$$\frac{d\vec{t}_1}{ds_1} = b'$$

$$\Rightarrow \frac{d\vec{t}_1}{ds_1} \frac{ds_1}{ds} = -\tau \vec{n}$$

$$\vec{t}_1' \frac{ds_1}{ds} = -\tau \vec{n}$$

$$k_1 \vec{n}_1 \frac{ds_1}{ds} = -\tau \vec{n}$$

$\Rightarrow$

$$\Rightarrow k_1 = \frac{\tau}{ds_1/ds} \rightarrow (3), \quad \vec{n}_1 = -\vec{n} \rightarrow (4)$$

$$\vec{b}_1 = \vec{t}_1 \times \vec{n}_1$$

put $\vec{t}_1 = \vec{b}$, $\vec{n}_1 = -\vec{n}$ from (1), (4)

$$\vec{b}_1 = \vec{b} \times -\vec{n} = -(\vec{b} \times \vec{n}) = \vec{t}$$

$$\vec{b}_1 = \vec{t}$$

Differentiate it w.r.t "s"

$$\frac{db_1}{ds} = \vec{t}' \qquad\qquad \vec{t} = \gamma'$$

$$\frac{db_1}{ds_1}\frac{ds_1}{ds} = \vec{\gamma}''$$

$$\frac{db_1}{ds_1}\frac{ds_1}{ds} = k\vec{n}$$

$$b_1' \frac{ds_1}{ds} = k\vec{n}$$

$$-\tau_1 \vec{n}_1 \frac{ds_1}{ds} = k\vec{n}$$

$$\Rightarrow \quad k = \tau_1 \frac{ds_1}{ds} \rightarrow (5), \quad \vec{n} = -\vec{n}_1 \rightarrow (6)$$

Multiply (4) and (5)

$$k_1 k = \frac{\tau}{\frac{ds_1}{ds}} \tau_1 \frac{ds_1}{ds}$$

$$kk_1 = \tau\tau_1 \frac{ds}{ds_1}\frac{ds_1}{ds}$$

$$\Rightarrow \quad kk_1 = \tau\tau_1$$

Question:-

Find the curvature and torsion of the locus of centre of spherical curvature

Sol:-

The equation of locus of centre of s. curvature is given by

$$\vec{\xi} = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$$

Differentiate it w.r.t "s"

$$\frac{d\xi}{ds} = \vec{r}' + \rho\vec{n}' + \rho'\vec{n} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'\vec{b}'$$

put $\vec{r}' = \vec{t}$, $\vec{n}' = \tau\vec{b} - k\vec{t}$, $\vec{b}' = -\tau\vec{n}$

$$\frac{d\vec{\xi}}{ds} = \vec{t} + \rho(\tau\vec{b} - k\vec{t}) + \rho'\vec{n} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'(-\tau\vec{n})$$

$$= \vec{t} + \rho\tau\vec{b} - \vec{t} + \rho'\vec{n} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} - \rho'\vec{n}$$

$$= \rho\tau\vec{b} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b}$$

$$\frac{d\vec{\xi}}{ds} = (\rho\tau + \sigma'\rho' + \sigma\rho'')\vec{b}$$

$$\frac{d\vec{\xi}}{ds_1}\frac{ds_1}{ds} = \left(\rho\tau + \frac{d}{ds}(\sigma\rho')\right)\vec{b}$$

$$\Rightarrow \vec{t}_1 \cdot \frac{ds_1}{ds} = \left(\rho\tau + \frac{d}{ds}(\sigma\rho')\right)\vec{b} \quad \text{where } \vec{t}_1 = \frac{d\vec{\xi}}{ds}$$

Comparing both sides

$$\Rightarrow \vec{t}_1 = \vec{b} \rightarrow (1) \quad \frac{ds_1}{ds} = \rho\tau + \frac{d}{ds}(\sigma\rho')$$

Differentiate (1) w.r.t "s"

$$\frac{d\vec{t}_1}{ds} = \vec{b}' \qquad \text{put } \vec{b}' = -\tau\vec{n}$$

$$\frac{d\vec{t}_1}{ds_1}\frac{ds_1}{ds} = -\tau\vec{n}$$

$$\vec{t}_1'\frac{ds_1}{ds} = -\tau\vec{n} \qquad \text{put } \vec{t}_1' = k_1\vec{n}_1$$

$$k_1\vec{n}_1\frac{ds_1}{ds} = -\tau\vec{n}$$

$$\Rightarrow k_1\frac{ds_1}{ds} = \tau \quad \text{and} \quad \vec{n}_1 = -\vec{n}$$

$$\Rightarrow k_1 = \frac{\tau}{ds_1/ds} \qquad \text{put } \frac{ds_1}{ds} = \rho\tau + \frac{d}{ds}(\sigma\rho')$$

$$\Rightarrow k_1 = \frac{\tau}{\rho\tau + \frac{d}{ds}(\sigma\rho')}$$

is curvature of centre of spherical curvature.

we know $\vec{b}_1 = \vec{t}_1 \times \vec{n}_1$ from (2) $\vec{t}_1 = \vec{b}, \vec{n}_1 = -\vec{n}$

$\vec{b}_1 = \vec{b} \times -\vec{n}_1 = -\vec{b} \times \vec{n} = -\vec{t}$ Diff w.r.t "s"

$$\frac{d\vec{b}_1}{ds} = \vec{t}' \quad \text{put } \vec{t}' = k\vec{n}$$

$$\frac{db_1}{ds_1}\frac{ds_1}{ds} = k\vec{n} \Rightarrow b_1' \frac{ds_1}{ds} = k\vec{n} \quad \text{put } \vec{b}_1' = -\tau_1 \vec{n}_1$$

$$\Rightarrow -\tau_1 \vec{n}_1 \frac{ds_1}{ds} = k\vec{n} \quad \text{comparing}$$

$$\Rightarrow \tau_1 \frac{ds_1}{ds} = k \quad , \quad \vec{n} = -\vec{n}_1$$

put $\frac{ds_1}{ds} = \rho\tau + \frac{d}{ds}(\sigma\rho')$

$$\Rightarrow \tau_1 = \frac{k}{\frac{ds_1}{ds}} = \frac{k}{\rho\tau + \frac{d}{ds}(\sigma\rho')}$$

is tortion of spherical curvature

Question:-

Prove that for a curve of constt curvature. The curvature and torsion of the locus of centre of spherical curvature are given by

$$k_1 = k \quad , \quad \tau_1 = k^2/\tau$$

Sol:-

Since, the curve is of constant curvature. So

$$\rho = \text{Constant}$$

$$\Rightarrow \rho' = 0$$

Then, Now, The equation of the locus of centre of curvature is

$$\vec{\xi} = \vec{r} + \rho\vec{n} + \rho'\sigma\vec{b}$$

put $\rho' = 0$

$$\vec{\xi} = \vec{r} + \rho\vec{n}$$

Differentiate it w.r.t "s"

$$\frac{d\vec{\xi}}{ds} = \vec{r}' + \rho\vec{n}'$$

put $\vec{r}' = \vec{t}$ and $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\frac{d\vec{\xi}}{ds} = \vec{t} + \rho(\tau\vec{b} - k\vec{t})$$

$$\frac{d\vec{\xi}}{ds_1}\frac{ds_1}{ds} = \vec{t} + \rho\tau\vec{b} - \rho k\vec{t}$$

$$= \vec{t} + \rho\tau\vec{b} - \vec{t}$$

$$\vec{t}_1\frac{ds_1}{ds} = \rho\tau\vec{b}$$

Comparing

$$\vec{t}_1 = \vec{b} \rightarrow (1) \quad , \quad \frac{ds_1}{ds} = \rho\tau$$

Differentiate (1)

$$\frac{d\vec{t}_1}{ds} = \vec{b}' \qquad \vec{b}' = -\tau\vec{n}$$

$$\frac{d\vec{t}_1}{ds_1}\frac{ds_1}{ds} = -\tau\vec{n}$$

$$\vec{t}_1'\frac{ds_1}{ds} = -\tau\vec{n} \qquad \text{put } \vec{t}_1' = k_1\vec{n}_1$$

$$k_1\vec{n}_1\frac{ds_1}{ds} = -\tau\vec{n}$$

$$\Rightarrow k_1\frac{ds_1}{ds} = -\tau \quad , \quad \vec{n}_1 = -\vec{n}$$

$$\text{put } \frac{ds_1}{ds} = \rho\tau$$

$$\Rightarrow k_1\rho\tau = \tau$$

$$\Rightarrow k_1 = \frac{1}{\rho} = k$$

$\Rightarrow k_1 = k$ Now we find τ_1

we know $\vec{b}_1 = \vec{t}_1 \times \vec{n}_1 = \vec{b} \times -\vec{n} = -\vec{b}\times\vec{n} = \vec{t}$

$$\frac{d\vec{b}_1}{ds} = \vec{t}' \qquad \text{put } \vec{t}' = k\vec{n}$$

$$\frac{d\vec{b}_1}{ds_1}\frac{ds_1}{ds} = k\vec{n}$$

$$\vec{b}_1'\frac{ds_1}{ds} = k\vec{n} \qquad \text{put } \vec{b}_1' = -\tau_1\vec{n}_1 \;,\; \frac{ds_1}{ds} = \rho\tau$$

$$-\tau_1\vec{n}_1\rho\tau = k\vec{n}$$

$$\Rightarrow \tau_1\rho\tau = k \quad , \quad \vec{n}_1 = -\vec{n}$$

$$\Rightarrow \tau_1 = \frac{k}{\rho\tau}$$

$$\Rightarrow \tau_1 = \frac{k^2}{\tau}$$

Question:-

If the radius of spherical curvature is constant for a curve. Then, prove that either the curve is of constant curvature or else the curve lies on the surface of a sphere.

Sol:-

The radius R of spherical curvature of a curve is given by

$$R^2 = \rho^2 + \sigma^2 \rho'^2$$

Differentiating both sides w.r.t "s"

$$0 = 2\rho\rho' + 2\sigma\sigma'\rho'^2 + 2\sigma^2\rho'\rho''$$

$$0 = 2\rho'(\rho + \sigma\rho'\sigma' + \sigma^2\rho'')$$

$$\Rightarrow 2\rho'(\rho + \sigma\sigma'\rho' + \sigma^2\rho'') = 0$$

$$\Rightarrow 2\rho'(\rho + \sigma(\sigma'\rho' + \sigma\rho'')) = 0$$

$$2\rho'\left(\rho + \frac{1}{2}\left(\frac{d}{ds}(\sigma\rho')\right)\right) = 0$$

$$\Rightarrow \rho'\left(\rho + \sigma\frac{d}{ds}(\sigma\rho')\right) = 0$$

Either $\rho' = 0 \rightarrow (1)$ or $\rho + \sigma\dfrac{d}{ds}(\sigma\rho') = 0 \rightarrow (2)$

If $\rho' = 0$

$\Rightarrow \rho = \text{Constant}$

$= k = \text{Constant}$

Hence, the curve is of Constant curvature.

Now, we consider the 2nd possibility

$$\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0 \rightarrow (2)$$

Now the equation of locus of centre of curvature is given by

$$\xi = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$$

Differentiate it w.r.t "s"

$$\frac{d\xi}{ds} = \vec{r}' + \rho'\vec{n} + \rho\vec{n}' + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'\vec{b}'$$

put $\vec{r}' = \vec{t}$, $\vec{n}' = \tau\vec{b} - k\vec{t}$ and $\vec{b}' = -\tau\vec{n}$

$$\frac{d\xi}{ds} = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - k\vec{t}) + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'(-\tau\vec{n})$$

$\because \rho = \frac{1}{k}$

$$\frac{d\xi}{ds} = \cancel{\vec{t}} + \cancel{\rho'\vec{n}} + \rho\tau\vec{b} - \cancel{\vec{t}} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} - \cancel{\sigma\rho'\tau\vec{n}}$$

$\tau = \frac{1}{\sigma}$

$$\frac{d\xi}{ds} = \rho\tau\vec{b} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b}$$

$$= (\rho\tau + \sigma'\rho' + \sigma\rho'')\vec{b}$$

$$\frac{d\xi}{ds} = \rho\,(\rho\tau + \frac{d}{ds}(\sigma\rho'))\vec{b}$$

$$\frac{d\xi}{ds} = (\rho/\sigma + \frac{d}{ds}(\sigma\rho'))\vec{b}$$

$$= \frac{1}{\sigma}\{\rho + \sigma\frac{d}{ds}(\sigma\rho')\}\vec{b}$$

$$= \frac{1}{\sigma}(0) \quad \text{by (2)}$$

$$\Rightarrow \frac{d\xi}{ds} = 0$$

$\Rightarrow \xi =$ constant.

Hence, the curve lies on the surface of a sphere.

Remark:-

The curve drawn on the surface of sphere oscollating sphere at any point is the same sphere. S_0, centre $\xi = 0$

Question:-

Let R be the radius of spherical curvature. Then, prove that

$$R^2 = \rho^4\sigma^2\vec{r}'''^2 - \sigma^2$$

Sol:-

$$\rho^4\sigma^2\,\vec{r}'''^2 - \sigma^2 \rightarrow (1)$$

we know $\vec{r}'' = k\vec{n}$

Differentiating w.r.t "s"

$$\vec{r}''' = k\vec{n}' + k'\vec{n}$$

put $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\vec{r}''' = k(\tau\vec{b} - k\vec{t}) + k'\vec{n}$$

$$\vec{r}''' = k\tau\vec{b} - k^2\vec{t} + k'\vec{n}$$

Now

$$\vec{r}'''\cdot\vec{r}''' = \vec{r}'''^2 = k^2\tau^2(\vec{b}\cdot\vec{b}) + k^4(\vec{t}\cdot\vec{t}) + k'^2(\vec{n}\cdot\vec{n})$$

$$\vec{r}'''^2 = k^2\tau^2 + k^4 + k'^2$$

put this in (1)

$$\rho^4\sigma^2\vec{r}'''^2 - \sigma^2 = \rho^4\sigma^2(k^2\tau^2 + k^4 + k'^2) - \sigma^2$$

$$= \rho^4\sigma^2k^2\tau^2 + \rho^4\sigma^2k^4 + k'^2\rho^4\sigma^2 - \sigma^2$$

we know $\rho = \frac{1}{k}$, $\tau = \frac{1}{\sigma}$

$$= \frac{1}{k^4}\sigma^2k^2\frac{1}{\sigma^2} + \frac{1}{k^4}\sigma^2k^4 + k'^2\rho^4\sigma^2 - \sigma^2$$

$$= \frac{1}{k^2} + \sigma^2 + \frac{1}{\rho'^2}\rho^4\sigma^2 - \sigma^2 \rightarrow (2)$$

$= \rho^2 +$ we know $\rho = \frac{1}{k}$

$$\rho' = -\frac{1}{k^2}k' \Rightarrow \rho'^2 = -\frac{1}{k^4}k'^2 = -\rho^4k'^2$$

put in (2)

$$= \rho^2 + \frac{1}{-\rho^4k'^2}\rho^4\sigma^2 = \rho^2 + \rho'^2\sigma^2 = R^2$$

$$R^2 = \rho^4\sigma^2\vec{r}'''^2 - \sigma^2$$

Question:-

For the curve

$\vec{r} = (4a\cos^3 u,\ 4a\sin^3 u,\ 3c\cos 2u)$

find the radius of spherical curvature.

Sol:-

$$R^2 = \rho^2 + \sigma^2 \rho'^2$$

$$\vec{k} = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|}$$

and
$$\tau = \frac{[\dot{\vec{r}}\ \ \ddot{\vec{r}}\ \ \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$R = \frac{6(a^2+c^2)}{ac}\sqrt{4a^2\cos^2(2u)+3c^2\cos^2(2u)+c^2}$$

R

Question:-

Prove that for the curves drawn on the surface of a sphere this equation holds

$$\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0$$

Sol:-

For the curve drawn on the Sure of a sphere, the oscallating sphere at any point is the same sphere. So, $\psi = 0$

Now, the equation of the locus of centre of spherical curvature is

$$\psi = \vec{r} + \rho\vec{n} + \rho'\sigma\vec{b}$$

Differentiate it w.r.t 's'.

$$0 = \vec{r}' + \rho'\vec{n} + \rho\vec{n}' + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'\vec{b}'$$

put $\vec{r}' = \vec{t}$, $\vec{n}' = (\tau\vec{b} - k\vec{t})$, $\vec{b}' = -\tau\vec{n}$

$$0 = \vec{t} + \rho'\vec{n} + \rho(\tau\vec{b} - k\vec{t}) + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} + \sigma\rho'(-\tau\vec{n})$$

$$0 = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho k\vec{t} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} - \sigma\rho'\tau\vec{n}$$

$$0 = \vec{t} + \rho'\vec{n} + \rho\tau\vec{b} - \rho\frac{1}{\rho}\vec{t} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b} - \sigma\rho'\frac{1}{\sigma}\vec{n}$$

$$0 = \vec{t} + \rho\tau\vec{b} + \sigma\rho''\vec{b} + \sigma'\rho'\vec{b} - \vec{t} + \rho'\vec{n} + \rho'\vec{n}$$

$$0 = \rho\tau\vec{b} + \sigma\rho''\vec{b} + \sigma'\rho'\vec{b}$$

$$0 = (\rho\tau + \sigma\rho'' + \sigma'\rho')\vec{b}$$

$$0 = \left(\rho\tau + \frac{d}{ds}(\sigma\rho')\right)\vec{b}$$

Since, $\vec{b} \neq 0$ $\quad \because \vec{b}$ is binormal

$$\Rightarrow \quad \rho\tau + \frac{d}{ds}(\sigma\rho') = 0$$

$$\Rightarrow \quad \rho/\sigma + \frac{d}{ds}(\sigma\rho') = 0$$

Helix:-

A curve drawn on the surface of a cylinder which cuts the generators of a cylinder and a constant angle is called a helix.

If the cylinder is a right 

cylinder, then it is (the curve) known as circular helix.

The equation of circular helix is

$$I = (a\cos u, \ a\sin u, \ bu)$$

Question:-

Prove that for a circular helix the radius of spherical curvature is equal to the radius of circular curvature $(R = \rho)$

Sol:-

Let $\Gamma \ \vec{r} = \vec{r}(s)$ be a circular helix where $\vec{r} = (a\cos u, \ a\sin u, \ bu)$

Differentiating both sides w.r.t "s"

$$r' = (-a\sin u, \ a\cos u, \ b)\frac{du}{ds} \to (i) \quad \because \vec{r}' = \vec{t}$$

Now

$$\vec{r}'.\vec{r}' = \sqrt{(a^2\sin^2 u + a^2\cos^2 u + b^2)\left(\frac{du}{ds}\right)^2}$$

$$\vec{t}\cdot\vec{t} = 1 = (a^2(\sin^2 u + \cos^2 u) + b^2)\left(\frac{du}{ds}\right)^2$$

$$1 = (a^2+b^2)\left(\frac{du}{ds}\right)^2$$

$$\Rightarrow \left(\frac{du}{ds}\right)^2 = \frac{1}{a^2+b^2}$$

$$\Rightarrow \frac{du}{ds} = \frac{1}{\sqrt{a^2+b^2}} = \text{constant}$$

Now again diff eq(1) w.r.t "s"

$$\vec{r}'' = (-a\cos u, -a\sin u, 0)\left(\frac{du}{ds}\right)^2 + 0.$$

$$\vec{r}'' = k\vec{n} = (-a\cos u, -a\sin u, 0)\left(\frac{du}{ds}\right)^2$$

$$\Rightarrow |k\vec{n}| = \sqrt{(a^2\cos^2 u + a^2\sin^2 u + 0)\left(\frac{du}{ds}\right)^4}$$

$$k|\vec{n}| = \sqrt{a^2(\cos^2 u + \sin^2 u)\left(\frac{du}{ds}\right)^4}$$

$$\Rightarrow k.1 = \sqrt{a^2(1)\left(\frac{du}{ds}\right)^4}$$

$$\Rightarrow k = a\left(\frac{du}{ds}\right)^2 \qquad \text{put } \left(\frac{du}{ds}\right)^2 = \frac{1}{a^2+b^2}$$

$$\Rightarrow k = \frac{a}{a^2+b^2}$$

which is a constant. So, k is constant

Since, $\rho = \frac{1}{k}$ = constant

Now, the radius of spherical curvature R is given by

$$R^2 = \rho^2 + \sigma^2\rho'^2$$

$\Rightarrow R^2 = \rho^2$ Since ρ = constant So, $\rho' = 0$

$\Rightarrow R = \rho$

Hence, the radius of spherical curvature is equal to radius of circular curvature.

Question:-

The curve is a helix, if and only if the curvature and torsion of a curve are in a constant ratio.

Sol:-

Suppose that the curve $\vec{r} = \vec{r}(s)$ is a helix which cuts the generators of the cylinder at a constant angle α.

Let $\vec{a}$ be a unit vector parallel to the generator of the cylinder, then

$$\vec{a} \cdot \vec{t} = |\vec{a}||\vec{t}| \cos\alpha$$

$$\vec{a} \cdot \vec{t} = \cos\alpha \rightarrow (i) \quad |\vec{a}| = |\vec{t}| = 1 \text{ (unit vectors)}$$

where α

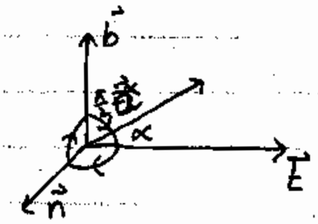
Differentiating (i) w.r.t "s"

$$\vec{a} \cdot \vec{t}' + \vec{a}' \cdot \vec{t} = -\sin\alpha \left(\frac{d\alpha}{ds}\right)$$

$\because \vec{a}$ is constt $\quad \because \frac{d\alpha}{ds} = 0$

$$\vec{a} \cdot \vec{t}' + 0 \cdot \vec{t} = 0$$

$\Rightarrow \quad \vec{a} \cdot \vec{t}' = 0 \qquad$ put $\vec{t}' = 0$



$\Rightarrow \quad \vec{a} \cdot k\vec{n} = 0$

$\Rightarrow \quad k(\vec{a} \cdot \vec{n}) = 0$

$\Rightarrow \quad k \neq 0, \quad \vec{a} \cdot \vec{n} = 0$

$\vec{a} \cdot \vec{n} = 0$

$\Rightarrow$ The component of $\vec{a}$ along $\vec{n}$ is zero

$\vec{a}$ component $= \vec{r} \cdot \frac{\vec{a}}{|\vec{a}|}$

Hence, the vector $\vec{a}$ lies in the plane of $\vec{t}$ and $\vec{b}$.

The component of $\vec{a}$ along $\vec{t}$ is $\cos\alpha$ from (i)

The component of $\vec{a}$ along $\vec{b}$ is $\sin\alpha$

$$\vec{a} \cdot \vec{b} = \cos\left(\frac{\pi}{2} - \alpha\right)$$

$$\vec{a} \cdot \vec{b} = \sin\alpha$$

Hence, $\vec{a} = \cos\alpha\,\vec{t} + 0\cdot\vec{n} + \sin\alpha\,\vec{b}$

$\vec{a} = \cos\alpha\,\vec{t} + \sin\alpha\,\vec{b} \rightarrow (2)$

Differentiating it w.r.t "s"

$\vec{0} = \cos\alpha\,\vec{t}' + 0 + \sin\alpha\,\vec{b}' + 0$

$\vec{0} = \cos\alpha\,\vec{t}' + \sin\alpha\,\vec{b}'$

put $\vec{t}' = k\vec{n}$, $\vec{b}' = -\tau\vec{n}$

$\vec{0} = \cos\alpha\,(k\vec{n}) + \sin\alpha\,(-\tau\vec{n})$

$\vec{0} = (k\cos\alpha - \tau\sin\alpha)\vec{n}$

$\because \vec{n} \neq 0$, $k\cos\alpha - \tau\sin\alpha = 0$

$\Rightarrow k\cos\alpha = \tau\sin\alpha$

$\Rightarrow k/\tau = \sin\alpha/\cos\alpha$

Hence, the curvature and torsion of a helix are in a constant ratio.

Conversly, suppose that the curvature and the torsion of a helix $\vec{r} = \vec{r}(s)$ is

$$\frac{k}{\tau} = \text{Constant}$$

$\because$ Range of $\tan\alpha = (-\infty, \infty)$

$$\Rightarrow \frac{k}{\tau} = \tan\alpha$$

$$\frac{k}{\tau} = \frac{\sin\alpha}{\cos\alpha}$$

$\Rightarrow k\cos\alpha = \tau\sin\alpha$

$\Rightarrow k\cos\alpha - \tau\sin\alpha = 0$

Multiplying both sides by $\vec{n}$

$(k\cos\alpha - \tau\sin\alpha)\vec{n} = 0$

$k\vec{n}\cos\alpha - \tau\vec{n}\sin\alpha = 0$

$\vec{t}'\cos\alpha + \vec{b}'\sin\alpha = 0$

Integrating both sides w.r.t "s"

$\vec{t}\cos\alpha + \vec{b}\sin\alpha = \text{constant vector}$

$\vec{t}\cos\alpha + \vec{b}\sin\alpha = \vec{a} \rightarrow (i)$

Taking dot (scalar) product with "t" on both sides of (ii)

$$\vec{a}\cdot\vec{t} = \cos\alpha(\vec{t}\cdot\vec{t}) + \sin\alpha(\vec{t}\cdot\vec{b})$$

$$\vec{a}\cdot\vec{t} = \cos\alpha$$

Hence, the curve $\vec{r}=\vec{r}(s)$ is a helix.

Question :- A curve is a helix if and only if $[\vec{r}''\ \vec{r}'''\ \vec{r}^{iv}] = 0$

Sol :- First, we suppose that the curve $\vec{r}=\vec{r}(s)$ is a helix, then it curvature and torsion are in a constant ratio.

(i-e) $\frac{k}{\tau} = \text{Constant}$ or $\frac{\tau}{k} = \text{Constant}$.

we know $\vec{r}'' = k\vec{n}$

Differentiate it w.r.t "s"

$$\vec{r}''' = k\vec{n}' + k'\vec{n}$$

put $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\vec{r}''' = k(\tau\vec{b} - k\vec{t}) + k'\vec{n}$$

$$\vec{r}''' = k\tau\vec{b} - k^2\vec{t} + k'\vec{n} \qquad \tau =$$

Again Differentiate it w.r.ts "s"

$$\vec{r}^{iv} = k'\tau\vec{b} + k\tau'\vec{b} + k\tau\vec{b}' - 2kk'\vec{t} - k^2\vec{t}' + k''\vec{n} + k'\vec{n}'$$

put $\vec{n}' = \tau\vec{b} - k\vec{t}$, $\vec{b}' = -\tau\vec{n}$, $\vec{t}' = k\vec{n}$

$$\vec{r}^{iv} = k'\tau\vec{b} + k\tau'\vec{b} + k\tau(-\tau\vec{n}) - 2kk'\vec{t} - k^2(k\vec{n}) + k''\vec{n} + k'(\tau\vec{b} - k\vec{t})$$

$$\vec{r}^{iv} = k'\tau\vec{b} + k\tau'\vec{b} - k\tau^2\vec{n} - 2kk'\vec{t} - k^3\vec{n} + k''\vec{n} + k'\tau\vec{b} - kk'\vec{t}$$

$$\vec{r}^{iv} = -3kk'\vec{t} + (2k'\tau + k\tau')\vec{b} + (k'' - k\tau^2 - k^3)\vec{n}$$

$$[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = \begin{vmatrix} 0 & k & 0 \\ -k^2 & k' & k\tau \\ -3kk' & k'' - k\tau^2 - k^3 & 2k'\tau + k\tau' \end{vmatrix}$$

$$= -k(-2k^2k'\tau + k^3\tau' + 3k^2k'\tau)$$
$$= -k(k^2k'\tau \pm k^3\tau')$$
$$= k\cdot k^2(-k'\tau + k\tau')$$

$$= k^3(k\tau' - k'\tau)$$
$$= \frac{k^2 \cdot k^3(k\tau' - k'\tau)}{k^2}$$

$$[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = k^5 \frac{d}{ds}\left(\frac{\tau}{k}\right) \rightarrow (1)$$

$$[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = 0$$

Since, $\frac{\tau}{k}$ is constant. So, $\frac{d}{ds}\left(\frac{\tau}{k}\right) = 0$.

Now conversly suppose that $[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = 0$

From (1)

$$[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = k^5 \frac{d}{ds}(\tau/k) = 0$$

$$\Rightarrow k \neq 0, \quad \frac{d}{ds}\left(\frac{\tau}{k}\right) = 0$$

Integrating w.r.t 's'

$$\Rightarrow \frac{\tau}{k} = \text{constant}.$$

Hence, the curvature and torsion of the curve are in constant ratio.

So, the curve is a helix.

($k \neq 0$ because curve is not st. line)

Question:-

Prove that the curve $r = (a\cos\theta, a\sin\theta$, $a\theta\cot\beta)$ is a helix

Proof:-

Let $\vec{r} = (a\cos\theta, a\sin\theta, a\theta\cot\beta)$

Differentiate it w.r.t "s"

$$\vec{r}' = \left(-a\sin\theta\frac{d\theta}{ds}, a\cos\theta\frac{d\theta}{ds}, a\cot\beta\frac{d\theta}{ds}\right)$$

$$\vec{r}' = (-a\sin\theta, a\cos\theta, a\cot\beta)\frac{d\theta}{ds} \rightarrow (1)$$

To find $\frac{d\theta}{ds}$, we have

$$\vec{r}'.\vec{r}' = (a^2\sin^2\theta + a^2\cos^2\theta + a^2\cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$$\vec{t}\cdot\vec{t} = a^2(\sin^2\theta + \cos^2\theta + \cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = a^2(1+\cot^2\beta)\left(\frac{d\theta}{ds}\right)^2$$

$\because 1+\cot^2\beta = \operatorname{cosec}^2\beta$

$$1 = a^2\operatorname{cosec}^2\beta\left(\frac{d\theta}{ds}\right)^2$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2\operatorname{cosec}^2\beta}$$

$$\frac{d\theta}{ds} = \frac{1}{a\operatorname{cosec}\beta} = \frac{1}{a}\sin\beta \quad \text{put in (1)}$$

$$\vec{r}' = (-a\sin\theta, a\cos\theta, a\cot\beta)\frac{1}{a}\sin\beta$$

$$\vec{r}' = (-\sin\theta, \cos\theta, \cot\beta)\sin\beta$$

$$\vec{r}'' = (-\cos\theta, -\sin\theta, 0)\frac{1}{a}\sin^2\beta$$

$$\vec{r}''' = (\sin\theta, -\cos\theta, 0)\frac{\sin^2\beta}{a}\frac{\sin\beta}{a}$$

$$\vec{r}''' = (\sin\theta, -\cos\theta, 0)\frac{\sin^3\beta}{a^2}$$

$$\vec{r}^{iv} = (\cos\theta, \sin\theta, 0)\frac{\sin^4\beta}{a^3}$$

Now

$$[\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = \begin{vmatrix} -\cos\theta & -\sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \end{vmatrix} \begin{vmatrix} \frac{\sin^2\beta}{a} \\ \frac{\sin^3\beta}{a^2} \\ \frac{\sin^4\beta}{a^3} \end{vmatrix}$$

As C_3 is zero. So determinant is zero.

$\Rightarrow [\vec{r}'' \ \vec{r}''' \ \vec{r}^{iv}] = 0$

$\Rightarrow$ The given curve is a helix

Fundamental theorem for space curves:-

A curve is uniquely determined except as to its position in space when its curvature and torsion are given as the functions of s (arc length)

Proof:-

Let c and c_1 be two curves in space having same curvature k and torsion τ for the same given value of "s" (arc-l)

Let $[\vec{t}, \vec{n}, \vec{b}]$ and $[\vec{t}_1, \vec{n}_1, \vec{b}_1]$ be their corresponding unit tangents, unit normal and unit binormal vectors.

Now Consider

$$\frac{d}{ds}(\vec{t}\cdot\vec{t}_1) = \vec{t}'\cdot\vec{t}_1 + \vec{t}\cdot\vec{t}_1'$$

put $\vec{t}' = \vec{r}'' = k\vec{n}$ ∵ As, curvature is same so $k = k_1$

$$\frac{d}{ds}(\vec{t}\cdot\vec{t}_1) = k\vec{n}\cdot\vec{t}_1 + \vec{t}\cdot k\vec{n}_1 \rightarrow (1)$$

Now $\frac{d}{ds}(\vec{b}\cdot\vec{b}_1) = \vec{b}'\cdot\vec{b}_1 + \vec{b}\cdot\vec{b}_1'$

Put $\vec{b}' = -\tau\vec{n}$

$$\frac{d}{ds}(\vec{b}\cdot\vec{b}_1) = -\tau\vec{n}\cdot\vec{b}_1 + \vec{b}\cdot(-\tau\vec{n}_1)$$

$$= -\tau(\vec{n}\cdot\vec{b}_1 + \vec{b}\cdot\vec{n}_1) \rightarrow (2)$$

Now Consider

$$\frac{d}{ds}(\vec{n}\cdot\vec{n}_1) = \vec{n}'\cdot\vec{n}_1 + \vec{n}\cdot\vec{n}_1'$$

put $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\frac{d}{ds}(\vec{n}\cdot\vec{n}_1) = (\tau\vec{b} - k\vec{t})\cdot\vec{n}_1 + (\vec{n}\cdot(\tau\vec{b}_1 - k\vec{t}_1)$$

$$= \tau\vec{b}\cdot\vec{n}_1 - k\vec{t}\cdot\vec{n}_1 + \vec{n}\tau\vec{b}_1 - k\vec{n}\vec{t}_1 \rightarrow (3)$$

Adding (1), (2) and (3)

$$\frac{d}{ds}(\vec{t}\cdot\vec{t}_1 + \vec{b}\cdot\vec{b}_1 + \vec{n}\cdot\vec{n}_1) = k\vec{n}\cdot\vec{t}_1 + \vec{t}k\vec{n}_1 - \tau\vec{n}\cdot\vec{b}_1$$
$$\tau\vec{b}\cdot\vec{n}_1 + \tau\vec{b}\cdot\vec{n}_1 - k\vec{t}\vec{n}_1 + \vec{n}\tau\vec{b}_1 - k\vec{n}\vec{t}_1$$

$$\Rightarrow \frac{d}{ds}(\vec{t}\cdot\vec{t}_1 + \vec{b}\cdot\vec{b}_1 + \vec{n}\cdot\vec{n}_1) = 0$$

Integrate w.r.t "s"

$$\Rightarrow \vec{t}\cdot\vec{t}_1 + \vec{b}\cdot\vec{b}_1 + \vec{n}\cdot\vec{n}_1 = A \text{ (constant)} \rightarrow (4)$$

and A is scalar constant because of dot product between two vectors.

Now, rotate the curve C_1 so that the initial points of C and C_1 coinsides with each other and again rotate the curve C_1 so that the principle planes of C and C_1 at initial point coinside with each other.

Hence $\vec{t} = \vec{t}_1$, $\vec{b} = \vec{b}_1$, $\vec{n} = \vec{n}_1$ at initial pt

Substituting values in eq (4)

$$\vec{t}\cdot\vec{t} + \vec{b}\cdot\vec{b} + \vec{n}\cdot\vec{n} = A$$
$$1 + 1 + 1 = A$$
$$A = 3 \quad \text{put in (4)}$$

$$\Rightarrow \vec{t}\cdot\vec{t}_1 + \vec{b}\cdot\vec{b}_1 + \vec{n}\cdot\vec{n}_1 = 3$$

which is possible only when $\vec{t} = \vec{t}_1$, $\vec{b} = \vec{b}_1$ and $\vec{n} = \vec{n}_1$ for all corresponding points of the curves.

Now at initial points of the curves c and c_1. Now as

$$\vec{t} = \vec{t}_1 \Rightarrow \frac{d\vec{r}}{ds} = \frac{d\vec{r}_1}{ds}$$

$$\Rightarrow \frac{d\vec{r}}{ds} - \frac{d\vec{r}_1}{ds} = 0 \Rightarrow \frac{d}{ds}(\vec{r} - \vec{r}_1) = 0$$

Integrate w.r.t 's'

$$\Rightarrow \vec{r} - \vec{r}_1 = B \rightarrow (S)$$

Now to find the value of B at initial points of the curve $\vec{r} = \vec{r}_1$ put in (S)

$$\Rightarrow \vec{r}_1 - \vec{r}_1 = B \Rightarrow B = 0 \quad \text{put in (S)}$$

$$\Rightarrow \vec{r} - \vec{r}_1 = 0$$

Hence, the position vectors of the corresponding points of the curve are the same.

Hence, the curves c and c_1 are the same curves.

Intrinsic equation of a curve :-

The equation $k = k(\vec{s})$ and $\tau = \tau(\vec{s})$ which represent the curvature and torsion of a curve as a function of arc length "s" are known as intrinsic equation of a curve, also called natural equation.

Examples :-

Intrinsic eqs of st. line are $k = 0$, $\tau = 0$

Intrinsic eqs of circle are $k = $ constt, $\tau = 0$

Intrinsic eqs of helix are $k = $ constt, $\tau = $ constt.

Spherical Indicatricts of tangent :-

The locus of a point whose position vector is unit tangent of the curve is known as spherical indicatricts of the tangent of the curve.

It is given by $\vec{r}_1 = \hat{t}$

Curvature and torsion of spherical indicatricts of tangent of the curve :-

The spherical indicatricts of tangent of curve is given by $\vec{r}_1 = \vec{t}$

Differentiate it w.r.t "s"

$$\frac{d\vec{r}_1}{ds} = \vec{t}' \quad \text{put } \vec{t}' = \vec{r}'' = k\vec{n}$$

$$\vec{r}_1' = k\vec{n}$$

$$\frac{d\vec{r}_1}{ds_1} \cdot \frac{d\vec{s}_1}{ds} = k\vec{n}$$

where s_1 is arc-length of spherical indicatricts

$$\Rightarrow \vec{t}_1 \cdot \frac{d\vec{s}_1}{ds} = k\vec{n} \quad \text{comparing}$$

$$\Rightarrow \vec{t}_1 = \vec{n} \rightarrow (1) \text{ and } k = \frac{ds_1}{ds} \rightarrow (2)$$

Diff eq (1) w.r.t "s"

$$\frac{d\vec{t}_1}{ds}' = \vec{n}'$$

$$\Rightarrow \frac{d\vec{t}_1}{ds_1} \cdot \frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$\frac{d\vec{t}_1}{ds_1} = \vec{t}_1' = k_1\vec{n}_1$$

$$\Rightarrow k_1\vec{n}_1 \cdot \frac{ds_1}{ds} = \tau\vec{b} - k\vec{t} \quad \text{put } \frac{ds_1}{ds} = k$$

$$\Rightarrow k_1\vec{n}_1 \cdot k = \tau\vec{b} - k\vec{t}$$

$$\Rightarrow k_1\vec{n}_1 = \frac{\tau\vec{b} - k\vec{t}}{k} \rightarrow (3)$$

squaring both sides

$$k_1^2 (\vec{n}_1 \cdot \vec{n}_1) = \frac{1}{k^2} [\tau^2 (\vec{b} \cdot \vec{b}) + k^2 (\vec{t} \cdot \vec{t})]$$

$$k_1^2 = \frac{1}{k^2} [\tau^2 + k^2]$$

$$k_1 = \sqrt{\frac{\tau^2 + k^2}{k^2}}$$

$$\Rightarrow k_1 = \frac{\sqrt{\tau^2 + k^2}}{k}$$

is the curvature of spherical indicatricts of tangent.

From eq. (3)

$$\vec{n}_1 = \frac{\tau \vec{b} - k\vec{t}}{k k_1}$$

put value of k_1

$$\vec{n}_1 = \frac{\tau \vec{b} - k\vec{t}}{k \frac{\sqrt{\tau^2 + k^2}}{k}}$$

$$\Rightarrow \vec{n}_1 = \frac{\tau \vec{b} - k\vec{t}}{\sqrt{\tau^2 + k^2}}$$

Now,

for binormal $\vec{b}_1 = \vec{t}_1 \times \vec{n}_1$ put values

$$\vec{b}_1 = \vec{n} \times \frac{\tau \vec{b} - k\vec{t}}{\sqrt{\tau^2 + k^2}}$$

$$= \frac{\tau(\vec{n} \times \vec{b}) - k(\vec{n} \times \vec{t})}{\sqrt{\tau^2 + k^2}}$$

$$\vec{b}_1 = \frac{\tau(-\vec{t}) - k(-\vec{b})}{\sqrt{\tau^2 + k^2}}$$

$$\vec{b}_1 = \frac{-\tau\vec{t} + k\vec{b}}{\sqrt{\tau^2 + k^2}}$$

$$\vec{b}_1 = \frac{k\vec{b} - \tau\vec{t}}{\sqrt{\tau^2 + k^2}}$$

is bi-normal of spherical indicatricts of tangent.

Now, to find the torsion of spherical indicatrix of tangent.

$$\tau_1 = \frac{1}{k_1^2}\left[\frac{d\vec{r}_1}{ds_1} \quad \frac{d^2\vec{r}_1}{(ds_1)^2} \quad \frac{d^3\vec{r}_1}{(ds_1)^3}\right]$$

we know

$$[\vec{r}' \ \vec{r}'' \ \vec{r}'''] = \frac{1}{k_1^2}\left[\frac{d\vec{r}_1}{ds}\frac{ds}{ds_1} \quad \frac{d^2\vec{r}_1}{(ds)^2}\frac{(ds)^2}{(ds_1)^2} \quad \frac{d^3\vec{r}_1}{(ds)^3}\frac{(ds)^3}{(ds_1)^3}\right]$$

$$= \frac{1}{k_1^2}\left(\frac{ds}{ds_1}\right)\left(\frac{ds}{ds_1}\right)^2\left(\frac{ds}{ds_1}\right)^3\left[\frac{d\vec{r}_1}{ds} \quad \frac{d^2\vec{r}_1}{(ds)^2} \quad \frac{d^3\vec{r}_1}{(ds)^3}\right]$$

$$\tau_1 = \frac{1}{k_1^2}\left(\frac{ds}{ds_1}\right)^6\left[\frac{d\vec{r}_1}{ds} \quad \frac{d^2\vec{r}_1}{(ds)^2} \quad \frac{d^3\vec{r}_1}{(ds)^3}\right]$$

we know, equation of spherical indicatrix of tangent of the curve is $\vec{r}_1 = \vec{t}$

and $\frac{ds}{ds_1} = \frac{1}{k}$ because $\frac{ds_1}{ds} = k$

$$\tau_1 = \frac{1}{k_1^2}\frac{1}{k^6}[\vec{t}' \ \vec{t}'' \ \vec{t}''']$$

put $[\vec{t}' \ \vec{t}'' \ \vec{t}'''] = k^3(k\tau' - k'\tau)$

$$\tau_1 = \frac{1}{k_1^2}\frac{1}{k^6}[k^3(k\tau' - \tau k')]$$

$$= \frac{1}{k_1^2}\left[k^3(k\tau' - \tau k')\frac{1}{k^6}\right] \qquad (s')^6 = \frac{1}{k^6}$$

$$\tau_1 = \frac{1}{k_1^2}\left[\frac{k\tau' - \tau k'}{k^3}\right]$$

put $k_1 = \sqrt{\frac{\tau^2 + k^2}{k}}$

$$\tau_1 = \frac{1}{\frac{\tau^2+k^2}{k^2}}\left[\frac{k\tau' - \tau k'}{k^3}\right]$$

$$\tau_1 = \frac{(k\tau' - \tau k')}{k(\tau^2 + k^2)}$$

Spherical indicatricts of binormal of a curve:-

The locus of a point whose position vector is unit binormal of the curve is called the spherical indicatricts of binormal of a curve.

If $\vec{r}_1$ is the position vector of any point of spherical indicatricts of binormal then the equation of spherical indicatricts of binormal is given by

$$\vec{r}_1 = \vec{b}$$

The curvature and Torsion of spherical indicatricts of binormal of a curve:-

If $\vec{r}_1$ is the position vector of any point of a curve, then the equation of spherical indicatricts is given by

$$\vec{r}_1 = \vec{b}$$

Differentiate w.r.t "s"

$$\frac{d\vec{r}_1}{ds} = \vec{b}'$$

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = -\tau\vec{n}$$

$$\vec{r}_1'\frac{ds_1}{ds} = -\tau\vec{n}$$

$$\vec{t}_1\frac{ds_1}{ds} = -\tau\vec{n}$$

comparing

$$\Rightarrow \vec{t}_1 = -\vec{n} \rightarrow (1) \qquad \frac{ds_1}{ds} = \tau \rightarrow (2)$$

Diff eq(1) w.r.t "s"

$$\frac{d\vec{t}_1}{ds} = -\vec{n}'$$

$$\frac{d\vec{t}_1}{ds_1}\frac{ds_1}{ds} = -(\tau\vec{b} - k\vec{t})$$

$$\Rightarrow \vec{t}_1'\frac{ds_1}{ds} = -\tau\vec{b} + k\vec{t}$$

put $\vec{t}_1' = \vec{r}_1'' = k_1\vec{n}_1$

$$\Rightarrow k_1 n_1 \frac{ds_1}{ds} = -\tau\vec{b} + k\vec{t}$$

put $\frac{ds_1}{ds} = \tau$

$$k_1 n_1 \tau = -\tau\vec{b} + k\vec{t} \rightarrow (3)$$

Squaring both sides

$$\tau^2 k_1^2 (n_1)^2 = \tau^2 + k^2$$

$$k_1^2 = \frac{\tau^2 + k^2}{\tau^2}$$

$$\Rightarrow k_1 = \sqrt{\frac{\tau^2 + k^2}{\tau^2}}$$

$$k_1 = \frac{\sqrt{\tau^2 + k^2}}{\tau}$$

From (3)

$$\vec{n}_1 = \frac{-\tau\vec{b} + k\vec{t}}{k_1\tau}$$

$$\vec{n}_1 = \frac{k\vec{t} - \tau\vec{b}}{\sqrt{\tau^2 + k^2}}$$

Now

$$\vec{b}_1 = \vec{t}_1 \times \vec{n}_1$$

put $\vec{t}_1 = -\vec{n}$

and $\vec{n}_1 = \frac{k\vec{t} - \tau\vec{b}}{\sqrt{\tau^2 + k^2}}$

$$\Rightarrow \quad \vec{b}_1 = -\vec{n} \times \frac{k\vec{t} - \tau\vec{b}}{\sqrt{\tau^2+k^2}}$$

$$= \frac{k(-\vec{n}\times\vec{t}) + \tau(\vec{n}\times\vec{b})}{\sqrt{\tau^2+k^2}}$$

$$\vec{b}_1 = \frac{k\vec{b} - \tau\vec{t}}{\sqrt{\tau^2+k^2}}$$

is binormal of spherical indicatricts.

$$\tau_1 = \frac{1}{k_1^2}\left[\frac{dr_1}{ds_1} \quad \frac{d^2r_1}{(ds_1)^2} \quad \frac{d^3r_1}{(ds_1)^3}\right]$$

$$= \frac{1}{k_1^2}\left[\frac{dr_1}{ds}\frac{ds}{ds_1} \quad \frac{d^2r_1}{(ds)^2}\frac{(ds)^2}{(ds_1)^2} \quad \frac{d^3r_1}{(ds)^3}\frac{(ds)^3}{(ds_1)^3}\right]$$

$$= \frac{1}{k_1^2}\left(\frac{ds}{ds_1}\right)\left(\frac{ds}{ds_1}\right)^2\left(\frac{ds}{ds_1}\right)^3\left[\frac{dr_1}{ds} \quad \frac{d^2r_1}{(ds)^2} \quad \frac{d^3r_1}{(ds)^3}\right]$$

$$\tau_1 = \frac{1}{k_1^2}\left(\frac{ds}{ds_1}\right)^6\left[\frac{dr_1}{ds} \quad \frac{d^2r_1}{(ds)^2} \quad \frac{d^3r_1}{(ds)^3}\right]$$

we know equation of spherical indicatricts of bi-normal of curve is $\vec{r}_1 = \vec{b}$

and $\frac{ds_1}{ds} = \tau$ so, $\tau_1 = \frac{1}{k_1^2}\frac{1}{\tau^6}[\vec{b}' \ \vec{b}'' \ \vec{b}''']$

$$\tau_1 = \frac{1}{k_1^2}\left[\frac{1}{\tau^6} \ \tau^3(\tau k' - k\tau')\right]$$

$$\because [\vec{b}' \ \vec{b}'' \ \vec{b}'''] = \tau^3(\tau k' - k\tau')$$

$$\tau_1 = \frac{1}{k_1^2}\left[\tau^3(\tau k' - k\tau')\frac{1}{(\tau)^6}\right] \quad \because \delta = \frac{1}{\tau}$$

$$\tau_1 = \frac{k^5}{k_1^2}\left(\frac{\tau k' - k\tau'}{\tau^3}\right)$$

put $k_1 = \sqrt{\frac{\tau^2+k^2}{\tau}}$

$$\tau_1 = \frac{k^5}{\frac{\tau^2+k^2}{\tau^2}}\left(\frac{\tau k' - k\tau'}{\tau^3}\right)$$

$$\tau_1 = \frac{\tau k' - k\tau'}{\tau(\tau^2+k^2)}$$

is torsion of spherical indicatricts of binormal of a curve.

Spherical indicatricts of principal normal of a curve:-

The locus of a point whose position vector is the unit principal normal of a curve is called the spherical indicatricts of the principal normal of a curve.

If $\vec{r}_1$ is the position vector of any point on the spherical indicatricts on a curve, then the equation of spherical indicatrict of principal normal is given by

$$\vec{r}_1 = \vec{n}$$

Question:-

Find the curvature and torsion of the spherical indicatricts of unit principal normal of a curve $\vec{r} = \vec{r}(s)$

Sol:-

If $\vec{r}_1$ is the position vector of any point on the spherical indicatricts on a curve, then the equation of spherical indicatricts of principle normal is given by

$$\vec{r}_1 = \vec{n}$$

Differentiate w.r.t "s"

$$\frac{d\vec{r}_1}{ds} = \vec{n}'$$

put $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$\vec{r}_1' \frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$\vec{t}_1 \frac{ds_1}{ds} = \tau\vec{b} - k\vec{t} \Rightarrow (1)$$

Taking dot product with itself

$$\vec{t}_1 \cdot \vec{t}_1 \left(\frac{ds_1}{ds}\right)^2 = (\tau\vec{b} - k\vec{t}) \cdot (\tau\vec{b} - k\vec{t})$$

$$\left(\frac{ds_1}{ds}\right)^2 = \tau^2 + k^2$$

$$\frac{ds_1}{ds} = \sqrt{\tau^2 + k^2}$$

Differentiate eq (1) w.r.t "s"

$$\frac{dt_1}{ds}\left(\frac{ds_1}{ds}\right)^2 + \vec{t}_1 \frac{d^2 s_1}{ds^2} = \tau\vec{b}' + \tau'\vec{b} - k\vec{t}' - k'\vec{t}$$

$$\frac{dt_1}{ds}\left(\frac{ds_1}{ds}\right)^2 + \vec{t}_1 \frac{d^2 s_1}{ds^2} = \tau(-\tau\vec{n}) + \tau'\vec{b} - k(k\vec{n}) - k'\vec{t}$$

$$\frac{dt_1}{ds}\left(\frac{ds_1}{ds}\right)^2 + \vec{t}_1 \frac{d^2 s_1}{ds^2} = -\tau^2\vec{n} + \tau'\vec{b} - k^2\vec{n} - k'\vec{t} \rightarrow (2)$$

we know $\frac{ds_1}{ds} = \sqrt{\tau^2 + k^2}$

$$\frac{d^2 s_1}{ds^2} = \frac{1}{2\sqrt{\tau^2 + k^2}}(2\tau\tau' + 2kk')$$

$$\frac{d^2 s_1}{ds^2} = \frac{\tau\tau' + kk'}{\sqrt{\tau^2 + k^2}}$$ put these in (2)

$$k_1 \vec{n}_1 \cdot (\tau^2 + k^2) + \vec{t}_1\left(\frac{\tau\tau' + kk'}{\sqrt{\tau^2 + k^2}}\right) = -k'\vec{t} - (\tau^2 + k^2)\vec{n} + \tau'\vec{b}$$

Now taking dot product

$$k_1^2 (\vec{n}_1 \cdot \vec{n}_1)(\tau^2 + k^2)^2 + (\vec{t}_1 \cdot \vec{t}_1)\left(\frac{\tau\tau' + kk'}{\sqrt{\tau^2 + k^2}}\right)^2 = k'^2(\vec{t} \cdot \vec{t}) +$$

$$(\tau^2 + k^2)^2 (\vec{n} \cdot \vec{n}) + \tau'^2 (\vec{b} \cdot \vec{b})$$

$$k_1^2 (\tau^2 + k^2)^2 + \frac{(\tau\tau' + kk')^2}{\tau^2 + k^2} = k'^2 + (\tau^2 + k^2)^2 + \tau'^2$$

$$k_1^2(\tau^2+k^2)^2 = k'^2 + (\tau^2+k^2)^2 + \tau'^2 - \frac{(\tau\tau'+kk')^2}{\tau^2+k^2}$$

$$= \frac{k'^2(\tau^2+k^2) + (\tau^2+k^2)^3 + \tau'^2(\tau^2+k^2) - (\tau\tau'+kk')^2}{(\tau^2+k^2)}$$

$$k_1^2 = k'^2\tau^2 + \cancel{k^2k'^2} + (\tau^2+k^2)^3 + \cancel{\tau^2\tau'^2} + \tau'^2k^2 - \cancel{\tau^2\tau'^2} - \cancel{k^2k'^2} + {} - 2\tau\tau'kk'$$

$$k_1^2 = \frac{(\tau^2+k^2)^3 + (k'^2\tau^2 + \tau'^2k^2 - 2\tau\tau'kk')}{(\tau^2+k^2)^3} \; (\tau^2+k^2)^3$$

$$k_1^2 = \frac{(\tau^2+k^2)^3 + (k\tau' - \tau k')^2}{(\tau^2+k^2)^3}$$

$$k_1 = \sqrt{\frac{(\tau^2+k^2)^3 + (k\tau' - \tau k')^2}{(\tau^2+k^2)^3}}$$

is curvature

$$\tau_1 = \frac{1}{k_1^2}\left[\frac{d\vec{r}_1}{ds_1} \quad \frac{d^2\vec{r}_1}{ds_1^2} \quad \frac{d^3\vec{r}_1}{ds_1^3}\right]$$

$$\tau_1 = \frac{1}{k_1^2}\left[\frac{d\vec{r}_1}{ds}\frac{ds}{ds_1} \quad \frac{d^2\vec{r}_1}{ds^2}\frac{ds^2}{ds_1^2} \quad \frac{d^3\vec{r}_1}{ds^3}\frac{ds^3}{ds_1^3}\right]$$

$$\tau_1 = \frac{1}{k_1^2}\left(\frac{ds}{ds_1}\right)^6\left[\frac{d\vec{r}_1}{ds} \quad \frac{d^2\vec{r}_1}{ds^2} \quad \frac{d^3\vec{r}_1}{ds^3}\right]$$ put $\vec{r}_1 = \vec{n}$

$$\tau_1 = \frac{1}{k_1^2}\left(\frac{ds}{ds_1}\right)^6[\vec{n}' \quad \vec{n}'' \quad \vec{n}'''] \rightarrow (3)$$

we know $\vec{n}' = \tau\vec{b} - k\vec{t}$, $\vec{n}'' = -\tau^2\vec{n} + \tau'\vec{b} - k^2\vec{n} - k'\vec{t}$

$\vec{n}'' = -k'\vec{t} - (\tau^2+k^2)\vec{n} + \tau'\vec{b}$

$\vec{n}''' = -3(kk'+\tau\tau')\vec{n} + (k^3 - k'' + k\tau^2)\vec{t} + (\tau'' - \tau^3 - k^2\tau)\vec{b}$

$$[\vec{n}' \quad \vec{n}'' \quad \vec{n}'''] = \begin{vmatrix} -k & 0 & \tau \\ -k' & -(\tau^2+k^2) & \tau' \\ k^3 - k'' + k\tau^2 & -3(kk'+\tau\tau') & \tau'' - \tau^3 - k^2\tau \end{vmatrix}$$

$$= -k(-\tau^2\tau'' + \tau^5 + \tau^3k^2 - k^2\tau'' + k^2\tau^3 + k^4\tau + 3\tau'kk' + 3\tau'^2\tau) + \tau(3k'^2k + 3k'\tau\tau' + \tau^2k^3 - \tau^2k'' + k\tau^4 + k^5 - k^2k'' + k^3\tau^2)$$

After simplifying

$[\vec{n}'\ \vec{n}''\ \vec{n}'''] = (\tau^2+k^2)(k\tau''-\tau k'') + 3(kk'+\tau\tau')(k'\tau-\tau'k)$

put this in (1) and value of k_1^2,

$$\tau_1 = \frac{\dfrac{(\tau^2+k^2)(k\tau''-\tau k'') + 3(kk'+\tau\tau')(k'\tau-\tau'k)}{(\tau^2+k^2)^3+(k\tau'-\tau k')^2} \quad (\tau^2+k^2)^3}{(\tau^2+k^2)^3}$$

$$\tau_1 = \frac{3(kk'+\tau\tau')(k'\tau-\tau'k)+(\tau^2+k^2)(k\tau''-\tau k'')}{(\tau^2+k^2)^3+(k\tau'-\tau k')^2}$$ is torsion

Question 8- Find the curvature and torsion of the three spherical indicatricts of the helix $\vec{r} = (a\cos u,\ a\sin u,\ bu)$

Skew curvature :-

The arc rate of rotation of the principle normal of a curve is known as skew curvature.

Now $\frac{d\vec{n}}{ds} = \vec{n}' = \tau\vec{b} - k\vec{t}$

$$\left|\frac{d\vec{n}}{ds}\right| = |\vec{n}'| = \sqrt{\tau^2 + k^2}$$

So the magnitude of skew curvature is given by $\sqrt{\tau^2+k^2}$

Question :-

Prove that the curvature of spherical indicatricts of tangent of a curve is the ratio of skew curvature and curvature of curve.

$$k_1 = \frac{\sqrt{\tau^2+k^2}}{k} = \frac{\text{skew curvature}}{\text{curvature}}$$

Question:-

Prove that the curvature of spherical indicatricts of binormal of a curve is ratio of skew curvature and torsion of a curve.

$$k_1 = \frac{\sqrt{\tau^2+k^2}}{\tau} = \frac{\text{Skew curvature}}{\text{Torsion}}$$

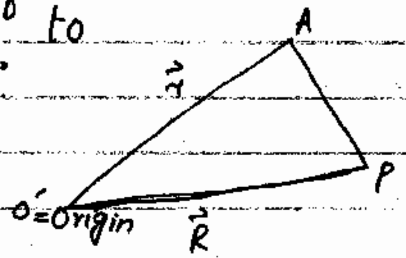
Tangent Surface:-

A surface generated by the tangent lines on a curve C is called a tangent surface to the curve C.

Equation of tangent surface:-

Let "A" be any point on a given curve C at an arcle distance s from a fixed point "O" on the curve C.

Let the position vector of A be $\vec{r}$ and let "P" be any point on the tangent surface. Let the position vector of P be $\vec{R}$. Now, the line $\overline{AP}$ is tangent to the curve C at point "A". Now, if u be the distance $\overline{AP}$ then



$$\overrightarrow{AP} = u\vec{t}$$

$u = u(s)$ and
$\vec{r} = \vec{r}(s)$ and
$\vec{R} = \vec{R}(t)$ and
$\vec{R} = \vec{R}(u) = R(s)$

Now. $\overline{OP} = \overline{OA} + \overline{AP}$

$\Rightarrow \vec{R} = \vec{r} + u\vec{t}$

$\Rightarrow \vec{R}(s,u) = \vec{r}(s) + u\vec{t}(s) \rightarrow (1)$

which is the equation of tangent surface.

Here, $u (= u(s))$ the function of s

if $u = \lambda(s)$ then (1) becomes

$$\vec{R}(s) = \vec{r}(s) + \lambda(s) \cdot \vec{t}(s)$$

which is the equation of any curve on tangent surface.

Involute:-

A curve which lies on the tangent surface to a given curve C and intersects the generators of the tangent surface orthogonally is called the involute of the given curve C. It is denoted by $\tilde{C}$

Equation of Involute:-

Let C be any given curve and $\tilde{C}$ be the involute of C. Let $\vec{r}_1$ be the position vector of any point P_1 on the involute $\tilde{C}$. Now, we know that the involute lies on the tangent surface to the given curve C and the equation of any curve on the tangent surface of the given curve is

$$\vec{R}(s) = \vec{r}(s) + \lambda(s)\cdot\vec{t}(s)$$

$$\Rightarrow \quad \vec{r}_1(s) = \vec{r}(s) + \lambda(s)\vec{t}(s) \rightarrow (1)$$

where $\vec{r}$ is the position vector of any point on the curve C and $\vec{r}_1$ is the position vector of point "P_1" which lies on the involute $\tilde{C}$ of the given curve C

Differentiating eq (1) w.r.t "s"

$$\frac{d\vec{r}_1(s)}{ds} = \frac{d\vec{r}(s)}{ds} + \lambda'(s)\vec{t}(s) + \lambda(s)\vec{t}'(s)$$

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = \frac{d\vec{r}}{d} + \lambda'\vec{t} + \lambda\vec{t}'$$

$$\Rightarrow \quad \vec{t}_1\frac{ds_1}{ds} = \vec{t} + \lambda'\vec{t} + \lambda(k\vec{n})$$

where tangent to involute is $\vec{t}_1$ which is orthogonally intersect (Perpandicular to $\vec{t}$). $\vec{t}_1$ is the unit tangent vector to involute $\tilde{C}$.

$$\Rightarrow \quad \vec{t}_1\cdot\vec{t}_1\left(\frac{ds_1}{ds}\right)^{*} = \vec{t}\cdot\vec{t} + \lambda'\,\vec{t}\cdot\vec{t} + \lambda k(\vec{n}\cdot\vec{t})$$

(because $\vec{t} \perp \vec{t}_1^{*}$)

$$0 = 1 + \lambda'(1) + \lambda k(0)$$

$$0 = 1 + \lambda'$$

$\Rightarrow \quad \lambda' = -1$

Integrate w.r.t "s"

$$\lambda = -s + c \rightarrow (2)$$

Subsititute this value in (1)

$\vec{r}_1(s) = \vec{r}(s) + \lambda(c-s)\vec{t}(s)$

$\Rightarrow \quad \vec{r}_1 = \vec{r} + \lambda(c-s)\vec{t}$

is the equation of involute to the given curve C

Note:-

Since C is a constant. So, there exist infinitely many involutes to a given curve C

Question:-

Find the curvature and torsion of the involute $\bar{C}$ of a given curve C.

Sol:-

The equation of involute is

$\vec{r}_1 = \vec{r} + (c-s)\vec{t} \rightarrow (A)$

where $\vec{r}_1$ is ~~a~~ position vector of any point on the involute $\bar{C}$ and $\vec{r}$ is position vector of any point on the curve C.

Differentiate eq (A) w.r.t "s"

$$\frac{d\vec{r}_1}{ds} = \vec{r}' + (c-s)\vec{t}' + \vec{t}(0-1)$$

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = \vec{r}' + (c-s)\vec{t}' - \vec{t}$$

$$\vec{t}_1 \frac{ds_1}{ds} = \vec{t} + (c-s)\vec{t}' - \vec{t}$$

$$\vec{t}_1 \frac{ds_1}{ds} = (c-s)\vec{t}' = (c-s)k\vec{n} \rightarrow (1)$$

squaring both sides

$$\vec{t}_1 \cdot \vec{t}_1 \left(\frac{ds_1}{ds}\right)^2 = (c-s)^2 k^2 (\vec{n}\cdot\vec{n})$$

$$\left(\frac{ds_1}{ds}\right)^2 = (c-s)^2k^2 \rightarrow (2)$$

Substitute this in (1)

Comparing both sides of eq (1)

$\Rightarrow \vec{t}_1 = \vec{n} \rightarrow (3)$ $\qquad \frac{ds_1}{ds} = (c-s)k \rightarrow (4)$

Differentiate (3) w.r.t "s"

$$\frac{d\vec{t}_1}{ds} = \vec{n}'$$

$$\frac{d\vec{t}_1}{ds_1}\frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$\vec{t}_1'\frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$k_1\vec{n}_1\frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$\Rightarrow k_1^2(\vec{n}_1\cdot\vec{n}_1)\left(\frac{ds_1}{ds}\right)^2 = \tau^2 + k^2$$

$$k_1^2\left(\frac{ds_1}{ds}\right)^2 = \tau^2 + k^2$$

$$\Rightarrow k_1^2k^2(c-s)^2 = \tau^2 + k^2 \quad \text{from } (2)$$

$$\Rightarrow k_1^2 = \frac{\tau^2 + k^2}{k^2(c-s)^2}$$

$$\Rightarrow k_1 = \frac{\sqrt{\tau^2 + k^2}}{k(c-s)}$$

which gives the curvature of the involute $\bar{C}$ of the given curve.

Now, For torsion

Consider $\vec{t}_1 = \vec{n}$

Differentiate w.r.t 's'

$$\frac{d\vec{t}_1}{ds} = \vec{n}'$$

$$\frac{d\vec{t}_1}{ds_1}\frac{ds_1}{ds} = \tau\vec{b} - k\vec{t}$$

$$\vec{t}_1' \left(\frac{ds_1}{ds}\right) = \tau\vec{b} - k\vec{t}$$

$$k_1\vec{n}_1 \left(\frac{ds_1}{ds}\right) = \tau\vec{b} - k\vec{t} \rightarrow (A)$$

put value of k_1 and $\frac{ds_1}{ds}$

$$\sqrt{\tau^2+k^2}\ \vec{n}_1\ k(c-s) = \tau\vec{b} - k\vec{t}$$

$$\Rightarrow \frac{k(c-s)}{\sqrt{\tau^2+k^2}}\ \vec{n}_1 = \tau\vec{b} - k\vec{t}$$

$$\Rightarrow \vec{n}_1 = \frac{\tau\vec{b} - k\vec{t}}{\sqrt{\tau^2+k^2}} \text{———} (A)$$

$$\vec{b}_1 = \vec{t}_1 \times \vec{n}_1$$

$$= \vec{n} \times \frac{1}{\sqrt{\tau^2+k^2}}(\tau\vec{b} - k\vec{t})$$

$$= \frac{1}{\sqrt{\tau^2+k^2}}(\tau(\vec{n}\times\vec{b}) - k(\vec{n}\times\vec{t}))$$

$\vec{n}\times\vec{t} = -\vec{b}$

$$\vec{b}_1 = \frac{1}{\sqrt{\tau^2+k^2}}(\tau\vec{t} + k\vec{b}) \rightarrow (S)$$

Differentiate w.r.t "s"

$$\frac{d\vec{b}_1}{ds} = \frac{\sqrt{\tau^2+k^2}\,(\tau\vec{t}' + \tau'\vec{t} + k\vec{b}' + k'\vec{b}) - \frac{1}{2\sqrt{\tau^2+k^2}}(2\tau\tau' + 2kk')(\tau\vec{b} - k\vec{t})}{\tau^2+k^2}$$

$$= \frac{\dfrac{2(\tau^2+k^2)(\tau k\vec{n} + \tau'\vec{t} - k(-\tau\vec{n}) - k'\vec{b}) - 2\tau^2\tau'\vec{t} + 2\tau k\tau'\vec{b} - 2kk'\tau\vec{t} + 2k^2k'\vec{b}}{2\sqrt{\tau^2+k^2}}}{\tau^2+k^2}$$

$$= \frac{2(\tau^2+k^2)(\tau k\vec{n} + \tau'\vec{t} + \tau k\vec{n} - k'\vec{b}) - 2[(\tau^2\tau' + kk'\tau)\vec{t} - (\tau\tau'k + k^2k')\vec{b}]}{2(\tau^2+k^2)^{3/2}}$$

$$= \frac{(\tau^2+k^2)(2\tau k\vec{n} + \tau'\vec{t} - k'\vec{b}) - 2(\tau\tau' + kk'\tau)\vec{t} + (\tau\tau'k + k^2k')\vec{b}}{(\tau^2+k^2)^{3/2}}$$

$$= 2\tau^3k\vec{n} + \tau^2\tau'\vec{t} - \tau^2k'\vec{b} + 2\tau k^3\vec{n} + \tau'k^2$$

From (5)

$$b_1 \sqrt{\tau^2+k^2} = \vec{t}\,\tau + k\vec{b}$$

$$\Rightarrow \quad b_1 k k_1 (c-s) = \vec{t}\,\tau + k\vec{b}$$

$$k k_1 (c-s) \frac{db_1}{ds_1}\frac{ds_1}{ds} + b_1 \frac{d}{ds}[k k_1 (c-s)]$$

$$= \tau \vec{t}' + \tau' \vec{t} + k' \vec{b} + k \vec{b}' \qquad \because \frac{ds_1}{ds} = k(c-s)$$

$\Rightarrow k^2$

$$\Rightarrow k^2 k_1 (c-s)^2 (-\tau_1 \vec{n}_1) + b_1 \frac{d}{ds}[k k_1 (c-s)] = \tau' \vec{t} + k' \vec{b} \quad \rightarrow (6)$$

From (A)(*)

$$k_1 k \vec{n}_1 (c-s) = \tau \vec{b} - k\vec{t} \quad \rightarrow (7)$$

Taking dot product of (6) and (7)

$$-k^3 k_1^2 \tau_1 (c-s)^3 = \tau k' - k\tau'$$

$$\tau_1 = \frac{\tau' k - \tau k'}{k_1^2 k^3 (c-s)^3}$$

put value of k_1^2

$$\tau_1 = \frac{(k\tau' - \tau k')(c-s)^2 k^2}{k^3 (c-s)^3 (\tau^2+k^2)}$$

$$= \frac{k\tau' - \tau k'}{k(c-s)(k^2+\tau^2)}$$

If we take

$$k = \frac{1}{\rho} \quad \text{and} \quad \tau = \frac{1}{\sigma}$$

$$k' = -\frac{\rho'}{\rho^2}, \quad \tau' = -\frac{\sigma'}{\sigma^2}$$

then $$\tau_1 = \frac{\rho(\sigma\rho' - \sigma'\rho)}{(\rho^2+\sigma^2)(c-s)}$$

Question:-

Prove that the unit tangent vector to involute is normal to the tangent vector $\vec{t}$.

(i-e) $\vec{t}_1 = \vec{n}$

Proof:-

The equation of involute is

$$\vec{r}_1 = \vec{r} + (c-s)\vec{t} \rightarrow (A)$$

where $\vec{r}_1$ is ~~position vector~~ of any point on the involute $\bar{C}$ and $\vec{r}$ is the position vector of any point on the curve C.

Differentiate eq (A) w.r.t "s"

$$\frac{d\vec{r}_1}{ds} = \vec{r}' + (c-s)\vec{t}' + (0-1)\vec{t}$$

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = \vec{r}' + (c-s)\vec{t}' - \vec{t}$$

$\because \vec{r}_1' = \vec{t}_1$

$$\vec{t}_1 \frac{ds_1}{ds} = \vec{t} + (c-s)\vec{t}' - \vec{t}$$

$$\vec{t}_1 \frac{ds_1}{ds} = (c-s)\vec{t}' \quad \text{put } \vec{t}' = k\vec{n}$$

$$\vec{t}_1 \frac{ds_1}{ds} = (c-s)k\vec{n} \rightarrow (i)$$

Taking dot product

$$(\vec{t}_1 \cdot \vec{t}_1)\left(\frac{ds_1}{ds}\right)^2 = (c-s)^2 k^2 (\vec{n} \cdot \vec{n})$$

$$\left(\frac{ds_1}{ds}\right)^2 = k^2 (c-s)^2 \quad \text{put in (i)}$$

$$\vec{t}_1 \frac{ds_1}{ds} = k(c-s) \quad \text{put in (i)}$$

$$\vec{t}_1 \, k(c-s) = k(c-s)\vec{n}$$

$$\Rightarrow \vec{t}_1 = \vec{n}$$

where $\vec{t}_1$ is unit tangent vector to involute.

Evolute of a curve:-

If $\tilde{C}$ is the involute of a given curve C, then the curve C is known as the evolute of $\tilde{C}$.

Theorem:-

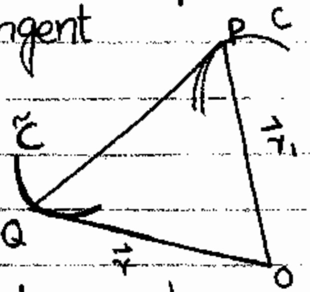
Let $\vec{r}=\vec{r}(s)$ be an involute of a curve C and let $[\vec{t}, \vec{n}, \vec{b}]$ be the moving trides at any point of the involute $\vec{r}=\vec{r}(s)$.
Let $\vec{r}_1$ be the position vector of any point on the curve C, then prove that

$$\vec{r}_1 = [\vec{r}+\rho\vec{n}+\rho \cot(\psi+c)\vec{b}$$

where $\psi = \int \tau ds$ and c is a constant.

Proof:-

Let P be any point on the curve C with position vector $\vec{r}_1$ corresponding to a point Q on $\tilde{C}$, where $\tilde{C}$ is involute of curve C.
Now the $\overline{QP}$ line $\overline{QP}$ is tangent at point 'p' to the curve C.



Since, $\overline{QP}$ is tangent at P to the curve C. So, it is perpandicular to the tangent at point Q of the curve $\tilde{C}$. So, $\overline{QP}$ lies in the normal plane at point Q.
Now, taking the co-ordinate system $[\vec{t}, \vec{n}, \vec{b}]$ at point Q we have

$$\vec{QP} = 0\cdot\vec{t}+\lambda\vec{n}+\mu\vec{b}$$

$$\vec{QP} = \lambda\vec{n}+\mu\vec{b}$$

where λ and μ are constant functions of "s" on the curve $\tilde{C}$ or on involute $\vec{r}=\vec{r}(s)$

$$\vec{OP} = \vec{OQ} + \vec{QP}$$

$$\Rightarrow \vec{r}_1 = \vec{r}+\lambda\vec{n}+\mu\vec{b} \rightarrow (1)$$

Now to determine the values of λ and μ.

Diff eq (i) w.r.t "s", we have

$$\frac{d\vec{r}_1}{ds} = \vec{r}' + \lambda\vec{n}' + \lambda'\vec{n} + \mu\vec{b}' + \mu'\vec{b}$$

$$\frac{d\vec{r}_1}{ds_1}\frac{ds_1}{ds} = \vec{t} + \lambda(\tau\vec{b} - k\vec{t}) + \lambda'\vec{n} + \mu(-\tau\vec{n}) + \mu'\vec{b}$$

$$= \vec{t} + \lambda\tau\vec{b} - \lambda k\vec{t} + \lambda'\vec{n} - \mu\tau\vec{n} + \mu'\vec{b}$$

$$\vec{t}_1 \frac{ds_1}{ds} = \vec{t}(1-\lambda k) + (\lambda' - \mu\tau)\vec{n} + (\lambda\tau + \mu')\vec{b} \to \text{(ii)}$$

Here $\vec{t}_1$ is unit tangent vector on the curve C at point "p". So, $\vec{t}_1\frac{ds_1}{ds}$ is the tangent at point "p" to the curve C. So, it lies in the normal plane at point Q on the curve $\bar{C}$ and hence it is parallel to

$\lambda\vec{n} + \mu\vec{b} \to$ (iii)

From eq (ii) and eq (iii)

$1 = k\lambda = 0 \to$ (iv) $\qquad \lambda' - \mu\tau = \lambda \to$ (v)

$\lambda\tau + \mu' = \mu \to$ (vi)

From (iv) $\quad 1 - k\lambda = 0 \Rightarrow 1 = k\lambda = \lambda = \frac{1}{k} = \rho$

From (v) and (vi) we have

$$\frac{\lambda' - \mu\tau}{\lambda} = 1, \quad \frac{\lambda\tau + \mu'}{\mu} = 1$$

$$\Rightarrow \frac{\lambda' - \mu\tau}{\lambda} = \frac{\lambda\tau + \mu'}{\mu}$$

$$\Rightarrow \mu(\lambda' - \mu\tau) = \lambda(\lambda\tau + \mu')$$

$$\Rightarrow \mu\lambda' - \mu^2\tau = \lambda^2\tau + \lambda\mu'$$

$$\Rightarrow \mu\lambda' - \lambda\mu' = \lambda^2\tau + \mu^2\tau$$

$$\Rightarrow \mu\lambda' - \lambda\mu' = (\lambda^2 + \mu^2)\tau$$

$$\frac{\mu\lambda' - \lambda\mu'}{\mu^2} = \left(\frac{\lambda^2 + \mu^2}{\mu^2}\right)\tau$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\lambda}{\mu}\right) = \left(\frac{\lambda^2}{\mu^2} + 1\right)\tau$$

$$\Rightarrow \frac{\frac{d}{ds}(\lambda/u)}{1+\frac{\lambda^2}{u^2}} = \tau$$

$$\Rightarrow \frac{d}{ds}\left(\tan^{-1}\left(\frac{\lambda}{u}\right)\right) = \tau$$

Taking $\int$ on both sides w.r.t s

$$\Rightarrow \int \frac{d}{ds}\left(\tan^{-1}\left(\frac{\lambda}{u}\right)\right) ds = \int \tau ds + c$$

$$\Rightarrow \tan^{-1}\left(\frac{\lambda}{u}\right) = \int \tau ds + c$$

where c is

$$\Rightarrow \frac{\lambda}{u} = \tan\left[\int \tau ds + c\right]$$

$$\Rightarrow \frac{\lambda}{u} = \tan[\psi + c]$$

$$\Rightarrow u = \frac{\lambda}{\tan(\psi + c)}$$

$$\Rightarrow u = \lambda \cot(\psi + c)$$

put $\lambda = \rho$

$$\Rightarrow u = \rho \cot(\psi + c)$$

put values of λ and u in (i)

$$\vec{r}_1 = \vec{r} + \rho \vec{n} + \rho \cot(\psi + c) \vec{b}$$

is equation of evolute

Question:-

Find the equation of involute of a circular helix $\vec{r} = (a\cos\theta, a\sin\theta, b\theta)$.

Sol:-

we know that the equation of involute is

$$\vec{r}_1 = \vec{r} + (c-s)\vec{t} \rightarrow (A)$$

Given that

$$\vec{r} = (a\cos\theta, a\sin\theta, b\theta)$$

Differentiate it w.r.t "s"

$$\frac{d\vec{r}}{ds} = (-a\sin\theta, a\cos\theta, b)\frac{d\theta}{ds} \rightarrow (1)$$

$$\vec{r}' = \vec{t} = (-a\sin\theta, a\cos\theta, b)\frac{d\theta}{ds}$$

$$\vec{t}\cdot\vec{t} = (a^2\sin^2\theta + a^2\cos^2\theta + b^2)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2(\sin^2\theta + \cos^2\theta) + b^2)\left(\frac{d\theta}{ds}\right)^2$$

$$1 = (a^2+b^2)\left(\frac{d\theta}{ds}\right)^2$$

$$\left(\frac{d\theta}{ds}\right)^2 = \frac{1}{a^2+b^2}$$

$$\Rightarrow \frac{d\theta}{ds} = \frac{1}{\sqrt{a^2+b^2}}$$

put $c = \sqrt{a^2+b^2}$

$$\frac{d\theta}{ds} = \frac{1}{c}$$

$$c = \frac{ds}{d\theta}$$

$$\Rightarrow c\,d\theta = ds$$

$$\Rightarrow c\theta = s$$

By (1)

$$\vec{t} = (-a\sin\theta,\ a\cos\theta,\ b)\left(\frac{1}{\sqrt{a^2+b^2}}\right).$$

$$\vec{t} = (-a\sin\theta,\ a\cos\theta,\ b)\left(\frac{1}{c}\right)$$

Put $S = c\theta$ and value of $\vec{t}$ in (A)

$$\vec{r}_1 = \vec{r} + (\lambda - S)\,\vec{t}$$

$$\Rightarrow \vec{r}_1 = (a\cos\theta,\ a\sin\theta,\ b\theta) + \frac{1}{c}(\lambda - c\theta)(-a\sin\theta,\ a\cos\theta,\ b)$$

PART II SURFACES

Surface:-

A surface is the locus of a point whose co-ordinates are functions of two independent parametres u and v.

Thus, if $P(x,y,z)$ is any point on a surface then

$$(x,y,z) = (x(u,v),\ y(u,v),\ z(u,v))$$

(i-e) $x = x(u,v)$ or $x = f_1(u,v) \rightarrow (1)$

$y = y(u,v)$ $\quad y = f_2(u,v) \rightarrow (2)$

$z = z(u,v)$ $\quad z = f_3(u,v) \rightarrow (3)$

These equations (1), (2), (3) are known as parametric equation of a surface, if we elliminate u and v from eq (1), (2) and (3) we obtain an equation in x, y, z

(i-e) $F(x,y,z) = 0$

which is known as the equation of a surface.

Example:-

The parametric equations of a sphere with centre at origin and radius "a" are

$$x = a\cos\theta\cos\phi \qquad y = a\cos\theta\sin\phi, \qquad z = a\sin\theta$$

If we elliminate θ and ϕ from these three equation, we obtain the equation of sphere with centre at origin and radius "a"

(i-e) $x^2+y^2+z^2 = a^2 \rightarrow (1)$

$$\Rightarrow x^2+y^2+z^2-a^2 = 0$$

$$\Rightarrow F(x,y,z) = 0$$

put values of x, y, z in (1)

$$x^2+y^2+z^2 = a^2\cos^2\theta\cos^2\phi + a^2\cos^2\theta\sin^2\phi + a^2\sin^2\theta$$

$$= a^2\cos^2\theta(\cos^2\phi+\sin^2\phi) + a^2\sin^2\theta$$

$$x^2+y^2+z^2 = a^2\cos^2\theta + a^2\sin^2\theta = a^2(\cos^2\theta+\sin^2\theta) = a^2$$

Example:-

The parametric equations of an ellipsiod with centre at origin and radius "r" are

$x = a\cos\theta\cos\phi$, $y = b\cos\theta\sin\phi$, $z = c\sin\theta$

By eliminating θ and ϕ from these 3 equations, we obtain an equation of ellipsiod

(i-e) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$\Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0$

$\Rightarrow F(x, y, z) = 0$

Tangent plane to a surface:-

The tangent to any point on a curve drawn on a surface is known as tangent line on a surface.

The tangent plane at any point P on a surface is the plane containing all tangent lines at that point P on the surface.

Equation of tangent plane to a surface:-

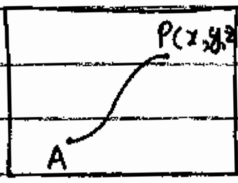
Let $F(x, y, z) = 0$ be a given surface. Let S be the arc-length of a curve drawn on this surface measured from a fixed point "A".

Let P be any point on this curve with position vector "$\vec{r}$"

Now Differentiating eq (1) w.r.t "S"

$F(x, y, z) = 0 \rightarrow (1)$

$$\frac{\partial F}{\partial x}\left(\frac{\partial x}{\partial s}\right)+\frac{\partial F}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial s}=0$$



$$\Rightarrow \left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)\cdot\left(\frac{\partial x}{\partial s},\frac{\partial y}{\partial s},\frac{\partial z}{\partial s}\right)=0$$

$$\Rightarrow \vec{\nabla}F\cdot\vec{r}\,'=0$$

$$\Rightarrow \vec{\nabla}F\cdot\vec{t}=0$$

$\Rightarrow$ $\vec{\nabla}F$ is perpandicular to the tangent line at point P on the surface.

Hence, $\vec{\nabla}F$ is perpandicular to all tangent lines at point $P(x,y,z)$ on the surface.

Hence, the equation of the tangent plane on the surface is

$$(\vec{R}-\vec{r})\cdot\vec{\nabla}F=0$$

where $\vec{R}$ is the position vector of any point on tangent plane to the surface at point $P(x,y,z)$ with position vector $\vec{r}$.

Now, if X, Y, Z are the co-ordinates of point with position vector $\vec{R}$, then the equation of the tangent plane becomes

$$(X-x,\ Y-y,\ Z-z)\cdot\left(\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right)=0$$

$$\Rightarrow (X-x)\frac{\partial F}{\partial x}+(Y-y)\frac{\partial F}{\partial y}+(Z-z)\frac{\partial F}{\partial z}=0$$

which is the equation of tangent plane at point $P(x,y,z)$ on the surface $F(x,y,z)=0$.

Question:-

Prove that the tangent plane to the surface $xyz=a^3$ and the co-ordinate planes bound a tetrahedron of a constant volume.

Sol:-

The given surface is $xyz = a^3 \to (1)$

$\Rightarrow xyz - a^3 = 0$

$\Rightarrow f(x,y,z) = xyz - a^3 = 0$

Now, $\frac{\partial f}{\partial x} = yz$, $\frac{\partial f}{\partial y} = xz$, $\frac{\partial f}{\partial z} = xy$

Now, the equation of tangent plane at any point $P(x,y,z)$ to the given surface is

$$(X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} + (Z-z)\frac{\partial f}{\partial z} = 0$$

$\Rightarrow (X-x)(yz) + (Y-y)(xz) + (Z-z)(xy) = 0$

$\Rightarrow Xyz - xyz + Yxz - xyz + xyZ - xyz = 0$

$\Rightarrow Xyz + Yxz + Zxy - 3xyz = 0$

put $xyz = a^3$ from (1)

$\Rightarrow Xyz + Yxz + Zxy - 3a^3 = 0 \to (2)$

which is the equation of tangent plane to the given surface.

Now, For X-intercept

put $Y = Z = 0$ put in (2)

$\Rightarrow Xyz + 0 + 0 - 3a^3 = 0$

$\Rightarrow Xyz = 3a^3$

$\Rightarrow X = \frac{3a^3}{yz}$

So, the point of intersection of the tangent plane and X-axis is

$\left(\frac{3a^3}{yz}, 0, 0\right)$

Now, For Y-intercept

put $X = Z = 0$ put in (2)

$\Rightarrow \quad 0 + yxz + 0 - 3a^3 = 0$

$\Rightarrow \quad yxz - 3a^3 = 0$

$yxz = 3a^3$

$\Rightarrow \quad y = \frac{3a^3}{xz}$

So, the point of intersection of the tangent plane and y-axis is $\left(0, \frac{3a^3}{xz}, 0\right)$

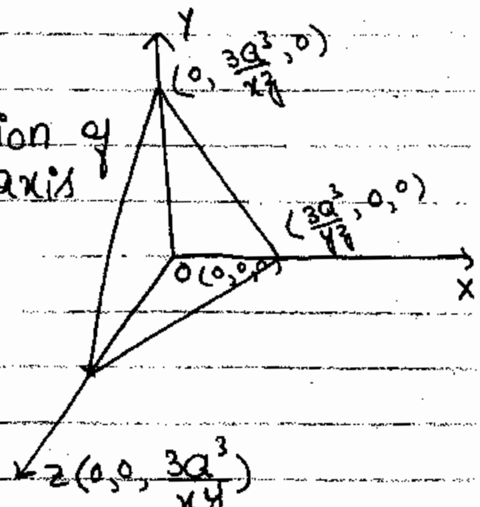
For z-intercept

$x = y = 0$ put in (2)

$\Rightarrow \quad 0 + 0 + zxy - 3a^3 = 0$

$\Rightarrow \quad zxy = 3a^3$

$\Rightarrow \quad z = \frac{3a^3}{xy}$



So, the point of intersection of the tangent plane and z-axis is $\left(0, 0, \frac{3a^3}{xy}\right)$

Now the volume of the tetrahedron to four points (x_1, y_1, z_1) (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) is

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \rightarrow (3)$$

Put all values in (3)

$$V = \frac{1}{6}\begin{vmatrix} \frac{3a^3}{xy} & 0 & 0 & 1 \\ 0 & \frac{3a^3}{xz} & 0 & 1 \\ 0 & 0 & \frac{3a^3}{yz} & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{1}{6}(1)^{4+4}\left[\frac{3a^3}{xy}\cdot\frac{3a^3}{xz}\cdot\frac{3a^3}{yz}\right]$$

$$= \frac{1}{6}\,\frac{27a^9}{x^2y^2z^2}$$

$$= \frac{9}{2}\,\frac{a^9}{(xyz)^2} = \frac{9}{2}\,\frac{a^9}{(a^3)^2}$$

$$= \frac{9}{2}\,\frac{a^9}{a^6} = \frac{9}{2}a^{9-6}$$

$$V = \frac{9}{2}a^3$$ which is a constant

Assignment :-

Find the equation of tangent plane to the surface $x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3}$ and prove that the sum of the square of the intercepts of the tangent plane with co-ordinate axis is constant

Sol :-

The given surface is

$$x^{2/3}+y^{2/3}+z^{2/3}=a^{2/3} \rightarrow (1)$$

$$\Rightarrow x^{2/3}+y^{2/3}+z^{2/3}-a^{2/3}=0$$

$$\Rightarrow F(x,y,z) = x^{2/3}+y^{2/3}+z^{2/3}-a^{2/3}=0$$

Now

$$\frac{\partial F}{\partial x} = \frac{2}{3}x^{-1/3}, \quad \frac{\partial F}{\partial y} = \frac{2}{3}y^{-1/3}, \quad \frac{\partial F}{\partial z} = \frac{2}{3}z^{-1/3}$$

Now the equation of tangent plane at any point $P(x, y, z)$ to the given surface is

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow (X-x)\left(\tfrac{2}{3}x^{-1/3}\right) + (Y-y)\left(\tfrac{2}{3}y^{-1/3}\right) + (Z-z)\left(\tfrac{2}{3}z^{-1/3}\right) = 0$$

$$\tfrac{2}{3}\left[x^{-1/3}X - x^{1-1/3} + y^{-1/3}Y - y^{1-1/3} + z^{-1/3}Z - z^{1-1/3}\right] = 0$$

$$\Rightarrow x^{-1/3}X - x^{2/3} + y^{-1/3}Y - y^{2/3} + z^{-1/3}Z - z^{2/3} = 0$$

$$\Rightarrow x^{-1/3}X + y^{-1/3}Y + z^{-1/3}Z - (x^{2/3} + y^{2/3} + z^{2/3}) = 0$$

put $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$

$$\Rightarrow x^{-1/3}X + y^{-1/3}Y + z^{-1/3}Z - a^{2/3} = 0 \rightarrow (2)$$ is eq of tangent plane

Now, for X-intercept

put $Y = Z = 0$ in (2)

$$x^{-1/3}X - a^{2/3} = 0$$

$$\Rightarrow x^{-1/3}X = a^{2/3}$$

$$\Rightarrow X = \frac{a^{2/3}}{x^{-1/3}} = a^{2/3}x^{1/3}$$

So, the point of intersection of the tangent plane and X-axis is $\left(\frac{a^{2/3}}{x^{1/2}}x^{1/3}, 0, 0\right)$

Now

For Y-intercept put $X = Z = 0$

$$\Rightarrow 0 + y^{-1/3}Y + 0 - a^{2/3} = 0$$

$$\Rightarrow y^{-1/3}Y - a^{2/3} = 0$$

$$\Rightarrow y^{-1/3}Y = a^{2/3}$$

$$\Rightarrow Y = \frac{a^{2/3}}{y^{-1/3}} = a^{2/3}y^{-1/3}$$

So, the point of intersection of the tangent plane and Y-axis is $\left(0, \frac{a^{2/3}}{y^{1}}y^{1/3}, 0\right)$

Now For Z-intercept

put $X = Y = 0$

$$\Rightarrow 0 + 0 + z^{-1/3}Z - a^{2/3} = 0$$

$$\Rightarrow z^{-1/3}Z = a^{2/3}$$

$$\Rightarrow Z = \frac{a^{2/3}}{z^{-1/3}} = a^{2/3}z^{1/3}$$

So, the point of intersection of the tangent plane and z-axis is

$$\left(0,\ 0,\ \frac{a^{2/3} z^{1/3}}{z^{1/2}}\right)$$

Now $OX = \left(\frac{a^{2/3} x^{1/3}}{x^{1/2}} - 0,\ 0-0,\ 0-0\right)$

$$OX = \frac{a^{2/3} x^{1/3}}{x^{1/2}}$$

Similarly, $OY = \frac{a^{2/3} y^{1/3}}{y^{1/2}}$, $OZ = \frac{a^{2/3} z^{1/3}}{z^{1/2}}$

squaring and adding them

$$(OX)^2 + (OY)^2 + (OZ)^2 = \left[\frac{a^{2/3} x^{1/3}}{x^{1/2}}\right]^2 + \left[\frac{a^{2/3} y^{1/3}}{y^{1/2}}\right]^2 + \left[\frac{a^{2/3} z^{1/3}}{z^{1/2}}\right]^2$$

$$(OX)^2 + (OY)^2 + (OZ)^2 = \frac{a^{4/3} x^{2/3}}{x} + \frac{a^{4/3} y^{2/3}}{y} + a^{4/3} z^{2/3}$$

$$(OX)^2 + (OY)^2 + (OZ)^2 = a^{4/3}\left[x^{2/3} + y^{2/3} + z^{2/3}\right]$$

put

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$$

$$(OX)^2 + (OY)^2 + (OZ)^2 = a^{4/3}\left(a^{2/3}\right)$$

$$= a^{4/3 + 2/3}$$

$$= a^{\frac{4+2}{3}}$$

$$= a^{6/3}$$

$$(OX)^2 + (OY)^2 + (OZ)^2 = a^2$$

which is a constant

Question:-

At a point common to the surface $a(xy+yz+xz)=xyz$ and a sphere whose centre is at origin. Prove that the tangent plane to the surface makes intercepts with the co-ordinate axis whose sum is constant.

Proof:-

The given surface is

$$F(x,y,z) = a(xy+yz+xz) - xyz = 0 \Rightarrow (1)$$

Now

$$\frac{\partial F}{\partial x} = ay - yz + az = a(y+z) - yz$$

$$\frac{\partial F}{\partial y} = ax + az - xz = a(x+z) - xz$$

$$\frac{\partial F}{\partial z} = ay + ax - xy = a(x+y) - xy$$

Now the equation of tangent plane at ~~any~~ point $P(x,y,z)$ common to the surface and the sphere is $x^2+y^2+z^2=b^2$

$$(X-x)\frac{\partial F}{\partial x} + (Y-y)\frac{\partial F}{\partial y} + (Z-z)\frac{\partial F}{\partial z} = 0$$

$$\Rightarrow (X-x)(a(y+z)-yz) + (Y-y)(a(x+z)-xz) + (Z-z)(a(x+y)-xy) = 0$$

$$\Rightarrow a(X-x)(y+z) - Xyz + xyz + a(Y-y)(x+z) - Yxz + xyz + a(Z-z)(x+y) - Zxy + xyz = 0$$

$$\Rightarrow aXy + aXz - axy - axz - Xyz + aYx + aYz - axy - ayz - Yxz + aZx + aZy - axz - ayz - Zxy + 3xyz = 0$$

$$\Rightarrow aXy + aXz - Xyz + aYx + aYz - Yxz + aZx + aZy - Zxy - axy - axy - axz - axz - ayz - ayz + 3xyz = 0$$

$$\Rightarrow (ay + az - yz)X + (ax + az - xz)Y + (az + ay - xy)Z$$

$-2axy - 2ayz - 2xz + 3xyz = 0$

$\Rightarrow x(a(y+z)-yz) + y(a(x+z)-xz) + z(a(x+y)-xy)$
$-2[a(xy+yz+xz)] + 3xyz = 0$

put $a(xy+yz+xz) = xyz$

$\Rightarrow x(a(y+z)-yz) + y(a(x+z)-xz) + z(a(x+y)-xy)$
$-2xyz + 3xyz = 0$

$\Rightarrow x(a(y+z)-yz) + y(a(x+z)-xz) + z(a(x+y)$
$-xy) + xyz = 0 \rightarrow (2)$

To find the intercept of the tangent plane with the co-ordinate axis.

For x - intercept

put $y = z = 0$ in (2)

$\Rightarrow x(a(y+z)-yz) + 0 + 0 + xyz = 0$

or intercept on x-axis is point on x-axis

$$\Rightarrow x = \frac{-xyz}{a(y+z)-yz} \quad \text{so } \left(\frac{-xyz}{a(y+z)-yz}, 0, 0\right)$$

For y - intercept

put $x = 0$, $z = 0$ in (2)

$\Rightarrow 0 + y(a(x+z)-xz) + 0 + xyz = 0$

$\Rightarrow y(a(x+z)-xz) = -xyz$

$$\Rightarrow y = \frac{-xyz}{a(x+z)-xz}$$

For z - intercept

put $x = y = 0$

$\Rightarrow 0 + 0 + z(a(x+y)-xy) + xyz = 0$

$\Rightarrow z(a(x+y)-xy) = -xyz$

$\Rightarrow z$

$$\Rightarrow z = \frac{-xyz}{a(x+y)-xy}$$

Now the sum of the intercepts

$$OX + OY + OZ = \frac{-xyz}{a(y+z)-yz} + \frac{-xyz}{a(x+z)-xz} + \frac{-xyz}{a(x+y)-xy}$$

$$OX + OY + OZ = -xyz\left[\frac{1}{a(y+z)-yz} + \frac{1}{a(x+z)-xz} + \frac{1}{a(x+y)-xy}\right]$$

$$= -xyz\left[\frac{x}{ax(y+z)-xyz} + \frac{y}{ay(x+z)-xyz} + \frac{z}{az(x+y)-xyz}\right]$$

$$= -xyz\left[\frac{x}{a(xy+xz)-xyz} + \frac{y}{a(xy+yz)-xyz} + \frac{z}{a(xz+yz)-xyz}\right]$$

From (1)

$$a(xy+xz)-xyz = -ayz$$

$$a(xy+yz)-xyz = -axz$$

$$a(xz+yz)-xyz = -axy$$

then

$$OX + OY + OZ = \left[\frac{-x^2yz}{-ayz} - \frac{xy^2z}{-axz} - \frac{xyz^2}{-axy}\right]$$

$$= \left[\frac{x^2}{a} + \frac{y^2}{a} + \frac{z^2}{a}\right]$$

$$= \frac{1}{a}\left[x^2 + y^2 + z^2\right]$$

$OX + OY + OZ = \frac{b^2}{a}$ is constant. Since the point is common to the surface and sphere.

Normal to a surface :-

The normal at any point $P(x, y, z)$ on a surface is defined as the line through point P and perpendicular to the tangent plane at point P.

Equation of the normal to a surface :-

Let $F(x, y, z) = 0 \rightarrow (1)$ be a given surface.

Since the normal at any point P on the surface is defined as the line through P and $\perp$ to the tangent plane at point P. So, the normal at point P is in the direction of $\vec{\nabla} F$.

Now, If $\vec{R}$ is the position vector of any point on the normal, then the equation of the normal to the surface (1) at a point P, with position vector $\vec{r}$ is

$$\vec{R} = \vec{r} + u \cdot \vec{\nabla} F \rightarrow (2)$$

Now, consider the co-ordinates of $\vec{R}(X, Y, Z)$ and $\vec{r}(x, y, z)$ then (2) becomes

$$\Rightarrow (X, Y, Z) = (x, y, z) + u\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

$$\Rightarrow (X - x, Y - y, Z - z) = u\left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$$

$$\Rightarrow (X - x, Y - y, Z - z) = \left(u\frac{\partial F}{\partial x}, u\frac{\partial F}{\partial y}, u\frac{\partial F}{\partial z}\right)$$

$$\Rightarrow X - x = u\frac{\partial F}{\partial x}, \quad Y - y = u\frac{\partial F}{\partial y}, \quad Z - z = u\frac{\partial F}{\partial z}$$

$$\Rightarrow \frac{X - x}{\frac{\partial F}{\partial x}} = u, \quad \frac{Y - y}{\frac{\partial F}{\partial y}} = u, \quad \frac{Z - z}{\frac{\partial F}{\partial z}} = u$$

$$\Rightarrow \frac{X - x}{\partial F/\partial x} = \frac{Y - y}{\partial F/\partial y} = \frac{Z - z}{\partial F/\partial z} = u$$

$$\Rightarrow \frac{X - x}{\partial F/\partial x} = \frac{Y - y}{\partial F/\partial y} = \frac{Z - z}{\partial F/\partial z}$$

which is the equation of normal to the surface at any point $P(x, y, z)$.

Question :-

The normal to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets at a point P meets the co-ordinate planes at points C_1, C_2, C_3 then, prove that the ratio $PC_1 : PC_2 : PC_3$ is constant

Proof :-

For the equation of normal at point P on the ellipsoid

$$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \rightarrow (1)$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2}, \quad \frac{\partial F}{\partial y}=\frac{2y}{b^2}, \quad \frac{\partial F}{\partial z}=\frac{2z}{c^2}$$

Equation of normal at any point on the ellipsoid is

$$\frac{X-x}{\partial F/\partial x}=\frac{Y-y}{\partial F/\partial y}=\frac{Z-z}{\partial F/\partial z}$$

$$\Rightarrow \frac{X-x}{\frac{2x}{a^2}}=\frac{Y-y}{\frac{2y}{b^2}}=\frac{Z-z}{\frac{2z}{c^2}}$$

$$\Rightarrow \frac{a^2(X-x)}{x}=\frac{b^2(Y-y)}{y}=\frac{c^2(Z-z)}{z}$$

Now,

The equation of the normal to the ellipsoid at point $P(x_1, y_1, z_1)$ is

$$\frac{a^2(X-x_1)}{x_1}=\frac{b^2(Y-y_1)}{y_1}=\frac{c^2(Z-z_1)}{z_1} \rightarrow (2)$$

Now, if the normal meets the XY-plane at a point C_1 then to find the co-ordinates of C_1 put $Z=0$ in (2)

$$\frac{a^2(X-x_1)}{x_1}=\frac{b^2(Y-y_1)}{y_1}=\frac{c^2(0-z_1)}{z_1}$$

$$\frac{a^2(X-x_1)}{x_1} = \frac{b^2(Y-y_1)}{y_1} = -c^2$$

$$\Rightarrow \frac{a^2(X-x_1)}{x_1} = -c^2, \quad \frac{b^2(Y-y_1)}{y_1} = -c^2$$

$$\Rightarrow a^2(X-x_1) = -c^2x_1 \qquad b^2(Y-y_1) = -c^2y_1$$

$$\Rightarrow a^2X - a^2x_1 = -c^2x_1, \quad b^2Y - b^2y_1 = -c^2y_1$$

$$\Rightarrow X = \frac{-c^2x_1 + a^2x_1}{a^2}, \quad Y = \frac{-c^2y_1 + b^2y_1}{b^2}$$

$$X = \frac{(a^2-c^2)x_1}{a^2} \qquad Y = \frac{(b^2-c^2)y_1}{b^2}$$

So, $C_1\left(\frac{a^2-c^2}{a^2}x_1, \frac{b^2-c^2}{b^2}y_1, 0\right)$

Now if the normal meets the YZ-plane at point C_2 then to find the co-ordinates of C_2 put $X=0$ in (2)

$$\frac{a^2(0-x_1)}{x_1} = \frac{b^2(Y-y_1)}{y_1} = \frac{c^2(Z-z_1)}{z_1}$$

$$-a^2 = \frac{b^2(Y-y_1)}{y_1} = \frac{c^2(Z-z_1)}{z_1}$$

$$\Rightarrow \frac{b^2(Y-y_1)}{y_1} = -a^2, \quad \frac{c^2(Z-z_1)}{z_1} = -a^2$$

$$\Rightarrow b^2(Y-y_1) = -a^2y_1, \quad c^2(Z-z_1) = -a^2z_1$$

$$\Rightarrow b^2Y - b^2y_1 = -a^2y_1, \quad c^2Z - c^2z_1 = -a^2z_1$$

$$\Rightarrow b^2Y = b^2y_1 - a^2y_1, \quad c^2Z = c^2z_1 - a^2z_1$$

$$\Rightarrow Y = \frac{(b^2-a^2)y_1}{b^2}, \quad Z = \frac{(c^2-a^2)z_1}{c^2}$$

So, $C_2\left(0, \frac{b^2-a^2}{b^2}y_1, \frac{c^2-a^2}{c^2}z_1\right)$

Now if the normal meets the xz-plane at point C_3 then to find the co-ordinates of C_3 put $Y=0$ in (2)

$$\frac{a^2(x-x_1)}{x_1} = \frac{b^2(0-y_1)}{y_1} = \frac{c^2(z-z_1)}{z_1}$$

$$\Rightarrow \frac{a^2(x-x_1)}{x_1} = -b^2 = \frac{c^2(z-z_1)}{z_1}$$

$$\Rightarrow \frac{a^2(x-x_1)}{x_1} = -b^2 \quad , \quad \frac{c^2(z-z_1)}{z_1} = -b^2$$

$$\Rightarrow a^2(x-x_1) = -b^2 x_1 \quad , \quad c^2(z-z_1) = -b^2 z_1$$

$$\Rightarrow a^2 x - a^2 x_1 = -b^2 x_1 \quad , \quad c^2 z - c^2 z_1 = -b^2 z_1$$

$$\Rightarrow a^2 x = a^2 x_1 - b^2 x_1 \quad , \quad c^2 z = c^2 z_1 - b^2 z_1$$

$$\Rightarrow x = \frac{(a^2-b^2)x_1}{a^2} \quad , \quad z = \frac{(c^2-b^2)}{c^2} z_1$$

So $C_3\left(\frac{a^2-b^2}{a^2}x_1 \,,\, 0 \,,\, \frac{c^2-b^2}{c^2} z_1\right)$

Now

we find PC_1, PC_2, and PC_3

$P(x_1, y_1, z_1)$

$$PC_1 = \sqrt{\left(\frac{a^2-c^2}{a^2}x_1 - x_1\right)^2 + \left(\frac{b^2-c^2}{b^2}y_1 - y_1\right)^2 + (0-z_1)^2}$$

$$= \sqrt{\left[\frac{(a^2-c^2)x_1 - a^2 x_1}{a^2}\right]^2 + \left[\frac{(b^2-c^2)y_1 - b^2 y_1}{b^2}\right]^2 + (-z_1)^2}$$

$$= \sqrt{\left[\frac{a^2x_1 - c^2x_1 - a^2x_1}{a^2}\right]^2 + \left[\frac{b^2y_1 - c^2y_1 - b^2y_1}{b^2}\right]^2 + z_1^2}$$

$$= \sqrt{\left(-\frac{c^2x_1}{a^2}\right)^2 + \left(-\frac{c^2y_1}{b^2}\right)^2 + z_1^2}$$

$$= \sqrt{\frac{c^4x_1^2}{a^4} + \frac{c^4y_1^2}{b^4} + z_1^2}$$

$$= \sqrt{\frac{c^4x_1^2}{a^4} + \frac{c^4y_1^2}{b^4} + \frac{c^4z_1^2}{c^4}} = \sqrt{c^4\left[\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\right]}$$

$$PC_1 = c^2\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}$$

$$PC_2 = \sqrt{(0-x_1)^2 + \left(\left(\frac{b^2-a^2}{b^2}\right)y_1 - y_1\right)^2 + \left(\left(\frac{c^2-a^2}{c^2}\right)z_1 - z_1\right)^2}$$

$$= \sqrt{x_1^2 + \left(\frac{b^2 y_1 - a^2 y_1 - b^2 y_1}{b^2}\right)^2 + \left(\frac{c^2 z_1 - a^2 z_1 - c^2 z_1}{c^2}\right)^2}$$

$$= \sqrt{x_1^2 + \left(\frac{-a^2 y_1}{b^2}\right)^2 + \left(\frac{-a^2 z_1}{c^2}\right)^2}$$

$$= \sqrt{\frac{a^4 x_1^2}{a^4} + \frac{a^4 y_1^2}{b^4} + \frac{a^4 z_1^2}{c^4}}$$

$$PC_2 = a^2 \sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}$$

$$PC_3 = \sqrt{\left(\left(\frac{a^2-b^2}{a^2}\right)x_1 - x_1\right)^2 + (0-y_1)^2 + \left(\left(\frac{c^2-b^2}{c^2}\right)z_1 - z_1\right)^2}$$

$$= \sqrt{\left(\frac{a^2 x_1 - b^2 x_1 - a^2 x_1}{a^2}\right)^2 + y_1^2 + \left(\frac{c^2 z_1 - b^2 z_1 - c^2 z_1}{c^2}\right)^2}$$

$$= \sqrt{\left(\frac{-b^2 x_1}{a^2}\right)^2 + y_1^2 + \left(\frac{-b^2 z_1}{c^2}\right)^2}$$

$$= \sqrt{\frac{b^4 x_1^2}{a^4} + \frac{b^4 y_1^2}{b^4} + \frac{b^4 z_1^2}{c^4}}$$

$$PC_3 = b^2 \sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}$$

Now we Prove that

$PC_1 : PC_2 : PC_3 = \text{Constant}$

$$PC_1 : PC_2 : PC_3 = c^2\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}} : a^2\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}} : b^2\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}$$

$PC_1 : PC_2 : PC_3 = c^2 : a^2 : b^2$ which is Constant

As

$$\frac{PC_1}{PC_2} = \frac{c^2\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}}{a^2\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+z_1^2/c^4}} = \frac{c^2}{a^2}$$

$$\frac{PC_2}{PC_1} = \frac{a^2\sqrt{x_1^2/a^4+y_1^2/b^4+z_1^2/c^4}}{b^2\sqrt{x_1^2/a^4+y_1^2/b^4+z_1^2/c^4}} = \frac{a^2}{b^2}$$

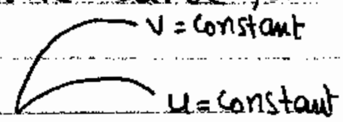
Curvilinear Co-ordinates for a surface and parametric curves:-

Curvilinear co-ordinate system is referred to a co-ordinate system whose co-ordinate axis are not straight lines. It is a system in which co-ordinate axis are curved lines

we know that a surface is defined as the locus of point whose position vector is a function of two independent parametres say u and v, and in this case, the rectangular co-ordinates of a point on a surface are functions of parametres u and v.

If we elliminate these parametres (from parametric eqs) we obtain a single equation, known as an implicite equation for a surface.

A curve on a surface along which one of the parametres remains Constant, is known as parametric curve for a surface.



The parametric curves

u = constant and v = constant generate the curvilinear co-ordinate system for a point on a surface.

Notations:-

The suffix "1" is used to indicate the partial derivatives w.r.t "u".

The suffix "2" is used to indicate the partial derivatives w.r.t "v"

(i-e) If $\vec{r}$ is the position vector of any point on a surface

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u}, \quad \vec{r}_2 = \frac{\partial \vec{r}}{\partial v}$$

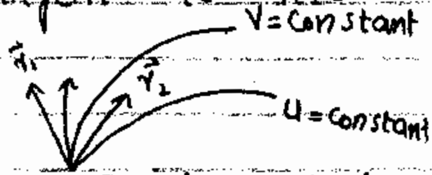
$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v}, \quad \vec{r}_{21} = \frac{\partial^2 \vec{r}}{\partial v \partial u}$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2}, \quad \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2}$$

$$\Rightarrow \vec{r}_{12} = \vec{r}_{21} = \frac{\partial^2 \vec{r}}{\partial u \partial v}$$

Remark:-

For parametric curve u = constant, v = constant on a surface, $\vec{r}_1$ is along the $\vec{r}_1$ tangent to the curve v = constant and $\vec{r}_2$ is along the tangent to the curve u = constant.



v = constant, tangent $\vec{r}_1 = \frac{\partial \vec{r}}{\partial u}$

u = constant, tangent $\vec{r}_2 = \frac{\partial \vec{r}}{\partial v}$

Metric on a surface, First Fundamental form for a surface, First Fundamental magnitude:-

Let $P(\vec{r})$ and $Q(\vec{r}+d\vec{r})$ be two neighbouring points on a surface with parametres u and v.

Now, $d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv$

$$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv$$

Since, the point P and Q are very close to each other, so, the elements ds of arc between P and Q is equal to $|d\vec{r}|$

$$\Rightarrow \quad |ds| = |d\vec{r}|$$

then $\quad (ds)^2 = |d\vec{r}|^2 = |\vec{r}_1 du + \vec{r}_2 dv|^2$

$$\Rightarrow (ds)^2 = (\vec{r}_1 du + \vec{r}_2 dv)\cdot(\vec{r}_1 du + \vec{r}_2 dv)$$

$$= \vec{r}_1^{\,2}(du)^2 + \vec{r}_1\cdot\vec{r}_2 du dv + \vec{r}_2\cdot\vec{r}_1 dv du + r_2^2 (dv)^2$$

$$(ds)^2 = r_1^2 (du)^2 + 2\vec{r}_1\cdot\vec{r}_2 du dv + r_2^2 (dv)^2$$

$$\Rightarrow (ds)^2 = E du^2 + 2F du dv + G dv^2 \rightarrow (A)$$

where $E = r_1^2$, $F = \vec{r}_1\cdot\vec{r}_2$, $G = r_2^2$

$$\Rightarrow ds = \sqrt{E du^2 + 2F du dv + G dv^2} \rightarrow (1)$$

Eq (A) is known as Ist Fundamental form for a surface or is known as a metric on a surface.

And $E = r_1^2$, $F = \vec{r}_1\cdot\vec{r}_2$, $G = r_2^2$ are known as Ist order magnitude or fundamental magnitudes of Ist order.

In case of parametric curve

$V =$ Constant, we have $dv = 0$ So, by (1)

$$ds = \sqrt{E du^2}$$

$$ds = \sqrt{E}\, du$$

$U =$ Constant, we have $du = 0$. So, by (1)

$$ds = \sqrt{G dv^2}$$

$$ds = \sqrt{G}\, dv$$

Hence, the element ds of arc length along parametric curves $V =$ constant and $u =$ constant are $\sqrt{E}\, du$ and $\sqrt{G}\, dv$ respectively.

Remarks:-

(i) $EG - F^2 = r_1^2 r_2^2 - (\vec{r}_1\cdot\vec{r}_2)^2$

If ω is the angle b/w $\vec{r}_1$ and $\vec{r}_2$ then

$$\vec{r}_1\cdot\vec{r}_2 = |\vec{r}_1||\vec{r}_2|\cos\omega$$

$$EG - F^2 = r_1^2 r_2^2 - r_1^2 r_2^2 \cos^2 \omega$$
$$EG - F^2 = r_1^2 r_2^2 (1 - \cos^2 \omega)$$
$$EG - F^2 = r_1^2 r_2^2 \sin^2 \omega$$
$$EG - F^2 = |\vec{r}_1^2 \times \vec{r}_2^2|^2 \geq 0$$
$$\Rightarrow \quad EG - F^2 \geq 0$$
and we denote $H = \sqrt{EG - F^2}$
$$\Rightarrow H = |\vec{r}_1 \times \vec{r}_2|$$
And if $\vec{a}$ and $\vec{b}$ are unit tangent vectors along the curves $u =$ constant and $v =$ constant then $\vec{a} = \dfrac{\vec{r}_2}{|\vec{r}_2|}$ and $\vec{b} = \dfrac{\vec{r}_1}{|\vec{r}_1|}$

put $|\vec{r}_1| = \sqrt{E}$, $|\vec{r}_2| = \sqrt{G}$
then
$$\vec{a} = \frac{\vec{r}_2}{\sqrt{G}}, \quad \vec{b} = \frac{\vec{r}_1}{\sqrt{E}}$$

(ii) If ω is the angle between the parametric curves
$U =$ Constant, $V =$ Constant on a Surface
then $\cos\omega = \dfrac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|}$
$$\cos\omega = \frac{F}{\sqrt{E}\sqrt{G}} = \frac{F}{\sqrt{EG}}$$
If the parametric curves cut at the right angle then $\cos\omega = 0$
$$\Rightarrow \frac{F}{\sqrt{EG}} = 0$$
$$\Rightarrow F = 0$$
If the parametric curves cut each other at the right angle then they form an orthogonal system
Hence, the necessary and sufficient Condition for parametric curve to be

orthogonal is $F=0$

Now $\quad \sin^2\omega = 1-\cos^2\omega$

$$= 1 - \frac{F^2}{EG}$$

$$\sin^2\omega = \frac{EG-F^2}{EG}$$

$$\sin\omega = \frac{\sqrt{EG-F^2}}{\sqrt{EG}}$$

$$\sin\omega = \frac{H}{\sqrt{EG}}$$

$$\Rightarrow \tan\omega = \frac{\sin\omega}{\cos\omega}$$

$$\tan\omega = \frac{\frac{H}{\sqrt{EG}}}{F/\sqrt{EG}}$$

$$\tan\omega = \frac{H}{F}$$

Question:-

For the surface given by $\vec{r} = (u\cos v, u\sin v, f(u))$

(i) Find the fundamental form of Ist order and Ist order mag

(ii) Prove that the parametric curves on the surface are orthogonal

Proof:-

$$\vec{r} = (u\cos v, u\sin v, f(u))$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = (\cos v, \sin v, f'(u))$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = (-u\sin v, u\cos v, 0)$$

we know

$$E = \vec{r}_1 \cdot \vec{r}_1$$

$$E = (\cos v, \sin v, f'(u)) \cdot (\cos v, \sin v, f'(u))$$

$\vec{E} = \cos^2 v + \sin^2 v + (f'(u))^2$

$E = 1 + f'^2(u)$

$F = \vec{r}_1 . \vec{r}_2$

$F = (\cos v, \sin v, f'(u)) . (-u\sin v, u\cos v, 0)$

$F = -u\sin v\cos v + u\sin v\cos v + 0$

$F = 0$

$G = \vec{r}_2 . \vec{r}_2$

$G = (-u\sin v, u\cos v, 0) . (-u\sin v, u\cos v, 0)$

$G = u^2\sin^2 v + u^2\cos^2 v = u^2(\sin^2 v + \cos^2 v) = u^2$

The fundamental form of Ist order for a Surface is

$(ds)^2 = Edu^2 + 2Fdudv + Gdv^2$

$= (1 + f'^2(u))du^2 + 2(0)dudv + u^2dv^2$

$(ds)^2 = (1 + f'^2(u))du^2 + u^2dv^2$

which is the fundamental form of Ist order for the given surface

Since, $F = 0$ So the parametric curves form an orthogonal system.

And the Ist order magnitude for the surface are

$E = 1 + f'^2(u) \quad F = 0 \quad G = u^2$

Question:-

Find the fundamental form of Ist order and Ist order magnitude for the surface $\vec{r} = (a\cos u\cos v, a\cos u\sin v, a\sin u)$

Proof:-

$\vec{r} = (a\cos u\cos v, a\cos u\sin v, a\sin u)$

$\vec{r}_1 = \dfrac{\partial \vec{r}}{\partial u} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

$\vec{r}_2 = \dfrac{\partial \vec{r}}{\partial v} = (-a\cos u\sin v, -a\cos u\cos v, 0)$

we know $E = \vec{r}_1 . \vec{r}_1$

$E = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)\cdot(-a\sin u\cos v, -a\sin u\sin v, a\cos u)$

$E = a^2\sin^2 u\cos^2 v + a^2\sin^2 u\sin^2 v + a^2\cos^2 u$

$E = a^2\sin^2 u(\cos^2 v + \sin^2 v) + a^2\cos^2 u$

$= a^2\sin^2 u + a^2\cos^2 u = a^2(\sin^2 u + \cos^2 u)$

$E = a^2$

$F = \vec{r}_1\cdot\vec{r}_2$

$= (-a\sin u\cos v, -a\sin u\sin v, a\cos u)\cdot(-a\cos u\sin v, a\cos u\cos v, 0)$

$F = a^2\sin u\cos u\sin v\cos v + a^2\sin u\cos u\sin v\cos v$

$F = 0$

$G = \vec{r}_2\cdot\vec{r}_2$

$= (-a\cos u\sin v, a\cos u\cos v, 0)\cdot(-a\cos u\sin v, a\cos u\cos v, 0)$

$= a^2\cos^2 u\sin^2 v + a^2\cos^2 u\cos^2 v + 0$

$= a^2\cos^2 u(\sin^2 v + \cos^2 v)$

$G = a^2\cos^2 u$

The fundamental form of Ist order for a surface is

$(ds)^2 = E du^2 + 2F du dv + G dv^2$

$= a^2 du^2 + 2(0) du dv + a^2\cos^2 u dv^2$

$(ds)^2 = a^2 du^2 + a^2\cos^2 u dv^2$

which is the fundamental form of Ist order ~~for~~ a given surface.

The Ist order magnitude for the given surface are

$E = a^2, \quad F = 0, \quad G = a^2\cos^2 u$

Since, $F = 0$

So the parametric curves of the given surface form an orthogonal system.

Directions on a surface:-

Any direction on a surface from a fixed point (u,v), $\vec{r}=\vec{r}(u,v)$ is determined by the increments du and dv in the parameters for the small displacement $d\vec{r}$ in that direction.

Sol:-

Let du and dv be the increments for the displacement $d\vec{r}$ in a direction of a surface and δu and δv be the incriments for the displacement $\delta\vec{r}$ in another direction on a surface.

Now, if ψ is the angle between these two directions which are taken above, then we have

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u}du + \frac{\partial \vec{r}}{\partial v}dv$$

$$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv \rightarrow (i)$$

$$\Rightarrow \quad \delta\vec{r} = \vec{r}_1 \delta u + \vec{r}_2 \delta v \rightarrow (ii)$$

Now $\quad d\vec{r}\cdot\delta\vec{r} = |d\vec{r}||\delta\vec{r}|\cos\psi$

$$\Rightarrow (\vec{r}_1 du+\vec{r}_2 dv)\cdot(\vec{r}_1\delta u+\vec{r}_2\delta v) = ds\,\delta s\cos\psi$$

$$\Rightarrow r_1^2 du\,\delta u + \vec{r}_1\cdot\vec{r}_2 du\,\delta v + \vec{r}_2\cdot\vec{r}_1 dv\,\delta u + r_2^2 dv\,\delta v = ds\,\delta s\cos\psi$$

we know $E=r_1^2$, $F=\vec{r}_1\cdot\vec{r}_2=\vec{r}_2\cdot\vec{r}_1$, $G=r_2^2$

$$\Rightarrow E du\,\delta u + F du\,\delta v + F dv\,\delta u + G dv\,\delta v = ds\,\delta s\cos\psi$$

$$\Rightarrow E du\,\delta u + F(du\,\delta v + dv\,\delta u) + G dv\,\delta v = ds\,\delta s\cos\psi \rightarrow (i)$$

Now

$$|d\vec{r}\times\delta\vec{r}| = |d\vec{r}||\delta\vec{r}|\sin\psi$$

$$\Rightarrow |d\vec{r}||\delta\vec{r}|\sin\psi = |d\vec{r}\times\delta\vec{r}|$$

$$ds\,\delta s\sin\psi = |(\vec{r}_1 du + \vec{r}_2 dv)\times(\vec{r}_1\delta u + \vec{r}_2\delta v)|$$

$$ds\,\delta s\sin\psi = |(0\cdot du\,\delta u) + (\vec{r}_1\times\vec{r}_2)du\,\delta v + 0\cdot dv\,\delta v + (\vec{r}_2\times\vec{r}_1)dv\,\delta u|$$

$$ds\delta s \sin\psi = |(\vec{r}_1 \times \vec{r}_2)(du\delta v - dv\delta u)|$$

$$= |\vec{r}_1 \times \vec{r}_2|(du\delta v - dv\delta u)$$

we know $H = |\vec{r}_1 \times \vec{r}_2|$

$$ds\delta s \sin\psi = H(du\delta v - dv\delta u) \rightarrow (2)$$

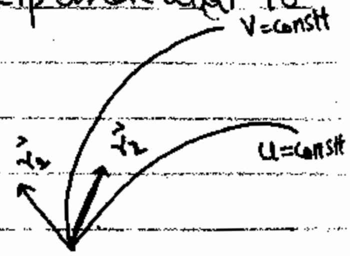
$$\frac{ds\delta s \sin\psi}{ds\delta s \cos\psi} = \frac{H(du\delta v - dv\delta u)}{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}$$

$$\frac{\sin\psi}{\cos\psi} = \frac{H(du\delta v - dv\delta u)}{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}$$

$$\tan\psi = \frac{H(du\delta v - dv\delta u)}{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}$$

Normal for a surface :-

The normal at any point on a surface is perpandicular to every tangent line through that point. Hence, the normal to a surface is perpandicular to the vectors $\vec{r}_1$ and $\vec{r}_2$.



Hence the normal is in the direction of $\vec{r}_1 \times \vec{r}_2$ and the positive direction of this vector is along the normal to the surface.

Hence, if $\vec{N}$ is the unit vector along the normal to the surface, then

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$$

And also $\vec{N} \cdot \vec{r}_1 = 0$ and $\vec{N} \cdot \vec{r}_2 = 0$ and

$$[\vec{N}\ \vec{r}_1\ \vec{r}_2] = \vec{N}\cdot(\vec{r}_1 \times \vec{r}_2)$$

$$= \vec{N}\cdot\vec{N}H$$

$$[\vec{N}\ \vec{r}_1\ \vec{r}_2] = 1\cdot H = H$$

$$\vec{N}\times(\vec{r}_1\times\vec{r}_2) = (\vec{N}\cdot\vec{r}_2)\vec{r}_1 - (\vec{N}\cdot\vec{r}_1)\vec{r}_2$$

$$\vec{N}\times\vec{r}_1 = \frac{\vec{r}_1\times\vec{r}_2}{H}\times\vec{r}_1$$

$$= \frac{1}{H}\{(\vec{r}_1\times\vec{r}_2)\times\vec{r}_1\}$$

$$= \frac{1}{H}\{(\vec{r}_1\cdot\vec{r}_1)\vec{r}_2 - (\vec{r}_2\cdot\vec{r}_1)\vec{r}_1\}$$

$$\vec{N}\times\vec{r}_1 = \frac{1}{H}\{E\vec{r}_2 - F\vec{r}_1\}$$

$$\vec{N}\times\vec{r}_2 = \frac{\vec{r}_1\times\vec{r}_2}{H}\times\vec{r}_2$$

$$= \frac{1}{H}\{(\vec{r}_1\times\vec{r}_2)\times\vec{r}_2\}$$

$$= \frac{1}{H}\{(\vec{r}_1\cdot\vec{r}_2)\vec{r}_2 - (\vec{r}_2\cdot\vec{r}_2)\vec{r}_1\}$$

$$\vec{N}\times\vec{r}_2 = \frac{1}{H}\{F\vec{r}_2 - G\vec{r}_1\}$$

Question:-

Find the tangent of the angle between two directions on a surface determined by a surface $Pdu^2 + Qdudv + Rdv^2 = 0$.

Sol:-

$$Pdu^2 + Qdudv + Rdv^2 = 0 \rightarrow (1)$$

Let $d\vec{r}$ be the displacement corresponding to the increment du and dv and $\delta\vec{r}$ be the displacement corresponding to the increments δu and δv in the parametres in two directions on the given surface determined by the eq (1)

Dividing eq (1) by dv^2, we have

$$P\left(\frac{du}{dv}\right)^2 + Q\frac{du\,dv}{dv^2} + R\frac{dv^2}{dv^2} = 0$$

$$P\left(\frac{du}{dv}\right)^2 + Q\frac{du}{dv} + R = 0 \rightarrow (2)$$

If $\frac{du}{dv}$ and $\frac{\delta u}{\delta v}$ are the roots of eq (2)

then, sum of the roots $= \frac{du}{dv} + \frac{\delta u}{\delta v} = -\frac{Q}{P} \rightarrow (3)$

and product of the roots $= \frac{du}{dv}\,\frac{\delta u}{\delta v} = \frac{R}{P} \rightarrow (4)$

Difference of the roots $= \left(\frac{du}{dv} - \frac{\delta u}{\delta v}\right)^2 = \left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right)^2 - 4\frac{du}{dv}\frac{\delta u}{\delta v}$

then

$$= \frac{du}{dv} - \frac{\delta u}{\delta v} = \sqrt{\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right)^2 - 4\frac{du}{dv}\frac{\delta u}{\delta v}}$$

$$= \sqrt{\left(\frac{Q}{P}\right)^2 - 4\frac{R}{P}}$$

$$\frac{du}{dv} - \frac{\delta u}{\delta v} = \sqrt{\frac{Q^2}{P^2} - \frac{4R}{P}} = \sqrt{\frac{Q^2 - 4RP}{P^2}}$$

Difference of roots $= \sqrt{Q^2 - 4RP}\,/P \rightarrow (5)$

If ψ is the angle between two directions on the surface then

$$ds\,\delta s\cos\theta = E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v \rightarrow (6)$$

$$ds\,\delta s\sin\theta = H(du\,\delta v - dv\,\delta u) \rightarrow (7)$$

Divinde (7) by (6)

$$\frac{\sin\theta}{\cos\theta} = \frac{H(du\,\delta v - dv\,\delta u)}{E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v}$$

Multiplying and dividing R.H.S by $dv\,\delta v$

$$\frac{\sin\theta}{\cos\theta} = \frac{H\left(\frac{du}{dv} - \frac{\delta u}{\delta v}\right)}{E\frac{du}{dv}\frac{\delta u}{\delta v} + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G} \quad \rightarrow (8)$$

Putting all values from (3), (4), (5) in (8)

$$\tan\theta = \frac{H\left(\frac{\sqrt{Q^2-4RP}}{P}\right)}{E\left(\frac{R}{P}\right) + F\left(-\frac{Q}{P}\right) + G}$$

$$= \frac{\frac{H\sqrt{Q^2-4RP}}{P}}{\frac{ER - QF + GP}{P}}$$

$$\tan\theta = \frac{H\sqrt{Q^2-4RP}}{ER - QF + GP}$$

Question:-

If θ is the angle between a direction on a surface and the curve $u =$ Constant, then prove that

$$\cos\theta = \frac{1}{\sqrt{G}}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right), \quad \sin\theta = \frac{H}{\sqrt{G}}\frac{du}{ds}$$

Proof:-

Let $d\vec{r}$ be the displacement corresponding to the increments du, dv in the parametre's u and v in a direction on a surface and $\delta\vec{r}$ be the displacement in the direction $u =$ Const.

Then, $d\vec{r} = \vec{r}_1\, du + \vec{r}_2\, dv$ → constant

$$\Rightarrow \quad \delta\vec{r} = \vec{r}_1\, \delta u + \vec{r}_2\, \delta v$$

$\delta u = 0$ because $u = $ constant

So, $\delta\vec{r} = \vec{r}_2\, \delta v \rightarrow (2)$

Now $\delta s = |\delta\vec{r}|$

$$\delta s = |\vec{r}_2|\, \delta v$$

$$\delta s = \sqrt{G}\, \delta v \rightarrow (3)$$

If θ is the angle between two directions then, we know that

$$ds\, \delta s \cos\theta = E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v$$

put $\delta s = \sqrt{G}\,\delta v$ From (3)

$$ds\,(\sqrt{G}\,\delta v)\cos\theta = E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v$$

put $\delta u = 0$

$$ds\,(\sqrt{G}\,\delta v)\cos\theta = 0 + F(du\,\delta v + 0) + G\,dv\,\delta v$$

$$\cos\theta = \frac{F\,du\,\delta v + G\,dv\,\delta v}{ds\,\sqrt{G}\,\delta v}$$

$$\cos\theta = \frac{1}{\sqrt{G}}\left[\frac{F\,du\,\delta v}{ds\,\delta v} + \frac{G\,dv\,\delta v}{ds\,\delta v}\right]$$

$$\cos\theta = \frac{1}{\sqrt{G}}\left[F\frac{du}{ds} + G\frac{dv}{ds}\right]$$

Similarly, For $\sin\theta$

$$ds\,\delta s \sin\theta = H(du\,\delta v - dv\,\delta u)$$

put $\delta s = \sqrt{G}\,\delta v$ and $\delta u = 0$

$$ds\,(\sqrt{G}\,\delta v)\sin\theta = H(du\,\delta v - 0)$$

$$\sin\theta = \frac{H\,du\,\delta v}{\sqrt{G}\,ds\,\delta v}$$

$$\sin\theta = \frac{H}{\sqrt{G}}\frac{du}{ds}$$ This θ: for $v = $ constt

2. **Second order magnitudes:-**

Second order magnitudes for a surface $\vec{r} = r(u,v)$ are determined by the resolved parts of the second order derivatives of $\vec{r} = \vec{r}(u,v)$ in the direction of the normal to the surface. and are denoted by L, M and N.

Here $L = \vec{N} \cdot \vec{r}_{11}$ where $\vec{N}$ is unit normal to the surface and $\vec{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{H}$

$M = \vec{N} \cdot \vec{r}_{12}$

$N = \vec{N} \cdot \vec{r}_{22}$

where $\vec{r}_{11} = \dfrac{\partial^2 \vec{r}_1}{\partial u^2}$, $\vec{r}_{12} = \dfrac{\partial^2 \vec{r}_1}{\partial u \partial v}$, $\vec{r}_{22} = \dfrac{\partial^2 \vec{r}_1}{\partial v^2}$

And we denote $T^2 = LN - M^2$

Remark:-

$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{11}] = \vec{r}_1 \cdot \vec{r}_2 \times \vec{r}_{11}$$
$$= (\vec{r}_1 \times \vec{r}_2) \cdot \vec{r}_{11}$$

put $\vec{r}_1 \times \vec{r}_2 = \vec{N} H$

$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{11}] = \vec{N} H \cdot \vec{r}_{11}$$
$$= H(\vec{N} \cdot \vec{r}_{11})$$
$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{11}] = HL$$

$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{12}] = \vec{r}_1 \cdot \vec{r}_2 \times \vec{r}_{12}$$
$$= (\vec{r}_1 \times \vec{r}_2) \cdot \vec{r}_{12}$$
$$= \vec{N} H \cdot \vec{r}_{12}$$
$$= H(\vec{N} \cdot \vec{r}_{12})$$
$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{12}] = HM$$

$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{22}] = \vec{r}_1 \cdot \vec{r}_2 \times \vec{r}_{22}$$
$$= (\vec{r}_1 \times \vec{r}_2) \cdot \vec{r}_{22}$$
$$= \vec{N} H \cdot \vec{r}_{22}$$
$$= H(\vec{N} \cdot \vec{r}_{22})$$
$$[\vec{r}_1 \ \vec{r}_2 \ \vec{r}_{22}] = HN$$

Question:-

Find the fundamental magnitudes for the surface given by the equations $x = u\cos\phi$, $y = u\sin\phi$, $z = c\phi$

where u and ϕ are the parametres.

Sol:-

$$\vec{r} = (x, y, z)$$

$$\vec{r} = (u\cos\phi, u\sin\phi, c\phi) \rightarrow (1)$$

we have to find E, F, G, L, M, N

we know $E = \vec{r}_1 \cdot \vec{r}_1 \rightarrow (2)$

$$F = \vec{r}_1 \cdot \vec{r}_2 \rightarrow (3) \qquad G = \vec{r}_2 \cdot \vec{r}_2 \rightarrow (4)$$

$$L = \vec{N} \cdot \vec{r}_{11} \rightarrow (5), \quad M = \vec{N} \cdot \vec{r}_{12} \rightarrow (6), \quad N = \vec{N} \cdot \vec{r}_{22} \rightarrow (7)$$

and $\vec{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{H} \rightarrow (8)$ and $H = |\vec{r}_1 \times \vec{r}_2| \rightarrow (9)$

Differentiate eq (1) w.r.t "u" and "ϕ"

So, $\vec{r}_1 = \dfrac{\partial \vec{r}}{\partial u} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = -u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + c\hat{k}$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = 0, \quad \vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial \phi^2} = -u\cos\phi\,\hat{i} - u\sin\phi\,\hat{j}$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u\,\partial\phi} = \frac{\partial}{\partial u}\left(\frac{\partial \vec{r}}{\partial \phi}\right) = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & 0 \\ -u\sin\phi & u\cos\phi & c \end{vmatrix}$$

$$= \hat{i}(c\sin\phi - 0) - \hat{j}(c\cos\phi - 0) + \hat{k}(u\cos^2\phi + u\sin^2\phi)$$

$$\vec{r}_1 \times \vec{r}_2 = c\sin\phi\,\hat{i} - c\cos\phi\,\hat{j} + u\hat{k}$$

$$H^2 = |\vec{r}_1 \times \vec{r}_2|^2 = c^2\sin^2\phi + c^2\cos^2\phi + u^2$$

$$H = \sqrt{c^2(\sin^2\phi + \cos^2\phi) + u^2}$$

$$H = \sqrt{c^2 + u^2}$$

put all values in (2), (3), (4), (5), (6), (7), (8), (9)

$$E = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) = \cos^2\phi + \sin^2\phi = 1$$

$$F = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot (-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + c\hat{k}) = 0$$

$$G = (-u\sin\phi\,\hat i + u\cos\phi\,\hat j + c\hat k)\cdot(-u\sin\phi\,\hat i + u\cos\phi\,\hat j + c\hat k)$$

$$G = u^2 + c^2$$

$$\vec N = \frac{c\sin\phi\,\hat i - c\cos\phi\,\hat j + u\hat k}{\sqrt{c^2+u^2}}$$

$$\vec N \cdot \vec r_{11} = \frac{c\sin\phi\,\hat i - c\cos\phi\,\hat j + u\hat k}{\sqrt{c^2+u^2}} \cdot 0 = 0$$

$$M = \vec N \cdot \vec r_{12} = \frac{c\sin\phi\,\hat i - c\cos\phi\,\hat j + u\hat k}{\sqrt{c^2+u^2}} \cdot (-\sin\phi\,\hat i + \cos\phi\,\hat j) = -c/\sqrt{c^2+u^2}$$

$$N = \vec N \cdot \vec r_{22} = \frac{c\sin\phi\,\hat i - c\cos\phi\,\hat j + u\hat k}{\sqrt{c^2+u^2}} \cdot (-u\cos\phi\,\hat i - u\sin\phi\,\hat j) = 0$$

Question:-

Find the fundamental magnitudes of the given surface

$x = a(u+v)$, $y = b(u+v)$, $z = uv$

Sol:-

$\vec r = (x, y, z)$

$\vec r = (a(u+v),\ b(u+v),\ uv) \rightarrow (1)$

we have to find E, F, G, L, M and N

we know

$E = \vec r_1 \cdot \vec r_1 \rightarrow (2)$, $F = \vec r_1 \cdot \vec r_2 \rightarrow (3)$, $G = \vec r_2 \cdot \vec r_2 \rightarrow (4)$

$L = \vec N \cdot \vec r_{11} \rightarrow (5)$, $M = \vec N \cdot \vec r_{12} \rightarrow (6)$, $N = \vec N \cdot \vec r_{22} \rightarrow (7)$

$$\vec N = \frac{\vec r_1 \times \vec r_2}{|\vec r_1 \times \vec r_2|} \rightarrow (8)$$

Differentiate eq (1) w.r.t "u", "v"

$$\vec r_1 = \frac{\partial \vec r}{\partial u} = (a(1+0),\ b(1+0),\ v)$$

$$\vec r_1 = a\hat i + b\hat j + v\hat k$$

$$\vec r_2 = \frac{\partial \vec r}{\partial v} = a(0+1)\hat i + b(0+1)\hat j + u\hat k = a\hat i + b\hat j + u\hat k$$

$$\vec r_{12} = \frac{\partial^2 \vec r}{\partial u \partial v} = \frac{\partial \vec r}{\partial u}\left(\frac{\partial \vec r}{\partial v}\right) = \frac{\partial}{\partial u}(a\hat i + b\hat j + u\hat k)$$

$$\vec{r}_{12} = \hat{k}$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial \vec{r}}{\partial u}\right) = \frac{\partial}{\partial u}(a\hat{i}+b\hat{j}+v\hat{k}) = 0$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial \vec{r}}{\partial v}\right) = \frac{\partial}{\partial v}(a\hat{i}+b\hat{j}+u\hat{k}) = 0$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & b & u \end{vmatrix}$$

$$= \hat{i}(bu-bv) - \hat{j}(au-av) + \hat{k}(ab-ab)$$

$$= \hat{i}\,b(u-v) - \hat{j}\,a(u-v) + 0$$

$$\vec{r}_1 \times \vec{r}_2 = b(u-v)\hat{i} - a(u-v)\hat{j} = (u-v)[b\hat{i}-a\hat{j}]$$

$$|r_1 \times \vec{r}_2| = \sqrt{b^2(u^2-v^2)^2 + a^2(u^2-v^2)^2}$$

$$= \sqrt{(u-v)^2(b^2+a^2)}$$

$$|\vec{r}_1 \times \vec{r}_2| = (u-v)\sqrt{b^2+a^2}$$

put all values in (2), (3), (4), (5), (6), (7)...

$$E = (a\hat{i}+b\hat{j}+v\hat{k})\cdot(a\hat{i}+b\hat{j}+v\hat{k})$$

$$E = a^2+b^2+v^2$$

$$F = (a\hat{i}+b\hat{j}+v\hat{k})\cdot(a\hat{i}+b\hat{j}+u\hat{k})$$

$$F = a^2+b^2+uv$$

$$G = (a\hat{i}+b\hat{j}+u\hat{k})\cdot(a\hat{i}+b\hat{j}+u\hat{k})$$

$$G = a^2+b^2+u^2$$

$$L = \vec{N}\cdot\vec{r}_{11}$$

and $\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{(u-v)[b\hat{i}-a\hat{j}]}{(u-v)\sqrt{b^2+a^2}}$

$$\vec{N} = \frac{b\hat{i} - a\hat{j}}{\sqrt{b^2+a^2}}$$

$$L = \frac{b\hat{i} - a\hat{j}}{\sqrt{b^2+a^2}} \cdot 0 = 0$$

$$M = \vec{N} \cdot \vec{r}_{12}$$

$$M = \frac{b\hat{i} - a\hat{j}}{\sqrt{b^2+a^2}} \cdot \hat{k} = 0$$

$$N = \vec{N} \cdot \vec{r}_{22}$$

$$N = \frac{b\hat{i} - a\hat{j}}{\sqrt{b^2+a^2}} \cdot 0 = 0$$

Question:-

If the parametric curves on a surface are orthogonal, then prove that the differential equation of a line on the surface cutting the parametric curve u = constant at a constant angle β is

$$\frac{du}{dv} = \sqrt{\frac{G}{E}} \tan\beta$$

Proof:-

Let $d\vec{r}$ be the displacement corresponding to the increments du, dv in parametres u and v and $\delta\vec{r}$ be the displacement corresponding to the increments δu and δv.

Then,
$$d\vec{r} = \vec{r}_1 du + \vec{r}_2 dv$$
$$\delta\vec{r} = \vec{r}_1 \delta u + \vec{r}_2 \delta v$$

If β is the angle between two directions then we know that

$$ds\,\delta s \cos\beta = E du\delta u + F(du\delta v + dv\delta u) + G dv\delta v \quad \rightarrow (1)$$

$$ds\,\delta s \sin\beta = H(du\,\delta v - dv\,\delta u) \rightarrow (2)$$

Since, the parametric curves are orthogonal

So $F = 0$

$$ds\,\delta s \cos\beta = E\,du\,\delta u + G\,dv\,\delta v \rightarrow (3)$$

If $d\vec{r}$ is in the direction of the line cutting the curve u = constant and $\delta\vec{r}$ is in the direction u = constant then $\delta u = 0$

So $\delta\vec{r} = \vec{r}_1 \cdot 0 + \vec{r}_2\,\delta v$

$\Rightarrow \delta\vec{r} = \vec{r}_2\,\delta v$

Now, $\delta s = |\delta\vec{r}| = |\vec{r}_2|\,\delta v$

$\Rightarrow \delta s = \sqrt{G}\,\delta v$

Divimding Eq (2) by (3)

$$\frac{ds\,\delta s \sin\beta}{ds\,\delta s \cos\beta} = \frac{H(du\,\delta v - dv\,\delta u)}{E\,du\,\delta u + G\,dv\,\delta v}$$

put $\delta u = 0$ because u = constant

$$\frac{\sin\beta}{\cos\beta} = \frac{H\,du\,\delta v}{G\,dv\,\delta v}$$

$$\tan\beta = \frac{H}{G}\frac{du}{dv}$$

Since $H = \sqrt{EG - F^2}$ $\because F = 0$

So $\dfrac{G}{H}\tan\beta = \dfrac{du}{dv}$

$$\frac{G}{\sqrt{EG - (0)^2}}\tan\beta = \frac{du}{dv}$$

$$\Rightarrow \frac{G}{\sqrt{EG}} = \frac{du}{dv}\frac{1}{\tan\beta}$$

$$\Rightarrow \frac{du}{dv} = \frac{\sqrt{G}}{\sqrt{E}}\tan\beta$$

$$\Rightarrow \frac{du}{dv} = \sqrt{\frac{G}{E}}\tan\beta$$

Weingarten Equations:-

Derivatives of $\vec{N}$:-

we will denote derivatives of u w.r.t $\vec{N}_1$ and derivative of v w.r.t $\vec{N}_2$ where u and v are parameters for the surface and $\vec{N}$ is the unit normal vector to the surface $\vec{r} = \vec{r}(u,v)$.

Since, the unit normal vector is perpendicular to the vectors $\vec{r}_1$ and $\vec{r}_2$.

So $\vec{N}\cdot\vec{r}_1 = 0 \rightarrow (1)$ $\quad \vec{N}\cdot\vec{r}_2 = 0 \rightarrow (2)$

Differentiating eq (1) w.r.t "u" we have

$$\vec{N}_1\cdot\vec{r}_1 + \vec{N}\cdot\vec{r}_{11} = 0$$

$$\Rightarrow \vec{N}_1\cdot\vec{r}_1 + L = 0$$

$$\Rightarrow \vec{N}_1\cdot\vec{r}_1 = -L \rightarrow (3)$$

Differentiating eq (2) w.r.t "u". we have

$$\vec{N}_1\cdot\vec{r}_2 + \vec{N}\cdot\vec{r}_{21} = 0$$

$$\vec{N}_1\cdot\vec{r}_2 + M = 0$$

$$\Rightarrow \vec{N}_1\cdot\vec{r}_2 = -M \rightarrow (4)$$

Differentiate eq (1) w.r.t "v"

$$\vec{N}_2\cdot\vec{r}_1 + \vec{N}\cdot\vec{r}_{12} = 0$$

$$\Rightarrow \vec{N}_2\cdot\vec{r}_1 + M = 0$$

$$\Rightarrow \vec{N}_2\cdot\vec{r}_1 = -M \rightarrow (5)$$

Differentiate eq (2) w.r.t "v"

$$\vec{N}_2\cdot\vec{r}_2 + \vec{N}\cdot\vec{r}_{22} = 0$$

$$\Rightarrow \vec{N}_2\cdot\vec{r}_2 + N = 0$$

$$\Rightarrow \vec{N}_2\cdot\vec{r}_2 = -N \rightarrow (6)$$

Also,

$$\vec{N}\cdot\vec{N} = 1$$

Diff w.r.t "u" on both sides

$\Rightarrow$

$$\Rightarrow 2\vec{N}\cdot\vec{N}_1 = 0$$

$$\Rightarrow \vec{N}\cdot\vec{N}_1 = 0$$

$\Rightarrow$ $\vec{N}_1$ is $\perp$ to $\vec{N}$

i-e) $\vec{N}_1$ is perpandicular to the $\vec{r}_1 \times \vec{r}_2$, it means N_1 lies in the plane of $\vec{r}_1$ and $\vec{r}_2$.

i-e) $\vec{N}_1$ is parallel to the vector plane of $a\vec{r}_1 + b\vec{r}_2$, where a and b are constants.

$$\Rightarrow \quad \vec{N}_1 = a\vec{r}_1 + b\vec{r}_2 \rightarrow (7)$$

Taking dot product with $\vec{r}_1$ on both sides

$$\vec{N}_1 \cdot \vec{r}_1 = a\vec{r}_1 \cdot \vec{r}_1 + b\vec{r}_2 \cdot \vec{r}_1$$

$$-L = aE + bF \rightarrow (8)$$

Taking dot product with $\vec{r}_2$ on both sides of eq (7)

$$\vec{N}_1 \cdot \vec{r}_2 = a\vec{r}_1 \cdot \vec{r}_2 + b\vec{r}_2 \cdot \vec{r}_2$$

$$-M = aF + bG \rightarrow (9)$$

Multiply eq (8) by F and eq (9) by E and then subtract it, we have

$$-LF = aEF + bF^2$$

$$\mp ME = \pm aEF \pm bGE$$

$$-LF + ME = bF^2 - bGE$$

$$-LF + ME = -b(GE - F^2)$$

$$LF - ME = b(EG - F^2)$$

$$\Rightarrow \quad b = \frac{LF - ME}{EG - F^2}$$

Multiplying eq (8) by G and eq (9) by F and then subtracting it

$$-GL = aEG + bGF$$

$$\mp MF = \pm aF^2 \pm bGF$$

$$MF - GL = aEG - aF^2$$

$$MF - GL = a(EG - F^2)$$

$$a = \frac{MF - GL}{EG - F^2}$$

Substitute values of "a" and "b" in eq (7)

$$H^2\vec{N}_1 = (MF-LG)\vec{r}_1 + (LF-ME)\vec{r}_2 \rightarrow (X)$$

Now again consider,

$$\vec{N}\cdot\vec{N} = 1$$

Differentiate both sides w.r.t "v", we have

$$2\vec{N}\cdot\vec{N}_2 = 0$$

$$\Rightarrow \vec{N}\cdot\vec{N}_2 = 0$$

$\Rightarrow \vec{N}_2$ is perpandicular to $\vec{N}$

Hence, $\quad \because \vec{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{H}$

$\vec{N}_2$ is $\perp$ to $\vec{r}_1 \times \vec{r}_2$ (vector)

i-e) $\vec{N}_2$ lies in the plane of $\vec{r}_1$ and $\vec{r}_2$

So,

$$\vec{N}_2 = c\vec{r}_1 + d\vec{r}_2 \rightarrow (11)$$

where "c" and "d" are constants.

Taking dot product with $\vec{r}_1$ on both sides of (11)

$$\vec{N}_2\cdot\vec{r}_1 = c\vec{r}_1\cdot\vec{r}_1 + d\,\vec{r}_2\cdot\vec{r}_1$$

$$-M = cE + dF \rightarrow (12)$$

Taking dot product with $\vec{r}_2$ on both sides,

$$\vec{N}_2\cdot\vec{r}_2 = c\vec{r}_1\cdot\vec{r}_2 + d\,\vec{r}_2\cdot\vec{r}_2 \rightarrow r$$

$$-N = cF + dG \rightarrow (13)$$

G eq(12) $-$ F eq(13)

$$NF - MG = c(EG - F^2)$$

$$c = \frac{NF-MG}{EG-F^2}$$

$$c = \frac{NF-MG}{H^2}$$

F eq(12) $-$ E eq(13)

$$-MF + NE = d(F^2 - EG)$$

$$\Rightarrow MF - NE = (EG - F^2)d$$

$$\Rightarrow d = \frac{MF-NE}{EG-F^2}$$

$$d = \frac{MF-NE}{H^2}$$

Substitute values of "c" and "d" in (11)

$$H^2\vec{N}_2 = (NF-MG)\vec{r}_1 + (MF-NG)\vec{r}_2 \to (14)$$

$$\Rightarrow \quad \vec{N}_2 = \frac{1}{H^2}\left[(NF-MG)\vec{r}_1 + (MF-NG)\vec{r}_2\right] \to (A)$$

From (x)

$$\vec{N}_1 = \frac{1}{H_2}\left[(MF-LG)\vec{r}_1 + (LF-ME)\vec{r}_2\right] \to (B)$$

The equations expressing $\vec{N}_1$ and $\vec{N}_2$ in term of E, F, G and L, M, N are called weingarten equations.

Normal section of a surface at a point:- The normal section of a surface at a point is a section of the surface by the plane containing the normal to the surface.

So, it is obvious that the principal normal to the normal section is in the direction of the normal to the surface, and we will adopt the convention that the principal normal to the normal section is the same as the normal to the surface.

Meunier's Theorem:-

If k is the curvature of any section (curve) of the surface at a point 'p' and consider the normal plane which touches this section at point 'p'. If k_n is the curvature of the normal section of the surface by the normal plane, then the angle θ between the planes of the two sections is given by

$$\cos\theta = \frac{k_n}{k}$$

Proof:-

The angle θ between two planes is the same as that of the angle between the principal normals of the two sections.

The unit principal normal to the normal section by the normal plane is $\vec{N}$ and the unit principal normal to the other section is $\frac{\vec{r}''}{k}$ (by Seret Feret Formula). Hence, the angle between these principal normals is

$$\vec{N}\cdot\frac{\vec{r}''}{k} = |\vec{N}|\left|\frac{\vec{r}''}{k}\right|\cos\theta$$

$$\vec{N}\cdot\frac{\vec{r}''}{k} = 1\cdot 1\cos\theta$$

$$\cos\theta = \vec{N}\cdot\frac{\vec{r}''}{k} \rightarrow (1)$$

Now $\vec{r}' = \dfrac{d\vec{r}}{ds} = \dfrac{\partial\vec{r}}{\partial u}\dfrac{du}{ds} + \dfrac{\partial\vec{r}}{\partial v}\dfrac{dv}{ds}$

$$\vec{r}'' = \frac{d}{ds}(\vec{r}') = \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{dv}{ds}\right)$$

$$\vec{r}'' = \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial u}\right)\frac{du}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial v}\right)\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\vec{r}'' = \left(\frac{\partial^2\vec{r}}{\partial u^2}\frac{du}{ds} + \frac{\partial^2\vec{r}}{\partial v\partial u}\frac{dv}{ds}\right)\frac{du}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \left(\frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\right.$$
$$\left. + \frac{\partial^2\vec{r}}{\partial v^2}\frac{dv}{ds}\right)\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\vec{r}'' = \frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du\,dv}{(ds)^2} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du\,dv}{(ds)^2}$$
$$+ \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\vec{r}'' = \frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2}$$
$$+ \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\vec{r}'' = \vec{r}_{11}(u')^2 + 2\vec{r}_{12}u'v' + \vec{r}_1 u'' + \vec{r}_{22}(v')^2 + \vec{r}_2 v''$$

Taking dot product on both sides with $\vec{N}$ we have

$$\vec{N}\cdot\vec{r}'' = \vec{N}\cdot\vec{r}_{11}(u')^2 + 2\vec{N}\cdot\vec{r}_{12}u'v' + \vec{N}\cdot\vec{r}_1 u'' + \vec{N}\cdot\vec{r}_{22}(v')^2$$
$$+ \vec{N}\cdot\vec{r}_2 v''$$

$\vec{N}\cdot\vec{\gamma}'' = \vec{N}\cdot\vec{\gamma}_{11}u'^2 + 2\vec{N}\cdot\vec{\gamma}_{12}u'v' + \vec{N}\cdot\vec{\gamma}_{22}v'^2$

$\vec{N}\cdot\vec{\gamma}'' = L(u')^2 + 2Mu'v' + N(v')^2 \rightarrow (2)$

Since, the values of u' and v' are the same for both sections at point "P" so from eq (2) the value of $\vec{N}\cdot\vec{\gamma}''$ is same for both sections

Now, the curvature of the normal section is given by $k_n = \frac{\vec{\gamma}''}{\vec{N}}$

$$k_n \vec{N} = \vec{\gamma}''$$

$$k_n(\vec{N}.\vec{N}) = \vec{\gamma}''.N$$

$$\Rightarrow k_n = \vec{\gamma}''.\vec{N} \rightarrow (3)$$

Now by eq (1) and (3), we have

$$\cos\theta = \frac{k_n}{k}$$

where $k_n = \vec{N}\cdot\vec{\gamma}'' = Lu'^2 + 2Mu'v' + Nv'^2$ is also normal curvature.

Remarks:-

$$k_n = Lu'^2 + 2Mu'v' + Nv'^2$$

$$k_n = L\left(\frac{d^2u}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

$$k_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{(ds)^2}$$

Now, by Ist Fundamental form, we have

$$(ds)^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$k_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Normal curvature and radius of normal curvature:-

The curvature of the normal

section of a surface is known as the normal curvature and its reciprocal is known as radius of normal curvature.
The normal curvature is usually denoted by k_n

Question:-

If L, M and N vanish at all points of a surface, then the surface is a plane.

Sol:-

The normal curvature at any point of the surface is given by

$$k_n = Lu'^2 + 2Mu'v' + Nv'^2$$

Since, $L = M = N = 0$ at all points of the surface. So,

$$k_n = 0 \qquad (\text{If } k=0, \text{ then curve is a st. line})$$

$\Rightarrow k_n = 0$ at all points of the surface.
i.e.) the normal curvature at all points of the surface is zero.
Hence, all the normal sections of the surface are straight lines
Hence, the surface is a plane.

Question:-

If $\frac{L}{E} = \frac{M}{F} = \frac{N}{G} = \alpha$ (constant) at all points of the surface, then prove that either the surface is a sphere or the surface is a plane.

Sol:-

The normal curvature at any point of the surface is given by

$$k_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \rightarrow (1)$$

$$\frac{L}{E} = \alpha \Rightarrow L = E\alpha$$

$$\frac{M}{F} = \alpha \Rightarrow M = F\alpha$$

$$\frac{N}{G} = \alpha \Rightarrow N = G\alpha$$

putting all values in (1)

$$k_n = \frac{E\alpha du^2 + 2F\alpha dudv + G\alpha dv^2}{Edu^2 + 2F dudv + Gdv^2}$$

$$k_n = \frac{\alpha\,(Edu^2 + 2Fdudv + Gdv^2)}{Edu^2 + 2Fdudv + dv^2}$$

$k_n = \alpha$

$\Rightarrow$ $k_n = \alpha$ at all points of the surface.

If $\alpha = 0$, then $k_n = 0$ at all points of the surface. ~~attencthen~~ the surface is a plane.

If $\alpha \neq 0$, then $k_n = \alpha$ (constant) which is non-zero constant.

Hence, the normal curvature at all points of the surface is constant.

So, the surface is a sphere.

Note:-

If $k_n = 0$ at all points of the surface then the surface is a plane. If $k_n = \alpha$ (constant) at all points of the surface then surface is a sphere.

Question:-

If $\frac{E}{L} = \frac{F}{M} = \frac{G}{N} = \alpha \neq 0$ at all points of the surface, then prove that either the surface is sphere or plane.

Sol:-

The normal curvature at any point of the surface is given by

$$k_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2} \rightarrow (1)$$

$$\frac{E}{L} = \alpha \Rightarrow E = L\alpha$$

$$\frac{F}{M} = \alpha \Rightarrow F = M\alpha$$

$$\frac{G}{N} = \alpha \Rightarrow G = N\alpha$$

put all these in (1)

$$k_n = \frac{Ldu^2 + 2Mdudv + Ndv^2}{L\alpha du^2 + 2M\alpha dudv + N\alpha dv^2}$$

$$k_n = \frac{1}{\alpha}\left(\frac{Ldu^2 + 2Mdudv + Ndv^2}{Ldu^2 + 2Mdudv + Ndv^2}\right)$$

$$k_n = \frac{1}{\alpha}$$

$\Rightarrow k_n = \frac{1}{\alpha}$ at all points of the surface.

If $\frac{1}{\alpha} = 0$ then $k_n = 0$ at all points of the surface and Hence, the surface is a plane.

If $\frac{1}{\alpha} \neq 0$ then $k_n = \frac{1}{\alpha}$ which is a non-zero constant, at all points of the surface and Hence, the surface is a sphere.

Question :-

Taking x, y as parameters, find the fundamental magnitudes and the unit normal to the surface $2z = ax^2 + 2hxy + by^2$

Sol :-

The Position vector $\vec{r}$ of any point on the surface is

$$\vec{r} = (x, y, z)$$

$$\vec{r} = \left(x, y, \frac{1}{2}(ax^2 + 2hxy + by^2)\right)$$

Differentiate $\vec{r}$ w.r.t "x"

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1, 0, \frac{1}{2}(2ax+2hy)\right)$$

$$\vec{r}_1 = (1, 0, ax+hy)$$

Differentiate $\vec{r}$ w.r.t "y"

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left(0, 1, \frac{1}{2}(2hx+2by)\right)$$

$$\vec{r}_2 = (0, 1, hx+by)$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \rightarrow (1) \qquad \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & ax+hy \\ 0 & 1 & hx+by \end{vmatrix}$$

$$\vec{r}_1 \times \vec{r}_2 = \hat{i}(0-(ax+hy)) - \hat{j}(hx+by-0) + \hat{k}(1-0)$$

$$\vec{r}_1 \times \vec{r}_2 = (-ax-hy)\hat{i} + (-hx-by)\hat{j} + \hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{(ax+hy)^2 + (hx+by)^2 + 1}$$

$$\vec{N} = \frac{(-ax-hy)\hat{i} + (-hx-by)\hat{j} + \hat{k}}{\sqrt{(ax+hy)^2 + (hx+by)^2 + 1}}$$

$$\vec{r}_{11} = \frac{\partial}{\partial x}(\vec{r}_1) = \frac{\partial}{\partial x}(1, 0, ax+hy) = (0, 0, a)$$

$$\vec{r}_{12} = \frac{\partial}{\partial x}(\vec{r}_2) = \frac{\partial}{\partial x}(0, 1, hx+by) = (0, 0, h)$$

$$\vec{r}_{22} = \frac{\partial}{\partial y}(\vec{r}_2) = \frac{\partial}{\partial y}(0, 1, hx+by) = (0, 0, b)$$

For Ist order magnitudes

Now $E = \vec{r}_1 \cdot \vec{r}_1$

$$= (1, 0, ax+hy) \cdot (1, 0, ax+hy)$$

$$= 1 + 0 + (ax+hy)^2$$

$$E = 1 + (ax+hy)^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$

$$= (1, 0, ax+hy) \cdot (0, 1, hx+by)$$

$$F = 0 + 0 + (ax+hy)(hx+by)$$
$$F = ahx^2 + abxy + hxy + bhy^2$$

$$G = \vec{r}_2 \cdot \vec{r}_2$$
$$= (0, 1, hx+by) \cdot (0, 1, hx+by)$$
$$= (0 + 1 + (hx+by)^2)$$
$$G = 1 + (hx+by)^2$$

Now, 2nd order magnitudes

$$L = \vec{N} \cdot \vec{r}_{11}$$
$$= \frac{(-ax-hy)\hat{i} + (-hx-by)\hat{j} + \hat{k}}{\sqrt{(ax+hy)^2 + (hx+by)^2 + 1}} \cdot (0, 0, a)$$
$$L = \frac{a}{\sqrt{1 + (ax+hy)^2 + (hx+by)^2}}$$
$$M = \vec{N} \cdot \vec{r}_{12}$$

$$= \frac{(-ax-hy)\hat{i} + (-hx-by)\hat{j} + \hat{k}}{\sqrt{(ax+hy)^2 + (hx+by)^2 + 1}} \cdot (0, 0, h)$$

$$M = \frac{h}{\sqrt{1 + (ax+hy)^2 + (hx+by)^2}}$$
$$N = \vec{N} \cdot \vec{r}_{22}$$

$$= \frac{(-ax-hy)\hat{i} + (-hx-by)\hat{j} + \hat{k}}{\sqrt{(ax+hy)^2 + (hx+by)^2 + 1}} \cdot (0, 0, b)$$

$$N = b / \sqrt{1 + (ax+hy)^2 + (hx+by)^2}$$

Ist order fundamental magnitudes are

$$E = 1 + (ax+hy)^2$$
$$F = ahx^2 + bhy^2 + (ab+h)xy$$
$$G = 1 + (hx+by)^2$$

2nd order fundamental magnitudes are

$L = M = N = 0$

and unit normal to surface is

$$\vec{N} = \frac{-(ax+hy)\hat{i} - (hx+by)\hat{j} + \hat{k}}{\sqrt{(ax+hy)^2+(hx+by)^2+1}}$$

Question:-

Find the fundamental magnitudes and unit normal $(\vec{N})$ to the surfaces

(i) $x = u\cos\phi$, $y = u\sin\phi$, $z = f(u)$

(ii) $x = u\cos\phi$, $y = u\sin\phi$, $z = f(\phi)$

(iii) $x = u\cos\phi$, $y = u\sin\phi$, $z = f(u) + c$

where u and ϕ are parametres for the surface.

Proof:-

(i) $x = u\cos\phi$, $y = u\sin\phi$, $z = f(u)$

The position vector $\vec{r}$ of any point on the surface is $\vec{r} = u\cos\phi\hat{i} + u\sin\phi\hat{j} + f(u)\hat{k} \rightarrow (1)$

Differentiate (1) w.r.t "u"

$$\vec{r}_1 = \frac{\partial\vec{r}}{\partial u} = \cos\phi\hat{i} + \sin\phi\hat{j} + f'(u)\hat{k}$$

Differentiate (1) w.r.t ϕ

$$\vec{r}_2 = \frac{\partial\vec{r}}{\partial\phi} = -u\sin\phi\hat{i} + u\cos\phi\hat{j} + 0\cdot\hat{k}$$

$$\vec{r}_2 = -u\sin\phi\hat{i} + u\cos\phi\hat{j}$$

$$\vec{r}_{11} = \frac{\partial}{\partial u}\left(\frac{\partial\vec{r}}{\partial u}\right) = \frac{\partial}{\partial u}\left(\cos\phi\hat{i} + \sin\phi\hat{j} + f'(u)\hat{k}\right)$$

$$= 0 + 0 + f''(u)\hat{k}$$

$$\vec{r}_{11} = f''(u)\hat{k}$$

$$\vec{r}_{12} = \frac{\partial}{\partial u}\left(\frac{\partial\vec{r}}{\partial\phi}\right) = \frac{\partial}{\partial u}\left(-u\sin\phi\hat{i} + u\cos\phi\hat{j}\right)$$

$$\vec{r}_{12} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$\vec{r}_{22} = \frac{\partial}{\partial\phi}\left(\frac{\partial\vec{r}}{\partial\phi}\right)$$

$$= \frac{\partial}{\partial\phi}\left(-u\sin\phi\hat{i} + u\cos\phi\hat{j}\right)$$

$$\vec{r}_{22} = -u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j}$$

The unit normal to the surface is

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & f'(u) \\ -u\sin\phi & u\cos\phi & 0 \end{vmatrix}$$

$$= \hat{i}(0 - u\cos\phi f'(u)) - \hat{j}(0 + u\sin\phi f'(u)) + \hat{k}(u\cos^2\phi + u\sin^2\phi)$$

$$\vec{r}_1 \times \vec{r}_2 = -u\cos\phi f'(u)\hat{i} - u\sin\phi f'(u)\hat{j} + u\hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{(u\cos\phi f'(u))^2 + (u\sin\phi f'(u))^2 + u^2}$$

$$= \sqrt{u^2\cos^2\phi f'^2(u) + u^2\sin^2\phi f'^2(u) + u^2}$$

$$= \sqrt{u^2 f'^2(u)(\cos^2\phi + \sin^2\phi) + u^2}$$

$$= \sqrt{u^2(f'(u))^2 + u^2}$$

$$= \sqrt{u^2(1 + (f'(u))^2}$$

$$\therefore |\vec{r}_1 \times \vec{r}_2| = u\sqrt{1 + (f'(u))^2}$$

$$\vec{N} = \frac{-u\cos\phi f'(u)\hat{i} - u\sin\phi f'(u)\hat{j} + u\hat{k}}{u\sqrt{1 + (f'(u))^2}}$$

$$\vec{N} = \frac{-\cos\phi f'(u)\hat{i} - \sin\phi f'(u)\hat{j} + \hat{k}}{\sqrt{1 + (f'(u))^2}}$$

Ist order fundamental magnitudes are

$$E = \vec{r}_1 \cdot \vec{r}_1$$

$E = (\cos\phi\hat{i} + \sin\phi\hat{j} + f'(u)\hat{k}).(\cos\phi\hat{i} + \sin\phi\hat{j} + f'(u)\hat{k})$

$E = \cos^2\phi\hat{i} + \sin^2\phi + (f'(u))^2$

$E = 1 + (f'(u))^2$

$F = \vec{r}_1 . \vec{r}_2$

$= (\cos\phi\hat{i} + \sin\phi\hat{j} + f(u)\hat{k}).(-u\sin\phi\hat{i} + u\cos\phi\hat{j})$

$F = -u\sin\phi\cos\phi + u\sin\phi\cos\phi = 0$

$\Rightarrow F = 0$

$G = \vec{r}_2 . \vec{r}_2$

$= (-u\sin\phi\hat{i} + u\cos\phi\hat{j}).(-u\sin\phi\hat{i} + u\cos\phi\hat{j})$

$G = u^2\sin^2\phi + u^2\cos^2\phi$

$= u^2(\sin^2\phi + \cos^2\phi)$

$G = u^2$

2nd order fundamental magnitudes are

$L = \vec{N} . \vec{r}_{11}$

$$L = \frac{-\cos\phi f(u)\hat{i} - \sin\phi f(u)\hat{j} + \hat{k}}{\sqrt{1 + (f'(u))^2}} . f''(u)\hat{k}$$

$$L = \frac{f''(u)}{\sqrt{1 + (f'(u))^2}}$$

$M = \vec{N} . \vec{r}_{12}$

$$= \frac{-\cos\phi f'(u)\hat{i} - \sin\phi f'(u)\hat{j} + \hat{k}}{\sqrt{1 + (f'(u))^2}} . -\sin\phi\hat{i} + \cos\phi\hat{j}$$

$$M = \frac{\sin\phi\cos\phi f'(u) - \sin\phi\cos\phi f'(u) + 0}{\sqrt{1 + (f'(u))^2}} = 0$$

$M = 0$

$N = \vec{N} . \vec{r}_{22}$

$$N = \frac{-\cos\phi f'(u)\hat{i} - \sin\phi f'(u)\hat{j} + \hat{k}}{\sqrt{1 + (f'(u))^2}} . -u\cos\phi\hat{i} - u\sin\phi\hat{j}$$

$$N = \frac{u\cos^2\phi\, f'(u) + u\sin^2\phi\, f'(u)}{\sqrt{1+(f'(u))^2}}$$

$$N = \frac{u f'(u)(\cos^2\phi + \sin^2\phi)}{\sqrt{1+(f'(u))^2}}$$

$$N = \frac{u f'(u)}{\sqrt{1+(f'(u))^2}}$$

(ii)

$x = u\cos\phi$, $y = u\sin\phi$, $z = f(\phi)$

The position vector of any point on the surface is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} + f(\phi)\hat{k} \rightarrow (1)$

Differentiate (1) w.r.t "u"

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = \frac{\partial}{\partial u}(u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} + f(\phi)\hat{k})$$

$$\vec{r}_1 = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

Differentiate (1) w.r.t ϕ

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = \frac{\partial}{\partial \phi}(u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} + f(\phi)\hat{k})$$

$$\vec{r}_2 = -u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + f'(\phi)\hat{k}$$

$$\vec{r}_{11} = \frac{\partial}{\partial u}(\vec{r}_1) = \frac{\partial}{\partial u}(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) = 0$$

$$\vec{r}_{12} = \frac{\partial}{\partial u}(\vec{r}_2) = \frac{\partial}{\partial u}(-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + f'(\phi)\hat{k})$$

$$\vec{r}_{12} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\vec{r}_{22} = \frac{\partial}{\partial \phi}(\vec{r}_2) = \frac{\partial}{\partial \phi}(-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + f'(\phi)\hat{k})$$

$$\vec{r}_{22} = -u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} + f''(\phi)\hat{k}$$

The unit normal to the surface is

$$N = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \rightarrow (2)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & 0 \\ -u\sin\phi & u\cos\phi & f'(\phi) \end{vmatrix}$$

$$= \hat{i}(\sin\phi f'(\phi) - 0) - \hat{j}(\cos\phi f'(\phi) - 0) + \hat{k}(u\cos^2\phi + u\sin^2\phi)$$

$$\vec{r}_1 \times \vec{r}_2 = \sin\phi f'(\phi)\hat{i} - \cos\phi f'(\phi)\hat{j} + u\hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{\sin^2\phi (f'(\phi))^2 + \cos^2\phi (f'(\phi))^2 + u^2}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{(f'(\phi))^2 + u^2}$$

put values in (2)

$$\vec{N} = \frac{\sin\phi f'(\phi)\hat{i} - \cos\phi f'(\phi)\hat{j} + u\hat{k}}{\sqrt{u^2 + (f'(\phi))^2}}$$

Ist order fundamental magnitudes are

$$E = \vec{r}_1 \cdot \vec{r}_1$$

$$E = (\cos\phi\hat{i} + \sin\phi\hat{j}) \cdot (\cos\phi\hat{i} + \sin\phi\hat{j})$$

$$E = \cos^2\phi + \sin^2\phi$$

$$E = 1$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$

$$= (\cos\phi\hat{i} + \sin\phi\hat{j}) \cdot (-u\sin\phi\hat{i} + u\cos\phi\hat{j} + f'(\phi)\hat{k})$$

$$F = -u\sin\phi\cos\phi + u\sin\phi\cos\phi + 0$$

$$F = 0$$

$$G = \vec{r}_2 \cdot \vec{r}_2$$

$$= (-u\sin\phi\hat{i} + u\cos\phi\hat{j} + f'(\phi)\hat{k}) \cdot (-u\sin\phi\hat{i} + u\cos\phi\hat{j} + f'(\phi)\hat{k})$$

$$G = u^2\sin^2\phi + u^2\cos^2\phi + (f'(\phi))^2$$

$$G = u^2(\sin^2\phi + \cos^2\phi) + (f'(\phi))^2$$

$$G = u^2 + (f'(\phi))^2$$

2nd order magnitudes are

$L = \vec{N} \cdot \vec{r}_{11}$, $M = \vec{N} \cdot \vec{r}_{12}$ and

$N = \vec{N} \cdot \vec{r}_{22}$

$$L = \frac{\sin\phi f'(\phi)\hat{i} - \cos\phi f'(\phi)\hat{j} + u\hat{k}}{\sqrt{u^2 + (f'(\phi))^2}} \cdot 0$$

$$L = 0$$

$$M = \vec{N} \cdot \vec{r}_{12}$$

$$M = \frac{\sin\phi f'(\phi)\hat{i} - \cos\phi f'(\phi)\hat{j} + u\hat{k}}{\sqrt{u^2 + (f'(\phi))^2}} \cdot (-\sin\phi\hat{i} + \cos\phi\hat{j})$$

$$M = \frac{-\sin^2\phi f'(\phi) - \cos^2\phi f'(\phi)}{\sqrt{u^2 + (f'(\phi))^2}}$$

$$M = \frac{-(\sin^2\phi + \cos^2\phi) f'(\phi)}{\sqrt{u^2 + (f'(\phi))^2}}$$

$$M = \frac{-f'(\phi)}{\sqrt{u^2 + (f'(\phi))^2}}$$

$$N = \vec{N} \cdot \vec{r}_{22}$$

$$N = \frac{\sin\phi f'(\phi)\hat{i} - \cos\phi f'(\phi)\hat{j} + u\hat{k}}{\sqrt{u^2 + (f'(\phi))^2}} \cdot (-u\cos\phi\hat{i} - u\sin\phi\hat{j} + f''(\phi)\hat{k})$$

$$N = \frac{-u\sin\phi\cos\phi f'(\phi) + u\sin\phi\cos\phi f'(\phi) + u f''(\phi)}{\sqrt{u^2 + (f'(\phi))^2}}$$

$$N = \frac{u f''(\phi)}{\sqrt{u^2 + (f'(\phi))^2}}$$

(iii)

$x = u\cos\phi$, $y = u\sin\phi$, $z = f(u) + c$

The position vector of any point on the surface is $\vec{r} = (x, y, z)$

$$\vec{r} = u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} + (f(u)+c)\hat{k} \rightarrow (1)$$

Differentiate (1) w.r.t u

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} + f'(u)\hat{k}$$

Differentiate (1) w.r.t ϕ

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = -u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j}$$

$$\vec{r}_{11} = \frac{\partial \vec{r}_1}{\partial \phi} = 0$$

$$\vec{r}_{12} = \frac{\partial}{\partial u}(\vec{r}_2) = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\vec{r}_{22} = \frac{\partial}{\partial \phi}(\vec{r}_2) = -u\cos\phi\,\hat{i} - u\sin\phi\,\hat{j}$$

The unit normal to the surface is

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \rightarrow (2)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & f'(u) \\ -u\sin\phi & u\cos\phi & 0 \end{vmatrix}$$

$$= \hat{i}(0 - u\cos\phi\, f'(u)) - \hat{j}(0 + u\sin\phi\, f'(u)) + \hat{k}(u\cos^2\phi + u\sin^2\phi)$$

$$\vec{r}_1 \times \vec{r}_2 = -u\cos\phi\, f'(u)\hat{i} - u\sin\phi\, f'(u)\hat{j} + u\hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{u^2\cos^2\phi(f'(u))^2 + u^2\sin^2\phi(f'(u))^2 + u^2}$$

$$= \sqrt{u^2(\cos^2\phi + \sin^2\phi)(f'(u))^2 + u^2}$$

$$= \sqrt{u^2(1 + (f'(u))^2)}$$

$$|\vec{r}_1 \times \vec{r}_2| = u\sqrt{1 + (f'(u))^2}$$

So, $$\vec{N} = \frac{-u\cos\phi\, f'(u)\hat{i} - u\sin\phi\, f'(u)\hat{j} + u\hat{k}}{u\sqrt{1+(f'(u))^2}}$$

$$\vec{N} = \frac{-\cos\phi\, f'(u)\hat{i} - \sin\phi\, f'(u)\hat{j} + \hat{k}}{\sqrt{1+(f'(u))^2}}$$

Ist order fundamental magnitudes are

$E = \vec{r}_1 \cdot \vec{r}_1$

$E = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + f'(u)\hat{k})\cdot(\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + f'(u)\hat{k})$

$E = \cos^2\phi + \sin^2\phi + (f'(u))^2$

$E = 1 + (f'(u))^2$

$F = \vec{r}_1 \cdot \vec{r}_2 = (\cos\phi\,\hat{i} + \sin\phi\,\hat{j} + f'(u)\hat{k})\cdot(-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j})$

$F = 0$

$G = \vec{r}_2 \cdot \vec{r}_2 = (-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j})\cdot(-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j})$

$G = u^2\sin^2\phi + u^2\cos^2\phi = u^2$

2nd order fundamental magnitudes are

$L = \vec{N}\cdot\vec{r}_{11}$

$$L = \frac{-\cos\phi\, f'(u)\hat{i} - \sin\phi\, f'(u)\hat{j} + \hat{k}}{\sqrt{1+(f'(u))^2}} \cdot 0$$

$L = 0$

$M = \vec{N}\cdot\vec{r}_{12}$

$$= \frac{-\cos\phi\, f'(u)\hat{i} - \sin\phi\, f'(u)\hat{j} + \hat{k}}{\sqrt{1+(f'(u))^2}} \cdot (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})$$

$M = +\sin\phi\cos\phi\, f'(u) - \sin\phi\cos\phi\, f'(u) + 0 = 0$

$\Rightarrow M = 0$

$N = \vec{N}\cdot\vec{r}_{22}$

$$= \frac{-\cos\phi\, f'(u)\hat{i} - \sin\phi\, f'(u)\hat{j} + \hat{k}}{\sqrt{1+(f'(u))^2}} \cdot (-u\cos\phi\,\hat{i} - u\sin\phi\,\hat{j})$$

$$N = \frac{u\cos^2\phi\, f'(u) + u\sin^2\phi\, f'(u) + 0}{\sqrt{1+(f'(u))^2}} = \frac{u f(u)}{\sqrt{1+(f'(u))^2}}$$

Question:-

On the surface generated by the unit principal bi-normal of a twisted curve, the position vector of the current point is $\vec{r}+u\vec{b}$. Find the fundamental magnitudes and unit normal to the surface, where $\vec{r}$ and $\vec{b}$ are functions of s.
Take u and s as parameters for the surface.

Sol:-

If $\vec{R}(u,s)$ is the position vector of current point on the surface, then $\vec{R} = \vec{r}+u\vec{b}$

$$\vec{R}_1 = \frac{\partial R}{\partial u} = 0+\vec{b} = \vec{b} \qquad \because \frac{d}{du}(\vec{r}(s))=0 \text{ and } \frac{d}{du}(\vec{b}(s))=0$$

$$\vec{R}_2 = \frac{\partial R}{\partial s} = \vec{r}\,'+u\vec{b}\,'$$
$$= \vec{t}+u(-\tau\vec{n})$$
$$\vec{R}_2 = \vec{t}-u\tau\vec{n}$$

$$E = \vec{R}_1\cdot\vec{R}_1 = \vec{b}\cdot\vec{b} = 1$$

$$F = \vec{R}_1\cdot\vec{R}_2 = \vec{b}\cdot(\vec{t}-u\tau\vec{n})$$
$$= \vec{b}\cdot\vec{t}-u\tau(\vec{b}\cdot\vec{n})$$
$$F = 0$$

$$G = \vec{R}_2\cdot\vec{R}_2 = (\vec{t}-u\tau\vec{n})\cdot(\vec{t}-u\tau\vec{n})$$
$$= \vec{t}\cdot\vec{t}+u^2\tau^2(\vec{n}\cdot\vec{n})$$
$$G = 1+u^2\tau^2$$

$$N = \frac{\vec{R}_1\times\vec{R}_2}{|\vec{R}_1\times\vec{R}_2|} \rightarrow (1)$$

$$\vec{R}_1\times\vec{R}_2 = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 0 & 0 & 1 \\ 1 & -u\tau & 0 \end{vmatrix}$$
$$= \vec{t}(0+u\tau)-\vec{n}(0-1)+\vec{b}(0-0)$$
$$\vec{R}_1\times\vec{R}_2 = u\tau\vec{t}+\vec{n}$$

$$|\vec{R}_1 \times \vec{R}_2| = \sqrt{u^2\tau^2+1}$$

put in (1)

$$\vec{N} = \frac{u\tau\vec{t}+\vec{n}}{\sqrt{1+u^2\tau^2}}$$

Now,

$$\vec{R}_{11} = \frac{\partial}{\partial u}(R_1) = \frac{\partial}{\partial u}(\vec{b}(s)) = 0$$

$$\vec{R}_{12} = \frac{\partial}{\partial u}(R_2) = \frac{\partial}{\partial u}(\vec{t} - u\tau\vec{n})$$

$$\vec{R}_{12} = 0 - \tau\vec{n} = -\tau\vec{n} = b'$$

$$\vec{R}_{22} = \frac{\partial}{\partial s}(\vec{R}_2) = \frac{\partial}{\partial s}(\vec{t} - u\tau\vec{n}) \quad \because \tau(s)$$

$$\vec{R}_{22} = \vec{t}' - u\tau\vec{n}' - u\tau'\vec{n}$$

Now,

$$L = \vec{N}\cdot\vec{R}_{11}$$

$$= \frac{u\tau\vec{t}+\vec{n}}{\sqrt{1+u^2\tau^2}}\cdot 0$$

$$L = 0$$

$$M = \vec{N}\cdot\vec{R}_{12}$$

$$= \frac{u\tau\vec{t}+\vec{n}}{\sqrt{1+u^2\tau^2}}\cdot(-\tau\vec{n})$$

$$M = \frac{-u\tau^2(\vec{t}\cdot\vec{n}) - \tau(\vec{n}\cdot\vec{n})}{\sqrt{1+u^2\tau^2}}$$

$$M = \frac{-u\tau^2\vec{0} - \tau}{\sqrt{1+u^2\tau^2}}$$

$$N = \vec{N}\cdot\vec{R}_{22}$$

$$= \frac{u\tau\vec{t} + \vec{n}}{\sqrt{1+u^2\tau^2}} \cdot (\vec{t}' - u\tau\vec{n}' - u\tau'\vec{n})$$

$$N = \frac{u\tau\vec{t} + \vec{n}}{\sqrt{1+u^2\tau^2}} \cdot (k\vec{n} - u\tau(+\tau\vec{b} - k\vec{t}) - u\tau'\vec{n})$$

$$= \frac{u\tau\vec{t} + \vec{n}}{\sqrt{1+u^2\tau^2}} \cdot ((k + u\tau^2)\vec{n} + u\tau k\vec{t} - u\tau'\vec{b})$$

$$N = \frac{u^2\tau^2 k + k - u\tau'}{\sqrt{1+u^2\tau^2}}$$

Question:-

On the surface generated by the unit tangent of the twisted curve. Find the fundamental magnitudes and unit normal to the surface.

Sol:-

Since, the given surface is generated by the unit tangent of a twisted curve. So, the position vector of the current point on the surface is

$$\vec{R} = \vec{r} + u\vec{t} \rightarrow (1)$$

where $\vec{r}$ and $\vec{t}$ are functions of "s" and u and s are parametres for the surface.

$$\vec{R}_1 = \frac{\partial \vec{R}}{\partial u} = \vec{t} \qquad \vec{R}_2 = \frac{\partial \vec{R}}{\partial s} = \vec{r}' + u\vec{t}'$$

put $\vec{r}' = \vec{t}$, $\vec{t}' = k\vec{n}$

$$\vec{R}_2 = \vec{t} + uk\vec{n}$$

$$\vec{R}_{11} = \frac{\partial \vec{R}_1}{\partial u} = 0$$

$$\vec{R}_{12} = \frac{\partial}{\partial u}(\vec{R}_2) = \vec{t}' = k\vec{n}$$

$$\vec{R}_{22} = \frac{\partial}{\partial s}(\vec{R}_2) = \vec{t}' + uk'\vec{n} + uk\vec{n}'$$

put $\vec{t}' = k\vec{n}$ and $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\vec{R}_{22} = k\vec{n} + uk'\vec{n} + uk(\tau\vec{b} - k\vec{t})$$

$$\vec{R}_{22} = (k + uk')\vec{n} + ukT\vec{b} - uk^2\vec{t}$$

$$\vec{R}_1 \times \vec{R}_2 = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 1 & 0 & 0 \\ 1 & uk & 0 \end{vmatrix} = \vec{t}(0 - uk) - \vec{n}(0-0) + \vec{b}(uk - 0)$$

$$\vec{R}_1 \times \vec{R}_2 = -uk\vec{t} + uk\vec{b}$$

$$|\vec{R}_1 \times \vec{R}_2| = \sqrt{u^2k^2 + u^2k^2} = \sqrt{2u^2k^2} = \sqrt{2}\,uk$$

$$N = \frac{\vec{R}_1 \times \vec{R}_2}{|\vec{R}_1 \times \vec{R}_2|} = \frac{-uk\vec{t} + uk\vec{b}}{\sqrt{2}\,uk} = \frac{\vec{b} - \vec{t}}{\sqrt{2}}$$

is unit normal to the surface.

Ist order fundamental magnitudes are

$$E = \vec{R}_1 \cdot \vec{R}_1 \quad , \quad F = \vec{R}_1 \cdot \vec{R}_2$$

$$E = \vec{t} \cdot \vec{t} = 1 \quad , \quad F = \vec{t} \cdot (\vec{t} + uk\vec{n}) = 1 + uk(\vec{t} \cdot \vec{n})$$

$$F = 1 + 0 = 1$$

$$G = \vec{R}_2 \cdot \vec{R}_2$$

$$G = (\vec{t} + uk\vec{n}) \cdot (\vec{t} + uk\vec{n}) = 1 + u^2k^2$$

2nd order fundamental magnitudes are

$$L = \vec{N} \cdot \vec{R}_{11}$$

$$L = \frac{\vec{b} - \vec{t}}{\sqrt{2}} \cdot 0 = 0$$

$$M = \vec{N} \cdot \vec{R}_{12}$$

$$M = \frac{\vec{b} - \vec{t}}{\sqrt{2}} \cdot (k\vec{n})$$

$$M = \frac{k(\vec{b} \cdot \vec{n}) - k(\vec{t} \cdot \vec{n})}{\sqrt{2}}$$

$$M = 0$$

$$N = \vec{N} \cdot \vec{R}_{22}$$

$$N = \frac{\vec{b} - \vec{t}}{\sqrt{2}} \cdot (k + uk')\vec{n} + ukT\vec{b} - uk^2\vec{t}$$

$$N = \frac{ukT + uk^2}{\sqrt{2}}$$

Questions:-

Consider the surface given by $z = f(x, y)$, if p and q are Ist order derivatives of z and r, s and t are 2nd order derivatives of z. Then, find the Ist and 2nd order magnitudes and the unit normal to the surface.

Sol:-

The position vector of the current point on the surface is given by

$$\vec{r} = (x, y, z)$$

$$\vec{r} = (x, y, f(x, y))$$

Given that

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial y^2} = t, \quad \frac{\partial^2 z}{\partial x \partial y} = s$$

Now

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = \left(1, 0, \frac{\partial z}{\partial x}\right) = (1, 0, p)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = \left(0, 1, \frac{\partial z}{\partial y}\right) = (0, 1, q)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial x}\right) = \left(0, 0, \frac{\partial^2 z}{\partial x^2}\right) = (0, 0, r)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \vec{r}}{\partial y}\right) = \left(0, 0, \frac{\partial^2 z}{\partial x \partial y}\right) = (0, 0, s)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \vec{r}}{\partial y}\right) = \left(0, 0, \frac{\partial^2 z}{\partial y^2}\right) = (0, 0, t)$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} \rightarrow (1)$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix} = \hat{i}(0-p) - \hat{j}(q-0) + \hat{k}(1-0)$$

$$(-p, -q, 1) \Leftarrow$$

$$\vec{r}_1 \times \vec{r}_2 = (1, 0, p) \times (0, 1, q)$$

$$= (-p, -q, 1)$$

$|\vec{r}_1 \times \vec{r}_2| = \sqrt{p^2+q^2+1}$

Put in (1)

$\vec{N} = \dfrac{-p\hat{i} - q\hat{j} + \hat{k}}{\sqrt{p^2+q^2+1}}$ is unit normal to the surface.

Ist order fundamental magnitudes are

$E = \vec{r}_1 \cdot \vec{r}_1$

$= (1, 0, p) \cdot (1, 0, p)$

$E = 1+p^2$

$F = \vec{r}_1 \cdot \vec{r}_2$

$= (1, 0, p) \cdot (0, 1, q)$

$F = pq$

$G = \vec{r}_2 \cdot \vec{r}_2$

$= (0, 1, q) \cdot (0, 1, q)$

$G = 1+q^2$

$H = \sqrt{EG - F^2} = \sqrt{(1+p^2)(1+q^2) - (pq)^2}$

$= \sqrt{1+p^2+q^2+p^2q^2-p^2q^2}$

$H = \sqrt{1+p^2+q^2}$

2nd order fundamental magnitudes are

$L = \vec{N} \cdot \vec{r}_{11}$

$L = \dfrac{-p\hat{i} - q\hat{j} + \hat{k}}{H} \cdot (0, 0, r)$

$L = \dfrac{r}{H} = \dfrac{r}{\sqrt{1+p^2+q^2}}$

$M = \vec{N} \cdot \vec{r}_{12}$

$$M = \frac{-p\hat{i} - q\hat{i} + \hat{k}}{H} \cdot (0, 0, s)$$

$$M = \frac{s}{H} = \frac{s}{\sqrt{1+p^2+q^2}}$$

$$N = \vec{N} \cdot \vec{\gamma}_{22}$$

$$N = \frac{-p\hat{i} - q\hat{i} + \hat{k}}{H} \cdot (0, 0, t) = \frac{t}{H} = \frac{t}{\sqrt{1+p^2+q^2}}$$

Question:-

Show that the curve $du^2 - (u^2 + c^2) d\phi^2 = 0$ form an orthogonal system for the surface $x = u\cos\phi$, $y = u\sin\phi$, $z = c\phi$

Sol:-

Two directions $\frac{du}{d\phi}$ and $\frac{\delta u}{\delta\phi}$ form an orthogonal system if $\frac{du}{d\phi}$

$$E \frac{du}{d\phi} \frac{\delta u}{\delta\phi} + F\left(\frac{du}{d\phi} + \frac{\delta u}{\delta\phi}\right) + G = 0 \rightarrow (A)$$

The given surface is

$$\vec{\gamma} = u\cos\phi\,\hat{i} + u\sin\phi\,\hat{j} + c\phi\,\hat{k}$$

$$\vec{\gamma}_1 = \frac{\partial\vec{\gamma}}{\partial u} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\vec{\gamma}_2 = \frac{\partial\vec{\gamma}}{\partial\phi} = -u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + c\hat{k}$$

$$E = \vec{\gamma}_1 \cdot \vec{\gamma}_1$$

$$= (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot (\cos\phi\,\hat{i} + \sin\phi\,\hat{j})$$

$$= \cos^2\phi + \sin^2\phi$$

$$E = 1$$

$$F = \vec{\gamma}_1 \cdot \vec{\gamma}_2$$

$$= (\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) \cdot (-u\sin\phi\,\hat{i} + u\cos\phi\,\hat{j} + c\hat{k})$$

$$F = -u\sin\phi\cos\phi + u\sin\phi\cos\phi$$

$$F = 0$$

$G = \vec{r}_2 . \vec{r}_2$

$= (-u\cos\phi\hat{i} + u\sin\phi\hat{j} + c\hat{k}) . (-u\cos\phi\hat{i} + u\sin\phi\hat{j} + c\hat{k})$

$G = u^2\cos^2\phi + u^2\sin^2\phi + c^2$

$= u^2(\cos^2\phi + \sin^2\phi) + c^2$

$G = u^2 + c^2$

The given curve is

$$du^2 + (u^2+c^2)d\phi^2 = 0$$

$$\Rightarrow du^2 - (u^2+c^2)\,d\phi^2 = 0 \rightarrow$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 = u^2+c^2 \rightarrow (1)$$

$\Rightarrow$ If $\frac{du}{d\phi}$ and $\frac{\delta u}{\delta\phi}$ are the roots of eq (1)

then $\frac{du}{d\phi} = \sqrt{u^2+c^2}$, $\frac{\delta u}{\delta\phi} = -\sqrt{u^2+c^2}$

Now the left hand side of eq (A) is

$$E\frac{du}{d\phi}\frac{\delta u}{\delta\phi} + F\left(\frac{du}{d\phi} + \frac{\delta u}{\delta\phi}\right) + G = 1(\sqrt{u^2+c^2})(-(\sqrt{u^2+c^2})) + 0 + u^2 + c^2$$

$$= -(u^2+c^2) + (u^2+c^2)$$

$$\Rightarrow E\frac{du}{d\phi}\frac{\delta u}{\delta\phi} + F\left(\frac{du}{d\phi} + \frac{\delta u}{\delta\phi}\right) + G = 0$$

Hence, the given curve $du^2 - (u^2+c^2)d\phi^2 = 0$ form an orthogonal system of directions for the surface $x = u\cos\phi$, $y = u\sin\phi$, $z = c\phi$

Question:-

For the surface generated by the unit principal normal of the twisted curve. Find the fundamental magnitudes and the unit normal to the surface.

Sol:-

The position vector $\vec{R}$ of the current point on the surface generated by the unit principal normal of a twisted curve is

$$\vec{R} = \vec{r} + u\vec{n}$$

where $\vec{r}$ and $\vec{n}$ are functions of s and $\vec{R}$ u and s

$$\vec{R}(u,s) = \vec{r}(s) + u\vec{n}(s)$$

$$\vec{R}_1 = \frac{\partial \vec{R}}{\partial u} = \vec{n}$$

$$\vec{R}_2 = \frac{\partial \vec{R}}{\partial s} = \vec{r}' + u\vec{n}'$$

put $\vec{r}' = \vec{t}$, $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\vec{R}_2 = \vec{t} + u(\tau\vec{b} - k\vec{t})$$

$$\vec{R}_2 = \vec{t} + u\tau\vec{b} - uk\vec{t}$$

$$\vec{R}_2 = (1-uk)\vec{t} + u\tau\vec{b}$$

$$\vec{R}_{11} = \frac{\partial \vec{R}_1}{\partial u} = 0$$

$$\vec{R}_{12} = \frac{\partial \vec{R}_2}{\partial u} = (0-k)\vec{t} + (1-uk)\vec{t}' + \tau\vec{b}$$

$$\vec{R}_{12} = -k\vec{t} + (1-uk)(+k\vec{n}) + \tau\vec{b}$$

$$\vec{R}_{12} = \tau\vec{b} - k\vec{t} + k\vec{n} - uk^2\vec{n} + \tau\vec{b}$$

$\tau(s)$, $k(s)$, $\vec{t}(s)$, $\vec{n}(s)$ are all functions of s.

$$\vec{R}_{22} = \frac{\partial \vec{R}_2}{\partial s} = (1-uk)\vec{t}' + u\tau\vec{b}' + u\tau'\vec{b} + (0-uk')\vec{t}$$

$$\vec{R}_{22} = -uk'\vec{t} + (1-uk)(k\vec{n}) + u\tau(-\tau\vec{n}) + u\tau'\vec{b}$$

$$= -uk'\vec{t} + k\vec{n} - uk^2\vec{n} - \tau^2 u\vec{n} + u\tau'\vec{b}$$

$$\vec{R}_{22} = -uk'\vec{t} + (k - uk^2 - u\tau^2)\vec{n} + u\tau'\vec{b}$$

$$\vec{N} = \frac{\vec{R}_1 \times \vec{R}_2}{|\vec{R}_1 \times \vec{R}_2|} \rightarrow (1)$$

$$\vec{R}_1 \times \vec{R}_2 = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 0 & 1 & 0 \\ 1-uk & 0 & u\tau \end{vmatrix}$$

$$\vec{R}_1 \times \vec{R}_2 = \vec{t}(u\tau - 0) - \vec{n}(0-0) + \vec{b}(1 - uk - 0)$$
$$= u\tau\vec{t} + (1-uk)\vec{b}$$

$$|\vec{R}_1 \times \vec{R}_2| = \sqrt{u^2\tau^2 + (1-uk)^2}$$

$$\vec{N} = \frac{u\tau\vec{t} + (1-uk)\vec{b}}{\sqrt{u^2\tau^2 + (1-uk)^2}}$$ is unit normal

Ist order fundamental magnitudes are

$$E = \vec{R}_1 \cdot \vec{R}_1$$
$$= \vec{n}_1 \cdot \vec{n}_1$$
$$E = 1$$
$$F = \vec{R}_1 \cdot \vec{R}_2$$
$$F = \vec{n} \cdot ((1-uk)\vec{t} + u\tau\vec{b})$$
$$= (1-uk)(\vec{n}\cdot\vec{t}) + u\tau(\vec{n}\cdot\vec{b})$$

$$F = 0$$
$$G = \vec{R}_2 \cdot \vec{R}_2$$
$$= ((1-uk)\vec{t} + u\tau\vec{b}) \cdot ((1-uk)\vec{t} + u\tau\vec{b})$$
$$= (1-uk)^2 + (u\tau)^2$$
$$G = (1-uk)^2 + u^2\tau^2$$

2nd order fundamental magnitudes are

$$L = \vec{N} \cdot \vec{R}_{11}$$

$$= \frac{u\tau\vec{t} - (1-uk)\vec{b}}{\sqrt{u^2\tau^2 + (1-uk)^2}} \cdot \vec{0}$$

$$L = 0$$
$$M = \vec{N} \cdot \vec{R}_{12}$$

$$M = \frac{u\tau\vec{t} - (1-uk)\vec{b}}{\sqrt{u^2\tau^2 + (1-uk)^2}} \cdot (\tau\vec{b} - k\vec{t})$$

$$M = \frac{u\tau\vec{t} - (1-uk)\vec{b}}{\sqrt{u^2\tau^2 + (1-uk)^2}} \cdot (-k\vec{t} + \tau\vec{b})$$

$$M = \frac{-uk\tau - (1-uk)\tau}{\sqrt{u^2\tau^2 + (1-uk)^2}}$$

$$N = \vec{N} \cdot \vec{R}_{22}$$

$$N = \frac{u\tau\vec{t} - (1-uk)\vec{b}}{\sqrt{u^2\tau^2 + (1-uk)^2}} \cdot (-uk'\vec{t} + (k - uk^2 - \tau^2 u)\vec{n} + u\tau'\vec{b})$$

$$N = \frac{-u^2\tau k' + 0 - u\tau'(1-uk)}{\sqrt{u^2\tau^2 + (1-uk)^2}}$$

$$N = \frac{-u^2\tau k' - u\tau' + u^2\tau' k}{\sqrt{u^2\tau^2 + (1-uk)^2}}$$

Principal directions and principal curvatures:-

The two principal directions through a point on a surface along which the normal curvature attains extreme values are known as principal directions on a surface, and these extreme values of normal curvature are denoted by k_a and k_b and are known as principal curvatures.

Differential equation for principal directions:

The normal curvature of a surface in the direction $\frac{du}{dv}$ is given by

$$k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \rightarrow (1)$$

$$k_n = \frac{L\left(\frac{du}{dv}\right)^2 + 2M\frac{du}{dv} + N}{E\left(\frac{du}{dv}\right)^2 + 2F\frac{du}{dv} + G}$$ by dividing dv^2

Let $\lambda = \frac{du}{dv}$

$$k_n = \frac{L\lambda^2 + 2M\lambda + N}{E\lambda^2 + 2F\lambda + G} \rightarrow (1)$$

Now To find extreme values of normal curvature k_n

Differentiate k_n w.r.t λ

$$\frac{dk_n}{d\lambda} = \frac{(E\lambda^2+2F\lambda+G)(2L\lambda+2M)-(L\lambda^2+2M\lambda+N)(2E\lambda+2F)}{(E\lambda^2+2F\lambda+G)^2}$$

$$\Rightarrow \frac{dk_n}{d\lambda} = \frac{2\{(E\lambda^2+2F\lambda+G)(L\lambda+M)-(L\lambda^2+2M\lambda+N)(E\lambda+F)\}}{(E\lambda^2+2F\lambda+G)^2} \rightarrow (A)$$

$\frac{dk_n}{d\lambda} = 0$ for extreme values

so

$(A) \Rightarrow (E\lambda^2+F\lambda+F\lambda+G)(L\lambda+M)-(L\lambda^2+2M\lambda+M\lambda+N)(E\lambda+F)=0$

$\Rightarrow \lambda(E\lambda+F)(L\lambda+M)+(F\lambda+G)(L\lambda+M)-\lambda(L\lambda+M)(E\lambda+F)$
$-(M\lambda+N)(E\lambda+F)=0$

$\Rightarrow (F\lambda+G)(L\lambda+M)-(M\lambda+N)(E\lambda+F)=0 \rightarrow (2)$

$FL\lambda^2+FM\lambda+GL\lambda+GM-ME\lambda^2 \pm MF\lambda \pm NE\lambda \pm NF=0$

$\Rightarrow (FL-ME)\lambda^2+(FM+GL+FM \pm NE)\lambda \pm GM \pm NF=0$

$\Rightarrow \lambda^2(LF-EM)+\lambda(GL-NE)+(GM-NE)=0$

put $\lambda = \frac{du}{dv}$

$\Rightarrow \left(\frac{du}{dv}\right)^2(LF-EM)+\frac{du}{dv}(GL-NE)+(GM-NE)=0$

$\Rightarrow du^2(LF-EM)+(GL-NE)\,du\,dv+(GM-NF)dv^2=0$

$\Rightarrow LF-EM)$

$\Rightarrow (GM-NE)dv^2+(GL-NE)du\,dv+(LF-EM)du^2=0$

$\rightarrow (3)$

$$\Rightarrow \begin{vmatrix} dv^2 & -du\,dv & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \rightarrow (4)$$

Eq (4) is known as differential equation for principal directions on a surface.

Now, From eq (2) we have

$$(F\lambda+G)(L\lambda+M) - (M\lambda+N)(E\lambda+F) = 0$$

$$\Rightarrow (F\lambda+G)(L\lambda+M) = (M\lambda+N)(E\lambda+F)$$

$$\frac{L\lambda+M}{E\lambda+F} = \frac{M\lambda+N}{F\lambda+G} = \alpha$$

$$\Rightarrow \frac{L\lambda+M}{E\lambda+F} = \alpha \quad , \quad \frac{M\lambda+N}{F\lambda+G} = \alpha$$

$$\Rightarrow L\lambda+M = \alpha(E\lambda+F) \rightarrow (5), \quad M\lambda+N = \alpha(F\lambda+G) \rightarrow (6)$$

put these values in (3)

$$k_n = \frac{L\lambda^2 + M\lambda + M\lambda + N}{E\lambda^2 + 2F\lambda + F\lambda + G}$$

$$k_n = \frac{\lambda(L\lambda+M) + (M\lambda+N)}{\lambda(E\lambda+F) + (F\lambda+G)} \rightarrow (7)$$

Now put values

$$k_n = \frac{\lambda(\alpha(E\lambda+F)) + \alpha(F\lambda+G)}{\lambda(E\lambda+F) + (F\lambda+G)}$$

$$k_n = \alpha\left[\frac{\lambda(E\lambda+F) + (F\lambda+G)}{\lambda(E\lambda+F) + (F\lambda+G)}\right]$$

$$k_n = \alpha(1)$$

$$\Rightarrow k_n = \alpha$$

put $\alpha = \dfrac{L\lambda+M}{E\lambda+F}$, $\alpha = \dfrac{M\lambda+N}{F\lambda+G}$

So $k_n = \dfrac{L\lambda+M}{E\lambda+F} = \dfrac{M\lambda+N}{F\lambda+G}$

$$\Rightarrow k_n = \frac{L\lambda + M}{E\lambda + F}, \quad k_n = \frac{M\lambda + N}{F\lambda + G}$$

$$\Rightarrow (E\lambda + F)k_n = L\lambda + M, \quad \Rightarrow (F\lambda + G)k_n = M\lambda + N$$
$$\Rightarrow Ek_n\lambda + Fk_n = L\lambda + M, \quad \Rightarrow Fk_n\lambda + Gk_n = M\lambda + N$$
$$\Rightarrow (Ek_n - L)\lambda = M - Fk_n, \Rightarrow (Fk_n - M)\lambda = N - Gk_n$$

$$\Rightarrow \lambda = \frac{M - Fk_n}{Ek_n - L} \rightarrow (8) \quad \Rightarrow \lambda = \frac{N - Gk_n}{Fk_n - M} \rightarrow (9)$$

Equating (8) and (9)

$$\frac{M - Fk_n}{Ek_n - L} = \frac{N - Gk_n}{Fk_n - M}$$

$$\Rightarrow (M - Fk_n)(Fk_n - M) = (N - Gk_n)(Ek_n - L)$$
$$\Rightarrow MFk_n - M^2 - F^2k_n^2 + MFk_n = NEk_n \pm GEk_n^2$$
$$- \cancel{GEk_n^2} - NL + GLk_n$$
$$\Rightarrow EGk_n^2 - F^2k_n^2 + 2MFk_n - NEk_n - LGk_n + LN - M^2 = 0$$

$$\Rightarrow (EG - F^2)k_n^2 + (2MF - NE - LG)k_n + LN - M^2 = 0$$

put $EG - F^2 = H^2$, $LN - M^2 = T^2$

$$\Rightarrow H^2k_n^2 + (2MF - NE - LG)k_n + T^2 = 0$$

Eq (10) gives the values of principal curvatures k_a and k_b

First Curvature:-

The first curvature at any point of a surface is defined as the sum of the principal curvatures. It is denoted by J, and

$J = k_a + k_b$ at that point.

Second Curvature:-

The second curvature at any point of the surface is defined as the product of the principal curvatures, it is denoted by k

i-e, $k = k_a k_b$

It is also known as specific curvature or Gauss's curvature or Gaussian curvature.

Amplitude and mean normal curvature:-

The ampelitude A of the normal curvature and the mean normal curvature B at any point of a surface are defined as

$$A = \frac{1}{2}(k_a - k_b)$$

and $$B = \frac{1}{2}(k_a + k_b)$$

Question:-

Find the principal directions and principal curvatures for the surface.

$x = u\cos\phi, \quad y = u\sin\phi, \quad z = c\phi$

Sol:-

$$\vec{r} = (u\cos\phi,\ u\sin\phi,\ c\phi)$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = (\cos\phi,\ \sin\phi)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = (-u\sin\phi,\ u\cos\phi,\ c)$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = (\cos\phi, \sin\phi)\cdot(\cos\phi, \sin\phi) = \cos^2\phi + \sin^2\phi$$

$$E = 1$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = (\cos\phi, \sin\phi)\cdot(-u\sin\phi, u\cos\phi, c)$$

$$F = -u\sin\phi\cos\phi + u\sin\phi\cos\phi + 0 = 0$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = (-u\sin\phi, u\cos\phi, c)\cdot(-u\sin\phi, u\cos\phi, c)$$

$$G = u^2\sin^2\phi + u^2\cos^2\phi + c^2 = u^2(\cos^2\phi + \sin^2\phi) + c^2 = u^2 + c^2$$

The differential equation for principal directions is

$$\begin{vmatrix} d\phi^2 & -du\,d\phi & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} d\phi^2 & -du\,d\phi & du^2 \\ 0 & \frac{-c}{\sqrt{u^2+c^2}} & 0 \\ 1 & 0 & u^2+c^2 \end{vmatrix} = 0$$

$$\Rightarrow d\phi^2\left(-c\sqrt{u^2+c^2} - 0\right) + du\,d\phi\,(0-0) + du^2\left(0 + \frac{c}{\sqrt{u^2+c^2}}\right) = 0$$

$$\Rightarrow -c\sqrt{u^2+c^2}\,d\phi^2 + c\,\frac{1}{\sqrt{u^2+c^2}}\,du^2 = 0$$

$$\Rightarrow \frac{1}{\sqrt{u^2+c^2}}\,du^2 - \sqrt{u^2+c^2}\,d\phi^2 = 0$$

$$\Rightarrow \frac{1}{\sqrt{u^2+c^2}}\,du^2 = \sqrt{u^2+c^2}\,d\phi^2$$

$$\Rightarrow \left(\frac{du}{d\phi}\right)^2 = \frac{du^2}{d\phi^2} = u^2+c^2$$

$$\Rightarrow \frac{du}{d\phi} = \pm\sqrt{u^2+c^2}$$

where are two principal direction

For Principal curvature

$$H^2 k_n^2 + (2MF - NE - LG)k_n + T^2 = 0 \rightarrow (1)$$

we know $H^2 = EG - F^2$, $T^2 = LN - M^2$

$H^2 = (1)(u^2+c^2) - 0$, $T^2 = 0 - \left(\frac{-c}{\sqrt{u^2+c^2}}\right)^2$

$H^2 = u^2+c^2$, $T^2 = \frac{-c^2}{u^2+c^2}$

put in (1)

$$(u^2+c^2)k_n^2 + (+0+0+0) - \frac{c^2}{u^2+c^2} = 0$$

$$\Rightarrow (u^2+c^2)k_n^2 = \frac{c^2}{u^2+c^2}$$

$$k_n^2 = \frac{c^2}{(u^2+c^2)^2}$$

$$k_n = \pm \frac{c}{u^2+c^2}$$

Question:-

Prove that the principal directions at every point of a surface are orthogonal.

Sol:-

Two directions on a surface are orthogonal if

$$E\frac{du}{dv}\frac{\delta u}{\delta v} + F\left(\frac{du}{dv} + \frac{\delta u}{\delta v}\right) + G = 0$$

where du/dv and $\delta u/\delta v$ are two directions.

Now the differential equation for principal directions is given by

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

$\Rightarrow$

$$\Rightarrow dv^2(MG-NF) + dudv(LG-NE) + du^2(LF-ME) = 0$$

Dividing throughout by dv^2

$$\Rightarrow (MG-NF) + \frac{du}{dv}(LG-NE) + \frac{du^2}{dv^2}(LF-ME) = 0$$

$$\Rightarrow \left(\frac{du}{dv}\right)^2(LF-EM) + \left(\frac{du}{dv}\right)(LG-NE) + (MG-NF) = 0 \quad (1)$$

If $\frac{du}{dv}$ and $\frac{\delta u}{\delta v}$ are the roots of eq (1). Then,

products of roots $= \frac{c}{a}$

sum of roots $= -\frac{b}{a}$

the product of roots

$$\frac{du}{dv}\frac{\delta u}{\delta v} = \frac{MG-NF}{LF-EM}$$

and sum of roots $\frac{du}{dv} + \frac{\delta u}{\delta v} = \frac{EN-LG}{LF-EM}$

Now Consider

$$E\frac{du}{dv}\frac{\delta u}{\delta v}+F\left(\frac{du}{dv}+\frac{\delta u}{\delta v}\right)+G=E\left(\frac{MG-NF}{LF-EM}\right)$$

$$+F\left(\frac{EN-LG}{LF-EM}\right)+G=0$$

$$\Rightarrow E\frac{du}{dv}\frac{\delta u}{\delta v}+F\left(\frac{du}{dv}+\frac{\delta u}{\delta v}\right)+G=\frac{E\cancel{MG}-\cancel{ENF}+\cancel{FEN}-\cancel{FLG}+\cancel{GLF}-\cancel{GEM}}{LF-EM}=0$$

$$\Rightarrow E\frac{du}{dv}\frac{\delta u}{\delta v}+F\left(\frac{du}{dv}+\frac{\delta u}{\delta v}\right)+G=0$$

Hence, the principal directions at every point of the surface are orthogonal.

Question:-

Find the principal directions and the curvatures for the surface $x=a(u+v)$, $y=b(u-v)$, $z=uv$

Sol:-

$$\vec{r}=(a(u+v),\ b(u-v),\ uv)$$

$$\vec{r}_1=\frac{\partial\vec{r}}{\partial u}=(a,\ b,\ v)$$

$$\vec{r}_2=\frac{\partial\vec{r}}{\partial v}=(a,\ -b,\ u)$$

$$\vec{r}_{11}=\frac{\partial^2\vec{r}}{\partial u^2}=(0,0,0)$$

$$\vec{r}_{12}=\frac{\partial^2\vec{r}}{\partial u\partial v}=(0,0,1)$$

$$\vec{r}_{22}=\frac{\partial^2\vec{r}}{\partial v^2}=(0,0,0)$$

$$E=\vec{r}_1\cdot\vec{r}_1=a^2+b^2+v^2$$

$$F=\vec{r}_1\cdot\vec{r}_2=a^2-b^2+uv$$

$G = \vec{r}_2 \cdot \vec{r}_2 = a^2+b^2+u^2$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & v \\ a & -b & u \end{vmatrix}$$

$$= \hat{i}(bu+bv) - \hat{j}(au-av) + \hat{k}(-ab-ab)$$

$$= b(u+v)\hat{i} - a(u-v)\hat{j} - 2ab\hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{b^2(u^2+v^2+2uv)+a^2(u^2+v^2-2uv)+4a^2b^2}$$

let $H = \sqrt{b^2u^2+b^2v^2+2b^2uv+a^2u^2+a^2v^2-2a^2uv+4a^2b^2}$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$$

$$\vec{N} = \frac{b(u+v)\hat{i} - a(u-v)\hat{j} - 2ab\hat{k}}{H}$$

Now

$$L = \vec{N} \cdot \vec{r}_{11} = 0$$

$$M = \vec{N} \cdot \vec{r}_{12} = \frac{b(u+v)\hat{i} - a(u-v)\hat{j} - 2ab\hat{k}}{H} \cdot (0,0,1)$$

$$M = \frac{-2ab}{H}$$

$$N = \vec{N} \cdot \vec{r}_{22} = 0$$

The differential equation for principal direction is

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ 0 & \frac{-2ab}{H} & 0 \\ a^2+b^2+v^2 & a^2-b^2+uv & a^2+b^2+u^2 \end{vmatrix} = 0$$

$$\Rightarrow dv^2\left(\frac{-2ab(a^2+b^2+u^2)}{H} - 0\right) + du\,dv(0-0)$$

$$+ du^2\left(0 + \frac{2ab(a^2+b^2+v^2)}{H}\right) = 0$$

$$\Rightarrow \frac{-2ab(a^2+b^2+u^2)dv^2}{H} + \frac{2ab(a^2+b^2+v^2)du^2}{H} = 0$$

$$\Rightarrow -(a^2+b^2+u^2)dv^2 + (a^2+b^2+v^2)du^2 = 0$$

$$\Rightarrow du^2(a^2+b^2+v^2) = (a^2+b^2+u^2)dv^2$$

$$\left(\frac{du}{dv}\right)^2 = \frac{a^2+b^2+u^2}{a^2+b^2+v^2}dv^2$$

$$\Rightarrow \frac{du}{dv} = \pm\sqrt{\frac{a^2+b^2+u^2}{a^2+b^2+v^2}}\,dv$$

The equation for principal curvature is

$$H^2k_n^2 + (2MF - NE - LG)k_n + T^2 = 0$$

$\therefore$ Now, First curvature J_a is k_b

$$J = k_a + k_b \qquad k_a + k_b = \text{Sum of roots} = \frac{-b}{a}$$

$$J = \frac{-2MF + NE + LG}{H^2} \rightarrow (1)$$

$$J = \frac{-2\left(\frac{-2ab}{H}\right)(a^2-b^2+uv) + 0 + 0}{H^2}$$

$$J = \frac{\frac{4ab(a^2-b^2+uv)}{H}}{H^2}$$

$$J = \frac{4ab(a^2-b^2+uv)}{H^3}$$

Now, the second curvature k is given by

$$k = k_a k_b$$

$$k = \frac{T^2}{H^2} \qquad \because H^2 = EG - F^2,\ T^2 = LN - M^2$$

$$k = \frac{LN - M^2}{EG - F^2}$$

$$s \cdot r = -\frac{b}{a}$$

$$p \cdot r = \frac{c}{a}$$

$$k = \frac{0 - \left(-\frac{2ab}{H}\right)^2}{(a^2+b^2 H^2)(a^2+b^2+u^2) - (a^2-b^2+uv)^2}$$

$$k = \frac{-\frac{4a^2b^2}{H^2}}{a\ H^2} = \frac{-4a^2b^2}{H^4}$$

where

$$H = \sqrt{b^2(u+v)^2 + a^2(u-v)^2 + 4a^2b^2}$$

Question:-

Find the principal curvatures and principal directions for the surface generated by the bi-normal of a twisted curve.

Sol:-

The equation for the surface generated by the bi-normal of a twisted curve is

$$\vec{R} = \vec{r} + u\vec{b} \to (1)$$

where u and s are parametres

$$\vec{R}_1 = \frac{\partial R}{\partial u} = \vec{b}$$

$$\vec{R}_2 = \frac{\partial \vec{R}}{\partial s} = \vec{r}' + u\vec{b}' = \vec{t} + u(-\tau\vec{n}) = \vec{t} - u\tau\vec{n}$$

$$\vec{R}_{11} = \frac{\partial^2 \vec{R}}{\partial u} = 0, \quad \vec{R}_{22} = \frac{\partial^2 \vec{R}}{\partial s^2} = \vec{t}' - u\tau'\vec{n} - u\tau\vec{n}' \quad \text{put } \vec{t}' = k\vec{n},\ \vec{n}' = \tau\vec{b} - k\vec{t}$$

$$\vec{R}_{22} = k\vec{n} - u\tau'\vec{n} - u\tau(\tau\vec{b} - k\vec{t})$$

$$\vec{R}_{22} = (k_1 - u\tau')\vec{n} - u\tau^2\vec{b} + u\tau k\vec{t}$$

$$\vec{R}_{12} = \frac{\partial \vec{R}_2}{\partial u} = \frac{\partial}{\partial u}(\vec{t} - u\tau\vec{n})$$

$$\vec{R}_{12} = -\tau\vec{n}$$

$$E = \vec{R}_1 \cdot \vec{R}_1 = \vec{b}\cdot\vec{b} = 1$$

$$F = \vec{R}_1 \cdot \vec{R}_2 = \vec{b}\cdot(\vec{t} - u\tau\vec{n}) = 0$$

$$G = \vec{R}_2 \cdot \vec{R}_2 = (\vec{t} - u\tau\vec{n})\cdot(\vec{t} - u\tau\vec{n})$$

$$G = 1 + u^2\tau^2$$

Also

$$\vec{N} = \frac{\vec{R}_1 \times \vec{R}_2}{|\vec{R}_1 \times \vec{R}_2|} \rightarrow (2)$$

$$\vec{R}_1 \times \vec{R}_2 = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 0 & 0 & 1 \\ 1 & -u\tau & 0 \end{vmatrix}$$

$$= \vec{t}(0 + u\tau) - \vec{n}(0 - 1) + \vec{b}(0 - 0)$$

$$= u\tau\vec{t} + \vec{n}$$

$$\vec{R}_1 \times \vec{R}_2 = u\tau\vec{t} + \vec{n} \Rightarrow |\vec{R}_1 \times \vec{R}_2| = \sqrt{1 + u^2\tau^2}$$

$$\vec{N} = \frac{k_1 - u\tau\vec{t} + u\tau^2 k}{\sqrt{1 + u^2\tau^2}} = \frac{k_1 - u\tau' + u^2\tau^2 k}{H}$$

$$L = \vec{N}\cdot R_{11} = 0$$

$$M = \vec{N}\cdot\vec{R}_{12} = \frac{u\tau\vec{t} + \vec{n}}{H}\cdot(-\tau n)$$

$$= \frac{-\tau}{H}$$

$$N = \vec{N}\cdot\vec{R}_{22} = \frac{u\tau\vec{t} + \vec{n}}{H}\cdot(k_1 - u\tau')\vec{n} - u\tau^2\vec{b} + u\tau k_1\vec{t}.$$

$$N = \frac{k_1 - u\tau' + u^2\tau^2 k}{H}$$

The differential equation of principal direction is

$$\begin{vmatrix} du^2 & -duds & ds^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} ds^2 & -duds & du^2 \\ 0 & -\tau/\sqrt{1+u^2\tau^2} & 0 \\ 1 & 0 & \end{vmatrix}$$

$$\begin{vmatrix} ds^2 & -duds & du^2 \\ 0 & -\tau/\sqrt{1+u^2\tau^2} & \dfrac{k-u\tau'+u^2\tau^2 k}{\sqrt{1+u^2\tau^2}} \\ 1 & 0 & 1+u^2\tau^2 \end{vmatrix} = 0$$

$$\Rightarrow ds^2\left(\frac{-\tau(1+u^2\tau^2)}{\sqrt{1+u^2\tau^2}} - 0\right) + duds\left(0 - \frac{k+u\tau'-u^2\tau^2 k}{\sqrt{1+u^2\tau^2}}\right)$$

$$\Rightarrow \quad + du^2\left(0 + \frac{\tau}{\sqrt{1+u^2\tau^2}}\right) = 0$$

$$\Rightarrow ds^2\left(-\tau(\sqrt{1+u^2\tau^2})\right) + duds\left(\frac{-k+u\tau'-u^2\tau^2 k}{\sqrt{1+u^2\tau^2}}\right) + du^2\left(\frac{\tau}{\sqrt{1+u^2\tau^2}}\right) = 0$$

$$\Rightarrow [-\tau(1+u^2\tau^2)]ds^2 + (-k+u\tau'-u^2\tau^2 k)duds + \tau du^2 = 0$$

$$H^2 k_n^2 + (2MF - NE - LG)k_n + T^2 = 0 \rightarrow (A)$$

$$T^2 = LN - M^2 = -\left(\frac{-\tau}{\sqrt{1+u^2\tau^2}}\right)^2 = \frac{-\tau^2}{1+u^2\tau^2} \quad \text{put in } (A)$$

$$(1+u^2\tau^2)k_n^2 + (-NE)k_n + T^2 = 0$$

$$\Rightarrow (1+u^2\tau^2)k_n^2 - \left(\frac{k-u\tau'+u^2\tau^2 k}{H}\right)k_n - \frac{\tau^2}{1+u^2\tau^2} = 0$$

$$\Rightarrow (1+u^2\tau^2)^2 k_n^2 - \left(\frac{k-u\tau'+u^2\tau^2 k}{\sqrt{1+u^2\tau^2}}\right)k_n - \tau^2 = 0$$

Now Ist curvature is $J = k_a + k_b = -\frac{b}{a}$

$$J = \frac{(k-u\tau'+u^2\tau^2 k)/(\sqrt{1+u^2\tau^2})}{(1+u^2\tau^2)^2}$$

$$= \frac{(k-u\tau'+u^2\tau^2 k)}{(1+u^2\tau^2)^{3/2}}$$

$$K = k_a \cdot k_b = c/a$$

$$K = -\tau^2/(1+u^2\tau^2)^2$$

Theorem:-

The necessary and sufficient condition for lines of curvature to be parametric curves is $F = M = 0$

(Lines of curvature are those along which normal directions are taken

Proof:-

Suppose that $F = M = 0$

The lines of curvatures are given by

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

Put $F = M = 0$

$$\Rightarrow \begin{vmatrix} dv^2 & -dudv & du^2 \\ L & 0 & N \\ E & 0 & G \end{vmatrix} = 0$$

$\Rightarrow dv^2(0-0) + dudv(LG - EN) + du^2(0-0) = 0$

$\Rightarrow (LG - EN)dudv = 0$

$\Rightarrow LG - EN \neq 0 \quad , \quad dudv = 0$

$\Rightarrow du = 0 \; , \; dv = 0$

Integrating

$\int du = \int 0 . du$

$u = \text{Constant}$

Hence, the lines of curvatures are parametric curves.

Now, Suppose that lines of curvatures are parametric curves

$u = \text{Constant}, \quad v = \text{Constant}$

$\Rightarrow du = 0 \quad , \quad dv = 0$

$\Rightarrow \quad du dv = 0 \rightarrow (1)$

Now, the lines of curvature are given by

$$\begin{vmatrix} dv^2 & -du dv & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

$\Rightarrow dv^2(MG - FN) + du dv(LG - EN) + du^2(LF - EM) = 0$

$\Rightarrow (MG - FN)dv^2 + (LG - EN)du dv + (LF - EM)du^2 = 0 \rightarrow (2)$

Comparing the co-efficients of du^2, $du dv$ and dv^2 in (1) and (2)

$\underset{(3)}{LF - EM = 0}, \quad \underset{(4)}{LG - EN = 1 \neq 0}, \quad \underset{(5)}{MG - FN = 0}$

$N \, Eq(3) + L \, Eq(5)$

$NLF - EMN = 0$

$-NLF + LMG = 0$

$0 + LMG - EMN = 0$

$M(LG - EN) = 0$

$\because LG - EN = 1) \quad M(1) = 0$

$\Rightarrow \quad M = 0$

Now,

$G \, Eq(3) + E \, Eq(5)$

$LFG - EMG = 0$

$-NFG + EMG = 0$

$LFG - NFE = 0$

$\Rightarrow F(LG - NE) = 0$

put $LG - NE = 1$ by (4)

$\Rightarrow F(1) = 0$

$\Rightarrow F = 0$

Euler's Theorem:-

If k_n is the normal curvature of a surface in any direction making an angle α with a principal direction, then $k_n = k_a \cos^2\alpha + k_b \sin^2\alpha$

Proof:-

Consider the lines of the curvature taken as parametric curves
$u =$ Constant and $v =$ Constant and the normal curvature being the principal curvature k_a is along $dv = 0$ or is of the curvature $dv = 0$ and the normal curvature k_b is of the curvature $du = 0$

Now the normal curvature of a surface in any direction $\frac{du}{dv}$ is given by

$$k_n = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} \Rightarrow (A)$$

Since, the lines of curvature are parametric curves so $F = M = 0$

$$k_n = \frac{L\,du^2 + 0 + N\,dv^2}{E\,du^2 + 0 + G\,dv^2}$$

$$k_n = \frac{L\,du^2 + N\,dv^2}{E\,du^2 + G\,dv^2}$$

Now, For k_a put $dv^2 = 0$

$$k_a = \frac{L\,du^2 + 0}{E\,du^2 + 0}$$

$$k_a = \frac{L}{E} \rightarrow (i)$$

and For k_b $\quad du^2 = 0$

$$k_b = \frac{0 + N dv^2}{0 + G dv^2}$$

$$k_b = \frac{N}{G} \rightarrow (2)$$

Now, Suppose that k_n is the normal curvature of the surface in any direction $\frac{du}{dv}$ making an angle α with principal direction $(du = 0, dv = 0)$ $dv = 0$. Then,

$$\cos\alpha = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right) = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right)$$

put $F = 0$;

$$\cos\alpha = \frac{1}{\sqrt{E}}\left(E\frac{du}{ds} + \frac{dv}{ds}\right)$$

$$\cos\alpha = \sqrt{E}\,\frac{du}{ds} \rightarrow (3)$$

Now $\sin\alpha = -\frac{H}{\sqrt{E}}\frac{dv}{ds} = -\frac{H}{\sqrt{E}}\frac{dv}{ds}$

put $H = \sqrt{EG - F^2}$

$$\sin\alpha = \frac{-\sqrt{EG - F^2}}{\sqrt{E}}\frac{dv}{ds} \quad \text{put } F = 0$$

$$= \frac{-\sqrt{EG}}{\sqrt{E}}\frac{dv}{ds}$$

$$\sin\alpha = -\sqrt{G}\,\frac{dv}{ds} \rightarrow (4)$$

Eq (3) and Eq (4) $\Rightarrow$

$$\frac{du}{ds} = \frac{1}{\sqrt{E}}\cos\alpha, \quad \frac{dv}{ds} = \frac{-1}{\sqrt{G}}\sin\alpha$$

Now, the normal curvature of a surface

in any direction is given by (A)

$$k_n = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

put $M = F = 0$

$$k_n = \frac{L du^2 + N dv^2}{E du^2 + G dv^2}$$

$$k_n = \frac{L\left(\frac{du}{ds}\right)^2 + N\left(\frac{dv}{ds}\right)^2}{E\left(\frac{du}{ds}\right)^2 + G\left(\frac{dv}{ds}\right)^2}$$

put $\frac{du}{ds} = \frac{1}{\sqrt{E}} \cos\alpha$, $\frac{dv}{ds} = \frac{-1}{\sqrt{G}} \sin\alpha$

$$k_n = \frac{L\left(\frac{1}{\sqrt{E}}\cos\alpha\right)^2 + N\left(-\frac{1}{\sqrt{G}}\sin\alpha\right)^2}{E\left(\frac{1}{\sqrt{E}}\cos\alpha\right)^2 + G\left(-\frac{1}{\sqrt{G}}\sin\alpha\right)^2}$$

$$= \frac{L\left(\frac{1}{E}\cos^2\alpha\right) + N\left(\frac{1}{G}\sin^2\alpha\right)}{E\left(\frac{1}{E}\cos^2\alpha\right) + G\left(\frac{1}{G}\sin^2\alpha\right)}$$

$$= \frac{\frac{L}{E}\cos^2\alpha + \frac{N}{G}\sin^2\alpha}{\cos^2\alpha + \sin^2\alpha}$$

$$k_n = \frac{L}{E}\cos^2\alpha + \frac{N}{G}\sin^2\alpha$$

put $\frac{L}{E} = k_a$ and $\frac{N}{G} = k_b$ from (1), (2)

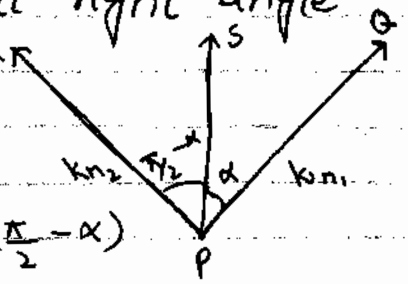
$$k_n = k_a \cos^2\alpha + k_b \sin^2\alpha$$

Corollary:-

The sum of the normal curvatures into two directions at right angle with each other is equal to the sum of the principal curvatures.

Proof:-

Let k_{n_1} and k_{n_2} be the normal curvatures for two directions at right angle with each other.



Then, by Euler's theorem

$$k_{n_1} = k_a \cos^2\alpha + k_b \sin^2\alpha \rightarrow (1)$$

$$k_{n_2} = k_a\left(\cos^2\left(\tfrac{\pi}{2} - \alpha\right)\right) + k_b \sin^2\left(\tfrac{\pi}{2} - \alpha\right)$$

$$k_{n_2} = k_a \sin^2\alpha + k_b \cos^2\alpha \rightarrow (2)$$

Adding (1) and (2)

$$k_{n_1} + k_{n_2} = k_a \sin^2\alpha + k_a \cos^2\alpha + k_b \cos^2\alpha + k_b \sin^2\alpha$$
$$= k_a$$
$$k_{n_1} + k_{n_2} = k_a + k_b$$

Question:-

Using Euler's theorem, Prove that

$$k_n = B - A\cos 2\alpha$$
$$k_n - k_a = 2A\sin^2\alpha$$
$$k_b - k_a = 2A\cos^2\alpha$$

where A and B are amplitude and mean curvature for the surface.

Sol:-

i) $k_n = B - A\cos 2\alpha$

$R.H.S = B - A\cos 2\alpha$

put $B = \frac{1}{2}(k_a + k_b)$, $A = \frac{1}{2}(k_b - k_a)$

and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$$B - A\cos 2\alpha = \frac{1}{2}(k_a + k_b) - \frac{1}{2}(k_b - k_a)(\cos^2\alpha - \sin^2\alpha)$$

$$B - A\cos 2\alpha = \frac{1}{2}k_a + \frac{1}{2}k_b - \frac{1}{2}k_b\cos^2\alpha + \frac{1}{2}k_b\sin^2\alpha$$

$$+\frac{1}{2}k_a\cos^2\alpha - \frac{1}{2}k_a\sin^2\alpha$$

$$B - A\cos 2\alpha = \frac{1}{2}\{k_b - k_b\cos^2\alpha + k_b\sin^2\alpha + k_a + k_a\cos^2\alpha - k_a\sin^2\alpha]$$

$$= \frac{1}{2}\{k_b(1-\cos^2\alpha) + k_b\sin^2\alpha + k_a(1-\sin^2\alpha) + k_a\cos^2\alpha]$$

$$= \frac{1}{2}\{k_b\sin^2\alpha + k_b\sin^2\alpha + k_a\cos^2\alpha + k_a\cos^2\alpha]$$

$$= \tfrac{1}{2}[2k_b\sin^2\alpha + 2k_a\cos^2\alpha]$$

$$B - A\cos 2\alpha = k_a\cos^2\alpha + k_b\sin^2\alpha$$

(ii) $k_n - k_a = 2A\sin^2\alpha$

R.H.S $= 2A\sin^2\alpha$

put $A = \frac{1}{2}(k_b - k_a)$

$$2A\sin^2\alpha = 2\left(\frac{1}{2}(k_b - k_a)\sin^2\alpha\right)$$

$$= (k_b - k_a)\sin^2\alpha$$

$$= k_b\sin^2\alpha - k_a\sin^2\alpha$$

$$= k_a\cos^2\alpha + k_b\sin^2\alpha - k_a\sin^2\alpha - k_a\cos^2\alpha$$

$$= k_n - k_a(\sin^2\alpha + \cos^2\alpha)$$

$$2A\sin^2\alpha = k_n - k_a$$

$$\Rightarrow k_n - k_a = 2A\sin^2\alpha$$

(iii)

$$k_b - k_a = 2A\cos^2\alpha$$

R.H.S $= 2A\cos^2\alpha$

put $A = \frac{1}{2}(k_b - k_a)$.

$$2A\cos^2\alpha = 2\left(\frac{1}{2}(k_b - k_a)\right)\cos^2\alpha$$

$$= (k_b - k_a)\cos^2\alpha$$

$$= k_b\cos^2\alpha - k_a\cos^2\alpha$$

Question:-

For the point of intersection of a paraboloid $xy = cz$ and a hyperboloid $x^2+y^2+z^2+c^2=0$. Find the principal radii of the paraboloid.

(the reciprocal of principal curvature is the principal radii of the paraboloid)

Sol:-

The given paraboloid is

$$\vec{r} = (x, y, z)$$

$$\vec{r} = (x, y, xy/c)$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = (1, 0, y/c)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0, 1, x/c)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, 0, 0)$$

$$\vec{r}_{22} = \quad = (0, 0, 0)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0, 0, 1/c)$$

$$E = \vec{r}_1 \cdot \vec{r}_1$$

$$= (1, 0, y/c) \cdot (1, 0, y/c)$$

$$= 1 + \frac{y^2}{c^2}$$

$$E = \frac{y^2+c^2}{c^2}$$

$$\vec{F} = \vec{r}_1 \cdot \vec{r}_2 = (1, 0, y/c) \cdot (0, 1, x/c)$$

$$F = xy/c^2$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = (0, 1, x/c) \cdot (0, 1, x/c)$$

$$G = 1 + \frac{x^2}{c^2} = \frac{x^2+c^2}{c^2}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y/c \\ 0 & 1 & x/c \end{vmatrix}$$

$$= \hat{i}(0 - y/c) - \hat{j}(x/c) + \hat{k}(1 - 0)$$

$$\vec{r}_1 \times \vec{r}_2 = (-y/c, -x/c, 1)$$

$$\vec{N} = \frac{(-y/c, -x/c, 1)}{H}$$

$$L = \vec{r}_{11} \cdot \vec{N}$$

$$= (0, 0, 0) \cdot \frac{(-y/c, -x/c, 1)}{H}$$

$$L = 0$$

$$M = \vec{r}_{12} \cdot N$$

$$= (0, 0, 1/c) \cdot \frac{(-y/c, -x/c, 1)}{H}$$

$$M = \frac{1/c}{H} = \frac{1}{CH}$$

$$N = \vec{r}_{22} \cdot N$$

$$N = (0, 0, 0) \cdot \frac{(-y/c, -x/c, 1)}{H}$$

$$N = 0$$

Now, the equation for principal curvatures is

$$H^2 k_n^2 + (2MF - NE - LG) k_n + T^2 = 0$$

put $N = 0$

$$H^2 k_n^2 + 2MF k_n + T^2 = 0$$

Dividing throughout by k_n^2

$$H^2 + 2MF \frac{1}{k_n} + T^2 \left(\frac{1}{k_n^2}\right) = 0$$

$$\Rightarrow \quad H^2 + 2MF \rho_n + T^2 \rho_n^2 = 0$$

$$\Rightarrow \rho_n^2 T^2 + 2MF\rho_n + H^2 = 0$$

put $M = \dfrac{1}{cH}$, $F = xy/c^2$

$$\Rightarrow \rho_n^2 T^2 + 2\left(\frac{1}{cH}\right)\left(\frac{xy}{c^2}\right)\rho_n + H^2 = 0$$

$$\Rightarrow \rho_n^2 T^2 + \frac{2xy}{c^3 H}\rho_n + H^2 = 0$$

put $T^2 = LN - M^2 \quad \because NL = 0$

$T^2 = -M^2 = -\left(\dfrac{1}{cH}\right)^2 = \dfrac{-1}{c^2H^2}$

$$\Rightarrow \rho_n^2\left(\frac{-1}{c^2H^2}\right) + \frac{2xy}{c^3H}\rho_n + H^2 = 0$$

Multiplying throughout by c^3H^2

$$\Rightarrow -c\rho_n^2 + 2xyH\rho_n + c^3H^4 = 0$$

$$\Rightarrow -c\rho_n^2 + 2xyH\rho_n + c^3H^4 = 0 \Rightarrow a = -c,\ b = +2xyH,\ c = +c^3H^4$$

$$\rho_n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\rho_n = \frac{-2xyH \pm \sqrt{4x^2y^2H^2 + 4c^4H^4}}{-2c} \rightarrow (1)$$

Now $H^2 = EG - F^2$

put $E = \dfrac{y^2+c^2}{c^2}$, $G = \dfrac{x^2+c^2}{c^2}$, $F = xy/c^2$

$$H^2 = \left(\frac{y^2+c^2}{c^2}\right)\left(\frac{x^2+c^2}{c^2}\right) - x^2y^2/c^4$$

$$= \frac{x^2y^2 + x^2c^2 + y^2c^2 + c^4 - x^2y^2}{c^4}$$

$$= \frac{(x^2+y^2)c^2 + c^4}{c^4}$$

$$H^2 = \frac{x^2+y^2+c^2}{c^2} \qquad \because x^2+y^2+c^2 = z^2$$

From hyperboloid put $x^2+y^2+c^2 = +z^2$

$$H^2 = \frac{-z^2}{c^2} \quad , \quad xy = cz$$

put in (i)

$$\rho_n = \frac{-2cz\left(+\frac{z}{c}\right) \pm \sqrt{4c^2z^2\left(-\frac{z^2}{c^2}\right) + 4c^4\left(-\frac{z^2}{c^2}\right)^2}}{-2c}$$

$$= \frac{-2z^2 \pm \sqrt{+4z^4 + 4z^4}}{-2c} = \frac{-2z^2 \pm \sqrt{8z^4}}{-2c}$$

$$\rho_n = \frac{-2z^2 \pm 2\sqrt{2}\,z^2}{-2c} = \frac{-z^2 \pm \sqrt{2}\,z^2}{c}$$

$$\rho_n = \frac{z^2(-1 \pm \sqrt{2})}{-c} = \frac{z^2(1 \pm \sqrt{2})}{c}$$

Question:-

Find the equation for principal curvature and the differential equation for the lines of curvature of the following curvatures.

(i) $2z = x^2/a + y^2/b$

(ii) $3z = ax^3 + by^3$

(iii) $z = c\tan^{-1}(y/x)$

Sol:-

(i) $$2z = \frac{x^2}{a} + \frac{y^2}{b}$$

As $\vec{r} = (x, y, z)$

$$\vec{r} = \left(x, y, \frac{x^2}{2a} + \frac{y^2}{2b}\right)$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = (1, 0, x/a)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0, 1, y/b)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, 0, 1/a)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0, 0, 0)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0, 1, 1/b)$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = 1 + \frac{x^2}{a^2} = \frac{a^2 + x^2}{a^2}$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = xy/ab$$

and $G = \vec{r}_2 \cdot \vec{r}_2$

$$G = 1 + \frac{y^2}{b^2} = \frac{b^2 + y^2}{b^2}$$

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & x/a \\ 0 & 1 & y/b \end{vmatrix}$$

$$\vec{r}_1 \times \vec{r}_2 = \hat{i}(-x/a) - \hat{j}(y/b) + \hat{k}(1) = \left(-\frac{x}{a}, -\frac{y}{b}, 1\right)$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}$$

$$\vec{N} = \frac{(-x/a, -y/b, 1)}{\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}}$$

$$L = \vec{N} \cdot \vec{r}_{11} = \frac{1}{a\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}} = \frac{1}{aH}$$

$$M = \vec{N} \cdot \vec{r}_{12} = 0$$

$$N = \vec{N} \cdot \vec{r}_{22}$$

$$N = \frac{1}{b\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}} = \frac{1}{bH}$$

Now, eq of principal curvature is

$$H^2 k_n^2 + (2MF - NE - LG)k_n + T^2 = 0$$

$$H^2 k_n^2 - (NE + LG)k_n + T^2 = 0 \quad \because M = 0$$

$$\rightarrow (1)$$

$$T^2 = LN - M^2$$

$$T^2 = \frac{1}{abH^2} - 0$$

$$T^2 = \frac{1}{abH^2}$$

$$(1) \Rightarrow \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1\right)k_n^2 - \left(\frac{(a^2+x^2)}{a^2 b\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}} + \frac{b^2+y^2}{b^2 a\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + 1}\right)}\right) + \frac{1}{abH^2} = 0$$

Now differential eq for lines of curvature is

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ \frac{1}{aH} & 0 & \frac{1}{bH} \\ \frac{a^2+x^2}{a^2} & \frac{xy}{ab} & \frac{b^2+y^2}{b^2} \end{vmatrix} = 0$$

$$\Rightarrow dy^2\left\{\frac{-xy}{ab^2H}\right\} + dxdy\left[\frac{b^2+y^2}{ab^2H} - \frac{a^2+x^2}{a^2bH}\right] + dx^2\left[\frac{xy}{a^2bH}\right] = 0$$

$$\Rightarrow dy^2\left[\frac{-xy}{ab^2H}\right] + dxdy\left[\frac{ab^2+ay^2-ba^2-bx^2}{a^2b^2H}\right] + dx^2\left[\frac{xy}{a^2bH}\right] = 0.$$

Surface of Revolution:-

A surface which is formed by the revolution of a plane curve about an axis in its plane is known as surface of revolution.

If z-axis is the axis of revolution and u denotes the distance of any point on the plane curves from z-axis, then the surface of revolution may be expressed as

$$\vec{r} = (u\cos\phi,\ u\sin\phi,\ f(u))$$

Minimal

Normal Surface:-

A surface on which the first curvature vanishes at all points is called the Minimal normal surface.

Question:-

If a surface of revolution is a normal surface, then show that

$$u\frac{d^2f}{du^2} + \frac{df}{du}\left\{1 + \left(\frac{df}{du}\right)^2\right\} = 0$$

Proof:-

The surface of revolution about z-axis is given by

$$\vec{r} = (u\cos\phi,\ u\sin\phi,\ f(u))$$

$$\vec{r}_1 = \frac{\partial\vec{r}}{\partial u} = (\cos\phi,\ \sin\phi,\ f'(u))$$

$$\vec{r}_2 = \frac{\partial\vec{r}}{\partial\phi} = (-u\sin\phi,\ u\cos\phi,\ 0)$$

$$\vec{r}_{11} = \frac{\partial^2\vec{r}}{\partial u^2} = (0,\ 0,\ f''(u))$$

$$\vec{r}_{12} = \frac{\partial^2\vec{r}}{\partial u\,\partial\phi} = (-\sin\phi,\ \cos\phi,\ 0)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial \phi^2} = (-u\cos\phi,\ -u\sin\phi,\ 0)$$

$\because f'(u) = f_1(u)$

$$E = \vec{r}_1 \cdot \vec{r}_1 = (\cos\phi,\ \sin\phi,\ f_1(u)) \cdot (\cos\phi,\ \sin\phi,\ f_1(u))$$
$$= \cos^2\phi + \sin^2\phi + f_1^2(u)$$
$$E = 1 + f_1^2(u)$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = (\cos\phi\ \sin\phi,\ f_1(u)) \cdot (-u\sin\phi,\ u\cos\phi,\ 0)$$
$$= -u\sin\phi\cos\phi + u\sin\phi\cos\phi + 0$$
$$F = 0$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = (-u\sin\phi,\ u\cos\phi\ 0) \cdot (-u\sin\phi,\ u\cos\phi,\ 0)$$
$$= u^2\sin^2\phi + u^2\cos^2\phi + 0$$
$$G = u^2$$

$$N = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & f_1(u) \\ -u\sin\phi & u\cos\phi & 0 \end{vmatrix}$$

$$= \hat{i}(0 - uf_1(u)\cos\phi) - \hat{j}(0 + uf_1(u)\sin\phi) + \hat{k}(u\cos^2\phi + u\sin^2\phi)$$

$$\vec{r}_1 \times \vec{r}_2 = -u\cos\phi f_1(u)\hat{i} - u\sin\phi f_1(u)\hat{j} + u\hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{u^2\cos^2\phi(f_1(u))^2 + u^2\sin^2\phi(f_1(u))^2 + u^2}$$
$$= \sqrt{u^2(f_1(u))^2 + u^2}$$
$$|\vec{r}_1 \times \vec{r}_2| = u\sqrt{1 + (f_1(u))^2}$$

$$N = \frac{(-u\cos\phi f_1(u),\ -u\sin\phi f_1(u),\ u)}{u\sqrt{1 + f_1^2(u)}}$$

$$N = \frac{(-\cos\phi f_1(u),\ -\sin\phi f_1(u),\ 1)}{\sqrt{1 + f_1^2(u)}}$$

$$L = \vec{N}\cdot\vec{r}_{11} = \frac{(-\cos\phi f_1(u), -\sin\phi f_1(u), 1)}{\sqrt{1+f_1^2(u)}}\cdot(0, 0, f_{11}(u))$$

$$L = \frac{f_{11}(u)}{\sqrt{1+f_1^2(u)}}$$

$$M = \vec{N}\cdot\vec{r}_{12} = \frac{(-\cos\phi f_1(u), -\sin\phi f_1(u), 1)}{\sqrt{1+f_1^2(u)}}\cdot(-\sin\phi, \cos\phi, 0)$$

$$= \frac{\sin\phi\cos\phi f_1(u) - \sin\phi\cos\phi f_1(u) + 0}{\sqrt{1+f_1^2(u)}}$$

$$M = 0$$

$$N = \vec{N}\cdot\vec{r}_{22} = \frac{(-\cos\phi f_1(u), -\sin\phi f_1(u), 1)}{\sqrt{1+f_1^2(u)}}\cdot(-u\cos\phi, -u\sin\phi, 0)$$

$$N = \frac{u\cos^2\phi f_1(u) + u\sin^2\phi f_1(u) + 0}{\sqrt{1+f_1^2(u)}}$$

$$N = \frac{u f_1(u)(\cos^2\phi + \sin^2\phi)}{\sqrt{1+f_1^2(u)}}$$

$$N = \frac{u f_1(u)}{\sqrt{1+f_1^2(u)}}$$

Now, equation of principal curvature is

$$H^2 k_n^2 + (2MF - NE - LG)k_n + T^2 = 0$$

$\because M = F = 0$

$$H^2 k_n^2 - (NE + LG)k_n + T^2 = 0$$

Ist curvature is

$$J = \frac{-b}{a} = \frac{NE + LG}{H^2}$$

Surface of revolution is a normal surface so $J = 0$

$$\frac{NE + LG}{H^2} = 0$$

$$\Rightarrow NE + LG = 0$$

$$\Rightarrow \left(\frac{u f_1(u)}{\sqrt{1+f_1^2(u)}}\right)(1+f_1^2(u)) + \frac{f_{11}(u)}{\sqrt{1+f_1^2(u)}}\, u^2 = 0$$

$$\Rightarrow \frac{u f_1(u)(1+f_1^2(u)) + u^2 f_{11}(u)}{\sqrt{1+f_1^2(u)}} = 0$$

$$\Rightarrow \frac{u f_1(u) + u f_1(u) f_1^2(u) + u^2 f_{11}(u)}{\sqrt{1+f_1^2(u)}} = 0$$

$$\Rightarrow u f_1(u) + u f_1^3(u) + u^2 f_{11}(u) = 0$$

$$\Rightarrow u\left(f_1(u) + f_1^3(u) + u f_{11}(u)\right) = 0$$

$$\Rightarrow f_1(u) + f_1^3(u) + u f_{11}(u) = 0$$

$$\Rightarrow u f_{11}(u) + f_1(u) + f_1^3(u) = 0$$

$$\Rightarrow u f_{11}(u) + f_1(u)\left[1 + f_1^2(u)\right] = 0$$

$$\Rightarrow u\frac{d^2 f}{du^2} + \frac{df}{du}\left[1 + \left(\frac{df}{du}\right)^2\right] = 0$$

Question:-

Show that on the surface formed by the revolution of parabola about its directrix one principle curvature is double than the other.

Proof:-

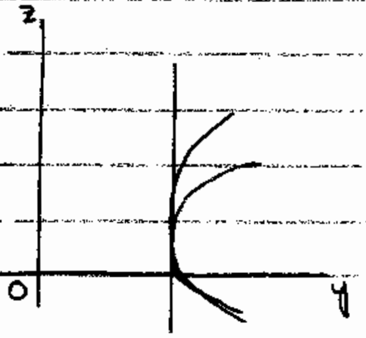
If yz is the plane of parabola and the directrix of parabola is along z-axis. Then, the given parabola is

$$z^2 = 4a(y-a)$$

Now, equation of surface of revolution becomes

$$z = 2\sqrt{a(y-a)}$$

$$\vec{r} = \left(y\cos\phi,\ y\sin\phi,\ 2\sqrt{a(y-a)}\right)$$



$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial y} = \left(\cos\phi,\ \sin\phi,\ 2\cdot\frac{1}{2}\left((a(y-a)\right)^{-1/2}(a)\right)$$

$$\vec{r}_1 = \left(\cos\phi,\ \sin\phi,\ \frac{a}{\sqrt{a(y-a)}}\right)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial \phi} = (-y\sin\phi,\ y\cos\phi,\ 0)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial y^2} = \left(0,\ 0,\ \frac{-a^2}{2(a(y-a))^{3/2}}\right)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial y \partial \phi} = (-\sin\phi,\ \cos\phi,\ 0)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial \phi^2} = (-y\cos\phi,\ -y\sin\phi,\ 0)$$

$$E = \vec{r}_1 \cdot \vec{r}_1$$

$$= \left(\cos\phi,\ \sin\phi,\ \frac{a}{\sqrt{a(y-a)}}\right)\cdot\left(\cos\phi,\ \sin\phi,\ \frac{a}{\sqrt{a(y-a)}}\right)$$

$$= \cos^2\phi + \sin^2\phi + \frac{a^2}{a(y-a)}$$

$$= 1 + \frac{a^2}{a(y-a)}$$

$$= 1 + \frac{a}{y-a}$$

$$= \frac{y-a+a}{y-a}$$

$$E = \frac{y}{y-a}$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$

$$= \left(\cos\phi,\ \sin\phi,\ \frac{a}{\sqrt{a(y-a)}}\right)\cdot(-y\sin\phi,\ y\cos\phi,\ 0)$$

$$F = -y\sin\phi\cos\phi + y\sin\phi\cos\phi + 0$$

$$F = 0$$

$$G = \vec{r}_2 \cdot \vec{r}_2$$
$$= (-y\sin\phi, y\cos\phi, 0)\cdot(-y\sin\phi, y\cos\phi, 0)$$
$$= y^2\sin^2\phi + y^2\cos^2\phi$$

$$G = y^2$$

Now $\vec{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\phi & \sin\phi & \dfrac{a}{\sqrt{a(y-a)}} \\ -y\sin\phi & y\cos\phi & 0 \end{vmatrix}$$

$$\vec{r}_1 \times \vec{r}_2 = \hat{i}\left(0 - \frac{ay\cos\phi}{\sqrt{a(y-a)}}\right) - \hat{j}\left(0 + \frac{ay\sin\phi}{\sqrt{a(y-a)}}\right) + \hat{k}\left(y\cos^2\phi + y\sin^2\phi\right)$$

$$\vec{r}_1 \times \vec{r}_2 = \frac{-ay\cos\phi}{\sqrt{a(y-a)}}\hat{i} - \frac{ay\sin\phi}{\sqrt{a(y-a)}}\hat{j} + y\hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2|^2 = \frac{a^2y^2\cos^2\phi + a^2y^2\sin^2\phi + y^2}{a(y-a)}$$

$$= \frac{a^2y^2 + y^2(ay - a^2)}{a(y-a)}$$

$$= \frac{a^2y^2 + ay^3 - a^2y^2}{a(y-a)}$$

$$= \frac{ay^3}{a(y-a)}$$

$$|\vec{r}_1 \times \vec{r}_2|^2 = \frac{y^3}{y-a}$$

$$\vec{r}_1 \times \vec{r}_2 = \sqrt{\frac{y^3}{y-a}} = y\sqrt{\frac{y}{y-a}} = \frac{y\sqrt{y}}{\sqrt{y-a}}$$

$$L = \vec{N} \cdot \vec{r}_{11}$$

$$L = \frac{\left(\frac{-ay\cos\phi}{\sqrt{a(y-a)}}, \frac{-ay\sin\phi}{\sqrt{a(y-a)}}, y \right)}{y\sqrt{\frac{y}{y-a}}} \cdot \left(0, 0, \frac{-a^2}{2(a(y-a))^{3/2}}\right)$$

$$L = \frac{-a^2 y}{2y\sqrt{\frac{y}{y-a}}\left(a^{3/2}(y-a)^{3/2}\right)} = \frac{-a^2\sqrt{y-a}}{2a^{3/2}(y-a)^{3/2}\sqrt{y}}$$

$$L = \frac{-a^{2-3/2}\,(y-a)^{1/2-3/2}}{2\sqrt{y}}$$

$$L = \frac{-a^{1/2}\,(y-a)^{-1}}{2\sqrt{y}}$$

$$L = \frac{-\sqrt{a}}{2(y-a)\sqrt{y}}$$

$$M = \vec{N} \cdot \vec{r}_{12}$$

$$= \frac{\left(\frac{-ay\cos\phi}{\sqrt{a(y-a)}}, \frac{-ay\sin\phi}{\sqrt{a(y-a)}}, y \right)}{y\sqrt{\frac{y}{y-a}}} \cdot (-\sin\phi, \cos\phi, 0)$$

$$M = 0$$

$$N = \vec{N} \cdot \vec{r}_{22}$$

$$= \frac{\left(\frac{-ay\cos\phi}{\sqrt{a(y-a)}}, \frac{-ay\sin\phi}{\sqrt{a(y-a)}}, y \right)}{y\sqrt{\frac{y}{y-a}}} \cdot (-y\cos\phi, -y\sin\phi, 0)$$

$$N = \frac{\frac{ay^2}{\sqrt{a(y-a)}}}{y\frac{\sqrt{y}}{\sqrt{y-a}}} = \frac{\frac{ay^2}{\sqrt{a}}}{y\sqrt{y}} = \frac{\frac{ay}{\sqrt{a}}}{\sqrt{y}} = \frac{ay}{\sqrt{a}\sqrt{y}}$$

$$N = \sqrt{ay}$$

By Euler's theorem, we have

$$k_a = \frac{L}{E} \quad , \quad k_b = \frac{N}{G}$$

$$k_a = \frac{\frac{-\sqrt{a}}{2\sqrt{y}(y-a)}}{y/y-a} = \frac{\frac{-\sqrt{a}}{2\sqrt{y}}}{y} = \frac{-\sqrt{a}}{2y^{3/2}}$$

and

$$k_b = \frac{\sqrt{ay}}{y^2} = \frac{\sqrt{a}}{y^{3/2}}$$

Question:-

Find the first and second curvature for the surface given by

$x = u\cos v, \quad y = u\sin v, \quad z = f(u) + cv$

Sol:-

$$\vec{r} = (x, y, z)$$

$$\vec{r} = (u\cos v, u\sin v, f(u) + cv)$$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial u} = (\cos v, \sin v, f'(u))$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial v} = (-u\sin v, u\cos v, c)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial u^2} = (0, 0, f''(u))$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial u \partial v} = (-\sin v, \cos v, 0)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial v^2} = (-u\cos v, -u\sin v, 0)$$

$$E = \vec{r}_1 \cdot \vec{r}_1$$

$$= (\cos v, \sin v, f'(u)) \cdot (\cos v, \sin v, f'(u))$$

$$= \cos^2 v + \sin^2 v + (f'(u))^2$$

$$E = 1 + (f'(u))^2$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$
$$= (\cos v, \sin v, f'(u)) \cdot (-u\sin v, u\cos v, c)$$
$$F = -u\sin v\cos v + u\sin v\cos v + c f'(u))$$
$$F = c f'(u)$$
$$G = \vec{r}_2 \cdot \vec{r}_2$$
$$= (-u\sin v, u\cos v, c) \cdot (-u\sin v, u\cos v, c)$$
$$= u^2\sin^2 v + u^2\cos^2 v + c^2$$
$$G = u^2 + c^2$$

Now

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & f'(u) \\ -u\sin v & u\cos v & c \end{vmatrix}$$

$$= \hat{i}(c\sin v - u\cos v f'(u)) - \hat{j}(c\cos v + u\sin v f'(u))$$
$$+ \hat{k}(u\cos^2 v + u\sin^2 v)$$
$$\vec{r}_1 \times \vec{r}_2 = (c\sin v - u\cos v f'(u))\hat{i} - (c\cos v + u\sin v f'(u))\hat{j}$$
$$+ u\hat{k}$$
$$|\vec{r}_1 \times \vec{r}_2|^2 = (c\sin v - u\cos v f'(u))^2 + (c\cos v + u\sin v f'(u))^2$$
$$+ u^2$$
$$= c^2\sin^2 v + u^2\cos^2 v(f'(u))^2 - 2cu\sin v\cos v f'(u) + c^2\cos^2 v$$
$$+ u^2\sin^2 v(f'(u))^2 + 2cu\sin v\cos v f'(u) + u^2$$
$$= c^2 + u^2(f'(u))^2 + u^2$$
$$|\vec{r}_1 \times \vec{r}_2|^2 = c^2 + u^2(1 + (f'(u))^2)$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{c^2 + u^2(1 + (f'(u))^2} = H$$

$$\vec{N} = \frac{(c\sin v - u\cos v f'(u), -c\cos v - u\sin v f'(u), u)}{H}$$

$$L = \vec{N} \cdot \vec{r}_{11}$$
$$L = \frac{u f''(u)}{H}$$

$M = \vec{N} \cdot \vec{r}_{12}$

$$M = \frac{(c\sin v - u\cos v f'(u), -c\cos v - u\sin v f'(u), u)}{H} \cdot (-\sin v, \cos v, 0)$$

$$M = \frac{-c\sin^2 v + u\sin v\cos v f'(u) + c\cos^2 v - u\sin v\cos v f'(u)}{H}$$

$$M = \frac{-c}{H}$$

$N = \vec{N} \cdot \vec{r}_{22}$

$$= \frac{(c\sin v - u\cos v f'(u), -c\cos v - u\sin v f'(u), u)}{H} \cdot (-u\cos v, -u\sin v, 0)$$

$$= \frac{-uc\sin v\cos v + u^2\cos^2 v f'(u) + uc\sin v\cos v + u^2\sin^2 v f'(u)}{H}$$

$$N = \frac{u^2 f'(u)}{H}$$

By Euler's theorem, we have

Ist curvature

$$k_a = \frac{L}{E}$$

$$= \frac{\frac{u f''(u)}{H}}{1+(f'(u))^2}$$

$$k_a = \frac{u f''(u)}{H(1+(f'(u))^2)}$$

2nd curvature

$$k_b = \frac{N}{G}$$

$$k_b = \frac{\frac{u^2 f'(u)}{H}}{u^2+c^2} \qquad \text{where } H = \sqrt{c^2+u^2(1+f'^2(u))}$$

$$k_b = \frac{u^2 f'(u)}{H(u^2+c^2)}$$

Question:-

For the surface generated by the tangent of a twisted curve. Find the principle curvatures and lines of curvatures.

Sol:-

The surface generated by the tangent of twisted curve is given by

$$\vec{R} = \vec{r} + u\vec{t}$$

where $\vec{r}$ and $\vec{t}$ are functions of s.

$$\vec{R_1} = \frac{\partial \vec{R}}{\partial u} = \vec{t}$$

$$\vec{R_2} = \frac{\partial \vec{R}}{\partial s} = \vec{r}' + u\vec{t}'$$

put $\vec{r}' = \vec{t}$ and $\vec{t}' = k\vec{n}$

$$\vec{R_2} = \vec{t} + uk\vec{n}$$

$$\vec{R_{11}} = \frac{\partial^2 \vec{R}}{\partial u^2} = 0$$

$$\vec{R_{12}} = \frac{\partial^2 \vec{R}}{\partial u \partial s} = \vec{t}' = k\vec{n}$$

$$\vec{R_{22}} = \frac{\partial^2 \vec{R}}{\partial s^2} = \vec{t}' + uk\vec{n}' + uk'\vec{n}$$

put $\vec{t}' = k\vec{n}$, $\vec{n}' = \tau\vec{b} - k\vec{t}$

$$\vec{R_{22}} = k\vec{n} + uk(\tau\vec{b} - k\vec{t}) + uk'\vec{n}$$

$$= k\vec{n} + uk'\vec{n} + uk\tau\vec{b} - uk^2\vec{t}$$

$$\vec{R_{22}} = -uk^2\vec{t} + (k + uk')\vec{n} + uk\tau\vec{b}$$

$$\vec{N} = \frac{\vec{R_1} \times \vec{R_2}}{|\vec{R_1} \times \vec{R_2}|}$$

$$\vec{R_1} \times \vec{R_2} = \begin{vmatrix} \vec{t} & \vec{n} & \vec{b} \\ 1 & 0 & 0 \\ 1 & uk & 0 \end{vmatrix}$$

$$= t(0) - \vec{n}(0) + \vec{b}(uk - 0)$$

$\vec{R}_1 \times \vec{R}_2 = uk\vec{b}$

$|\vec{R}_1 \times \vec{R}_2| = \sqrt{u^2k^2}$

$|\vec{R}_1 \times \vec{R}_2| = uk$

$\vec{N} = \dfrac{uk\vec{b}}{uk}$

$\vec{N} = \vec{b}$

$E = \vec{R}_1 \cdot \vec{R}_1 = \vec{t} \cdot \vec{t} = 1$

$F = \vec{R}_1 \cdot \vec{R}_2 = \vec{t} \cdot (\vec{t} + uk\vec{n})$

$= \vec{t} \cdot \vec{t} + uk(\vec{t} \cdot \vec{n})$

$= 1 + k(0)$

$F = 1$

$G = \vec{R}_2 \cdot \vec{R}_2 = (\vec{t} + uk\vec{n}) \cdot (\vec{t} + uk\vec{n})$

$= (\vec{t} \cdot \vec{t}) + u^2k^2(\vec{n} \cdot \vec{n})$

$G = 1 + u^2k^2$

$L = \vec{N} \cdot \vec{R}_{11}$

$L = \vec{b} \cdot \vec{0} = 0$

$M = \vec{N} \cdot \vec{R}_{12}$

$= \vec{b} \cdot (k\vec{n})$

$= k(\vec{b} \cdot \vec{n})$

$M = 0$

$N = \vec{N} \cdot \vec{R}_{22}$

$= \vec{b} \cdot (-uk^2\vec{t} + (k + uk')\vec{n} + uk\tau\vec{b})$

$= uk\tau(\vec{b} \cdot \vec{b}) + 0 + 0$

$N = uk\tau$

Differential equation for lines of curvature is given by

$$\begin{vmatrix} ds^2 & -du\,ds & du^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

$$\begin{vmatrix} ds^2 & -du\,ds & du^2 \\ 0 & 0 & ukT \\ 1 & 1 & 1+u^2k^2 \end{vmatrix} = 0$$

$\Rightarrow ds^2(0-ukT) + du\,ds(0-ukT) + du^2(0-0) = 0$

$\Rightarrow -ukT\,ds^2 - ukT\,du\,ds = 0$

$\Rightarrow -ukT(ds^2 + du\,ds) = 0$

$\Rightarrow ds^2 + du\,ds = 0$

$\Rightarrow ds(ds + du) = 0$

$\Rightarrow ds = 0 \quad , \quad ds + du = 0$

$\Rightarrow S = \text{Constt} \qquad u + S = \text{Constt}$ (By Integrat)

These are equations for lines of curvature.

Now, equation of principal curvature is

$H^2 k_n^2 + (2MF - NE - LG)k_n + T^2 = 0$

$u^2k^2k_n^2 + (2(0) - ukT - 0)k_n + LG - M^2 = 0$

$\because L = 0,\ M = 0$

$u^2k^2k_n^2 - ukTk_n + 0 = 0$

$\Rightarrow u^2k^2k_n^2 - ukTk_n = 0$

$\Rightarrow k_n(u^2k^2k_n - ukT) = 0$

Ist curvature is

$$J = -\frac{b}{a} = \frac{ukT}{u^2k^2} = \frac{T}{uk}$$

2nd curvature is

$$k = \frac{c}{a} = \frac{0}{u^2k^2} = 0$$

Question

Show that the lines of curvature of the paraboloid $xy = cz$ lie on the surface $\sinh^{-1}\frac{x}{c} \pm \sinh^{-1}\frac{y}{c} = \text{Constant}$

Sol:-

Equation of surface is

$\bar{r} = (x, y, z)$

$\vec{r} = (x, y, xy/c)$

$$\vec{r}_1 = \frac{\partial \vec{r}}{\partial x} = (1, 0, y/c)$$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0, 1, x/c)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0, 0, 0)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0, 0, 1/c)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0, 0, 0)$$

Now $\vec{N} = \dfrac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \dfrac{\vec{r}_1 \times \vec{r}_2}{H}$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y/c \\ 0 & 1 & x/c \end{vmatrix}$$

$$= \hat{i}(0 - y/c) - \hat{j}(x/c - 0) + \hat{k}(1-0)$$

$$\vec{r}_1 \times \vec{r}_2 = -y/c\,\hat{i} - x/c\,\hat{j} + \hat{k}$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{\frac{y^2}{c^2} + \frac{x^2}{c^2} + 1}$$

$$= \sqrt{\frac{y^2 + x^2 + c^2}{c^2}}$$

$$H = |\vec{r}_1 \times \vec{r}_2| = \frac{\sqrt{x^2 + y^2 + c^2}}{c}$$

$$\vec{N} = \frac{(-y/c, -x/c, 1)}{\sqrt{x^2+y^2+c^2}/c} = \frac{\frac{1}{c}(-y, -x, c)}{\frac{\sqrt{x^2+y^2+c^2}}{c}}$$

$$\vec{N} = \frac{(-y, -x, c)}{\sqrt{x^2+y^2+c^2}}$$

$$E = \vec{r}_1 \cdot \vec{r}_1 = \left(1, 0, \frac{y}{c}\right) \cdot \left(1, 0, \frac{y}{c}\right)$$

$$= 1 + \frac{y^2}{c^2}$$

$$E = \frac{c^2 + y^2}{c^2}$$

$$F = \vec{r}_1 \cdot \vec{r}_2 = (1, 0, y/c) \cdot (0, 1, x/c)$$

$$F = \frac{xy}{c^2}$$

$$G = \vec{r}_2 \cdot \vec{r}_2 = (0, 1, x/c) \cdot (0, 1, x/c)$$

$$= 1 + \frac{x^2}{c^2}$$

$$G = \frac{c^2 + x^2}{c^2}$$

Now $L = \vec{N} \cdot \vec{r}_{11}$

$$= \left(\frac{-y, -x, c}{\sqrt{y^2 + x^2 + c^2}}\right) \cdot (0, 0, 0)$$

$$L = 0$$

$$M = \vec{N} \cdot \vec{r}_{12}$$

$$= \left(\frac{-y, -x, c}{\sqrt{y^2 + x^2 + c^2}}\right) \cdot (0, 0, 1/c)$$

$$M = \frac{0 + 0 + 1}{\sqrt{y^2 + x^2 + c^2}}$$

$$M = \frac{1}{\sqrt{x^2 + y^2 + c^2}} = \frac{1}{H}$$

$$N = \vec{N} \cdot \vec{r}_{22}$$

$$= \left(\frac{-y, -x, c}{\sqrt{x^2 + y^2 + c^2}}\right) \cdot (0, 0, 0)$$

$N = 0$

Differential equation for lines of curvature is

$$\begin{vmatrix} dy^2 & -dxdy & dx^2 \\ L & M & N \\ E & F & G \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} dy^2 & -dxdy & dx^2 \\ 0 & 1/H & 0 \\ \frac{c^2+y^2}{c^2} & xy/c^2 & \frac{c^2+x^2}{c^2} \end{vmatrix} = 0$$

$$\Rightarrow dy^2\left(\frac{1}{H}\left(\frac{c^2+x^2}{c^2}\right)\right) + dxdy(0-0) + dx^2\left(0 - \frac{1}{H}\left(\frac{c^2+y^2}{c^2}\right)\right) = 0$$

$$\Rightarrow dy^2\left(\frac{1}{\sqrt{x^2+y^2+c^2}}\left(\frac{c^2+x^2}{c^2}\right)\right) - dx^2\left(\frac{1}{\sqrt{x^2+y^2+c^2}}\left(\frac{c^2+y^2}{c^2}\right)\right) = 0$$

$$\Rightarrow \frac{(c^2+x^2)\,dy^2 - (c^2+y^2)\,dx^2}{c^2\sqrt{x^2+y^2+c^2}} = 0$$

$$\Rightarrow (c^2+x^2)\,dy^2 - (c^2+y^2)\,dx^2 = 0$$

$$\Rightarrow (x^2+c^2)\,dy^2 - (y^2+c^2)\,dx^2 = 0$$

$$\Rightarrow (x^2+c^2)\,dy^2 = (y^2+c^2)\,dx^2$$

$$\frac{dy^2}{y^2+c^2} = \frac{dx^2}{x^2+c^2}$$

$$\Rightarrow \frac{dy}{\sqrt{y^2+c^2}} = \pm \frac{dx}{\sqrt{x^2+c^2}}$$

Integrating both sides

$$\int \frac{dy}{\sqrt{y^2+c^2}} = \pm \int \frac{dx}{\sqrt{x^2+c^2}}$$

$\pm \sinh^{-1}\frac{y}{c} = \pm \sinh^{-1}\frac{x}{c} + \text{Constant}$

or

$\Rightarrow \sinh^{-1}\frac{x}{c} = \pm \sinh^{-1}\frac{y}{c} + \text{Constant}$

$\Rightarrow \sinh^{-1}\frac{x}{c} \pm \sinh^{-1}\frac{y}{c} = \text{Constant}$

One Parametre family of surface :- The equation $F(x, y, z, a) = 0$ where "a" is a constant represent a surface corresponding to different values of "a", this equation represent different surface.

The set of all surfaces obtained by taking different values of "a" is known as one parametre family of surfaces with parametric value "a".

For example, $x^2+y^2+z^2 = a^2$ is equation of sphere with centre at origin and radius "a".

Taking different values of "a". we obtain one parametre family of sphere with centre at origin and different radii.

$F(x, y, z, a) = x^2+y^2+z^2-a^2 = 0$

represent one parametre family of spheres with centre at origin and parametre value "a".

Note :-

we will take $F(a) = 0$ in the place of $F(x, y, z, a) = 0$

Characteristic of surface:-

The curve of intersection of these consective surfaces is known as characteristic of surface.

Let $F(a) = 0 \rightarrow (1)$ and $F(a+\delta a) = 0 \rightarrow (2)$ be two surfaces of the same family. Then, the curve of intersection of these two surfaces is determined by these two equations (1) and (2)

From (1) and (2)

$$F(a+\delta a) - F(a) = 0$$

$$\Rightarrow \frac{F(a+\delta a) - F(a)}{\delta a} = 0 \rightarrow (3)$$

when $\delta a \rightarrow 0$

Then, these two surfaces become consective surfaces of the family and from eq (3)

$$\lim_{\delta a \to 0} \frac{F(a+\delta a) - F(a)}{\delta a} = 0$$

$$\Rightarrow \frac{\partial F(a)}{\partial a} = 0$$

Envelop:-

The locus of all characteristics is called envelop and it is a surface whose equation is obtained by eliminating "a" from $F(a) = 0$ and

$$\frac{\partial F(a)}{\partial a} = 0$$

Question:-

Find the equation of envelop.

for the family of sphere with constant radius "b" and having centres on a fixed circle $x^2+y^2=a^2$ and $z=0$

Sol:-

The co-ordinates of any point on the given circle are

$x = a\cos\theta, \quad y = a\sin\theta, \quad z=0$

The equation of a sphere with radius b and centred at $(a\cos\theta, a\sin\theta, 0)$ is

$$(x-a\cos\theta)^2+(y-a\sin\theta)^2+(z-0)^2=b^2$$

The given family of sphere is $F(x,y,z,\theta)$

$$F(x,y,z,\theta) = (x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2-b^2=0$$

$\Rightarrow F(\theta)=0$

$\Rightarrow (x-a\cos\theta)^2+(y-a\sin\theta)^2+z^2-b^2=0$

$\Rightarrow x^2+a^2\cos^2\theta+2ax\cos\theta+y^2+a^2\sin^2\theta+2ay\sin\theta+z^2-b^2=0$

$\Rightarrow x^2+y^2+z^2+a^2-b^2-2ax\cos\theta-2ay\sin\theta=0 \rightarrow (1)$

Differentiate w.r.t θ we have

$$\frac{\partial F(\theta)}{\partial\theta} = 2ax\sin\theta-2ay\cos\theta \rightarrow (2)$$

The eq of the envelop is determined by eliminating θ from $F(\theta)=0$ and $\frac{\partial F(\theta)}{\partial\theta}=0$

(2) $\Rightarrow$

$2ax\sin\theta = 2ay\cos\theta$

$\Rightarrow x\sin\theta = y\cos\theta$

$$\sin\theta = \frac{y}{x}\cos\theta$$

$$\Rightarrow \sin^2\theta = \frac{y^2}{x^2}\cos^2\theta$$

$$\Rightarrow \sin^2\theta = \frac{y^2}{x^2}(1-\sin^2\theta)$$

$$\Rightarrow \sin^2\theta + \frac{y^2}{x^2}\sin^2\theta = \frac{y^2}{x^2}$$

$$\Rightarrow \sin^2\theta\left(1+\frac{y^2}{x^2}\right) = \frac{y^2}{x^2}$$

$$\sin^2\theta\left(\frac{x^2+y^2}{x^2}\right) = \frac{y^2}{x^2}$$

$$\Rightarrow \sin^2\theta = \frac{y^2}{x^2+y^2}$$

$$\cos^2\theta = 1-\sin^2\theta$$

$$= 1-\frac{y^2}{x^2+y^2} = \frac{x^2+y^2-y^2}{x^2+y^2} = \frac{x^2}{x^2+y^2}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$ put in eq (1)

$$x^2+y^2+z^2+a^2+b^2 = 2ax\cos\theta + 2ay\sin\theta$$

$$= 2ax\left(\frac{x}{\sqrt{x^2+y^2}}\right) + 2ay\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{2ax^2+2ay^2}{\sqrt{x^2+y^2}} = \frac{2a(x^2+y^2}{\sqrt{x^2+y^2}}$$

$$x^2+y^2+z^2+a^2-b^2 = 2a\sqrt{x^2+y^2}$$

squaring both sides we have

$$(x^2+y^2+z^2+a^2-b^2)^2 = 4a(x^2+y^2)$$

Question:-

Find the envelop of the family of surfaces given by

$$x^2+y^2 = 4a(z-a) \text{ with "a" (parameter)}$$

Sol:-

Here we have

$$F(x,y,z,a) = x^2+y^2-4a(z-a)$$

$$F(a) = x^2+y^2-4a(z-a) \rightarrow (1)$$

Differentiating w.r.t "a", we have

$$\frac{\partial F}{\partial a} = -4z + 8a$$

The equation of envelope is determined by eliminating "a" from $F(a) = 0$

$$\Rightarrow \quad F_a(a) = \frac{\partial F}{\partial a} = 0$$

put $\frac{\partial F}{\partial a} = 0$

$$\Rightarrow -4z + 8a = 0 \Rightarrow 8a = 4z$$

$$\Rightarrow z = 2a$$

$$\Rightarrow \frac{z}{2} = a$$

using this value in eq (1)

$$0 = x^2 + y^2 - 4\left(\frac{z}{2}\right)\left(z - \frac{z}{2}\right)$$

$$0 = x^2 + y^2 - 2z\left(\frac{2z - z}{2}\right)$$

$$0 = x^2 + y^2 - 2z\left(\frac{1 \cdot z}{2}\right)$$

$$0 = x^2 + y^2 - z^2$$

$$\Rightarrow x^2 + y^2 - z^2 = 0$$

So, the envelope of the family of surfaces (which is given $x^2 + y^2 = 4a(z-a)$) is

$$x^2 + y^2 - z^2 = 0$$

Edge of regression:-

The locus of the ultimate intersection of consective characteristics of family of surface is called the edge of regression.

Equation of edge of regression:-

Let $F(x, y, z, a) = 0$ be a one parametre family of surface with parametre "a".

Now, equations of characteristics with parametres "a" and $a+\delta a$ are

$F(x, y, z, a) = 0$, $F_a(x, y, z, a) = 0$

and

$F(x, y, z, a+\delta a) = 0$, $F_a(x, y, z, a+\delta a) = 0$

$\Rightarrow F_a(x, y, z, a+\delta a) - F_a(x, y, z, a) = 0$

$$\Rightarrow \frac{F_a(x, y, z, a+\delta a) - F_a(x, y, z, a)}{\delta a} = 0$$

when $\delta a \to 0$, we have

$$\lim_{\delta a \to 0} \frac{F_a(x, y, z, a+\delta a) - F_a(x, y, z, a)}{\delta a} = 0$$

$\Rightarrow F_{aa} = 0$ $\qquad \because r_1 = 2$

$r_{11} = 0$

$\Rightarrow F_{aa}(x, y, z, a) = 0$ $\qquad$ double ~~[illegible]~~ value

Hence, we have three equations

$F(x, y, z, a) = 0 \rightarrow (1)$

$F_a(x, y, z, a) = 0 \rightarrow (2)$

$F_{aa}(x, y, z, a) = 0 \rightarrow (3)$

The equations of edge of regression is obtained by eliminating "a" from eq (1), (2), (3).

Questions:-

Find the edge of regression of the family of ellipsoid

$$c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{z^2}{c^2} = 1$$

where c, b are parametres

Developable Surface:-

(i) Intersection of two planes is a straight line
(ii) Intersection of two curves is a point
(iii) Intersection of two surfaces is a curve.

Developable surfaces:-

The characteristics of one parametre family of planes are straight lines, being the intersection of two planes. Hence, in the case of one parametre family of planes be straight lines (characteristics) are generators of the envelop for this family. And this envelop is known as a developable surface.

Remark:-

(i) The envelop of one parametre family of planes is generated by straight lines which are the characteristics for that family. Hence, the envelop will be an unrolled surface or a developed surface without streches and for this reason, it is named as developable surface.

(ii) The tangent plane at any point of a developable surface depends only on one parametre.

Question:-

Find the conditions for a surface $z = f(x,y)$ to be a developable surface.

Sol:-

The given surface is

$$z = f(x,y)$$

$$\Rightarrow z - f(x,y) = 0$$

$\Rightarrow F(x, y, z) = z - f(x, y) = 0$

$$\frac{\partial F}{\partial x} = -f_x, \quad \frac{\partial F}{\partial y} = -f_y, \quad \frac{\partial F}{\partial z} = 1$$

Now, the equation of tangent plane at a point (x, y, z) of a given surface is

$$\frac{\partial F}{\partial x}(X - x) + \frac{\partial F}{\partial y}(Y - y) + \frac{\partial F}{\partial z}(Z - z) = 0$$

$$\Rightarrow -f_x(X - x) - f_y(Y - y) + (Z - z) = 0$$

Since, for a developable surface the tangent plane depends only on one parametre, So there must be a relation between f_x and f_y.

Let

$$f_x = \phi(f_y) \rightarrow (1)$$

Differentiating both sides w.r.t "x"

$$f_{xx}^{2} = \phi'(f_y) f_{yx} \rightarrow (2)$$

Differentiating eq (1) w.r.t "y"

$$f_{xy} = \phi'(f_y) f_{yy} \rightarrow (3)$$

From (2) and (3)

$$\frac{f_{xx}}{f_{xy}} = \frac{f_{yx}}{f_{yy}}$$

$$\Rightarrow f_{xx} f_{yy} = f_{xy} f_{yx}$$

$$\Rightarrow f_{xx} f_{yy} = (f_{xy})^2$$

$$\Rightarrow f_{x^2} f_{y^2} = (f_{xy})^2$$

which is the required condition for a surface $z = f(x,y)$ to be a developable surface.

Questions:-

Check ~~whether~~ the surface is developable or not.

$(z-a)^2 = xy$

Sol:-

The given surface is

$$(z-a)^2 = xy$$

$$\Rightarrow z - a = \sqrt{xy}$$

$$\Rightarrow z = a + \sqrt{xy} = f(x,y)$$

$$\Rightarrow f(x,y) = a + \sqrt{xy}$$

$$f_x = \frac{1}{2\sqrt{xy}} y = \frac{y}{2\sqrt{xy}} = \frac{1}{2}\sqrt{\frac{y}{x}}$$

$$f_{xx} = \frac{1}{2}\left(\frac{\sqrt{xy} \cdot 0 - y \frac{1}{2\sqrt{xy}} y}{\sqrt{(xy)^2}}\right)$$

$$= \frac{1}{2}\left(\frac{-y^2/2\sqrt{xy}}{xy}\right)$$

$$= \frac{-y^2}{4\, xy \cdot \sqrt{xy}} = \frac{-y^2}{4(xy)^{3/2}}$$

$$f_{xy} = \frac{1}{2\sqrt{xy}} * \frac{1}{2\sqrt{y}}$$

$$f_{xy} = \frac{1}{4\sqrt{x}\sqrt{y}}$$

$$f_{xy} = \frac{1}{4\sqrt{xy}}$$

$$f_y = \frac{1}{2\sqrt{xy}} \, x$$

$$f_{yy} = \frac{x}{2}\left(-\frac{1}{2}(xy)^{-3/2} x\right)$$

$$= -\frac{1}{4} x^2 (xy)^{-3/2}$$

$$f_{yy} = \frac{-x^2}{4(xy)^{3/2}}$$

So for developable surface

$$f_{x^2} \cdot f_{y^2} = (f_{xy})^2$$

$$\left(\frac{-y^2}{4(xy)^{3/2}}\right)\left(\frac{-x^2}{4(xy)^{3/2}}\right) = \left(\frac{1}{4\sqrt{xy}}\right)^2$$

$$\frac{x^2 y^2}{16(xy)^{3/2+3/2}} = \frac{1}{16xy}$$

$$\Rightarrow \frac{x^2 y^2}{16(xy)^3} = \frac{1}{16xy}$$

$$\Rightarrow \frac{x^2 y^2}{16x^3 y^3} = \frac{1}{16xy}$$

$\Rightarrow \frac{1}{16xy} = \frac{1}{16xy}$ So, the given surface is developable.

Question:-

Prove that a surface is a developable surface iff the specific curvature is zero at all points.

Sol:-

Suppose that the surface

$z = f(x,y)$ is a developable surface then

$$\vec{r} = (x,\ y,\ f(x,y))$$

Let $\dfrac{\partial z}{\partial x} = p$, $\dfrac{\partial z}{\partial y} = q$, $\dfrac{\partial^2 z}{\partial x^2} = r$, $\dfrac{\partial^2 z}{\partial x \partial y} = s$

and $\dfrac{\partial^2 z}{\partial y^2} = t$

Now, $\vec{r}_1 = \dfrac{\partial \vec{r}}{\partial x} = (1,\ 0,\ p)$

$$\vec{r}_2 = \frac{\partial \vec{r}}{\partial y} = (0,\ 1,\ q)$$

$$\vec{r}_{11} = \frac{\partial^2 \vec{r}}{\partial x^2} = (0,\ 0,\ r)$$

$$\vec{r}_{12} = \frac{\partial^2 \vec{r}}{\partial x \partial y} = (0,\ 0,\ s)$$

$$\vec{r}_{22} = \frac{\partial^2 \vec{r}}{\partial y^2} = (0,\ 0,\ t)$$

Now

$$\vec{N} = \frac{\vec{r}_1 \times \vec{r}_2}{|\vec{r}_1 \times \vec{r}_2|} = \frac{\vec{r}_1 \times \vec{r}_2}{H}$$

$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & p \\ 0 & 1 & q \end{vmatrix}$$

$$= \hat{i}(0-p) - \hat{j}(q-0) + \hat{k}(1-0)$$

$$\vec{r}_1 \times \vec{r}_2 = -p\hat{i} - q\hat{j} + \hat{k}$$

$$H^2 = |\vec{r}_1 \times \vec{r}_2|^2 = p^2 + q^2 + 1$$

$$\vec{N} = \frac{(-p,\ -q,\ 1)}{H}$$

$L = \vec{N} \cdot \vec{r}_{11}$

$L = \dfrac{(-p, -q, 1)}{H} \cdot$

$L = \dfrac{r}{H}$

$M = \dfrac{s}{H}$, $N = \dfrac{t}{H}$

$T^2 = LN - M^2$

$T^2 = \dfrac{rt}{H^2} - \dfrac{s^2}{H^2} = \dfrac{rt - s^2}{H^2}$

Now, the equation of principal curvature is

$H^2 k_n^2 + (2MF - EN - LG) k_n + T^2 = 0$

Now

Specific curvature k

$$k = \frac{T^2}{H^2}$$

$$= \frac{rt - s^2}{H^2} \Big/ H^2$$

$$k \Rightarrow \frac{rt - s^2}{H^4} \rightarrow (1)$$

Now the given surface is developable

~~$f_x^2 f_y^2 = (f_{xy})^2$~~

$$r \cdot t = s^2$$

$$\Rightarrow rt - s^2 = 0$$

put in eq (1)

$\Rightarrow k = \dfrac{0}{H^4} = 0$

$k = 0$

$\Rightarrow$ Specific curvature is zero at all points.

Now, Conversly suppose that specific curvature

$$k = 0$$

$$\frac{rt - s^2}{H^4} = 0$$

$$\Rightarrow rt - s^2 = 0$$

$$\Rightarrow rt = s^2$$

$$\Rightarrow f_{x^2} f_{y^2} = (f_{xy})^2$$

$\Rightarrow$ The given surface is a developable surface.

Hence, a surface is a developable surface iff its specific curvature is zero at all points.

Geodesics or Geodesics line on a surface:-

A Geodesics or a Geodesics line on a surface is the curve of shortest distance on a surface between given points.

Remark:-

From the definition of Geodesics, we may deduce the property that the principle normal to the Geodesics concide with the normal to the surface.

Geodesics curvature vector:-

If $\vec{r} = \vec{r}(s)$ is the position vector of any point P on the curve on a surface, then $\vec{r}'$ can be expressed as

$\vec{r} = \vec{r}(u,v)$ $\quad \vec{r}'' = k_n \vec{N} + \lambda \vec{r}_1 + \mu \vec{r}_2$

where the vector $\lambda \vec{r}_1 + \mu \vec{r}_2$ with components (λ, μ) is tangential to the surface and is called the geodesics curvature vector at point p and is denoted as

i.e) $\vec{k_g} = \lambda \vec{r_1} + \mu \vec{r_2}$

Theorem:-

A curve on a surface is a geodesics iff the geodesics curvature vector is zero.

Proof:-

Let $\vec{r} = \vec{r}(s)$ be a geodesics on a surface $\vec{r} = \vec{r}(u,v)$. Then,

$$\vec{r}'' = k_n N + \lambda \vec{r_1} + \mu \vec{r_2} \rightarrow (1)$$

we suppose that the curve is a geodesics then by the property of a geodesics on a surface the principle normal to the geodesics concides with the surface normal or normal to the surface.

If $\vec{n}$ is principle normal (geodesics) to the curve $\vec{r} = \vec{r}(s)$ and $\vec{N}$ is the surface normal. Then, by the property of geodesics

$$\vec{n} = \vec{N}$$

Then, from eq (1)

$$\vec{r}'' = k_n \vec{N} + \lambda \vec{r_1} + \mu \vec{r_2}$$

$$\Rightarrow k \vec{N} = k_n \vec{N} + \lambda \vec{r_1} + \mu \vec{r_2}$$

Comparing co-efficients on both sides

$$\lambda = \mu = 0$$

$$\Rightarrow \vec{k_g} = \lambda \vec{r_1} + \mu \vec{r_2}$$

$$\Rightarrow \vec{k_g} = 0$$

Now Conversly suppose that the geodesics curvature vector with components (λ, μ) is zero.

i-e) $\vec{k}_g = \lambda \vec{r}_1 + \mu \vec{r}_2 = 0$

Now from eq (1) we have

$\vec{r}'' = k_n \vec{N} + \lambda \vec{r}_1 + \mu \vec{r}_2$

$\vec{r}'' = k_n \vec{N} \qquad \because \lambda \vec{r}_1 + \mu \vec{r}_2 = 0$

$\Rightarrow k \vec{n} = k_n \vec{N}$

$\Rightarrow \vec{n} = \frac{k_n}{k} \vec{N}$

$\Rightarrow$ The principal normal to the given curve is parallel to the surface normal which is the property of geodesics.

Hence, the given curve is a geodesics.

Theorem:-

The geodesics curvature vector of a curve is orthogonal to the given curve.

Proof:-

If $\vec{r} = \vec{r}(s)$ is a curve on a surface or if $\vec{r} = \vec{r}(s)$ is the position vector of any point on a curve on a surface. Then.

$\vec{r}'' = k_n \vec{N} + \lambda \vec{r}_1 + \mu \vec{r}_2$

$\Rightarrow k \vec{n} = k_n \vec{N} + \lambda \vec{r}_1 + \mu \vec{r}_2 \rightarrow (1)$

Now, Since, the tangent $\vec{t}$ at point with position vector $\vec{r}$ is perpendicular to both principal normal $\vec{n}$ and the surface normal $\vec{N}$. So

$\vec{n} \cdot \vec{t} = \vec{N} \cdot \vec{t} = 0$

Taking dot product with $\vec{t}$ on both sides of (1)

$k(\vec{n} \cdot \vec{t}) = k_n (\vec{N} \cdot \vec{t}) + (\lambda (\vec{r}_1 + \mu \vec{r}_2) \cdot \vec{t}$

$0 = 0 + (\lambda \vec{r}_1 + \mu \vec{r}_2) \cdot \vec{t}$

$\Rightarrow (\lambda \vec{r}_1 + \mu \vec{r}_2) \cdot \vec{t} = 0$

$\Rightarrow \vec{k}_g \cdot \vec{t} = 0$

Hence, the geodesics curvature vector of any curve is orthogonal to the given curve.

Geodesics curvature:-

The magnitude of geodesics curvature vector with proper sign is known as geodesics curvature. It can be positive or negative according as the angle between geodesics curvature vector and the curve is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$. If $\lambda\vec{r}_1 + \mu\vec{r}_2$ is the geodesics curvature vector at any point $\vec{r} = \vec{r}(s)$ of a curve on a surface, then the geodesics curvature $k_g = \pm\sqrt{\lambda^2 + \mu^2}$

Theorem:-

The geodesics curvature of a geodesics on a surface is zero and conversly

Proof:-

Let $\vec{r} = \vec{r}(s)$ be the position vector of any point of a geodesics on a surface. Then, we know that

$$\vec{r}'' = k_n\vec{N} + \lambda\vec{r}_1 + \mu\vec{r}_2 \rightarrow (1)$$

Now, by the property of geodesics, we take $\vec{n} = \vec{N}$

Then, (1) $\Rightarrow$ $k\vec{n} = k_n\vec{N} + \lambda\vec{r}_1 + \mu\vec{r}_2$

$$k\vec{N} = k_n\vec{N} + \lambda\vec{r}_1 + \mu\vec{r}_2 \rightarrow (2)$$

$\Rightarrow$ comparing co-effients on both sides

$\Rightarrow k = k_n,\ \lambda = 0,\ \mu = 0$

$\Rightarrow k_g = \pm\sqrt{\lambda^2 + \mu^2}$

$k_g = 0 \Rightarrow$ geodesics curvature at point is zero

Now Conversly, Suppose that for a curve $\vec{r} = \vec{r}(s)$, geodesics curvature is zero.

i.e) $k_g = 0$

$$\Rightarrow \pm\sqrt{\lambda^2 + \mu^2} = 0 \rightarrow (3)$$

where (λ, μ) are components of geodesics curvature vector.

(3) $\Rightarrow \lambda = 0, \ \mu = 0$

Now Consider

$$\vec{r}'' = k_n\vec{N} + \lambda\vec{r}_1 + \mu\vec{r}_2$$

$$\Rightarrow k\vec{n} = k_n\vec{N} + 0 + 0$$

$$\Rightarrow k\vec{n} = k_n\vec{N}$$

$\Rightarrow$ The principal normal to the curve $\vec{r} = \vec{r}(s)$ is parallel to the surface normal which is the property of geodesics.

Hence, the curve is a geodesics.

Question:-

For a curve $\vec{r} = \vec{r}(s)$ on a surface. Prove that (i) $k_g = [\vec{N} \ \ \vec{r}' \ \ \vec{r}'']$

(ii) $k_g = (\dot{s})^{-3}[\vec{N} \ \ \dot{\vec{r}} \ \ \ddot{\vec{r}}]$

where $\cdot$ represent the derivative w.r.t parametre "t"

Sol:-

(i) we know that geodesics curvature vector at any point with position vector $\vec{r} = \vec{r}(s)$ on a curve on a surface is orthogonal to the unit tangent vector $\vec{t}$. and $\vec{t} = \vec{r}' = \dfrac{d\vec{r}}{ds}$ at point with position vector $\vec{r} = \vec{r}(s)$. Also, if (λ, μ) are the components of geodesics curvature vector $\vec{r} = \vec{r}(s)$. Then, the geodesics curvature vector $\lambda \vec{r}_1 + \mu \vec{r}_2$ lies in the tangent plane at point $\vec{r} = \vec{r}(s)$ and Hence is $\perp$ (orthogonal) to the surface normal at that point.

Hence, the geodesics curvature vector is $\perp$ to both $\vec{r}'$ and $\vec{N}$ (surface normal and Hence is parallel to the unit vector $\vec{N} \times \vec{r}'$.

Now, since k_g is the magnitude of the geodesics curvature vector. So, the geodesics curvature vector $\lambda \vec{r}_1 + \mu \vec{r}_2$ can be written as

$$k_g (\vec{N} \times \vec{r}') \rightarrow (1)$$

Also, we know that

$$\vec{r}'' = k_n \vec{N} + \lambda \vec{r}_1 + \mu \vec{r}_2$$

$$\Rightarrow \vec{r}'' = k_n \vec{N} + k_g (\vec{N} \times \vec{r}')$$

Taking dot product with $\vec{N} \times \vec{r}'$ on both sides, we have

$$(\vec{N} \times \vec{r}') \cdot \vec{r}'' = k_n (\vec{N} \times \vec{r}') \cdot \vec{N} + k_g (\vec{N} \times \vec{r}') \cdot (\vec{N} \times \vec{r}')$$

$$\vec{N} \cdot \vec{r}' \times \vec{r}'' = k_n (0) + k_g (1)$$

$$\Rightarrow [\vec{N} \ \vec{r}' \ \vec{r}''] = k_g$$

$$\Rightarrow k_g = [\vec{N} \ \vec{r}' \ \vec{r}'']$$

(ii)

$$\vec{r}' = \frac{d\vec{r}}{ds}$$

$$= \frac{d\vec{r}}{dt} \frac{dt}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{1}{\frac{ds}{dt}} = \frac{\dot{\vec{r}}}{\dot{s}}$$

$$\vec{r}'' = \frac{d^2\vec{r}}{ds^2} \ldots \quad \vec{r}'' = \frac{d}{ds}(\vec{r}') = \frac{d}{ds}\left(\frac{\dot{\vec{r}}}{\dot{s}}\right)$$

$k_g \vec{q}'' = \left[\dfrac{d}{dt}\left(\dfrac{\dot{\vec{r}}}{\dot{s}}\right) \Big/ \dfrac{1}{\dot{s}}\dot{\vec{r}}\right]$

$= \left[N \dfrac{1}{\dot{s}} \left[\dfrac{d\vec{r}}{dt}\dfrac{\dot{\vec{r}}}{(\dot{s})^2} \dfrac{d}{dt}\right]\right]$

$\vec{r}' = \dot{s}\ddot{\vec{r}} - \dot{\vec{r}}\ddot{s}$

$= [N \quad \dfrac{d\vec{r}}{dt} (\dot{s})^2 \dfrac{d^2}{}] \dfrac{dt}{} \dfrac{d^2}{}$

Now $\vec{r}' \times \vec{r}'' = \dot{\vec{r}} \times [\dot{s}\ddot{\vec{r}} - \dot{\vec{r}}\ddot{s}]$

$= \left[N \dfrac{d\vec{r}}{dt} \quad \dfrac{d^2\vec{r}}{dt^2} \dfrac{\dot{s}(\dot{\vec{r}} \times \ddot{\vec{r}})}{(\dot{s})^4} \dfrac{d(\dot{\vec{r}} \times \dot{\vec{r}})\ddot{s}}{(\dot{s})^4}\right] \left(\dfrac{dt}{}\right)(\dot{s}^3)^2$

$= \left\{N \dot{\vec{r}} \dfrac{d\vec{r}'}{dt} \quad \dfrac{d^2\vec{r}}{dt^2} \dfrac{\dot{\vec{r}} \times \ddot{\vec{r}}}{(\dot{s})^3}\right\}^3 - 0$

k_g put $\vec{N} \cdot \dfrac{d\vec{r}}{dt} \times \vec{r}''$ and $\dfrac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$

$= \vec{N} \dfrac{d\vec{r}}{dt}$

$= \vec{N} \cdot \dot{\vec{r}} \times \ddot{\vec{r}}$

$k_g = \left[N \quad \dot{\vec{r}}(\dot{s})^3 \ddot{\vec{r}}\right] \left(\dfrac{ds}{dt}\right)^{-3}$

$= (\dot{s})^{-3} [\vec{N} \cdot \dot{\vec{r}} \times \ddot{\vec{r}}]$

$k_g = [\vec{N} \quad \dot{\vec{r}} \quad \ddot{\vec{r}}] \, (\dot{s})^{-3}$

$k_g = (\dot{s})^{-3} [\vec{N} \quad \dot{\vec{r}} \quad \ddot{\vec{r}}]$

Differential equation for geodesics on a surface :-

By the property of geodesics on a surface.

we know that the principle normal at any point with position vector $\vec{r}$ on a geodesics on a surface is parallel to the normal to the surface.

Now, by Seret Frenet Formula, we know that $\vec{r}'' = k\vec{n}$

Now, $\vec{r}' = \vec{r}_1 u' + \vec{r}_2 v'$ $\quad \because \dfrac{d\vec{r}}{ds} = \dfrac{\partial \vec{r}}{\partial u}$

$$\frac{d\vec{r}}{ds} = \frac{\partial\vec{r}}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial\vec{r}}{\partial v}\frac{\partial v}{\partial s}$$

$\vec{r}'' = \vec{r}_{11}(u')^2 + \vec{r}_{12}u'v' + \vec{r}_1 u'' + \vec{r}_{22}v'^2 + \vec{r}_{12}u'v' + \vec{r}_2 v''$

$\Rightarrow k_1\vec{n} = \vec{r}_{11}u'^2 \quad \because \vec{r}_{12} = \vec{r}_{21}$

So $\vec{r}'' = \vec{r}_{11}u'^2 + 2\vec{r}_{12}u'v' + \vec{r}_1 u'' + \vec{r}_2 v'' + \vec{r}_{22}(v')^2$

$\Rightarrow k_1\vec{n} = \vec{r}_{11}u'^2 + 2\vec{r}_{12}u'v' + \vec{r}_1 u'' + \vec{r}_2 v'' + \vec{r}_{22}(v')^2$

Taking dot product with $\vec{r}_1$ and $\vec{r}_2$ on both sides

$\Rightarrow k_1\vec{n}\cdot\vec{r}_1 = (\vec{r}_1\cdot\vec{r}_{11})u'^2 + 2(\vec{r}_1\cdot\vec{r}_{12})u'v' + (\vec{r}_1\cdot\vec{r}_1)u'' + (\vec{r}_1\cdot\vec{r}_2)v''$
$+ (\vec{r}_1\cdot\vec{r}_{22})v'^2$

Put $\vec{n}\cdot\vec{r}_1 = 0$

(Principle normal $\vec{n}$ is $\parallel$ to surface normal $\vec{N}$ which is perpandicular to both $\vec{r}_1$ and $\vec{r}_2$. So, $\vec{n}\cdot\vec{r}_1 = \vec{N}\cdot\vec{r}_1 = \vec{n}\cdot\vec{r}_2 = \vec{N}\cdot\vec{r}_2 = 0$)

Put $\vec{r}_1\cdot\vec{r}_1 = E$ and $\vec{r}_1\cdot\vec{r}_2 = F$

$\Rightarrow 0 = (\vec{r}_1\cdot\vec{r}_{11})u'^2 + 2(\vec{r}_1\cdot\vec{r}_{12})u'v' + Eu'' + Fv'' + (\vec{r}_1\cdot\vec{r}_{22})v'^2 \rightarrow (1)$

Now dot product with $\vec{r}_2$

$k\vec{n}\cdot\vec{r}_2 = (\vec{r}_2\cdot\vec{r}_{11})u'^2 + 2(\vec{r}_2\cdot\vec{r}_{12})u'v' + (\vec{r}_2\cdot\vec{r}_1)u'' + (\vec{r}_2\cdot\vec{r}_2)v''$
$+ (\vec{r}_2\cdot\vec{r}_{22})v'^2$

$\Rightarrow 0 = (\vec{r}_2\cdot\vec{r}_{11})u'^2 + 2(\vec{r}_2\cdot\vec{r}_{12})u'v' + Fu'' + Gv'' + (\vec{r}_2\cdot\vec{r}_{22})v'^2$

Since, the curve is a geodesics $\rightarrow (2)$

So $\vec{n}\cdot\vec{r}_1 = \vec{n}\cdot\vec{r}_2 = 0$

Now $E_1 = \frac{\partial}{\partial u}(E)$

$= \frac{\partial}{\partial u}(\vec{r}_1\cdot\vec{r}_1) = \vec{r}_{11}\cdot\vec{r}_1 + \vec{r}_1\cdot\vec{r}_{11} = 2\vec{r}_1\cdot\vec{r}_{11}$

$\Rightarrow \vec{r}_1\cdot\vec{r}_{11} = \frac{1}{2}E_1$

$E_2 = \frac{\partial E}{\partial v} = \frac{\partial}{\partial v}(\vec{r}_1\cdot\vec{r}_1) = \vec{r}_1\cdot\vec{r}_{12} + \vec{r}_{12}\cdot\vec{r}_1$

$E_2 = 2\vec{r}_1\cdot\vec{r}_{12}$

$\Rightarrow \vec{r}_1\cdot\vec{r}_{12} = \frac{1}{2}E_2$

$$F_1 = \frac{\partial F}{\partial u} = \frac{\partial}{\partial u}(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_{11} \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_{21}$$

$$F_1 = \vec{r}_1 \cdot \vec{r}_{12} + \vec{r}_{11} \cdot \vec{r}_2 \rightarrow (3)$$

$$F_2 = \frac{\partial F}{\partial v} = \frac{\partial}{\partial v}(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_{12} \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_{22}$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_{12} \cdot \vec{r}_2 \rightarrow (4)$$

$$G_1 = \frac{\partial G}{\partial u}$$

$$= \frac{\partial}{\partial u}(\vec{r}_2 \cdot \vec{r}_2) = \vec{r}_2 \cdot \vec{r}_{12} + \vec{r}_{12} \cdot \vec{r}_2$$

$$G_1 = 2\,\vec{r}_2 \cdot \vec{r}_{12}$$

$$\vec{r}_2 \cdot \vec{r}_{12} = \frac{1}{2} G_1$$

$$G_2 = \frac{\partial G}{\partial v} = \frac{\partial}{\partial v}(\vec{r}_2 \cdot \vec{r}_2)$$

$$G_2 = \vec{r}_2 \cdot \vec{r}_{22} + \vec{r}_{22} \cdot \vec{r}_2$$

$$G_2 = 2\,\vec{r}_2 \cdot \vec{r}_{22}$$

$$\vec{r}_2 \cdot \vec{r}_{22} = \frac{1}{2} G_2$$

Put the values in $\vec{r}_1 \cdot \vec{r}_{12} = \frac{1}{2} E_2$ in (3); $\vec{r}_{12} \cdot \vec{r}_2 = \frac{1}{2} G_1$ in (4)

$$F_1 = \vec{r}_1 \cdot \vec{r}_{12} + \vec{r}_{11} \cdot \vec{r}_2$$

$$F_1 = \frac{1}{2} E_2 + \vec{r}_{11} \cdot \vec{r}_2$$

$$F_1 - \frac{1}{2} E_2 = \vec{r}_{11} \cdot \vec{r}_2$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_{12} \cdot \vec{r}_2$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \frac{1}{2} G_1$$

$$\vec{r}_1 \cdot \vec{r}_{22} = F_2 - \frac{1}{2} G_1$$

Substituting all values in (1) and (2) we have.

$$0 = \frac{1}{2}E_1 u'^2 + E_2 u'v' + Eu'' + Fv'' + (F_2 - \frac{1}{2}G_1)v'^2$$

$$0 = (F_1 - \frac{1}{2}E_2)u'^2 + G_1 u'v' + Fu'' + Gv'' + \frac{1}{2}G_2 v'^2$$

$$\Rightarrow Eu'' + Fv'' = -\frac{1}{2}E_1 u'^2 - E_2 u'v' - (F_2 - \frac{1}{2}G_1)v'^2$$

$$Fu'' + Gv'' = -(F_1 - \frac{1}{2}E_2)u'^2 - G_1 u'v' - \frac{1}{2}G_2 v'^2$$

$$\Rightarrow Eu'' + Fv'' = (\frac{1}{2}G_1 - F_2)v'^2 - E_2 u'v' - \frac{1}{2}E_1 u'^2 \rightarrow (a)$$

$$Fu'' + Gv'' = (\frac{1}{2}E_2 - F_1)u'^2 - G_1 u'v' - \frac{1}{2}G_2 v'^2 \rightarrow (b)$$

Add $E_1 u'^2$, $E_2 u'v'$, $F_1 u'v'$ and $F_2 v'^2$ in (a)

$$\Rightarrow E_1 u'^2 + E_2 u'v' + Eu'' + F_1 u'v' + F_2 v'^2 + Fv'' = (\frac{1}{2}G_1 - F_2)v'^2 - E_2 u'v' + E_2 u'v' + E_1 u'^2 + F_1 u'v' + F_2 v'^2 - \frac{1}{2}E_1 u'^2$$

$$\Rightarrow \frac{d}{ds}(Eu' + Fv') = \frac{1}{2}E_1 u'^2 + \frac{1}{2}G_1 v'^2 + F_1 u'v' \rightarrow (c)$$

Now adding $F_2 u'v'$, $F_1 u'^2 + G_1 u'v' + G_2 v'^2$

$$\Rightarrow Fu'' + Gv'' + F_2 u'v' + F_1 u'^2 + G_1 u'v' + G_2 v'^2 = (\frac{1}{2}E_2 - F_1)u'^2 - G_1 u'v' + G_1 u'v' + F_1 u'^2 + \frac{1}{2}G_2 v'^2 = \frac{1}{2}G_2 v'^2 + F_2 u'v'$$

$$\Rightarrow \frac{d}{ds}(Fu' + Gv') = \frac{1}{2}G_2 v'^2 + \frac{1}{2}E_2 u'^2 + F_2 u'v'$$

$$\frac{d}{ds}(Fu' + Gv') = \frac{1}{2}E_2 u'^2 + \frac{1}{2}G_2 v'^2 + F_2 u'v' \rightarrow (d)$$

The equations (c) and (d) are know as general differential equations for a geodesics.

Christoffel symbols of Ist kind:-

Find Γ_{ijk} for all values of i, j, k where $i, j, k = 1, 2$.

Sol:-

$$\Gamma_{ijk} = \frac{1}{2}\left[(\vec{r}_i \cdot \vec{r}_j)_k + (\vec{r}_i \cdot \vec{r}_k)_j - (\vec{r}_j \cdot \vec{r}_k)_i\right]$$

For $i = 1$, $j = 1$ and $k = 1$

$$\Gamma_{111} = \frac{1}{2}\left[(\vec{r}_1 \cdot \vec{r}_1)_1 + \cancel{(\vec{r}_1 \cdot \vec{r}_1)_1} - \cancel{(\vec{r}_1 \cdot \vec{r}_1)_1}\right]$$

$$\Gamma_{111} = \frac{1}{2}\left[(\vec{r}_1 \cdot \vec{r}_1)_1\right] \qquad \because \vec{r}_1 \cdot \vec{r}_1 = E$$

$$E_1 = \frac{\partial E}{\partial u} = (\vec{r}_1 \cdot \vec{r}_1)_1 = \frac{\partial}{\partial u}(\vec{r}_1 \cdot \vec{r}_1)$$

$$= \vec{r}_{11} \cdot \vec{r}_1 + \vec{r}_1 \cdot \vec{r}_{11} = \vec{r}_1 \cdot \vec{r}_{11} + \vec{r}_1 \cdot \vec{r}_{11}$$

$$E_1 = (\vec{r}_1 \cdot \vec{r}_1)_1 = 2\vec{r}_1 \cdot \vec{r}_{11}$$

$$\Rightarrow E_1 = 2\vec{r}_1 \cdot \vec{r}_{11}$$

$$\Gamma_{111} = \frac{1}{2}E_1 \Rightarrow \vec{r}_1 \cdot \vec{r}_{11} = \frac{1}{2}E_1$$

$$\Rightarrow \Gamma_{111} = \frac{1}{2}E_1$$

For $i = 1$, $j = 1$, $k = 2$

$$\Gamma_{112} = \vec{r}_1 \cdot \vec{r}_{12}$$

$$\Gamma_{112} = \frac{1}{2}E_2$$

$$E_2 = \frac{\partial}{\partial v}(E) = \frac{\partial}{\partial v}(\vec{r}_1 \cdot \vec{r}_1) = \vec{r}_1 \cdot \vec{r}_{12} + \vec{r}_{12} \cdot \vec{r}_1 = 2\vec{r}_1 \cdot \vec{r}_{12}$$

$$\Rightarrow \vec{r}_1 \cdot \vec{r}_{12} = \frac{1}{2}E_2$$

$$\Rightarrow \Gamma_{112} = \frac{1}{2}E_2$$

For $i = 1$, $j = 2$, $k = 1$

$$\Gamma_{121} = \vec{r}_1 \cdot \vec{r}_{21}$$

$$\Gamma_{121} = \frac{1}{2}E_2 \quad \text{because } \vec{r}_{21} = \vec{r}_{12}$$

So $\Gamma_{112} = \Gamma_{121}$

For $i=1$, $j=2$, $k=2$

$\Gamma_{122} = \vec{r}_1 \cdot \vec{r}_{22}$

$F_2 = \frac{\partial F}{\partial v} = \frac{\partial}{\partial v}(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_{21} \cdot \vec{r}_2$

$F_2 - \vec{r}_2 \cdot \vec{r}_{21} = \vec{r}_1 \cdot \vec{r}_{22}$

and $G_1 = \frac{\partial}{\partial u}(\vec{r}_2 \cdot \vec{r}_2)$

$= \vec{r}_{21} \cdot \vec{r}_2 + \vec{r}_2 \cdot \vec{r}_{21} = 2\vec{r}_2 \cdot \vec{r}_{21}$

$\frac{1}{2} G_1 = \vec{r}_2 \cdot \vec{r}_{21}$

then $\vec{r}_1 \cdot \vec{r}_{22} = F_2 - \frac{1}{2} G_1 \Rightarrow (a)$

$\Rightarrow \Gamma_{122} = F_2 - \frac{1}{2} G_1$

For $i=2$, $j=1$, $k=1$

$\Gamma_{211} = \vec{r}_2 \cdot \vec{r}_{11}$

$F_1 = \frac{\partial}{\partial u}(\vec{r}_1 \cdot \vec{r}_2) = \vec{r}_{11} \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_{12}$

$E = \vec{r}_1 \cdot \vec{r}_1$

$E_2 = \frac{\partial E}{\partial v} = \vec{r}_1 \cdot \vec{r}_{12} + \vec{r}_{12} \cdot \vec{r}_1 = 2\vec{r}_1 \cdot \vec{r}_{12}$

$\frac{1}{2} E_2 = \vec{r}_1 \cdot \vec{r}_{12}$

$F_1 = \vec{r}_{11} \cdot \vec{r}_2 + \frac{1}{2} E_2$

$\Rightarrow \vec{r}_{11} \cdot \vec{r}_2 = F_1 - \frac{1}{2} E_2$

$\Rightarrow \vec{r}_2 \cdot \vec{r}_{11} = F_1 - \frac{1}{2} E_2$

$\Rightarrow \Gamma_{211} = F_1 - \frac{1}{2} E_2$

For $i=2,\ j=2,\ k=1$

$$\Gamma_{221} = \vec{r}_2 \cdot \vec{r}_{21}$$

$$G = \vec{r}_2 \cdot \vec{r}_2$$

$$G_1 = \frac{\partial G}{\partial u} = \vec{r}_2 \cdot \vec{r}_{21} + \vec{r}_{21} \cdot \vec{r}_2$$

$$\frac{1}{2} G_2 = \vec{r}_2 \cdot \vec{r}_{21}$$

$$\Gamma_{221} = \frac{1}{2} G_2$$

For $i=2,\ j=2,\ k=2$

$$\Gamma_{222} = \vec{r}_2 \cdot \vec{r}_{12}$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$

$$F_2 = \frac{\partial F}{\partial v} = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_{12} \cdot \vec{r}_2$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_2 \cdot \vec{r}_{12}$$

From (a) $\vec{r}_1 \cdot \vec{r}_{22} = F_2 - \frac{1}{2} G_1$

$$F_2 = F_2 - \frac{1}{2} G_1 + \vec{r}_2 \cdot \vec{r}_{12}$$

$$\Rightarrow \frac{1}{2} G_1 = \vec{r}_2 \cdot \vec{r}_{12}$$

$$\Rightarrow \Gamma_{212} = \Gamma_{221}$$

For $i=2,\ j=2,\ k=2$

$$\Gamma_{222} = \vec{r}_2 \cdot \vec{r}_{22}$$

$$G_2 = \frac{\partial G}{\partial v} = \vec{r}_2 \cdot \vec{r}_{22} + \vec{r}_{22} \cdot \vec{r}_2$$

$$= 2\vec{r}_2 \cdot \vec{r}_{22}$$

$$\frac{1}{2} G_2 = \vec{r}_2 \cdot \vec{r}_{22}$$

$$\Rightarrow \Gamma_{222} = \frac{1}{2} G_2$$

Christoffel symbols of second kind:-

Γ^i_{jk} is christoffel symbol of second kind.

For $i=1$ and $i=2$

$$\Gamma^1_{jk} = H^{-2}(G\,\Gamma_{1jk} - F\,\Gamma_{2jk})$$

$$\Gamma^2_{jk} = H^{-2}(E\,\Gamma_{2jk} - F\,\Gamma_{1jk})$$

For $i=1$, $j=1$, $k=1$

$$\Gamma^1_{11} = H^{-2}(G\,\Gamma_{111} - F\,\Gamma_{211}) \rightarrow (1)$$

$$\Gamma_{111} = \vec{r}_1 \cdot \vec{r}_{11}$$

$$\frac{1}{2}E_1 = \vec{r}_1 \cdot \vec{r}_{11} \rightarrow (a)$$

$$\Gamma_{211} = \vec{r}_2 \cdot \vec{r}_{11}$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$

$$F_1 = \frac{\partial F}{\partial u} = \vec{r}_1 \cdot \vec{r}_{21} + r_{11} \cdot \vec{r}_2$$

$$= \vec{r}_1 \cdot \vec{r}_{12} + \vec{r}_{11} \cdot \vec{r}_2$$

$$\frac{1}{2}E_2 = \vec{r}_1 \cdot \vec{r}_{12}$$

$$F_1 = \frac{1}{2}E_2 + \vec{r}_2 \cdot \vec{r}_{11}$$

$$F_1 - \frac{1}{2}E_2 = \vec{r}_2 \cdot \vec{r}_{11}$$

$$\Gamma_{211} = F_1 - \frac{1}{2}E_2 \rightarrow (b)$$

put in (1)

$$\Gamma^1_{11} = \frac{1}{H^2}\left[G\left(\frac{1}{2}E_1\right) - F\left(F_1 - \frac{1}{2}E_2\right)\right]$$

$$= \frac{1}{H^2}\left[\frac{1}{2}GE_1 - FF_1 + \frac{1}{2}FE_2\right]$$

$$\Gamma_{11}^{1} = \frac{1}{2H^2}\left[GE_1 - 2FF_1 + FE_2\right]$$

For $i=1,\ j=2,\ k=1$

$$\Gamma_{21}^{1} = H^{-2}\left[G\,\Gamma_{121} - F\,\Gamma_{221}\right] \to (2)$$

$$\Gamma_{121} = \vec{r}_1 \cdot \vec{r}_{21} = \vec{r}_1 \cdot \vec{r}_{12}$$

$$\frac{1}{2}E_2 = \vec{r}_1 \cdot \vec{r}_{12}$$

$$\Rightarrow \Gamma_{121} = \frac{1}{2}E_2$$

$$\Gamma_{221} = \vec{r}_2 \cdot \vec{r}_{21}$$

$$\Rightarrow G_1 = \frac{\partial G}{\partial u} = \vec{r}_2 \cdot \vec{r}_{21} + \vec{r}_{21} \cdot \vec{r}_2 = 2\,\vec{r}_2 \cdot \vec{r}_{21}$$

$$\Rightarrow \frac{1}{2}G_1 = \vec{r}_2 \cdot \vec{r}_{21}$$

$$\Gamma_{221} = \frac{1}{2}G_1 \quad \text{put in } (2)$$

$$\Gamma_{21}^{1} = \frac{1}{H^2}\left[G\left(\frac{1}{2}E_2\right) - F\left(\frac{1}{2}G_1\right)\right]$$

$$\Gamma_{21}^{1} = \frac{1}{2H^2}\left[GE_2 - FG_1\right]$$

For $i=1,\ j=1,\ k=2$

$$\Gamma_{12}^{1} = H^{-2}\left[G\,\Gamma_{112} - F\,\Gamma_{212}\right] \to (3)$$

$$\Rightarrow \Gamma_{112} = \vec{r}_1 \cdot \vec{r}_{12}$$

$$\Gamma_{112} = \frac{1}{2}E_2$$

$$\Gamma_{212} = \vec{r}_2 \cdot \vec{r}_{12} = \vec{r}_2 \cdot \vec{r}_{21}$$

$$\Gamma_{212} = \frac{1}{2}G_1$$

put in (3)

$$\Gamma_{12}^{1} = \frac{1}{H^2}\left[G\left(\frac{1}{2}E_2\right) - F\left(\frac{1}{2}G_1\right)\right]$$

$$\Gamma_{12}^{1} = \frac{1}{2H^2}\left[GE_2 - FG_1\right] = \Gamma_{21}^{1}$$

For $i=1,\ j=2,\ k=2$

$$\Gamma_{22}^{1} = H^{-2}\left[G\Gamma_{122} - F\Gamma_{222}\right] \rightarrow (4)$$

$$\Gamma_{122} = \vec{r}_1 \cdot \vec{r}_{22}$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_{21} \cdot \vec{r}_2$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_2 \cdot \vec{r}_{21}$$

$$\vec{r}_2 \cdot \vec{r}_{21} = \frac{1}{2}G_1$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \frac{1}{2}G_1$$

$$\Rightarrow \vec{r}_1 \cdot \vec{r}_{22} = F_2 - \frac{1}{2}G_1$$

$$\Gamma_{222} = \vec{r}_2 \cdot \vec{r}_{22} = \frac{1}{2}G_2$$

put in (4)

$$\Gamma_{22}^{1} = \frac{1}{H^2}\left[G\left(F_2 - \frac{1}{2}G_1\right) - F\left(\frac{1}{2}G_2\right)\right]$$

$$= \frac{1}{H^2}\left[GF_2 - \frac{1}{2}GG_1 - \frac{1}{2}FG_2\right]$$

$$\Gamma_{22}^{1} = \frac{1}{2H^2}\left[2GF_2 - GG_1 - FG_2\right]$$

Now, $i=2,\ j=1,\ k=1$

$$\Gamma_{11}^{2} = H^{-2}\left[E\Gamma_{211} - F\Gamma_{111}\right] \rightarrow (5)$$

$$\Gamma_{211} = \vec{r}_2 \cdot \vec{r}_{11}$$

$$F = \vec{r}_1 \cdot \vec{r}_2$$

$$F_1 = \vec{r}_{11} \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_{21}$$

$$F_1 = \vec{r}_{11} \cdot \vec{r}_2 + \vec{r}_1 \cdot \vec{r}_{12}$$

$$\vec{r}_1 \cdot \vec{r}_{12} = \frac{1}{2} E_2$$

$$F_1 = \vec{r}_2 \cdot \vec{r}_{11} + \frac{1}{2} E_2$$

$$\vec{r}_2 \cdot \vec{r}_{11} = F_1 - \frac{1}{2} E_2$$

$$\Gamma_{211} = F_1 - \frac{1}{2} E_2$$

$$\Gamma_{111} = \vec{r}_1 \cdot \vec{r}_{11}$$

$$\Gamma_{111} = \frac{1}{2} E_1$$

put in (5)

$$\Gamma^2_{11} = \frac{1}{H^2}\left[E\left(F_1 - \frac{1}{2} E_2\right) - F\left(\frac{1}{2} E_1\right)\right]$$

$$= \frac{1}{H^2}\left[EF_1 - \frac{1}{2} EE_2 - \frac{1}{2} FE_1\right]$$

$$\Gamma^2_{11} = \frac{1}{2H^2}\left[2EF_1 - EE_2 - FE_1\right]$$

For $i=2$, $j=2$, $k=1$

$$\Gamma^2_{21} = H^{-2}\left[E\,\Gamma_{221} - F\,\Gamma_{121}\right] \rightarrow (6)$$

$$\Gamma_{221} = \vec{r}_2 \cdot \vec{r}_{21} = \vec{r}_2 \cdot \vec{r}_{12} = \frac{1}{2} G_1$$

$$\Gamma_{121} = \vec{r}_1 \cdot \vec{r}_{21} = \vec{r}_1 \cdot \vec{r}_{12} = \frac{1}{2} E_2 \quad \text{put in (6)}$$

$$\Gamma^2_{21} = \frac{1}{H^2}\left[E\left(\frac{1}{2} G_1\right) - F\left(\frac{1}{2} E_2\right)\right]$$

$$\Gamma_{21}^{2} = \frac{1}{H^2}\left[\frac{1}{2}EG_1 - \frac{1}{2}FE_2\right]$$

$$\Gamma_{21}^{2} = \frac{1}{2H^2}\left[EG_1 - FE_2\right]$$

For $i=2,\quad j=1,\quad k=2$

$$\Gamma_{12}^{2} = H^{-2}\left[E\Gamma_{212} - F\Gamma_{112}\right] \to (7)$$

$$\Gamma_{212} = \vec{r}_2 \cdot \vec{r}_{12} = \vec{r}_2 \cdot \vec{r}_{21} = \frac{1}{2}G_1$$

$$\Gamma_{112} = \Gamma_{121} = \frac{1}{2}E_2 \quad \text{put in (7)}$$

$$\Gamma_{12}^{2} = \frac{1}{H^2}\left[E\left(\frac{1}{2}G_1\right) - F\left(\frac{1}{2}E_2\right)\right]$$

$$\Gamma_{12}^{2} = \frac{1}{2H^2}\left[EG_1 - FE_2\right] = \Gamma_{21}^{2}$$

For $i=2,\quad j=2,\quad k=2$

$$\Gamma_{22}^{2} = H^{-2}\left[E\Gamma_{222} - F\Gamma_{122}\right] \to (8)$$

$$\Gamma_{222} = \vec{r}_2 \cdot \vec{r}_{22} = \frac{1}{2}G_2$$

$$\Gamma_{122} = \vec{r}_1 \cdot \vec{r}_{22}$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_{21} \cdot \vec{r}_2$$

$$= \vec{r}_1 \cdot \vec{r}_{22} + \vec{r}_2 \cdot \vec{r}_{21}$$

$$F_2 = \vec{r}_1 \cdot \vec{r}_{22} + \frac{1}{2}G_1$$

$$\vec{r}_1 \cdot \vec{r}_{22} = F_2 - \frac{1}{2}G_1$$

$$\Gamma_{122} = F_2 - \frac{1}{2}G_1 \quad \text{put in (8)}$$

$$\Gamma_{22}^{2} = \frac{1}{H^2}\left[E\left(\frac{1}{2}G_2\right) - F\left(F_2 - \frac{1}{2}G_1\right)\right]$$

$$\Gamma_{22}^{2} = \frac{1}{2H^2}\left[EG_2 - 2FF_2 + FG_1\right]$$