Notes of Complex Analysis

by

Ms. Iqra Liaqat

Partial Contents

These are handwritten notes. We are very thankful to Ms. Iqra Liaqat for sending these notes.

CHAPTER No 0

Introduction :-

Real numbers :-

The set of rational and irrational numbers form the set of real numbers.

$$\mathbb{R} = Q \cup Q'$$

Complex numbers :-

The numbers which are in i(iota) form or which have two parts "real" and "imaginary" are called complex numbers.

(i-e) $z = x + iy$ is a complex number in which x is real and y is imaginary part.

Difference between Real and Complex numbers :-

(i) The set of real numbers is denoted by "$\mathbb{R}$ or R" and the set of complex numbers is denoted by "C".

(ii) Functions whose domain and range are both subsets of "R" are called real functions and the functions whose domain and range are both subsets of "C" are called complex functions.

(iii) Real numbers can be written in order form while complex numbers can not be written in order form.

(iv) Complex numbers can be real numbers when imaginary part is zero. but real numbers can not be complex number.

Note :-

If $z = x + iy$

then $\arg z = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Ex :-

Find the locus of z, where

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\lambda}{3}$$

Sol :-

$$\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$$

$$= \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \frac{[(x-1)+iy][(x+1)-iy]}{(x+1)^2-(iy)^2}$$

$$= \frac{(x-1)[(x+1)-iy]+iy[(x+1)-iy]}{(x+1)^2-i^2y^2}$$

$$= \frac{x^2-1+iy-\cancel{ixy}+\cancel{ixy}+iy-i^2y^2}{(x+1)^2-i^2y^2}$$

$\because i^2 = -1$

$$= \frac{x^2-1+2iy-(-1)y^2}{(x+1)^2-(-1)y^2}$$

$$= \frac{x^2-1+2iy+y^2}{(x+1)^2+y^2}$$

$$\frac{z-1}{z+1} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}$$

$$\arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\left(\frac{\cancel{\frac{2y}{(x+1)^2+y^2}}}{\frac{x^2+y^2-1}{\cancel{(x+1)^2+y^2}}}\right) = \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

$$\arg\left(\frac{z-1}{z+1}\right) = \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

Since $\arg\left(\frac{z-1}{z+1}\right) = \frac{\lambda}{3}$

$$\frac{\lambda}{3} = \tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)$$

$$\tan\frac{\lambda}{3} = \frac{2y}{x^2+y^2-1} \qquad \because \tan\frac{\lambda}{3} = \sqrt{3}$$

$$\sqrt{3} = \frac{2y}{x^2+y^2-1}$$

$$\sqrt{3}(x^2+y^2-1) = 2y$$

$$x^2+y^2-1 = \frac{2}{\sqrt{3}}y$$

$$\Rightarrow x^2+y^2-\frac{2}{\sqrt{3}}y-1 = 0$$

which is equivilent to the circle.

Ex :-

If $z = \frac{(1+it)^2 - it}{1+t^2}$ then prove that the locus of z is an ellipse. Find the semi-major and semi-minor axis.

Sol :-

$$z = \frac{(1+it)^2 - it}{1+t^2}$$

We know $z = x+iy$

$$x+iy = \frac{(1+it)^2 - it}{1+t^2}$$

$$x+iy = \frac{(1+i^2t^2+2it) - it}{1+t^2} \qquad \therefore i^2 = -1$$

$$x+iy = \frac{(1-t^2+2it) - it}{1+t^2}$$

$$x+iy = \frac{1-t^2+2it-it}{1+t^2}$$

$$x+iy = \frac{1-t^2+it}{1+t^2}$$

$$(x+iy)(1+t^2) = 1-t^2+it$$

$$x(1+t^2)+iy(1+t^2) = 1-t^2+it$$

$$x+xt^2+iy+it^2y = 1-t^2+it$$

$$(x+xt^2)+(y+yt^2)i = (1-t^2)+it$$

$\Rightarrow$ $x+xt^2 = 1-t^2 \rightarrow (1)$

$\Rightarrow$ $y+yt^2 = t \rightarrow (2)$

From (1) $\quad xt^2+t^2 = 1-x$

$$(1+x)t^2 = 1-x$$

$$t^2 = \frac{1-x}{1+x}$$

$$\Rightarrow t = \sqrt{\frac{1-x}{1+x}}$$

put t - value in (2)

$$y + y\left(\sqrt{\frac{1-x}{1+x}}\right)^2 = \sqrt{\frac{1-x}{1+x}}$$

$$y + y\left(\frac{1-x}{1+x}\right) = \sqrt{\frac{1-x}{1+x}}$$

is locus of z

$$y\left(1 + \frac{1-x}{1+x}\right) = \sqrt{\frac{1-x}{1+x}}$$

$$y\left(\frac{1+x+1-x}{1+x}\right) = \sqrt{\frac{1-x}{1+x}}$$

$$y\left(\frac{2}{1+x}\right) = \sqrt{\frac{1-x}{1+x}}$$

squaring both sides

$$y^2 \frac{4}{(1+x)^2} = \frac{1-x}{1+x}$$

$$\frac{4y^2}{1+x} = 1-x$$

$$4y^2 = (1+x)(1-x)$$

$$4y^2 = 1-x^2$$

$$x^2 + 4y^2 = 1$$

$$\frac{x^2}{1^2} + \frac{y^2}{(1/2)^2} = 1$$

is an ellipse whose semi-major and semi-minor axes are "1" and $1/2$ respectively.

Prove that locus of z is circle, Find radius and centre of circle

if $z = \frac{(1+i)+(3+2i)t}{1+it}$

Sol:-

$$z = \frac{(1+i)+(3+2i)t}{1+it}$$

we know $z = x+iy$

$$x+iy = \frac{(1+i)+(3+2i)t}{1+it}$$

$$(x+iy)(1+it) = (1+i)+(3+2i)t$$

$$x(1+it)+iy(1+it) = 1+i+3t+2ti$$

$$x+xti+yi+i^2yt = (1+3t)+(1+2t)i \quad \because i^2=-1$$

$$x+xti+yi-yt = (1+3t)+(1+2t)i$$

$$(x-yt)+(xt+y)i = (1+3t)+(1+2t)i$$

$$\Rightarrow \quad x-yt = 1+3t \rightarrow (1)$$

$$xt+y = 1+2t \rightarrow (2)$$

From (1) $\quad x-1 = 3t+yt$

$$x-1 = (3+y)t$$

$$\frac{x-1}{y+3} = t$$

put $t = \frac{x-1}{y+3}$ in (2)

$$x\left(\frac{x-1}{y+3}\right)+y = 1+2\left(\frac{x-1}{y+3}\right)$$

$$\frac{x^2-x}{y+3}+y = 1+\frac{2x-2}{y+3}$$

$$x^2-x+y(y+3) = y+3+2x-2$$

$$x^2-x+y^2+3y = 2x+y+1$$

$$x^2+y^2-x-2x+3y-y-1 = 0$$

$$x^2+y^2-3x+2y-1 = 0$$

is locus of z and the locus of z is circle

$x^2+y^2+fx+gy+c=0$ is general equation of circle with centre $(-f,-g)$ and radius $\sqrt{f^2+g^2-c}$

Now, we find circle and radius of

$$x^2+y^2-3x+2y-1=0$$

centre $(-f,-g)=(+3,-2)$

Dividing by 2

centre $\left(\frac{3}{2},-1\right)$

radius $=\sqrt{\left(\frac{3}{2}\right)^2+(-1)^2-(-1)}$

radius $=\sqrt{\frac{9}{4}+1+1}$

radius $=\sqrt{\frac{9+4+4}{4}}$

radius $=\sqrt{\frac{17}{4}}=\frac{\sqrt{17}}{2}$

Example :-

Find the locus of z where $z=a\cos t+b\sin t$ where t is real parametric and a, b are complex constants.

Sol :-

Given $z=a\cos t+b\sin t$

we know $z=x+iy$

$$a=a_1+ia_2$$

$$b=b_1+ib_2$$

$$x+iy=(a_1+ia_2)\cos t+(b_1+ib_2)\sin t$$

$$x+iy=a_1\cos t+a_2\cos t\, i+b_1\sin t+b_2\sin t\, i$$

$$x+iy=(a_1\cos t+b_1\sin t)+(a_2\cos t+b_2\sin t)i$$

$$\Rightarrow \quad x = a_1 \cos t + b_1 \sin t \Rightarrow a_1 \cos t + b_1 \sin t - x = 0$$

$$y = a_2 \cos t + b_2 \sin t \Rightarrow a_2 \cos t + b_2 \sin t - y = 0$$

$$\frac{\cos t}{+b_2 x + (-b_1 y)} = \frac{-\sin t}{-a_1 y + a_2 x} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$\frac{\cos t}{b_2 x - b_1 y} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$\cos t = \frac{b_2 x - b_1 y}{a_1 b_2 - a_2 b_1} \rightarrow (1)$$

$$\frac{-\sin t}{-(a_1 y - a_2 x)} = \frac{1}{a_1 b_2 - a_2 b_1}$$

$$\sin t = \frac{a_1 y - a_2 x}{a_1 b_2 - a_2 b_1} \rightarrow (2)$$

Squaring and adding eq (1) and (2)

$$\cos^2 t + \sin^2 t = \frac{(b_2 x - b_1 y)^2}{(a_1 b_2 - a_2 b_1)^2} + \frac{(a_1 y - a_2 x)^2}{(a_1 b_2 - a_2 b_1)^2}$$

$$\therefore \cos^2 t + \sin^2 t = 1$$

$$1 = \frac{b_2^2 x^2 + b_1^2 y^2 - 2xy b_1 b_2 + a_1^2 y^2 + a_2^2 x^2 - 2xy a_1 a_2}{(a_1 b_2 - a_2 b_1)^2}$$

$$\Rightarrow \cancel{(a_1 b_2 - a_2 b_1)^2} \times \frac{a_2^2 x^2 + b_2^2 x^2 + a_1^2 y^2 + b_1^2 y^2 - 2xy a_1 a_2 - 2xy b_1 b_2}{\cancel{(a_1 b_2 - a_2 b_1)^2}}$$

$$= (a_1 b_2 - a_2 b_1)^2 \times 1$$

$$\Rightarrow (a_2^2 + b_2^2) x^2 + (a_1^2 + b_1^2) y^2 - 2xy (a_1 a_2 + b_1 b_2) = (a_1 b_2 - a_2 b_1)^2 \rightarrow (3)$$

is locus of (ellipse) z which is an ellipse if $h^2 - ab < 0$

$(a_1a_2 + b_1b_2)^2 - (b_2^2 + a_2^2)(b_1^2 + a_1^2) < 0$

$\Rightarrow \cancel{a_1^2a_2^2} + \cancel{b_1^2b_2^2} - 2a_1a_2b_1b_2 - \cancel{b_1^2b_2^2} - a_1^2b_2^2 - a_2^2b_1^2$
$- \cancel{a_1^2a_2^2} < 0$

$\Rightarrow \; -[2a_1a_2b_1b_2 + a_1^2b_2^2 + a_2^2b_1^2] < 0$

$\Rightarrow \; -[a_1b_2 - a_2b_1]^2 < 0$ which is -ve
so is ellipse

Note :-

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is an ellipse if $h^2 - ab < 0$.

(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(iii) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$

Prove that $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Proof :-

Let z_1 and z_2 be two complex numbers with modolui r_1 and r_2, with orguments θ_1 and θ_2.

then $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)$

$\overline{z_1 + z_2} = (x_1 + x_2) - (iy_1 + iy_2)$

$\overline{z_1 + z_2} = (x_1 - iy_1) + (x_2 - iy_2)$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

Note :-

(i) $|\bar{z}| = |z|$

(ii) $|z|^2 = z\bar{z}$

(iii) $|z_1 z_2| = |z_1||z_2|$

(iv) $|z_1 + z_2| \leq |z_1| + |z_2|$

(v) $|z_1 - z_2| \geq |z_1| - |z_2|$

(vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

(vii)

Theorem :-

The argument of the product of any n number of complex numbers is equal to the sum of their arguments

Proof :-

Suppose, $z_1, z_2, ----, z_n$ be non-zero complex numbers.

Further, let $r_1, r_2, ----, r_n$ be the moduly and $\theta_1, \theta_2, ----, \theta_n$ be the arguments we have

$$z_1 z_2 ---- z_n = r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) ---- r_n(\cos\theta_n + i\sin\theta_n)$$

$$= (r_1 r_2 ---- r_n)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) ------ (\cos\theta_n + i\sin\theta_n)$$

$$= (r_1 r_2 ---- r_n)(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) ---- (\cos\theta_n + i\sin\theta_n))$$

$$z_1 z_2 ---- z_n = (r_1 r_2 ---- r_n)(\cos(\theta_1+\theta_2+ ---- +\theta_n) + i\sin(\theta_1+\theta_2+ ---- +\theta_n))$$

$$\operatorname{Arg}(z_1 z_2 ---- z_n) = \theta_1 + \theta_2 + ---- + \theta_n$$

$$\operatorname{Arg}(z_1 z_2 ---- z_n) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) + ----- + \operatorname{Arg}(z_n)$$

Theorem :-

The argument of the quotient of two complex numbers is equal to the difference of their respective arguments

Proof :-

Let z_1 and z_2 be non-zero complex numbers with modolui r_1 and r_2 and arguments

θ_1 and θ_2

then $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{\cos^2\theta_2 - i^2\sin^2\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\frac{(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))}{\cos^2\theta_2 - (-1)\sin^2\theta_2} \quad \because i^2 = -1$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\frac{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}$$

Note:- if $z = re^{i\theta}$
$\arg z = \theta$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$$

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2)$$

Ex :-

prove that $\left|\dfrac{az+b}{b\bar{z}+a}\right| = 1$ if $|z| = 1$

Sol :-

$|z|^2 = 1$

$\Rightarrow \quad z\bar{z} = 1$

$\Rightarrow \quad \bar{z} = \dfrac{1}{z}$

Consider $\left| \dfrac{az+b}{b\bar{z}+a} \right|$

$$= \left| \frac{az+b}{b \cdot \frac{1}{z} + a} \right| = \left| \frac{az+b}{\frac{b+az}{z}} \right|$$

$$= \left| \frac{az+b}{az+b} \cdot z \right| \qquad \therefore |z| \leq 0$$

$$\Rightarrow |z| = 1$$

Ex :-

Prove that

$$\left| \frac{a+b}{1+\bar{a}b} \right| \leq 1 \quad \text{if} \quad |a| < 1 \quad \text{and} \quad |b| < 1$$

When does equality hold?

Sol :-

$$\left| \frac{a+b}{1+\bar{a}b} \right| \leq 1$$

$$\frac{|a+b|}{|1+\bar{a}b|} \leq 1$$

$$\Rightarrow |a+b| \leq |1+\bar{a}b|$$

L.H.S.

$$|a+b|^2 = (a+b)\overline{(a+b)} \qquad \therefore a\bar{a} = |a|^2$$

$$= (a+b)(\bar{a}+\bar{b})$$

$$= a\bar{a} + a\bar{b} + \bar{a}b + b\bar{b}$$

$$|a+b|^2 = |a|^2 + a\bar{b} + \bar{a}b + |b|^2 \rightarrow (1)$$

R.H.S.

$$|1+\bar{a}b|^2 = (1+\bar{a}b)\overline{(1+\bar{a}b)} \qquad \therefore \bar{\bar{a}} = a$$

$$= (1+\bar{a}b)(1+a\bar{b})$$

$$|1+\bar{a}b|^2 = 1 + a\bar{b} + \bar{a}b + a\bar{a}b\bar{b}$$

$|1+\bar{a}b|^2 = 1+\bar{a}b+a\bar{b}+|a|^2|b|^2$

Adding and subtract $|a|^2, |b|^2$

$|1+\bar{a}b|^2 = (|a|^2+|b|^2+a\bar{b}+\bar{a}b)+1-|a|^2-|b|^2$

By using Result (1) $+|a|^2|b|^2$

$= |a+b|^2+1(1-|a|^2)+(-1)(|b|^2-|a|^2|b|^2)$

$= |a+b|^2+1(1-|a|^2)-|b|^2(1-|a|^2)$

$|1+\bar{a}b|^2 = |a+b|^2+(1-|a|^2)(1-|b|^2)$

$\Rightarrow |a+b|^2 \leq |1+\bar{a}b|^2$

$\because 1-|a|^2>0, \quad 1-|b|^2>0$

$\therefore \left|\dfrac{a+b}{1+\bar{a}b}\right| < 1$ If inequality hold when $|a|^2=1, |b|^2=1$

$1-|a|^2=0, \; 1-|b|^2=0 \Rightarrow$ or $|a|=|b|=1$

Ex:-

Prove that $|a+b|^2+|a-b|^2 = 2\{|a|^2+|b|^2\}$ where a and b are complex constants

Sol:-

$|a+b|^2+|a-b|^2$

$= (a+b)(\overline{a+b})+(a-b)(\overline{a-b})$

$= (a+b)(\bar{a}+\bar{b})+(a-b)(\bar{a}-\bar{b})$

$= a\bar{a}+\cancel{a\bar{b}}+\cancel{\bar{a}b}+b\bar{b}+a\bar{a}+b\bar{b}-\cancel{a\bar{b}}-\cancel{\bar{a}b}$

$= |a|^2+|b|^2+|a|^2+|b|^2 \qquad \because a\bar{a}=|a|^2$

$= 2|a|^2+2|b|^2$

$= 2\{|a|^2+|b|^2\}$

Hence, $|a+b|^2+|a-b|^2 = 2\{|a|^2+|b|^2\}$

Note:-

(i) In complex analysis, limit is used in plane but in real analysis, limit is used in line.

(ii) A function which is analytic will be differentiable, will be continuous and its limit exist.

CHAPTER NO ONE

LIMITS, CONTINUITY AND DIFFERENTIABILITY

Limit :-

Let function f be defined in a domain D except perhaps at the point z_0 in D. We say that the limit of function f as z approaches z_0 equals to ω_0 and we write

$$\lim_{z \to z_0} f(z) = \omega_0 \quad \text{or}$$

$$f(z) \to \omega_0 \quad \text{as} \quad z \to z_0$$

if for every $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$|f(z) - \omega_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

Example :-

(i) $\lim\limits_{z \to 1} \dfrac{z^2 - 1}{z - 1}$

$$= \lim_{z \to 1} \frac{\cancel{(z-1)}(z+1)}{\cancel{(z-1)}}$$

$$= \lim_{z \to 1} (z+1)$$

$$= 1 + 1 = 2$$

(ii)

$$\lim_{z \to 3i} \frac{z^2 + 9}{z - 3i} = 6i$$

Sol :-

$$\lim_{z \to 3i} \frac{z^2 + 9}{z - 3i} \qquad \because z^2 + 9 = (z - 3i)(z + 3i)$$

$$= \lim_{z \to 3i} \frac{(z - 3i)(z + 3i)}{(z - 3i)}$$

$$= \lim_{z \to 3i} (z + 3i)$$

$$\Rightarrow 3i + 3i = 6i$$

Example:-

Prove that

$$\lim_{z\to 1}\left(\frac{z^n-1}{z-1}\right)=n$$

(i) n is a positive integer
(ii) n is negative integer
(iii) n is a fraction

Sol:-

(i) When n is a positive integer

$$\lim_{z\to 1}\left(\frac{z^n-1}{z-1}\right)=\lim_{z\to 1}\left(z^{n-1}+z^{n-2}+\cdots+z+1\right)$$

$$=\lim_{z\to 1}(z^{n-1})+\lim_{z\to 1}(z^{n-2})+\cdots+\lim_{z\to 1} z+1$$

$$=(1)^{n-1}+(1)^{n-2}+\cdots+1+1$$

$$=1+1+\cdots+1+1 \quad (n \text{ times})$$

$$\lim_{z\to 1}\left(\frac{z^n-1}{z-1}\right)=n$$

(ii) When n is a negative integer

Let $n=-m$ where m is +ve integer

$$\lim_{z\to 1}\left(\frac{z^n-1}{z-1}\right)=\lim_{z\to 1}\left(\frac{z^{-m}-1}{z-1}\right)$$

$$=\lim_{z\to 1}\left(\frac{\frac{1}{z^m}-1}{z-1}\right)$$

$$=\lim_{z\to 1}\left(\frac{\frac{1-z^m}{z^m}}{z-1}\right)$$

$$=\lim_{z\to 1}\left(\frac{1-z^m}{z^m(z-1)}\right)$$

$$\lim_{z\to1}\left(\frac{z^n-1}{z-1}\right) = -\lim_{z\to1}\frac{(z^m-1)}{z-1} \times \lim_{z\to1}\left(\frac{1}{z^m}\right)$$

$$= -m \times 1$$

$$= -m$$

$$\lim_{z\to1}\left(\frac{z^n-1}{z-1}\right) = n$$

(iii) when n is a fraction

when n is a fraction of the form $n = \frac{p}{q}$ where p and q are integers, $q \neq 0$

$$\lim_{z\to1}\left(\frac{z^n-1}{z-1}\right) = \lim_{z\to1}\left(\frac{z^{p/q}-1}{z-1}\right)$$

put $z^{1/q} = t$ (i-e) $z^{p/q} = t^p$

$z = t^q$ when $z \to 1$, $t \to 1$

$$\lim_{z\to1}\left(\frac{z^{p/q}-1}{z-1}\right) = \lim_{t\to1}\left(\frac{t^p-1}{t^q-1}\right)$$

$$= \lim_{t\to1}\left(\frac{\frac{t^p-1}{t-1}}{\frac{t^q-1}{t-1}}\right)$$

$$= \lim_{t\to1}\left(\frac{t^p-1}{t-1} \times \frac{1}{\left(\frac{t^q-1}{t-1}\right)}\right)$$

$$= \frac{p}{q} = n$$

Note :-

If a function is differentiable at a point of domain then it may or may not be analytic at this point or at its neighbouring point.

Example :-

prove that $\lim_{z\to 2i} \frac{z^2+4}{z-2i} = 4i$

Sol :-

$$\lim_{z\to 2i} \frac{z^2+4}{z-2i}$$

$$= \lim_{z\to 2i} \frac{\cancel{(z-2i)}(z+2i)}{\cancel{z-2i}} \qquad z^2+4 = (z-2i)(z+2i)$$

$$= \lim_{z\to 2i} z+2i$$

$$= 2i+2i$$

$$= 4i$$

2nd method :-

$$\lim_{z\to 2i} \frac{z^2+4}{z-2i}$$

$$= \lim_{z\to 2i} \left(\frac{(z^2+2i)(z-2i)}{z-2i} - 4i \right) < \epsilon$$

$$= |z-2i| < \varepsilon = \delta$$

(i-e) $|f(z) - 4i| < \varepsilon$ where

$0 < |z-2i| < \delta$

According to definition

$$\lim_{z\to 2i} \frac{z^2+4}{z-2i} = 4i$$

Theorem :-

If a limit of a function $w = f(z)$ exists, then it is unique

Proof :-

Suppose $\lim_{z \to z_0} f(z) = w_0$

and

$$\lim_{z \to z_0} f(z) = w_1$$

$$|f(z) - w_0| < \frac{\varepsilon}{2}, \quad 0 < |z - z_0| < \delta_1 \rightarrow (1)$$

$$|f(z) - w_1| < \frac{\varepsilon}{2}, \quad 0 < |z - z_0| < \delta_2 \rightarrow (2)$$

Let $\delta = \min(\delta_1, \delta_2)$

$$|w_0 - w_1| = |f(z) + w_0 - w_1 - f(z)|$$
$$|w_0 - w_1| \leq |f(z) - w_1| + |-(f(z) - w_0)|$$
$$|w_0 - w_1| \leq |f(z) - w_1| + |f(z) - w_0|$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \text{by (1) and (2)}$$

$$|w_0 - w_1| \leq \varepsilon \quad 0 < |z - z_0| < \delta$$

Since, ε is an arbitrary small +ve number if it follows that

$$|w_0 - w_1| = 0$$
$$\Rightarrow \quad w_0 - w_1 = 0$$
$$\Rightarrow \quad w_0 = w_1$$

We conclude that our supposition is incorrect.

Hence, limit is unique

Continuity :-

Let the function f be defined in a domain D containing the point z_0. Function f is said to be continuous at the point z_0 if

$$\lim_{z \to z_0} f(z) = f(z_0)$$

or

$$|f(z) - f(z_0)| < \varepsilon \ , \quad 0 < |z - z_0| < \delta$$

Note :-

(i) Every function is continuous at every point of domain (Continuous function)

(ii) In Complex value function, if a function is continuous, then its real and imaginary parts are also continuous

(iii) If $y = f(x)$

then $w = f(z)$

$$w = z^2$$

$$u(x,y) + i\,v(x,y) = (x+iy)^2$$

$$= x^2 + i^2 y^2 + 2xyi \qquad \because i^2 = -1$$

$$u(x,y) + i\,v(x,y) = x^2 - y^2 + 2xyi$$

$$\Rightarrow \quad u(x,y) = x^2 - y^2$$

and $v(x,y) = 2xy$

Theorem :-

If $w = f(z)$ be complex valued function and is continuous at a point $z = z_0$ than prove that its real and imaginary parts are also continuous.

Proof :-

Given $f(z)$ is continuous and is complex valued function

To prove $\lim_{z \to z_0} u(x,y) = u_0$

$$\lim_{z \to z_0} v(x,y) = v_0$$

Since, $f(z)$ is continuous at $z = z_0$

$$|f(z) - f(z_0)| < \varepsilon \ , \quad 0 < |z - z_0| < \delta$$

$$|u(x,y) + iv(x,y) - u_0 - iv_0| < \varepsilon, \quad 0 < |z - z_0| < \delta$$

$$|u(x,y) - u_0 + i(v(x,y) - v_0)| < \varepsilon, \quad 0 < |z - z_0| < \delta$$

$$|u(x,y) - u_0| < \varepsilon, \quad 0 < |z - z_0| < \delta$$

$$|v(x,y) - v_0| < \varepsilon, \quad 0 < |z - z_0| < \delta$$

$\Rightarrow \lim\limits_{z \to z_0} u(x,y) = u_0$ is continous

and

$\Rightarrow \lim\limits_{z \to z_0} v(x,y) = v_0$ is continous

Example:-

Discuss the continuity of $f(z) = |z|^2$ at $z = z_0$

Sol:-

The value of $f(z) = |z|^2$ at $z = z_0$ is $f(z_0) = |z_0|^2$ which is finite.

$$\lim_{z \to z_0} (|z|^2) = |z_0|^2$$

Limit and value of $f(z)$ at $z = z_0$ are equal (do agree), therefore function $f(z) = |z|^2$ is continuous at $z = z_0$.

Example:-

Discuss the continuity of $f(z) = e^{1/z^2}$ at $z = 0$

Sol:-

Given $f(z) = e^{1/z^2}$ at $z_0 = 0$

as $z \to 0$, $\frac{1}{z} \to \infty \quad \therefore e^{1/z^2} \to \infty$

The value of function is infinite and the first condition is violated, therefore the function $f(z) = e^{1/z^2}$ is discontinuous at $z = 0$

Example :-

Discuss the continuity of

$$f(z) = \frac{z^2+4}{z+2i} \quad , \quad z \neq -2i$$

$$f(z) = -5i \quad , \quad z = -2i$$

Sol :-

(i) The value of function is

$f(z) = -5i$ (given)

(ii)

Limit

$$\lim_{z \to -2i} \left(\frac{z^2+4}{z+2i} \right) = \lim_{z \to -2i} \left(\frac{(z+2i)(z-2i)}{z+2i} \right)$$

$$= \lim_{z \to -2i} (z - 2i)$$

$$= -2i - 2i$$

$$= -4i$$

(iii)

Limit of function $\neq$ value of function

(i-e) $-4i \neq -5i$

The last condition is not true, therefore the function is discontinuous at $z = -2i$

Uniform continuity :- A function $f(z)$ is said to be uniformly continuous in the domain D_f if for given $\varepsilon > 0$, it is possible to find $\delta > 0$ depending upon ε such that

$|f(z_1) - f(z_2)| < \varepsilon$ for every pair of points z_1 and z_2 of the domain D_f

such that $|z_1 - z_2| < \delta$

or $|f(z) - f(z_0)| < \varepsilon$, $0 < |z - z_1| < \delta$

Example:-

Let R consists of the set of all points z such that $0 < |z| \leq 1$

Let $f(z) = z^2$

Verify that $f(z)$ is uniformly continuous on R

Sol:-

Given $f(z) = z^2$

and $0 < |z| \leq 1$

$z_1, z_2 \in R$

(i-e) $0 < |z_1| \leq 1$ and $0 < |z_2| \leq 1$

Consider $|f(z_1) - f(z_2)| = |z_1^2 - z_2^2|$

$= |(z_1 - z_2)(z_1 + z_2)|$

$= |z_1 - z_2| \cdot |z_1 + z_2| \rightarrow (1)$

Since, $0 < |z_1| \leq 1$ and $0 < |z_2| \leq 1$

(1) becomes

$|f(z_1) - f(z_2)| \leq 2 |z_1 - z_2| \rightarrow (2)$

(equality will hold when $|z_1| = 1$ and $|z_2| = 1$)

If we take $\delta = \varepsilon/2$ then for any two points z_1 and z_2 of R such that $|z_1 - z_2| < \delta$,

putting this value in (2) then

$|f(z_1) - f(z_2)| = 2 \cdot \delta$

$= 2 \cdot \varepsilon/2$

$|f(z_1) - f(z_2)| = \varepsilon$

(i-e) $|f(z_1) - f(z_2)| < \varepsilon \rightarrow (3)$

From (3) we conclude that $f(z) = z^2$ is uniformly continuous on R

Note:-

$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ at origin '0'

If two limits of function exists, then the function is not differentiable

Uniformly continuous :-

A function f is defined in a domain D is said to be uniformly continuous in D, if for every $\epsilon>0$, there exists a number $\delta=\delta(\epsilon)>0$ such that $\forall$ z_1 and z_2 in D

$|f(z_1)-f(z_2)|<\epsilon$ whenever $|z_1-z_2|<\delta$.

Example :-

Let $R=\{z \mid 0<|z|<1\}$
Let function f be defined by $f(z)=\frac{1}{z}$ $\forall$ $z\in R$, show that $f(z)$ is not uniform continuous.

Sol :-

Continuity :-

$$f(z)=\frac{1}{z} \quad ; \quad 0<|z|<1 \quad \forall z\in R$$

Clearly, f is continuous on R.
We shall now show that f is not uniformly continuous on R.

Suppose that for definiteness, we take $\epsilon=\frac{1}{10}$. We would like to show that there exists no $\delta>0$ satisfying the definition of uniform continuity.

Suppose on contrary that a δ, $0<\delta<1$, can be found in accordance with definition, we choose two points of R to be $z_1=\delta$ and $z_2=(\frac{9}{10})\delta$
then, $|z_1-z_2|=|\delta-\frac{9}{10}\delta|$

$$=\left|\frac{10\delta-9\delta}{10}\right|$$

$$|z_1 - z_2| = \left|\frac{\delta}{10}\right|$$

$$|z_1 - z_2| = \frac{\delta}{10} < \delta$$

For these two points, we have

$$|f(z_1) - f(z_2)| = \left|\frac{1}{z_1} - \frac{1}{z_2}\right|$$

$$= \left|\frac{1}{\delta} - \frac{10}{9\delta}\right|$$

$$= \left|\frac{9-10}{9\delta}\right|$$

$$= \left|\frac{-1}{9\delta}\right|$$

$$|f(z_1) - f(z_2)| = \frac{1}{9\delta} > \frac{1}{10}$$

Since, $0 < \delta < 1$

Thus, contradicting the definition of uniform continuity.

Differentiability :-

Let function f be defined in a domain D and Let z_0 be any fixed point in D. Then, the function f is said to have a derivative at point z_0. If the following limit exists

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0) = \text{number}$$

$$= f'(z) = \text{function}$$

Note :-

(i) A function which is not continuous but its limit exists.

(ii) $\frac{d}{dz}(f(z) \pm g(z)) = \frac{d}{dz}(f(z)) + \frac{d}{dz}(g(z)) = f'(z) + g'(z)$

(iii) For given $\varepsilon > 0$, there exists a number $\delta > 0$, depending upon ε and z_0 such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \quad 0 < |z - z_0| < \delta$$

(iv) Constant complex number is $1+2i$ or $3+4i$ not $x+iy$

Rules :-

(i) $\frac{d}{dz}(c) = 0$

(ii) $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\frac{df}{dz} - f(z)\frac{dg}{dz}}{[g(z)]^2} \quad (g(z) \neq 0)$

(iii)

(iv) If n is a positive integer. Then,

$$\frac{dw}{dz} = nz^{n-1} \quad \text{where } w = z^n$$

$$\frac{d}{dz}(z^n) = nz^{n-1}$$

Theorem :-

Prove that if $w = f(z)$ is differentiable then $f(z)$ is continuous.

Proof :-

Given $f(z)$ is differentiable say at $z = z_0$

then $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$

to prove $f(z)$ is continuous at $z = z_0$

(i-e) $\lim_{z \to z_0} f(z) = f(z_0)$

Consider $[f(z) - f(z_0)] = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0)$

$$\lim_{z \to z_0} [f(z) - f(z_0)] = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0)$$

$$\lim_{z \to z_0} f(z) - f(z_0) = f'(z_0) \times 0$$

$$\lim_{z \to z_0} f(z) - f(z_0) = 0$$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

Note :-

(i) A function which is continuous may or may not be differentiable

(ii) Every differentiable function is continuous

Example :-

Test the continuity and differentiability of the following functions

(i) $W = f(z) = \bar{z}$ at $z = z_0$

(ii) $W = f(z) = |z|^2$ at $z = z_0$

Sol :-

(i) Continuity at $z = z_0$

The value of function $f(z) = \bar{z}$ at $z = z_0$ is

$$f(z_0) = \bar{z}_0$$

Limit of the function $f(z) = \bar{z}$ at $z = z_0$ is

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \bar{z}$$

$$= \bar{z}_0$$

Since functional value and limiting value of the function $f(z) = \bar{z}$ exists at $z = z_0$

Therefore the function is continuous

Differentiability at $z = z_0$

$$f'(z_0) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \right]$$

$$f'(z_0) = \lim_{z \to z_0} \left[\frac{\bar{z} - \bar{z}_0}{z - z_0} \right]$$

$$f'(z_0) = \lim_{z \to z_0} \left\{ \frac{\overline{z - z_0}}{z - z_0} \right\}$$

which shows that limit does not exist then $f(z) = \bar{z}$ is not differentiable at $z = z_0$.

Note :-

$z - z_0$ may be real or imaginary. If $z - z_0$ is real number then answer of limit $(z - z_0)$ is $f'(z_0) = 1$ or if $z - z_0$ is imaginary then $f'(z_0) = -1$

(ii)

Continuity at $z = z_0$:-

The value of function $f(z) = |z|^2$ at $z = z_0$ is

$$f(z) = |z|^2 \qquad \because |z|^2 = z\bar{z}$$

$$f(z) = z\bar{z}$$

$$f(z_0) = z_0 \bar{z}_0$$

Limit of function $f(z) = |z|^2$ at $z = z_0$ is

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} (|z|^2)$$

$$= \lim_{z \to z_0} (z\bar{z})$$

$$\lim_{z \to z_0} f(z) = z_0 \bar{z}_0$$

Since, the functional value and limiting value of the function $f(z) = |z|^2$ exists at $z = z_0$.

Therefore, the function is continuous.

Differentiability at $z = z_0$

$$f'(z_0) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} \right] \rightarrow (1)$$

$$f(z) = |z|^2 = z\bar{z}$$

$$f(z_0) = |z_0|^2 = z_0 \bar{z}_0$$

$$f'(z_0) = \lim_{z \to z_0} \left[\frac{z\bar{z} - z_0\bar{z}_0}{z - z_0}\right]$$

which shows that limit does not exists. $f(z) = |z|^2$ is not differentiable at $z = z_0$

In real analysis

A function which is continuous but not differentiable is $f(x) = |x|$

Continuity at $x = 0$

Value of function at $x = 0$

$$f(x) = |x|$$

$$f(0) = |0| = 0$$

Limit of function at $x = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x|$$

$$\Rightarrow |0| = 0.$$

Since, the functional value and the limiting value of function exists at $x = 0$, therefore the function is continuous.

Differentiability at $x = 0$

$$f'(x) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$f'(0) = \lim_{x \to 0} \frac{|x| - 0}{x - 0}$$

$$f'(0) = \lim_{x \to 0} \frac{|x|}{x}$$

$f'(0) = \frac{|x|}{x}$ does not tend to a definite quantity as $x \to 0$.

or $\lim_{x \to 0+} \frac{|x|}{x} = 1$ and

$$\lim_{x \to 0-} \frac{|x|}{x} = -1$$

Analytic or Regular or holomorphic functions

Analytic function :-

A complex valued function $f(z)$ is said to be analytic at a point z_0 in a domain D, if it has derivative at z_0 and also all points in some neighbourhood of z_0. (regular function)

Note :-

(i) Every differentiable function may or may not be analytic but every analytic function is differentiable.

(ii) If a function is differentiable at all points of domain, then function is differentiable at neighbouring points

(iii) $f(z) = \bar{z}$ is not analytic

(iv) If a function is continuous then it is not necessary that its derivative is also continuous.

Theorem :-

Prove that a necessary and sufficient condition of a function $f(z)$ defined $f(z) = u(x,y) + i\,v(x,y)$ to be analytic in a domain D is that the 4th partial derivatives u_x, u_y, v_x, v_y exist are continuous and satisfy the cauchy-Riemann conditions $u_x = v_y$, $u_y = -v_x$ at each point of D.

Proof :-

Necessary condition :-

Given $f'(z)$ exists

(i.e) $f'(z) = \lim_{\delta z \to 0} \dfrac{f(z+\delta z) - f(z)}{\delta z}$

To prove $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Consider $f'(z) = \lim_{\delta z \to 0} \left[\frac{u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y) - u(x,y) - iv(x,y)}{\delta x + i\delta y}\right] \to (A)$

$\delta z = \delta x + i\delta y$
$\because z = x+iy$

Take the path along x-axis

$$f'(z) = \lim_{\delta x \to 0} \left[\frac{u(x+\delta x, y) + iv(x+\delta x, y) - u(x,y) - iv(x,y)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \left[\frac{u(x+\delta x, y) - u(x,y)}{\delta x}\right] + i \lim_{\delta x \to 0} \left[\frac{v(x+\delta x, y) - v(x,y)}{\delta x}\right]$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \to (B)$$

Take the path along y-axis

$$f'(z) = \lim_{\delta y \to 0} \left[\frac{u(x, y+\delta y) + iv(x, y+\delta y) - u(x,y) - iv(x,y)}{i\delta y}\right]$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i\partial v}{i\partial y}$$

$$f'(z) = \frac{i}{i^2}\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y} \qquad i^2 = -1$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \to (C)$$

Comparing B, C

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Sufficient condition :-

Given $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\delta v}{\delta x}$

To prove $f'(z)$ exists

$$f(z+\delta z)-f(z) = [u(x+\delta x, y+\delta y)+iv(x+\delta x, y+\delta y)-u(x,y) - i\,v(x,y)]$$

$$= [u(x+\delta x, y+\delta y)-u(x, y+\delta y)+u(x,y+\delta y)-u(x,y)] + i[v(x+\delta x, y+\delta y)-v(x+\delta x, y)+v(x+\delta x,y)-v(x,y)]$$

$$= [\delta x\left(\frac{\partial u}{\partial x}+\varepsilon_1\right)+\delta y\left(\frac{\partial u}{\partial y}+\varepsilon_2\right)]+[i(\delta y\left(\frac{\partial v}{\partial y}+\varepsilon_3\right)) + i\,\delta x\left(\frac{\partial v}{\partial x}+\varepsilon_4\right)]$$

at limit we put $= \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$

we know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f(z+\delta z)-f(z) = (\delta x+i\delta y)\frac{\partial u}{\partial x}+i(\delta x+i\delta y)\frac{\partial v}{\delta x} + \delta x(\varepsilon_1+i\varepsilon_4)+\delta y(\varepsilon_2+i\varepsilon_3)$$

$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{\delta z}{\delta z}\frac{\partial u}{\partial x} + i\frac{\delta z}{\delta z}\frac{\partial v}{\delta x}+\frac{\delta x}{\delta z}\delta_1 + \frac{\delta y}{\delta z}\delta_2$$

$$\frac{f(z+\delta z)-f(z)}{\delta z} = \frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}+\delta_1\frac{\delta x}{\delta z}+\delta_2\frac{\delta y}{\delta z}$$

$$\left|\frac{f(z+\delta z)-f(z)}{\delta z}-\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\delta x}\right)\right| = \left|\delta_1\frac{\delta x}{\delta z}+\delta_2\frac{\delta y}{\delta z}\right|$$

$$\leq \left|\delta_1\frac{\delta x}{\delta z}\right|+\left|\delta_2\frac{\delta y}{\delta z}\right|$$

$$< \delta_1 + \delta_2 \qquad \because \left|\frac{\delta x}{\delta z}\right| < 1$$

$$\left| \frac{f(z+\delta z) - f(z)}{\delta z} - \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \right| < \varepsilon \qquad \because \delta x < \delta z$$

which is differentiable

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow (1)$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \rightarrow (2)$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \rightarrow (3)$$

So, the sufficient condition is true
∴ If the function is analytic then its derivative can be taken by equations (1), (2) and (3).

Note:-

(i) C-R is true in every analytic function.
(ii) Trignometric and hyperbolic functions are continuous
(iii) If C.R is satisfied, then there is no garenty that the function is analytic or not.

Ex

Show that function $f(z) = z^2$ is analytic in domain $D = C$

Sol:-

$$f(z) = z^2$$

$$u(x,y) + iv(x,y) = (x+iy)^2$$
$$= x^2 - y^2 + 2ixy$$

$$u = x^2 - y^2, \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \quad , \quad \frac{\partial v}{\partial y} = 2x$$

By Cauchy - Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2x = 2x \quad , \quad -2y = -2y$$

C.R is satisfied

The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist, are continuous and C.R equations are true $\forall z \in C$

Hence, the function $f(z) = z^2$ is analytic in domain $D = C$

Ex

Show that function $f(z) = |z|^2$ is analytic in domain $D = C$

Sol:-

$$f(z) = |z|^2$$

$$u(x,y) + iv(x,y) = x^2 + y^2$$

$$\Rightarrow u(x,y) = x^2 + y^2 , \quad v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial y} = 2y \quad , \quad \frac{\partial v}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

Since C.R equations are not true

Hence, $f(z) = |z|^2$ is not analytic function.

Note:-

(i) $f(z) = \bar{z}$ is not analytic function

(ii) $f(z) = |z|^2 = z\bar{z}$ is not analytic function.

Ex:-

Test the analyticity of function
$W = C$

Sol:-

$W = C$ is a constant function

Let $W = C = C_1 + iC_2$

$u(x,y) + iV(x,y) = C_1 + iC_2$

$\Rightarrow \quad u(x,y) = C_1 \,, \quad V(x,y) = C_2$

$\frac{\partial u}{\partial x} = 0 \,, \quad \frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0 \,, \quad \frac{\partial v}{\partial y} = 0$

$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

C.R equations are satisfied

Hence, the constant function is analytic

Note:-

Constant function is continuous.

Ex:-

Test the function $f(z) = \sin(x + 3iy)$ is analytic or not.

Sol:-

$f(z) = \sin(x + 3iy)$

$f(z) = \sin(x + 3iy) = \sin(2z - \bar{z})$

$u(x,y) + iV(x,y) = \sin x \cosh 3y + i \cos x \sinh 3y$ :: if $z = x + iy$

Since the function involves $\bar{z} = x - iy$

$\bar{z}$, so, function is not analytic

$2z - \bar{z} = 2x + 2iy - x + iy$

$(2z - \bar{z}) = x + 3iy$

Ex

Test the analyticity of function

$W = \frac{z+1}{z-1}$

Sol:-

$$W = \frac{z+1}{z-1}$$

$$W = \frac{(x+1)+iy}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

$$W = \frac{(x+1)[(x-1)-iy]+iy[(x-1)-iy)]}{(x-1)^2-(iy)^2}$$

$$W = \frac{(x^2-1)-\cancel{ixy}-iy+\cancel{ixy}-iy-i^2y^2}{(x-1)^2-i^2y^2}$$

$$W = \frac{x^2-1-2iy-(-1)y^2}{(x-1)^2-(-1)y^2}$$

$$W = \frac{x^2+y^2-1-2iy}{(x-1)^2+y^2}$$

$$u(x,y)+iv(x,y) = \frac{x^2+y^2-1}{(x-1)^2+y^2} - i\frac{2y}{(x-1)^2+y^2}$$

$$u = \frac{x^2+y^2-1}{(x-1)^2+y^2}, \quad v = \frac{-2y}{(x-1)^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{[(x-1)^2+y^2][2x]-[x^2+y^2-1][2(x-1)]}{[(x-1)^2+y^2]^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x((x-1)^2+y^2)-2(x-1)(x^2+y^2-1)}{((x-1)^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{[(x-1)^2+y^2][2y]-[x^2+y^2-1][2y]}{((x-1)^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y((x-1)^2+y^2) - 2y(x^2+y^2-1)}{((x-1)^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{((x-1)^2+y^2) - (-2y)(2(x-1))}{((x-1)^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x-1)^2+y^2+4y(x-1)}{((x-1)^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{((x-1)^2+y^2)(-2) - (-2y)(2y)}{((x-1)^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-2((x-1)^2+y^2)+4y^2}{((x-1)^2+y^2)^2}$$

Ex

prove that the function

$f(z) = 0 , \quad z = 0$

$f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2} , \quad z \neq 0$

Sol :-

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} , \quad z \neq 0$$

$$f(z) = 0 \qquad z = 0$$

Cauchy Riemann are satisfied at the origin but $f'(0)$ fails to exist

Sol :-

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} , \quad z \neq 0$$

$$f(z) = \frac{x^3 + ix^3 - y^3 + iy^3}{x^2+y^2}$$

$$u(x,y) + iv(x,y) = \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{x^3+y^3}{x^2+y^2}$$

$$u(x,y) = \frac{x^3-y^3}{x^2+y^2} , \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

C.R equations at the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x}$$

$$= \lim_{x\to 0} \frac{x}{x}$$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \left(\frac{-y^3/y^2}{y}\right)$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \left(-\frac{y}{y}\right)$$

$$\frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{V(x,0) - V(0,0)}{x}$$

$$= \lim_{x\to 0} \left(\frac{x^3/x^2}{x}\right)$$

$$= \lim_{x\to 0} \frac{x}{x}$$

$$\frac{\partial v}{\partial x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{V(0,y) - V(0,0)}{y}$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \left(\frac{y^3/y^2}{y}\right)$$

$$= \lim_{y\to 0} \left(\frac{y}{y}\right)$$

$$\frac{\partial v}{\partial y} = 1$$

We know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Therefore, C.R equations are satisfied at origin $(0,0)$

Now

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$f'(0) = \lim_{z\to 0}\left(\frac{\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} - 0}{x+iy-0}\right)$$

Let z vary along the line $y=x$ then, putting $y=x$

$$f'(0) = \lim_{x\to 0}\frac{2ix^3/2x^2}{x+ix}$$

$$f'(0) = \lim_{x\to 0}\frac{ix}{x(1+i)}$$

$$f'(0) = \frac{i}{1+i} \rightarrow (1)$$

Let z approches the origin along x-axis then put $y=0$, $x\to 0$.

$$f'(0) = \lim_{x\to 0}\left(\frac{x^3+ix^3/x^2}{x}\right)$$

$$f'(0) = \lim_{x\to 0}\frac{x^3(1+i)}{x^3}$$

$$f'(0) = \lim_{x\to 0}\frac{x^3}{x^3}(1+i)$$

$$f'(0) = 1+i \rightarrow (2)$$

From (1) and (2)

We note that $f'(0)$ does not exists. Hence, it is obvious that CR equations are true but the function is not analytic at the origin.

Harmonic functions or Potential functions:-

A real value function $U = U(x, y)$ is said to be harmonic in a domain D if $\forall x, y \in D$, the second partial derivatives of U (with respect to x and y) exist are continuous and satisfy the laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0 \qquad U_{xx} + U_{yy} = 0$$

Ex:-

Prove that $u(x,y) = x^3 - 3xy^2$ is harmonic Find its corresponding conjugate and the function f(z)

Sol:-

Given $u(x, y) = x^3 - 3xy^2$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightarrow (1)$$

Substitute the values in (1)

$$6x + (-6x) = 0$$

$$6x - 6x = 0$$

$$0 = 0$$

Laplace's equation is satisfied.
Hence the function is harmonic
As we know that C.R conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$$

$$\Rightarrow \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

Integrate it w.r.t "y"

$$v(x, y) = 3x^2y - 3\frac{y^3}{3}$$

$$v(x, y) = 3x^2y - y^3 + g(x) \rightarrow (1)$$

Differentiate w.r.t "x"

$$\frac{\partial v}{\partial x} = 6xy + g'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + g'(x)$$

$$6xy = 6xy + g'(x)$$

$$\Rightarrow \quad g'(x) = 0$$

Integrate it w.r.t "x"

$g(x) = \text{constant } A$ put in (1)

$$V = 3x^2y - y^3 + A$$

$$f(z) = u + iv$$

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3 + A)$$

$$f(z) = x^3 - 3xy^2 + 3ix^2y - iy^3 + iA$$

$$f(z) = (x+iy)^3 + iA$$

$\therefore iA = B$

$$f(z) = z^3 + B$$

If we take $B = 0$

$$f(z) = z^3$$

Note :-

In harmonic function, cauchy Riemann equations satisfy

Ex :-

Find the original function $f(z)$, if $u(x,y) = x^3 - 3xy^2$

Sol :-

$$f(z) = \int (u_x(z,0) - i u_y(z,0))\, dz \rightarrow (1)$$

$$u(x,y) = x^3 - 3xy^2$$

$$u_x = \frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$u_y = \frac{\partial u}{\partial y} = -3x(2y)$$

$$u_y = -6xy$$

$$(u_x)_{(z,0)} = 3z^2 - 0$$

$$u_{x(z,0)} = 3z^2$$

$$u_y(z,0) = 0$$

then

$$f(z) = \int (3z^2 - i(0))\, dz$$

$$f(z) = \int 3z^2\, dz$$

$$f(z) = 3\int z^2\, dz$$

$$f(z) = 3\frac{z^3}{3} + A$$

$$f(z) = z^3 + A$$

where A is constant of integration

Ex :-

(i) Prove that $u(x,y)$ given by the following is harmonic, obtain its corresponding conjugate and the original function

(ii) Find the original function without finding the corresponding conjugate

Sol :-

i) (a) $U(x,y) = e^x \cos y$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad , \quad \frac{\partial u}{\partial y} = e^x(-\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad , \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$e^x \cos y - e^x \cos y = 0$$

$$0 = 0$$

Laplace's equation is satisfied

So, the function is harmonic

As we know, C.R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

Integrate w.r.t 'y'

$$V = e^x \int \cos y \, dy$$

$$V = e^x \sin y + g(x)$$

Differentiate w.r.t 'x'

$$\frac{\partial v}{\partial x} = e^x \sin y + g'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$-\frac{\partial u}{\partial y} = e^x \sin y + g'(x)$$

$$e^x \sin y = e^x \sin y + g'(x)$$

$$\Rightarrow g'(x) = 0$$

Integrate it w.r.t 'x'

$$g(x) = \text{constant}$$

then $v(x,y) = e^x \sin y + c$

we know

$$f(z) = u(x,y) + iv(x,y)$$

$$f(z) = e^x \cos y + i(e^x \sin y + c)$$

$$f(z) = e^x \cos y + ie^x \sin y + ic$$

$$f(z) = e^x (\cos y + i \sin y) + B$$

$$f(z) = e^x \cdot e^{iy} + B$$

$$f(z) = e^{x+iy} + B$$

$$f(z) = e^z + B$$

(ii) without corresponding conjugate

$$f(z) = \int (U_x(z,0) - i U_y(z,0))\,dz$$

$$U = e^x \cos y$$

$U_x = e^x \cos y$ $\qquad$ $U_y = e^x(-\sin y)$

$U_x(z,0) = e^z \cos(0)$ $\qquad$ $U_y(z,0) = e^z(-\sin(0))$

$U_x(z,0) = e^z$ $\qquad$ $U_y(z,0) = 0$

$$f(z) = \int (e^z - i(0))\,dz$$

$$f(z) = \int e^z\,dz$$

$$f(z) = e^z + B$$

(b)

i) $u(x,y) = e^x(x \cos y - y \sin y)$

$$\frac{\partial u}{\partial x} = e^x(\cos y - 0) + e^x(x\cos y - y\sin y)$$

$$\frac{\partial u}{\partial x} = e^x\cos y + xe^x\cos y - ye^x\sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x\cos y + e^x\cos y + xe^x\cos y - ye^x\sin y$$

$$\frac{\partial^2 u}{\partial x^2} = 2e^x\cos y + xe^x\cos y - ye^x\sin y \rightarrow (1)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x((2+x)\cos y - y\sin y)$$

$$\frac{\partial u}{\partial y} = xe^x(-\sin y) - (e^x\sin y + ye^x(\cos y))$$

$$\frac{\partial u}{\partial y} = -xe^x\sin y - e^x\sin y - ye^x\cos y$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x\cos y - e^x\cos y - e^x\cos y - ye^x(-\sin y)$$

$$\frac{\partial^2 u}{\partial y^2} = -xe^x\cos y - 2e^x\cos y + ye^x\sin y \rightarrow (2)$$

Laplace's equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Putting values from (1) and (2)

$$2e^x\cancel{\cos y} + xe^x\cancel{\cos y} - ye^x\cancel{\sin y} - 2e^x\cancel{\cos y} - xe^x\cancel{\cos y} + ye^x\cancel{\sin y} = 0$$

$$\Rightarrow \quad 0 = 0$$

Laplace's equation is satisfied.

Hence the function is harmonic.

As we know C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = e^x\cos y + xe^x\cos y - y\,e^x\sin y = \frac{\partial v}{\partial y}$$

$$\Rightarrow V_y = e^x\cos y + xe^x\cos y - ye^x\sin y$$

Integrate it w.r.t 'y'

$$V(x,y) = e^x \sin y + xe^x \sin y - e^x (y(-\cos y) - \int 1\cdot(-\cos y)\,dy)$$

$$V(x,y) = e^x \sin y + xe^x \sin y + ye^x \cos y - e^x \int \cos y\,dy$$

$$V(x,y) = e^x \sin y + xe^x \sin y + ye^x \cos y - e^x \sin y + g(x)$$

Differentiate w.r.t "x"

$$\frac{\partial v}{\partial x} = e^x \sin y + e^x \sin y + xe^x \sin y + ye^x \cos y - e^x \sin y + g'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y + e^x \cancel{\sin y} + xe^x \sin y + ye^x \cos y - e^x \cancel{\sin y} + g'(x)$$

$$-\frac{\partial u}{\partial y} = xe^x \sin y + ye^x \cos y + e^x \sin y + g'(x)$$

put $\frac{\partial u}{\partial y} = -(xe^x \sin y + ye^x \cos y + e^x \sin y)$

$$xe^x \sin y + ye^x \cos y + e^x \sin y = xe^x \sin y + ye^x \cos y + e^x \sin y + g'(x)$$

$$\Rightarrow \quad g'(x) = 0$$

Integrate it w.r.t 'x'

$$\Rightarrow g(x) = \text{constant}$$

then

$$V(x,y) = e^x \cancel{\sin y} + xe^x \sin y + ye^x \cos y - e^x \cancel{\sin y} + C$$

$$V(x,y) = e^x[x \sin y + y \cos y] + C$$

we know

$$f(z) = u(x,y) + iV(x,y)$$

$$f(z) = e^x[x \cos y - y \sin y] + i[e^x[x \sin y + y \cos y] + C]$$

$$f(z) = e^x[x \cos y - y \sin y + ix \sin y + iy \cos y + iC]$$

$$f(z) = e^x[(\cos y + i \sin y)x - (\cos y + i \sin y)iy + iC]$$

$$f(z) = e^x[(\cos y + i\sin y)(x+iy) + c']$$

$$f(z) = [e^x \cdot e^{iy} \ (x+iy) + c']$$

$$f(z) = e^{x+iy} \ (x+iy) + c'$$

$$f(z) = e^z \ (z) + c'$$

$$f(z) = ze^z + c'$$

(ii) without corresponding conjugate

$$f(z) = \int (u_x(z,0) - i\, u_y(z,0))dz$$

$$u_x = \frac{\partial u}{\partial x} = e^x \cos y + xe^x \cos y - ye^x \sin y$$

$$u_x(z,0) = e^z \cos(0) + ze^z \cos(0) - 0$$

$$u_x(z,0) = e^z + ze^z$$

$$u_y = -xe^x \sin y - e^x \sin y - ye^x \cos y$$

$$u_y(z,0) = -ze^z \sin(0) - e^z \sin(0) - 0$$

$$u_y(z,0) = 0$$

$$f(z) = \int((e^z + ze^z) - 0)\, dz = ze^z + c'$$

(c) (i)

$$u(x,y) = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \quad , \quad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \quad 2 - 2 = 0$$

$$\Rightarrow \quad 0 = 0$$

Laplace's equation is satisfied.
Hence, the function is harmonic
As we know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

Integrate it w.r.t 'y'

$$v(x,y) = 2xy + g(x)$$

Differentiate it w.r.t 'x'.

$$\frac{\partial v}{\partial x} = 2y + g'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2y + g'(x)$$

$$-(-2y) = 2y + g'(x)$$

$$2y = 2y + g'(x)$$

$$\Rightarrow g'(x) = 0$$

$$\Rightarrow g(x) = \text{Constant}$$

$$v(x,y) = 2xy + \text{Constant}$$

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f(z) = x^2 - y^2 + i(2xy + c)$$

$$f(z) = x^2 - y^2 + 2ixy + ic$$

$$f(z) = (x+iy)^2 + c'$$

$$f(z) = z^2 + c'$$

(ii) without corresponding conjugate:-

$$f(z) = \int (u_x(z,0) - i\,u_y(z,0))\,dz$$

$u_x = 2x$ $\qquad$ $u_y = -2y$

(u_x) ... $= 2z$ $\qquad$ $u_y(z,0) = 0$

$$f(z) = \int (2z - 0)\,dz$$

$$f(z) = 2\int z\,dz$$

$$f(z) = 2\frac{z^2}{2} + c'$$

$$f(z) = z^2 + c'$$

(d)

i) $u(x,y) = \sin x \cosh y$

$$\frac{\partial u}{\partial x} = \cos x \cosh y \quad , \quad \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y \quad , \quad \frac{\partial^2 u}{\partial y^2} = \sin x \cosh y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$-\sin x \cosh y + \sin x \cosh y = 0$$

$$\Rightarrow 0 = 0$$

Laplace's equation is satisfied.
Hence, the function is harmonic.

As we know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$$

Integrate it w.r.t "y"

$$v(x,y) = \cos x \sinh y + g(x)$$

Differentiate it w.r.t 'x'

$$\frac{\partial v}{\partial x} = -\sin x \sinh y + g'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\sin x \sinh y + g'(x)$$

$$-\sin x \sinh y = -\sin x \sinh y + g'(x)$$

$\Rightarrow \quad g'(x) = 0$

Integrate it w.r.t 'x'

$g(x) = \text{constant}$

then $\quad V(x,y) = \cos x \sinh y + A$

$f(z) = u(x,y) + iv(x,y)$

$f(z) = \sin x \cosh y + i(\cos x \sinh y + A)$

$f(z) = \sin x \cosh y + i \cos x \sinh y + iA$

$f(z) = \sin(x+iy) + A'$

$f(z) = \sin z + A'$

(ii) **without corresponding conjugate :-**

$f(z) = \int (u_x(z,0) - i\, u_y(z,0))\, dz$

$u_x = \cos x \cosh y \quad , \quad u_y = \sin x \sinh y$

$u_x(z,0) = \cos z \cosh(0) \quad , \quad u_y(z,0) = \sin z \sinh(0)$

$u_x(z,0) = \cos z \quad , \quad u_y(z,0) = 0.$

$f(z) = \int (\cos z - 0)\, dz$

$= \int \cos z\, dz$

$f(z) = \sin z + A'$

(e)

(i) $\quad u(x,y) = e^x (y \sin x + x \cos y)$

$\frac{\partial u}{\partial x} = ye^x \sin x + ye^x \cos x + e^x \cos y + xe^x \cos y$

$\frac{\partial^2 u}{\partial x^2} = ye^x \cancel{\sin x} + ye^x \cos x + ye^x \cos x - ye^x \cancel{\sin x} + e^x \cos y + e^x \cos y + xe^x \cos y$

$$\frac{\partial_2 u}{\partial x^2} = 2ye^x \cos x + 2e^x \cos y + xe^x \cos y.$$

$$\frac{\partial u}{\partial y} = e^x(\sin x + x(-\sin y))$$

$$\frac{\partial u}{\partial y} = e^x(\sin x - x\sin y)$$

$$\frac{\partial_2 u}{\partial y^2} = -e^x \cos y$$

$$\frac{\partial_2 u}{\partial x^2} + \frac{\partial_2 u}{\partial y^2} = 0$$

$$2ye^x\cos x + 2e^x\cos y + xe^x\cos y - x\cos y \neq 0$$

So Laplace's equation does not satisfy. Hence, the function is not harmonic.

(ii) without corresponding conjugate :-

$$f(z) = \int (u_x(z,0) + iu_y(z,0))\,dz$$

$$u_x = ye^x\sin x + ye^x\cos x + e^x\cos y + xe^x\cos y$$

$$u_x(z,0) = 0 + 0 + e^z\cos(0) + ze^z\cos(0)$$

$$u_x(z,0) = e^z + ze^z$$

$$u_y = e^x\sin x - x\sin y$$

$$u_y(z,0) = e^z\sin z - z\sin(0)$$

$$u_y(z,0) = e^z\sin z$$

$$f(z) = \int(e^z + ze^z + ie^z\sin z)\,dz$$

$$f(z) = e^z + i\left(e^z(-\cos z) - \int e^z(-\cos z)\,dz\right)$$

$$f(z) = e^z - ie^z\cos z + \int e^z\cos z\,dz.$$

(f) $U(x,y) = \dfrac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x(2x+0)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial_2 u}{\partial x^2} = \frac{(x^2+y^2)^2(0-2x)-(y^2-x^2)\,(2(x^2+y^2)(2x))}{((x^2+y^2)^2)^2}$$

$$\frac{\partial_2 u}{\partial x^2} = \frac{-2x\,(x^2+y^2)^2-4x(x^2+y^2)(y^2-x^2)}{(x^2+y^2)^4}$$

$$\frac{\partial_2 u}{\partial x^2} = \frac{-2x(x^2+y^2)^2-4x\,((y^2)^2-(x^2)^2)}{(x^2+y^2)^4}$$

$$\frac{\partial_2 u}{\partial x^2} = \frac{-2x(x^2+y^2)^2-4x\,(y^4-x^4)}{(x^2+y^2)^4}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(\cdot 0)-x(0+2y)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial_2 u}{\partial y^2} = \frac{(x^2+y^2)^2(-2x)-(-2xy)(2(x^2+y^2)(2y))}{((x^2+y^2)^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2x(x^2+y^2)^2 + 8xy^2(x^2+y^2)}{(x^2+y^2)^4}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{-2x(x^2+y^2)^2 - 4x(y^4-x^4)}{(x^2+y^2)^4} + \frac{-2x(x^2+y^2)^2 + 8xy^2(x^2+y^2)}{(x^2+y^2)^4} \neq 0$$

Laplace's equation does not satisfy

Hence the function is not harmonic

(ii) no corresponding conjugate :-

$$f(z) = \int (u_x(z,0) - i u_y(z,0))\, dz$$

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$u_x(z,0) = \frac{0 - z^2}{(z^2-0)^2}$$

$$u_x(z,0) = \frac{-z^2}{(z^2)^2}$$

$$u_x(z,0) = -\frac{z}{z^2} = \frac{-1}{z^2}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$u_y(z,0) = 0$$

$$f(z) = \int \left(-\frac{1}{z^2} - 0\right) dz$$

$$f(z) = -\int \frac{1}{z^2}\, dz$$

$$f(z) = -\int z^{-2}\, dz$$

$$f(z) = -\frac{z^{-2+1}}{-2+1} + C$$

$$f(z) = \frac{-z^{-1}}{-1} + C$$

$$f(z) = \frac{1}{z} + C$$

Ex :-

Prove that a harmonic function satisfies the formal differential equation

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

Sol :-

Since, u is a harmonic function. So, it must satisfy the laplace's equation

$$\frac{\partial_2 u}{\partial x^2} + \frac{\partial_2 u}{\partial y^2} = 0$$

Now, $z = x + iy$

$\bar{z} = x - iy$

$$x = \frac{1}{2}(z + \bar{z})$$

$$y = \frac{1}{2i}(z - \bar{z})$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}$$

$$\frac{\partial_2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y \partial x \partial z}\frac{\partial y}{}\right] - \frac{1}{2i}\left[\left(\frac{\partial^2 u}{\partial x \partial y} \cdot \frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial z}\right]\right.$$

$$= \frac{1}{2}\left(\frac{1}{2}\frac{\partial^2 u}{\partial x^2} - \frac{i}{2}\frac{\partial^2 u}{\partial y \partial x}\right) - \frac{1}{2i}\left(\frac{\partial^2 u}{\partial x \partial y} \cdot \frac{1}{2}\right.$$

$$\left. + \frac{1}{2i}\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{i}{4}\frac{\partial^2 u}{\partial y \partial x} + \frac{i}{4}\frac{\partial^2 u}{\partial x \partial y} + \frac{1}{4}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

we know laplace's equations.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}(0)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

(or)

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial u}{\partial y}\left(\frac{-1}{2i}\right)$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar{z}}\right) = \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial z}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial z}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial y^2}\left(+\frac{1}{2i}\right)\right]$$

$$= \frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{1}{4i^2}\frac{\partial^2 u}{\partial y^2}$$

$$= \frac{1}{4}\frac{\partial^2 u}{\partial x^2} + \frac{1}{4}\frac{\partial^2 u}{\partial y^2}$$

$$= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\because \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$= \frac{1}{4}(0)$$

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$$

Ex :-

Find analytic function of which real part is

$e^{-x}[(x^2-y^2)\cos y + 2xy \sin y]$

Sol :-

$$u(x,y) = e^{-x}[(x^2-y^2)\cos y + 2xy \sin y]$$

$$f(z) = \int (u_x(z,0) - i u_y(z,0))\,dz$$

$$u_x = e^{-x}[2x\cos y + 2y \sin y] - e^{-x}[(x^2-y^2)\cos y + 2xy \sin y]$$

$$u_x(z,0) = e^{-z}[2z\cos(0) + 0] - e^{-z}[(z^2-0)\cos(0) + 0]$$

$$u_x(z,0) = 2ze^{-z} - z^2e^{-z}$$

$$= e^{-z}[2z - z^2]$$

$$u_x(z,0) = \frac{2z - z^2}{e^z}$$

$$u_y = e^{-x}[(x^2-y^2)(-\sin y) + (0-2y)\cos y + 2x \sin y + 2xy\cos y]$$

$$u_y(z,0) = e^{-z}[(z^2-0)(-\sin(0)) + (-2(0) + 2\cancel{z}\sin(0) + 2(0)]$$

$$u_y(z,0) = e^{-z}(0)$$

$$u_y(z,0) = 0$$

$$f(z) = \int \left(\frac{2z - z^2}{e^z} - 0\right) dz$$

$$f(z) = \int (2e^{-z}z - e^{-z}z^2)\,dz$$

$$f(z) = 2\int ze^{-z}\,dz - \int z^2e^{-z}\,dz$$

$$f(z) = 2\left[z\frac{e^{-z}}{-1} - \int 1 \cdot \frac{e^{-z}}{-1}dz\right] - \left[z^2\frac{e^{-z}}{-1} - \int 2z \cdot \frac{e^{-z}}{-1}dz\right]$$

$$= -2ze^{-z} + 2\int e^{-z}dz - [-z^2e^{-z} + 2\int ze^{-z}dz]$$

$$f(z) = -2ze^{-z} + 2\frac{e^{-z}}{-1} + z^2e^{-z} - 2\left(z\frac{e^{-z}}{-1} - \int 1\cdot\frac{e^{-z}}{-1}dz\right)$$

$$= \cancel{-2ze^{-z}} - 2e^{-z} + z^2e^{-z} + \cancel{2ze^{-z}} - 2\int e^{-z}dz$$

$$f(z) = -2e^{-z} + z^2e^{-z} - 2\frac{e^{-z}}{-1} + c$$

$$f(z) = \cancel{-2e^{-z}} + z^2e^{-z} + \cancel{2e^{-z}} + c$$

$$= \quad + z^2e^{-z} + c$$

$$= e^{-z}[0 + z^2] + c$$

$$f(z) = \frac{0 + z^2}{e^z} + c$$

$$f(z) = z^2e^{-z} + c$$

Ex:-

If $u(x,y) = x^3 + 3x^2 - 3xy^2 - 3y^2 + 1$ Show that $u(x,y)$ is harmonic, find its corresponding conjugate and original function.

Sol:

$$u(x,y) = x^3 + 3x^2 - 3xy^2 - 3y^2 + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 + 6x - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial u}{\partial y} = -(6xy + 6y)$$

$$\frac{\partial^2 u}{\partial y^2} = -(6x + 6)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$6x + 6 - 6x - 6 = 0$$

$$\Rightarrow \quad 0 = 0$$

Laplace's equation is satisfied.

Hence, the function is harmonic

As we know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = 3x^2 + 6x - 3y^2 = \frac{\partial v}{\partial y}$$

Integrate it w.r.t "y"

$$V(x,y) = 3x^2y + 6xy - 3\frac{y^3}{3} + g(x)$$

$$V(x,y) = 3x^2y + 6xy - y^3 + g(x) \rightarrow (1)$$

Differentiate it w.r.t 'x'

$$\frac{\partial v}{\partial x} = 6xy + 6x + g'(x)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 6xy + 6x + g'(x)$$

$$= (-(-(6xy + 6x)) = 6xy + 6x + g'(x)$$

$$\Rightarrow 6xy + 6x = 6xy + 6x + g'(x)$$

$$\Rightarrow g'(x) = 0$$

$$\Rightarrow g(x) = \text{constant}$$

$$V(x,y) = 3x^2y + 6xy - y^3 + c$$

$$f(z) = u(x,y) + iV(x,y)$$

$$f(z) = (x^3 + 3x^2 - 3xy^2 - 3y^2 + 1) + i(3x^2y + 6xy - y^3 + c)$$

$$f(z) = x^3 + 3x^2 - 3xy^2 - 3y^2 + 1 + 3ix^2y + 6ixy - iy^3 + ic$$

$$f(z) = (x^3 - 3xy^2 + 3ix^2y - iy^3) + 3(x^2 - y^2 + 2ixy) + (1 + ic)$$

$$f(z) = z^3 + 3z^2 + A'$$

because $z^3 = (x+iy)^3$ and $z^2 = (x+iy)^2$

Note:-

(i) In harmonic function there is not $\bar{z}$, then the function is analytic and in analytic function Cauchy Riemann equations satisfy.

(ii) If $\frac{\partial f}{\partial \bar{z}} \neq 0$ then function is not analytic.

Ex:-

If $w = f(z)$ is an analytic then $\frac{\partial f}{\partial \bar{z}} = 0$

Sol:-

Let $z = x + iy$

$\bar{z} = x - iy$

$x = \frac{1}{2}(z + \bar{z})$

$y = \frac{1}{2i}(z - \bar{z})$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \cdot \frac{\partial x}{\partial \bar{z}} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \cdot \frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \cdot \frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y}$$

we know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial v}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\left(-\frac{\partial v}{\partial x}\right) - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial v}{\partial y} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = 0$$

Ex:-

If $\frac{\partial f(z)}{\partial \bar{z}} = 0$, then cauchy Riemann equations are true

Sol:-

It is given $\frac{\partial f(z)}{\partial \bar{z}} = 0$

Let $z = x + iy$

$\bar{z} = x - iy$

$x = \frac{1}{2}(z + \bar{z})$

$y = \frac{1}{2i}(z - \bar{z})$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\cdot\frac{\partial x}{\partial \bar{z}} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\cdot\frac{\partial y}{\partial \bar{z}}$$

$$\frac{\partial f}{\partial \bar{z}} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{1}{2}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right)$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} - \frac{i}{2i^2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y}$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y}$$

put $\frac{\partial f}{\partial \bar{z}} = 0$

$$0 = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial v}{\partial y} + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\frac{1}{2}\frac{\partial v}{\partial y} - \frac{i}{2}\frac{\partial v}{\partial x} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y}$$

$$\cancel{\tfrac{1}{2}}\left[\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}\right] = \cancel{\tfrac{1}{2}}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$$

Comparing real and imaginary parts

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1)$$

$$-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow (2)$$

Hence, it is proved that if $\frac{\partial f}{\partial \bar{z}} = 0$ then Cauchy Riemann equations are true

Eq (1) and Eq (2) are Cauchy Riemann equations

Ex :-

$f(z) = \sin(3x + 5iy)$

then show that function is analytic

Sol :-

$f(z) = \sin(3x+5iy)$

let $4z = 4(x+iy)$, $\bar{z} = x - iy$

$\sin(3x+5iy) = \sin(4z - \bar{z})$

$\Rightarrow f(z) = \sin(4z - \bar{z})$

The given function involves $\bar{z}$, So, the function is not analytic

Ex :-

Derive cauchy Riemann equations in polar form from cartesian form

Proof :-

We know that

$f(z) = u(x,y) + iv(x,y)$

$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow r^2 = x^2+y^2$

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$$

$$\frac{\cancel{r}\sin\theta}{\cancel{r}\cos\theta} = \frac{y}{x}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$f(r,\theta) = u(r,\theta) + iv(r,\theta)$

We know that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$\because\ 2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{r}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\cdot\left(-\frac{\sin\theta}{r}\right) \rightarrow (1)$$

$\because\ \frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$

$\frac{\partial \theta}{\partial x} = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)$

$\frac{\partial \theta}{\partial x} = \frac{-r\sin\theta}{r^2}$

$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$\cancel{2}r\frac{\partial r}{\partial y} = \cancel{2}y$, $\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$

$\frac{\partial r}{\partial y} = \frac{r\sin\theta}{r}$, $\frac{\partial \theta}{\partial y} = \frac{x^2}{x^2+y^2}\,\frac{1}{x}$

$\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial \theta}{\partial y} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}(\sin\theta) + \frac{\partial v}{\partial \theta}\left(\frac{\cos\theta}{r}\right) \rightarrow (2)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \rightarrow (3)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cdot\cos\theta + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} \rightarrow (4)$$

By C.R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial r}\cos\theta + \frac{\partial u}{\partial \theta}\left(\frac{-\sin\theta}{r}\right) - \left(\frac{\partial v}{\partial r}\sin\theta + \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r}\right) = 0$$

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} - \frac{\partial v}{\partial r}\sin\theta - \frac{\partial v}{\partial \theta}\frac{\cos\theta}{r} = 0 \rightarrow (A)$$

$$\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} + \frac{\partial v}{\partial r}\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta}{r} = 0 \rightarrow (B)$$

$\cos\theta$ Eq (A) + $\sin\theta$ Eq (B)

$$\frac{\partial u}{\partial r}\cos^2\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r} - \frac{\partial v}{\partial r}\sin\theta\cos\theta - \frac{\partial v}{\partial \theta}\frac{\cos^2\theta}{r} = 0$$

$$\frac{\partial u}{\partial r}\sin^2\theta + \frac{\partial u}{\partial \theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial v}{\partial r}\sin\theta\cos\theta - \frac{\partial v}{\partial \theta}\frac{\sin^2\theta}{r} = 0$$

$$\frac{\partial u}{\partial r}(\cos^2\theta + \sin^2\theta) - \frac{\partial v}{\partial \theta}\frac{1}{r}(\cos^2\theta + \sin^2\theta) = 0$$

$$\frac{\partial u}{\partial r}(1) - \frac{1}{r}\frac{\partial v}{\partial \theta}(1) = 0.$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \rightarrow (c)$$

$\sin\theta$ Eq (A) $-$ $\cos\theta$ Eq (B)

$$\frac{\partial u}{\partial r}\sin\theta\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin^2\theta}{r} - \frac{\partial v}{\partial r}\sin^2\theta - \frac{\partial v}{\partial \theta}\frac{\sin\theta\cos\theta}{r} = 0$$

$$\underset{-}{+}\frac{\partial u}{\partial r}\sin\theta\cos\theta \underset{-}{+}\frac{\partial u}{\partial \theta}\frac{\cos^2\theta}{r} \underset{-}{+}\frac{\partial v}{\partial r}\cos^2\theta \mp \frac{\partial v}{\partial \theta}\frac{\sin\theta\cos\theta}{r} = 0$$

$$-\frac{\partial u}{\partial \theta}\frac{1}{r}(\sin^2\theta + \cos^2\theta) - \frac{\partial v}{\partial r}(\sin^2\theta + \cos^2\theta) = 0$$

$$\Rightarrow \quad -\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}$$

$$\Rightarrow \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \rightarrow (d)$$

(c) (d) are C.R equations in polar form.

Ex :-

Derive the Laplace's equations in polar form

Proof :-

We know that Cauchy Riemann equations in polar form are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \rightarrow (1)$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \rightarrow (2)$$

Differentiate eq (1) w.r.t "r" and eq (2) w.r.t "θ"

$$\frac{\partial^2 u}{\partial r^2} = \frac{-1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} \rightarrow (3)$$

$$\frac{\partial^2 v}{\partial r \partial \theta} = -\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \rightarrow (4)$$

$\frac{1}{r}$ Eq (4) + Eq (3)

$$\cancel{\frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \cancel{\frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\cdot\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

We know $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, by (1)

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

is laplace's equation in polar form

Ex :-

Prove that $u(r,\theta) = r^n \cos n\theta$ is harmonic. Also find $v(r,\theta)$ and original function $f(z)$

Sol :-

$$u(r,\theta) = r^n \cos n\theta$$

We know laplace's equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \rightarrow (1)$$

$$\frac{\partial u}{\partial r} = n r^{n-1} \cos n\theta$$

$$\frac{\partial^2 u}{\partial r^2} = n(n-1) r^{n-2} \cos n\theta$$

$$\frac{\partial u}{\partial \theta} = r^n (-n \sin n\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = r^n (-n^2 \cos n\theta)$$

put all values in (1)

$$n(n-1) r^{n-2} \cos n\theta + \frac{1}{r} n r^{n-1} \cos n\theta + \frac{1}{r^2}(-n^2 r^n \cos n\theta) = 0$$

$$(n^2 - n) r^{n-2} \cos n\theta + n r^{n-1-1} \cos n\theta - n^2 r^{n-2} \cos n\theta = 0$$

$$\Rightarrow n^2 r^{n-2} \cancel{\cos n\theta} - n r^{n-2} \cancel{\cos n\theta} + n r^{n-2} \cancel{\cos n\theta} - n^2 r^{n-2} \cancel{\cos n\theta} = 0$$

$$0 = 0$$ " equation satisfy

Hence, the function is harmonic

As we know

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{(By C.R equation)}$$

$$n r^{n-1} \cos n\theta = \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = n r^{n-1+1} \cos n\theta$$

$$\frac{\partial v}{\partial \theta} = n r^n \cos n\theta$$

Integrate it w.r.t "θ"

$$V(r, \theta) = n r^n \frac{\sin n\theta}{n} + g(r)$$

$$V(r, \theta) = r^n \sin n\theta + g(r)$$

Differentiate it w.r.t "r"

$$\frac{\partial v}{\partial r} = n r^{n-1} \sin n\theta + g'(r)$$

C.R eq is $\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = nr^{n-1}\sin n\theta + g'(r)$$

$$-\frac{1}{r}(-nr^n \sin n\theta) = nr^{n-1}\sin n\theta + g'(r)$$

$$nr^{n-1}\sin n\theta = nr^{n-1}\sin n\theta + g'(r)$$

$\Rightarrow \quad g'(r) = 0$

$\Rightarrow \quad g(r) = \text{constant}$

then

$$V(r,\theta) = r^n \sin n\theta + \text{constant}$$

$$f(z) = f(r,\theta) = U(r,\theta) + iV(r,\theta)$$

$$f(z) = r^n \cos n\theta + i(r^n \sin n\theta + \text{constant} = A)$$

$$f(z) = r^n \cos n\theta + ir^n \sin n\theta + iA$$

$$f(z) = r^n[\cos n\theta + i\sin n\theta] + A'$$

$$f(z) = r^n e^{in\theta} + A'$$

$$f(z) = Z^n + A'$$

Ex :-

Prove that $U(r,\theta) = r^2\cos 2\theta$ is harmonic find $V(r,\theta)$ and original function.

Sol :-

$$U(r,\theta) = r^2 \cos 2\theta$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta$$

$$\frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta$$

$$\frac{\partial u}{\partial \theta} = r^2 2(-\sin 2\theta)$$

$$\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$$

$$\frac{\partial^2 u}{\partial \theta^2} = -2r^2(2\cos 2\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -4r^2\cos 2\theta$$

Laplace's equation in polar form is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta) = 0$$

$$2\cos 2\theta + 2\cos 2\theta - 4\cos 2\theta = 0$$

$$4\cos 2\theta - 4\cos 2\theta = 0$$

$0 = 0$ Eq satisfy.

Hence, the function is harmonic

As we know C.R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$2r\cos 2\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta$$

Integrate it w.r.t "θ"

$$V(r,\theta) = 2r^2\int \cos 2\theta \, d\theta$$

$$V(r,\theta) = 2r^2\frac{\sin 2\theta}{2} + g(r)$$

$$V(r,\theta) = r^2\sin 2\theta + g(r)$$

Differentiate it w.r.t "r"

$$\frac{\partial V}{\partial r} = 2r\sin 2\theta + g'(r)$$

we know, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ (C.R equation)

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = 2r\sin 2\theta + g'(r)$$

$$-\frac{1}{r}(-2r^2\sin 2\theta) = 2r\sin 2\theta + g'(r)$$

$$2r\sin 2\theta = 2r\sin 2\theta + g'(r)$$

$\Rightarrow \quad g'(r) = 0 \quad$ Integrate it w.r.t 'r'

$\Rightarrow \quad g(r) = \text{constant} = A$

then $\quad V(r,\theta) = r^2\sin 2\theta + A$

$$f(z) = U(r,\theta) + iV(r,\theta)$$

$$f(z) = r^2\cos 2\theta + i(r^2\sin 2\theta + A)$$

$$f(z) = r^2\cos 2\theta + ir^2\sin 2\theta + iA$$

$$f(z) = r^2(\cos 2\theta + i\sin 2\theta) + A'$$

$$f(z) = r^2 e^{i2\theta} + A'$$

$$f(z) = z^2 + A'$$

Ex:-

Prove that following functions are harmonic, find $V(r,\theta)$ and $f(z)$

(i) $U(r,\theta) = \left(\frac{r^2+1}{r}\right)\cos\theta$

(ii) $U(r,\theta) = \frac{2r^3+\cos 3\theta}{3}$

Sol:-

$$U(r,\theta) = \left(\frac{r^2+1}{r}\right)\cos\theta$$

$$\frac{\partial U}{\partial r} = \left(\frac{r(2r+0) - 1\cdot(r^2+1)}{r^2}\right)\cos\theta$$

$$\frac{\partial U}{\partial r} = \left(\frac{2r^2 - r^2 - 1}{r^2}\right)\cos\theta$$

$$\frac{\partial U}{\partial r} = \left(\frac{r^2-1}{r^2}\right)\cos\theta$$

$$\frac{\partial^2 U}{\partial r^2} = \left(\frac{r^2(2r-0) - 2r(r^2-1)}{(r^2)^2}\right)\cos\theta$$

$$\frac{\partial^2 U}{\partial r^2} = \left(\frac{2r^3 - 2r^3 + 2r}{r^4}\right)\cos\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{2r}{r^4}\cos\theta$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{2\cos\theta}{r^3}$$

$$\frac{\partial u}{\partial \theta} = \left(\frac{r^2+1}{r}\right)(-\sin\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = -\left(\frac{r^2+1}{r}\right)\cos\theta$$

Laplace equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{2\cos\theta}{r^3} + \frac{1}{r}\left(\frac{r^2-1}{r^2}\right)\cos\theta + \frac{1}{r^2}\left(\frac{r^2+1}{r}\right)(-\cos\theta) = 0$$

$$\frac{2\cos\theta}{r^3} + \frac{(r^2-1)\cos\theta}{r^3} - \frac{(r^2+1)}{r^3}\cos\theta = 0$$

$$2\cos\theta + (r^2-1)\cos\theta - (r^2+1)\cos\theta = 0$$

$$2\cos\theta + r^2\cos\theta - \cos\theta - r^2\cos\theta - \cos\theta = 0$$

$$2\cos\theta - 2\cos\theta = 0$$

$0 = 0$ Eq satisfy fn is harmoni

As we know, C·R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\left(\frac{r^2-1}{r^2}\right)\cos\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = r\frac{(r^2-1)}{r^2}\cos\theta$$

$$\frac{\partial v}{\partial \theta} = \left(\frac{r^2-1}{r}\right)\cos\theta$$

Integrate it w.r.t "θ"

$$V(r,\theta) = \left(\frac{r^2-1}{r}\right)\sin\theta + g(r)$$

Differentiate it w.r.t "r"

$$\frac{\partial v}{\partial r} = \frac{r(2r-0)-1.(r^2-1)}{r^2}\sin\theta + g'(r)$$

$$\frac{\partial V}{\partial r} = \frac{2r^2 - r^2 + 1}{r^2} \sin\theta + g'(r)$$

$$\frac{\partial V}{\partial r} = \frac{r^2+1}{r^2} \sin\theta + g'(r)$$

we know By C.R equation

$$\frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{r^2+1}{r^2} \sin\theta + g'(r)$$

$$-\frac{1}{r}\,\frac{r^2+1}{r}(-\sin\theta) = \frac{r^2+1}{r^2}\sin\theta + g'(r)$$

$$\frac{r^2+1}{r^2}\sin\theta = \frac{r^2+1}{r^2}\sin\theta + g'(r)$$

$$\Rightarrow \quad g'(r) = 0$$

$$\Rightarrow \quad g(r) = \text{Constant} = B$$

then $V(r,\theta) = \left(\frac{r^2-1}{r}\right)\sin\theta + B$

$$f(z) = U(r,\theta) + iV(r,\theta)$$

$$f(z) = \left(\frac{r^2+1}{r}\right)\cos\theta + i\left[\left(\frac{r^2-1}{r}\right)\sin\theta + B\right]$$

$$f(z) = \frac{r^2+1}{r}\cos\theta + i\left(\frac{r^2-1}{r}\right)\sin\theta + iB$$

$$f(z) = \frac{r^2+1}{r}\cos\theta + i\,\frac{r^2-1}{r}\sin\theta + B'$$

(ii)

$$u(r,\theta) = \frac{2r^3 + \cos 3\theta}{r^3}$$

$$\frac{\partial u}{\partial r} = \frac{r^3(6r^2+0) - 3r^2(2r^3 + \cos 3\theta)}{(r^3)^2}$$

$$\frac{\partial u}{\partial r} = \frac{6r^5 - 6r^5 - 3r^2\cos 3\theta}{r^6}$$

$$\frac{\partial u}{\partial r} = -\frac{3r^2\cos 3\theta}{r^6}$$

$$\frac{\partial u}{\partial r} = \frac{-3\cos 3\theta}{r^4}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{-3(r^4(0) - (4r^3)\cos 3\theta)}{(r^4)^2}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{-3(-4r^3\cos 3\theta)}{r^8}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{12r^3\cos 3\theta}{r^8}$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{12\cos 3\theta}{r^5}$$

$$\frac{\partial u}{\partial \theta} = \frac{1}{r^3}(0 - 3\sin 3\theta)$$

$$\frac{\partial u}{\partial \theta} = \frac{-3\sin 3\theta}{r^3}$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{-3}{r^3}(3\cos 3\theta)$$

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{-9\cos 3\theta}{r^3}$$

Laplace's equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$\frac{12\cos 3\theta}{r^5} + \frac{1}{r}\left(\frac{-3\cos 3\theta}{r^4}\right) + \frac{1}{r^2}\left(-\frac{9\cos 3\theta}{r^3}\right) = 0$$

$$\frac{12\cos 3\theta}{r^5} - \frac{3\cos 3\theta}{r^5} - \frac{9\cos 3\theta}{r^5} = 0$$

$$12\cos 3\theta - 3\cos 3\theta - 9\cos 3\theta = 0$$

$$\Rightarrow \quad 12\cos 3\theta - 12\cos 3\theta = 0$$

$\Rightarrow \quad 0 = 0$ eq. satisfy.

Hence, the function is harmonic

As we know C.R equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\frac{-3\cos 3\theta}{r^4} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$r \frac{-3\cos 3\theta}{r^4} = \frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial \theta} = \frac{-3\cos 3\theta}{r^3}$$

Integrate it w.r.t "θ"

$$V(r,\theta) = -\frac{3}{r^3}\int \cos 3\theta \, d\theta$$

$$V(r,\theta) = -\frac{\cancel{3}}{r^3}\,\frac{\sin 3\theta}{\cancel{3}} + g(r)$$

$$V(r,\theta) = -\frac{\sin 3\theta}{r^3} + g(r)$$

Differentiate it w.r.t 'r'

$$\frac{\partial v}{\partial r} = -\left[\frac{r^3(0) - 3r^2\sin 3\theta}{(r^3)^2}\right] + g'(r)$$

$$\frac{\partial v}{\partial r} = \frac{3r^2 \sin 3\theta}{r^6} + g'(r)$$

$$\frac{\partial v}{\partial r} = \frac{3\sin 3\theta}{r^4} + g'(r)$$

we know $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$$-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{3\sin 3\theta}{r^4} + g'(r)$$

$$-\frac{1}{r}\left(-\frac{3\sin 3\theta}{r^3}\right) = \frac{3\sin 3\theta}{r^4} + g'(r)$$

$$\frac{3\sin 3\theta}{r^4} = \frac{3\sin 3\theta}{r^4} + g'(r)$$

$\Rightarrow$ $g'(r) = 0$

$\Rightarrow$ $g(r) = \text{constant} = C$

then

$$V(r,\theta) = -\frac{\sin 3\theta}{r^3} + C$$

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$f(z) = \frac{2r^3+\cos 3\theta}{r^3} + i\left(-\frac{\sin 3\theta}{3} + c\right)$$

$$= \frac{2r^3+\cos 3\theta}{r^3} - \frac{i\sin 3\theta}{3} + ic$$

$$f(z) = \frac{2r^3+\cos 3\theta}{r^3} - \frac{i\sin 3\theta}{3} + c'$$

Ex: 3.5 Q No 1 → 6, 8 page: 104
Ex: 3.6 Q No 1 → 8 page 107] **Pennisi's book**

Ex:-

Find an analytic function in the disk $|z-1|<1$ whose real part is $\log\sqrt{x^2+y^2}$

Sol:-

$$u(x,y) = \log\sqrt{x^2+y^2}$$

$$u(x,y) = \frac{1}{2}\log(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}(2x)$$

$$u_x(x,y) = \frac{x}{x^2+y^2}$$

$$u_y(x,y) = \frac{y}{x^2+y^2}$$

$\because |z-1|<1$
$|x+iy-1|<1$
$|(x-1)+i(y-0)|<1$
$\Rightarrow (x-1)^2+(y-0)^2<1$
is a circle with centre $(1,0)$
$\because |z-(1+i)|<1$
is a circle with centre $(1,1)$

$$f(z) = \int (u_x(z,0) - iu_y(z,0))\,dz$$

$$f(z) = \int \left(\frac{z}{z^2} - 0\right)dz$$

$$f(z) = \int \frac{1}{z}\,dz$$

$$f(z) = \log(z) + A$$

"Analytic function is a regular function"

Ex :-

Prove that $\nabla^2[\operatorname{Re} f(z)]^2 = 2|f'(z)|^2$, where $f(z)$ is an analytic function.

Sol :-

$L.H.S = \nabla^2[\operatorname{Re} f(z)]^2$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$f(z) = u + iv$$

$$\operatorname{Re} f(z) = u$$

$$[\operatorname{Re} f(z)]^2 = u^2$$

$$\nabla^2[\operatorname{Re} f(z)]^2 = \nabla^2 u^2$$

$$= \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial u^2}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial u^2}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[2u\frac{\partial u}{\partial y}\right]$$

$$= \left[2\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2}\right)\right] + \left[2\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial y} + u\frac{\partial^2 u}{\partial y^2}\right)\right]$$

$$= [2(u_x u_x + u u_{xx})] + [2(u_y u_y + u u_{yy})]$$

$$= 2[u_x^2 + u u_{xx} + u_y^2 + u u_{yy}]$$

$$\nabla^2[\operatorname{Re} f(z)]^2 = 2[u_x^2 + u_y^2 + u(u_{xx} + u_{yy})] \rightarrow (1)$$

Since, the function $f(z)$ is analytic so, its real part $u(x,y)$ must be harmonic. Therefore, laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u_{xx} + u_{yy} = 0$$

putting this in (1)

$$\nabla^2[\operatorname{Re} f(z)]^2 = 2(u_x^2 + u_y^2 + u(0))$$

$$= 2(u_x^2 + u_y^2)$$

$R.H.S = 2|f'(z)|^2$

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ $\quad\because \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ By C-R equations

$f'(z) = \frac{\partial f(z)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}$

$|f'(z)|^2 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$2|f'(z)|^2 = 2[u_{xx} + u_{yy}]$

$\Rightarrow \nabla^2[Re f(z)]^2 = 2[f'(z)]^2$

Ex :-

If $|f(z)|$ is constant. Prove that $f(z)$ is also constant where $f(z)$ is an analytic function

Sol :-

$f(z) = u + iv$

$|f(z)| = \sqrt{u^2+v^2}$

It is given $|f(z)|$ is constant

$|f(z)| = \sqrt{u^2+v^2} = c \quad ; c > 0$

$\Rightarrow u^2 + v^2 = c^2 \rightarrow (1)$

Differentiate w.r.t "x"

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$

$uu_x + vv_x = 0 \rightarrow (2)$

Differentiate (1) w.r.t "y"

$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$

$uu_y + vv_y = 0 \rightarrow (3)$

* if $c = 0$ then Sol does not exist
* if $c < 0$ then $|f(z)|$ should be negative, which is impossible because $|f(z)|$ should not be "-ve" $|f(z)| \not< 0$

By C.R equation

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

put $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ in (2)

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \rightarrow (A)$$

$\because f(z)$ is analytic function.

$$u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0 \rightarrow (B)$$

Multiply Eq(A) by u and Eq(B) by v and then squaring and adding.

$$u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} + uv\frac{\partial u}{\partial y} + v^2\frac{\partial u}{\partial x} = 0$$

$$(u^2+v^2)\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} = 0$$

$\Rightarrow$ u is independent of "x"

By C.R equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

then $\frac{\partial v}{\partial y} = 0$

$\Rightarrow$ v is independent of "y"

$-v$ Eq(A) $+ u$ Eq(B)

$$-uv\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial y} + u^2\frac{\partial u}{\partial y} + uv\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow (u^2+v^2)\frac{\partial u}{\partial y} = 0 \qquad u^2+v^2 \neq 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0$$

$\Rightarrow$ u is independent of "y"

By C.R equation

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\Rightarrow \frac{\partial v}{\partial x} = 0$$

$\Rightarrow$ v is independent of "x"

U is independent of x and y and V is also independent of x and y.
Hence, $f(z) = U + iV$ is constant because so, $f(z)$ is independent of x and y

Ex:-

If $U = (x-1)^3 - 3xy^2 + 3y^2$. Determine V so that $U + iV$ is a regular (analysis) function of $x + iy$

Sol:-

$$f(z) = \int (U_x(z,0) - iU_y(z,0))\,dz$$

$$U = (x-1)^3 - 3xy^2 + 3y^2$$

$$\frac{\partial U}{\partial x} = 3(x-1)^2(1) - 3y^2$$

$$U_x(z,0) = 3(z-1)^2 - 3(0)$$

$$U_x(z,0) = 3(z-1)^2$$

$$\frac{\partial U}{\partial y} = -6xy + 6y$$

$$\left(\frac{\partial U}{\partial y}\right)_{(z,0)} = 0$$

$$\Rightarrow U_y(z,0) = 0$$

$$f(z) = \int (3(z-1)^2)\,dz$$

$$f(z) = 3\frac{(z-1)^3}{3} + A$$

$$f(z) = (z-1)^3 + A$$

$$f(z) = z^3 - 3z^2 + 3z - 1 + A$$

$$f(z) = (x+iy)^3 - 3(x+iy)^2 + 3(x+iy) - 1 + A$$

$$= x^3 + 3x^2yi + 3x(iy)^2 + (iy)^3 - 3(x^2 + 2xyi + (iy)^2) + 3x + 3yi - 1 + A$$

$$f(z) = x^3 + 3x^2yi + 3x(-1)y^2 - iy^3 - 3x^2 - 6xyi + 3y^2 + 3x + 3yi - 1 + A$$

$$f(z) = x^3 - 3xy^2 - 3x^2 + 3y^2 + 3x + 3x^2yi - y^3i - 6xyi + 3yi - 1 + A$$

$$f(z) = (x^3 - 3x^2 + 3x - 3xy^2 + 3y^2) + i(-y^3 + 3x^2y - 6xy + 3y) + A - 1$$

$$f(z) = (x^3 - 3x^2 + 3x - 3xy^2 + 3y^2) + i(-y^3 + 3x^2y - 6xy + 3y) + A'$$

$$f(z) = u(x,y) + iV(x,y)$$

then

$$V(x,y) = -y^3 + 3x^2y - 6xy + 3y$$

$$V(x,y) = 3x^2y - 6xy + 3y - y^3$$

$$V(x,y) = 3(x^2 - 2x + 1)y - y^3$$

$$V(x,y) = 3(x-1)^2y - y^3$$

Assignment

Complex Analysis by Louis-L-Permisi

Exercise No 3.5 page 104

Use C.R condition to show function is analytic in D = C

QNo1:-

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$$

Sol:-

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$$

$$u(x,y) + iV(x,y) = x^3 - 3xy^2 + i(3x^2y - y^3)$$

$$\Rightarrow \quad u(x,y) = x^3 - 3xy^2, \quad V(x,y) = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial V}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial V}{\partial y} = 3x^2 - 3y^2$$

C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

So, $3x^2 - 3y^2 = 3x^2 - 3y^2$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

and $-6xy = -6xy$

$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ equations satisfied

The partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

exist, are continuous, and C.R equations are true for the given function $f(z)$

Hence, the function $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ is analytic in domain $D = \mathbb{C}$

Q.No2:- $\sin z = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y$

$$f(z) = \sin x \cosh y + i \cos x \sinh y$$

Sol:-

$f(z) = \sin x \cosh y + i \cos x \sinh y$

$u(x,y) + iv(x,y) = \sin x \cosh y + i \cos x \sinh y$

$\Rightarrow u(x,y) = \sin x \cosh y \; , \; v(x,y) = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cosh y \quad , \quad \frac{\partial v}{\partial x} = -\sin x \sinh y$

$\frac{\partial u}{\partial y} = \sin x \sinh y \quad , \quad \frac{\partial v}{\partial y} = \cos x \cosh y$

C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\cos x \cosh y = \cos x \cosh y \quad , \quad \sin x \sinh y = \sin x \sinh y$

C.R equations are satisfied

Partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

exist, are continuous and C.R equations are true for the given function

Hence $f(z) = \sin x \cosh y + i \cos x \sinh y$ is an analytic function in domain $D = C$

Q.No3:-

$$f(z) = e^x \cos y + x + i(e^x \sin y + y)$$

Sol:-

$$f(z) = e^x \cos y + x + i(e^x \sin y + y)$$

$$U(x,y) + iV(x,y) = e^x \cos y + x + i(e^x \sin y + y)$$

$$\Rightarrow \quad U(x,y) = e^x \cos y + x$$

$$V(x,y) = e^x \sin y + y$$

$$\frac{\partial U}{\partial x} = e^x \cos y + 1 \quad , \quad \frac{\partial V}{\partial x} = e^x \sin y$$

$$\frac{\partial U}{\partial y} = e^x(-\sin y) \quad , \quad \frac{\partial V}{\partial y} = e^x \cos y + 1$$

C.R equations are

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad , \quad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$$e^x \cos y + 1 = e^x \cos y + 1 \quad , \quad -e^x \sin y = -e^x \sin y$$

C.R equations are satisfied

Partial derivatives $\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}$

exist, are continuous and C.R equations are true for the given function.

Hence $f(z) = e^x \cos y + x + i(e^x \sin y + y)$ is an analytic function in domain $D = C$.

Q.No4:-

$$f(z) = e^x(x \cos y - y \sin y) + ie^x(x \sin y + y \cos y)$$

Sol:-

$$f(z) = e^x(x \cos y - y \sin y) + ie^x(x \sin y + y \cos y)$$

$$u(x,y)+iv(x,y) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y)$$

$$\Rightarrow u(x,y) = e^x(x\cos y - y\sin y)$$

$$v(x,y) = e^x(x\sin y + y\cos y)$$

$$u(x,y) = e^x x\cos y - e^x y\sin y$$

$$\frac{\partial u}{\partial x} = e^x x\cos y + e^x\cos y - e^x y\sin y$$

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$\frac{\partial u}{\partial y} = xe^x(-\sin y) - (e^x\sin y + ye^x\cos y)$$

$$\frac{\partial u}{\partial y} = -xe^x\sin y - e^x\sin y - ye^x\cos y$$

$$\frac{\partial u}{\partial y} = -e^x[x\sin y + \sin y + y\cos y]$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y + 0)$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x\sin y$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y + y(-\sin y) + \cos y)$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y)$$

C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$e^x(x\cos y - y\sin y + \cos y) = e^x(x\cos y - y\sin y + \cos y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$-e^x[x\sin y + \sin y + y\cos y] = -e^x[x\sin y + \sin y + y\cos y]$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

C.R equations are satisfied.

Partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ exist, are continuous and C·R equations are true for the given function.

Hence, $f(z) = e^x(x\cos y - y\sin y) + ie^x(x\sin y + y\cos y)$ is analytic function in domain $D = \mathbb{C}$

Q no 5:-

$$f(z) = (\sin x + \cos x)\cosh y + i(\cos x - \sin x)\sinh y$$

Sol:-

$$f(z) = (\sin x + \cos x)\cosh y + i(\cos x - \sin x)\sinh y$$

$$u(x,y) + iv(x,y) = (\sin x + \cos x)\cosh y + i(\cos x - \sin x)\sinh y$$

$$\Rightarrow u(x,y) = (\sin x + \cos x)\cosh y$$

$$v(x,y) = (\cos x - \sin x)\sinh y$$

$$\frac{\partial u}{\partial x} = (\cos x - \sin x)\cosh y$$

$$\frac{\partial u}{\partial y} = (\sin x + \cos x)\sinh y$$

$$\frac{\partial v}{\partial x} = (-\sin x - \cos x)\sinh y$$

$$\frac{\partial v}{\partial x} = -(\sin x + \cos x)\sinh y$$

$$\frac{\partial v}{\partial y} = (\cos x - \sin x)\cosh y$$

C·R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$(\cos x - \sin x)\cosh y = (\cos x - \sin x)\cosh y$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$-(\sin x + \cos x)\sinh y = -(\sin x + \cos x)\sinh y$$

$\Rightarrow \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ C.R equations are satisfied

Partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ exist, are continuous and C.R equations are satisfied (true) for the given function. Hence, $f(z) = (\sin x + \cos x)\cosh y + i(\cos x - \sin x)\sinh y$ is analytic function in domain $D = \mathbb{C}$

Q.No6:-

$$f(z) = (\cos y \sinh x)^2 - (\sin y \cosh x)^2 + i(2\cos y \sin y \cosh x \sinh x)$$

Sol:-

$$f(z) = (\cos y \sinh x)^2 - (\sin y \cosh x)^2 + i(2\cos y \sin y \cosh x \sinh x)$$

$$f(z) = (\cos y \sinh x + i \sin y \cosh x)^2$$

$$u(x,y) + iv(x,y) = (\cos y \sinh x)^2 - (\sin y \cosh x)^2 + i(2\cos y \sin y \cosh x \sinh x)$$

$$\Rightarrow u(x,y) = (\cos y \sinh x)^2 - (\sin y \cosh x)^2$$
$$v(x,y) = 2\cos y \sin y \cosh x \sinh x$$

$$\frac{\partial u}{\partial x} = 2\cos y \sinh x \overset{\cos y}{\cosh x} - 2\sin y \cosh x \overset{\sin y}{\sinh x}$$
$$= 2\sinh x \cosh x(\cos^2 y - \sin^2 y) = \sinh 2x \cos 2y$$

$$\frac{\partial u}{\partial y} = 2\cos y \overset{\sinh x}{\sinh x}(-\sin y) - 2\sin y \cosh x \cos y$$

$$\frac{\partial u}{\partial y} = -2\sin y \cos y(\sinh^2 x + \cosh^2 x) = -\sin 2y \cosh 2x$$

$$\frac{\partial v}{\partial x} = 2\cos y \sin y(\sinh x \sinh x + \cosh x \cosh x)$$

$$\frac{\partial v}{\partial x} = 2\cos y \sin y(\sinh^2 x + \cosh^2 x)$$

$\frac{\partial v}{\partial x} = 2\sin y\cos y\,(\sin^2 hx + 1 + \sin^2 hx)$

$\because \cosh^2 x - \sinh^2 x = 1$

$\frac{\partial v}{\partial x} = 2\sin y\cos y\,(1 + 2\sinh^2 x) = \sin 2y(\cosh 2x)$

$\frac{\partial v}{\partial y} = 2\sinh x\cosh x\,(\cos y\cos y + (-\sin y)\sin y)$

$= 2\sinh x\cosh x\,(\cos^2 y - \sin^2 y) = \sinh 2x\cos 2y$

$\because \cos^2 y + \sin^2 y = 1$

$\frac{\partial v}{\partial y} = 2\sinh x\cosh x\,(\cos^2 y - (1 - \cos^2 y))$

$\frac{\partial v}{\partial y} = 2\sinh x\cosh x\,(\cos^2 y - 1 + \cos^2 y)$

$\frac{\partial v}{\partial y} = 2\sinh x\cosh x\,(2\cos^2 y - 1)$

C.R equations are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

~~2 sinhx coshx(cosy − siny) = 2 sinhx coshx~~

$\ell\ \sinh 2x\cos 2y = \sinh 2x\cos 2y$, satisfy.

$\sin 2y\cosh 2x = -2\sin 2y\cosh 2x$

C R Equations are true $\forall\, z \in C$

Hence, the function is an analytic function.

QNo3 :- **Ex 3.6**

$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{y}{x}\int \frac{1}{2}\,d\alpha - \frac{y}{x}\int \frac{1}{2}\cos 2\alpha\, d\alpha$

$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{2}{x}\alpha - \frac{2}{x}\frac{\sin 2\alpha}{2} + g(x)$

$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{2}{x}\alpha - \frac{\sin 2\alpha}{x} + g(x)$ $\quad \alpha = \tan^{-1}(y/x)$

Differentiate it w.r.t "x"

$\frac{\partial V}{\partial x} = 2\left(\frac{-1}{x^2}\right)\frac{y}{1+y^2/x^2}\left(\frac{-1}{x^2}\right) - 2\left(\frac{-1}{x^2}\right)\alpha - \sin 2\alpha\left(\frac{-1}{x^2}\right) + g'(x)$

$\frac{\partial V}{\partial x} = \left(-\frac{2}{x^2}\right)\left(\frac{-x^2 y}{x^2+y^2}\right)\left(\frac{1}{x^2}\right) + \frac{2}{x^2}\alpha + \frac{\sin 2\alpha}{x^2} + g'(x)$

$\frac{\partial V}{\partial x} = \frac{-2y}{x^2(x^2+y^2)} + \frac{2\alpha}{x^2} + \frac{\sin 2\alpha}{x^2} + g'(x)$

By C-R equation $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

Q.No 8:-

Show that each function f defined as indicated is not analytic in any domain D

Sol:-

(a) $f(z) = y + ix$

$u(x,y) + iv(x,y) = y + ix$

$\Rightarrow u(x,y) = y \quad , \quad v(x,y) = x$

$\frac{\partial u}{\partial x} = 0 \quad , \quad \frac{\partial v}{\partial x} = 1$

$\frac{\partial u}{\partial y} = 1 \quad , \quad \frac{\partial v}{\partial y} = 0$

C.R equations are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$0 = 0 \quad , \quad 1 \neq -1$ not true

$\Rightarrow$ C.R equations are not true

Hence, the function $f(z) = y + ix$ is not analytic in domain D.

(b) $f(z) = x^2 - y^2 - i2xy$

$u(x,y) + iv(x,y) = x^2 - y^2 - i2xy$

$\Rightarrow u(x,y) = x^2 - y^2 \quad , \quad v(x,y) = -2xy$

$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial x} = -2y$

$\frac{\partial u}{\partial y} = -2y \quad , \quad \frac{\partial v}{\partial y} = -2x$

C.R equations are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$2x \neq -2x \quad , \quad -2y \neq 2y$

not satisfied

$\Rightarrow$ C.R equations are not true.
Hence, the function $f(z) = x^2 - y^2 - i2xy$ is not analytic.

(c) $f(z) = e^x \cos y - i e^x \sin y$

$u(x,y) + iv(x,y) = e^x \cos y - i e^x \sin y$

$\Rightarrow u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial x} = -e^x \sin y$

$\frac{\partial u}{\partial y} = e^x(-\sin y)$, $\frac{\partial v}{\partial y} = -e^x \cos y$

C.R equations are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$e^x \cos y \neq -e^x \cos y$, $-e^x \sin y \neq e^x \sin y$

C.R equations are not true
Hence, the function $f(z) = e^x \cos y - i e^x \sin y$ is not analytic

(d) $f(z) = e^x \cos x + i e^x \sin x$

$u(x,y) + iv(x,y) = e^x \cos x + i e^x \sin x$

$\Rightarrow u(x,y) = e^x \cos x$, $v(x,y) = e^x \sin x$

$\frac{\partial u}{\partial x} = e^x \cos x - e^x \sin x$

$\frac{\partial v}{\partial x} = e^x \sin x + e^x \cos x$

$\frac{\partial u}{\partial y} = 0$ $\frac{\partial v}{\partial y} = 0$

C.R eqs are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow e^x(\cos x - \sin x) \neq 0$ $e^x(\sin x + \cos x) \neq 0$

C.R equations are not true.
Hence, the function $f(z) = e^x \cos x + i e^x \sin x$ is not analytic.

Exercise 3.6

Show that u is harmonic find its corresponding conjugate v

Q No 1:-

$u(x,y) = x^4 - 6x^2y^2 + y^4$; $D = \mathbb{C}$

Sol:-

$$u(x,y) = x^4 - 6x^2y^2 + y^4$$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$= 12(x^2 - y^2)$$

$$\frac{\partial u}{\partial y} = -12x^2y + 4y^3$$

$$\frac{\partial^2 u}{\partial y^2} = -12x^2 + 12y^2$$

$$\frac{\partial^2 u}{\partial y^2} = 12(y^2 - x^2)$$

Laplace's equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$12(x^2 - y^2) + 12(y^2 - x^2) = 0$$

$$12(x^2 - y^2) - 12(x^2 - y^2) = 0$$

$\Rightarrow$ $0 = 0$ Eq satisfy.

Hence, $u(x,y) = x^4 - 6x^2y^2 + y^4$ is harmonic

$$f(z) = \int (u_x(z,0) - i\,u_y(z,0))\,dz$$

$$u_x(z,0) = 4z^3 - 0$$

$$u_x(z,0) = 4z^3$$

$$u_y(z,0) = 0 + 0 = 0$$

$$f(z) = \int (4z^3 - 0)\,dz$$

$$f(z) = 4\int z^3\,dz$$

$$f(z) = 4\,\frac{z^4}{4} + A$$

$$f(z) = z^4 + A$$
$$f(z) = (x+iy)^4 + A$$
$$f(z) = [(x+iy)^2]^2 + A$$

$$f(z) = [x^2 - y^2 + 2ixy]^2 + A$$

$$f(z) = (x^2)^2 + (y^2)^2 + (2ixy)^2 + 2(x^2(-y^2)) + 2(x^2(2ixy))$$
$$+ 2(-y^2(2ixy)) + A$$
$$f(z) = x^4 + y^4 + 4i^2x^2y^2 + (-2x^2y^2) + 4ix^3y$$
$$- 4ixy^3 + A$$
$$f(z) = x^4 + y^4 - 4x^2y^2 - 2x^2y^2 + i\,4x^3y - i\,4xy^3 + A$$

$$f(z) = x^4 + y^4 - 6x^2y^2 + i(4x^3y - 4xy^3) + A$$
$$u(x,y) + iV(x,y) = x^4 + y^4 - 6x^2y^2 + i(4x^3y - 4xy^3) + A$$

$\Rightarrow$ $V(x,y) = 4x^3y - 4xy^3 + A$

2nd method:-

As we know $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ (C.R equation)

$$4x^3 - 12xy^2 = \frac{\partial V}{\partial y}$$

Integrate it w.r.t "y"

$$V(x,y) = 4x^3y - 12x\frac{y^3}{3} + g(x)$$

$$V(x,y) = 4x^3y - 4xy^3 + g(x) \rightarrow (1)$$

Differentiate w.r.t 'x'

$$\frac{\partial V}{\partial x} = 12x^2y - 4y^3 + g'(x)$$

By C.R equation $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

$$-\frac{\partial u}{\partial y} = 12x^2y - 4y^3 + g'(x)$$

$$-(-12x^2y + 4y^3) = 12x^2y - 4y^3 + g'(x)$$

$$12x^2y - 4y^3 = 12x^2y - 4y^3 + g'(x)$$
$$\Rightarrow \quad g'(x) = 0$$
$$\Rightarrow \quad g(x) = \text{Constant } A$$
then $\quad V(x, y) = 4x^3y - 4xy^3 + A$

QNo2:-

$$u(x, y) = x^2 - y^2 + 3x + 4 \ ; D = C$$

Sol:-

$$u(x, y) = x^2 - y^2 + 3x + 4$$
$$\frac{\partial u}{\partial x} = 2x + 3 \quad , \quad \frac{\partial u}{\partial y} = -2y$$
$$\frac{\partial^2 u}{\partial x^2} = 2 \quad , \quad \frac{\partial^2 u}{\partial y^2} = -2$$

Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$2 - 2 = 0$$
$$0 = 0 \quad \text{Eq satisfied.}$$

Hence, $u(x, y) = x^2 - y^2 + 3x + 4$ is harmonic.

As we know C.R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$2x + 3 = \frac{\partial v}{\partial y} \quad \text{Integrate it w.r.t 'y'}$$
$$V(x, y) = 2xy + 3y + g(x) \rightarrow (1)$$

Differentiate w.r.t 'x'

$$\frac{\partial v}{\partial x} = 2y + g'(x)$$

By C.R equation $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$-\frac{\partial u}{\partial y} = 2y + g'(x)$$
$$-(-2y) = 2y + g'(x)$$
$$2y = 2y + g'(x)$$
$$\Rightarrow \quad g'(x) = 0$$
$$\Rightarrow \quad g(x) = \text{constant } B$$

$V(x,y) = 2xy + 3y + B$

Q No 3:-

$u(x,y) = 2x/x^2+y^2 \quad ; D = C - \{0,0\}$

Sol:-

$$u(x,y) = \frac{2x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left\{\frac{(x^2+y^2)^2(0-2x) - (y^2-x^2)(2(x^2+y^2)(2x))}{((x^2+y^2)^2)^2}\right\}$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\frac{-2x(x^2+y^2)^2 - 4x(x^2+y^2)(y^2-x^2)}{(x^2+y^2)^4}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[\frac{-2x(x^2+y^2)^2 - 4x(y^4-x^4)}{(x^2+y^2)^4}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \left[2(2x^5-6xy^4-4x^3y^2)/(x^2+y^2)^4\right]$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - 2x(0+2y)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)^2(-4x) - 4xy(2(x^2+y^2)(2y))}{((x^2+y^2)^2)^2}$$

$$= \frac{-4x(x^2+y^2)^2 - 16xy^2(x^2+y^2)}{(x^2+y^2)^4}$$

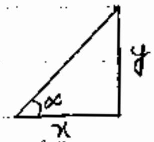
$$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{2}{x}\int d\alpha - \frac{2}{x}\int \cos 2\alpha\, d\alpha + g(x)$$

$$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{2}{x}\,\alpha - \frac{2}{x}\,\frac{\sin 2\alpha}{2} + g(x)$$

put $\alpha = \tan^{-1}(y/x)$

$$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{2}{x}\tan^{-1}(y/x) - \frac{1}{x}\sin 2\alpha + g(x)$$

$$V(x,y) = \frac{-1}{x}\, 2\sin\alpha\cos\alpha + g(x)$$



$$V(x,y) = -\frac{1}{x}\,\frac{2\tan\alpha}{1+\tan^2\alpha} + g(x)$$

put $\alpha = \tan^{-1}(y/x)$

$2\sin\alpha\cos\alpha = \sin 2\alpha$

$$= -\frac{2}{x}\,\frac{\tan\tan^{-1}(y/x)}{1+\tan^2(\tan^{-1}(y/x))} + g(x)$$

$2\,\frac{\sin\alpha}{\cos\alpha}\cos^2\alpha = \sin 2\alpha$

$\frac{2\tan\alpha}{\sec^2\alpha} = \sin 2\alpha$

$$= -\frac{2}{x}\,\frac{y/x}{1+\frac{y^2}{x^2}} + g(x) = -\frac{2}{x}\,\frac{y/x}{\frac{x^2+y^2}{x^2}} + g(x)$$

$\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

$$V(x,y) = \frac{-2y}{x^2+y^2} + g(x)$$

$\Rightarrow 2\sin\alpha\cos\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$

$$\frac{\partial V}{\partial x} = \frac{(x^2+y^2)(0) - (2y)(2x)}{(x^2+y^2)^2} + g'(x) \Rightarrow \frac{\partial V}{\partial x} = \frac{+4xy}{(x^2+y^2)^2} + g'(x)$$

By C R eq $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

$$-\frac{\partial u}{\partial y} = +\frac{4xy}{(x^2+y^2)^2} = \frac{4xy}{(x^2+y^2)^2} + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = \text{Constant } A$$

$$V(x,y) = -2y/x^2+y^2+A$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-4x(x^2+y^2)^2 - 16xy^2(x^2+y^2)}{(x^2+y^2)^4} = -2\left[\frac{2x^5+6xy^4-4x^3y^2}{(x^2+y^2)^4}\right.$$

Laplace's equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$-\frac{4x(x^2+y^2)^2 - 8x(y^4-x^4)}{(x^2+y^2)^4} - \frac{4x(x^2+y^2)^2 - 16xy^2(x^2+y^2)}{(x^2+y^2)^4} = 0$$

$$2\left[\frac{2x^5+6xy^4-4x^3y^2}{(x^2+y^2)^4}\right] - 2\left[\frac{2x^5+6xy^4-4x^3y^2}{(x^2+y^2)^4}\right] = 0$$

$0 = 0$ Eq satisfy

Laplace's equation is satisfied.

Hence function is harmonic

As we know C.R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{2(y^2-x^2)}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = \frac{2(y^2-x^2)}{(x^2+y^2)^2}$$ Integrate w.r.t "y"

$$V(x,y) = 2\int\left(\frac{y^2}{(x^2+y^2)^2} - \frac{x^2}{(x^2+y^2)^2}\right)dy$$

$$= 2\int\left(\frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2x^2}{(x^2+y^2)^2}\right)dy$$

$$= 2\int\frac{dy}{(x^2+y^2)} - 4x^2\int\frac{dy}{(x^2+y^2)^2}$$

$$= \frac{2}{x}\tan^{-1}(y/x) - 4x^2\int\frac{dy}{(x^2+y^2)^2}$$

put $y = x\tan\alpha$

$dy = x\sec^2\alpha\, d\alpha$

$$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - 4x^2\int\frac{x\sec^2\alpha\, d\alpha}{x^4\sec^4\alpha}$$

$$= \frac{2}{x}\tan^{-1}(y/x) - \frac{4}{x}\int\cos^2\alpha\, d\alpha$$

$$V(x,y) = \frac{2}{x}\tan^{-1}(y/x) - \frac{4}{x}\int\left(\frac{1+\cos 2\alpha}{2}\right)d\alpha$$

Now above Q No 8 Ex 3.5

Qn/04:-

$u(x,y) = e^x(x\cos y - y\sin y)$; $D = C$

Sol:-

$u(x,y) = e^x(x\cos y - y\sin y)$

$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x(\cos y)$

$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x\cos y + e^x(\cos y)$

$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y + 2\cos y) - \sin y)$

$\frac{\partial u}{\partial y} = e^x(x(-\sin y) - (y\cos y + \sin y))$

$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - (y(-\sin y) + \cos y) - \cos y)$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y + y\sin y - \cos y - \cos y)$

$\frac{\partial^2 u}{\partial y^2} = -e^x[x\cos y - y\sin y + 2\cos y]$

Laplace's equation is:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$e^x(x\cos y - y\sin y + 2\cos y) - e^x(x\cos y - y\sin y + 2\cos y = 0$

$\Rightarrow \quad 0 = 0$ Eq. satisfy.

Hence, the function is harmonic.

As we know, CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y) = \frac{\partial u}{\partial x}$

Integrate it w.r.t "y"

$v(x,y) = e^x(x\sin y - (y(-\cos y) - \int 1\cdot(-\cos y)dy) + \sin y)$

$= e^x(x\sin y + y\cos y \pm \sin y + \sin y) + g(x)$

Differentiate it w.r.t "x"

$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + 2\sin y) + e^x\sin y + g'(x)$

$\because \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ By C.R equations.

$-[e^x(-x\sin y - y\cos y - \sin y)] = e^x(x\sin y + y\cos y + \sin y] + g'(x)$

$\Rightarrow \quad g'(x) = 0$

$\Rightarrow \quad g(x) = \text{Constant } A.$

then $V(x,y) = e^x(x\sin y + y\cos y) + A$

QNo5:-

$u(x,y) = \cos x \cosh y \quad ; D = \mathbb{C}$

Sol:-

$u(x,y) = \cos x \cosh y$

$\dfrac{\partial u}{\partial x} = (-\sin x)\cosh y$

$\dfrac{\partial^2 u}{\partial x^2} = -\cos x \cosh y$

$\dfrac{\partial u}{\partial y} = \cos x \sinh y$

$\dfrac{\partial^2 u}{\partial y^2} = \cos x \cosh y$

Laplace's equation is

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

$-\cos x \cosh y + \cos x \cosh y = 0$

$\Rightarrow \quad 0 = 0$ satisfy.

Hence, $u(x,y) = \cos x \cosh y$ is harmonic.

As we know C.R equation are.

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$

$-\sin x \cosh y = \dfrac{\partial v}{\partial y}$

Integrate it w.r.t 'y'

$V(x,y) = -\sin x \sinh y + g(x)$

Differentiate it w.r.t 'x'

$\frac{\partial V}{\partial x} = -\cos x \sinh y + g'(x)$

By C.R equations $\frac{\partial V}{\partial x} = -\frac{\partial u}{\partial y}$

$-\frac{\partial u}{\partial y} = -\cos x \sinh y + g'(x)$

$-\cos x \sinh y = -\cos x \sinh y + g'(x)$

$\Rightarrow \quad g'(x) = 0$

$\Rightarrow \quad g(x) = \text{constant } C$.

then

$V(x,y) = -\sin x \sinh y + C$

QNo6:-

$u(x,y) = x \sin x \cosh y - y \cos x \sinh y \; ; \; D = \mathbb{C}$

Sol:-

$u(x,y) = x \sin x \cosh y - y \cos x \sinh y$

$\frac{\partial u}{\partial x} = x \cos x \cosh y + \sin x \cosh y - y(-\sin x) \sinh y$

$\frac{\partial u}{\partial x} = x \cos x \cosh y + \sin x \cosh y + y \sin x \sinh y$

$\frac{\partial^2 u}{\partial x^2} = x(-\sin x)\cosh y + \cos x \cosh y + \cos x \cosh y + y \cos x \sinh y$

$\frac{\partial^2 u}{\partial x^2} = -x \sin x \cosh y + 2\cos x \cosh y + y \cos x \sinh y$

$\frac{\partial u}{\partial y} = x \sin x \sinh y - \cos x \sinh y - y \cos x \cosh y$

$\frac{\partial^2 u}{\partial y^2} = x \sin x \cosh y - \cos x \cosh y - \cos x \cosh y - y \cos x \sinh y$

$\frac{\partial^2 u}{\partial y^2} = x \sin x \cosh y - 2\cos x \cosh y - y \cos x \sinh y$

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$-x\sin x\cosh y+2\cos x\cosh y+y\cos x\sinh y+x\sin x\cosh y$
$-2\cos x\cosh y-y\cos x\sinh y=0$

$\Rightarrow \quad 0=0$ Eq satisfy

Hence, $U(x,y)=x\sin x\cosh y-y\cos x\sinh y$ is harmonic

As we know C.R equations are:

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

$$x\cos x\cosh y+\sin x\cosh y+y\sin x\sinh y=\frac{\partial v}{\partial y}$$

Integrate it w.r.t "y"

$V(x,y)=x\cos x\sinh y+\sin x\sinh y+\sin x(y\cosh y - \int 1\cdot\cosh y\,dy)$

$V(x,y)=x\cos x\sinh y+\sin x\sinh y+\sin x(y\cosh y-\sinh y)+g(x)$

$V(x,y)=x\cos x\sinh y+\sin x\sinh y+y\sin x\cosh y-\sin x\sinh y+g(x)$

$V(x,y)=x\cos x\sinh y+y\sin x\cosh y+g(x)$

Differentiate w.r.t "x"

$\frac{\partial v}{\partial x}=\cos x\sinh y-x\sin x\sinh y+y\cos x\cosh y+g'(x)$

By C.R equation $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

$-\frac{\partial u}{\partial y}=\cos x\sinh y-x\sin x\sinh y+y\cos x\cosh y+g'(x)$

$-x\sin x\sinh y+\cos x\sinh y+y\cos x\cosh y=-x\sin x\sinh y+\cos x\sinh y+y\cos x\cosh y+g'(x)$

$\Rightarrow \quad g'(x)=0$

$\Rightarrow \quad g(x)=$ Constant A

then $V(x,y)=x\cos x\sinh y+y\sin x\cosh y+A$

QNo7:-

$U(x,y)=\ln[(x-x_0)^2+(y-y_0)^2]$; $D=C-\{x_0,y_0\}$

Sol:-

$U(x,y)=\ln[(x-x_0)^2+(y-y_0)^2]$

$\frac{\partial u}{} = 2(x-x_0)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{[(x-x_0)^2+(y-y_0)^2]2 - 2(x-x_0)2(x-x_0)}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+x_0^2-2xx_0+y^2+y_0^2-2yy_0)-(4x-4x_0)(x-x_0)}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2x^2+2x_0^2-4xx_0+2y^2+2y_0^2-4yy_0-4x^2+4xx_0+4xx_0-4x_0^2}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2x^2-2x_0^2+2y^2+2y_0^2-4yy_0+4xx_0}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$\frac{\partial u}{\partial y} = \frac{2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{[(x-x_0)^2+(y-y_0)^2]2 - 2(y-y_0)2(y-y_0)}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$= \frac{2(x^2+x_0^2-2xx_0+y^2+y_0^2-2yy_0)-4(y-y_0)^2}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$= \frac{2x^2+2x_0^2-4xx_0+2y^2+2y_0^2-4yy_0-4(y^2+y_0^2-2yy_0)}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2+2x_0^2-4xx_0+2y^2+2y_0^2-4yy_0-4y^2+8yy_0-4y_0^2}{[(x-x_0)^2+(y-y_0)^2]^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2x^2+2x_0^2-4xx_0-2y^2-2y_0^2+4yy_0}{[(x-x_0)^2+(y-y_0)^2]^2}$$

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$-\left(\frac{2x^2+2x_0^2-2y^2-2y_0^2+4yy_0-4xx_0}{[(x-x_0)^2+(y-y_0)^2]}\right)+\left(\frac{2x^2+2x_0^2-2y^2-2y_0^2}{[(x-x_0)^2+(y-y_0)^2]}+4yy_0-4xx_0\right)=0$$

$$\Rightarrow \quad 0=0$$

Laplace's equation satisfy.
Hence, $U(x,y) = \ln[(x-x_0)^2 + (y-y_0)^2]$ is harmonic

As we know that C.R-equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{2(x-x_0)}{(x-x_0)^2+(y-y_0)^2} = \frac{\partial v}{\partial y}$$

Integrate it w.r.t "y"

$$V(x,y) = \int \frac{2x}{(x-x_0)^2+(y-y_0)^2}\,dy - 2\int \frac{x_0}{(x-x_0)^2+(y-y_0)^2}\,dy$$

$$= \frac{1}{(x-x_0)}\, 2(x-x_0) \tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) + g(x)$$

$$V(x,y) = 2\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) + g(x)$$

Partially differentiate w.r.t "x"

$$\frac{\partial v}{\partial x} = 2\,\frac{1}{1+\left(\frac{y-y_0}{x-x_0}\right)^2}\,\frac{d}{dx}\left(\frac{y-y_0}{x-x_0}\right) + g'(x)$$

$$= \frac{2\cancel{(x-x_0)^2}}{(x-x_0)^2+(y-y_0)^2}\cdot\frac{(x-x_0)(0)-(y-y_0)}{\cancel{(x-x_0)^2}} + g'(x)$$

$$-\frac{\partial u}{\partial y} = \frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} + g'(x)$$

$$\frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} = \frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2} + g'(x)$$

$\Rightarrow \quad g'(x) = 0$

$\Rightarrow \quad g(x) = \text{Constant } C$

So,

$$V(x,y) = 2\tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) + A = C$$

Q No 8 :-
$u(x,y) = e^{x^2-y^2}\cos(2xy)$; $D = C$

Sol:-

$$u(x,y) = e^{x^2-y^2}\cos(2xy)$$

$$\frac{\partial u}{\partial x} = e^{x^2-y^2}(2x)\cos(2xy) + e^{x^2-y^2}(-\sin(2xy))(2y)$$

$$\frac{\partial u}{\partial x} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy).$$

$$\frac{\partial^2 u}{\partial x^2} = 2[e^{x^2-y^2}\cos(2xy) + xe^{x^2-y^2}(2x)\cos(2xy) + xe^{x^2-y^2}(-\sin(2xy)(2y))] - 2y[e^{x^2-y^2}(2x)\sin 2xy + e^{x^2-y^2}\cos(2xy)(2y)]$$

$$\frac{\partial^2 u}{\partial x^2} = 2e^{x^2-y^2}\cos(2xy) + 4x^2e^{x^2-y^2}\cos(2xy) - 4xye^{x^2-y^2}\sin(2xy) - 4xye^{x^2-y^2}\sin(2xy) - 4y^2e^{x^2-y^2}\cos(2xy)$$

$$\frac{\partial u}{\partial y} = e^{x^2-y^2}(-2y)\cos(2xy) + e^{x^2-y^2}(-\sin(2xy)(2x))$$
$$= -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy)$$

$$\frac{\partial^2 u}{\partial y^2} = -2[e^{x^2-y^2}\cos(2xy) + ye^{x^2-y^2}(-2y)\cos(2xy) + ye^{x^2-y^2}(-\sin(2xy))(2x)] - 2x[e^{x^2-y^2}(-2y)\sin(2xy) + e^{x^2-y^2}\cos(2xy)(2x)]$$

$$\frac{\partial^2 u}{\partial y^2} = -2e^{x^2-y^2}\cos(2xy) + 4y^2e^{x^2-y^2}\cos(2xy) + 4xye^{x^2-y^2}\sin(2xy) + 4xye^{x^2-y^2}\sin(2xy) - 4x^2e^{x^2-y^2}\cos(2xy)$$

Laplace's equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$2e^{x^2-y^2}\cos(2xy) + 4x^2e^{x^2-y^2}\cos(2xy) - 4xye^{x^2-y^2}\sin(2xy) - 4xye^{x^2-y^2}\sin(2xy) - 4y^2e^{x^2-y^2}\cos(2xy) - 2e^{x^2-y^2}\cos(2xy) + 4y^2e^{x^2-y^2}\cos(2xy) + 4xye^{x^2-y^2}\sin(2xy) + 4xye^{x^2-y^2}\sin(2xy) - 4x^2e^{x^2-y^2}\cos(2xy) = 0$$

$\Rightarrow$ $0 = 0$ Equation satisfy.

Hence, $u(x,y)$ is harmonic.

As we know C.R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) = \frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y} = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy)$$

Integrate it w.r.t 'y'

$$V(x,y) = 2x\left[e^{x^2-y^2}\frac{(\sin 2xy)}{2x} - \int e^{x^2-y^2}(-2y)\frac{(\sin 2xy)}{2x}\right]$$

$$- \int e^{x^2-y^2}(2y)\sin 2xy$$

$$V(x,y) = \frac{2xe^{x^2-y^2}\sin(2xy)}{2x} + 2y\int e^{x^2-y^2}\sin(2xy)$$

$$- 2y\int e^{x^2-y^2}\sin(2xy)$$

$$V(x,y) = e^{x^2-y^2}\sin(2xy) + g(x)$$

Differentiate it w.r.t "x"

$$\frac{\partial v}{\partial x} = e^{x^2-y^2}(2x)\sin(2xy) + e^{x^2-y^2}2y\cos(2xy) + g'(x)$$

$$\frac{\partial v}{\partial x} = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy) + g'(x)$$

By C R equation $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$-\frac{\partial u}{\partial y} = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy) + g'(x)$$

$$-\left[-2xe^{x^2-y^2}\sin(2xy) - 2ye^{x^2-y^2}\cos(2xy)\right] = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy)$$

$$\Rightarrow 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy) = 2xe^{x^2-y^2}\sin(2xy) + g'(x)$$
$$+ 2ye^{x^2-y^2}\cos(2xy) + g'(x)$$

$\Rightarrow$ $g'(x) = 0$

integrate it w.r.t "x"

$g(x) = \text{Constant } C$

then

$$V(x,y) = e^{x^2-y^2}\sin(2xy) + C$$

Level curves :-

Let the function f be defined in a domain D. The set of all points z in D such that

$$|f(z)| = M \; ; \; M > 0$$

is known as level curve or contour curve of f.

Observe that by giving different positive values, we obtain a one parametre family (or system) of level curves.

Ex :-

Sketch the level curves if $u = c_1$, and $v = c_2$ when $f(z) = z^2 \; ; \; z \neq 0$

Sol :-

$$f(z) = z^2$$

$$f(z) = (x+iy)^2$$

$$f(z) = u + iv = x^2 - y^2 + 2ixy$$

$$\Rightarrow u = x^2 - y^2 \quad , \quad v = 2xy$$

Given $u = c_1$, $v = c_2$

then $x^2 - y^2 = c_1$, $2xy = c_2$

$$x^2 - y^2 = c_1 \rightarrow (1) \qquad xy = c_2/2 = c_3$$

$$\Rightarrow xy = c_3 \rightarrow (2)$$

Eq (1) and (2) are rectangular hyperbolas.

Now, (1) and (2) level curves are rectangular hyperbolas.

Orthogonal system :-

Two families of curve $u(x,y) = c_1$ and $v(x,y) = c_2$ are said to form an orthogonal system, if the curves intersect at right angles at each of their points of intersection.

Consider $u(x,y) = c_1$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

Slope of curve $= m_1 = \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y}$

$V(x,y) = C_2$

$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$

Slope of curve $= m_2 = \frac{dy}{dx} = \frac{-\partial v/\partial x}{\partial v/\partial y}$

The two families of curves will intersect orthogonal if

$$m_1 m_2 = -1$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \rightarrow (a)$$

when Eq(a) satisfy then families (level of curves) are orthogonal.

Ex:-

Let function f defined by $f(z) = \frac{1}{z}$; $D = \mathbb{C} - \{0\}$ such that level curves $u = c_1$ and $v = c_2$ are orthogonal circles which pass through origin and have their centres on x-axis and y-axis respectively.

Sol:-

$f(z) = \frac{1}{z}$; $D = \mathbb{C} - \{0\}$ $\because f(z) = \frac{1}{z}$ is undefined at '0'.

$f(z) = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$f(z) = \frac{x-iy}{x^2+y^2}$

$u + iv = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$\Rightarrow u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$

Given $u = c_1$, $v = c_2$

$$\Rightarrow \frac{x}{x^2+y^2} = c_1, \quad \frac{-y}{x^2+y^2} = c_2$$

$$c_1(x^2+y^2) - x = 0$$

$$(c_1x^2 - x) + c_1y^2 = 0$$

$$c_1\left(x^2 - \frac{x}{c_1}\right) + c_1(y-0)^2 = 0$$

$$c_1\left(x^2 - 2x\left(\frac{1}{2c_1}\right) + \left(\frac{1}{2c_1}\right)^2\right) + c_1(y-0)^2 = \left(\frac{1}{2c_1}\right)^2$$

$$c_1\left(x - \frac{1}{2c_1}\right)^2 + c_1(y-0)^2 = \frac{1}{4c_1^2}$$

$$\Rightarrow \left(x - \frac{1}{2c_1}\right)^2 + (y-0)^2 = \frac{1}{4c_1^3}$$

is equation of circle which passes through origin and centre at x-axis which is $\left(\frac{1}{2c_1}, 0\right)$

$$c_2(x^2+y^2) + y = 0$$

$$c_2x^2 + (c_2y^2 + y) = 0$$

$$c_2(x-0)^2 + c_2\left(y^2 + \frac{y}{c_2}\right) = 0$$

$$c_2(x-0)^2 + c_2\left(y^2 + 2y\left(\frac{1}{2c_2}\right) + \left(\frac{1}{2c_2}\right)^2\right) = \left(\frac{1}{2c_2}\right)^2$$

$$c_2(x-0)^2 + c_2\left(y + \frac{1}{2c_2}\right)^2 = \frac{1}{4c_2^2}$$

$$\Rightarrow (x-0)^2 + \left(y + \frac{1}{2c_2}\right)^2 = \frac{1}{4c_2^3}$$

is equation of circle which passes through origin and centre at y-axis which is $\left(0, -\frac{1}{2c_2}\right)$

Now, for orthogonal system we have.

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0)-x(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$$ put all values in (1)

$$\left(\frac{y^2-2x^2}{(x^2+y^2)^2}\right)\left(\frac{+2xy}{(x^2+y^2)^2}\right)+\frac{(-2xy)}{(x^2+y^2)^2}\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)=$$

$\Rightarrow$ $0 = 0$ Eq satisfy.

There fore, level curves are orthogonal.

Ex:-
If $f = \frac{z+4}{z-4}$, find level curves $u = c_1$, $v = c_2$. Also verify that level curves $u = c_1$, $v = c_2$ form an orthogonal system of circle.

Sol:-

$$f(z) = \frac{(x+iy)+4}{(x+iy)-4}$$

$$f(z) = \frac{(x+4)+iy}{(x-4)+iy} \times \frac{(x-4)-iy}{(x-4)-iy}$$

$$f(z) = \frac{[(x+4)+iy][(x-4)-iy]}{(x-4)^2+y^2}$$

$$f(z) = \frac{(x^2-16+y^2)+i(xy-4y-xy-4y)}{(x-4)^2+y^2}$$

$$u+iv = \frac{x^2-16+y^2}{(x-4)^2+y^2} + i\frac{-8y}{(x-4)^2+y^2}$$

$$\Rightarrow u = \frac{x^2-16+y^2}{(x-4)^2+y^2}, \quad v = \frac{-8y}{(x-4)^2+y^2}$$

Given $u = c_1$, $v = c_2$

$$\frac{x^2-16+y^2}{(x-4)^2+y^2} = c_1, \quad \frac{-8y}{(x-4)^2+y^2} = c_2$$

$\Rightarrow x^2-16+y^2 = c_1((x-4)^2+y^2))$

$\Rightarrow x^2-16+y^2 = c_1[x^2-8x+16+y^2]$

$\Rightarrow c_1(x^2+y^2-8x+16)-x^2-y^2+16=0$

$\Rightarrow x^2(c_1-1)+y^2(c_1-1)-8c_1x+16c_1+16=0 \rightarrow (1)$

which is equation of circle

$$\frac{-8y}{(x-4)^2+y^2} = c_2$$

$-8y = c_2[x^2-8x+16+y^2]$

$8y-8y = c_2[x^2-8x+16+y^2]+8y$

$\Rightarrow \quad C_2(x^2+y^2-8x+16)+8y=0 \rightarrow (2)$

$\Rightarrow$ is also equation of circle

$$U = \frac{x^2+y^2-16}{(x-4)^2+y^2}, \qquad V = \frac{-8y}{(x-4)^2+y^2}$$

$$\frac{\partial U}{\partial x} = \frac{[(x-4)^2+y^2](2x) - 2(x^2+y^2-16)(x-4)}{[(x-4)^2+y^2]^2}$$

$$= \frac{2x^3-16x^2+32x+2xy^2-2x^3-2xy^2+32x+8x^2+8y-128}{[(x-4)^2+y^2]^2}$$

$$\frac{\partial U}{\partial x} = \frac{8y^2-8x^2+64x-128}{[(x-4)^2+y^2]^2} \rightarrow (a)$$

$$\frac{\partial U}{\partial y} = \frac{\cancel{[(x-4)^2+y^2]2y-2y(x^2+y^2-16)}}{[(x-4)^2+y^2]^2}$$

$$\frac{\partial U}{\partial y} = \frac{2x^2y-16xy+32y+2y^3-2x^2y-2y^3+32y}{[(x-4)^2+y^2]^2}$$

$$\frac{\partial U}{\partial y} = \frac{16y(4-x)}{[(x-4)^2+y^2]^2} = \frac{64y-16xy}{[(x-4)^2+y^2]^2} = \frac{-(16xy-64y)}{[(x-4)^2+y^2]^2} \rightarrow (b)$$

$$\frac{\partial V}{\partial x} = \frac{8y\cdot 2(x-4)}{[(x-4)^2+y^2]^2}$$

$$\frac{\partial V}{\partial x} = \frac{16xy-64y}{[(x-4)^2+y^2]^2} \rightarrow (3)$$

$$\frac{\partial V}{\partial y} = \frac{[(x-4)^2+y^2](-8)+8y\cdot 2y}{[(x-4)^2+y^2]^2}$$

$$\frac{\partial V}{\partial y} = \frac{8y^2-8x^2+64x-128}{[(x-4)^2+y^2]^2} \rightarrow (4)$$

Now, for orthogonal system we have

$$\frac{\partial U}{\partial x}\cdot\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\cdot\frac{\partial V}{\partial y} = 0$$

$$\left[\frac{8y^2-8x^2+64x-128}{[(x-4)^2+y^2]^2}\right]\left[\frac{16xy-64y}{[(x-4)^2+y^2]^2}\right] - \left[\frac{8y^2-8x^2+64x-128}{[(x-4)^2+y^2]^2}\right]$$

$$\frac{[16xy - 64y]}{[(x-4)^2 + y^2]} = 0$$

$\Rightarrow$ $0 = 0$ Equation satisfy.

Because eqs (a), (4) are same and (b) = $-(3)$, therefore it forms an orthogonal system of circles.

Ex:-

Let f be defined by $f(z) = z^2 + 2z + 1$; $D = C - \{-i\}$ Show that level curves $U(x, y) = C_1$, $V(x, y) = C_2$ form an orthogonal system of rectangular hyperbolas.

Sol:-

$f(z) = z^2 + 2z + 1$; $D = C - \{-i\}$

$f(z) = (x + iy)^2 + 2(x + iy) + 1$

$= x^2 - y^2 + 2ixy + 2x + 2iy + 1$

$f(z) = x^2 - y^2 + 2x + 1 + 2ixy + 2iy$

$u + iv = (x^2 - y^2 + 2x + 1) + i(2xy + 2y)$

$\Rightarrow$ $u = x^2 - y^2 + 2x + 1$, $V = 2xy + 2y$

Given $u(x, y) = c_1$, $V(x, y) = c_2$

$\Rightarrow$ $x^2 - y^2 + 2x + 1 = c_1$, $2y + 2xy = c_2$

$(x^2 + 2x) - (y - 0)^2 + 1 = c_1$

$(x^2 + 2x \cdot \frac{1}{2} + (\frac{1}{2})^2) - (y - 0)^2 = c_1 - 1 + (\frac{1}{2})^2$

$$(x + \tfrac{1}{2})^2 - (y - 0)^2 = c_1 - 1 + \frac{1}{4}$$

$$(x + \tfrac{1}{2})^2 - (y - 0)^2 = c_1 - \frac{4 + 1}{4}$$

$$(x + \tfrac{1}{2})^2 - (y - 0)^2 = c_1 - \frac{3}{4} \rightarrow (1)$$

is equation of hyperbola.

$2y + 2xy = c_2$

$(1 + x)2y = c_2$

$\Rightarrow (1 + x)y = c_2/2 \rightarrow (2)$

is equation

Eq (1) and (2) are rectangular hyperbolas.
For orthogonal system, we have:

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0 \rightarrow (9)$$

$$\frac{\partial u}{\partial x} = 2x+2 \quad , \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial y} = 2x+2 \quad \frac{\partial v}{\partial x} = 2y$$

$$(2x+2)(2y) + (-2y)(2x+2) = 0$$

$$4xy + 4y - 4xy - 4y = 0$$

$$0 = 0$$ Equation satisfied.

Hence, Eq (1) and (2) are rectangular hyperbolas of orthogonal system.

CH No 2

Ex:-

Sum of two discontinuous function may or may be continuous.

Sol:-

CHAPTER No 3

ELEMENTARY TRANSCENDENTAL FUNCTIONS

Elementry function:-

We consider here various elementry functions studied in calculus and define corresponding functions of a complex variable.

— To be specific, we define analytic function of a complex variable Z which reduce to those elementry functions when $Z = x + i0$. We start by defining the complex exponential function and then use it to develop the others.

Exponential function:-

If $Z = x + iy$, then complex exponential e^z is given by

$$e^z = e^{x+iy}$$

$$e^z = e^x . e^{iy}$$

$$e^z = e^x [\cos y + i \sin y]$$

Theorem:-

If $Z_1 = x_1 + iy_1$, $Z_2 = x_2 + iy_2$ are two complex numbers, then

$$e^{Z_1+Z_2} = e^{Z_1} . e^{Z_2}$$

Proof:-

Let $Z_1 = x_1 + iy_1$ and $Z_2 = x_2 + iy_2$

we have to prove

$$e^{Z_1+Z_2} = e^{Z_1} . e^{Z_2}$$

$R.H.S = e^{Z_1} . e^{Z_2}$ $\qquad \because e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$= e^{(x_1+iy_1)} . e^{(x_2+iy_2)} \rightarrow (1)$

$$e^{x_1+iy_1} = 1 + \frac{(x_1+iy_1)}{1!} + \frac{(x_1+iy_2)^2}{2!} + \cdots + \frac{(x_1+iy_1)^n}{n!} + \cdots$$

$$e^{x_2+iy_2} = 1 + \frac{(x_2+iy_2)}{1!} + \frac{(x_2+iy_2)^2}{2!} + \cdots + \frac{(x_2+iy_2)^n}{n!} + \cdots$$

$$e^{(x_1+iy_1)}\cdot e^{(x_2+iy_2)} = \left[1+\frac{(x_1+iy_1)}{1!}+\frac{(x_1+iy_1)^2}{2!}+\cdots\right]\left[1+\frac{(x_2+iy_2)}{1!}+\frac{(x_2+iy_2)^2}{2!}+\cdots\right]$$

$$e^{z_1}\cdot e^{z_2} = \left[1+((x_1+iy_1)+(x_2+iy_2))+\left(\frac{(x_1+iy_1)^2}{2!}+\frac{(x_2+iy_2)^2}{2!}\right)+\cdots\right]$$

$$e^{z_1}\cdot e^{z_2} = \left[1+[(x_1+iy_1)+(x_2+iy_2)]+\left[\frac{(x_1+iy_1)^2}{2!}+\frac{(x_2+iy_2)^2}{2!}+(x_1+iy_1)(x_2+iy_2)\right]+\cdots\right]$$

$$\begin{aligned}
L.H.S &= e^{z_1+z_2}\\
&= e^{(x_1+iy_1)+(x_2+iy_2)}\\
&= \left[1+\frac{[(x_1+iy_1)+(x_2+iy_2)]}{1!}+\frac{[(x_1+iy_1)+(x_2+iy_2)]^2}{2!}+\cdots\right]\\
&= \left[1+[(x_1+iy_1)+(x_2+iy_2)]+\left[\frac{(x_1+iy_1)^2}{2}+\frac{(x_2+iy_2)^2}{2}+\frac{2}{2}(x_1+iy_1)(x_2+iy_2)\right]+\cdots\right]
\end{aligned}$$

$$e^{z_1+z_2} = \left[1+[(x_1+iy_1)+(x_2+iy_2)]+\left[\frac{(x_1+iy_1)^2}{2!}+\frac{(x_2+iy_2)^2}{2!}+(x_1+iy_1)(x_2+iy_2)\right]+\cdots\right]$$

$$\Rightarrow\quad e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$$

(OR)

$$e^{z_1+z_2} = e^{z_1}\cdot e^{z_2}$$

$$\begin{aligned}
R.H.S &= e^{z_1}\cdot e^{z_2}\\
&= e^{x_1+iy_1}\cdot e^{x_2+iy_2}\\
&= e^{x_1}\cdot e^{iy_1}\cdot e^{x_2}\cdot e^{iy_2}\\
&= e^{x_1}\cdot e^{x_2}\cdot e^{iy_1}\cdot e^{iy_2}\\
&= e^{x_1+x_2}(\cos y_1+i\sin y_1)(\cos y_2+i\sin y_2)\\
&= e^{x_1+x_2}(\cos y_1\cos y_2+i\sin y_2\cos y_1+i\sin y_1\cos y_2+i^2\sin y_1\sin y_2)\\
&= e^{x_1+x_2}[(\cos y_1\cos y_2-\sin y_1\sin y_2)+i(\sin y_1\cos y_2+\sin y_2\cos y_1)]\\
&= e^{x_1+x_2}[\cos(y_1+y_2)+i\sin(y_1+y_2)]\\
&= e^{x_1+x_2}\cdot e^{i(y_1+y_2)}\\
&= e^{x_1}\cdot e^{x_2}\cdot e^{iy_1}\cdot e^{iy_2}\\
&= e^{x_1+iy_1}\cdot e^{x_2+iy_2} = e^{(x_1+iy_1)+(x_2+iy_2)}\\
&= e^{z_1+z_2} = L.H.S
\end{aligned}$$

Ex :-

Prove that e^z is an analytic and also show that $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic.

Sol :-

$$f(z) = e^z$$
$$f(x+iy) = e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x(\cos y + i\sin y)$$

$$U(x,y) + iV(x,y) = e^x \cos y + ie^x \sin y$$

$$\Rightarrow \quad U(x,y) = e^x \cos y \,, \quad V(x,y) = e^x \sin y.$$

We know C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = e^x \cos y \,, \quad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y) \,, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$e^x \cos y = e^x \cos y \,, \quad -e^x \sin y = -e^x \sin y.$$

C.R equations are true.

Hence, $f(z) = e^z$ is analytic function.

$$f(z) = e^z$$

$$f(z) = e^x \cos y + ie^x \sin y \qquad \because f(z) = u + iv$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$f'(z) = e^x \cos y - i(-e^x \sin y)$$

$\because$ CR equations are true because $f(z)$ is analytic

So put $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$f'(z) = e^x \cos y + ie^x \sin y \qquad f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad U = e^x \cos y \,, \quad V = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \,, \quad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = e^x(-\sin y) \quad , \quad \frac{\partial v}{\partial y} = e^x \cos y$$

C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$e^x \cos y = e^x \cos y \quad , \quad e^x \sin y = -(-e^x \sin y)$$

$$e^x \sin y = e^x \sin y$$

C.R equations are true

Hence, $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is analytic

Ex:-

Prove ~~**that**~~ **if e^z is analytic function then show that $\frac{d}{dz}(e^z) = e^z \ \forall\ z$.**

Sol:-

$$f(z) = e^z = e^{x+iy}$$

$$f(z) = u + iv = e^x(\cos y + i\sin y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\Rightarrow u = e^x \cos y \quad , \quad v = e^x \sin y$

Since, e^z is analytic so C.R equations are true $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \text{ is also analytic} \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

Now, we have to prove $\quad \frac{\partial u}{\partial y} = -e^x \sin y$

$$\frac{d}{dz}(e^z) = e^z \ \forall\ z$$

putting all values in (1)

$$f'(z) = \frac{d}{dz}(f) = \frac{d}{dz}(e^z) = e^x \cos y - i(-e^x \sin y)$$

$$= e^x \cos y + ie^x \sin y$$

$$= e^x(\cos y + i\sin y)$$

$$\frac{d}{dz}(e^z) = e^x \cdot e^{iy}$$

$$\frac{d}{dz}(e^z) = e^{x+iy}$$

$$\frac{d}{dz}(e^z) = e^z$$

Ex:-

For all $z = x+iy$

(i) $e^z \neq 0$ (ii) $|e^{iy}| = 1$ and $|e^z| = e^x$

(iii) a necessary and sufficient condition that $e^z = 1$ is that $z = 2k\pi i$ where k is an integer

(iv) a necessary and sufficient condition that $e^{z_1} = e^{z_2}$ is that $z_1 - z_2 = 2k\pi i$; k is integer

Sol:-

(i) $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$

$e^z \cdot e^{-z} = e^{z-z}$ "$-z$" is inverse of z

$= e^0 = 1 \neq 0$

$\Rightarrow e^z \neq 0$

(ii)

$e^z = e^{x+iy}$

$e^z = e^x \cdot e^{iy}$

$e^{iy} = \cos y + i\sin y$

$|e^{iy}| = \sqrt{\cos^2 y + \sin^2 y}$

$|e^{iy}| = \sqrt{1} = 1$

$|e^z| = e^x$

$e^z = e^{x+iy} = e^x \cdot e^{iy}$

$|e^z| = |e^x(\cos y + i\sin y)|$

$\because \cos^2 y + \sin^2 y = 1$

$|e^z| = \sqrt{(e^x)^2(\cos^2 y + \sin^2 y)}$

$|e^z| = e^x$

(iii) Necessary:-

$e^z = 1$

$e^{x+iy} = e^x \cdot e^{iy}$

$= e^x(\cos y + i\sin y) = 1$

$\Rightarrow e^x\cos y + ie^x\sin y = 1$

$\Rightarrow e^x\cos y = 1 \rightarrow (1)$, $e^x\sin y = 0$

$e^x\sin y = 0$

$e^x \neq 0$, $\sin y = 0$

$\Rightarrow y = n\pi$; n is integer

put $y = n\pi$ in (1)

$e^x\cos(n\pi) = 1$ $\because \cos n\pi = (-1)^n$

$\Rightarrow e^x(-1)^n = 1$

$\Rightarrow e^x = 1 = e^0 \Rightarrow \ln e^x = \ln(1)$

we take $n = 2k$ So, $(-1)^n = 1$

$$z = x + iy$$
$$z = 0 + (2k\pi)i$$
$$\Rightarrow z = 2k\pi i \quad ; k \text{ is integer.}$$

Sufficient :-

Suppose $z = 2k\pi i$

$$e^z = e^{2k\pi i}$$
$$= (\cos(2k\pi) + i\sin(2k\pi)) \qquad \because \cos(2\pi) = 1$$
$$= 1 + i \cdot 0 \qquad \sin(2\pi) = 0$$
$$e^z = 1$$

Hence, $e^z = 1$ iff $z = 2k\pi i$; k is integer.

(iv)

Necessary :- $e^{z_1} = e^{z_2}$

$$e^{z_1 - z_2} = 1$$
$$e^{(x_1+iy_1)-(x_2+iy_2)} = 1$$
$$e^{(x_1-x_2)+i(y_1-y_2)} = 1$$
$$e^{x_1-x_2}[\cos(y_1-y_2) + i\sin(y_1-y_2)] = 1 + 0\cdot i$$
$$\Rightarrow e^{x_1-x_2}\cos(y_1-y_2) + ie^{x_1-x_2}\sin(y_1-y_2) = 1 + 0\cdot i$$
$$\Rightarrow e^{x_1-x_2}\cos(y_1-y_2) = 1 \rightarrow (a) , \quad e^{x_1-x_2}\sin(y_1-y_2) = 0$$
$$e^{x_1-x_2} \neq 0, \quad \sin(y_1-y_2) = 0$$
$$\Rightarrow y_1 - y_2 = n\pi.$$
$$z_1 - z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2) \rightarrow (1)$$

put $y_1 - y_2 = n\pi$ in (a)

$$e^{x_1-x_2}\cos(n\pi) = 1 \qquad \because \cos(\pi) = 0$$
$$e^{x_1-x_2}(-1)^n = 1 \quad \text{when } n = 2k ; n \text{ is even}$$
$$\Rightarrow e^{x_1-x_2} = e^0$$
$$\text{or } \ln e^{x_1-x_2} = \ln(1)$$
$$x_1 - x_2 = 0$$
$$\Rightarrow x_1 = x_2$$

put all values in (1)

$$z_1 - z_2 = 0 + in\pi \qquad n = 2k$$

$\Rightarrow \quad z_1 - z_2 = 2k\pi i$

Sufficient :-

$$z_1 - z_2 = 2k\pi i$$

$$e^{z_1 - z_2} = e^{2k\pi i}$$

$$= \cos(2k\pi) + i\sin(2k\pi)$$

$$= \cos(n\pi) + i\sin(n\pi)$$

$$= 1 + 0 \cdot i$$

$$e^{z_1 - z_2} = 1$$

Hence, $e^{z_1 - z_2} = 1$ iff $z_1 - z_2 = 2k\pi i$; k is integer.

Ex :-

To prove the exponential

$$\frac{\exp z_1}{\exp z_2} = \exp(z_1 - z_2)$$

Sol :-

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

L.H.S $e^{z_2} = e^{z_1}/e^{z_2}$

$$= \frac{e^{x_1 + iy_1}}{e^{x_2 + iy_2}} = \frac{e^{x_1} \cdot e^{iy_1}}{e^{x_2} \cdot e^{iy_2}} = \frac{e^{x_1}(\cos y_1 + i\sin y_1)}{e^{x_2}(\cos y_2 + i\sin y_2)}$$

$$= e^{x_1} \cdot e^{-x_2}\left[\frac{\cos y_1 + i\sin y_1}{\cos y_2 + i\sin y_2} \times \frac{\cos y_2 - i\sin y_2}{\cos y_2 - i\sin y_2}\right]$$

$$= e^{x_1 - x_2}\left[\frac{\cos y_1 \cos y_2 - i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 - i^2 \sin y_1 \sin y_2}{(\cos y_2)^2 - (i\sin y_2)^2}\right]$$

$$= e^{x_1 - x_2}\,\frac{(\cos y_1 \cos y_2 + \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 - \cos y_1 \sin y_2)}{\cos^2 y_2 - i^2 \sin^2 y_2}$$

$$= e^{x_1 - x_2}\,\frac{\cos(y_1 - y_2) + i\sin(y_1 - y_2)}{\cos^2 y_2 + \sin^2 y_2}$$

$$= e^{x_1 - x_2}\,\frac{e^{i(y_1 - y_2)}}{1} \qquad \because \cos^2 y_2 + \sin^2 y_2$$

$$= e^{x_1 - x_2} \cdot e^{iy_1 - iy_2}$$

$$= e^{x_1 - x_2 + iy_1 - iy_2}$$

$$= e^{(x_1 + iy_1) - (x_2 + iy_2)}$$

$$= e^{z_1 - z_2} = \text{R.H.S}$$

$$\Rightarrow \quad \frac{\exp z_1}{\exp z_2} = \exp(z_1 - z_2)$$

Periodic function:-

Let function f be defined in a domain $D \subseteq \mathbb{C}$ and let $\lambda \neq 0$ be a constant. Suppose that $\forall\ z \in D$ also $z+\lambda \in D$ then function f is said to be periodic function of period λ in D if $\forall\ z \in D$

$$f(z+\lambda) = f(z)$$

Ex:-

Show $f(z) = e^z$ is periodic function

Sol:-

$$f(z) = e^z$$

$$f(z+\lambda) = e^{z+\lambda} \quad \text{(For period)}$$

For periodic function

$$f(z+\lambda) = f(z)$$

$$e^{z+\lambda} = e^z \quad \text{which is possible only}$$

if $z = 0$

then $e^{0+\lambda} = e^0$

$$\Rightarrow \quad e^{\lambda} = 1$$

Take $\lambda = \alpha + i\beta$

$$e^{\lambda} = e^{\alpha+i\beta} = e^{\alpha} \cdot e^{i\beta}$$

$$e^{\lambda} = e^{\alpha}[\cos\beta + i\sin\beta] = 1$$

$$e^{\alpha}\cos\beta + ie^{\alpha}\sin\beta = 1$$

$$\Rightarrow e^{\alpha}\cos\beta = 1, \quad e^{\alpha}\sin\beta = 0$$

$e^{\alpha} \neq \quad \sin\beta = 0 \Rightarrow \beta = n\pi$; n is integer.

$e^{\alpha}\cos(n\pi) = 1 \Rightarrow e^{\alpha}(-1)^n = 1$ Take $n = 2k$.

$$\Rightarrow e^{\alpha} = e^0 \Rightarrow \alpha = 0, \quad \beta = n\pi$$

$\lambda = 0 + in\pi \quad ; n = 2k$

$\lambda = 2\pi k i$ is period of exponential function.

Hence, $f(z) = e^z$ is periodic function because

$$e^{z+2\pi k i} = e^z$$

$$\Rightarrow \quad e^{z+\lambda} = e^z$$

$$f(z+\lambda) = f(z)$$

Ex:-

Prove that $\overline{(e^{z})} = e^{\bar{z}}$

Show that $e^{\bar{z}}$ is not analytic function.

Sol:-

$\overline{(e^{z})} = e^{\bar{z}}$ $\qquad \because z = x+iy$

L.H.S $= \overline{(e^{z})}$ $\qquad \bar{z} = x-iy$

$e^{z} = e^{x+iy} = e^{x}.e^{iy} = e^{x}(\cos y + i\sin y)$

$\overline{e^{z}} = \overline{e^{x+iy}} = e^{x}(\cos y - i\sin y) = e^{x}\cdot e^{-iy}$

$\overline{e^{z}} = e^{x-iy}$

$\overline{e^{z}} = e^{\bar{z}} = R.H.S$

$\Rightarrow \overline{e^{z}} = e^{\bar{z}}$

$e^{\bar{z}}$ involves $\bar{z}$, so $e^{\bar{z}}$ is not analytic function.

Trignometric function:-

The function involves only Sine and cosine are called Trignometric functions and also called sinosidal function.

From the equation.

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

It follows that

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

For every real number "x". It is therefore natural to define the sine and cosine function of complex variable z by means of equations.

Ex:-

Prove that $\sin z$ and $\cos z$ are periodic functions with primitive (fundamental) period 2π.

Sol:-

We have to prove

(i) $\sin(z+2\pi) = \sin z$

put $z=0$ For period $\sin(z+\lambda)=\sin z$

$\sin(0+\lambda) = \sin(0)$

$\sin\lambda = 0$

$\Rightarrow \lambda = n\pi \quad \forall$ n is integer.

For periodic function we have

$\sin(z+n\pi) = (-1)^n \sin z$

It will be equal to $\sin z$ if $n = \text{even} = 2k$

$\sin(z+2\pi) = (-1)^2 \sin z$, Thus $\lambda = 2k\pi$; $k = 0, \pm 1, \pm \text{---}$

$\Rightarrow \sin(z+2\pi) = \sin z$

and its preimitive period is 2π

(ii)

$\cos(z+2\pi) = \cos z$

put $z=0$ For period

$\cos(z+\lambda) = \cos z$

$\cos(0+\lambda) = \cos(0)$

$\cos\lambda = 1$

$\lambda = \cos^{-1}(1)$

$\lambda = n\pi$; n is integer.

For Periodic function

$f(z+\lambda) = f(z)$

$\Rightarrow \cos(z+\lambda) = \cos z$

$\Rightarrow$ Equality holdz if $n = 2k$

$\cos(z+n\pi) = (-1)^n \cos z$

$\cos(z+\lambda) = (-1)^{2k} \cos z$

$\cos(z+2k\pi) = 1 \cdot \cos z$; $k = 0, \pm 1, \pm 2, \pm 3 \text{---}$

$\cos(z+2\pi) = \cos z$.

Note :-

(i) If $f(z)$ is analytic in domain then C.R equations are true in domain.

(ii) If $f(z)$ is analytic, then it is differentiable at a point z_0 then C.R equations are true.

(iii) Periodic function does not lie in the real domain because its period (λ) is imaginary part.

Ex :-

Show that $\sin z$, $\cos z$ are analytic

Sol :-

(i) $\sin z$ is analytic :-

$$f(z) = \sin z$$
$$= \sin(x+iy)$$
$$u+iv = \sin x \cosh y + i \cos x \sinh y$$
$$\Rightarrow u = \sin x \cosh y, \quad v = \cos x \sinh y$$

C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

$$\frac{\partial u}{\partial y} = -\sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\cos x \cosh y = \cos x \cosh y, \quad -\sin x \sinh y = \sin x \sinh y$$

$\Rightarrow$ CR equations are true

Hence, $f(z) = \sin z$ is analytic.

(ii) Cos z is analytic :-

$$f(z) = \cos z$$
$$= \cos(x+iy)$$
$$u+iv = \cos x \cosh y - i \sin x \sinh y$$
$$\Rightarrow \quad u = \cos x \cosh y, \quad v = -\sin x \sinh y$$

C.R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial v}{\partial x} = -\cos x \sinh y$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y, \quad \frac{\partial v}{\partial y} = -\sin x \cosh y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$-\sin x \cosh y = -\sin x \cosh y, \quad \cos x \sinh y = +\cos x \sinh y$$

$\Rightarrow$ C.R equations are true

Hence, $f(z) = \cos z$ is analytic function

Ex :-

If $f(z) = \sin z$ is analytic function then show that $\frac{d}{dz}(\sin z) = \cos z$

Sol :-

$$f(z) = \sin z$$
$$f(z) = \sin(x+iy)$$
$$u+iv = \sin x \cosh y + i \cos x \sinh y$$
$$\Rightarrow \quad u = \sin x \cosh y, \quad v = \cos x \sinh y$$

$$\frac{\partial u}{\partial x} = \cos x \cosh y$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow (1)$$

Since, $f(z)$ is analytic function.

So, C.R equations are true

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

then $\frac{d}{dz}(f(z)) = \frac{d}{dz} f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

Putting all values

$$\frac{d}{dz}(\sin z) = \cos x \cosh y - i \sin x \sinh y$$
$$= \cos(x+iy)$$

$$\frac{d}{dz}(\sin z) = \cos z$$

Ex:-

If $f(z) = \cos z$ is analytic function then show that $\frac{d}{dz}(\cos z) = \sin z$

Sol:-

$$f(z) = \cos z$$
$$f(z) = \cos(x+iy)$$
$$u+iv = \cos x \cosh y - i \sin x \sinh y$$
$$\Rightarrow \quad u = \cos x \cosh y \quad , \quad v = -\sin x \sinh y$$
$$\frac{\partial u}{\partial x} = -\sin x \cosh y$$
$$\frac{\partial u}{\partial y} = \cos x \sinh y$$

$$\frac{d}{dz}(f(z)) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Since, $f(z) = \sin z$ is analytic function

So, C.R equations are true.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

then $\frac{d}{dz}(f(z)) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

putting all values

$$\frac{d}{dz}(\cos z) = -\sin x \cosh y - i \cos x \sinh y$$

$$\frac{d}{dz}(\cos z) = -(\sin x \cosh y + i \cos x \sinh y)$$
$$= -\sin(x+iy)$$

$$\frac{d}{dz}(\cos z) = -\sin z$$

Ex:-

Find zeros of $\sin z$ and $\cos z$

Sol:-

Zeros of $\sin z$:-

zeros of a function mean the values of z for which $f(z)=0$ are called zeros of $f(z)$

$$f(z) = \sin z$$
put $f(z) = 0$
$$\Rightarrow \sin z = 0$$
$$\sin(x+iy) = 0$$
$$\sin x \cosh y + i \cos x \sinh y = 0 + i0$$
$$\Rightarrow \sin x \cosh y = 0 \rightarrow (1), \quad \cos x \sinh y = 0 \rightarrow (2)$$

From (1)

Either $\sin x = 0$ or $\cosh y = 0$

But $\cosh y \neq 0$ because $\cosh y = \frac{e^y + e^{-y}}{2}$

Therefore, we take $\sin x = 0$

$$\Rightarrow x = \sin^{-1}(0)$$
$$\Rightarrow x = n\pi \text{ ; } n \text{ is integer}$$

putting $x = n\pi$ in (2)

$$\cos(n\pi)\sinh y = 0$$
$$(-1)^n \sinh y = 0$$
$$\Rightarrow \sinh y = 0$$
$$\frac{e^y - e^{-y}}{2} = 0$$
$$\Rightarrow e^y - e^{-y} = 0$$

$\Rightarrow \quad e^{y} = e^{-y}$

$\Rightarrow \quad \dfrac{e^{y}}{e^{-y}} = 1$

$\Rightarrow \quad e^{y}\cdot e^{y} = 1$

$\Rightarrow \quad e^{2y} = 1$

$\ln e^{2y} = \ln(1)$

$2y = 0$

$\Rightarrow \quad y = 0$

$z = x + iy$

$z = n\pi + i(0)$

$z = n\pi$ is zeros of $\sin z$.

Note :-

Zeros of $\tan z$ are same as that of $\sin z$ because $\tan z = \dfrac{\sin z}{\cos z}$

Take $\tan z = 0$

$\dfrac{\sin z}{\cos z} = 0 \rightarrow (1)$ then solve it by Taking $\sin z = 0$

(ii) Zeros of $\cos z$:-

$f(z) = \cos z$

put $f(z) = 0$

$\Rightarrow \quad \cos z = 0$

$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y = 0$

$\Rightarrow \cos x \cosh y - i \sin x \sinh y = 0 + i0$

$\Rightarrow \quad \cos x \cosh y = 0 \rightarrow (1) \qquad \sin x \sinh y = 0 \rightarrow (2)$

From (2) Either $\sin x = 0$ or $\sinh y = 0$

But we take $\sin x = 0$

$\Rightarrow x = n\pi$; n is an integer

put $x = n\pi$ in (1)

$\cos n\pi \cosh y = 0$; [illegible]

$\Rightarrow \quad x = [illegible]\pi$; n is [illegible]

$\cosh y = 0$ (impossible)

because $\frac{e^y+e^{-y}}{2}=0$

$\Rightarrow e^y+\frac{1}{e^y}=0$

$\Rightarrow e^{2y}+1=0$

$\Rightarrow e^{2y}=-1$ (which is wrong) $\because e^x>0$

Now from (1) $\cos x \cosh y=0$

Either $\cos x=0$ or $\cosh y=0$ but $\cosh y\neq 0$

$\therefore$ we take $\cos x=0$

$\Rightarrow x=\left(\frac{2n+1}{2}\right)\pi$; n is integer

put this in (2) $\Rightarrow \sin\left(\left(\frac{2n+1}{2}\right)\pi\right)\sinh y=0$

$\Rightarrow (-1)^n \sinh y=0 \Rightarrow \sinh y=0$ because $(-1)^n\neq 0$

So, $\sinh y=\frac{e^y-e^{-y}}{2}=0$

$\Rightarrow e^{2y}-1=0 \Rightarrow e^{2y}=1$

$\Rightarrow e^{2y}=e^0 \Rightarrow 2y=0 \Rightarrow y=0$

So, $z=x+iy=\left(\frac{2n+1}{2}\right)\pi+i\cdot 0=(n+\frac{1}{2})\pi$

Ex:-

Find all values of z for which $\cos z=2$

Sol:-

$\cos z=2$ — In real $\cos x\neq 2$ about real line $-1\leq x\leq 1$

$\cos(x+iy)=2$

$\cos x\cosh y-i\sin x\sinh y=2+i\cdot 0$

$\Rightarrow \cos x\cosh y=2 \rightarrow (1)$

$\sin x\sinh y=0\rightarrow (2)$

From (2) $\sin x\sinh y=0$

Either $\sin x=0$ or $\sinh y=0$

we take $\sin x=0$ $\quad \frac{e^y-e^{-y}}{2}=\sinh y$

$\Rightarrow x=n\pi$; n is integer

put value of x in (1)

$\cos(n\pi)\cosh y=2$

$(-1)^n\frac{e^y+e^{-y}}{2}=2$

when is even

$e^y+e^{-y}=4$

$$e^y + \frac{1}{e^y} = 4$$

$$e^y \cdot e^y + 1 = 4e^y$$

$$e^{2y} + 1 = 4e^y$$

$$e^{2y} - 4e^y + 1 = 0$$

Let $t = e^y$

$$t^2 - 4t + 1 = 0$$

$a = 1$, $b = -4$, $c = 1$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{4 \pm \sqrt{16 - 4}}{2(1)}$$

$$t = \frac{4 \pm \sqrt{4(3)}}{2}$$

$$t = \frac{4 \pm 2\sqrt{3}}{2}$$

$$t = \frac{2(2 \pm \sqrt{3})}{2}$$

$$t = 2 \pm \sqrt{3}$$

$$e^y = 2 \pm \sqrt{3} \quad \because e^y = t$$

$$\ln e^y = \ln(2 \pm \sqrt{3})$$

$$y = \ln(2 \pm \sqrt{3})$$

$$z = x + iy$$

$$z = n\pi + i\ln(2 \pm \sqrt{3})$$

When n is odd then $e^y - e^{-y} = -4$

$$e^{2y} + 1 = -4e^y$$

$$e^{2y} + 4e^y + 1 = 0$$

Let $t = e^y$

$$t^2 + 4t + 1 = 0$$

$a = 1$, $b = 4$, $C = 1$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-4 \pm \sqrt{16-4}}{2(1)}$$

$$t = \frac{-4 \pm \sqrt{4(3)}}{2}$$

$$t = \frac{-4 \pm 2\sqrt{3}}{2}$$

$$t = -2 \pm \sqrt{3}$$

$$e^y = -2 \pm \sqrt{3}$$

$$\ln e^y = \ln(-2 \pm \sqrt{3})$$

$$y = \ln(-2 \pm \sqrt{3})$$

$$z = x + iy$$

$$z = n\pi + i\ln(-2 \pm \sqrt{3})$$

Ex:-

Find values of z for which $\sin z = 2i$

Sol:-

$$f(z) = \sin z$$

$$= \sin(x+iy) = 2i$$

$$\Rightarrow \sin x \cosh y + i \cos x \sinh y = 0 + 2i$$

$\Rightarrow \sin x \cosh y = 0 \rightarrow (1)$, $\cos x \sinh y = 2 \rightarrow (2)$

From (1) either $\sin x = 0$ or $\cosh y = 0$

$\frac{e^y + e^{-y}}{2} = 0$ (i-e) $\cosh y \neq 0$

$\therefore \sin x = 0$

$\Rightarrow x = n\pi$; n is integer.

From (2) put $x = n\pi$

$$\cos(n\pi) \sinh y = 2$$

$$(-1)^n \sinh y = 2 \quad ; n = 2k$$

$$\sinh y = 2$$

$$\frac{e^y - e^{-y}}{2} = 2$$

$$e^y - e^{-y} = 4$$

$$e^y - \frac{1}{e^y} = 4$$

$$e^{2y} - 1 = 4e^y$$

$$e^{2y} - 4e^{y} - 1 = 0$$

put $e^{y} = t$

$$t^2 - 4t - 1 = 0$$

$a = 1,\quad b = -4,\quad c = -1$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{4 \pm \sqrt{16 + 4}}{2}$$

$$t = \frac{4 \pm 2\sqrt{5}}{2}$$

$$t = 2 \pm \sqrt{5}$$

$$e^{y} = 2 \pm \sqrt{5}$$

$$\ln e^{y} = \ln(2 \pm \sqrt{5})$$

$$y = \ln(2 \pm \sqrt{5})$$

$$Z = x + iy = n\pi + i\ln(2 \pm \sqrt{5})$$

$$Z = 2k\pi + i\ln(2 \pm \sqrt{5})$$

Ex:-

Find all roots of equation

$$\sin z = \cosh 4$$

Sol:-

$$\sin z = \cosh 4 \rightarrow (1)$$

we know

$$\cosh y = \frac{e^{y} + e^{-y}}{2} \qquad \because \cosh 4 \text{ is constant}$$

$$\cosh 4 = \frac{e^{4} + e^{-4}}{2}$$

$$\cosh 4 = \frac{54.598 + 0.018}{2}$$

$$\cosh 4 = \frac{54.616}{2}$$

$\cosh y = 27.308$

; put this in (1)

$\sin z = 27.308$

$\sin(x+iy) = 27.308$

$\sin x \cosh y + i \sinh y \cos x = 27.308 + i \cdot 0$

$\Rightarrow \sin x \cosh y = 27.308 \to (1)$, $\sinh y \cos x = 0 \to (2)$

From (2) $\sinh y \cos x = 0$

Either $\sinh y = 0$ or $\cos x = 0$

$\Rightarrow x = (n+\frac{1}{2})\pi$; n is integer

Put $x = (n+\frac{1}{2})\pi$ in (1) we have

$\sin(n+\frac{1}{2})\pi \cosh y = 27.308$

$\Rightarrow (-1)^n \cosh y = 27.308$

$\Rightarrow (-1)^n \dfrac{e^y + e^{-y}}{2} = 27.308.$

When n is even:-

when n is even then $(-1)^n = 1$

and $\dfrac{e^y + e^{-y}}{2} = 27.308$

$\Rightarrow e^y + e^{-y} = 54.616$

$\Rightarrow e^{2y} + 1 = 54.616 e^y$

put $t = e^y$

$\Rightarrow t^2 + 1 = 54.616t$

$\Rightarrow t^2 - 54.616t + 1 = 0$

$$t = \frac{54.616 \pm \sqrt{(-54.616)^2 - 4}}{2}$$

$$t = \frac{54.616 \pm \sqrt{2978.907}}{2}$$

$$e^y = \frac{54.616 \pm \sqrt{2978.907}}{2}$$

$$y = \ln\left[\frac{54.616 \pm \sqrt{2978.907}}{2}\right]$$

$$z = x + iy$$

then $z = (n+\frac{1}{2})\pi + i\ln\left[\frac{54.616 \pm \sqrt{2978.907}}{2}\right]$

when n is odd :-

when n is odd then $(-1)^n = -1$

$$\Rightarrow \quad -1\left(\frac{e^y + e^{-y}}{2}\right) = 27.308$$

$$-e^y - e^{-y} = 54.616$$

$$e^y + e^{-y} = -54.616$$

$$\Rightarrow \quad e^y + \frac{1}{e^y} = -54.616$$

$$\Rightarrow \quad e^{2y} + 1 = -54.616e^y$$

put $t = e^y$

$$\therefore \quad t^2 + 1 + 54.616t = 0$$

$$\Rightarrow \quad t^2 + 54.616t + 1 = 0$$

$$t = \frac{-54.616 \pm \sqrt{(54.616)^2 - 4(1)(1)}}{2(1)}$$

$$t = \frac{-54.616 \pm \sqrt{2978.907}}{2}$$

put $t = e^y$

$$e^y = \frac{-54.616 \pm \sqrt{2978.907}}{2}$$

$$y = \ln\left[\frac{-54.616 \pm \sqrt{2978.907}}{2}\right]$$

$$z = x + iy$$

then

$$z = (n+\frac{1}{2})\pi + i\ln\left[\frac{-54.616 \pm \sqrt{2978.907}}{2}\right]$$

Ex:-

Prove that $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

Sol:-

(i) $\sin z = \sin(x+iy)$

$= \sin x \cosh y + i \cos x \sinh y$

$|\sin z|^2 = \sin^2 x \cos^2 hy + \cos^2 x \sin^2 hy$

$\because 1 = \cos^2 hy - \sin^2 hy$

put $\cos^2 hy = 1 + \sin^2 hy$

$|\sin z|^2 = \sin^2 x(1+\sin^2 hy) + \cos^2 x \sin^2 hy$

$= \sin^2 x + \sin^2 x \sin^2 hy + \cos^2 x \sin^2 hy$

$= \sin^2 x + \sin^2 hy(\sin^2 x + \cos^2 x)$

$|\sin z|^2 = \sin^2 x + \sin^2 hy$

(ii)

$\cos z = \cos(x+iy)$

$\cos z = \cos x \cosh y - i \sin x \sinh y$

$|\cos z|^2 = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y$

$\because \cos^2 hy - \sin^2 hy = 1$

put $\cos^2 hy = 1 + \sin^2 hy$

$|\cos z|^2 = \cos^2 x(1+\sin^2 hy) + \sin^2 x \sin^2 hy$

$= \cos^2 x + \cos^2 x \sinh^2 y + \sin^2 x \sin^2 hy$

$= \cos^2 x + \sinh^2 y(\cos^2 x + \sin^2 x)$

$|\cos z|^2 = \cos^2 x + \sin^2 hy$

Ex:-

Show that

(i) $\overline{\sin z} = \sin \bar{z}$

(ii) $\overline{\cos z} = \cos \bar{z}$

(iii) $\cos \overline{iz} = \overline{\cos iz}$ $\forall z$

(iv) $\sin \overline{iz} = \overline{\sin iz}$ iff $z = n\pi i$, n is integer

Sol:-

(i) $\overline{\sin z} = \sin \bar{z}$

$\sin z = \sin(x+iy)$

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\overline{\sin z} = \sin x \cosh y - i \cos x \sinh y$

$R \cdot H \cdot S = \sin \bar{z}$

$= \sin (\overline{x + iy})$

$= \sin(x - iy)$

$= \sin x \cosh y - i \cos x \sinh y = L \cdot H \cdot S$

$\Rightarrow \quad \overline{\sin z} = \sin \bar{z}$

(ii)

$\overline{\cos z} = \cos \bar{z}$

$L \cdot H \cdot S = \overline{\cos z}$

$\cos z = \cos(x + iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$\overline{\cos z} = \cos x \cosh y + i \sin x \sinh y$

$R \cdot H \cdot S = \cos \bar{z}$

$= \cos(\overline{x + iy})$

$= \cos(x - iy)$

$= \cos x \cosh y + i \sin x \sinh y = L \cdot H \cdot S$

$\Rightarrow \quad \overline{\cos z} = \cos \bar{z}$

(iii)

$\cos \overline{iz} = \overline{\cos iz} \quad \forall z$

$R \cdot H \cdot S = \overline{\cos iz}$

$\Rightarrow \cos iz = \cos i(x + iy)$

$= \cos(ix + i^2 y)$

$= \cos(ix - y)$

$= \cos[-(y - ix)]$

$= \cos(y - ix)$

$\cos iz = \cos y \cosh x + i \sin y \sinh x$

$\overline{\cos iz} = \cos y \cosh x - i \sin y \sinh x$

$L \cdot H \cdot S = \cos \overline{iz}$

$iz = i(x + iy)$

$= (ix + i^2 y)$

$iz = (ix - y)$

$\overline{iz} = (-ix-y)$
$\overline{iz} = -(ix+y)$
$\cos(\overline{iz}) = \cos(-(ix+y))$
$\cos\overline{iz} = \cos(y+ix)$
$\cos\overline{iz} = \cos y\cosh x - i\sin y\sinh x = R.H.S$
$\Rightarrow \cos\overline{iz} = \overline{\cos iz} \quad \forall z$

(iv)
$\sin\overline{iz} = \overline{\sin iz}$ iff $z = n\pi i$; n is integer

Let $\sin iz = \sin i(x+iy)$
$= \sin(ix-y)$
$= \sin(-y+ix)$
$= \sin(-y)\cos(ix) + \cos(-y)\sin(ix)$
$\sin iz = -\sin y\cosh x + i\cos y\sinh x$

R.H.S $\overline{\sin iz} = -\sin y\cosh x - i\cos y\sinh x$

L.H.S $\sin\overline{iz}$
$\overline{iz} = \overline{i(x+iy)}$
$= (ix+i^2y)$
$iz = (ix-y)$
$\overline{iz} = -ix-y$
$\overline{iz} = -(ix+y)$
$\overline{iz} = -(y+ix)$
$\sin(\overline{iz}) = \sin(-(y+ix))$
$= -\sin(y+ix)$
$= -(\sin y\cosh x + i\cos y\sinh x)$
$\sin(\overline{iz}) = -\sin y\cosh x - i\cos y\sinh x$
$\Rightarrow \sin(\overline{iz}) = \overline{\sin iz}$

Hyperbolic function:-

The hyperbolic Sine and Cosine of a complex variable defined as they are with a real variable that is

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

Ex:-

Show that $\sinh z$ and $\cosh z$ are analytic functions.

Sol:-

(i) we have to prove $\sinh z$ is analytic function

$$f(z) = \sinh z$$

$$\frac{d}{dz}(f(z)) = f'(z) = \frac{d}{dz}(\sinh z)$$

$$\frac{d}{dz}(f(z)) = \cosh z$$

$$\frac{d}{dz}(\sinh z) = \cosh z$$

and $$\frac{d}{dz}(\cosh z) = \sinh z$$

Derivatives exist Therefore, $\sinh z$ and $\cosh z$ are analytic functions

Ex:-

If $\cosh z$ is analytic function then show that $\frac{d}{dz}(\cosh z) = \sinh z$

Sol:-

$$f(z) = \cosh z$$

$$f(z) = \cos(iz)$$

$$= \cos(i(x+iy))$$

$$= \cos(ix - y)$$

$$f(z) = \cos(y)\cosh x + i \sin y \sinh x$$

$$U+iV = \cos y \cosh x + i \sin y \sinh x$$

$$\Rightarrow \quad U = \cos y \cosh x \quad , \quad V = \sin y \sinh x$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Since $f(z) = \cosh z$ is analytic so, C.R equations are true

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

then

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = \cos y \sinh x \quad , \quad \frac{\partial u}{\partial y} = -\sin y \cosh x$$

put in (1)

$$f'(z) = \cos y \sinh x - i(-\sin y \cosh x)$$

$$\frac{d}{dz}(\cosh z) = \cos y \sinh x + i \sin y \cosh x$$

$$= \sin(ix+y) = \sinh x \cos(-y) - i \sin(-y) \cosh x)$$

$$\frac{d}{dz}(\cosh z) = \sinh z$$

Ex :-

If $\sinh z$ is analytic function then show that $\frac{d}{dz}(\sinh z) = \cosh z$

Sol :-

$$f(z) = \sinh z$$

$$= \frac{\sin(iz)}{i}$$

$$= \frac{\sin i(x+iy)}{i}$$

$$= \frac{\sin(ix-y)}{i}$$

$$f(z) = \frac{-\sin(y-ix)}{i}$$

$$f(z) = \frac{+(\sin y \cosh x - i \cos y \sinh x)}{i}$$

$$f(z) = -\frac{1}{i} \sin y \cosh x + \cos y \sinh x$$

$$f(z) = \cos y \sinh x - \frac{1}{i} \sin y \cosh x$$

$$u + iv = \cos y \sinh x - \frac{1}{i} \sin y \cosh x$$

$$\Rightarrow u = \cos y \sinh x \quad , \quad iv = -\frac{1}{i} \sin y \cosh x$$

$$u = \cos y \sinh x \quad , \quad v = \frac{-1}{i^2} \sin y \cosh x$$

$$u = \cos y \sinh x \quad , \quad v = \sin y \cosh x$$

$$\frac{\partial u}{\partial x} = \cos y \cosh x$$

$$\frac{\partial u}{\partial y} = -\sin y \sinh x$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Since, $f(z) = \sinh z$ is analytic function
So, C.R equations are true

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y}$$

Putting all values

$$\frac{d}{dz}(f(z)) = \cos y \cosh x - i(-\sin y \sinh x)$$

$$= \cos y \cosh x + i \sin y \sinh x$$

$$= \cos i(x + iy)$$

$$\frac{d}{dz}(\sinh z) = \cosh z$$

Ex :-

Show that $\sinh z$ and $\cosh z$ are periodic functions

Sol :-

First, we have to show $\sinh z$ is periodic function.

For period

$\sinh(z+\lambda) = \sinh z \rightarrow (1)$

put $z=0$

$\sinh(0+\lambda) = \sinh(0)$

$\sinh(\lambda) = 0$

$\Rightarrow \lambda = 2n\pi i$; n is integer.

For periodic function

put $\lambda = 2n\pi i$ in (1)

$\sinh(z \pm 2n\pi i) = \sinh z$ where $n = 0, \pm 1, \pm 2 \ldots$

$\Rightarrow \sinh(z+\lambda) = \sinh z$

Hence, $\sinh z$ is periodic function.

Now, we have to show $\cosh z$ is periodic function.

For period put $z=0$

$\cosh(z+\lambda) = \cosh z \rightarrow (2)$

$\cosh(0+\lambda) = \cosh(0)$

$\cosh(\lambda) = \cos(0)$

$\cosh(\lambda) = 1$

$\Rightarrow \lambda = 2n\pi i$; n is integer

For periodic function

put $\lambda = 2n\pi i$ in (2)

$\cosh(z+2n\pi i) = \cosh z$; $n = 0, \pm 1, \pm 2, \ldots$

$\cosh(z+\lambda) = \cosh z$

Hence, $\cosh z$ is periodic function.

Therefore, the fundamental period of $\sinh z$ and $\cosh z$ is $\lambda = 2\pi i$

Ex :-

Solve the followings

(1) $\sinh z = 0$

(ii) $\cosh z = 0$

Sol :-

$\Rightarrow$ $\sinh z = 0$

$\dfrac{\sin(iz)}{i} = 0$

$\Rightarrow \sin(iz) = 0$

$\Rightarrow \sin i(x+iy) = 0$

$\Rightarrow \sin(ix + i^2y) = 0$

$\Rightarrow \sin(ix - y) = 0$

$\Rightarrow \sin(-(y - ix)) = 0$

$\Rightarrow -\sin(y - ix) = 0$

$\Rightarrow -[\sin y \cosh x - i\cos y \sinh x] = 0$

$\Rightarrow -\sin y \cosh x + i\cos y \sinh x = 0$

$\Rightarrow -\sin y \cosh x = 0 \rightarrow (1)$ $\cos y \sinh x = 0 \rightarrow (2)$

From (1)

$\sin y \cosh x = 0$

Either $\sin y = 0$ or $\cosh x = 0$

but $\cosh x \neq 0$

$\sin y = 0$

$\Rightarrow y = n\pi$; n is integer

put $y = n\pi$ in (2)

$\cos(n\pi) \sinh x = 0$

$(-1)^n \sinh x = 0$ $\therefore n = 2k$

$\sinh x = 0$

$\dfrac{e^x - e^{-x}}{2} = 0$

$e^x - e^{-x} = 0$

$e^x - \dfrac{1}{e^x} = 0$

$e^{2x} - 1 = 0$

$\Rightarrow e^{2x} = 1$

$= \ln e^{2x} = \ln(1)$

$$2x = 0$$
$$\Rightarrow \quad x = 0$$
then $\quad z = x+iy$
$$z = 0 + i(n\pi) \qquad \therefore n = 2k$$
$$\Rightarrow \quad z = 2k\pi i$$

(ii)

$$\cosh z = 0$$
$$\cos(iz) = 0$$
$$\cos(i(x+iy)) = 0$$
$$\cos(ix - y) = 0$$
$$\cos(-(y - ix)) = 0$$
$$\Rightarrow \quad \cos(y - ix) = 0$$
$$\Rightarrow \quad \cos y \cosh x + i \sin y \sinh x = 0$$
$$\Rightarrow \quad \cos y \cosh x = 0 \rightarrow (1), \quad \sin y \sinh x = 0 \rightarrow (2)$$

From (1)

$$\cos y \cosh x = 0$$

Either $\cos y = 0$ or $\cosh x = 0$

but $\quad \cosh x \neq 0$

$$\cos y = 0$$
$$\Rightarrow \quad y = \left(\frac{2n+1}{2}\right)\pi \qquad ; n \text{ is integer}$$

put in (2)

$$\sin\left(\left(\frac{2n+1}{2}\right)\pi\right) \sinh x = 0$$
$$(-1)^{n} \sinh x = 0 \qquad \text{(} n \text{ is equal to } 2k\text{)}$$
$$\sinh x = 0 \qquad (-1)^n \neq 0$$
$$e^{x} - e^{-x} = 0$$
$$e^{x} = e^{-x}$$
$$e^{x} = \frac{1}{e^{x}}$$
$$e^{2x} = 1$$
$$\ln e^{2x} = \ln(1)$$
$$2x = 0$$
$$\Rightarrow \quad x = 0$$
$$\Rightarrow \quad z = x + iy$$

$$z = 0 + i\left(\frac{2n+1}{2}\right)\pi \qquad \therefore n = 2k$$

$$z = \left(n+\tfrac{1}{2}\right)\pi i$$

$$z = \left(2k+\tfrac{1}{2}\right)\pi i$$

Show that

(i) $\sinh(-z) = -\sinh z$

(ii) $\cosh(-z) = \cosh z$

(iii) $\sin(-z) = -\sin z$

(iv) $\cos(-z) = \cos z$

(v) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

(vi) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$

(vii) $\sin^2 z + \cos^2 z = 1$

(viii) $1 + \tan^2 z = \sec^2 z$

(ix) $1 + \cot^2 z = \csc^2 z$

(x) $\cos(2z) = 1 - 2\sin^2 z = 2\cos^2 z - 1$

(xi) $\cos(3z) = 4\cos^3 z - 3\cos z$ (xii) $3\sin z - 4\sin^3 z = \sin 3z$

Sol:-

(i) $\sinh(-z) = -\sinh z$

Sol:-

$\sinh(-z) = \sinh z$

$$L.H.S = \sinh(-z)$$

$$= \frac{\sin i(-z)}{i}$$

$$= \frac{\sin i(-(x+iy))}{i}$$

$$= -\frac{\sin i(x+iy)}{i}$$

$$= -\frac{\sin(ix-y)}{i}$$

$$= -\frac{\sin(-(y-ix))}{i}$$

$$\sinh(-z) = \frac{\sin(y-ix)}{i}$$

$$\text{R.H.S} = -\sinh z$$
$$= -\frac{\sin(iz)}{i}$$
$$= -\frac{\sin i(x+iy)}{i}$$
$$= -\frac{\sin(ix-y)}{i}$$
$$= +\frac{\sin(y-ix)}{i}$$

$$-\sinh z = \frac{\sin(y-ix)}{i}$$

$$\Rightarrow \quad \sinh(-z) = -\sinh z$$

(ii)

$$\cosh(-z) = \cosh z$$
$$\cos i(-(x+iy)) = \cos i(x+iy)$$
$$\cos i(x+iy) = \cos i(x+iy)$$

(or)

$$\cosh(-z) = \cos i(-(x+iy))$$
$$= \cos i(x+iy)$$
$$= \cos(ix-y)$$
$$= \cos(-(y-ix))$$
$$= \cos(y-ix)$$
$$\cos(h(-z)) = \cos y \cosh x + i \sin y \sinh x$$
$$\cosh z = \cos i(z)$$
$$= \cos i(x+iy)$$
$$= \cos(ix-y)$$
$$= \cos(-(-ix+y))$$
$$= \cos(y-ix)$$
$$= \cos y \cosh x + i \sin y \sinh x$$
$$\Rightarrow \quad \cosh(-z) = \cosh z$$

(iii) $\sin(-z) = -\sin z$

$L.H.S = \sin(-z)$

$= \sin(-z)$

$= \sin(-(x+iy))$ $= \sin(-x)\cosh(-y) + i\cos(-x)\sinh(-y)$

$= -\sin x\cosh y - i\cos x\sinh y$

$= -[\sin x\cosh y + i\cos x\sinh y]$

$= \sin(x+iy)$

$\sin(-z) = -\sin z$

(iv) $\cos(-z) = \cos z$

$L.H.S = \cos(-z)$

$= \cos(-(x+iy))$

$= \cos(-x)\cosh(-y) - i\sin(-x)\sinh(-y)$

$= \cos x\cosh y - i\sin x\sinh y$

$= \cos(x+iy)$

$\cos(-z) = \cos z$

(v)

$|\sinh z|^2 = \sinh^2 x + \sin^2 y$

$L.H.S = |\sinh z|^2$

$\sinh z = \frac{\sin iz}{i}$

$= \frac{\sin i(x+iy)}{i}$

$= \frac{\sin(ix-y)}{i}$

$= \frac{\sin(-(y-ix))}{i}$

$\sinh z = -\frac{\sin(y-ix)}{i}$

$$\sinh z = -\frac{1}{i}\left[\sin y \cosh x - i \cos y \sinh x\right]$$

$$= -\frac{i}{i^2}\left[\sin y \cosh x - i \cos y \sinh x\right]$$

$$= i\left[\sin y \cosh x - i \cos y \sinh x\right]$$

$$= i \sin y \cosh x - i^2 \cos y \sinh x$$

$$\sinh z = i \sin y \cosh x + \cos y \sinh x$$

$$\sinh z = \cos y \sinh x + i \sin y \cosh x$$

$$|\sinh z|^2 = \cos^2 y \sinh^2 x + \sin^2 y \cosh^2 x$$
$$= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$
$$= \sinh^2 x \cos^2 y + (1 + \sinh^2 x)\sin^2 y$$
$$\because \cosh^2 x - \sinh^2 x = 1$$
$$= \sinh^2 x \cos^2 y + \sin^2 y + \sinh^2 x \sin^2 y$$
$$= \sinh^2 x(\cos^2 y + \sin^2 y) + \sin^2 y$$
$$|\sinh z|^2 = \sinh^2 x + \sin^2 y = R.H.S$$
$$\Rightarrow |\sinh z|^2 = \sinh^2 x + \sin^2 y$$

(vi)

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

$$L.H.S = |\cosh z|^2$$

$$\cosh z = \cos iz$$
$$= \cos i(x+iy)$$
$$= \cos(ix - y)$$
$$= \cos(-(y - ix))$$
$$= \cos(y - ix)$$
$$\cosh z = \cos y \cosh x + i \sin y \sinh x$$

$$|\cosh z|^2 = \cos^2 y \cosh^2 x + \sin^2 y \sinh^2 x$$

put $\cosh^2 x = 1 + \sinh^2 x$

$$|\cosh z|^2 = \cos^2 y(1+\sinh^2 x) + \sin^2 y \sinh^2 x$$
$$= \cos^2 y + \sinh^2 x \cos^2 y + \sinh^2 x \sin^2 y$$
$$= \cos^2 y + \sinh^2 x(\cos^2 y + \sin^2 y)$$
$$= \cos^2 y + \sinh^2 x \quad = R.H.S$$
$$\Rightarrow |\cosh z|^2 = \sinh^2 x + \cos^2 y$$

(vii)

$$\sin^2 z + \cos^2 z = 1$$

$$L.H.S = \sin^2 z + \cos^2 z$$
$$\sin z = \sin(x+iy)$$
$$= \sin x \cosh y + i \cos x \sinh y$$
$$\sin^2 z = [\sin x \cosh y + i\cos x \sinh y]^2$$
$$\sin^2 z = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y + 2i \sin x \cosh y \cos x \sinh y \rightarrow (1)$$

$$\cos z = \cos(x+iy)$$
$$= \cos x \cosh y - i \sin x \sinh y$$
$$\cos^2 z = [\cos x \cosh y - i \sin x \sinh y]^2$$

$$\cos^2 z = \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y - 2i \cos x \cosh y \sin x \sinh y \rightarrow (2)$$

Adding (1) and (2)

$$\sin^2 z + \cos^2 z = \sin^2 x \cosh^2 y - \cos^2 x \sinh^2 y + \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y + 2i \cos x \cosh y \sin x \sinh y - 2i \cos x \cosh y \sin x \sinh y$$

$$\sin^2 z + \cos^2 z = \sin^2 x \cosh^2 y + \cos^2 x \cosh^2 y - \cos^2 x \sinh^2 y + \sin^2 x \sinh^2 y$$
$$= (\sin^2 x + \cos^2 x)\cosh^2 y - \sinh^2 y(\sin^2 x + \cos^2 x)$$

$$= 1 \cdot \cosh^2 y - \sinh^2 y$$
$$\sin^2 z + \cos^2 z = 1 \quad = R.H.S$$
$$\Rightarrow \sin^2 z + \cos^2 z = 1$$

(viii) $1+\tan^2 z = \sec^2 z$

$$\text{L.H.S} = 1+\tan^2 z$$

$$\tan^2 z = \frac{\sin^2 z}{\cos^2 z}$$

$$1+\tan^2 z = \frac{\sin^2 z}{\cos^2 z} + 1$$

$$1+\tan^2 z = \frac{\sin^2 z + \cos^2 z}{\cos^2 z}$$

$$1+\tan^2 z = \frac{1}{\cos^2 z} \qquad \therefore \sin^2 z + \cos^2 z = 1$$

$$\therefore \frac{1}{\cos^2 z} = \sec^2 z$$

$$1+\tan^2 z = \sec^2 z$$

(ix)

$$1+\cot^2 z = \operatorname{cosec}^2 z$$

$$\cot^2 z = \frac{\cos^2 z}{\sin^2 z}$$

$$1+\cot^2 z = 1+\frac{\cos^2 z}{\sin^2 z}$$

$$= \frac{\sin^2 z + \cos^2 z}{\sin^2 z} \qquad \therefore \sin^2 z + \cos^2 z = 1$$

$$\therefore \frac{1}{\sin^2 z} = \operatorname{cosec}^2 z$$

$$= \frac{1}{\sin^2 z}$$

$$1+\cot^2 z = \operatorname{cosec}^2 z$$

(x)

$$\cos 2z = 1-2\sin^2 z = 2\cos^2 z - 1$$

$$\text{L.H.S} = \cos 2z$$

$$\cos(2z) = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1-2\sin^2 z$$

$$\cos(2z) = \frac{e^{i(2z)} + e^{-i(2z)}}{2}$$

$=$

$$\cancel{\cos^2 x + \sin^2 x} \quad \cancel{\cos^2 hy + \sin^2 hy}$$

$$\cos^2 z - \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= \frac{e^{2iz}+e^{-2iz}+2}{4} + \frac{e^{2iz}+e^{-i2z}-2}{4}$$

$$= \frac{e^{2iz} + e^{-2iz} + e^{2iz} + e^{-2iz}}{4}$$

$$= \frac{2e^{2iz} + 2e^{-2iz}}{4}$$

$$\cos^2 z - \sin^2 z = \frac{e^{2iz} + e^{-2iz}}{2} = \cos(2z)$$

$$2\cos^2 z - 1 = 2\left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - 1$$

$$= \frac{\left(e^{iz} + e^{-iz}\right)^2}{2} - 1$$

$$= \frac{e^{2iz} + e^{-2iz} + 2}{2} - 1$$

$$= \frac{e^{2iz} + e^{-2iz} + 2 - 2}{2}$$

$$2\cos^2 z - 1 = \frac{e^{2iz} + e^{-2iz}}{2} = \cos(2z)$$

$$1 - 2\sin^2 z = 1 - 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= 1 + \frac{e^{2iz} + e^{-2iz} - 2}{2}$$

$$1 - 2\sin^2 z = \frac{2 + e^{2iz} + e^{-2iz} - 2}{2}$$

$$1 - 2\sin^2 z = \frac{e^{2iz} + e^{-2iz}}{2} = \cos(2z).$$

(xi) $\cos(3z) = 4\cos^3 z - 3\cos z = 3\sin z - 4\sin^3 z$

Sol:-

$$\text{L.H.S } \cos(3z) = \frac{e^{i(3z)} + e^{-i(3z)}}{2}$$

$$\text{R.H.S} = 4\cos^3 z - 3\cos z$$

$$= \cos z\left(4\cos^2 z - 3\right)$$

$$= \left(\frac{e^{iz} + e^{-iz}}{2}\right)\left(4\left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - 3\right)$$

$$4\cos^3 z - 3\cos z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(e^{2iz}+e^{-2iz}+2-3\right)$$

$$= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(e^{2iz}+e^{-2iz}-1\right)$$

$$= \frac{e^{2iz}e^{iz}+e^{-2iz}e^{iz}-e^{iz}+e^{2iz}\cdot e^{-iz}+e^{-2iz}\cdot e^{-iz}-e^{-iz}}{2}$$

$$= \frac{e^{3iz}+\cancel{e^{-iz}}-\cancel{e^{iz}}+\cancel{e^{iz}}+e^{-3iz}-\cancel{e^{-iz}}}{2}$$

$$= \frac{e^{3iz}+e^{-3iz}}{2}$$

$$4\cos^3 z - 3\cos z = \cos(3z)$$

(xii)

$$\sin(3z) = 3\sin z - 4\sin^3 z$$

$$= \sin z\,(3-4\sin^2 z)$$

$$= \frac{e^{iz}-e^{-iz}}{2i}\left(3-4\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2\right)$$

$$= \frac{e^{iz}-e^{-iz}}{2i}\left(3+4\left(\frac{e^{2iz}+e^{-2iz}-2}{4}\right)\right)$$

$$= \frac{e^{iz}-e^{-iz}}{2i}\left(3+e^{2iz}+e^{-2iz}-2\right)$$

$$= \frac{e^{iz}-e^{-iz}}{2i}\left(e^{2iz}+e^{-2iz}+1\right)$$

$$= \frac{e^{2iz}\cdot e^{iz}+e^{-2iz}\cdot e^{iz}+e^{iz}-e^{2iz-iz}-e^{-2iz}e^{-iz}-e^{-iz}}{2i}$$

$$= \frac{e^{3iz}+\cancel{e^{-iz}}+\cancel{e^{iz}}+\cancel{e^{iz}}-e^{-3iz}-\cancel{e^{-iz}}}{2i}$$

$$= \frac{e^{3iz}-e^{-3iz}}{2i}$$

$$\sin(3z) = \sin(3z)$$

Show that

(i) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

(ii) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \pm \sin z_1 \sin z_2$

(iii) $\cos^2 h z - \sin^2 h z = 1$

(iv) $\sinh 2z = 2 \sinh z \cosh z$

(v) $\cosh 2z = \cos^2 h z + \sin^2 h z$

Sol:-

(i) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

$L.H.S = \sin(z_1 \pm z_2)$

$$= \frac{e^{i(z_1 \pm z_2)} - e^{-i(z_1 \pm z_2)}}{2i}$$

$R.H.S = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

$$= \left(\frac{e^{iz_1} - e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2} + e^{-iz_2}}{2}\right) \pm \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2} - e^{iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1 + iz_2} + e^{iz_1 - iz_2} - e^{-iz_1 + iz_2} - e^{-iz_1 - iz_2}}{4i} - \frac{e^{iz_1 + iz_2} - e^{iz_1 - iz_2} + e^{iz_1 + iz_2} - e^{-iz_1 - iz_2}}{4i}$$

$$= \cancel{e^{iz_1 + iz_2}} + e^{iz_1 - iz_2} - e^{-iz_1 + iz_2} - \cancel{e^{-iz_1 - iz_2}} - \cancel{e^{iz_1 + iz_2}} + e^{iz_1 - iz_2} + e^{-iz_1 + iz_2} - \cancel{e^{-iz_1 - iz_2}} / 4i$$

$$= \frac{2e^{iz_1 - iz_2} - 2e^{-iz_1 + iz_2}}{4i}$$

$$= \frac{e^{i(z_1 - z_2)} - e^{-(z_1 - z_2)i}}{2i}$$

$$= \frac{e^{i(z_1 - z_2)} - e^{-i(z_1 - z_2)}}{2i} = L.H.S$$

$\Rightarrow \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

(ii)

(a) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(b) $\cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

(a) $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$L.H.S = \cos(z_1+z_2)$

$$= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2}$$

$R.H.S = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$$= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) - \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1+iz_2}+e^{iz_1-iz_2}+e^{-iz_1+iz_2}+e^{-iz_1-iz_2}}{4} - \frac{e^{iz_1+iz_2}-e^{-iz_1-iz_2}-e^{-iz_1+iz_2}+e^{-iz_1-iz_2}}{4i^2}$$

$$= e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} + e^{-i(z_1-z_2)} + e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{-i(z_1-z_2)} + e^{-i(z_1+z_2)} / 4$$

$$= \frac{2e^{i(z_1+z_2)} + 2e^{-i(z_1+z_2)}}{4}$$

$$= \frac{e^{i(z_1+z_2)} + e^{-i(z_1+z_2)}}{2} = \cos(z_1+z_2)$$

$\Rightarrow \cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(b)

$\cos(z_1-z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

$L.H.S = \cos(z_1-z_2)$

$$= \frac{e^{i(z_1-z_2)} + e^{-i(z_1-z_2)}}{2}$$

$R.H.S = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$

$$= \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) + \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right)$$

$$= \frac{e^{iz_1+iz_2}+e^{iz_1-iz_2}+e^{-iz_1+iz_2}+e^{-iz_1-iz_2}}{4} - e^{iz_1+iz_2} - e^{iz_1-iz_2}$$

$$\frac{-e^{-iz_1+iz_2}+e^{-iz_1-iz_2}}{4}$$

$$R.H.S = \cancel{e^{i(z_1+z_2)}} + e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} + \cancel{e^{-i(z_1+z_2)}} - \cancel{e^{i(z_1+z_2)}} + e^{i(z_1-z_2)} + e^{-i(z_1-z_2)} + \cancel{e^{-i(z_1+z_2)}} / 4$$

$$= \frac{2e^{i(z_1-z_2)} + 2e^{-i(z_1-z_2)}}{4}$$

$$= \frac{e^{i(z_1-z_2)} + e^{-i(z_1-z_2)}}{2} = \cos(z_1 - z_2)$$

$$\Rightarrow \cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$$

(iii)

$$\cos^2 hz - \sin^2 hz = 1$$

$$L.H.S = \cos^2 hz - \sin^2 hz$$

$$= \left(\frac{e^{z} + e^{-z}}{2}\right)^2 - \left(\frac{e^{z} - e^{-z}}{2}\right)^2$$

$$= \frac{e^{2z} + e^{-2z} + 2}{4} - \frac{e^{2z} + e^{-2z} - 2}{4}$$

$$= \frac{e^{2z} + e^{-2z} + 2 - e^{2z} - e^{-2z} + 2}{4}$$

$$= \frac{4}{4} = 1 = R.H.S$$

$$\Rightarrow \cos^2 hz - \sin^2 hz = 1$$

(iv)

$$\sinh 2z = 2 \sinh z \cosh z$$

$$L.H.S = \sinh 2z$$

$$= \frac{e^{2z} - e^{-2z}}{2}$$

$$R.H.S = 2 \sinh z \cosh z$$

$$= 2\left(\frac{e^{z} - e^{-z}}{2}\right)\left(\frac{e^{z} + e^{-z}}{2}\right)$$

$$= 2\left[\frac{e^{z+z} + e^{z-z} - e^{-z+z} - e^{-2z}}{4}\right]$$

$$= 2\left[\frac{e^{2z} + 1 - 1 - e^{-2z}}{4}\right]$$

$$= \frac{e^{2z} - e^{-2z}}{2} = \sinh(2z)$$

$$\Rightarrow \quad \sinh 2z = 2 \sinh z \cosh z$$

(V)

$$\cosh 2z = \cos^2 hz + \sin^2 hz$$

$$L.H.S = \cosh 2z$$
$$= \frac{e^{2z} + e^{-2z}}{2}$$

$$R.H.S = \cos^2 hz + \sin^2 hz$$
$$= \left(\frac{e^{z} + e^{-z}}{2}\right)^2 + \left(\frac{e^{z} - e^{-z}}{2}\right)^2$$
$$= \frac{e^{2z} + e^{-2z} + 2}{4} + \frac{e^{2z} + e^{-2z} - 2}{4}$$
$$= \frac{e^{2z} + e^{-2z} + 2 + e^{2z} + e^{-2z} - 2}{4}$$
$$= \frac{2e^{2z} + 2e^{-2z}}{4}$$
$$= \frac{e^{2z} + e^{-2z}}{2} = \cosh 2z$$

$$\Rightarrow \cosh 2z = \cos^2 hz + \sin^2 hz$$

Solve it

(i) $\sinh(z_1 \pm z_2)$

(ii) $\cosh(z_1 \pm z_2)$

(iii) $\sinh z = i$

Sol:-

(i) $\sinh(z_1 \pm z_2)$

$$= \sin i(z_1 \pm z_2)$$
$$= \sin(iz_1 \pm iz_2)$$

we know that

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\sinh(z_1 \pm z_2) = \sin(iz_1 \pm iz_2)$$
$$= \sin(iz_1)\cos(iz_2) \pm \cos(iz_1)\sin(iz_2)$$

$$\sinh(z_1 \pm z_2) = \sin i(z_1) \cos i(z_2) \pm \cos i(z_1) \sin i(z_2)$$
$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

(ii)

$\cosh(z_1 \pm z_2)$

(a) $\cosh(z_1 + z_2)$

(b) $\cosh(z_1 - z_2)$

Sol:-

(a) $\cosh(z_1 + z_2)$

$= \cos i(z_1 + z_2)$

$= \cos(iz_1 + iz_2)$

we know $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

$\cosh(z_1 + z_2) = \cos(iz_1)\cos(iz_2) - \sin(iz_1)\sin(iz_2)$

$= \cos i(z_1) \cos i(z_2) - \sin i(z_1) \sin i(z_2)$

$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2$

(b)

$\cosh(z_1 - z_2)$

$= \cos i(z_1 - z_2)$

$= \cos(iz_1 - iz_2)$

$= \cos(iz_1)\cos(iz_2) + \sin(iz_1)\sin(iz_2)$

$= \cos i(z_1) \cos i(z_2) + \sin i(z_1) \sin i(z_2)$

$= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

(iii)

$\sinh z = i$

$\dfrac{(\sin iz)}{i} = i$

$\sin(iz) = -1$

$\Rightarrow \sin[i(x+iy)] = -1$

$\sin(ix - y) = -1$

$-\sin(y - ix) = -1$

$-[\sin y \cosh x - i \cos y \sinh x] = -1$

$\Rightarrow -\sin y \cosh x + i \cos y \sinh x = -1 + 0 \cdot i$

$\Rightarrow \sinh x \cos y = 0, \quad -\cosh x \sin y = -1$

$\cosh x \sin y = 1 \rightarrow (1), \quad \sinh x \cos y = 0 \rightarrow (2)$

From eq (2)

$$\sinh x \cos y = 0$$

Either $\sinh x = 0$ or $\cos y = 0$

Take $\cos y = 0$

$$y = (n+\tfrac{1}{2})\pi$$

put in (1)

$$\cosh x \sin((n+\tfrac{1}{2})\pi) = 1$$

$$\cosh x \; (-1)^{n/2} = 1$$

Case I :-

when n is even

$$\cosh x = 1$$

$$\frac{e^x + e^{-x}}{2} = 1$$

$\Rightarrow e^x + e^{-x} = 2$

$\Rightarrow e^{2x} + 1 = 2e^x$

$\Rightarrow e^{2x} - 2e^x + 1 = 0$ let $e^x = t$

$$t^2 - 2t + 1 = 0$$

$$t = \frac{+2 \pm \sqrt{4 - 4(1)}}{2(1)}$$

$$t = \frac{+2 \pm \sqrt{0}}{2} \Rightarrow t = \frac{+2}{2} = +1$$

put $t = e^x$

$\Rightarrow e^x = 1 \Rightarrow \ln e^x = \ln 1 \Rightarrow x = 0 \quad \therefore z = x+iy$

$\Rightarrow z = [(n+\tfrac{1}{2})\pi]i$

Case II :- when n is odd

then $-1 \cosh x = 1$

$$e^x + e^{-x} = -2$$

$$e^{2x} + 1 = -2e^x$$

$$e^{2x} + 2e^x + 1 = 0$$

$$e^x = \frac{-2 \pm \sqrt{4-4}}{2}$$

$e^x = -2/2 = -1 \Rightarrow \ln e^x = \ln(-1) \Rightarrow x = \pi i$

$z = x + iy$

$z = \pi i + i(n+\tfrac{1}{2})\pi$

Find all roots of equation

(i) $\cosh z = 1/2$

(ii) $\cosh z = -2$

Sol:-

(i) $\cosh z = \frac{1}{2}$

$\cos(iz) = \frac{1}{2}$

$\cos[i(x+iy)] = 1/2$

$\cos(y-ix) = 1/2$

$\cos y \cosh x + i \sin y \sinh x = 1/2 + i0$

$\Rightarrow \cos y \cosh x = 1/2 \rightarrow (1) \quad \sin y \sinh x = 0 \rightarrow (2)$

From (2)

Either $\sin y = 0$ or $\sinh x = 0$

$\sin y = 0 \Rightarrow y = n\pi$

put in (1)

$\cos(n\pi)\cosh x = 1/2$

$(-1)^n \cosh x = 1/2$

when n is even

$\cosh x = 1/2 \Rightarrow \frac{e^x + e^{-x}}{2} = 1/2$

$\Rightarrow e^{2x} + 1 = e^x$

$\Rightarrow e^{2x} - e^x + 1 = 0 \quad$ put $e^x = t \Rightarrow t^2 - t + 1 = 0$

$t = \frac{1 \pm \sqrt{1-4}}{2} \quad t = \frac{1 \pm \sqrt{-3}}{2} \Rightarrow e^x = \frac{1 \pm \sqrt{-3}}{2}$

$x = \ln\left(\frac{1 \pm \sqrt{-3}}{2}\right) \Rightarrow z = \ln\left(\frac{1 \pm \sqrt{3}i}{2}\right) + in\pi$

When n is odd

then $\quad -1 \cosh x = 1/2 \Rightarrow e^x + e^{-x} = -1$

$e^{2x} + 1 = -e^{+x} \Rightarrow e^{2x} + e^x + 1 = 0 \quad$ put $e^x = t$

$t^2 + t + 1 = 0 \Rightarrow t = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$

$e^x = \frac{-1 \pm \sqrt{-3}}{2} \Rightarrow x = \ln\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$

$z = \ln\left(\frac{-1 \pm \sqrt{-3}}{2}\right) + in\pi$ or $\ln\left(\frac{-1 \pm \sqrt{3}i}{2}\right) + n\pi i$

(ii) $\cosh z = -2$

$\cos iz = -2$

$\cos i(x+iy) = -2$

$\cos(ix - y) = -2 \Rightarrow \cos(y - ix) = -2$

$\cos y \cosh x + i \sin y \sinh x = -2 + i0$

$\Rightarrow \cos y \cosh x = -2 \rightarrow (1)$, $\sin y \sinh x = 0 \rightarrow (2)$

From (2) Either $\sin y = 0$ or $\sinh x = 0$

Take $\sin y = 0 \Rightarrow y = n\pi$; n is integer

Eq (2) becomes $\cos(n\pi)\cosh x = -2$

$(-1)^n \cosh x = -2$

when n is even number

then $\cosh x = -2$

$\frac{e^x + e^{-x}}{2} = -2$

$\Rightarrow e^x + e^{-x} = -4$

$\Rightarrow e^{2x} + 1 = -4e^x \Rightarrow e^{2x} + 4e^x + 1 = 0$

put $t = e^x \Rightarrow t^2 + 4t + 1 = 0$

$t = \frac{-4 \pm \sqrt{16 - 4(1)(1)}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2}$

$t = -2 \pm \sqrt{3}$

put $t = e^x \Rightarrow e^x = -2 \pm \sqrt{3} \Rightarrow x = \ln(-2 \pm \sqrt{3})$

$z = \ln(-2 \pm \sqrt{3}) + in\pi$

when n is even number then

$-1 \cosh x = -2$

$\cosh x = 2 \Rightarrow e^x + e^{-x} = 4$

$\Rightarrow e^{2x} + 1 = 4e^x \Rightarrow e^{2x} - 4e^x + 1 = 0$ put $e^x = t$

$t^2 - 4t + 1 = 0 \Rightarrow t = \frac{4 \pm \sqrt{16 - 4}}{2}$

$t = \frac{4 \pm \sqrt{12}}{2}$

$t = \frac{4 \pm 2\sqrt{3}}{2}$

$t = 2 \pm \sqrt{3}$ put $t = e^x$

$e^x = 2 \pm \sqrt{3}$

$\Rightarrow x = \ln(2 \pm \sqrt{3})$

$z = \ln(2 \pm \sqrt{3}) + in\pi$; n is integer.

Logarithmic function :-

Let $\ln r$ denote the natural logarithm of a positive real number "r" [(1, 2, ---) non-negative] (0, 1, 2, --- ; $\ln 0 = \infty$)

$\therefore$ Logarithm of -ve number does not exists.

We now define the logarithm function of a non-zero complex variable $z = re^{i\theta}$ by means of equation.

$$\ln z = \ln r + i\theta \rightarrow (1)$$

$$\ln z = \ln|z| + i\arg(z) \rightarrow (2)$$

The value of θ used in the experior $z = re^{i\theta}$ for specific non-zero number z, of course not unique.

Note :-

(i) $f(z) = \ln z$ is analytic, if $\arg(z)$ is principal value ; if $-\pi < \arg(z) \leq \pi$

(ii) In complex number, $-\pi < \theta \leq \pi$, then θ is principal value.

EX :-

If Θ denote the principal value $(-\pi < \Theta \leq \pi)$ of θ. We can write

$$\theta = \Theta + 2n\pi \rightarrow (3) \quad ; n = 0, \pm 1, \pm 2, ---$$

$$\ln z = \ln r + i(\Theta + 2n\pi) \quad ; n = 0, \pm 1, \pm 2, ---$$

It is clear that function $\ln z$ is multi-valued with infinitely many values. These values all have the same real part, and their imaginary parts differ by integral multiple of 2π.

The principal value of $\ln z$ is obtained from eq (3) when $n = 0$

$$\ln z = \ln r + i\Theta$$

$$\log z = \log|z| + i\arg(z) \rightarrow (4)$$

This $\log z$ is analytic function (single value function) when we use value of $arg(z)$ as principal value; $(-\pi < arg(z) \leq \pi)$ otherwise, it is not analytic

$$\log z = \log(x+iy)$$

According to eq (4)

$$\log z = \log(\sqrt{x^2+y^2}) + i\tan^{-1}(y/x)$$

Ex :-

If $\log z = \sqrt{x^2+y^2} + i\tan^{-1}(y/x)$ then show that $\frac{d}{dz}(\log z) = 1/z$

Sol :-

$$z = x+iy$$

$$|z| = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

$$u = \log|z|$$

$$u = \log(\sqrt{x^2+y^2})$$

$$u = \frac{1}{2}\log(x^2+y^2)$$

$$v = \theta = \tan^{-1}(y/x)$$

$$f(z) = \log z = \log|z| + i\,arg(z)$$

$$\Rightarrow u+iv = \log(|z|) + i\,arg(z)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow (1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{2x}{(x^2+y^2)}$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) = \frac{1}{\frac{x^2+y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$$

$$f'(z) = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{\bar{z}}{|z|^2}$$

$$f'(z) = \frac{\bar{z}}{z\bar{z}} \qquad \because |z|^2 = z\bar{z}$$

$$\frac{d}{dz}(\log z) = \frac{1}{z}$$

Ex :-

Find Log z if

(i) $z = i$, (ii) $z = -i$ (iii) $z = -1+i$ (iv) $z = -1-i$

Sol :-

(i) $z = i$

$z = x+iy = i$

$\Rightarrow x = 0, \quad y = 1$

$\log z = \log |z| + i\arg(z) \qquad |z| = \sqrt{1} = 1$

$\log z = 0 + i\tan^{-1}(1/0) \qquad \arg(z) = \tan^{-1} y/x$

$= 0 + i\tan^{-1}(\infty)$

$= i\frac{\pi}{2}$

$\log z = \frac{\pi}{2} i$

(ii)

$z = -i$

$z = x+iy = -i$

$\Rightarrow x = 0, \quad y = -1$

$\log z = \log |z| + i\arg(z) \rightarrow (1)$

$|z| = \sqrt{x^2+y^2}$

$|z| = \sqrt{(0)^2+(-1)^2} = \sqrt{1} = 1 \qquad \log 1 = 0$

$\arg(z) = \tan^{-1}(y/x)$

$= \tan^{-1}(-\frac{1}{0})$

$\arg(z) = \tan^{-1}(-\infty)$

putting all values in (1)

$$\log z = 0 + i\left(-\frac{\pi}{2}\right)$$

$$\log z = -\frac{\pi}{2} i$$

(iii)

$$Z = -1 + i$$

$$z = x + iy = -1 + i$$

$$\Rightarrow x = -1, \quad y = 1$$

$$|z| = \sqrt{x^2 + y^2}$$

$$|z| = \sqrt{1+1} = \sqrt{2}$$

$$\arg(z) = \tan^{-1}(y/x)$$

$$= \tan^{-1}(1/-1)$$

$$= \tan^{-1}(-1)$$

$$\arg(z) = -\frac{\pi}{4}$$

$$\log z = \log|z| + i\arg(z)$$

$$= \log\sqrt{2} + i\left(-\frac{\pi}{4}\right)$$

$$= \tfrac{1}{2}\log(2) - \frac{\pi}{4} i$$

$$\log z = \frac{1}{2}\log(2) - \frac{\pi}{4} i$$

lo = ?

$$\log z = \frac{1}{2}\log(2) - \frac{\pi}{4} i$$

(iv)

$$Z = -1 - i$$

$$z = x + iy = -1 - i$$

$$\Rightarrow x = -1, \quad y = -1$$

$$|z| = \sqrt{1+1} = \sqrt{2}$$

$$\arg(z) = \tan^{-1}\left(-\frac{1}{-1}\right)$$

$$= \tan^{-1}(1)$$

$$\arg(z) = \frac{\pi}{4}$$

$$\log z = \log|z| + i\arg(z)$$

$$\log z = \log\sqrt{2} + i\left(\frac{\pi}{4}\right)$$

$$\log z = \frac{1}{2}\log(2) + \frac{\pi}{4}i$$

$$\log z = \frac{1}{2}\log(2) + \frac{\pi}{4}i$$

Ex :-

If $z_1 = -1$, $z_2 = -1$

prove that $\log z_1 z_2 = 2\pi i$

Sol:-

$$\log z_1 z_2$$

$$= \log z_1 + \log z_2$$

$$= \log(-1) + \log(-1) \rightarrow (1)$$

$$z_1 = -1 \Rightarrow x_1 = -1, \quad y_1 = 0$$

$$\log z_1 = \log|z_1| + i\arg(z_1)$$

$$\log(-1) = \log|-1| + i\arg(-1)$$

$\log|-1| = \log\sqrt{1} = \log 1 = 0$

$$= 0 + i\tan^{-1}(-0/1)$$

$$= 0 + i\tan^{-1}(0)$$

$$\log(-1) = +\pi i$$

$$z_2 = -1$$

$$\Rightarrow x_2 + iy_2 = -1$$

$$\Rightarrow x_2 = -1, \quad y_2 = 0$$

$$\log z_2 = \log|z_2| + i\arg(z_2)$$

$$= 0 + i\tan^{-1}(0)$$

$$\log z_2 = \pi i$$

putting all values in (1)

$$\log z_1 z_2 = \pi i + \pi i$$

$\ln(-1) = \pi i$
In real number its value does not exist.

$$= 2(\pi i)$$

$$= 2\pi i$$

Ex:-

Prove that $\left(\frac{ia-1}{ia+1}\right)^{ib} = e^{-2b\cot^{-1}(a)}$

where a and b are real

Sol:-

Consider

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = Exp\left[\log\left(\frac{ia-1}{ia+1}\right)^{ib}\right]$$

$$= Exp\left[ib\log\left(\frac{ia-1}{ia+1}\right)\right]$$

$$= Exp[ib(\log(ia-1) - \log(ia+1))] \rightarrow (1)$$

$$\log(ia-1) = \log|ia-1| + i\arg(ia-1)$$

$|ia-1| = \sqrt{1+a^2}$

$$\log(ia-1) = \log\sqrt{1+a^2} + i\tan^{-1}\left(\frac{a}{-1}\right)$$

$\arg(ia-1) = \tan^{-1}\left(\frac{a}{-1}\right)$

$$\log(ia+1) = \log\sqrt{1+a^2} + i\tan^{-1}(a)$$

$$\log(ia-1) - \log(ia+1) = \log\sqrt{1+a^2} + i\tan^{-1}(-a) - [\log\sqrt{1+a^2} + i\tan^{-1}(a)]$$

$$\log(ia-1) - \log(ia+1) = i\tan^{-1}(-a) - i\tan^{-1}(a)$$

$$= i[\tan^{-1}(-a) - \tan^{-1}(a)]$$

$$= i[\pi - \tan^{-1}(a) - \tan^{-1}(a)]$$

$$= i[\pi - 2\tan^{-1}(a)]$$

$$= 2i\left[\frac{\pi}{2} - \tan^{-1}(a)\right]$$

$$= 2i\cot^{-1}(a)$$

put this in (1)

$$\left(\frac{ia-1}{ia+1}\right)^{ib} = Exp[2i\cot^{-1}(a)](ib)$$

$$= e^{ib(2i\cot^{-1}(a))}$$

$$= e^{-2b\cot^{-1}(a)} \qquad \because i^2 = -1$$

Ex:-

Prove that $\tan^{-1}(-a) = \pi - \tan^{-1}(a)$

Sol:-

Let $\tan^{-1}(-a) = z$

$\Rightarrow -a = \tan z$

$-a = \dfrac{\sin z}{\cos z}$

$-a = \dfrac{\sin(\pi - z)}{-\cos(\pi - z)}$

$-a = -\tan(\pi - z)$

$a = \tan(\pi - z)$

$\tan^{-1}(a) = \pi - z$

$\pi - z = \tan^{-1}(a)$

$\pi - \tan^{-1}(-a) = \tan^{-1}(a)$

$\Rightarrow \tan^{-1}(-a) = \pi - \tan^{-1}(a)$

Ex:-

Show that $\cot^{-1}(a) = \frac{\pi}{2} - \tan^{-1}(a)$

Sol:-

Let $\cot^{-1}(a) = z$

$a = \cot z$

$a = \dfrac{\cos z}{\sin z}$

$a = \dfrac{\sin(\pi/2 - z)}{\cos(\pi/2 - z)}$

$a = \tan\left(\frac{\pi}{2} - z\right)$

$\tan^{-1}(a) = \frac{\pi}{2} - z$

$\tan^{-1}(a) = \frac{\pi}{2} - \cot^{-1}(a)$

$\cot^{-1}(a) = \frac{\pi}{2} - \tan^{-1}(a)$

Ex :-

Prove that $Re[(1+i)^{\log(1+i)}] = e^{-\pi^2/16}\, 2^{1/4 \log 2}$

Sol :-

$$Z = (1+i)^{\log(1+i)}$$

Consider

$$(1+i)^{\log(1+i)} = Exp[\log(1+i)^{\log(1+i)}] \rightarrow (1)$$

$$\log(1+i)^{\log(1+i)} = \log(1+i)\log(1+i)$$

$$= [\log(1+i)]^2 \rightarrow (2)$$

$$[\log(1+i)]^2 = [\log|1+i| + i \arg(1+i)]^2$$

$$|1+i| = \sqrt{1+1} = \sqrt{2}$$

$$\arg(1+i) = \tan^{-1}(1/1) = \frac{\pi}{4}$$

$$[\log(1+i)]^2 = [\log\sqrt{2} + i\frac{\pi}{4}]^2$$

$$= [\frac{1}{2}\log(2) + i\frac{\pi}{4}]^2$$

$$= \frac{1}{4}(\log(2))^2 - \frac{\pi^2}{16} + i\log(2)\frac{\pi}{4}$$

$$[\log(1+i)]^2 = \frac{1}{4}\log(2)\log(2) - \frac{\pi^2}{16} + i\frac{\pi}{4}\log(2)$$

put in (1)

$$(1+i)^{\log(1+i)} = Exp[\frac{1}{4}\log(2)\log(2) - \frac{\pi^2}{16} + i\frac{\pi}{4}\log(2)]$$

$$Re[(1+i)^{\log(1+i)}] = Re[e^{\frac{1}{4}\log(2)\log(2) - \pi^2/16}\, \boxed{e^{i\frac{\pi}{4}\log(2)}}]$$

$$= Re[e^{\log(2)\frac{1}{4}\log(2)}\, e^{-\pi^2/16}\, \cancel{(\cos(\pi/4\log(2)) + i\sin(\pi/4\log(2)))}$$

$$Re[(1+i)^{\log(1+i)}] = Re[2^{\frac{1}{4}\log(2)} \cdot e^{-\pi^2/16}\, \boxed{\cos(\pi/4\log(2))}]]$$

Note :-

(i) $(-i)^{i} = e^{i\ln(-i)} = e^{i(-\frac{\pi}{2}i)} = e^{\pi/2}$

(ii) $(-1)^{2i} = e^{2i\ln(-1)} = e^{2i\times\pi i} = e^{-2\pi}$

(iii) If real part of function $f(z)$ is harmonic then $f(z)$ is analytic ; realpart $= u(x,y)$.

(i) and (ii) are obtained from

If z^{α}

where z is complex and α is also complex number then

$$z^{\alpha} = e^{\alpha \ln(z)}$$

Churchill's book (page 112)

Inverse Trignometric and hyperbolic function :-

Inverses of trignometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function $\sin^{-1}z$

we write $w = \sin^{-1}z$ where $z = \sin w$

(i-e) $w = \sin^{-1}z$ when

$$z = \frac{e^{iw} - e^{-iw}}{2i} \rightarrow (1)$$

If we put this equation in this form

$$(e^{iw})^2 - 2iz\,e^{iw} - 1 = 0$$

because (1) implies

$$2iz = e^{iw} - e^{-iw}$$

$$2iz = e^{iw} - \frac{1}{e^{+iw}}$$

$$2ize^{iw} = (e^{iw})^2 - 1$$

$$\Rightarrow (e^{iw})^2 - 2ize^{iw} - 1 = 0$$

which is quadratic in e^{iw} and solve it for e^{iw}

$a = 1, \quad b = -2iz, \quad c = -1$

$$e^{iw} = \frac{2iz \pm \sqrt{4i^2z^2 - 4(1)(-1)}}{2(1)}$$

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2+4}}{2}$$

$$e^{iw} = \frac{2iz \pm \sqrt{4(1-z^2)}}{2}$$

$$e^{iw} = \frac{2iz \pm 2\sqrt{1-z^2}}{2}$$

$$e^{iw} = \frac{2(iz \pm \sqrt{1-z^2})}{2}$$

$$e^{iw} = iz \pm [1-z^2]^{1/2} \rightarrow \text{(ii)}$$

where $(1-z^2)^{1/2}$ is, of course, a double valued function of z.

Taking logarithms of each side of (ii)

$$\log e^{iw} = \log[iz \pm [1-z^2]^{1/2}]$$

$$iw = \log[iz \pm [1-z^2]^{1/2}]$$

put $w = \sin^{-1} z$

$$\sin^{-1} z = -i \log[iz \pm (1-z^2)^{1/2}]$$

$\sin^{-1} z$ is multiple - valued function with infinitely many values at each point z.

Note :-

Usual formula solves the quadratic eq $az^2+bz+c=0$ $(a \neq 0)$ where a, b and c are complex numbers (co-efficient). Specifically, by completing the square on left-hand side. Quadratic formula is

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$ where both square roots are to be considered when $b^2-4ac \neq 0$

Ex :-

(i) $(-i)^{i} = e^{i\ln(-i)} = e^{\pi/2}$

(ii) $(-1)^{2i} = e^{2i\ln(-1)} = e^{-2\pi}$

Sol :-

(i) $(-i)^{i} = e^{\pi/2}$

we know that $z^{\alpha} = e^{\alpha\ln(z)}$

$(-i)^{i} = e^{i\ln(-i)} \rightarrow (1)$

$\ln z = \ln|z| + i\arg(z)$ where $z = -i$

$\ln(-i) = \ln(1) + i\tan^{-1}(-1/0)$ $|z| = \sqrt{(-i)^2} = \sqrt{1} = 1$

$\ln(-i) = 0 + i\tan^{-1}(-\infty) = 0 + i(\pi/2)$

put this result in (1)

$(-i)^{i} = e^{i(0+i\pi/2)}$

$= e^{-i^2\pi/2}$

$(-i)^{i} = e^{\pi/2}$

(ii) $(-1)^{2i} = e^{-2\pi}$

we know $z^{\alpha} = e^{\alpha\ln(z)} \Rightarrow (-1)^{2i} = e^{2i\ln(-1)} \rightarrow (a)$

$z = -1$

$x+iy = -1 + i\cdot 0 \Rightarrow x = -1,\ y = 0$

$|z| = \sqrt{(-1)^2} = 1$

$\arg(z) = \tan^{-1}(y/x)$

$= \tan^{-1}(0/-1)$

$\arg(z) = \tan^{-1}(0) = \pi$

$\ln z = \ln|z| + i\arg(z)$

putting all values

$\ln(-1) = \ln(1) + i(\pi)$

$\ln(-1) = 0 + \pi i$

putting this in (a)

$(-1)^{2i} = e^{2i(\pi i)}$

$(-1)^{2i} = e^{-2\pi}$

—— x —— x —— x —— x —— x —— x ——

Q: Show that Sin z, Cos z, Sinh z, Cosh z are not analytic and Sinx Sinhy v.v.v. Imp

Chapter No 4

Conformal Representation

Mappings or Transformations:-

If $f(x)$ is a real valued function of a real variable "x", then the equation $y = f(x)$ establishes a correspondence between points on x-axis and points on y-axis.

We can show the relationship $y = f(x)$ by means of a curve drawn on xy-plane.

When $f(z)$ is a complex valued function of a complex variable z, no such graphical representation is easily available for $w = f(z)$ because to represent each variable $z(x, y) = x + iy$ and $w(u, v) = u + iv$; we need separate planes for z and w.

Ex:-

Prove that a line $y = x - 1$ is mapped into a circle $u^2 + v^2 - u - v = 0$, under the transformation $w = \frac{1}{z}$. Locate the centre and radius of circle.

Sol:-

The transformation is $w = \frac{1}{z}$

$$w = \frac{\bar{z}}{z\bar{z}} \qquad \text{As } w = \frac{1}{z}$$

$$w = \frac{1}{x+iy}$$

$$w = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$w = \frac{x-iy}{x^2+y^2}$$

$$u + iv = \frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}$$

$$\Rightarrow \quad u = \frac{x}{x^2+y^2} \rightarrow (1), \qquad v = \frac{-y}{x^2+y^2} \rightarrow (2)$$

Eqs (1) and (2) are transformation equations.

$$\frac{u}{v} = \frac{\frac{x}{x^2+y^2}}{\frac{-y}{x^2+y^2}}$$

$$\frac{u}{v} = -x/y \rightarrow (3)$$

The given line is $y = x-1 \rightarrow (4)$

$$\frac{u}{v} = \frac{-x}{x-1} \qquad \because \; y = x-1$$

$$\Rightarrow \quad u(x-1) + xv = 0$$

$$\Rightarrow \quad ux + vx - u = 0$$

$$\Rightarrow \quad x(u+v) = u$$

$$x = \frac{u}{u+v}$$

put this in (4)

$$y = \frac{u}{u+v} - 1$$

$$y = \frac{u-(u+v)}{u+v}$$

$$y = \frac{u-u-v}{u+v}$$

$$y = \frac{-v}{u+v}$$

putting values of x and y in (1)

$$u = \frac{\frac{u}{u+v}}{\left(\frac{u}{u+v}\right)^2 + \left(\frac{-v}{u+v}\right)^2}$$

$$U = \frac{u/u+v}{\frac{u^2}{(u+v)^2} + \frac{v^2}{(u+v)^2}}$$

$$U = \frac{\frac{u}{u+v}}{\frac{u^2+v^2}{(u+v)^2}}$$

$$U = \frac{u(u+v)}{u^2+v^2}$$

$$U = \frac{u^2+uv}{u^2+v^2}$$

$$U^3 + Uv^2 = u^2 + uv$$

$$u^3 + u v^2 - u^2 - uv = 0$$

$$u(u^2+v^2 - u^2 - uv) = 0$$

$$\Rightarrow u^2 + v^2 - u - v = 0$$

$$\Rightarrow \left[u^2 - 2(u)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2\right] + \left[v^2 - 2v\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2\right] - u - v = 0 + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2$$

$$\Rightarrow \left(u - \tfrac{1}{2}\right)^2 + \left(v - \tfrac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4}$$

$$\Rightarrow \left(u - \tfrac{1}{2}\right)^2 + \left(v - \tfrac{1}{2}\right)^2 = 2\,\frac{1}{4}$$

$$\left(u - \tfrac{1}{2}\right)^2 + \left(v - \tfrac{1}{2}\right)^2 = \left(\tfrac{1}{\sqrt{2}}\right)^2$$

is equation of circle with centre $\left(\frac{1}{2}, \frac{1}{2}\right)$ and radius $\frac{1}{\sqrt{2}}$

Ex :-

Transform $x = a$ through transformation $w = z^2$. Find the curve in (u, v) plane and discuss the nature of the curve.

Sol:-

The transformation is $w = z^2$

$$u + iv = (x+iy)^2$$

$$u + iv = x^2 - y^2 + 2ixy$$

$$\Rightarrow u = x^2 - y^2 \rightarrow (1), \quad v = 2xy \rightarrow (2)$$

From (2)

$$y = v/2x \quad \text{and} \quad x = a$$

$$y = v/2a$$

put the values of x and y in (1)

$$u = a^2 - v^2/4a^2$$

$$u = \frac{4a^4 - v^2}{4a^2}$$

$$4a^2 u = 4a^4 - v^2$$

$$v^2 = 4a^4 - 4a^2 u$$

$$v^2 = 4a^2(a^2 - u)$$

The curve is parabola.

Ex :-

Transform $|z| = 1$ under transformation $w = \frac{1}{z-1}$ Discuss the nature of curve

Sol :-

The transformation is $w = \frac{1}{z-1}$; $z = x+iy$

$$w = \frac{1}{x+iy-1}$$

$$w = \frac{1}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy}$$

$$w = \frac{(x-1)-iy}{(x-1)^2-(iy)^2}$$

$$w = \frac{(x-1)-iy}{(x-1)^2+y^2}$$

$$U+iV = \frac{(x-1)-iy}{(x-1)^2+y^2}$$

$$\Rightarrow U = \frac{x-1}{(x-1)^2+y^2} \rightarrow (1), \quad V = \frac{-y}{(x-1)^2+y^2} \rightarrow (2)$$

are transformation equations.

$$U = \frac{x-1}{x^2-2x+1+y^2}, \quad V = \frac{-y}{x^2-2x+1+y^2}$$

$$U = \frac{x-1}{x^2+y^2-2x+1}, \quad V = \frac{-y}{x^2+y^2-2x+1}$$

put $x^2+y^2=1 \quad \because |z|=1$

$$U = \frac{x-1}{1+1-2x}, \quad V = \frac{-y}{1+1-2x}$$

$$U = \frac{x-1}{2-2x}$$

$$U = \frac{x-1}{-2(x-1)}$$

$$U = -\frac{1}{2}$$

the equation is parallel to V-axis and is straight line.

Imp Conformal and Isogonal Transformation:-

Let C_1 and C_2 be two curves in the z-plane and α is the angle between them. Further, Let $w=f(z)$ be the transformation under this transformation; C_1' and C_2' be two image curves in the w-plane and β is angle between them.

If $\alpha=\beta$ in magnitude and direction, then the transformation is called conformal.

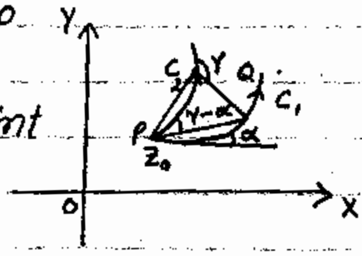
If $\alpha = \beta$ in magnitude but the direction is not preserved then the transformation is called Isogonal.

Theorem:-

Let $f(z)$ be an analytic function of z in a region D of the z-plane. Then, prove that the transformation is affected by $w = f(z)$ is conformal at all points of D if $f'(z) \neq 0$ in D

Proof :-

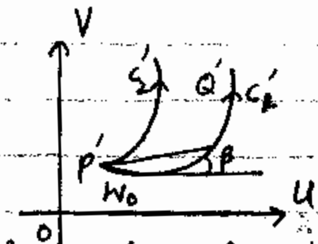
Step I :-



Let C_1 and C_2 be two curves at point $p(z_0)$ in z-plane, $p'(z)$ be any point on C_1 such that

$$z - z_0 = re^{i\alpha} \rightarrow (A)$$

where "r" is absolute value (i-e) the chord pp' makes an angle α with x-axis



Likewise, C_1' and C_2' be the images of these two curves in W-plane at a corresponding point $Q(w_0)$. Let $Q'(w)$ be a point on C_1' such that

$$W - W_0 = r'e^{i\beta} \rightarrow (B)$$

(i-e) the chord QQ' makes an angle β with u-axis. As the point $p'(z)$ moves in one plane its image $Q'(w)$ moves in other plane.

Step II :-

We know that $w = f(z)$ and $w_0 = f(z_0)$

Also $\quad f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

$$f'(z_0) = \lim_{z \to z_0} \frac{w - w_0}{z - z_0} \to (1)$$

Let $f'(z_0) = \rho e^{i\theta}$

$|f'(z_0)| = \rho$

Eq (1) takes the form

$$\rho e^{i\theta} = \lim_{z \to z_0} \frac{r' e^{i\beta}}{r e^{i\alpha}}$$

$$\rho e^{i\theta} = \lim_{z \to z_0} \left(\frac{r'}{r}\right) e^{i(\beta - \alpha)}$$

or

$$\rho = \lim_{z \to z_0} \left(\frac{r'}{r}\right) \to (2) \quad , \quad \theta = \lim_{z \to z_0} (\beta - \alpha) \to (3)$$

Likewise, if $p'(z)$ and $Q(w)$ are any two points on C_2 and C_2' respectively and γ and δ are the angles which the chords PP' and QQ' make with real-axis, then

$$\rho = \lim_{z \to z_0} \left(\frac{r'}{r}\right) \quad , \quad \theta = \lim_{z \to z_0} (\delta - \gamma) \to (4)$$

From (3) and (4) $\quad \delta - \gamma = \beta - \alpha \to (5)$

If $f'(x)$ exist, then the limit is unique.

$$\alpha - \gamma = \beta - \delta \to (6)$$

From (6), it is clear that angle between C_1 and C_2 in z-plane is same as the angle between C_1' and C_2' in w-plane in magnitude and direction.

Therefore, the mapping is affected by $w = f(z)$ is conformal provided $f'(z) \neq 0$ in D.

Remark :-

The orthogonal curves in one-plane (z-plane) give rise to orthogonal curves in other plane (w-plane) under conformal mapping.

Ex :-

Discuss the transformation $W = z^n$ (OR) Illustrate the mapping affected by $W = z^n$

Sol :-

Step I :-

Transform equation is $W = z^n$

Let $z = re^{i\theta}$

$W = \rho e^{i\phi}$

$$\therefore \rho e^{i\phi} = (re^{i\theta})^n \qquad \because W = z^n$$

$$\rho e^{i\phi} = r^n e^{in\theta}$$

$$\rho = r^n \rightarrow (1)$$

$$\phi = n\theta \rightarrow (2)$$

Eq (1) and (2) are transformation equations

Step II :-

From equation (1)

$$\rho = r^n$$

If r = Constant, then a circle in the z-plane is mapped into a circle in the W-plane with the radius as a power of n of radius in the z-plane $\qquad \because r = \text{Constt}$

If $r = 1$, then a circle in z-plane is mapped into a circle in the W-plane with radius as a the radius in z-plane. $\qquad x^2+y^2 = \text{Constt}$

If $r = 0$, then a point circle in z-plane is mapped into a point circle in W-plane

Now from eq (2)

$$\phi = n\theta$$

A line in the z-plane is mapped into a line in W-plane with angle as n-times the angle in the z-plane and passing through origin.

$\therefore \theta = \tan^{-1} y/x$

$y/x = \tan\theta$

$y = x\tan\theta$

$y = mx$

is eq of line where $m = \tan\theta$

Ex:-

Discuss the Transformation affected by $W = \frac{1}{2}(z + \frac{1}{z})$

Sol:-

StepI:- Transformation equations are

$$W = \frac{1}{2}(z + \frac{1}{z})$$

$$u + iv = \frac{1}{2}(re^{i\theta} + \frac{1}{r}e^{-i\theta})$$

$$= \frac{1}{2}[(r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)]$$

$$= \frac{1}{2}[(r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta]$$

$$\Rightarrow \{u = \frac{1}{2}(r + \frac{1}{r})\cos\theta, \quad v = \frac{1}{2}(r - \frac{1}{r})\sin\theta\} \rightarrow (L)$$

$$\cos\theta = \frac{2u}{r + \frac{1}{r}} \rightarrow (1), \qquad \sin\theta = \frac{2v}{r - \frac{1}{r}} \rightarrow (2)$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{4u^2}{(r + \frac{1}{r})^2} + \frac{4v^2}{(r - \frac{1}{r})^2} = 1 \rightarrow (A)$$

From (1) and (2)

$$r + \frac{1}{r} = \frac{2u}{\cos\theta}, \qquad r - \frac{1}{r} = \frac{2v}{\sin\theta}$$

squaring and subtracting

$$\left(\frac{2u}{\cos\theta}\right)^2 - \left(\frac{2v}{\sin\theta}\right)^2 = r^2 + \frac{1}{r^2} + 2 - r^2 - \frac{1}{r^2} + 2$$

$$4\left(\frac{u^2}{\cos^2\theta} - v^2/\sin^2\theta\right) = 4$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1 \rightarrow (B)$$

Step II :-

Discussion :-

From eq (A)

$$\frac{u^2}{\frac{1}{4}(r+\frac{1}{r})^2} + \frac{v^2}{\frac{1}{4}(r-\frac{1}{r})^2} = 1$$

If $r = \text{constt}$, a circle in z-plane is mapped into an ellipse in w-plane with focie at $(\pm ae, 0)$.

(i-e) $ae = \sqrt{a^2 - b^2}$

Foci are at $(\pm 1, 0) = \sqrt{\frac{1}{4}(c+\frac{1}{c})^2 - \frac{1}{4}(c-\frac{1}{c})^2}$

$$= \sqrt{\frac{1}{4}(c^2+\frac{1}{c^2}+2) - \frac{1}{4}(c^2+\frac{1}{c^2}-2)}$$

$\sqrt{\frac{1}{2}+\frac{1}{2}} = \sqrt{1} = 1$

Focal length $= 2$ units

If r is a parametre, then a family of concentric circles mape into a family of confocal ellipse.

If $r = 1$, then a unit circle in the z-plane is mapped into a pair of straight lines in the w-plane.

$r = 1$, $\frac{v^2}{1-1} = 0$

then Eq (A) becom

$\frac{4u^2}{4} = 1$

$u^2 = 1$

$\Rightarrow u = \pm 1$

Now, From equation (B)

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 1$$

If $\theta = \text{constant}$, then a straight line in z-plane through the origin is mapped into a hyperbola in the w-plane with foci are at $(\pm 1, 0)$, focal length $= 2$ units

If θ is a parametre, then a system of lines through the origin in the z-plane is mapped into confocal hyperbolas in the w-plane.

Ex:- $f(z) = w = \frac{1}{2}\left(z + \frac{1}{z}\right)$

In order to find the points at which this transformation is not conformal, we put

$$f'(z) = 0$$

$$\frac{1}{2}\left(1 - \frac{1}{z^2}\right) = 0$$

$$\frac{z^2 - 1}{z^2} = 0 \Rightarrow z^2 - 1 = 0$$

$$z^2 = 1$$

$$\Rightarrow z = \pm 1$$

Then, the given transformation is not conformal at the points $0, \pm 1$

This function is not analytic because its derivative at point '0' does not exist.

Critical Point:-

A point z_0 is said to be critical point of analytic mapping $w = f(z)$ if

$$f'(z_0) = 0$$

Analytic mapping is analytic function

$$w = f(z) \text{ or } f(z) = z^2$$

Ex:-

Under the transformation $(a-b)w^2 - 2zw + (a+b) = 0$ Prove that (i) If w maps a circle then z maps an ellipse. The concentric circles in the w-plane are mapped into confocal ellipse in z-plane.

(ii) If w ~~xxx~~ maps a straight line, then

z maps a hyperbola, further a system of lines through the origin in the w-plane is mapped into confocal hyperbolas in z-plane.

Sol:-

The transformation is

$$(a-b)w^2 - 2zw + (a+b) = 0$$

$$(a-b)w^2 + (a+b) = 2zw$$

$$z = \frac{(a-b)w^2}{2w} + \frac{(a+b)}{2w}$$

$$z = \frac{(a-b)}{2}w + \frac{(a+b)}{2w} \rightarrow (1)$$

we know $w = re^{i\theta}$, $z = x+iy$

then $z = x+iy = \left(\frac{a-b}{2}\right)re^{i\theta} + \left(\frac{a+b}{2re^{i\theta}}\right)$

$$\Rightarrow x+iy = \left(\frac{a-b}{2}\right)(r(\cos\theta + i\sin\theta)) + \left(\frac{a+b}{2r}\right)(\cos\theta - i\sin\theta)$$

$$= \left(\frac{a-b}{2}\right)(r\cos\theta + ri\sin\theta) + \frac{1}{r}\left(\frac{a+b}{2}\right)(\cos\theta - i\sin\theta)$$

$$x+iy = \frac{1}{2}\left[(a-b)r + \frac{(a+b)}{r}\right]\cos\theta + \frac{i}{2}\left[(a-b)r - \frac{(a+b)}{r}\right]\sin\theta$$

$$\Rightarrow x = \frac{1}{2}\left[(a-b)r + \frac{a+b}{r}\right]\cos\theta, \quad y = \frac{1}{2}\left[(a-b)r - \frac{a+b}{r}\right]\sin\theta$$

$$\cos\theta = \frac{x}{\frac{1}{2}\left[(a-b)r + \frac{a+b}{r}\right]}, \quad \sin\theta = \frac{y}{\frac{1}{2}\left[(a-b)r - \frac{a+b}{r}\right]}$$

Squaring and adding values of $\cos\theta$. $\sin\theta$

$$\cos^2\theta + \sin^2\theta = \frac{x^2}{\left[\frac{1}{2}\left((a-b)r + \frac{a+b}{r}\right)\right]^2} + \frac{y^2}{\left[\frac{1}{2}\left((a-b)r - \frac{(a+b)}{r}\right)\right]^2} \rightarrow (A)$$

$$1 = \frac{x^2}{(a-b)^2\left(r+\frac{1}{r}\right)^2} + \frac{y^2}{(a+b)^2\left(r-\frac{1}{r}\right)^2}$$

Squaring and subtracting (1) and (2) we get

$$\frac{x^2}{\cos^2\theta} - \frac{y^2}{\sin^2\theta} = \frac{1}{4}(a^2-b^2)$$

$$\Rightarrow \frac{x^2}{\frac{1}{4}(a^2-b^2)\cos^2\theta} - \frac{y^2}{\frac{1}{4}(a^2-b^2)\sin^2\theta} = 1 \rightarrow (B)$$

From equation (A)

(i) If r = constant = C in w-plane, then a circle $r = C$ in w-plane is mapped into an ellipse (A) in z-plane with foci are at

$$\left(\pm\sqrt{A^2-B^2},\, 0\right)$$

$$= \sqrt{\frac{1}{4}\left[(a-b)r + \frac{a+b}{r}\right]^2 - \frac{1}{4}\left[(a-b)r - \left(\frac{a+b}{r}\right)\right]^2}$$

$$= \sqrt{a^2-b^2}$$

Foci are at $\left(\pm\sqrt{a^2-b^2},\, 0\right)$

Focal length $= 2\sqrt{a^2-b^2}$

(ii) If $r = C$ and if C is a parametre then concentric circles in w-plane are mapped into confocal ellipses in the z-plane.

From equation (B)

(i) If θ = constant = C then a line through the origin in the w-plane is mapped into a hyperbola (B) in the z-plane with

Foci are at $(\pm Ae, 0)$

where $Ae = \sqrt{A^2+B^2}$

$$= \sqrt{\frac{1}{4}(a^2-b^2)\cos^2\theta + \frac{1}{4}(a^2-b^2)\sin^2\theta}$$

$$= \sqrt{\frac{1}{4}(a^2-b^2)(\cos^2\theta + \sin^2\theta)}$$

$\because \cos^2\theta + \sin^2\theta = 1$

$$= \sqrt{\frac{1}{4}(a^2-b^2)(1)}$$

Foci $Ae = \frac{1}{2}\sqrt{a^2-b^2}$

Foci are at $\left(\pm \frac{1}{2}\sqrt{a^2-b^2}, 0\right)$

Focal length $= \sqrt{a^2-b^2}$

(ii)

If C is a parametre then a net of straight lines through origin in the w-plane is mapped into confocal hyperbolas in the z-plane.

Ex :-

Under the transformation $(w+1)^2 z = 4$. Prove that if w describes a unit circle then z describes a parabola.

Sol:-

The transformation is

$$(w+1)^2 z = 4$$

$\therefore \; z = re^{i\theta}$, $w = e^{i\phi}$ and w describes unit circle.

$$(w+1)^2 z = 4$$

$$\Rightarrow \quad z = \frac{4}{(w+1)^2}$$

$$re^{i\theta} = \frac{4}{(e^{i\phi}+1)^2} \quad \text{or} \quad \frac{1}{r} e^{-i\theta} = (e^{i\phi}+1)^2$$

$$r(\cos\theta + i\sin\theta) = \frac{4}{(e^{i\phi})^2 + 2e^{i\phi} + 1}$$

$$r(\cos\theta + i\sin\theta) = \frac{4}{(\cos\phi + i\sin\phi)^2 + 2(\cos\phi + i\sin\phi) + 1}$$

Taking Reciprocal:

$$\frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1 + \cos^2\phi - \sin^2\phi + 2\cos\phi + 2i\sin\phi + 2i\sin\phi\cos\phi}{4}$$

Comparing real and imaginary parts.

$$-\frac{1}{r}\sin\theta = \frac{2\sin\phi + 2\sin\phi\cos\phi}{4} \rightarrow (1)$$

$$-\frac{1}{r}\sin\theta = 2\sin\phi\,(1 + \cos\phi)/4 \rightarrow (1)$$

$$\frac{1}{r}\cos\theta = \frac{1+\cos^2\phi + 2\cos\phi - \sin^2\phi}{4} = \frac{2\cos\phi\,(1+\cos\phi)}{4} \rightarrow (2)$$

squaring and adding (1) and (2)

$$\frac{1}{r^2}(\cos^2\theta + \sin^2\theta) = \frac{1}{4}(1+\cos\phi)^2[\cos^2\phi + \sin^2\phi]$$

$$\frac{1}{r^2} = (1+\cos\phi)^2/4$$

$$\text{or} \quad r^2 = \frac{4}{(1+\cos\phi)^2}$$

$$\Rightarrow \quad r = \frac{2}{1+\cos\phi} \rightarrow (3)$$

Eq (3) is equation of parabola in z-plane.

JOUKOWSKI'S TRANSFORMATION :-

Ex :-

Consider the transformation $w = z + \frac{a^2}{z}$, a being real and positive. At what finite points this will not conformal, apply this transformation to the circle $|z| = k$. Discuss when (i) $k >> a$ (ii) $k = a$ (iii) $k << a$

Sol :-

Step I :-

The transformation is $w = z + \frac{a^2}{z}$

Transformation equations are

$$u + iv = (x+iy) + \frac{a^2}{x+iy}$$

Given $|z| = k \Rightarrow x^2+y^2 = k^2$

$$u + iv = (x+iy) + \frac{a^2(x-iy)}{x^2+y^2}$$

$$u+iv = x + \frac{a^2 x}{x^2+y^2} + i\left(y - \frac{a^2 y}{x^2+y^2}\right)$$

put $x^2+y^2 = k^2$

$$u+iv = x\left(1+\frac{a^2}{k^2}\right) + i\,y\left(1-\frac{a^2}{k^2}\right)$$

$$\Rightarrow u = x\left(1+\frac{a^2}{k^2}\right) \rightarrow (1), \quad v = y\left(1-\frac{a^2}{k^2}\right) \rightarrow (2)$$

The points at which the transformation is not conformal, we put $\frac{dw}{dz} = 0$

$$1 - \frac{a^2}{z^2} = 0$$

$$\Rightarrow 1 = \frac{a^2}{z^2}$$

$$\Rightarrow z^2 = a^2$$

$$\Rightarrow z = \pm a$$

$\therefore$ The points at which transformation is not conformal at $0, \pm a$

Step II :-

From (1) and (2) we obtain x and y

$$x = \frac{u}{\ldots} \rightarrow (a), \quad y = \frac{v}{\ldots} \rightarrow (b)$$

squaring and adding (a) and (b)

$$x^2+y^2 = \frac{u^2}{\left(1+\frac{a^2}{k^2}\right)^2} + \frac{v^2}{\left(1-\frac{a^2}{k^2}\right)^2} \qquad \because x^2+y^2=k^2$$

$$k^2 = \frac{u^2}{\frac{(a^2+k^2)^2}{k^4}} + \frac{v^2}{\frac{(k^2-a^2)^2}{k^4}}$$

$$1 = \frac{u^2}{\frac{k^2(a^2+k^2)^2}{k^4}} + \frac{v^2}{\frac{k^2(k^2-a^2)^2}{k^4}}$$

$$1 = \frac{u^2}{\left(\frac{a^2+k^2}{k}\right)^2} + \frac{v^2}{\left(\frac{k^2-a^2}{k}\right)^2} \rightarrow (A)$$

is equation of ellipse with centre $(0,0)$, semi-major axis $= \frac{a^2+k^2}{k}$ and semi-minor axis $= \frac{k^2-a^2}{k}$ ← if $k<a$ then $\left(\frac{a^2-k^2}{k}\right)$

Step III :-

Case I :-

If $k >> a$

then a^2/k^2 will be very small (i-e) zero. then equation A will become approximately $u^2+v^2=k^2$ (circle) passing through origin and radius k

Case II :-

If $k = a$

then $u = 2x$, $v = 0$

$u = 2k \Rightarrow u^2 = 4k^2$

$\Rightarrow u = \pm 2k$

Eq (A) will become that part as the real axis from $u = -2k$ to $u = 2k$

Case III :-

If $K << a$

then a^2/K^2 will be greater than '1' and eq (A) will become equation of ellipse. If $a = 1$, then the transformation

$w = z + \frac{a^2}{z}$ is called JOUKOWSIKS TRANSFORMATION.

Ex :-

Consider the transformation $w = \frac{a^2}{z} + z$

(i) If z is mapped into a circle $x^2+y^2 = a^2$ Prove that its mapping in w-plane is a straight line. Find the length of st. line.

(ii) If z mapped a circle $x^2+y^2 = b^2$ and $b > a$, then its maps an ellipse in the w-plane with the foci as the extrimities of the line in path.

Sol :-

Transformation equation's is are

$$w = \frac{a^2}{z} + z$$

$$u + iv = \frac{a^2}{x+iy} + (x+iy)$$

$$u + iv = \frac{a^2(x-iy)}{x^2+y^2} + x + iy$$

$$u + iv = \left(x + \frac{a^2 x}{x^2+y^2}\right) + i\left(y - \frac{a^2 y}{x^2+y^2}\right)$$

$$\Rightarrow \left. u = x\left(1 + \frac{a^2}{x^2+y^2}\right), \quad v = y\left(1 - \frac{a^2}{x^2+y^2}\right)\right] \rightarrow (A)$$

(i) $\Rightarrow$ From (A)

$$\left. x = \frac{u}{\left(1 + \frac{a^2}{x^2+y^2}\right)}, \quad y = \frac{v}{\left(1 - \frac{a^2}{x^2+y^2}\right)}\right] \rightarrow (B)$$

Given $x^2+y^2 = a^2$

$$x = \frac{u}{1+1} \quad , \quad y = \frac{v}{1-1}$$

$$x = \frac{u}{4} \quad , \quad y = 0$$

Squaring and adding both

$$x^2+y^2 = \frac{u^2}{4} - 0$$

put $x^2+y^2 = a^2$

$$u^2 = 4a^2$$

$\Rightarrow u = \pm 2a$ is equation of st. line

length of st. line $= 2(2a) = 4a$

(ii) From (B) Given $x^2+y^2=b^2$

$$x = \frac{u}{1+\frac{a^2}{b^2}} \quad , \quad y = \frac{v}{1-\frac{a^2}{b^2}}$$

squaring and adding

$$x^2+y^2 = \frac{u^2}{\left(1+\frac{a^2}{b^2}\right)^2} + \frac{v^2}{\left(1-\frac{a^2}{b^2}\right)^2} \rightarrow (C)$$

put $x^2+y^2=b^2$

$$b^2 = \frac{u^2}{\frac{(b^2+a^2)^2}{b^4}} + \frac{v^2}{\frac{(b^2-a^2)^2}{b^4}}$$

$$1 = \frac{u^2}{\left(\frac{b^2+a^2}{b}\right)^2} + \frac{v^2}{\left(\frac{b^2-a^2}{b}\right)^2}$$

is equation of an ellipse with centre $C(0,0)$, semi-major axis $= \frac{b^2+a^2}{b}$ and semi-minor axis $= \frac{b^2-a^2}{b}$

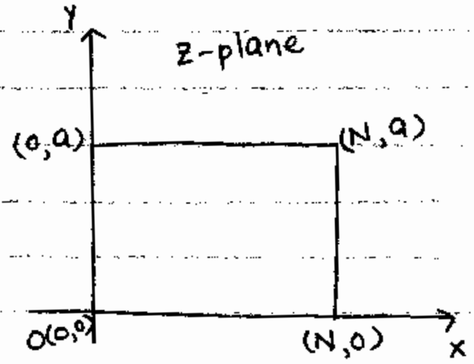
Given that $b>a$, then $\frac{a^2}{b^2}$ is very small (i-e) zero and equation (C) will become approximately $u^2+v^2=b^2$.

Ex :-

If $w = \cosh\left(\frac{\pi z}{a}\right)$ is transformation where a is a real and positive. Find the region in the w-plane corresponding to the rectangle bounded by the lines $x=0,\ y=0,\ x=N>0,\ y=a$

Sol :-

Step I :-



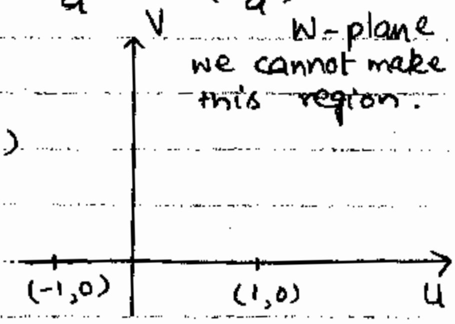
Transformation is

$$w = \cosh\left(\frac{\pi z}{a}\right)$$

Transformation equations are $u+iv = \cosh\left(\frac{\pi}{a}(x+iy)\right)$

$$u+iv = \cosh\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right) + i\sinh\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

$$\Rightarrow \left.\begin{aligned} u &= \cosh\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{a}\right) \\ v &= \sinh\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right) \end{aligned}\right\} \rightarrow (A)$$



Step II :-

Under the transformation equation (A).

we get $x=0,\ y=0,$

$u(0,0) = 1, \quad v(0,0) = 0$

So, $O'(1,0)$

$x = N>0,\ y=0$

$u(N,0) = \cosh\left(\frac{\pi N}{a}\right), \quad v(N,0) = 0$

So, $A'\left(\cosh\left(\frac{N\pi}{a}\right), 0\right)$

At $B(N,a)$

$u(N,a) = -\cosh\left(\frac{N\pi}{a}\right), \quad v(N,a) = 0$

So, $B'\left(-\cosh\left(\frac{N\pi}{a}\right), 0\right)$

at $C(0,a)$

$u(0,a) = -1, \quad v(0,a) = 0$

$U(0,0)=1$ z-plane   w-plane

$V(0,0)=0$ $O(0,0)$, $O'(1,0)$

$U(N,0)=\cosh\left(\frac{N\pi}{a}\right)$ $A(N,0)$, $A'\left(\cos\frac{N\pi}{a},0\right)$

$V(N,0)=0$ $B(N,a)$, $B'\left(-\cosh\left(\frac{N\pi}{a}\right),0\right)$

$C(0,a)$, $C'(-1,0)$

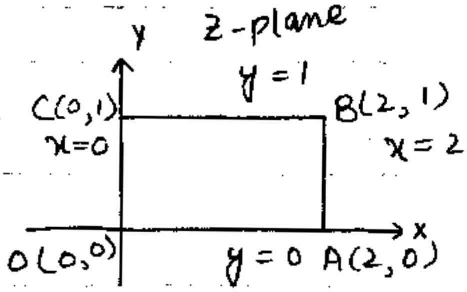
Ex :-

Let the rectangular Region R in the z-plane be bounded by $x=0$, $y=0$, $x=2$, $y=1$. Determine the region R of the w-plane into which R is mapped under the transformation

(i) $W=z+(1-2i)$

(ii) $W=\sqrt{2}\,e^{i\pi/4}z$

(iii) $W=\sqrt{2}\,e^{i\pi/4}z+(1-2i)$



Sol :-

(i) $W=z+(1-2i)$

Step I :-

$$W=z+(1-2i)$$
$$U+iV=(x+iy)+(1-2i)$$
$$U+iV=(x+1)+i(y-2)$$
$$\Rightarrow U=x+1 \rightarrow (1), \quad V=y-2 \rightarrow (2)$$

Step II :-

Using (1) and (2)

The line at $(0,0)$

$U=1$, $V=-2$

So $O'(1,-2)$

At $A(2,0)$

$U=3$, $V=-2$

So, $A'(3,-2)$

At $B(2,1)$

$u = 3, \quad v = -1$

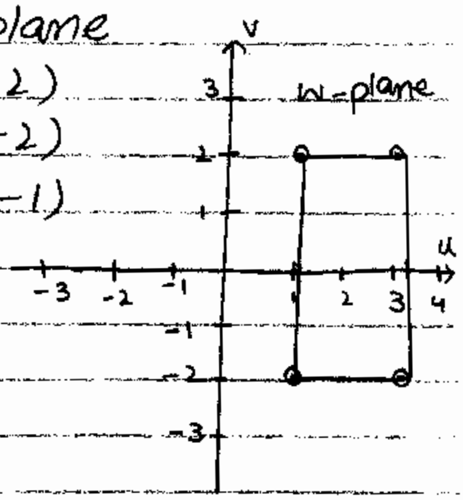
So, $\quad B'(3, -1)$

At, $\quad C(0, 1)$

$u = 1, \quad v = -1$

So, $\quad C'(1, -1)$

The z-plane	w-plane
$O(0,0)$	$O'(1, -2)$
$A(2,0)$	$A'(3, -2)$
$B(2,1)$	$B'(3, -1)$
$C(0,1)$	$C'(1, -1)$

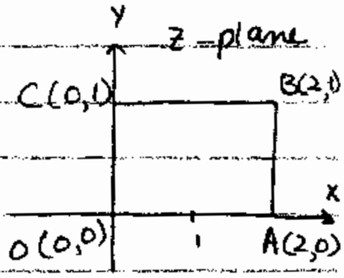


(ii) $\quad w = \sqrt{2}\, e^{i\pi/4} z$

Step I :-

$$w = \sqrt{2}\, e^{i\pi/4} z \qquad \because e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$$

$$w = \sqrt{2}\left(\frac{1}{\sqrt{2}}(1+i)(x+iy)\right)$$



$$u + iv = (1+i)(x+iy)$$

$$u + iv = x + iy + ix - y$$

$$u + iv = (x-y) + i(x+y)$$

$\Rightarrow \quad u = x - y \rightarrow (1), \quad v = x + y \rightarrow (2)$

Step II :-

From (1) and (2)

At $O(0,0)$

$u = 0, \quad v = 0$

So $\quad O'(0, 0)$

At $\quad A(2, 0)$

$u = 2, \quad v = 2$

So $\quad A'(2, 2)$

At $\quad B(2, 1)$

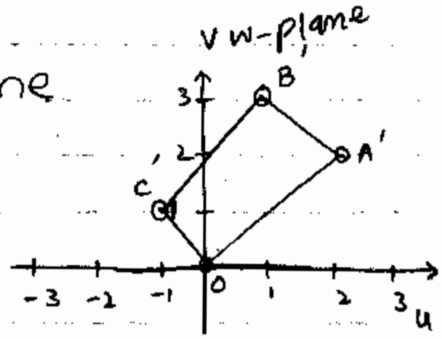
$U = 1, \quad V = 3$

So, $B'(1, 3)$

At $C(0, 1)$

$U = -1, \quad V = 1$

So, $C'(-1, 1)$



Z - plane	W-plane
$O(0,0)$	$O'(0,0)$
$A(2,0)$	$A'(2,2)$
$B(2,1)$	$B'(1,3)$
$C(0,1)$	$C'(-1,1)$

(iii) $W = \sqrt{2}\, e^{i\pi/4} z + (1-2i)$

Step I :-

$$W = \sqrt{2}\, e^{i\pi/4} z + (1-2i)$$

$$W = \sqrt{2}\left(\frac{1}{\sqrt{2}}(1+i)(x+iy)\right) + (1-2i)$$

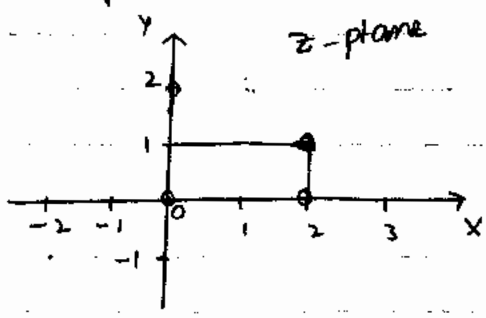
$$W = (1+i)(x+iy) + (1-2i)$$

$$W = x - y + ix + iy + 1 - 2i$$

$$U + iV = (x-y+1) + i(x+y-2)$$

$\Rightarrow U = x - y + 1 \rightarrow (1), \quad V = x + y - 2 \rightarrow (2)$

Step II :-



From (1) and (2)

At $O(0,0)$

$U = 1, \quad V = -2$

So, $O'(1, -2)$

At $A(2, 0)$

$U = 3, \quad V = 0$

So, $A'(3, 0)$

At $B(2, 1)$

$U = 2, \quad V = 1$

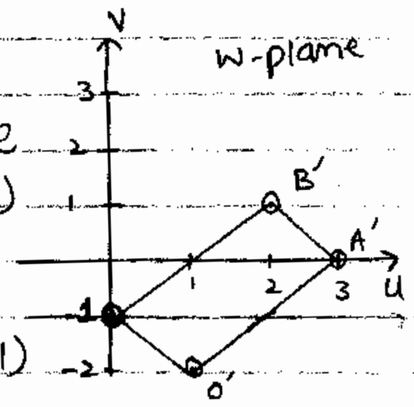
So, $B'(2, 1)$

At $C(0, 1)$

$U = 0, \quad V = -1$

So, $C'(0, -1)$

z-plane	w-plane
$O(0,0)$	$O'(1,-2)$
$A(2,0)$	$A'(3,0)$
$B(2,1)$	$B'(2,1)$
$C(0,1)$	$C'(0,-1)$



Ex:-

Show that the transformation $w = 2z - 3i\bar{z} + 5 - 4i$ is equivalent to $u = 2x + 3y + 5, \; v = 2y - 3x - 4$

Sol:-

Transformation is $w = 2z - 3i\bar{z} + 5 - 4i$

we know $z = x + iy$, $\bar{z} = x - iy$

$$u + iv = 2(x+iy) - 3i(x-iy) + 5 - 4i$$

$$u + iv = 2x + 2iy - 3ix + 3i^2 y + 5 - 4i$$

$\because i^2 = -1$

$$u + iv = 2x + 3(-1)y + 5 + 2iy - 3ix - 4i$$

$$u + iv = (2x - 3y + 5) + i(2y - 3x - 4)$$

$\Rightarrow$ $u = 2x - 3y + 5$

$v = 2y - 3x - 4$

Define Linear Transformation and its properties:

Linear Transformation :-

A transformation of the form $w = Az + B$, where A and B are complex constants and $A \neq 0$, is called a linear transformation.

note: If $A = 0$ then $w = B$ (constant) is not transformation.

Properties of Linear Transformation :-

(i) Translation :-

If $A = 1$, then the transformation becomes $W = z + B$. Each point z is displaced through the vector B that is, if $z = x + iy$, $W = u + iv$, $B = b_1 + ib_2$ then the image of any point (x, y) in the z-plane is the point $(x + b_1, y + b_2)$ in the W-plane.

(ii) Rotation or Magnification :-

If $B = 0$ then the transformation becomes $W = Az$ then where $z = re^{i\theta}$, $W = Re^{i\varphi}$, $A = ae^{i\alpha}$

$W = Az$ becomes

$Re^{i\varphi} = ae^{i\alpha} re^{i\theta}$

∴ Linear Transformations are $W = 3z$, $W = iz + 4i$

$\Rightarrow \quad R = ar \rightarrow (1)$, $\quad \varphi = \alpha + \theta \rightarrow (2)$

The figure will be magnified (larger or shorter) by the amount of $a = |A|$ by eq (1).

The figure will be rotated by the amount of α in the W-plane through or by eq (2).

Theorem :-

Prove that a linear transformation is a conformal mapping.

Sol :-

Given a linear transformation

$W = Az + B \quad ; \quad A \neq 0$

$\frac{dW}{dz} = A$

Therefore, According to the theorem, a transformation affected $W = f(z)$ is analytic in a region D is conformal, provided $f'(z) \neq 0$ in D.

Ex :-

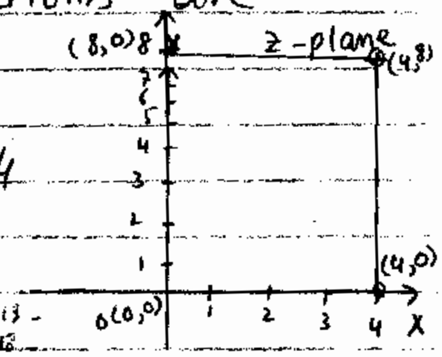
A rectangle is formed by $x=0$, $y=0$, $x=4$, $y=8$ in z-plane. Find the image curve in the w-plane under transformation

$$W = -2iz + 4$$

Sol:-

I: Transformation equations are

$$U + iV = -2i(x + iy) + 4$$
$$U + iV = -2ix - 2i^2 y + 4$$
$$U + iV = -2ix - 2(-1)y + 4$$
$$U + iV = -2ix + 2y + 4$$
$$U + iV = (2y + 4) - i(2x)$$



$\Rightarrow$ $U = 2y + 4 \rightarrow (1)$

$V = -2x \rightarrow (2)$

II:

At $O(0,0)$

at $x=0 \Rightarrow V=0$,

at $y=0 \Rightarrow U=4$

then $O'(4,0)$

At $A(4,0)$

at $x=4$

$V = -2(4)$

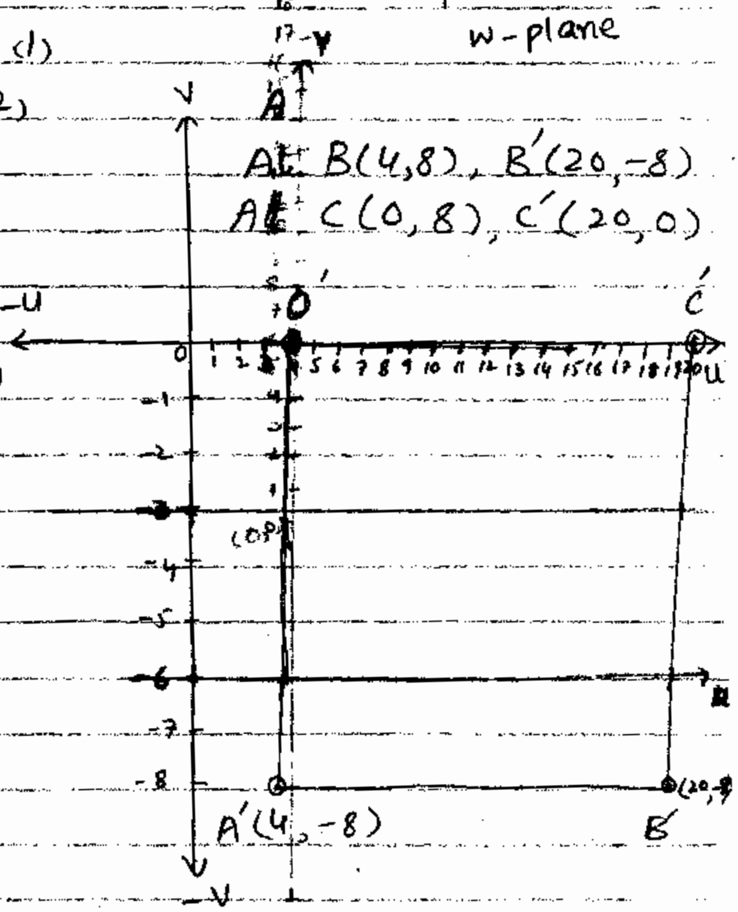
$V = -8$

at $y=0$

$U = 2(0)+4$

$U = 0+4$

$U = 4$



So, $A'(4, -8)$

The figure of transformation in z-plane and w-plane is shown

Ex :-

A rectangle is formed by $x=0$, $y=0$, $x=3$ and $y=5$ in the z-plane. Find the image curve in w-plane under the transformation $w=(1+i)z+2+i$

Sol :-

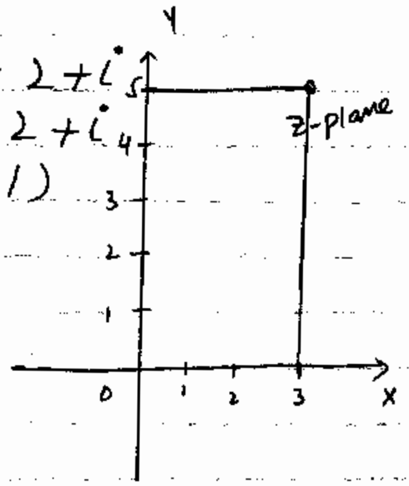
$$w=(1+i)z+2+i$$

Step I :-

$$u+iv=(1+i)(x+iy)+2+i$$

$$= x+iy+ix-y+2+i$$

$$u+iv=(x-y+2)+i(x+y+1)$$



$\Rightarrow u = x-y+2 \rightarrow (1)$

$v = x+y+1 \rightarrow (2)$

Step II :-

From (1) and (2)

At $O(0,0)$

at $x=0 \Rightarrow u=-y+2, \quad v=y+1 \Rightarrow u+v=3 \rightarrow (1)$

at $y=0 \Rightarrow u=x+2, \quad v=x+1 \Rightarrow u-v=1 \rightarrow (2)$ (A)

At $A(3,5)$

at $x=3 \Rightarrow u=5-y, \quad v=4+y \Rightarrow u+v=9 \rightarrow (3)$

at $y=5 \Rightarrow u=x+3, \quad v=x+6 \Rightarrow v-u=3 \rightarrow (4)$

Step III :-

Step II can also be taken from eq (A)

$O(0,0)$ in z-plane maps in w-plane

At $O(0,0)$

$u=2 \quad v=1$

So, $O'(2,1)$

At $A(3,0)$

$u = 5, \quad v = 4$

So, $A'(5,4)$

At $B(3,5)$

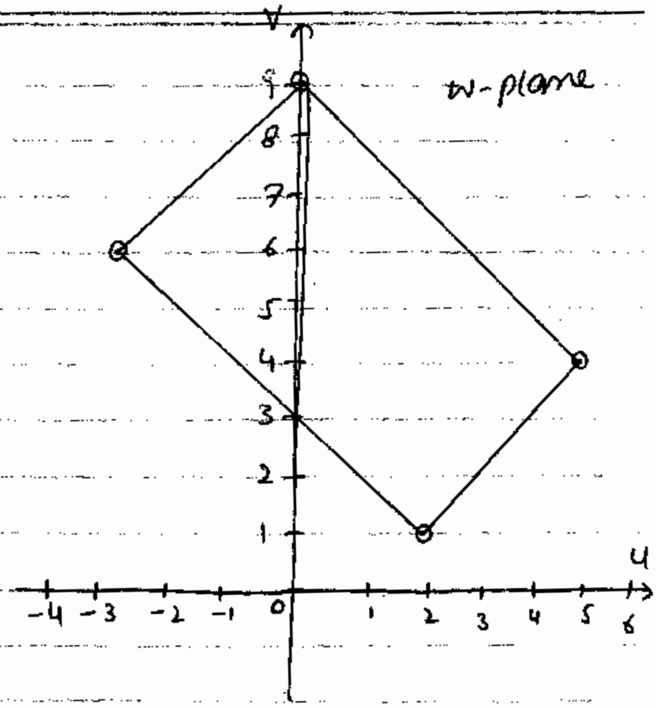
$u = 0, \quad v = 9$

So, $B'(0,9)$

At $C(0,5)$

$u = -3, \quad v = 6$

So, $C'(-3,6)$



Ex :-

Find region of w-plane by lines $x=0$, $y=0$, $x=1$, $y=2$ under transformation $w = z+2-i$

Sol :-

$$w = z+2-i$$

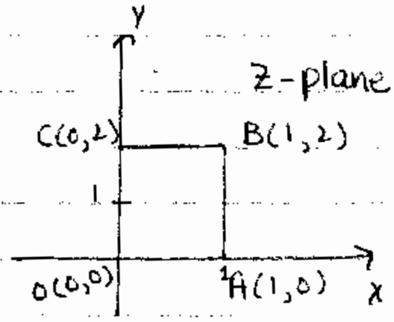
Step I :-

$$w = z+2-i$$

$$u+iv = (x+iy)+2-i$$

$$u+iv = (x+2)+i(y-1)$$

$$\Rightarrow \left.\begin{aligned} u &= x+2 \\ v &= y-1 \end{aligned}\right] \rightarrow (A)$$



Step II :-

From Eq (A)

At $O(0,0)$

$u = 2, \quad v = -1$

$O'(2,-1)$

At $A(1,0)$

$u = 3, \quad v = -1$

So, $A'(3,-1)$

At $B(1,2)$

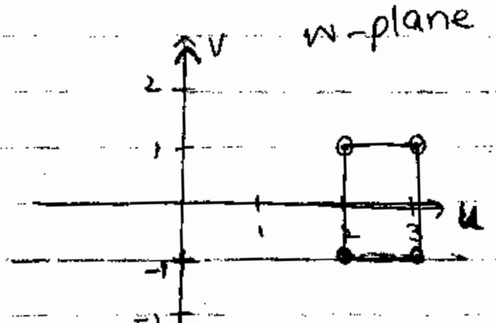
$u = 3, \quad v = 1$

So, $B'(3,1)$

At $C(0,2)$

$u = 2, \quad v = 1$

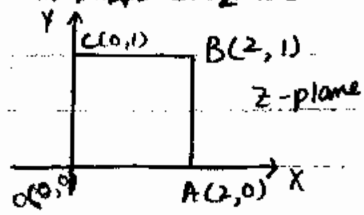
So, $C'(2,1)$



Ex:-

Consider the transformation $w = \sqrt{2}\, e^{i\pi/4} z + (1-2i)$

Find its image in U-V-plane



Sol:-

$$W = \sqrt{2}\, e^{i\pi/4} z + (1-2i)$$

Step I:-

$$u + iv = \sqrt{2}\left(\frac{1}{\sqrt{2}}(1+i)(x+iy)\right) + (1-2i)$$

$$= (1+i)(x+iy) + (1-2i)$$

$$= (x-y) + i(x+y) + (1-2i)$$

$$u + iv = (x-y+1) + i(x+y-2i)$$

$\Rightarrow$ $u = x - y + 1 \rightarrow (1)$

$v = x + y - 2 \rightarrow (2)$

Step II:-

From (1) and (2)

At $O(0,0)$

$u = 1,\ v = -2$

So, $O'(1,-2)$

At $A(2,0)$

$u = 3,\ v = -4$

So, $A'(3,-4)$

At $B(2,1)$

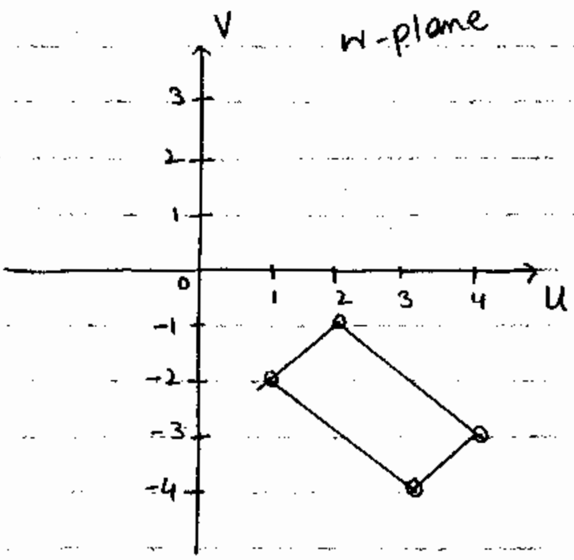
$u = 4,\ v = -3$

So, $B'(4,-3)$

At $C(0,1)$

$u = 2,\ v = -1$

So, $C'(2,-1)$



V.Imp

Ex:-

Consider the transformation $w = e^{i\pi/4} z$ and determine the region in the w-plane corresponding to the triangle region bounded by the lines $x=0$, $y=0$, and $x+y=1$ in the z-plane.

Sol:-

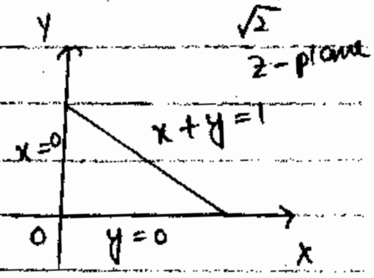
StepI:-

The given transformation is

$$w = e^{i\pi/4} z$$

Transformation equations are $\quad \because e^{i\pi/4} = \frac{1}{\sqrt{2}}(1+i)$

$$u+iv = \frac{1}{\sqrt{2}}(1+i)(x+iy)$$



$$u+iv = \frac{1}{\sqrt{2}}(x+iy+ix-y)$$

$$u+iv = \frac{1}{\sqrt{2}}((x-y)+i(x+y))$$

$$\Rightarrow u = \frac{1}{\sqrt{2}}(x-y), \quad v = \frac{1}{\sqrt{2}}(x+y) \Big] \rightarrow (A)$$

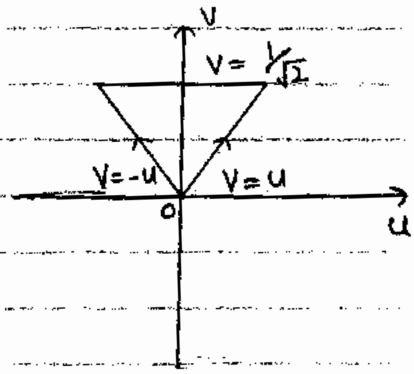
StepII:-

From Eq (A)

The line $x=0$ gives

$$u = \frac{-y}{\sqrt{2}}, \quad v = \frac{y}{\sqrt{2}}$$

or $u = -v$ or $v = -u$



The line $y=0$ gives

$$u = \frac{x}{\sqrt{2}}, \quad v = \frac{x}{\sqrt{2}}$$

$\Rightarrow \quad u = v$

The line $x+y=1$ gives

$$v = \frac{1}{\sqrt{2}}$$

Hence, the given triangle region in the z-plane is transformed into the triangular region in the w-plane bounded by the lines $v=u$, $v=-u$, and $v=\frac{1}{\sqrt{2}}$

Ex :-

Given triangle T in z-plane with vertices at i, $1-i$, $1+i$ $(0,1)$, $(1,-1)$, $(1,1)$. Determine triangle T into which T is mapped under the transformations

(a) $W = 3z + 4 - 2i$

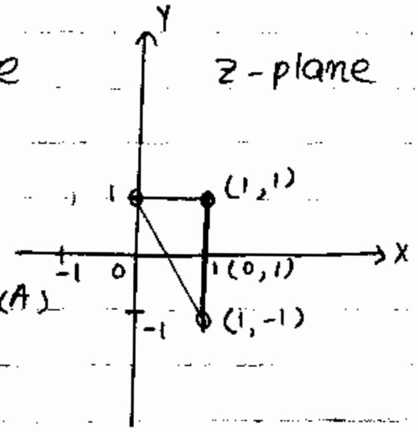
(b) $W = iz + 2 - i$

(c) $W = 8e^{i\pi/4} z - 2 + 4i$, $W = 5e^{i\pi/3} z - 2 + 4i$

Sol :-

(a) The given transformation is

$$W = 3z + 4 - 2i$$



Transformation equations are

$$U + iV = 3(x + iy) + 4 - 2i$$
$$U + iV = 3x + 3iy + 4 - 2i$$
$$U + iV = (3x + 4) + i(3y - 2)$$
$$\Rightarrow \quad U = 3x + 4, \quad V = 3y - 2 \quad \Big] \rightarrow (A)$$

Using eq (A)

At $A(0, 1) = i$

at $x = 0 \Rightarrow U = 4$

at $y = 1 \Rightarrow V = 1 \Rightarrow A'(4, 1)$

At $B(1, -1) = 1 - i$

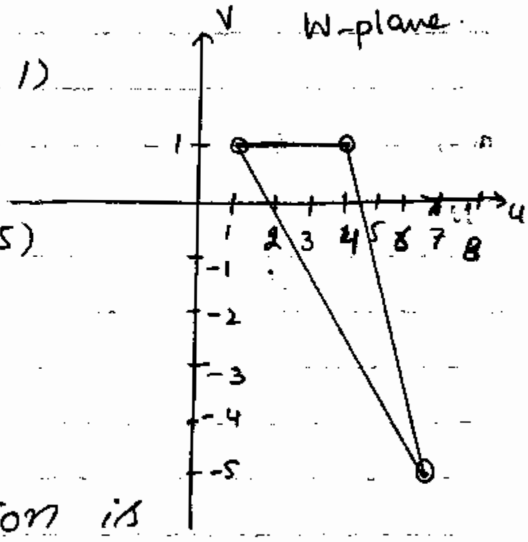
at $x = 1 \Rightarrow U = 7$

at $y = -1 \Rightarrow V = -5 \Rightarrow B'(7, -5)$

At $C(1, 1) = 1 + i$

at $x = 1 \Rightarrow U = 7$

at $y = 1 \Rightarrow V = 1 \Rightarrow C'(1, 1)$



(b)

The given transformation is

$$W = iz + 2 - i$$

Transformation equations are

$$U + iV = i(x + iy) + 2 - i$$
$$= ix - y + 2 - i$$
$$U + iV = 2 - y + i(x - 1)$$

$\Rightarrow U = 2-y$, $V = x-1 \to (A)$

Using eq (A)

at $A(0,1) = i$

at $x=0$, $V=-1$

at $y=1$, $U=1$ $\Rightarrow A'(1,-1)$

At $B(1,-1) = 1-i$

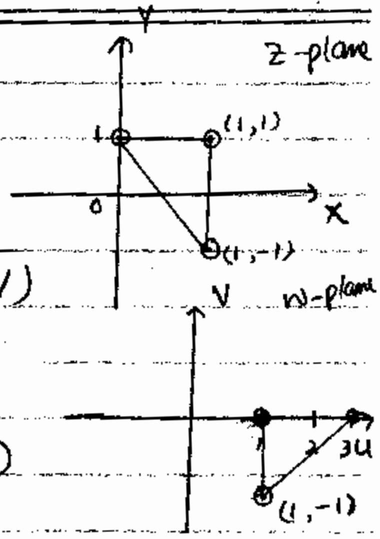
at $x=1 \Rightarrow V=0$

at $y=-1 \Rightarrow U=3 \Rightarrow B'(3,0)$

At $C(1,1) = 1+i$

at $x=1 \Rightarrow V=0$

at $y=1 \Rightarrow U=1 \Rightarrow C'(1,0)$



(c)

The given transformation is

$$W = 5e^{i\pi/3} z - 2 + 4i$$

$$U+iV = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)(x+iy) - 2 + 4i$$

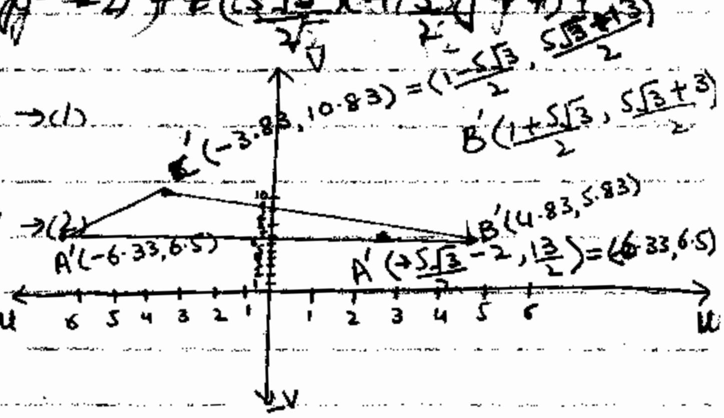
$$U+iV = 5\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(x+iy) - 2 + 4i$$

$$U+iV = \frac{5x}{2} + \frac{5\sqrt{3}}{2}ix + \frac{5}{2}iy - \frac{5\sqrt{3}}{2}y - 2 + 4i$$

$$U+iV = \left(\frac{5}{2}x - \frac{5\sqrt{3}}{2}y - 2\right) + i\left(\frac{5\sqrt{3}}{2}x + \frac{5}{2}y + 4\right)$$

$\Rightarrow U = \frac{5}{2}x - \frac{5\sqrt{3}}{\sqrt{2}}y - 2 \to (1)$

$V = \frac{5\sqrt{3}}{\sqrt{2}}x + \frac{5}{\sqrt{2}}y + 4 \to (2)$



At $A(0,1) = i$

at $x=0$, $y=1$

$U = -\frac{5\sqrt{3}}{2} - 2$, $V = \frac{5}{2} + 4 = \frac{5+8}{2} = \frac{13}{2}$

At $B(1,-1) \Rightarrow x=1$, $y=-1$

$U = \frac{5}{2} + \frac{5\sqrt{3}}{2} - 2 = \frac{5+5\sqrt{3}-4}{2} = \frac{1+5\sqrt{3}}{2}$, $V = \frac{5\sqrt{3}+3}{2}$

At $C(1,1) \Rightarrow x=1$, $y=1$

$U = \frac{5}{2} - \frac{5\sqrt{3}}{2} - 2$, $U = \frac{1-5\sqrt{3}}{2}$, $V = \frac{5\sqrt{3}+13}{2}$

Ex:-

Determine the eq of curve in the w-plane into which st. line $x+y=1$ is mapped under the transformations

(i) $w = z^2$

(ii) $w = 1/z$

Sol:-

(i) The given transformation is

$$w = z^2$$

Transformation equations are

$$u+iv = (x+iy)^2$$

$$u+iv = x^2-y^2+2ixy$$

$$\Rightarrow \quad u = x^2-y^2 \rightarrow (1) \qquad v = 2xy \rightarrow (2)$$

$$u = (x+y)(x-y) \qquad \text{Given } x+y=1$$

$$u = x-y$$

is a straight line.

Bilinear, Fractional (Linear fractional) or Mobius Transformation :-

The transformation T defined by $W = T(z) = \dfrac{Az+B}{Cz+D} \rightarrow (1)$

where A, B, C and D are complex constants and $AD - BC \neq 0$

is called bilinear, linear fractional or Mobius transformation.

Determinant :-

If $W = \dfrac{Az+B}{Cz+D}$

$$(Cz+D)W = Az+B$$
$$CWz + DW = Az + B$$
$$\Rightarrow CWz + DW - Az - B = 0$$
$$\Rightarrow CWz - Az + DW - B = 0$$
$$\Rightarrow (CW - A)z + DW - B = 0$$
$$\Rightarrow (CW - A)z = -DW + B$$
$$z = \frac{-DW+B}{CW-A}$$

"Replace A, D with opposite sign then we obtain inverse of Trans"

$$z = T^{-1}(W) = \frac{-DW+B}{CW-A} \; ; \; AD - BC \neq 0$$

is inverse of W

then $AD - BC$ is called the determinant of the transformation. Further, if $AD - BC = 1$ then the transformation $W = \dfrac{Az+B}{Cz+D}$ is called normalized.

Simple Transformation :-

A transformation $W = f(z)$ is said to be simple, if

(i) It is analytic

(ii) It is one - one

[one-one means distinct objects have distinct images]

Theorem :-

Prove that bilinear transformation is one - one transformation.

Proof :-

A bilinear transformation is

$$W = \frac{AZ+B}{CZ+D} \quad ; \quad AD - BC \neq 0$$

(i) First, we prove that it is analytic.

$AD - BC \neq 0$ shows that all the complex constants are non-zero.

So, their derivatives exists and continuous. So, $W = \frac{AZ+B}{CZ+D}$ is analytic.

(ii) Now, we prove that $W = \frac{AZ+B}{CZ+D}$ is 1-1 transformation.

Let $Z_1, Z_2 \in Z$

So that $W_1, W_2 \in W$

we have to prove that

if $W_1 = W_2$ then $Z_1 = Z_2$

$$W_1 = \frac{AZ_1+B}{CZ_1+D} \quad , \quad W_2 = \frac{AZ_2+B}{CZ_2+D} \quad ; \; AD-BC \neq 0$$

If $W_1 = W_2$

$$\frac{AZ_1+B}{CZ_1+D} = \frac{AZ_2+B}{CZ_2+D}$$

$$(AZ_1+B)(CZ_2+D) = (AZ_2+B)(CZ_1+D)$$

$$\Rightarrow \cancel{ACZ_1Z_2} + ADZ_1 + BCZ_2 + \cancel{BD} = \cancel{ACZ_1Z_2} + ADZ_2 + BCZ_1 + \cancel{BD}$$

$$\Rightarrow ADZ_1 - BCZ_1 = ADZ_2 - BCZ_2$$

$$\Rightarrow (AD-BC)Z_1 = (AD-BC)Z_2$$

$$\Rightarrow Z_1 = Z_2$$

Therefore, bilinear transformation is 1-1 Transformation.

Remarks :-

(1) $W = \dfrac{AZ+B}{CZ+D}$; $AD-BC \neq 0$

It has 4 constants namely A, B, C and D but actually it has three effective constants.

$$W = \frac{(A/B)z+1}{(C/B)z + D/B} = \frac{\alpha z+1}{\beta z+\gamma} \quad ; \quad \alpha\gamma - \beta \neq 0$$

(2) Critical points of the transformation are those points where $\dfrac{dw}{dz} = 0$ or ∞

$$W = \frac{Az+B}{Cz+D}$$

$$\frac{dw}{dz} = \frac{(Cz+D)(A(1)+0)-(Az+B)(C(1)+0)}{(Cz+D)^2}$$

$$\frac{dw}{dz} = \frac{ACz+AD-ACz-BC}{(Cz+D)^2}$$

$$\frac{dw}{dz} = \frac{AD-BC}{(Cz+D)^2} \rightarrow (A)$$

If $\dfrac{dw}{dz} = 0$

then $AD-BC = 0$ or $Cz+D = \infty$ or $z = \infty$ because C and D are constants

If $\dfrac{dw}{dz} = \infty$

then $Cz+D = 0$, then $z = -D/C$.

Therefore, $z = \infty$, $z = -D/c$ are critical points of a bilinear transformation

(3) Since $AD-BC \neq 0$ in the bilinear transformation then $\dfrac{dw}{dz} \neq 0$ from eq (A).

Every bilinear transformation represents a one-to-one conformal mapping of the whole closed z-plane onto the whole closed w-plane.

Product of two bilinear transformations:

Consider two transformations T_1 and T_2 defined by

$$\theta = T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \; ; \; a_1 d_1 - b_1 c_1 \neq 0$$

and

$$W = T_2(\theta) = \frac{a_2 \theta + b_2}{c_2 \theta + d_2} \; ; \; a_2 d_2 - b_2 c_2 \neq 0$$

$$W = T_2(\theta)$$

$$= T_2(T_1(z))$$

$$W = T_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right)$$

$$W = a.$$

$$W = \frac{a_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + b_2}{c_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + d_2}$$

$$W = \frac{\dfrac{a_1 a_2 z + a_2 b_1 + b_2 c_1 z + b_2 d_1}{c_1 z + d_1}}{\dfrac{a_1 c_2 z + b_1 c_2 + c_1 d_2 z + d_1 d_2}{c_1 z + d_1}}$$

$$W = \frac{(a_1 a_2 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)}$$

$$W = T_2(T_1(z)) = \frac{\alpha z + \beta}{\gamma z + \delta} \; , \; \alpha\delta - \beta\gamma \neq 0$$

is a bilinear transformation.

Ex:-

Consider the transformation

$W = T_1(z) = \frac{z+2}{z+3}$ and $W = T_2(z) = \frac{z}{z+1}$

Find $T_1^{-1}(W)$, $T_2^{-1}(W)$, $T_2T_1(z)$, $T_1T_2(z)$, $T_2^{-1}T_1(z)$

Sol:-

(i) $W = T_1(z) = \frac{z+2}{z+3}$

$$(z+3)W = z+2$$
$$zW + 3W = z+2$$
$$zW - z = 2 - 3W$$
$$z(W-1) = -3W + 2$$
$$z = \frac{-3W+2}{W-1}$$
$$T_1^{-1}(W) = \frac{-3W+2}{W-1}$$

(ii) $W = T_2(z) = \frac{z}{z+1}$

$$(z+1)W = z$$
$$zW + W = z$$
$$zW - z = -W$$
$$z(W-1) = -W$$
$$z = \frac{-W}{W-1}$$
$$T_2^{-1}(W) = \frac{-W}{W-1}$$

(iii) $T_2 T_1(z)$

$T_1(z) = \frac{z+2}{z+3}$, $T_2(z) = \frac{z}{z+1}$

$$T_2T_1(z) = T_2(T_1(z))$$
$$= T_2\left(\frac{z+2}{z+3}\right)$$

$$T_2 T_1(z) = \frac{\frac{z+2}{z+3}}{\frac{z+2}{z+3}+1}$$

$$= \frac{\frac{z+2}{z+3}}{\frac{z+2+z+3}{z+3}}$$

$$T_2 T_1(z) = \frac{z+2}{2z+5}$$

(iv) $T_1 T_2(z)$

$$T_1 T_2(z) = T_1(T_2(z))$$

$$= T_1\left(\frac{z}{z+1}\right)$$

$$T_1 T_2(z) = \frac{\frac{z}{z+1}+2}{\frac{z}{z+1}+3}$$

$$= \frac{\frac{z+2z+2}{z+1}}{\frac{z+3z+3}{z+1}}$$

$$T_1 T_2(z) = \frac{3z+2}{4z+3}$$

(v) $T_2^{-1} T_1(z)$

$$T_2^{-1}(T_1(z)) = T_2^{-1}\left(\frac{z+2}{z+3}\right)$$

$$= \frac{\frac{z+2}{z+3}}{1 - \frac{z+2}{z+3}}$$

$$= \frac{\frac{z+2}{\cancel{z+3}}}{\frac{\cancel{z}+3-\cancel{z}-2}{z+3}} = z+2$$

V.V.Imp

Theorem :-

Prove that the set of all bilinear transformation forms a non-abelian group under the product of transformation :-

Proof :-

(i) Closure property :-

We have already proved that the product of any two bilinear transformations is also a bilinear transformation.

(i-e) $T_1 T_2(z) = T_3(z)$

So that closure property is satisfied.

(ii) Associative law :-

The associative law $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ holds in general because this law holds for arbitrary transformations.

(iii) Existence of Identity :-

The identity transformation $I(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z \\ 1 \end{bmatrix} = \frac{z}{1}$ is a bilinear transformation. So that I serves as an identity element.

(iv) Existence of Inverse :-

Given $w = T(z) = \dfrac{az+b}{cz+d}$

$$z = T^{-1}(w) = \frac{dw-b}{-cw+a}$$

$$T^{-1}T(z) = T^{-1}\left(\frac{az+b}{cz+d}\right)$$

$$T^{-1}T(z) = \frac{d\left(\frac{az+b}{cz+d}\right)-b}{-c\left(\frac{az+b}{cz+d}\right)+a}$$

$$= \frac{\frac{adz+bd-bcz-bd}{cz+d}}{\frac{-acz-bc-acz+ad}{cz+d}}$$

$$= \frac{(ad-bc)z}{ad-bc} = z$$

$\Rightarrow \quad T^{-1}T(z) = I(z) = z$

Likewise

$$TT^{-1}(w) = I(z) = z$$

$$TT^{-1} = T^{-1}T = I$$

Hence, all four axioms are satisfied, therefore a set of bilinear transformations forms a group under multiplication.

Remarks:-

Since in general

$$T_1T_2(z) \neq T_2T_1(z) \quad \text{or} \quad T_1T_2 \neq T_2T_1$$

The commutative property is not true in general therefore, the group is non-abelian.

Theorem:-

Prove that a bilinear transformation maps circles and st. line into circle or st. line.

Proof:-

The general equation of a circle and straight line in the z-plane may be written as

$$AZ\bar{Z} + BZ + \bar{B}\bar{Z} + C = 0 \rightarrow (1)$$

where A and C are real and B is a Complex constants.

Equation (1) is a circle if $A \neq 0$ and $B\bar{B} - AC \geq 0$ and equation (1) is a straight line if $A = 0$, $B \neq 0$

The bilinear Transformation is

$$W = \frac{AZ + B}{CZ + D}, \quad AD - BC \neq 0$$

$$Z = \frac{-DW + B}{CW - A} \rightarrow (2)$$

$$\bar{Z} = \frac{-D\bar{W} + \bar{B}}{C\bar{W} - A} \rightarrow (3)$$

putting value of Z, $\bar{Z}$ in (1)

$$A\left[\left(\frac{-DW+B}{CW-A}\right)\left(\frac{-D\bar{W}+\bar{B}}{C\bar{W}-A}\right)\right] + B\left[\frac{-DW+B}{CW-A}\right] + \bar{B}\left[\frac{-D\bar{W}+\bar{B}}{C\bar{W}-A}\right] + C = 0$$

which is of the form $\alpha W\bar{W} + \beta\bar{W} + \bar{\beta}\bar{W} + \gamma = 0$

Hence, it is clear that it is circle if $\alpha \neq 0$, $\beta\bar{\beta} - \alpha\gamma \geq 0$ and it is a st-line if $\alpha = 0$ and $\beta \neq 0$

Remark :-

Bilinear Transformation is

$$W = \frac{AZ + B}{CZ + D}; \quad AD - BC \neq 0$$

$$W = \frac{\frac{C(AZ+B)}{C} - AD + AD}{CZ + D}$$

$$W = \frac{ACZ + BC - AD + AD}{C(CZ + D)}$$

$$W = \frac{ACz + AD + BC - AD}{C(Cz+D)}$$

$$Cz + D \,\overline{)\, Az + B}\quad \frac{A}{C}$$

$$\pm Az \pm \frac{AD}{C}$$

$$B - \frac{AD}{C} \Rightarrow \frac{BC - AD}{C}$$

$$W = \frac{A(Cz+D) + BC - AD}{C(Cz+D)}$$

$$W = \frac{A}{C} + \frac{BC - AD}{C(Cz+D)}$$

$$W = \frac{A}{C} + \frac{BC-AD}{C} \cdot \frac{1}{Cz+D}$$

Take $A/C = \alpha$, $\frac{BC-AD}{C} = \beta$

$z_1 = Cz + D$

$z_2 = \frac{1}{z_1}$

then $W = \alpha + \beta z_2$

Ex :-

Under the bi-linear transformation $W = \frac{2z+3}{z-4}$. Discuss the mapping of a circle $x^2 + (y-2)^2 = 4$

Sol :-

$$W = \frac{2z+3}{z-4}$$

$$W = \frac{2z+8-8+3}{z-4}$$

$$W = \frac{2z-8+11}{z-4}$$

$$W = \frac{2(z-4)+11}{z-4}$$

$$W = 2 + \frac{11}{z-4}$$

$z_1 = z - 4 \;, \rightarrow (1)$

$$Z_2 = \frac{1}{Z-4} \rightarrow (2)$$

$$W = 2 + 11 Z_2 \rightarrow (3)$$

Consider the equation (1) $Z_1 = Z - 4$

$\Rightarrow x_1 + iy_1 = (x+iy) - 4$

$\Rightarrow x_1 + iy_1 = (x-4) + iy$

$\Rightarrow x_1 = x - 4 \quad , \quad y_1 = y$

Equation of circle $x^2 + (y-2)^2 = 4 \rightarrow (A)$

put value of x and y

$$(x_1+4)^2 + (y_1-2)^2 = 4$$

which is equation of circle at centre $(-4, 2)$ and radius is "2".

Consider the equation (2)

$$Z_2 = \frac{1}{Z-4}$$

$$Z_2 = \frac{1}{Z_1}$$

$$x_2 + iy_2 = \frac{1}{x_1+iy_1} \times \frac{x_1 - iy_1}{x_1 - iy_1}$$

$$x_2 + iy_2 = \frac{x_1 - iy_1}{x_1^2 + y_1^2}$$

$$\Rightarrow x_2 + iy_2 = \frac{x_1}{x_1^2+y_1^2} - i\frac{y_1}{x_1^2+y_1^2}$$

$\Rightarrow x_2 = \frac{x_1}{x_1^2+y_1^2} \quad , \quad y_2 = \frac{-y_1}{x_1^2+y_1^2}$ put in (A)

$$\left(\frac{x_1}{x_1^2+y_1^2} - 0\right)^2 + \left(\frac{y_1}{x_1^2+y_1^2} - 2\right)^2 = 4$$

which is equation of circle with centre at $(0, 2)$ and radius "2".

Consider the equation (3).

$$W = 2 + 11 Z_2$$

$$U + iV = 2 + 11(x_2 + iy_2)$$

$$U + iV = 2 + 11x_2 + i11y_2$$

$$\Rightarrow U = 2 + 11x_2 \quad , \quad V = 11y_2$$

$$x_2 = \frac{U-2}{11} \quad , \quad y_2 = \frac{V}{11} \quad \text{put in (A)}$$

$$\left(\frac{U-2}{11}\right)^2 + \left(\frac{V}{11} - 2\right)^2 = 4$$

$$\frac{(U-2)^2}{121} + \frac{(V-22)^2}{121} = 4$$

$$(U-2)^2 + (V-22)^2 = (2 \times 11)^2$$

is equation of circle, centre at $(2, 22)$ and radius "22".

Theorem:-

If $W = \frac{Az+B}{Cz+D}$; $AD - BC \neq 0$ is a bilinear transformation further if w_1, w_2, w_3 and w_4 are the images of z_1, z_2, z_3 and z_4 under this bilinear transformation, then prove that the cross ratio of 4 points remains invariant

$$\{W_1, W_2, W_3, W_4\} = \{z_1, z_2, z_3, z_4\}$$

$$\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_2)(w_3 - w_4)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$$

or $$\frac{(w_1 - w_4)(w_2 - w_3)}{(w_1 - w_3)(w_2 - w_4)} = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)}$$

Proof:-

Given the bilinear transformation

$$\left.\begin{aligned} W_1 &= \frac{Az_1 + B}{Cz_1 + D} \quad , \quad W_2 = \frac{Az_2 + B}{Cz_2 + D} \\ W_3 &= \frac{Az_3 + B}{Cz_3 + D} \quad , \quad W_4 = \frac{Az_4 + B}{Cz_4 + D} \end{aligned}\right] AD - BC \neq 0$$

To prove $\frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_2)(w_3 - w_4)} = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$

$$L\cdot H\cdot S = \frac{(\omega_1-\omega_3)(\omega_2-\omega_4)}{(\omega_1-\omega_2)(\omega_3-\omega_4)} \rightarrow (A)$$

$$\omega_1-\omega_3 = \left(\frac{Az_1+B}{Cz_1+D}\right)-\left(\frac{Az_3+B}{Cz_3+D}\right)$$

$$= \frac{(Az_1+B)(Cz_3+D)-(Az_3+B)(Cz_1+D)}{(Cz_1+D)(Cz_3+D)}$$

$$= ACz_1z_3+ADz_1+BCz_3+BD-(ACz_1z_3+ADz_3+BCz_1+BD)/(Cz_1+D)(Cz_3+D)$$

$$\omega_1-\omega_3 = \frac{ADz_1-ADz_3+BCz_3-BCz_1}{(Cz_1+D)(Cz_3+D)}$$

$$\omega_1-\omega_3 = \frac{AD(z_1-z_3)-BC(z_1-z_3)}{(Cz_1+D)(Cz_3+D)}$$

$$\omega_1-\omega_3 = \frac{(AD-BC)(z_1-z_3)}{(Cz_1+D)(Cz_3+D)}$$

Similarly

$$\omega_2-\omega_4 = \frac{(AD-BC)(z_2-z_4)}{(Cz_2+D)(Cz_4+D)}$$

$$\omega_1-\omega_2 = \frac{(AD-BC)(z_1-z_2)}{(Cz_1+D)(Cz_2+D)}$$

$$\omega_3-\omega_4 = \frac{(AD-BC)(z_3-z_4)}{(Cz_3+D)(Cz_4+D)}$$

putting all values in (A)

$$\frac{\left(\frac{(AD-BC)(z_1-z_3)}{(Cz_1+D)(Cz_3+D)}\right)\left(\frac{(AD-BC)(z_2-z_4)}{(Cz_2+D)(Cz_4+D)}\right)}{\left(\frac{(AD-BC)(z_1-z_2)}{(Cz_1+D)(Cz_2+D)}\right)\left(\frac{(AD-BC)(z_3-z_4)}{(Cz_3+D)(Cz_4+D)}\right)}$$

$$= \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_2)(z_3-z_4)} = R\cdot H\cdot S$$

$$\Rightarrow L\cdot H\cdot S = R\cdot H\cdot S$$

Similarly

$$\frac{(\omega_1-\omega_4)(\omega_2-\omega_3)}{(\omega_1-\omega_3)(\omega_2-\omega_4)} = \frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_2-z_4)}$$

Remarks:-

With the help of invariance of cross ratios of 4 points under the bilinear transformation $W = \frac{AZ+B}{CZ+D}$; $AD - BC \neq 0$

$\{W_1, W_2, W_3, W_4\} = \{Z_1, Z_2, Z_3, Z_4\}$

We can uniquelly determined with the three given pair of points when

$$W = \frac{\alpha Z + 1}{\beta Z + \gamma} \quad \{W, W_1, W_2, W_3\} = \{Z, Z_1, Z_2, Z_3\}$$

Ex :-

Find the bilinear transformation with help of following pair of points $z = \pm i, -1$ $W = \pm 1, i$

Sol :-

$Z = \pm i, -1 \Rightarrow Z_1 = i,\ Z_2 = -i,\ Z_3 = -1$

$W = \pm 1, i \Rightarrow W_1 = 1,\ W_2 = -1,\ W_3 = i$

Cross ratio of four points can be written as $\{W, W_1, W_2, W_3\} = \{Z, Z_1, Z_2, Z_3\}$

$$\text{or} \quad \frac{(W-W_1)(W_2-W_3)}{(W_1-W_2)(W_3-W)} = \frac{(Z-Z_1)(Z_2-Z_3)}{(Z_1-Z_2)(Z_3-Z)}$$

$$\frac{(W-1)(-1-i)}{(1+1)(i-W)} = \frac{(Z-i)(-i+1)}{(i+i)(-1-Z)}$$

$$\frac{(W-1)(-1-i)}{-2(W-i)} = \frac{(Z-i)(1-i)}{-2i(Z+1)}$$

$$\frac{-W - Wi + 1 + i}{-2W+2i} = \frac{Z - iZ - i - 1}{-2iZ - 2i}$$

$(-2iZ - 2i)(-W - Wi + 1 + i) = (-2W + 2i)(Z - iZ - i - 1)$

$\Rightarrow \cancel{2iWZ} - \cancel{2WZ} - 2iZ + \cancel{2Z} + \cancel{2iW} - 2W - \cancel{2i} + \cancel{2} = -\cancel{2WZ} + \cancel{2iWZ} + \cancel{2iW}$
$\qquad + 2W + 2iZ + \cancel{2Z} - \cancel{2i} + \cancel{2}$

$\Rightarrow \quad -2iZ - 2W = 2W + 2iZ$

$\quad -2iZ - 2iZ = 2W + 2W$

$iW = 4Wz + 4iz = 0$

$$W = -iz$$

For $w_1 = 1,\ z_1 = i$

$1 = -i(i) = 1$ Hence, $w = -iz$

Assignment :-

Ex :-

Find the bilinear transformation using the given data $z_i = \pm 2, i$, $w_i = \pm i, 1$

$i = 1, 2, 3$

Sol :-

$z_i = \pm 2, i \Rightarrow z_1 = 2,\ z_2 = -2,\ z_3 = i$

$w_i = \pm i, 1 \Rightarrow w_1 = i,\ w_2 = -i,\ w_3 = 1$

Cross ratio of 4 points can be written as $\{w, w_1, w_2, w_3\} = \{z, z_1, z_2, z_3\}$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z_2)}$$

$$\Rightarrow \frac{(w-i)(-i-1)}{(i+i)(i-w)} = \frac{(z-2)(-2-i)}{(2+2)(i-z)}$$

$$\Rightarrow \frac{(-wi-w-1+i)}{2i(1-w)} = \frac{(-2z-iz+4+2i)}{4(i-z)}$$

$\Rightarrow (-wi-w+1+i)(4i-4z) = (-2z-iz+4+2i)(2i-2iw)$

$\Rightarrow 4w+4iwz-4iw+4wz-4i+4z-4-4iz = -4iz+2z+8i-4+4iwz-2wz-8iw$

$\Rightarrow 4w-4iw+4wz-4i+4z = 2z+8i+4w-2wz-8iw$

$\Rightarrow -4iw+8iw+4wz+2wz-4i-8i+4z-2z = 0$

$\Rightarrow +4iw+6wz-12i+2z = 0$

$\Rightarrow +4iw-12i = -2z-6wz$

$\Rightarrow 2z+6wz = 12i+14iw$

$\Rightarrow 6wz+4iw = 12i+2z$

$\Rightarrow 2w(3z+2i) = 12i-2z$

$\Rightarrow 2w(3z+2i) = 2(6i-z)$

$$w(3z+2i) = (6i-z)$$

$$w = \frac{6i-z}{3z+2i}$$

Ex :-

Find the linear fractional (bilinear) transformation that maps the points

(i) $z_1 = 1$, $z_2 = -1$, $z_3 = i$ onto the points $w_1 = 0$, $w_2 = \infty$, $w_3 = 1$,

(ii) $z_1 = 2$, $z_2 = i$, $z_3 = -2$ onto the points $w_1 = 1$, $w_2 = i$, $w_3 = -1$

Sol :-

(i) $z_1 = 1$, $z_2 = -1$, $z_3 = i$, $w_1 = 0$, $w_2 = \infty$, $w_3 = 1$

Cross ratios of 4 points can be written as $\{w, w_1, w_2, w_3\} = \{z, z_1, z_2, z_3\}$

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_3)(z_3-z_2)}$$

$$\frac{(w-0)(\infty-1)}{(\cancel{w}-\cancel{\infty})(0-\cancel{\infty})} = \frac{(z-1)(-1-i)}{(z+1)(i+z)}$$

$$\frac{\cancel{\infty}(w-0)(1-\frac{1}{\infty})}{\cancel{\infty}(0-1)(1-w)} = \frac{(z-1)(-1-i)}{(2(i-z))}$$

$(2z-2i)(w-0) = (w-1)(-z-iz+1+i)$

$2w(w-0)(1-0)$ [illegible] $2w - zw$ [illegible] $zi + i + iz + iz - i$

[illegible] $+ (1-w)$ [illegible] $2(i-z)$

$$\frac{w}{[illegible]} = \frac{-z-zi+1+i}{2(i-z)}$$

[illegible] $w-1$

$(2i-2z)w = (w-1)(-z-zi+1+i)$

$\Rightarrow 2iw - 2wz = -wz - wzi + w + wi + z + zi - 1 - i$

$\Rightarrow 2iw - iw - 2wz + wz = -wzi + w + z + zi - 1 - i$

$\Rightarrow 2iw - wz = -wzi + w + z + zi - 1 - i$

$iw - wz + wzi - w = z + zi - 1 - i$

$w(i - z + zi - 1) = z + zi - 1 - i$

$$w = \frac{zi + z - i - 1}{zi - z + i - 1}$$ is bilinear transformation

$$\therefore 0 = \frac{\cancel{i} + \cancel{1} - \cancel{i} - \cancel{1}}{i - 1 + i - 1} = \frac{0}{2(i-1)}$$

(ii) $z_1 = 2 \,,\; z_2 = i \,,\; z_3 = -2$

$w_1 = 1 \,,\; w_2 = i \,,\; w_3 = -1$

Cross ratio of four points can be written as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\frac{wi+w-i-1}{-1-w+i+wi} = \frac{zi+2z-2i-4}{-4-2z+2i+iz}$$

$$(-4-2z+2i+iz)(wi+w-i-1) = (zi+2z-2i-4)(-1-w+i+iw)$$

$\Rightarrow -4\cancel{wi}-4w+4i+4-2wzi-\cancel{2zw}+2\cancel{iz}+2z+2w+2\cancel{iw}+\cancel{z}$
$-2i-\cancel{wz}+iwz+z-iz = -zi-2z+2i+4-wzi-$
$2\cancel{wz}+2\cancel{iw}+4w-z+2\cancel{iz}+\cancel{z}-4i-\cancel{wz}+2iwz+2w-\cancel{4iw}$

$\Rightarrow -4w+4i+\cancel{4}-2wzi+2z-2w-2i+iwz+z-iz$
$= -zi-2z+2i+\cancel{4}-wzi+4w-z-4i+2iwz+2w$

$\Rightarrow -4w-4w-2w-2w+4i-2i+4i-2i+2z+z+z+2z$
$-2wzi+wzi+wzi-2wzi-iz+iz = 0$

$\Rightarrow -12w+4i+6z-2wzi = 0$

$-12w-2wzi = -6z-4i$

$-2w(6+iz) = -2(3z+2i)$

$$w = \frac{3z+2i}{(6+iz)}$$

$$w = \frac{3z+2i}{16+2iz}$$

$$1 = \frac{6+2i}{16+4i}$$

$1 = 1$

Hence, $w = \dfrac{3z+2i}{6+iz}$ is bilinear Transformation.

Ex:-

Find a bilinear transformation which transforms the unit circle $|z|=1$ into the real axis in such a way that the point $z_1 = 1$ is mapped into $w_1 = 0$, $z_2 = i$ is mapped into $w_2 = 1$ and $z_3 = -1$ is mapped into $w_3 = \infty$

Sol:-

Step I:-

The points $z_1 = 1$, $z_3 = -1$, $z_2 = i$ all lie on the circle $|z| = 1$ and the points $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$ lie on the real axis of the w-plane.

Cross ratios of 4-points can be written as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z-1)(i+1)}{(1-i)(-1-z)}$$

$$\frac{w(1-\frac{1}{0})}{(1-0)(w-\frac{1}{0})} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w}{1} = \frac{(z-1)(i+1)^2}{(z+1)(i^2-1)}$$

$$w = \frac{(z-1)(i^2+1+2i)}{(z+1)(-1-1)}$$

$$w = \frac{(z-1)(-1+1+2i)}{(z+1)(-2i)}$$

$$w = \frac{+2i(z-1)}{-2(z+1)}$$

$$w = \frac{-i(z-1)}{z+1}$$

$$w = \frac{i(1-z)}{z+1}$$

Step II :-

The inverse transformation can be obtained as

$$W = \frac{i(1-z)}{z+1}$$

$$zW + W = i - iz$$

$$zW + iz = i - W$$

$$z(W+i) = i - W$$

$$z = \frac{i-W}{W+i}$$

So that $|z| = \left|\frac{i-W}{W+i}\right| = 1$ Given $|z| = 1$

Hence, $|z| = 1$ is transformed into

$$\left|\frac{i-W}{W+i}\right| = 1$$

$$\frac{|W-i|}{|W+i|} = 1$$

$$|W-i| = |W+i|$$

$$|u+iv-i| = |u+iv+i|$$

$$|u+i(v-1)| = |u+i(v+1)|$$

$\Rightarrow \quad u^2 + (v-1)^2 = u^2 + (v+1)^2$

$\Rightarrow \quad (v-1)^2 = (v+1)^2$

$\Rightarrow \quad (v+1)(v+1) = (v-1)^2$

$\Rightarrow \quad v^2 + 2v + 1 = v^2 - 2v + 1$

$\Rightarrow \quad 2v + 2v = 0$

$\quad 4v = 0$

$\Rightarrow \quad v = 0$ which is the real axis of W-plane (i-e) u-axis.

Ex :- (Page 373 Pennsi)

Find the linear transformation which maps the points $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ onto the points $w_1 = -i$, $w_2 = 1$, $w_3 = i$

Sol:-

$z_1 = -1, \quad z_2 = 0, \quad z_3 = 1$

$w_1 = -i, \quad w_2 = 1, \quad w_3 = i$

Cross product of four points can be written as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w+i)(1-i)}{(-i-1)(i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\frac{w-iw+i+1}{+1+iw-i+w} = \frac{-z-1}{-1+z}$$

$$(z-1)(w-iw+i+1) = (-z-1)(w-i+iw+1)$$

$$wz-\cancel{iwz}+\cancel{iz}+z-\cancel{w}+iw-i-\cancel{1} = -wz+\cancel{iz}-\cancel{iwz}-z$$
$$-\cancel{w}+i-iw-\cancel{1}$$

$$wz+wz+z+z+iw+iw-i-i = 0$$

$$2wz+2z+2iw-2i = 0$$

$$2(wz+iw+z-i) = 0$$

$$wz+iw = i-z$$

$$w(i+z) = i-z$$

$$w = \frac{i-z}{i+z}$$ is bilinear transformation

$$-i = \frac{i+1}{i-1} \times \frac{i-1}{i-1}$$

$$-i = \frac{i^2-1}{(i-1)^2}$$

$$-i = \frac{-1-1}{i^2-2i+1}$$

$$-i = \frac{-2}{-\cancel{1}-2i+\cancel{1}}$$

$$-i = \frac{\cancel{-2}}{\cancel{-2}i}$$

$-i = i$, $i = -i$

Ex :-

To find linear fractional transformation which maps the points (i) $z_1 = -1$, $z_2 = 0$, $z_3 = 1$ onto the points $w_1 = -i$, $w_2 = 1$, $w_3 = i$ respecively

(ii) $z_1 = 2$, $z_2 = i$, $z_3 = -2$ onto the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$ respectively.

(iii) $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ onto the points $w_1 = 0$, $w_2 = i$, $w_3 = \infty$ respectively.

Sol :-

(i) $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, $w_1 = -i$, $w_2 = 1$, $w_3 = i$

Cross ratios of four points are

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w+i)(1-i)}{(-i-1)(i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}$$

$$\Rightarrow \frac{w+i-wi+1}{1+iw-i+w} = \frac{-z-1}{-1+z}$$

$$\Rightarrow (z-1)(w-wi+i+1) = (-z-1)(1+w+iw-i)$$

$$\Rightarrow (zw - \cancel{zwi} + \cancel{zi} + z - \cancel{w} + wi - i - \cancel{1}) = (-z - wz - i\cancel{wz} + \cancel{iz} - \cancel{1} - \cancel{w} - iw + i)$$

$$zw + zw + z + z + wi + wi - i - i = 0$$

$$2zw + 2z + 2wi - 2i = 0$$

$$zw + z + wi - i = 0.$$

$$(zw + wi) + (z - i) = 0$$

$$w(z+i) = -z + i$$

$$w = \frac{-z+i}{z+i}$$

(ii) $z_1 = 2$, $z_2 = i$, $z_3 = -2$, $w_1 = 1$, $w_2 = i$, $w_3 = -1$

Cross ratios of 4 points are

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$$\frac{wi-i+w-1}{-1-w+i+iw} = \frac{zi+2z-2i-4}{-4-2z+2i+zi}$$

$\Rightarrow (-4-2z+2i+zi)(wi-i+w-1) = (zi+2z-2i-4)(-1-w+i+iw)$

$\Rightarrow -4wi+4i-4w+4-2iwz+2iz-2wz+2z-2w+2+2iw-2i+zw+z+izw-zi = -zi-2z+2i+4-wzi-2wz+2iw+4w-z+2iz+z-4i-wz+2iwz+2w-4iw$

$\Rightarrow iwz-4w-2w+2z+z+zw = -2z-z+4w+2w+2iwz-iwz-wz+2iwz-4i$

$\Rightarrow -6w+3z+iwz+iwz-3z+6w+2iwz-4i$

$\Rightarrow 3z-6w+3z-6w-2iwz+4i=0$

$\Rightarrow 6z-12w=2iwz-4i$

$-2iwz-12w=-6z-4i$

$-2iwz-12w=-2(3z+2i)$

$-(2iz+12)w = \frac{-2(3z+2i)}{(z-iz-12)}$

$w = \frac{-2(3z+2i)}{-2(6+iz)}$ is bilinear transformation.

$w=\frac{3z+2i}{6+zi}$ $\quad 1 = \frac{3(1)+2i}{6+2i}$

$1=1$

(iii)

$z_1=\infty,\quad z_2=i,\quad z_3=0$

$w_1=0,\quad w_2=i,\quad w_3=\infty$

Cross ratio of four points can be written as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$$

$$\frac{\cancel{\infty}(w)(\frac{i}{\infty}-1)}{\cancel{\infty}(-i)(1-\frac{w}{\infty})} = \frac{\cancel{\infty}(\frac{z}{\infty}-1)(i)}{\cancel{\infty}(1-\frac{i}{\infty})(-z)}$$

$$\frac{w(0-1)}{-i(1-0)} = \frac{(0-1)(i)}{(1-0)(-z)}$$

$$\frac{-w}{-i} = \frac{-i}{-z}$$

$$\frac{w}{i} = +\frac{i}{z}$$

$$w = \frac{i^2}{z}$$

$w = -\frac{1}{z}$ is bilinear transformation

$$0 = -\frac{1}{\infty}$$

$$\Rightarrow 0 = 0$$

Ex :-

Find the linear fractional transformation which maps the points $z_1 = 0$, $z_2 = 0$, $z_3 = -1$ onto $w_1 = i$, $w_2 = \infty$, $w_3 = -1$ respectively

Sol :-

$z_1 = 0$, $z_2 = 0$, $z_3 = -1$

$w_1 = i$, $w_2 = \infty$, $w_3 = -1$

Cross ratio of four points can be written as

$$\frac{(w-w_1)(w_3-w_2)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(\infty+1)}{(i-\infty)(-1-w)} = \frac{(z-0)(0+1)}{(0-0)(-1-z)}$$

$$\frac{\cancel{\infty}(w-i)(1+\frac{1}{\infty})}{\cancel{\infty}(\frac{i}{\infty}-1)(-w-1)} = \frac{z}{0}$$

$$\frac{(w-i)(1+0)}{(0-1)(-w-1)} = \frac{z}{0}$$

$$\frac{w-i}{w+1} = \frac{z}{0}$$

$$0(w-i) = z(w+1)$$

$$-z(w+1) = 0$$

$$-zw - z = 0$$

$$\Rightarrow -zw = z$$

$$\Rightarrow w = z$$

$$w = -1 \quad = \frac{0(z)+(-1)}{1-0}$$

Ex:-

Find linear fractional transformation which maps the points $z_1 = 1$, $z_2 = 0$, $z_3 = -1$ onto $w_1 = i$, $w_2 = \infty$, $w_3 = 1$ respectively.

Sol:-

$z_1 = 1 \qquad z_2 = 0 \qquad z_3 = -1$

$w_1 = i \qquad w_2 = \infty \qquad w_3 = 1$

Cross ratio of four points can be written as

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(w-i)(\infty-1)}{(i-\infty)(1-w)} = \frac{(z-1)(0+1)}{(1-0)(-1-z)}$$

$$\frac{\cancel{\infty}(w-i)(1-0)}{\cancel{\infty}(0-1)(1-w)} = \frac{z-1}{-1-z}$$

$$\frac{w-i}{-1+w} = \frac{z-1}{-1-z}$$

$$(W-i)(-1-z) = (z-1)(W-1)$$
$$\cancel{-W} - Wz + i + iz = Wz - z - \cancel{W} + 1$$
$$-Wz - Wz = 1 - z - i - iz$$
$$-2Wz = 1 - z - i(1+z)$$
$$W = \frac{1-z+i(1+z)}{-2z}$$ is bilinear transformation

$$i = \frac{\cancel{1}-\cancel{1}+i(1+1)}{-2}$$
$$i = \frac{-2i}{-2}$$
$$i = i$$

Ex:- Most Imp

Find bilinear transformation which maps the points $z_1 = i$, $z_2 = -i$, $z_3 = 1$ of the z-plane into $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ of w-plane respectively.

Sol:-

$$z_1 = i, \quad z_2 = -i, \quad z_3 = 1$$
$$w_1 = 0, \quad w_2 = 1, \quad w_3 = \infty$$

Cross ratio of four points can be written as

$$\frac{(W-W_1)(W_2-W_3)}{(W_1-W_2)(W_3-W)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\frac{(W-0)(+1-\infty)}{(0-1)(\infty-W)} = \frac{(z-i)(-i-1)}{(i+i)(1-z)}$$

$$\frac{\cancel{\infty}(W)\left(\frac{1}{\infty}-1\right)}{\cancel{\infty}(-1)\left(1-\frac{W}{\infty}\right)} = \frac{(z-i)(-i-1)}{2i(1-z)}$$

$$\frac{W(0-1)}{-1(1-0)} = \frac{-iz-z-1+i}{2i-2iz}$$

$$\frac{\cancel{-}W}{\cancel{-1}} = \frac{-z(i+1)-1+i}{2(i-iz)}$$

$W = \frac{-z(1+i)+i-1}{2i(1-z)}$ is bilinear transf

Ex:- $= -z+iz+i-1$

Prove that transformation $w = \frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ onto the st. line $4u+3=0$ and explain why curve obtaine is not a circle.

Sol:-

Step I:- The given transformation is $w = \frac{2z+3}{z-4}$

$\Rightarrow wz - 4w = 2z+3$

$wz - 2z = 4w+3$

$z = \frac{4w+3}{w-2} \rightarrow (1)$ is inverse of $w = \frac{2z+3}{z-4}$

$z = x+iy, \quad \bar{z} = x-iy$

$z\bar{z} = x^2+y^2, \quad z+\bar{z} = 2x$

Eq of circle $x^2+y^2-4x=0$ can be written as $z\bar{z} - 2(z+\bar{z}) = 0 \rightarrow (2)$

Step II:- putting $z = \frac{4w+3}{w-2}, \quad \bar{z} = \frac{4\bar{w}+3}{\bar{w}-2}$ in (2)

$$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right) - 2\left(\frac{4w+3}{w-2} + \frac{4\bar{w}+3}{\bar{w}-2}\right) = 0$$

$$\frac{(4w+3)(4\bar{w}+3) - 2(4w+3)(\bar{w}-2) - 2(4\bar{w}+3)(w-2)}{(w-2)(\bar{w}-2)} = 0$$

$\Rightarrow 16w\bar{w} + 12w + 12\bar{w} + 9 - 2(4w\bar{w} - 8w + 3\bar{w} - 6) - 2(4w\bar{w} - 8\bar{w} + 3w - 6) = 0$

$\Rightarrow \cancel{16w\bar{w}} + 12w + 12\bar{w} + 9 - \cancel{8w\bar{w}} + 16w - 6\bar{w} + 12 - \cancel{8w\bar{w}} + 16\bar{w} - 6w + 12 = 0$

$22w + 22\bar{w} + 33 = 0$

$11(2w + 2\bar{w} + 3) = 0$

$\Rightarrow 2w + 2\bar{w} + 3 = 0$

$\Rightarrow 2(w+\bar{w}) + 3 = 0$

$\Rightarrow 4u + 3 = 0$ which is st. line.

Since, st. line is a particular case of a circle when co-efficient of $w\bar{w}$ is zero, which is ... under ... bilinear transformation.

Ex:- Find the mobius transformation that maps the points z_1 onto the points $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$ respective

Sol:-

$$z_1, z_2, z_3 \text{ onto } w_1 = 0, w_2 = 1, w_3 = \infty$$

Cross ratio of four points can be written as

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{w_3(w-w_1)(\frac{w_2}{w_3}-1)}{w_3(\frac{w}{w_3}-1)(w_1-w_2)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{(w-0)(\frac{w_2}{\infty}-1)}{(\frac{w}{\infty}-1)(0-1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{(w-0)(0-1)}{(0-1)(0-1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$\frac{-w}{1} = \frac{(z_1-z_1)(z_2-z_3)}{(z-z_3)(z_1-z_2)}$$

$$-w = \frac{z\,z_2 - z_1 z_3 - z_1 z_2 + z_1 z_3}{z z_1 - z z_2 - z_1 z_3 + z_2 z_3}$$

$$w = \frac{-(z(z_2-z_3) - z_1(z_2-z_3))}{z(z_1-z_2) - z_3(z_1-z_2)}$$

or

$$w = \frac{z(z_3-z_2) + (z_1 z_2 - z_1 z_3)}{z(z_1-z_2) + (z_2 z_3 - z_1 z_3)}$$

Fixed or Invariant points of Bilinear Transformation:-

The points which concide with their transformation under a bilinear transformation are called the fixed points of the transformation

Consider any bilinear transformation

$$w = \frac{az+b}{cz+d} \quad ; \quad ad-bc \neq 0$$

The fixed points of transformation can be obtained by putting $w = z$

$$z = \frac{az+b}{cz+d}$$

$$(cz+d)z = az+b$$

$$cz^2+dz = az+b$$

$$cz^2+dz-az-b = 0$$

$$cz^2+(d-a)z-b = 0$$

If Disc $= 0$ then You will get a single non-~~in~~finite fixed point.

If Disc > 0 " " " two non-~~in~~finite fixed points.

If Disc < 0 " " " " "

If $c = 0$, $d \neq 0$

Then $w = a/d \, z + b/d$

For fixed point $w = z$

$$z = a/d \, z + b/d$$

$$z - a/d \, z = b/d$$

$$z\left(1-\frac{a}{d}\right) = \frac{b}{d}$$

$$z\frac{(d-a)}{d} = \frac{b}{d}$$

$$z(d-a) = b$$

$$\Rightarrow \quad z = \frac{b}{d-a}$$

one of the fixed point is ∞ if $d=a$. If $d-a \neq 0$, then the transformation has fixed point $\frac{b}{d-a}$.

Theorem:-

Prove that every bilinear transformation with a single finited fixed point say α can be written as

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$$

Proof:-

Let the bilinear transformation be

$$w = \frac{az+b}{cz+d} \; ; \quad ad-bc \neq 0 \rightarrow (1)$$

which has a single fixed point say α

Replace w by z, we get

$$z = \frac{az+b}{cz+d}$$

or $\quad cz^2 + (d-a)z - b = 0 \rightarrow (2)$

Since, (1) has a single point say α and $c \neq 0$

$$cz^2 + (d-a)z - b = c(z-\alpha)^2$$

$$cz^2 + (d-a)z - b = c(z^2 - 2\alpha z + \alpha^2)$$

$$cz^2 + (d-a)z - b = cz^2 - 2\alpha c z + c\alpha^2$$

Comparing both sides (co-efficients

$$d-a = -2\alpha c$$

$$\Rightarrow a = d + 2\alpha c$$

$$-b = c\alpha^2$$

$$b = -c\alpha^2$$

Putting values of a and b in (1)

$$w = \frac{(d+2\alpha c)z - \alpha^2 c}{cz+d}$$

$$w-\alpha = \frac{(d+2\alpha c)z - \alpha^2 c}{cz+d} - \alpha$$

$$= \frac{(d+2\alpha c)z - \alpha^2 c - \alpha(cz+d)}{cz+d}$$

$$= \frac{(d+2\alpha c)z - \alpha cz - \alpha^2 c - \alpha d}{cz+d}$$

$$= \frac{(d+\alpha c)z - \alpha(\alpha c+d)}{cz+d}$$

$$\frac{1}{w-\alpha} = \frac{cz+d}{(d+\alpha c)z - \alpha(\alpha c+d)}$$

$$= \frac{cz+d}{(z-\alpha)(\alpha c+d)}$$

$$\frac{1}{w-\alpha} = \frac{d}{(z-\alpha)(\alpha c+d)}\left[1+\frac{cz+d}{\alpha c+d}\right] = \frac{1}{(z-\alpha)}\left[1+\frac{c(z-\alpha)}{\alpha c+d}\right]$$

$$= \frac{1}{z-\alpha} + \frac{c}{\alpha c+d}$$

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k \quad ; \quad k = \frac{c}{\alpha c+d}$$

So, we can say that a bilinear transformation has a single fixed point then we can write this transformation in the form $\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$.

Ex :-

Prove that every bilinear transformation with two fixed points α, β can be written in the form.

$$\frac{W-\alpha}{W-\beta} = K\left(\frac{Z-\alpha}{Z-\beta}\right)$$

which is known as the normal form.

Proof:-

Consider any bilinear transformation with α, β as fixed points and suppose it transforms a point γ into the point δ, then the points α, β, γ, Z are mapped into the points α, β, δ, W respectively.

Since the cross ratio of four points is preserved under the bilinear transformation

$$\therefore (W, \alpha, \beta, \delta) = (Z, \alpha, \beta, \gamma)$$

or

$$\frac{(W-\alpha)(\delta-\beta)}{(\alpha-\delta)(\beta-W)} = \frac{(Z-\alpha)(\gamma-\beta)}{(\alpha-\gamma)(\beta-Z)}$$

or $$\frac{W-\alpha)}{W-\beta} = \frac{Z-\alpha}{Z-\beta}\left(\frac{\gamma-\beta}{\alpha-\gamma}\right)\left(\frac{\alpha-\delta}{\delta-\beta}\right)$$

or $$\frac{W-\alpha}{W-\beta} = K\left(\frac{Z-\alpha}{Z-\beta}\right)$$ where $K = \left(\frac{\gamma-\beta}{\alpha-\gamma}\right)\left(\frac{\alpha-\delta}{\delta-\beta}\right)$

Ex:-

Find the fixed points and the normal form of the bilinear transformation.

$$W = \frac{3iz+1}{z+i}$$

Sol:-

Step I:-

$$W = \frac{3iz+1}{z+i}$$

The fixed points are given by $W = z$

$$z = \frac{3iz+1}{z+i}$$

$$z(z+i) = 3iz+1$$
$$z^2+zi = 3iz+1$$
$$z^2-3zi-zi-1=0$$
$$z^2-2zi-1=0$$
$$z^2-2zi+i^2=0$$
$$(z-i)^2=0$$

Thus $z=i$ is only the fixed point.

Step II:-

Normal form

$$w = \frac{3iz+1}{z+i}$$
$$wz+iw=3iz+1$$
$$(wz-1)+i(w-3z)=0$$
$$(w-i)(z-i)+iw+iz+iw-3iz=0$$
$$(w-i)(z-i)+2iw-2iz=0$$
$$(w-i)(z-i)+2iw-2i^2-2iz+2i^2=0$$
$$(w-i)(z-i)+2i(w-i)-2i(z-i)=0$$

Dividing by $(w-i)(z-i)$

$$1+\frac{2i}{z-i}-\frac{2i}{w-i}=0$$
$$\frac{2i}{w-i}=\frac{2i}{z-i}+1 \quad \text{Dividing by } 2i$$
$$\frac{1}{w-i}=\frac{1}{z-i}+\frac{1}{2i}$$
$$\frac{1}{w-i}=\frac{1}{z-i}+\frac{i}{2i^2}$$
$$\frac{1}{w-i}=\frac{1}{z-i}-\frac{i}{2} \quad \text{is normal form.}$$

Ex:-

Find the fixed points of transformation

(i) $w = 2z-5$

Sol:-

$$W = \frac{2z-5}{z+4}$$

Fixed points of the given transformation is obtained by putting $W = z$

$$z = \frac{2z-5}{z+4}$$

$$z(z+4) = 2z-5$$

$$z^2+4z = 2z-5$$

$$z^2+4z-2z+5=0$$

$$z^2+2z+5=0$$

$$a=1,\ b=2,\ c=5$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$z = \frac{-2 \pm \sqrt{-16}}{2}$$

$$z = \frac{-2 \pm 4i}{2}$$

$$z = -1 \pm 2i$$

$z = [-1+2i,\ -1-2i]$ fixed points of W.

CHAPTER NO 5

COMPLEX INTEGRATION

Introduction:-

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally consine and powerful, and most of the proofs are simple. The theory is also noted for its great utility in Applied Mathematics.

Basic Definitions :-

Locus of a point :-

It is the path traced by the point moving under certain given conditions. **Curve:** It is the locus of a point whose co-ordinates can be expressed in the form of single parametre.

e.g
$$\left.\begin{array}{l} x = a\cos t \\ y = a\sin t \end{array}\right] \Rightarrow x^2+y^2=a^2$$

$$\left.\begin{array}{l} x = a\cos t \\ y = b\sin t \end{array}\right] \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Smooth curve :-

A curve traced by $z=f(t)$ such that $f'(t) \neq 0$ in the interval $[a,b]$ then, this curve is called a smooth curve.

Piecewise smooth a sectional smooth curve or contour :-

If $z=f(t)$ be a complex valued

function tracing a curve in the interval $I_k = (a_k, b_k)$; $k = 1, 2, 3, \ldots$
and if the curve is smooth then we say that the curve traced by $z = f(t)$ is sectionally smooth.

Closed curve :-

A curve traced by a function $z = f(t)$ such that the initial point and the terminal point are the same then the curve is called a closed curve.

closed curve

Simple curve :-

A curve traced by $z = f(t)$ such that $f(t_1) \neq f(t_2)$ for $t_1 \neq t_2$ then curve is called simple curve.

(e.g) $f(t) = t^2$

$f(1) = 1$ as $1 \neq -1$ but $f(1) = f(-1)$

$f(-1) = 1$

This is not a simple curve

Inverse of a curve :-

If the curve C is traced by $z = f(t)$ in the interval $I = (a, b)$, then the inverse of C is denoted by $\bar{C}$ which is traced by $z = g(t)$ such that

$g(t) = f(a+b-t)$

Jorden curve :-

A curve which is simple as well as closed is called a Jorden curve

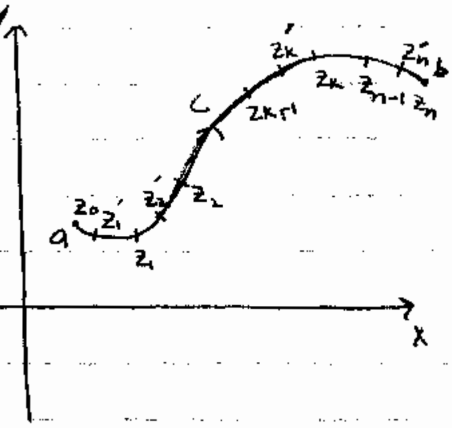
(e.g) Circle, polygon

Complex Line Integrals :-

Let $f(z)$ be continuously at all points of the curve C which we shall assume has a finite length

(i-e) C is a rectifiable curve

Subdivide C into n-parts by means of $z_1, z_2, \ldots, z_{n-1}$, chosen arbitrarily, and call

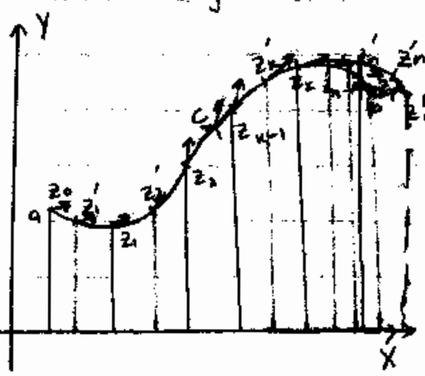


$a = z_0$, $b = z_n$ on each arc joining z_{k-1} to z_k [where K goes from 1 to n]

Choose a point z_k'

From the sum

$$S_n = f(z_1')(z_1 - a) + f(z_2')(z_2 - z_1) + \cdots + f(z_n')(b - z_{n-1}) \to (1)$$



on writing $z_k - z_{k-1} = \Delta z_k$

$$S_n = \sum_{k=1}^{n} f(z_k') \Delta z_k$$

Let

Let the number of sub-division n increase in such a way that the larger of the chord length $|\Delta z_k|$ approaches to zero. Then, the sum S_n approaches a limit which does not depend on the mode of sub-division and we denote this limit by $\int_a^b f(z)dz$ or $\int f(z)dz$

is called a complex line integral or it is also called line integral of $f(z)$ along the curve C or also called definite integral $f(z)$ from a to b along the curve C.

Applications :-

Evaluate $\int_a^b z\,dz$

Sol :-

$$\int_a^b z\,dz$$

$$I = \sum_{k=1}^{n} f(z_k')(z_k - z_{k-1}) \rightarrow (1)$$

$$= \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1}) \rightarrow (2)$$

$$= \sum_{k=1}^{n} f(z_{k-1})(z_k - z_{k-1}) \rightarrow (3)$$

put $f(z) = z$

$$I = \sum_{k=1}^{n} z_k(z_k - z_{k-1})$$

$$I = \sum_{k=1}^{n} z_{k-1}(z_k - z_{k-1})$$

Adding both

$$2I = \sum_{k=1}^{n} z_k(z_k - z_{k-1}) + \sum_{k=1}^{n} z_{k-1}(z_k - z_{k-1})$$

$$= \sum_{k=1}^{n} (z_k^2 - z_{k-1}^2)$$

$$= \cancel{z_1^2} - z_0^2 + \cancel{z_2^2} - \cancel{z_1^2} + \cancel{z_3^2} - \cancel{z_2^2} + \cdots + z_{n-1}^2 + \cancel{z_{n-1}^2}$$

$$= z_n^2 - z_0^2$$

$$2I = b^2 - a^2$$

$$I = \frac{b^2 - a^2}{2}$$

Evaluate $\int_\alpha^\beta dz$

Sol:-

$$\int_\alpha^\beta 1 \cdot dz$$

$$I = \sum_{k=1}^{n} f(z_k)(z_k - z_{k-1})$$

Put $f(z) = 1$

$$= \sum_{k=1}^{n} (z_k - z_{k-1})$$

$$I = \cancel{z_1} - z_0 + \cancel{z_2} - \cancel{z_1} + \cancel{z_3} - \cancel{z_2} + \cdots + z_n - \cancel{z_{n-1}}$$

$$= z_n - z_0$$

$I = \beta - \alpha$

Another Representation of $\int_c f(z)dz$

$\int_c f(z)dz$ We know that

$w = f(z) = U(x,y) + iV(x,y)$

$z = x+iy$

$dz = dx + idy$

Therefore, the given integral $\int_c f(z)dz$ can be re-written as

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy)$$

$$= \int_c ((udx - vdy) + i(udy + vdx))$$

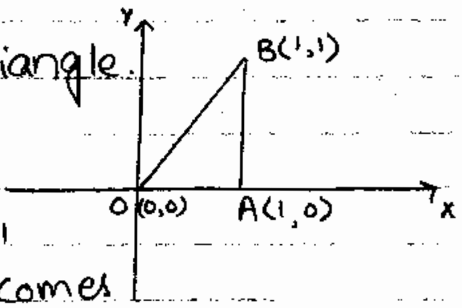
$$= \int_c (udx - vdy) + i\int_c (vdx + udy)$$

Ex :-

Evaluate $\int_0^{1+i} z^2 dz$, where c (curve) consists of OA, AB, OAB, OB and OABO

Sol :-

The contour C is triangle.



Step I :-

Integrate along OA.

put $x = t$, $y = 0$; $0 \le t \le 1$

The given integral becomes

$$\int_0^{1+i} z^2 dz = \int_0^{1+i} ((x^2+iy^2))^2 (dx+idy)$$

$$= \int_0^{1+i} ((x^2 - y^2) + 2ixy)(dx+idy)$$

$$= \int_0^1 ((t^2 - 0) + 2it(0))(dt + i(0))$$

$$= \int_0^1 t^2 dt$$

$$= \frac{t^3}{3}\Big|_0^1$$

$$= \frac{1}{3}(1-0)$$

$$= \frac{1}{3}$$

Step II :-

Integrate along AB

put $x = 1$, $y = t$; $0 \le t \le 1$

The given integral becomes

$$\int_0^{1+i} z^2 dz = \int_0^{1+i} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_0^1 ((1-t^2) + 2it)(idt)$$

$$= i\int_0^1 (1-t^2)dt + 2i^2\int_0^1 t\,dt$$

$$= i\int_0^1 1\,dt - i\int_0^1 t^2 dt - 2\int_0^1 t\,dt$$

$$= i\,t\Big|_0^1 - i\frac{t^3}{3}\Big|_0^1 - 2\frac{t^2}{2}\Big|_0^1$$

$$= i(1-0) - \frac{i}{3}(1-0) - (1-0)$$

$$= i - \frac{i}{3} - 1$$

$$= \frac{3i - i}{3} - 1$$

$$= \frac{2i}{3} - 1$$

Step III :-

Integrate along OB

put $x = t$, $y = t$; $0 \le t \le 1$

The given integral becomes

$$\int_0^{1+i} z^2 dz = \int_0^{1+i} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_0^1 2it^2 (dt + idt)$$

$$= 2i(1+i)\int_0^1 t^2 dt$$

$= 2i(1+i)\left(\frac{t^3}{3}\right|_0^1$

$= (2i + 2i^2)\left(\frac{1}{3}(1-0)\right)$

$= (2i - 2)\left(\frac{1}{3}\right)$

$= \frac{2}{3}i - \frac{2}{3}$

$= \frac{2}{3}(i-1)$

Step IV :-

Integrate along OAB

Integrate along OAB = Integrate along OA + Integrate along AB

$= \frac{1}{3} + \frac{2}{3}i - 1$

$= \frac{2}{3}i + \frac{1}{3} - 1$

$= \frac{2}{3}i - \frac{2}{3} = \frac{2}{3}(i-1)$

Step V :-

Integrate along OABO

Integrate along OABO = Integrate along OAB + Integrate along BO

Integrate along OB = −(Integrate along BO)

$= \frac{2}{3}(i-1) - \frac{2}{3}(i-1)$

Integrate along OABO = 0

Evaluate $\int_0^{2+i} z\,dz$ along C where C is (i) OA (ii) AB (iii) OB

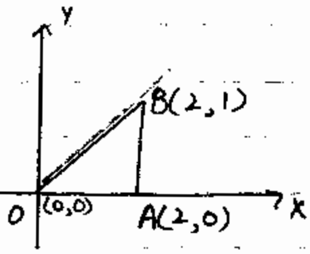
Sol :-

(i)

Integrate along OA

Put $x = t$, $y = 0$, $0 \le t \le 2$

The given integral becomes



$\int_0^{2+i} z\,dz = \int_0^{2+i} (x+iy)(dx+idy)$

$$= \int_0^2 (t+i(0))(dt+i(0))$$

$$= \int_0^2 t\,dt$$

$$= \frac{t^2}{2}\Big|_0^2$$

$$= \frac{1}{2}(4-0)$$

$$= 2$$

~~Integrate~~ along AB

put $x = 2$, $y = t$ $\quad 0 \le t \le 1$

The given integral becomes

$$\int_0^{2+i} (x+iy)(dx+idy)$$

$$\int_0^1 (2+it)(i\,dt)$$

$$= i\int_0^1 (2+it)\,dt$$

$$= i\left[2t + i\frac{t^2}{2}\right|_0^1$$

$$= i\left[2(1-0) + \frac{i}{2}(1-0)\right]$$

$$= 2i - \frac{1}{2}$$

Integrate along OB

Equation of line passing through two points is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

Equation of line OB

$$\frac{x-0}{2-0} = \frac{y-0}{1-0}$$

$$\Rightarrow x = 2y$$

$$\Rightarrow y = x/2$$

$$dy = \frac{1}{2}dx$$

The given integral becomes

$$\int_0^{2+i}(x+iy)(dx+idy)$$

$$=\int_0^2\left(x+i\frac{x}{2}\right)\left(dx+i\frac{dx}{2}\right)$$

$$=\int_0^2 xdx+\frac{i}{2}\int_0^2 xdx+\frac{i}{2}\int_0^2 xdx-\frac{1}{4}\int_0^2 xdx$$

$$=\left(1+\frac{i}{2}+\frac{i}{2}-\frac{1}{4}\right)\int_0^2 xdx$$

$$=\left(1-\frac{1}{4}+\frac{i}{2}+\frac{i}{2}\right)\int_0^2 xdx$$

$$=\left(\frac{3}{4}+i\right)\left(\frac{x^2}{2}\right|_0^2$$

$$=\left(\frac{3}{4}+i\right)\left(\frac{1}{2}(2^2-0)\right)$$

$$=\left(\frac{3}{4}+i\right)\left(\frac{1}{2}(4)\right)$$

$$=\left(\frac{3}{4}+i\right)(2)$$

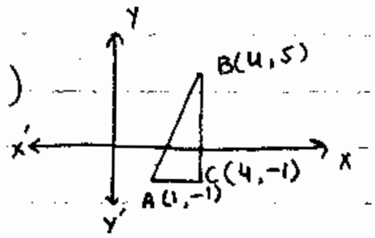
$$=\frac{3}{2}+2i$$

Ex:-

Evaluate the integral $\int_c z\,dz$ where c is a St. line $y=2x-3$ having end points $1-i\,(1,-1)$, $4+5i\,(4,5)$

Sol:-

(i) $$\int_c z\,dz=\int_c(x+iy)(dx+idy)$$



$$y=2x-3$$
$$dy=2dx$$

The given integral becomes

$$=\int(x+i(2x-3))(dx+i2dx))$$

$$= \int_1^4 (x - 3i + 2ix)(1 + 2i)\,dx$$

$$= \int_1^4 (x + 2ix - 3i + 6i + 2ix - 4x)\,dx$$

$$= \int_1^4 (4ix - 3x - 3i + 6i)\,dx$$

$$= 4i\frac{x^2}{2}\Big|_1^4 - 3\frac{x^2}{2}\Big|_1^4 - 3ix\Big|_1^4 + 6ix\Big|_1^4$$

$$= 2i(4^2 - 1^2) - \frac{3}{2}(4^2 - 1^2) - 3i(4 - 1) + 6i(4 - 1)$$

$$= 2i(16 - 1) - \frac{3}{2}(16 - 1) - 3i(3) + 6i(3)$$

$$= 32i - 2i - \frac{3}{2}(15) - 9i + 18i$$

$$= 30i - \frac{45}{2} - 9i + 18i$$

$$= 30i - 9i^2 - \frac{45}{2} + 18$$

$$\int_c z\,dz = 21i - \frac{9}{2}$$

(ii)

$$\int_c z\,dz = \int_c (x + iy)(dx + idy)$$

$$x = \frac{y+3}{2}, \quad dx = \frac{1}{2}dy$$

The given integral becomes

$$= \int_{-1}^{5} \left(\frac{y}{2} + \frac{3}{2} + iy\right)\left(\frac{1}{2}dy + idy\right)$$

$$= \int_{-1}^{5} \left(\frac{y + 3 + 2iy}{2}\right)\left(\frac{1 + 2i}{2}\right)dy$$

$$= \frac{1}{4}\int_{-1}^{5} (y + 3 + 2iy)(1 + 2i)\,dy$$

$$= \frac{1}{4}\int_{-1}^{5} (y + 2iy + 3 + 6i + 2iy - 4y)\,dy$$

$$= \frac{1}{4}\left[\frac{y^2}{2}\Big|_{-1}^{5} + 2i\frac{y^2}{2}\Big|_{-1}^{5} + 3y\Big|_{-1}^{5} + 6iy\Big|_{-1}^{5} + 2i\frac{y^2}{2}\Big|_{-1}^{5} - 4\frac{y^2}{2}\Big|_{-1}^{5}\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}(5^2 - (-1)^2) + i(5^2 - (-1)^2) + 3(5 - (-1)) + 6i(5 - (-1)) + i(5^2 - (-1)^2) - 2(5^2 - (-1)^2)\right]$$

$$= \frac{1}{4}\left[\frac{1}{2}(25 - 1) + i(25 - 1) + 3(5 + 1) + 6i(5 + 1)\right.$$

$$= \frac{1}{4}\left[12 + i(25 - 1) - 2(25 - 1)\right]$$

$$= [illegible] \Rightarrow 21i - [illegible]$$

$$\int_C z\,dz = \frac{1}{4}\left[\frac{1}{2}(24) + i(24) + 3(6) + 6i(6) + i(24) - 2(24)\right]$$
$$= \frac{1}{4}\left[12 + 24i + 18 + 36i + 24i - 48\right]$$
$$=$$
$$= \frac{1}{4}\left[12 + 18 - 48 + 24i + 24i + 36i\right]$$
$$= \frac{1}{4}\left[-18 + 84i\right]$$
$$\int_C z\,dz = 21i - 9/2$$

Theorem:-

If C is continuous curve and $f(z)$ is defined everywhere on C. Then, prove that $\left|\int_C f(z)dz\right| \le ML$ where $M = |f(z)|$ and L is length of curve C.

Proof:-

Since $f(z)$ is defined everywhere on C, we know that

$$\int_C f(z)dz = \sum_{k=1}^{n} f(z_k)\,\Delta_k z_k$$

$$\left|\int_C f(z)dz\right| = \left|\sum_{k=1}^{n} f(z_k)\,\Delta_k z_k\right| \le |f(z_k)| \sum_{k=1}^{n} |\Delta_k z| = ML$$

where $M = |f(z_k)|$, $L = \sum_{k=1}^{n} |\Delta_k z_k| = \int_C ds =$ Length of Contour C

where $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

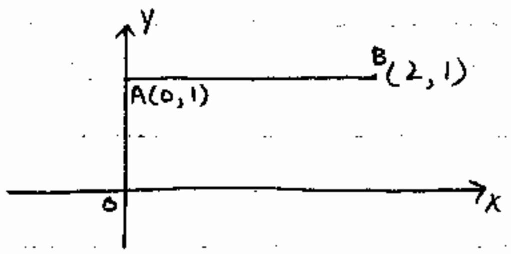
Prove that $\left|\int_i^{2+i} \frac{1}{z^2}\,dz\right| \le 2$

Sol:-

$$\left|\int_i^{2+i} \frac{1}{z^2}\,dz\right|$$

Step I:-

$$f(z) = \frac{1}{z^2}$$



Contour C is a straight line AB having end points $A(0,1)$ and $B(2,1)$

Integrate along AB

$y=1, \quad x=t, \quad ; 0 \leq t \leq 2$

$\frac{dy}{dt}=0, \quad \frac{dx}{dt}=1$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$ds = \sqrt{1+0}$

$ds = 1$

$L = \int_c ds$

$L = \int_0^2 1 \cdot dt$

$= t \Big|_0^2$

$= 2-0$

$L = 2$

Step II :-

$z^2 = x^2-y^2+2ixy$

$z^2 = (t^2-1)+2it$

$|z^2|^2 = (t^2-1)^2+(2t)^2$

$|z^2| = \sqrt{(t^2)^2-2t^2+1+4t^2}$

$= \sqrt{(t^2)^2+2t^2+1}$

$= \sqrt{(t^2+1)^2}$

$|z^2| = t^2+1$

$|z^2| = 1+t^2 \qquad \because |z^2| \geq 1$

$\frac{1}{|z^2|} \leq 1$

$\therefore \quad M = \frac{1}{|z^2|} \leq 1$

$\therefore \left|\int_i^{2+i} \frac{1}{z^2}\,dz\right| \leq ML \leq 1.2 \leq 2$

$\Rightarrow \left|\int^{2+i} \frac{1}{z^2}\,dz\right| \leq 2$

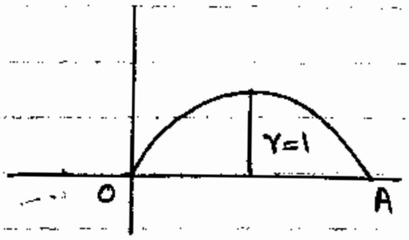
Ex:-

Evaluate $\left|\int_c \bar{z}\,dz\right|$ where c is a semi unit circle

Sol:-

The contour C is a semi unit circle.
The parametric equations of the circle

$$\left.\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}\right] \quad 0 \le t \le \pi$$



$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t}$$

$$ds = 1$$

$$L = \int_0^{\pi} ds$$

$$L = \int_0^{\pi} 1 \cdot dt$$

$$L = t\Big|_0^{\pi}$$

$$L = \pi - 0$$

$$L = \pi$$

$$\bar{z} = x - iy$$

$$|\bar{z}| = \sqrt{x^2 + y^2}$$

$$= \sqrt{\cos^2 t + \sin^2 t}$$

$$|\bar{z}| = 1$$

$$M = |\bar{z}| = 1$$

we know that "if c is continuous curve and $f(z)$ is defined everywhere on C. Then,

$$\left|\int f(z)\,dz\right| \le ML$$

So, $\left|\int_c \bar{z}\,dz\right| \le 1 \cdot \pi$

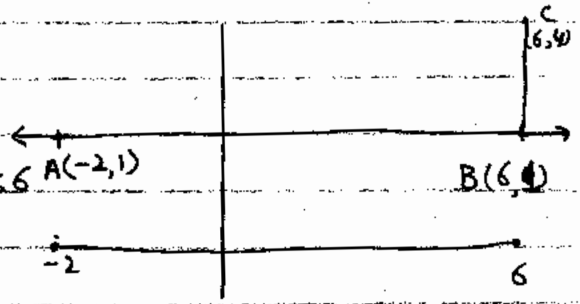
$$\left|\int_c \bar{z}\,dz\right| \le \pi$$

Ex:-

Evaluate $\left|\int_c \bar{z}^2 dz\right|$ where c is contour from $-2+i$ to $6+i$ and then $6+4i$

Sol:-

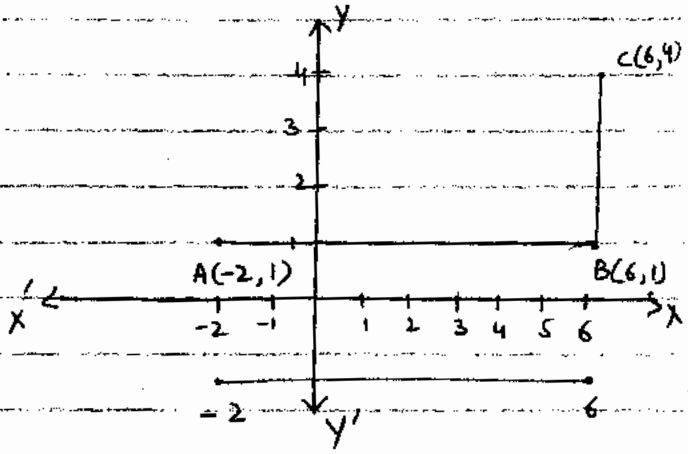
First, we integrate along AB



$$x_1 = t, \quad y_1 = 1 \; ; \; -2 \le t \le 6$$

$$ds = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dy_1}{dt}\right)^2}$$

$$ds = \sqrt{1+0} = 1$$



$$L_1 = \int_{-2}^{6} ds$$

$$L_1 = \int_{-2}^{6} 1 \cdot dt$$

$$L_1 = t \Big|_{-2}^{6}$$

$$L_1 = 6-(-2)$$

$$= 6+2$$

$$L_1 = 8$$

$$z_1 = x+iy$$

$$\bar{z}_1 = x-iy$$

$$\bar{z}_1^2 = x^2 \pm y^2 + 2ixy$$

$$z_1^2 = t^2 - 1 + 2it$$

$$M_1 = |z_1^2|$$

$$= \sqrt{(t^2-1)^2 + (2t)^2}$$

$$M_1 = \sqrt{t^4 + 1 - 2t^2 + 4t^2}$$

$$M_1 = \sqrt{(t^2)^2 + 2t^2 + 1}$$
$$M_1 = \sqrt{(t^2+1)^2}$$
$$M_1 = t^2+1$$
$$M_1 \leq (6)^2+1 = 36+1 = 37$$

Now, we will integrate along BC

$$x_2 = 6, \quad y_2 = t \quad ; 1 \leq t \leq 4$$

$$ds = \sqrt{\left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dy_2}{dt}\right)^2}$$

$$ds = \sqrt{0+(1)^2}$$

$$ds = 1$$

$$L_2 = \int ds$$

$$L_2 = \int_1^4 1 \cdot dt$$

$$L_2 = t \Big|_1^4$$

$$L_2 = 4-1 = 3$$

$$z_2^2 = x^2 - y^2 + 2ixy$$
$$z_2^2 = 36 - t^2 + 12it$$

$$|z_2^2| = \sqrt{(36-t^2)^2 + (12t)^2}$$

$$|z_2^2| = \sqrt{(36)^2 + t^4 - 72t^2 + 144t^2} = \sqrt{36 + 72t^2 + t^4}$$
$$|z_2^2| = \sqrt{(36+t^2)^2} = 36+t^2$$
$$M_2 \leq 36+(4)^2$$
$$\leq 36+16$$
$$M_2 \leq 52$$
$$\therefore \left|\int_C z^2 dz\right| \leq M_1L_1 + M_2L_2$$

$$\left|\int_c z^2 dz\right| \le 37(8)+52(3) = 296+156$$

$$\left|\int_c z^2 dz\right| \le 452$$

Ex:-

Evaluate $\int \bar{z}\, dz$, where $z = t^2 + it$ and c is from $z=0$ to $z=4+2i$ along

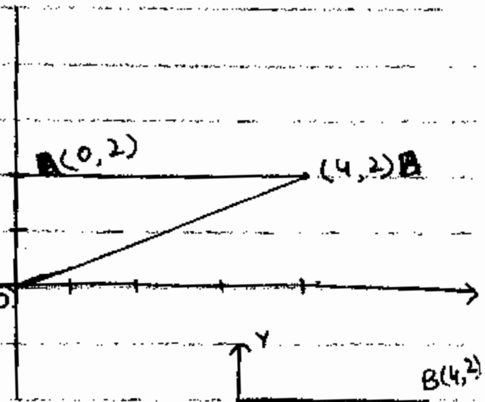
(i) The line from $z=0$ to $z=2i$

(ii) The line from $z=2i$ to $4+2i$

Sol:-

$$\int_c \bar{z}\, dz \quad \text{where } z = t^2 + it$$

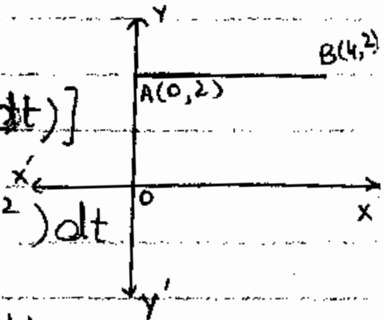
$$\bar{z} = t^2 - it, \quad dz = (2t+i)dt$$



$$\int_c \bar{z}\, dz = \int_c (t^2 - it)(2t+i)dt$$

$$= \int_c (2t^3 + it^2 dt - 2t^2 i + t)dt$$

$$= \int_c [(2t^3+t)dt + i(t^2 dt - 2t^2 dt)]$$



$$= \int_c (2t^3+t)dt + i\int_c (t^2 - 2t^2)dt$$

$$= \int_c (2t^3+t)dt + i\int_c (-t^2)dt$$

$$\int_c \bar{z}\, dz = \int_c (2t^3+t)dt - i\int_c t^2 dt$$

(i) Integration along OA (line from $z=0$ to $z=2i$)

then $\int_c \bar{z}\, dz = \int_c (2t^3+t)dt - i\int_c t^2 dt$ and $0 \le t \le 2$

$$\int_c \bar{z}\, dz = \int_0^2 (2t^3+t)dt - i\int_0^2 t^2 dt$$

$$\int_c \bar{z}\, dz = \left(\frac{2t^4}{4} + \frac{t^2}{2}\right)\Big|_0^2 - i\,\frac{t^3}{3}\Big|_0^2$$

$$\int_C \bar{z}\,dz = \left(\frac{t^4}{2} + \frac{t^2}{2}\right)\Big|_0^2 - i\,\frac{t^3}{3}\Big|_0^2$$

$$= \frac{1}{2}(2^4 - 0) + \frac{1}{2}(2^2 - 0) - \frac{i}{3}(2^3 - 0)$$

$$= \frac{1}{2}(16 - 0) + \frac{1}{2}(4 - 0) - \frac{i}{3}(8 - 0)$$

$$= 8 + 2 - \frac{8}{3}i$$

$$\int_C \bar{z}\,dz = 10 - \frac{8}{3}i$$

(ii) Integration along AB (line from $z = 2i$ to $z = 4 + 2i$)

$$\int_C \bar{z}\,dz = \int_C (2t^3 + t)\,dt - i\int t^2\,dt \quad \text{and } 0 \le t \le 4$$

$$= \int_0^4 (2t^3 + t)\,dt - i\int_0^4 t^2\,dt$$

$$= \left(2\frac{t^4}{4} + \frac{t^2}{2}\right)\Big|_0^4 - i\,\frac{t^3}{3}\Big|_0^4$$

$$= \left(\frac{t^4}{2} + \frac{t^2}{2}\right)\Big|_0^4 - \frac{i}{3}\,t^3\Big|_0^4$$

$$= \frac{1}{2}(4^4 - 0) + \frac{1}{2}(4^2 - 0) - \frac{i}{3}(4^3 - 0)$$

$$= \frac{1}{2}(256) + \frac{1}{2}(16) - \frac{i}{3}(64)$$

$$= 128 + 8 - \frac{64i}{3}$$

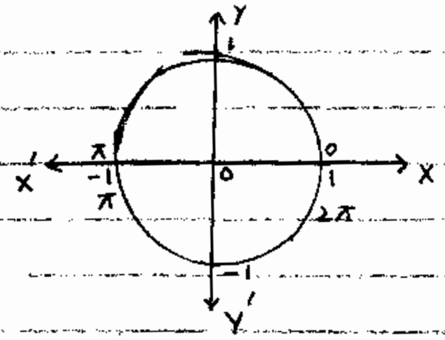
$$\int_C \bar{z}\,dz = 136 - \frac{64}{3}i$$

Ex :-

Prove that the value of $\int_C \frac{1}{z}\,dz$ where C is a semicircle arc $|z| = 1$ from -1 to 1 is $-\pi i$ to πi according as the arc lies above or below the real axis.

Sol :-

Consider, the circle



$|z| = 1$

$\Rightarrow x^2 + y^2 = 1$

$\Rightarrow r = 1$

$z = re^{i\theta}$

$z = e^{i\theta}$ in polar form

$dz = ie^{i\theta} d\theta$

Above real axis :-

$x = r\cos\theta, \quad y = r\sin\theta \quad ; \; r = 1$

then $x = \cos\theta$, $y = \sin\theta$ where θ varies from π to 0

$$I = \int_{\pi}^{0} \frac{1}{z}\, dz$$

$$= \int_{\pi}^{0} \frac{ie^{i\theta} d\theta}{e^{i\theta}}$$

$$= \int_{\pi}^{0} i\, d\theta$$

$$= i\theta \Big|_{\pi}^{0}$$

$$= i(0 - \pi)$$

$$I = -\pi i$$

Below real axis :-

$x = \cos\theta, \; y = \sin\theta \quad ; \; r = 1$ where θ varies from π to 2π

$$I = \int_{\pi}^{2\pi} \frac{1}{z}\, dz$$

$$= \int_{\pi}^{2\pi} \frac{ie^{i\theta} d\theta}{e^{i\theta}}$$

$$= \int_{\pi}^{2\pi} i\, d\theta$$

$$I = i\theta \Big|^{2\pi}$$

$I = i(2\pi - \pi)$

$I = \pi i$

Cauchy's fundamental theorem :-

Statement :-

If $f(z)$ is a complex valued function such that (i) $f(z)$ is analytic on and within Closed contour C

(ii) $f'(z)$ is continuous. Then, $\oint_C f(z)dz = 0$

Proof :-

Green's theorem

If $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous within a domain D and if C is any Closed contour in D. Then,

$$\oint (u dx + v dy) = \iint_D (V_x - U_y) dx\, dy \rightarrow (1)$$

Since, $f(z)$ is analytic on and within closed contour C, $f(z) = u + iv$

$z = x + iy$

$dz = dx + i dy$

Cauchy Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\oint_C f(z) dz = \oint_C (u + iv)(dx + i dy)$$

$$= \oint_C (u dx + u i dy + i v dx - v dy)$$

$$= \oint_C [(u dx - v dy) + i(u dy + v dx)]$$

$$\oint_C f(z) dz = \oint_C (u dx - v dy) + i \oint_C (v dx + u dy)$$

Using Green's theorem

$$\oint_C f(z)\,dz = \iint_D (-v_x - u_y)\,dx\,dy + i\iint_D (u_x - v_y)\,dx\,dy$$

Using C·R equation.

$$\oint_C f(z)\,dz = \iint_D (-v_x + v_x)\,dx\,dy + i\iint_D (v_y - v_y)\,dx\,dy$$

$$\therefore \oint_C f(z)\,dz = 0$$

Applications:-

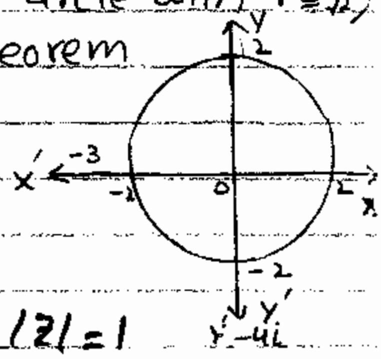
Evaluate $\int_C f(z)\,dz$, where $f(z) = \dfrac{\sinh z + e^z + z}{(z+3)(z+4i)}$ and $C : |z| = 2$

Sol:-

Since, $f(z) = \dfrac{\sinh z + e^z + z}{(z+3)(z+4i)}$

is analytic on and within a closed contour $C : |z| = 2$ (is eq of circle with $r = \sqrt{2}$)

$\therefore$ By Cauchy's fundamental theorem



$$\int_C \frac{\sinh z + e^z + z}{(z+3)(z+4i)} = 0$$

Evaluate $\displaystyle\int_C \frac{\sinh z + e^z + z}{(z^2-4)(z^2+9)}\,dz$ $\quad C : |z| = 1$

Sol:-

Since, $f(z) = \dfrac{\sinh z + e^z + z}{(z^2-4)(z^2+9)}$

is analytic on and within a closed contour $C : |z| = 1$

$\therefore$ By Cauchy's fundamental theorem.

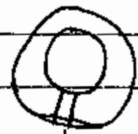
$$\int \frac{\sinh z + e^z + z}{(z^2-4)(z^2+9)} = 0$$

Simply connected Domain:-

A region in which every closed curve can be shrunk to a point without passing out of the region then the region is called simply connected region otherwise is said to be multiply connected region.

Doubly connected Domain:-

If there is a hole in the domain, is called doubly connected region and if there are (n-1) holes in it, then the domain is called n-ply connected domain.



doubly connected domain

Ex:-

Evaluate $\int z^3 dz$; $C = |z| = 1$

The function $f(z)$ is analytic on and within contour $C: |z| = 1$.

So, by Cauchy's fundamental theorem

$$\int_C z^3 dz = 0$$

Ex:-

Evaluate $\int_C |(z-1)||dz|$; C is unit circle

Sol:-

Since, the function $f(z) = |z-1|$ is not analytic

put $z = e^{i\theta}$; $r = 1$

$$z - 1 = e^{i\theta} - 1$$

$$= (\cos\theta + i\sin\theta) - 1$$

$$= (\cos\theta - 1) + i\sin\theta$$

$$|z-1| = \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$= \sqrt{\cos^2\theta + 1 - 2\cos\theta + \sin^2\theta}$$

$$= \sqrt{1 + 1 - 2\cos\theta}$$

$$= 2 - 2\cos\theta$$

$$= \sqrt{2(1 - \cos\theta)}$$

$$= \sqrt{2 \cdot (2\sin^2\frac{\theta}{2})}$$

$$= \sqrt{4\sin^2\theta/2}$$

$$|z-1| = 2\sin\theta/2$$

$$z = e^{i\theta}d$$

$$dz = ie^{i\theta}d\theta$$

$$|dz| = |ie^{i\theta}d\theta|$$

$$= |i(\cos\theta + i\sin\theta)d\theta| = |i| (1) \cdot |d\theta|$$

$$dz = |d\theta| \qquad |i| = \sqrt{0 + (1)^2} = 1$$

$$|d\theta| = d\theta \qquad \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\int_C |z-1||dz| = \int_0^{2\pi} 2\sin\theta/2 \, d\theta$$

$$\int_0^{2\pi} 2\sin\theta/2 \, d\theta = 2\left(-\frac{\cos\theta/2}{1/2}\right)\Big|_0^{2\pi}$$

$$= 4\left(-\cos\frac{\theta}{2}\right)\Big|_0^{2\pi}$$

$$= -4\left[\cos\frac{2\pi}{2} - \cos(0)\right]$$

$$= -4(\cos\pi - 1) \qquad \because \cos\pi = -1$$

$$= -4(-1-1) = -4(-2)$$

$$\int_0^{2\pi} 2\sin\theta/2 \, d\theta = 8$$

Consequences of Cauchy's Fundamental Theorem:-

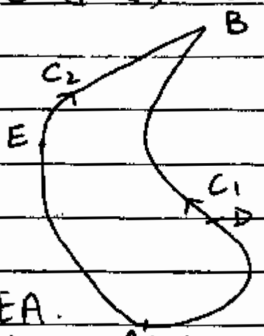
1 Statement:-

Let $f(z)$ be analytic function in a domain D having interval (a,b). C_1 and C_2 are two paths from A to B. Then, prove that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

(OR) If $f(z)$ is analytic function in a domain containing the interval a to b forming the contour C_1 and C_2 then prove that $\int_a^b f(z)\,dz$ is independent of the path from A to B (i-e) in the interval a to b.

Proof:-



Consider the curve ADBEA which is closed, $f(z)$ is analytic on and within closed contour ADBEA.

Therefore, by Cauchy's fundamental theorem,

$$\int_{ADBEA} f(z)\,dz = 0$$

$$\int_{ADB} f(z)\,dz + \int_{BEA} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

'They are independent'

Since, $f(z)$ is analytic function in domain containing the interval a to b forming the contour C_1 and C_2 then $\int_a^b f(z)\,dz$ is independent of path from A to B.

2 Theorem:-

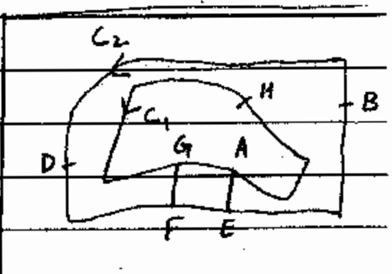
Statement:-

If $f(z)$ is complex valued function defined in a domain D and is analytic on and within D, D is consisting of two contour C_1 and C_2 (i-e D is doubly connected domain) then Prove that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Proof:-

Given $f(z)$ is analytic on and within a domain D, which is doubly connected can be made simply connected by a cross cut



Now, after making it simply connected the closed contour is AEBDFGHA

Therefore, By Cauchy's Fundamental theorem

$$\int_{AEBDFGHA} f(z)\,dz = 0$$

$$\int_{AE} f(z)dz + \int_{EBDF} f(z)dz + \int_{FG} f(z)dz + \int_{GHA} f(z)dz = 0$$

Since, AE is the inverse of FG

So, $$\int_{C_2} f(z)dz - \int_{C_1} f(z)dz = 0$$

$$\Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Ex:-

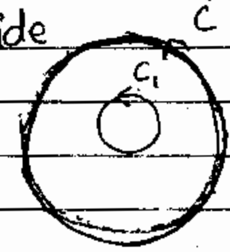
Evaluate $\int_C \frac{1}{z-a}\,dz$

(i) C is a contour containing the point $z=a$

(ii) C is a contour when $z=a$ is outside the contour

Sol:-

(i) Suppose, point "a" is inside C and C_1 is a circle of radius ε (with $\varepsilon \to 0$) centre at $z=a$



Now by theorem 2

" If $f(z)$ is complex valued function defined in domain D and is analytic on and within D ; (D is doubly connected domain) ; D is consisting of two contour C_1 and C_2 (i-e) D is doubly connected domain, then prove that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

$$\int_C \frac{1}{z-a}\,dz = \int_{C_1} \frac{1}{z-a}\,dz$$

Let $I = \int_{C_1} \frac{1}{z-a}\,dz$

Now on C_1, $|z-a| = \varepsilon$

or $z - a = \varepsilon e^{i\theta}$

$z = a + \varepsilon e^{i\theta}$; $0 \leq \theta \leq 2\pi$

$dz = \varepsilon i e^{i\theta} d\theta$

$$\therefore \int_C \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} d\theta$$

$$= i\int_0^{2\pi} 1 \cdot d\theta$$

$$= i\,\theta \Big|_0^{2\pi}$$

$$\int_C \frac{1}{z-a}\,dz = 2\pi i$$

(ii)

Since, $f(z)$ is analytic on and within contour C. The point, where $f(z)$ is not analytic is $z = a$, which lies outside the contour C.

So, by Cauchy's fundamental theorem $\int_C f(z)\,dz = 0$

$$\int_C \frac{1}{z-a}\,dz = 0$$

Note:-

(i) $(1,1) = 1+i$

$|z-(1+i)| = 1$ is circle with centre $(1,1)$ and $r = 1$

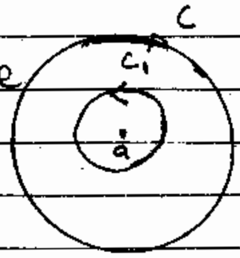
(ii) $|z-(1-2i)| = 1$ is circle with centre $(1,-2)$ and $r = 1$

Ex:-

Evaluate $\int_C \frac{1}{(z-a)^n}\,dz$, where C is closed contour enclosing the point $z=a$, interpret the result when $n=1,2,3,\ldots$

Sol:-

Suppose, point "a" is inside C and C_1 is a circle of radius ε (with $\varepsilon \to 0$), centre at $z=a$



By theorem 2

$$\int_C \frac{1}{(z-a)^n}\,dz = \int_{C_1} \frac{1}{(z-a)^n}\,dz$$

Take $n=1$

then $\int_C \frac{1}{z-a}\,dz = \int_{C_1} \frac{1}{z-a}\,dz$

$|z-a| = \varepsilon$ is C_1 circle

$$z-a = \varepsilon e^{i\theta}$$

$$z = a+\varepsilon e^{i\theta}$$

$$dz = ie^{i\theta}\varepsilon\, d\theta \quad ; \quad 0 \le \theta \le 2\pi$$

$$\int_{C_1} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}}\, ie^{i\theta}\varepsilon\, d\theta$$

$$= i\int_0^{2\pi} 1 \cdot d\theta$$

$$= i\,\theta\Big|_0^{2\pi}$$

$$= i\,(2\pi - 0)$$

$$\int_{C_1} \frac{1}{z-a}\,dz = 2\pi i$$

Take $n=2$

$$\int_C \frac{1}{(z-a)^2}\,dz = \int_{C_1} \frac{1}{(z-a)^2}\,dz$$

$|z-a| = \varepsilon$

$$|z-a = \varepsilon e^{i\theta} \quad , dz = i\varepsilon e^{i\theta} d\theta$$
$$(z-a)^2 = \varepsilon^2 e^{2i\theta}$$
$$\int_{C_1} \frac{1}{(z-a)^2} dz = \int_0^{2\pi} \frac{1}{\varepsilon^2 e^{2i\theta}} i\varepsilon e^{i\theta} d\theta$$
$$= \int_0^{2\pi} \frac{i}{\varepsilon} e^{i\theta - 2i\theta} d\theta$$
$$= \frac{i}{\varepsilon} \int_0^{2\pi} e^{-i\theta} d\theta$$
$$= \frac{i}{\varepsilon} \left. \frac{e^{-i\theta}}{-i} \right|_0^{2\pi}$$
$$= \frac{-1}{\varepsilon} (e^{-2\pi i} - e^{0}) \qquad \because e^{-2\pi i} = 1$$
$$\because \cos(-2\pi) + i\sin(-2\pi)$$
$$= \frac{-1}{\varepsilon} (1-1) \qquad \Rightarrow \cos(2\pi) + i\sin(2\pi)$$
$$\int_{C_1} \frac{1}{(z-a)^2} dz = 0 \qquad \Rightarrow$$

Take $n=3$

$$\int_{C} \frac{1}{(z-a)^3} dz = \int_{C_1} \frac{1}{(z-a)^3} dz$$
$$\int_{C_1} \frac{1}{(z-a)^3} dz = |z-a| = \varepsilon$$
$$z-a = \varepsilon e^{i\theta} \quad , dz = i\varepsilon e^{i\theta} d\theta$$
$$(z-a)^3 = \varepsilon^3 e^{3i\theta}$$
$$\int_{C_1} \frac{1}{(z-a)^3} dz = \int_{0_1}^{2\pi} \frac{1}{\varepsilon^3 e^{3i\theta}} i\varepsilon e^{i\theta} d\theta$$
$$= \int_{0_1}^{2\pi} \frac{i}{\varepsilon^2} e^{i\theta - 3i\theta} d\theta$$
$$= \int_{0_1}^{2\pi} \frac{i}{\varepsilon^2} e^{-2i\theta} d\theta$$
$$= \frac{i}{\varepsilon^2} \left. \frac{e^{-2i\theta}}{-2i} \right|_0^{2\pi}$$
$$= \frac{i}{} \frac{-1}{2\varepsilon} (e^{-2i(2\pi)} - e^{-0}) \qquad \because e^{-4i\pi} = 1$$
$$\int_{C_1} \frac{1}{(z-a)^3} dz = -\frac{1}{2\varepsilon} (1-1) = 0$$

Now Take $n = n$

$$|z-a| = \varepsilon$$

$$z - a = \varepsilon e^{i\theta}, \quad dz = i\varepsilon e^{i\theta} d\theta$$

$$(z-a)^n = \varepsilon^n e^{ni\theta}$$

$$\int_c \frac{1}{(z-a)^n} dz = \int_{c_1} \frac{1}{(z-a)^n} dz$$

$$\int_{c_1} \frac{1}{(z-a)^n} dz = \int_0^{2\pi} \frac{1}{\varepsilon^n e^{ni\theta}} i\varepsilon e^{i\theta} d\theta$$

$$= \frac{i}{\varepsilon^{n-1}} \int_0^{2\pi} e^{i\theta - ni\theta} d\theta$$

$$= \frac{i}{\varepsilon^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta} d\theta$$

$$= \frac{i}{\varepsilon^{n-1}} \left. \frac{e^{(1-n)i\theta}}{(1-n)i} \right|_0^{2\pi}$$

$$= \frac{1}{(1-n)\varepsilon^{n-1}} \left(e^{(1-n)2\pi i} - e^0\right) \qquad \because e^{(1-n)2\pi i} = 1$$

$$\int_{c_1} \frac{1}{(z-a)^n} dz = \frac{1}{(1-n)\varepsilon^{n-1}} (1-1) = 0$$

Hence,

$$\int_c \frac{1}{(z-a)^n} dz = 0 \quad \forall\ n \text{ except } n = 1$$

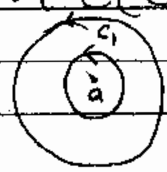
$$= 2\pi i \quad \text{for } n = 1$$

Ex:-

Evaluate $\displaystyle\int_c \frac{1}{(z-a)^5} dz$; **a is inside c**

Sol:-

Suppose, point "a" is inside c.
C_1 is the circle with radius ε $(\varepsilon \to 0)$
and centre at $z = a$



then $\displaystyle\int_c \frac{1\,dz}{(z-a)^5}$ then $\displaystyle\int_{c_1} \frac{dz}{(z-a)^5}$

$$\therefore \int_C \frac{1}{(z-a)^5}\,dz = \int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\,d\theta}{(\varepsilon e^{i\theta})^5} \qquad \Rightarrow z-a=\varepsilon e^{i\theta}$$

$$\Rightarrow z = a+\varepsilon e^{i\theta}$$

$$dz = i\varepsilon e^{i\theta}d\theta$$

$$= \frac{i}{\varepsilon^4}\int_0^{2\pi} e^{-4i\theta}d\theta \qquad 0 \le \theta \le 2\pi$$

$$\int_C \frac{1}{(z-a)^n}\,dz = \frac{i}{\varepsilon^4}\,\frac{e^{-4i\theta}}{-4i}\Bigg|_0^{2\pi}$$

Take $n=5$ $= \frac{-1}{4\varepsilon^4}(1-1) = 0$

So, $\int_{C_1} \frac{1}{(z-a)^5}\,dz = 0$ By last Example.

Cauchy's Integral Formula:-

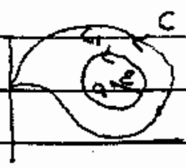
If $f(z)$ is a complex valued function, analytic on and within closed Contour C.
$z=a$ is point inside C. Then, Prove that

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Proof:-

Since $f(z)$ is analytic on and within C, $z=a$ is a point inside C.

Now $\frac{f(z)}{z-a} = \phi(z)$



$\phi(z)$ is analytic on and within C except at point $z=a$.

In order to overcome this difficulty, draw a circle Contract at $z=a$ and and radius ε where $\varepsilon \to 0$.

Now, the domain is doubly connected, we made it Simply connected by a Cross cut.

Therefore, by theorem

If $f(z)$ is complex valued function defined in a domain D and is analytic on and within D; D is consisting of two contour C_1 and C_2 (i.e) D is doubly connected domain). So, $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz$$

Step II :-

Conside the integral

$$I = \int_{C_1} \frac{f(z)}{z-a}\, dz$$

$$= \int_0^{2\pi} \left[\frac{(f(z)-f(a)) + f(a)}{z-a} \right] dz$$

$$= \int_0^{2\pi} \frac{f(a)}{z-a}\, dz + \int_0^{2\pi} \frac{f(z)-f(a)}{z-a}\, dz$$

$$= I_1 + I_2 \rightarrow (1)$$

Now consider

$$I_1 = \int_0^{2\pi} \frac{f(a)}{z-a}\, dz$$

$$= f(a) \int_0^{2\pi} \frac{1}{z-a}\, dz$$

put $z - a = \varepsilon e^{i\theta}$

$$dz = i\varepsilon e^{i\theta} d\theta$$

$$I_1 = f(a) \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}}\, i\varepsilon e^{i\theta} d\theta$$

$$= i f(a) \int_0^{2\pi} 1 \cdot d\theta$$

$$= i f(a)\, \theta \Big|_0^{2\pi}$$

$$= i f(a) [2\pi - 0]$$

$$I_1 = 2\pi i f(a)$$

* $z - a = \varepsilon e^{i\theta}$, because C_1 is the circle with radius ε $(\varepsilon \to 0)$ and centre at $z = a$

So, $|z-a| = \varepsilon$

$$z - a = \varepsilon e^{i\theta}$$

$$z = a + \varepsilon e^{i\theta}$$

$$|I_2| = \left| \int_0^{2\pi} \frac{f(z)-f(a)}{z-a}\, dz \right|$$

$$\leq \int_0^{2\pi} \frac{|f(z)-f(a)|\,|dz|}{|z-a|}$$

Since $f(z)$ is continuous at $z=a$

$\therefore |f(z)-f(a)| < \varepsilon, \quad |z-a| < \delta$

$\int_{c_1} |dz| = 2\pi\varepsilon$ is arc length where $r=\varepsilon$

$$|I_2| < \frac{\varepsilon}{\delta} \cdot 2\pi\varepsilon$$

$\Rightarrow I_2 \to 0$ as $\varepsilon \to 0$

putting values of I_1 and I_2

$$\int_c \frac{f(z)}{z-a}\, dz = I_1 + I_2$$
$$= 2\pi i f(a) + 0$$
$$\int_c \frac{f(z)}{z-a}\, dz = 2\pi i f(a)$$

Ex :-

Evaluate $\displaystyle\int_c \frac{\cos z + \cosh 4z}{z}\, dz$, $c: |z| = 2$

Sol :-

$$\int_c \frac{\cos z + \cosh 4z}{z}\, dz$$

$f(z) = \cos z + \cosh z$

$f(0) = \cos(0) + \cosh(0)$

$f(0) = 1+1$

$f(0) = 2$

$\therefore$ Since $\frac{\cos z + \cosh 4z}{z}$ is analytic

on and within a contour $C: |z| = 2$

and $z=0$ is inside the contour.

So, by Cauchy's integral formula

$$\int_c \frac{\cos z + \cosh 4z}{z}\, dz = 2\pi i f(0)$$

$$\int_C \left(\frac{\cos z + \cosh 4z}{z}\right) dz = 2 \times 2\pi i = 4\pi i$$

Ex:-

$$\int_C \frac{9z^2 - iz + 4}{z(z^2+1)}\, dz \quad ; \quad C : |z| = 2$$

Sol:-

Consider

$$\frac{9z^2 - iz + 4}{z(z^2+1)} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i} \rightarrow (a)$$

Multiplying both sides by $z(z^2+1)$ $\because z^2 - i^2 = z^2 + 1$

$$9z^2 - iz + 4 = A(z^2+1) + B(z(z+i)) + C(z(z-i)) \rightarrow (1)$$

put $z = 0$ in (1)

$$0 + 0 + 4 = A(1) + B(0) + C(0)$$

$$4 = A \quad \text{or} \quad A = 4$$

put $z - i = 0$ or $z = i$ in (1)

$$9i^2 - i(i) + 4 = A(i^2+1) + B(i(i+i)) + C(0)$$

$$9(-1) - (-1) + 4 = A(-1+1) + B(i^2 + i^2)$$

$$-9 + 1 + 4 = B(2i^2)$$

$$-4 = -2B$$

$$B = 2$$

put $z + i = 0$ or $z = -i$ in (1)

$$9(-i)^2 - i(-i) + 4 = A((-i)^2+1) + B(-i(-i+i)) + C(-i(-i-i))$$

$$9(i)^2 + i^2 + 4 = A(i^2+1) + B(+i^2 - i^2) + C(+i^2 + i^2)$$

$$9(-1) + (-1) + 4 = A(-1+1) + B(0) + C(-1-1)$$

$$-9 - 1 + 4 = 0 + 0 + C(-2)$$

$$-6 = -2C$$

$$C = 3$$

putting all values in (a)

$$\frac{9z^2 - iz + 4}{z(z^2+1)} = \frac{4}{z} + \frac{2}{z-i} + \frac{3}{z+i}$$

$$\therefore \int_C \frac{8z^2 - iz + 4}{z(z^2+1)}\,dz = 4\int_C \frac{1}{z}\,dz + 2\int \frac{1}{z-i}\,dz + 3\int \frac{1}{z+i}\,dz$$

$$= 4\times 2\pi i + 2\times 2\pi i + 3\times 2\pi i$$

$$= 8\pi i + 4\pi i + 6\pi i$$

$$\int_C \frac{8z^2 - iz + 4}{z(z^2+1)}\,dz = 18\pi i$$

Ex :-

$$\int_C \frac{1}{z^2+1}\,dz \; ; \; C : |z| = 2$$

Sol :-

Consider $\dfrac{1}{z^2+1} = \dfrac{A}{z-i} + \dfrac{B}{z+i} \rightarrow (a)$

Multiplying both sides by $\underset{z^2+1}{:}$ $(z-i)(z+i) = z^2 + \cancel{iz} - \cancel{iz} - i^2 \Rightarrow z^2+1$

$1 = A(z+i) + B(z-i) \rightarrow (1)$

put $z - i = 0$ or $z = i$ in (1)

$1 = A(i+i) + B(0)$

$1 = 2iA$

$A = \dfrac{1}{2i}$

put $z + i = 0$ in (1) or $z = -i$

$1 = A(0) + B(-i-i)$

$1 = B(-2i)$

$B = -\dfrac{1}{2i}$

putting all values in (a)

$$\frac{1}{z^2+1} = \frac{\frac{1}{2i}}{(z-i)} - \frac{\frac{1}{2i}}{z+i}$$

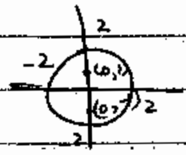
$$\frac{1}{z^2+1} = \frac{1}{2i(z-i)} - \frac{1}{2i(z+i)}$$

$$\therefore \int_C \frac{1}{z^2+1}\,dz = \frac{1}{2i}\int \frac{1}{(iz-i^2)}\,dz - \frac{1}{2i}\int \frac{i}{iz+i}\,dz$$

$$= \frac{1}{2i}\int_C \frac{1}{(iz+i)}\,dz - \frac{1}{2i}\int_C \frac{1}{(iz+i)}\,dz$$

$z = i$ inside the contour C and $z = -i$ outside the contour

So, by Cauchy's integral formula



$$\int_C \frac{1}{z-i}\, dz = 2\pi i$$

By, cauchy's fundamental theorem

$$\int_C \frac{1}{z+i}\, dz = c2\pi i$$

Then,

$$\int_C \frac{1}{z^2+1}\, dz = \frac{1}{2i} \times 2\pi i - \frac{1}{2i} \times 2\pi i$$

$$= \pi - 0$$

$$\int_C \frac{1}{z^2+1}\, dz = a$$

Ex:-

Evaluate $\int_C \frac{1}{z^2-1}\, dz \; ; \; C : |z| = 2$

Sol:- Since, $f(z) = 1$

Consider $\frac{1}{z^2-1} = \frac{A}{z+1} + \frac{B}{z-1} \rightarrow (a)$

Multipling both sides by z^2-1

$1 = A(z-1) + B(z+1) \rightarrow (1)$

put $z+1 = 0$ or $z = -1$ in (1)

$1 = A(-1-1) + B(0)$

$1 = -2A$

$\Rightarrow A = -1/2$

put $z-1 = 0$ or $z = 1$ in (1)

$1 = A(0) + B(1+1)$

$1 = 2B$

$B = 1/2$

putting all values in (a)

$$\frac{1}{z^2-1} = \frac{-1/2}{z+1} + \frac{1/2}{z-1}$$

$$\frac{1}{z^2-1} = \frac{-1}{2(z+1)} + \frac{1}{2(z-1)}$$

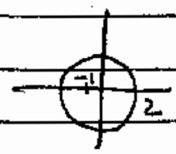
$$\int_C \frac{1}{z^2-1}\,dz = -\int_C \frac{1}{2(z+1)}\,dz + \int_C \frac{1}{2(z-1)}\,dz$$

$z = -1$ outside the contour C so, by cauchy's fundamental theorem

$$\int_C \frac{1}{2(z+1)}\,dz = \frac{0}{2}(2\pi i)$$

$z = 1$ inside the contour C so, by cauchy's integral formula

$$\int_C \frac{1}{2(z-1)} = \frac{1}{2}2\pi i$$



$$\int_C \frac{1}{z^2-1}\,dz = -\frac{1}{2}(0) + \frac{1}{2}(2\pi i)$$

$$\int_C \frac{1}{z^2-1}\,dz = \pi i$$

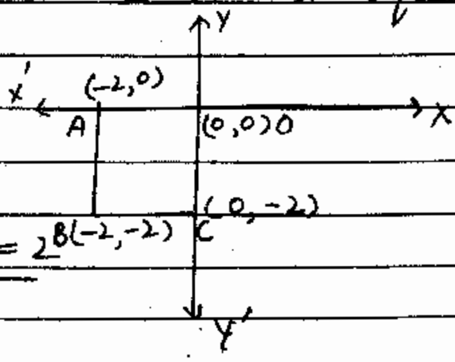
Ex:-

Evaluate $\int_C \frac{1}{z^2+2z+2}\,dz$ where C is a square having corners $(0,0)$, $(-2,0)$, $(-2,-2)$ and $(0,-2)$

Sol:-

Since, $f(z) = 1$ and C is a square.

Consider,



$$z^2+2z+2=0$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$a=1$, $b=2$, $c=2$

$$z = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$Z = \frac{-2 \pm \sqrt{-4}}{2}$$

$$= \frac{-2 \pm 2i}{2}$$

$Z = -1 + i$ or $Z + 1 \pm i$

or the factors are $(Z+1+i)(Z+1-i)$

$$\frac{1}{(Z+1+i)(Z+1-i)} = \frac{A}{Z+1+i} + \frac{B}{Z+1-i} \rightarrow (a)$$

Multiplying both sides by $(Z+1-i)(Z+1+i)$

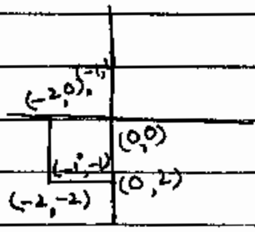
$1 = A(Z+1-i) + B(Z+1+i) \rightarrow (1)$

put $Z+1+i = 0$ or $Z = -1-i$ in (1)

$1 = A(-1-i+1-i) + B(0)$

$1 = A(-2i)$

$$A = \frac{-1}{2i} = \frac{i^2}{2i}$$



$$A = \frac{i}{2}$$

put $Z+1-i = 0$ or $Z = -1+i$ in (1)

$1 = A(0) + B(-1+i+1+i)$.

$1 = B(2i)$

$$B = \frac{1}{2i}$$

putting all values in (a)

$$\frac{1}{(Z+1+i)(Z+1-i)} = \frac{\frac{-1}{2i}}{(Z+1+i)} + \frac{\frac{1}{2i}}{Z+1-i}$$

$$\int_C \frac{1}{Z^2+2Z+2}\,dZ = \frac{-1}{2i}\int_C \frac{1}{Z+1+i}\,dZ + \frac{1}{2i}\int_C \frac{dZ}{Z+1-i}$$

Since, $Z = -1-i$ lies inside the square. So, by cauchy's integral formula

$$\int_C \frac{dZ}{Z+1+i} = 2\pi i$$

Since $Z = -1+i$ lies outside the square, So, by cauchy's fundamental theorem $\int_C \frac{dz}{z+1-i} = 0$

then $\int_C \frac{dz}{z^2+2z+2} = -\frac{1}{2i}(2\pi i) + \frac{1}{2i}(0)$

$$\int_C \frac{dz}{z^2+2z+2} = -\pi$$

Ex:-

Evaluate $\int_C \frac{\sin\pi z^2 + \cos\pi z^3}{(z-1)(z-2)}$; $C: |z| = 3$

Sol:-

$f(z) = \sin\pi z^2 + \cos\pi z^3$

$f(0) = \sin\pi(0) + \cos\pi(0)$

$= \sin(0) + \cos(0)$

$f(0) = 1$

Consider $\frac{\sin\pi z^2 + \cos\pi z^3}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \rightarrow (a)$

Multiplying both sides by $(z-1)(z-2)$

$\sin\pi z^2 + \cos\pi z^3 = A(z-2) + B(z-1) \rightarrow (1)$

put $z-1=0$ or $z=1$ in (1)

$\sin\pi(1)^2 + \cos\pi(1)^3 = A(1-2) + B(0)$

$\sin\pi + \cos\pi = A(-1)$

$0 + (-1) = -A$

$-1 = -A$

$\Rightarrow A = 1$

put $z-2=0$ or $z=2$ in (1)

$\sin\pi(2)^2 + \cos\pi(2)^3 = A(0) + B(2-1)$

$\sin\pi(4) + \cos\pi(8) = 0 + B(1)$

$\sin(720) + \cos(1620) = B$

$0 + (+1) = B$

$\Rightarrow B = +1$

putting all values in (a)

$$\frac{\sin\pi z^2 + \cos\pi z^3}{(z-1)(z-2)} = \frac{1}{z-1} + \frac{(+1)}{z-2}$$

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^3}{(z-1)(z-2)}\,dz = \int_c \frac{1}{z-1}\,dz + \int_c \frac{1}{z-2}\,dz$$

Since, $z=1$ and $z=2$ lies inside the contour $C: |z| = 3$

So, by cauchy's integral formula.

$$\int_c \frac{1}{z-1}\,dz = 2\pi i \quad , \quad \int_c \frac{1}{z-2}\,dz = 2\pi i$$

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^3}{(z-1)(z-2)}\,dz = 2\pi i + 2\pi i$$

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^3}{(z-1)(z-2)}\,dz = 4\pi i$$

Ex :-

Evaluate $\int_c \frac{z}{(z-1)(z+2i)}\,dz$ (i) $c: |z| = 1/2$ (ii) $c: |z| = 3/2$

(iii) C is rectangle having vertices $(2,1), (-1,1), (-1,-3), (2,-3)$

(iv) $C: |z-3| = 1$

Sol :-

Consider $\frac{1}{(z-1)(z+2i)} = \frac{A}{(z-1)} + \frac{Bz+C}{z+2i} \to (a)$

Multiplying both sides by $(z-1)(z+2i)$

$$1 = A(z+2i) + (Bz+C)(z-1) \to (1)$$

put $z-1=0$ or $z=1$ in (1)

$$1 = A(1+2i) + (Bz+C)(0)$$

$$1 = A(1+2i)$$

$$A = \frac{1}{1+2i}$$

From (1) $z+2i = 0$ or $z = -2i$ in (1)

$$1 = Az + 2iA + Bz^2 - Bz + Cz - C$$

$$0\cdot z^2 + 0\cdot z + 1 = Bz^2 + (A - B + C)z + (2iA - C)$$

1 = Comparing both sides

$0 \cdot z^2 (= B z^2 i) B - (2i+1)C$

$\Rightarrow B = 0$

$1 = 2iA - C$

$C = 2iA - 1$

$C = 2i\left(\frac{1}{1+2i}\right) - 1$

$C = \frac{2i - (1+2i)}{1+2i}$

$C = \frac{\cancel{2i} - 1 - \cancel{2i}}{1+2i}$

$C = \frac{-1}{1+2i}$

putting all values in (a).

$$\frac{1}{(z-1)(z+2i)} = \frac{\frac{1}{1+2i}}{z-1} + \frac{0(z) - \frac{1}{1+2i}}{z+2i}$$

$$\int_C \frac{1}{(z-1)(z+2i)}\,dz = \int_C \frac{1}{(1+2i)(z-1)}\,dz - \int_C \frac{1}{(1+2i)(z+2i)}\,dz$$

$$\int_C \frac{1}{(z-1)(z+2i)}\,dz = \frac{1}{1+2i}\int_C \frac{1}{z-1}\,dz - \frac{1}{1+2i}\int_C \frac{1}{z+2i}\,dz$$

(i)

$C : |z| = 1/2$

Since $z = 1$ and $z = -2i$ lies outside the contour $C : |z| = 1/2$

So, by cauchy's fundamental theorem

$$\int_C \frac{1}{z-1}\,dz = 0, \quad \int_C \frac{1}{z+2i} = 0$$

$$\int_C \frac{1}{(z-1)(z+2i)}\,dz = 0$$

(ii) $C : |z| = 3/2$

Since $z = 1$ lies inside the contour $C : |z| = 3/2$ and $z = -2i$ lies outside the contour

So, by cauchy's integral formula

$$\int_C \frac{1}{z-1}\, dz = 2\pi i$$

By cauchy's fundamental theorem

$$\int_C \frac{1}{z+2i}\, dz = 0$$

$$\int_C \frac{1}{(z-1)(z+2i)}\, dz = \frac{1}{1+2i}(2\pi i) + 0$$

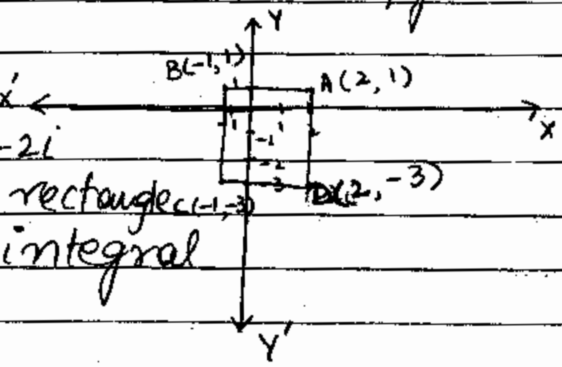
$$\int_C \frac{1}{(z-1)(z+2i)}\, dz = \frac{2\pi i}{1+2i}$$

(iii)

C is rectangle as shown in figure

Since $z = 1$ lies inside the rectangle and $z = -2i$ lies inside the rectangle

So, by cauchy's integral formula



$$\int_C \frac{1}{z-1}\, dz = 2\pi i \qquad \int_C \frac{1}{z+2i}\, dz = 2\pi i$$

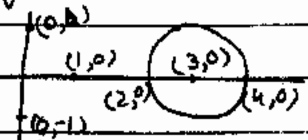
$$\int_C \frac{1}{(z-1)(z+2i)}\, dz = \frac{1}{1+2i}(2\pi i) + \frac{1}{1+2i}(2\pi i)$$

$$\int_C \frac{1}{(z-1)(z+2i)} = \frac{4\pi i}{1+2i}$$

(iv) $C : |z-3| = 1$

Since, $z = 1$ is on the contour $C : |z-3| = 1$ So, by cauchy's integral formula

$$\int_C \frac{1}{z-1}\, dz = 2\pi i$$



Since, $z = -2i$ is outside the contour $C : |z-3| = 1$ So, by cauchy's fundamental theorem.

$$\int_C \frac{1}{z+2i}\, dz = 0 .$$

$$\int_C \frac{1}{(z-1)(z+2i)}\, dz = \frac{1}{1+2i}(2\pi i) + 0$$

$$\int_C \frac{1}{(z-1)(z+2i)}\, dz = \frac{2\pi i}{1+2i}$$

Ex:-

Evaluate $\int_C \frac{dz}{z^2+1}$ where C is the path of parabola $y = 4 - x^2$ from $A(2,0)$, $B(-2,0)$

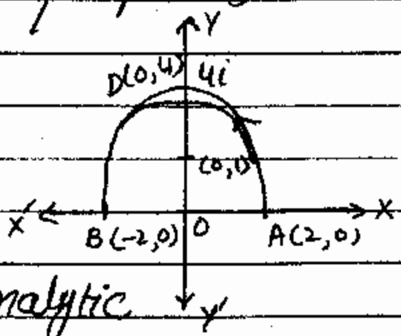
Sol:-

Since, C is the part of parabola $y = 4 - x^2$

when $x = 0$, $y = 4$.

$x = 2$, $y = 0$

$x = -2$, $y = 0$



The points where the function is non-analytic are $z^2 = -1 = i^2$

$$z^2 = i^2$$

$$z = \pm i$$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$\frac{1}{(z+i)(z-i)} = \frac{A}{z+i} + \frac{B}{z-i} \rightarrow (a)$$

Multiplying both sides by $(z+i)(z-i)$

$$1 = A(z-i) + B(z+i) \rightarrow (1)$$

put $z+i=0$ or $z=-i$ in (1)

$$1 = A(-i-i) + B(0)$$

$$1 = -2iA$$

$$\Rightarrow A = \frac{-1}{2i}$$

put $z-i=0$ or $z=i$ in (1)

$$1 = A(0) + B(i+i)$$

$$1 = B(2i)$$

$$B = \frac{1}{2i}$$

putting all values in (a)

$$\frac{1}{(z+i)(z-i)} = \frac{-1/2i}{z+i} + \frac{1/2i}{z-i}$$

$$\int_C \frac{1}{z^2+1}\,dz = \frac{-1}{2i}\int_C \frac{1}{z+i}\,dz + \frac{1}{2i}\int_C \frac{1}{z-i}\,dz$$

Since, $z=-i$ lies out side the contour C

So, by cauchy's fundamental theorem

$$\int_C \frac{1}{z+i}\,dz = 0$$

Since, $z=i$ lies inside the contour

So, by cauchy's integral formula.

$$\int_C \frac{1}{z-i}\,dz = 2\pi i$$

$$\int_C \frac{1}{z^2+1}\,dz = 0 + \frac{1}{2i}(2\pi i)$$

$$\int_C \frac{1}{z^2+1}\,dz = \pi$$

Now, $\int_C \frac{1}{z^2+1} dz = \int_{ADB} \frac{1}{z^2+1} dz + \int_{BOA} \frac{1}{z^2+1} dz$

$\therefore \int_{ADB} \frac{1}{z^2+1} dz = \pi - \int_{BOA} \frac{1}{z^2+1} dz - \int_{BOA} \frac{1}{z^2+1} dz$

$\int_{ADB} \frac{1}{z^2+1} dz = \pi - \int_{BOA} \frac{1}{z^2+1} dz$

Consider $\int_{BOA} \frac{1}{z^2+1} dz$ along BOA

then $z = x$, $dz = dx$ $\quad -2 \le x \le 2$

$$\int_{BOA} \frac{1}{z^2+1} dz = \int_{-2}^{2} \frac{1}{x^2+1} dx$$

$$= \tan^{-1} x \Big|_{-2}^{2}$$

$$= \tan^{-1} x \quad = \tan^{-1}(2) - \tan^{-1}(-2)$$
$$= \tan^{-1}(2) + \tan^{-1}(2)$$
$$= 2\tan^{-1}(2)$$

$$\therefore \int_{ADB} \frac{dz}{z^2+1} = \pi - 2\tan^{-1} 2$$

Theorem:-

Let $f(z)$ be analytic on and within boundary of C of a simply connected region D, and let 'a' be any point within $C_{(n)}$ then

$$f(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \quad \text{or}$$

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Proof:- Since, $f(z)$ is analytic on and

within C and $z=a$ is a point in C.
So, by cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int \frac{f(z)}{z-a}dz \rightarrow (1)$$

Likewise if $\overset{c}{z} = a+h$ be a point inside "C"
Then $f(a+h) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a-h}\, dz \Rightarrow (2)$

Also

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \rightarrow (3)$$

using (1), (2), (3)

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\int_c \frac{f(z)}{z-a-h}dz - \int_c \frac{f(z)}{z-a}dz\right]$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\int_c \left(\frac{f(z)}{z-a-h} - \frac{f(z)}{z-a}\right)dz\right]$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\int_c \frac{z-h-z+a+h\, f(z)\, dz}{(z-a-h)(z-a)}\right]$$

$$= \lim_{h\to 0} \frac{1}{2\pi i h}\left[\int_c \frac{hf(z)}{(z-a-h)(z-a)}dz\right]$$

$$= \lim_{h\to 0} \frac{1}{2\pi i}\left[\int_c \frac{hf(z)}{(z-a-h)(z-a)}dz\right]$$

$$= \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(z-a)^2}$$

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}dz \rightarrow (4)$$

Likewise

$$f'(a+h) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a-h)^2}dz \rightarrow (5)$$

$$f''(a) = \lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h}$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h}\left[\int_c \left(\frac{1}{(z-a-h)^2} - \frac{1}{(z-a)^2} \right) f(z)\,dz \right]$$

$$= \lim_{h \to 0} \frac{1}{2\pi i h}\left[\int_c \frac{h(2z-2a-h)}{(z-a-h)^2(z-a)^2} f(z)\,dz \right]$$

$$= \lim_{h \to 0} \frac{1}{2\pi i}\left[\int_c \frac{(2z-2a-h)}{(z-a-h)^2(z-a)^2} f(z)\,dz \right]$$

$$f''(a) = \frac{1}{2\pi i} \int \frac{2(z-a)}{(z-a)^2(z-a)^2} f(z)\,dz$$

$$= \frac{2}{2\pi i} \int \frac{f(z)}{(z-a)^3}\,dz$$

Like-wise,

$$f'''(a) = \frac{6}{2\pi i} \int_c \frac{f(z)}{(z-a)^4}\,dz = \frac{3!}{2\pi i} \int_c \frac{f(z)}{(z-a)^4}\,dz$$

$$f'''(a) = \frac{3!}{2\pi i} \int_c \frac{f(z)}{(z-a)^4}\,dz$$

when, it is continuous, then

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Ex :-

If $f(z)$ is a complex valued function, analytic on and within a contour C. Further C encloses a point $z=a$. Then, prove that

$$f^{(4)}(a) = \frac{4!}{2\pi i} \int_c \frac{1}{(z-a)^5} f(z)\,dz$$

Sol:-

Since, $f(z)$ is analytic on and within C and $z=a$ is a point in C. So, by Cauchy's integral formula.

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz \rightarrow (1)$$

Likewise, if $z = a+h$ is point in C

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a-h}\,dz \rightarrow (2)$$

After last Q.

Also

$$f'(a) = \lim_{h\to 0} \int_C \frac{f(a+h) - f(a)}{h} \rightarrow (3)$$

put the values of $f(a+h)$ and $f(a)$

$$f'(a) = \lim_{h\to 0}\int_C \frac{\left(\frac{1}{2\pi i}\left(\frac{f(z)}{z-a-h}\right)dz - \frac{1}{2\pi i}\frac{f(z)}{z-a}\right)dz}{h\,(z-a)} \qquad \because z-a=h$$

$$= \lim_{h\to 0}\frac{1}{2\pi i h}\left[\int_C\left(\frac{f(z)}{z-a-h} - \frac{f(z)}{z-a}\right)dz\right]$$

$$= \lim_{h\to 0}\frac{1}{2\pi i h}\int_C\left[\frac{\cancel{z}-\cancel{a}-\cancel{z}+\cancel{a}+h}{(z-a)(z-a-h)}\right]f(z)\,dz.$$

$$= \lim_{h\to 0}\frac{1}{2\pi i h}\int_C \frac{h}{(z-a)(z-a-h)}f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)(z-a)}\,dz$$

Ex:-

Evaluate $\int_c \frac{6z^2+5}{(z-1)^3}\,dz$, $c: |z|=2$

Sol:-

$$\int_c \frac{6z^2+5}{(z-1)^3}\,dz \quad ; \quad c: |z|=2$$

$$f(z) = 6z^2+5$$
$$f'(z) = 12z$$
$$f''(z) = 12$$
$$f''(1) = 12$$

$\therefore$ By theorem

"Let $f(z)$ be analytic on and within boundary of C of a simply connected region D. And let "a" be any point within C then $\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$

$$\therefore \int_c \frac{6z^2+5}{(z-1)^3}\,dz = \frac{2\pi i}{2!} \times 12$$
$$= 12\pi i$$

Ex:-

(i) Evalute $\int_c \frac{e^{2z}+\cos z}{z^4}\,dz$; $c: |z|=1$

(ii) $\int_c \frac{e^{2z}+\sin z}{(z-1)^4}$; $c: |z|=2$

Sol:-

(i)
$$\int_c \frac{e^{2z}+\cos z}{(z-0)^4}\,dz$$

$$f(z) = e^{2z}+\cos z$$

$$a=0, \quad n=3 \quad \therefore n+1=4$$

then by last theorem.

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{3!} f^{(n)}(a) \rightarrow (1)$$

$$f'(z) = 2e^{2z} - \sin z$$

$f''(z) = 4e^{2z} - \cos z$

$f'''(z) = 8e^{2z} + \sin z$

$f'''(0) = 8e^{(0)} + \sin(0)$

$= 8 + 0$

$f'''(0) = 8$, $n = 3$

put this in (1)

$$\int_c \frac{f(z)}{(z-a)^{3+1}} dz = \frac{2\pi i}{3!} f^{(3)}(0)$$

$$\int_c \frac{f(z)}{z^4} dz = \frac{2\pi i (8)}{6} = \frac{8\pi i}{3}$$

(ii)

$$\int_c \frac{e^{2z} + \sin z}{(z-1)^4}$$

$f(z) = e^{2z} + \sin z$, $z = 1$, $n = 3$

By theorem

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{2!} f^{(n)}(a) \quad (1)$$

$f'(z) = 2e^{2z} + \cos z$

$f''(z) = 4e^{2z} - \sin z$

$f'''(z) = 8e^{2z} - \cos z$

$f'''(1) = 8e^{2(1)} - \cos(1)$

$f'''(1) = 8e^2 - \cos(1)$

putting all values in (1)

$$\int_c \frac{f(z)}{(z-a)^{3+1}} dz = \frac{2\pi i}{3!} f^{(3)}(a)$$

$$= \frac{2\pi i}{3!} f^{(3)}(1)$$

$$= \frac{2\pi i}{3!} (8e^2 - \cos(1))$$

$$\int_c \frac{f(z)}{(z-a)^4} dz = \frac{2\pi i}{3 \cdot 2} (8e^2 - \cos(1))$$

Cauchy's Inequality Theorem:-

If $f(z)$ is a complex valued function analytic on and within $C: |z-a| = r$ and $f(z)$ is bounded (i.e) $|f(z)| \le M$, then prove that

$$|f^{(n)}(a)| \le \frac{n!\, M}{r^n}$$

Proof:-

Since, $f(z)$ is analytic on and with $C: |z-a| = r$

" Let $f(z)$ be analytic on and within C of a simply connected region 'D'. let 'a' be any point within C then

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Now, $$|f^{(n)}(a)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz \right|$$

$$\le \frac{n!}{2\pi i} \int_C \frac{|f(z)|}{|(z-a)^{n+1}|}\, |dz| \qquad |i| = 1$$

$|dz|$ = circular arc $= 2\pi r$

$$\le \frac{n!}{2\pi} \frac{M}{r^{n+1}} \times 2\pi r$$

$$\le \frac{n!\, M}{r^{n+1-1}}$$

$$\le \frac{n!\, M}{r^n}$$

Entire function:-

A function which is analytic everywhere in the complex

plane is called entire function.
(i-e) (i) Sinz is entire function
(ii) logz is not entire but analytic.
(iii) A polynomial and complex functions are ~~fun~~ entire functions.
(iv) All Transcendental functions are entire.

Liouvilles theorem:-

Prove that entire bounded function is constant (OR)
Let f(z) be analytic in the entire complex planed and is bounded then prove that f(z) is constant.

Proof:-

Since, f(z) is analytic everywhere in the complex plane and is bounded, Therefore, by cauchy's inequality formula.

$$|f^{(n)}(a)| \leq \frac{n!\,M}{r^n} \quad ; \quad C : |z-a| = r$$

$$|f'(a)| \leq \frac{M}{r}$$

$$|f'(z)| \leq \frac{M}{r}$$

Since, f(z) is an entire function.
So, when $r \to \infty$

$$f'(z) = 0$$

$$f(z) = \text{Constant}$$

Remarks:-

(i) e^z, $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ are entire functions.

(ii) $f(z) = \dfrac{z^2+1}{(z+1)(z-3)}$ is not entire but analytic without $-1, 3$

(iii) $f(z) = z^2+1$ is entire function

(iv) The function which is non-analytic or discontinuous at three points is

$$f(z) = \frac{1}{z(z-1)(z-3)} \quad \text{at } z = 0, 1, 3$$

(v) Every entire function is analytic but every analytic function may or may not entire function.

Morera's theorem:- *Morera's Theorem:-*

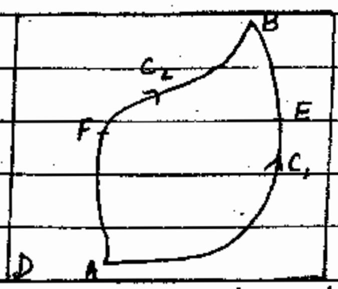
Statement:-

Let $f(z)$ be continuous in a simply connected domain D further let $\int_C f(z) = 0$, where C is closed curve D, then prove that $f(z)$ is analytic in D.

Proof:-

Let "a" be a fixed point and z be a variable point.

Let C_1 and C_2 be any two continuous curves in D. Joining a to z. Let C be the closed contour consisting of C_1 and C_2.



Then, by consequences of cauchy's fundamental theorem

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

This shows that integral along every curve in D. Joining a to z is the same. Hence Taking "t" as the variable of integration we may write

$$F(z) = \int_a^z f(t)\,dt \rightarrow (1)$$

Further, let $z+h$ be any point near to the point z. Then

$$f(z+h) = \int_a^{z+h} f(t)\,dt \rightarrow (2)$$

$$f(z+h) - f(z) = \int_a^{z+h} f(t)\,dt - \int_a^{z} f(t)\,dt$$

$$= \int_a^{z+h} f(t)\,dt + \int_z^{a} f(t)\,dt$$

$$= \int_z^{a} f(t)\,dt + \int_a^{z+h} f(t)\,dt$$

$$f(z+h) - f(z) = \int_z^{z+h} f(t)\,dt \rightarrow (3)$$

$$\frac{f(z+h) - f(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(t)\,dt - \frac{f(z)\,h}{h} \quad \because h = \int_z^{z+h} dt$$

$$= \frac{1}{h}\left[\int_z^{z+h} f(t)\,dt - f(z)\int_z^{z+h} dt\right] \rightarrow (4)$$

$$= \frac{1}{h}\left[\int_z^{z+h} (f(t) - f(z))\,dt\right] \rightarrow (4)$$

Since, $f(t)$ is continuous at z, Therefore, a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(t) - f(z)| < \varepsilon \rightarrow (5)$

For every ε, such that $|t - z| < \delta \rightarrow (6)$

$$\left|\frac{f(z+h) - f(z)}{h} - f(z)\right| < \frac{1}{|h|}\,\varepsilon\int_z^{z+h} |dt|$$

$$< \frac{1}{|h|}\,\varepsilon\,|t|\Big|_z^{z+h}$$

$$< \frac{1}{|h|} \varepsilon (|z+h-z|)$$

$$< \frac{1}{|h|} \varepsilon |h|$$

$$< \varepsilon \rightarrow (7)$$

Since ε is an arbitrary, we conclude from eq (7)

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f(z)$$

Hence, $f'(z) = f(z)$ exists, $\forall z \in D$ we conclude that $f(z)$ has the derivative at every point of $z \in D$ and consequently $F(z)$ is analytic in D.

But the derivative of an analytic function is also analytic.

Assignment:-

Example:-

Evaluate $\int_{(0,3)}^{(2,4)} (2y+x^2)\,dx + (3x-y)\,dy$

along (i) parabola $x = 2t,\ y = t^2+3$

(ii) st. line from $(0,3)$ to $(2,3)$ and then $(2,3)$ to $(2,4)$

(iii) s.t line from $(0,3)$ to $(2,4)$

Sol:-

$$F(x,y) = (2y+x^2)\,dx + (3x-y)\,dy$$

(i) parabola $x = 2t,\ y = t^2+3 \Rightarrow dx = 2dt,\ dy = 2t\,dt$

At $(0,3)$ $x = 0,\ t = 0$

$y = 3,\ t^2 = 0$ $\quad 3 = t^2+3$

At $(2,4)$ $x = 2,\ t = 1$

$y = 4,\ t = \pm 1$ $\quad 4-3 = t^2$

Then, the given integral becomes $\quad 1 = t^2$

$$\int_0^1 (2(t^2+3)+(2t)^2)dt + (3(2t)-(t^2+3))2t\,dt)$$

$$= \int_0^1 2(2t^2+6+4t^2)dt + ((6t-t^2-3)2t))dt$$

$$= 2\int_0^1((6t^2+6)\,dt + \frac{1}{2}(12t^2-2t^3-6t)\,dt)$$

$$= 2\int_0^1(6t^2+6)dt + \frac{1}{2}\int_0^1(12t^2-2t^3-6t)\,dt$$

$$= 2\left[6\frac{t^3}{3}\Big|_0^1 + 6t\Big|_0^1\right] + \frac{1}{2}\left[12\frac{t^3}{3}\Big|_0^1 - 2\frac{t^4}{4}\Big|_0^1 - 6\frac{t^2}{2}\Big|_0^1\right]$$

$$= 2\left[2t^3\Big|_0^1 + 6(1-0)\right] + \frac{1}{2}\left[4t^3\Big|_0^1 - \frac{1}{2}t^4\Big|_0^1 - 3t^2\Big|_0^1\right]$$

$$= 4(1-0) + 12(1-0) + 4(1-0) - \frac{1}{2}(1-0) - 3(1-0)$$

$$= 4+12+4-\frac{1}{2}-3$$

$$= 20 - \frac{1}{2} - 3$$

$$= \frac{40-1-6}{2} = \frac{33}{2}$$

(ii)

straight line from $(0,3)$ to $(2,3)$ and then $(2,3)$ to $(2,4)$

(a) from $(0,3)$ to $(2,3)$

put $x = t$, $y = 3$; $0 \le t \le 2$

$dx = dt$, $dy = 0$

The given integral becomes

$$\int_0^2 (2(3)+t^2)dt + (3t-3)(0)$$

$$= \int_0^2 (6+t^2)dt$$

$$= 6t\Big|_0^2 + \frac{t^3}{3}\Big|_0^2$$

$$= 6(2-0) + \frac{1}{3}(2^3-0)$$

$$= 12 + \frac{1}{3}(8-0)$$

$$= 12 + \frac{8}{3}$$

$= \frac{36+8}{3}$

$= 44$

(b) from $(2, 3)$ to $(2,4)$

$x = 2, \quad y = t \quad ; \quad 3 \le t \le 4$

$dx = 0, \quad dy = dt$

Integral becomes $\int_3^4 (2(t)+(2)^2)(0) + (3(2)-t)\,dt$

$= \int_3^4 0 + (6-t)\,dt = \int_3^4 (6-t)\,dt$

$= 6t\Big|_3^4 - \frac{t^2}{2}\Big|_3^4 = 6(4-3) - \frac{1}{2}(4^2-3^2)$

$= 6(1) - \frac{1}{2}(16-9)$

$= 6 - \frac{1}{2}(7)$

$= 6 - \frac{7}{2} = \frac{12-7}{2} = \frac{5}{2} \Rightarrow \int_0^2 F(x,y)\,dx + \int_3^4 F(x,y)\,dx$

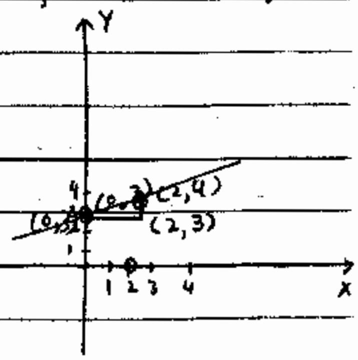
$= \frac{44}{3} + \frac{5}{2} = \frac{88+15}{6} = \frac{103}{6}$

(iii) Straight line from $(0,3)$ to $(2,4)$

Eq of st. line passes through $(0,3)$ and $(2,4)$

$\frac{y_2 - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$\frac{y-3}{4-3} = \frac{x-0}{2-0}$



$y = \frac{x}{2} + 3 \qquad \frac{y-3}{1} = \frac{x}{2}$

$y = \frac{x+6}{2}$

$dy = \frac{1}{2}dx \Rightarrow 2(y-3) = x$

$\Rightarrow x = 2y - 6, \quad dx = 2dy$

$\int_{(0,3)}^{(2,4)} (2y + (2y-6)^2)(2dy) + (3(2y-6)-y)\,dy = \int_0^2 (2y\cancel{x+\frac{4x}{2}})^2$

$\cancel{-14y\,x^2)\,dx\,dy(3xy - \frac{x}{2} - y3)dy}\ dx \quad \int^4 (2y(4y^2-24y+36)\,dy$

So we solve one side $+(6y-18-y)\,dy$

$= \int_0^2 \cancel{((4y+6)-y\,x^2 4)\,dx + \frac{dy - x + 6}{2}) \frac{1}{2}dx)}\,dy$

$= \int_0^2 \cancel{H((x^2+4x+6)^2 + 8(\frac{6x - 7x}{4} + 6)))\,dx}\,dy \quad (18-\ldots)\,dy$

$$= \int_0^2 \left(\int_2 \left(4x^2 + 4xy + \frac{5x + 6y + 6}{4} \right) dy \right) dy + \int (2y^2 - 44y$$

$$= \int_0^2 \left(\frac{4x^2 + 4x + 24 + 15x + 6}{4} \right) dx$$

$$= \frac{1}{4} \int_0^2 (4x^2 + 9x + 30)\, dx$$

$$= \frac{1}{4}\left[4\int_0^2 x^2\, dx + 9 \int_0^2 x\, dx + 30 \int_0^2 1\cdot dx \right]$$

$$= \int_0^2 x^2\, dx + \frac{9}{4} \int_0^2 x\, dx + \frac{15}{2} \int_0^2 dx$$

$$= \frac{x^3}{3}\Big|_0^2 + \frac{9}{4}\frac{x^2}{2}\Big|_0^2 + \frac{15}{2} x \Big|_0^2$$

$$= \frac{1}{3}(2^3 - 0) + \frac{9}{8}(2^2 - 0) + \frac{15}{2}(2 - 0)$$

$$= \frac{1}{3}(8 - 0) + \frac{9}{8}(4 - 0) + \frac{15}{2}(2)$$

$$= \frac{8}{3} + 9 + 15$$

$$= \frac{8 + 27 + 45}{3}$$

$$= \frac{80}{3}$$

CHAPTER # 06

POWER SERIES

An infinite series is said to be convergent if the partial sums tend to be finite and definite limit as n is very very large.

Ex:-

Consider the series

$$S = 1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$$

$$\because S_\infty = \frac{a}{1-r}$$

$$\lim_{n\to\infty} S_n = S_\infty = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{4-1}{4}} = \frac{1}{3/4} = 4/3$$

Ex:-

Consider the series

$$S = 1^2 + 2^2 + 3^2 + \cdots$$

$$S_n = \frac{n(n+1)(n+2)}{6} = \Sigma n^2$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)(n+2)}{6} = \infty$$

Note:-

An infinite series is said to be divergent if the partial sums tend to $+\infty$ or $-\infty$; as n is very very large.

Remark:-

Consider the series

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Harmonic Series.

$$S = \sum_{n=1}^{\infty} \frac{1}{n}$$

Although it has finite and definite limit yet it can be prove by camparison test that it diverges.

Ex:-

Using camparison test, Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Sol:-

Let $S = \sum_{n=1}^{\infty} \frac{1}{n}$

we have to prove that $S = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

Although it has finite and definite limit, yet it can be proved by camparison test that it diverges

Consider $1 > \frac{1}{2}$

$$\frac{1}{2} + \frac{1}{3} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2} \text{ etc.}$$

Therefore, $1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}\right) + \cdots > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

Now Consider

$$S = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

$$S_n = \frac{n}{2}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{n}{2}$$

$$\lim_{n \to \infty} S_n = \frac{\infty}{2} = \infty$$

By camparison test, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Comparison Test:-

(a) If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ are two positive term series such that $U_n \leq V_n$ then $\sum_{n=1}^{\infty} U_n$ converges if $\sum_{n=1}^{\infty} V_n$ Converges

(b) If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ are two positive term series and such that $U_n \geq V_n$ then if $\sum_{n=1}^{\infty} V_n$ diverges then $\sum_{n=1}^{\infty} U_n$ diverges.

Oscillatory series:-

A series is said to be oscillatory if neither the partial sums tend to finite and definite limit nor tend to $+\infty$ or $-\infty$, rather oscillate between two numbers, is called oscillatory series.

Absolute convergence-

A series $\sum_{n=1}^{\infty} U_n$ is said to be absolute convergent if $\sum_{n=1}^{\infty} |U_n|$ is convergent If $\sum_{n=1}^{\infty} |U_n|$ is divergent then $\sum_{n=1}^{\infty} U_n$ is conditionally convergent.

Ex:-

Consider $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

$$= \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$\sum_{n=1}^{\infty} |U_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is infinite and definite

Therefore, $\sum_{n=1}^{\infty} |U_n|$ is divergent.

So, $\sum_{n=1}^{\infty} |U_n|$ is conditionally convergent.

Ex:-

Consider $\sum_{n=1}^{\infty} U_n = \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$\sum_{n=1}^{\infty} |U_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\cdots = \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$ Harmonic series as $n \to \infty$

$\sum_{n=1}^{\infty} |U_n|$ is divergent.

This series is conditionally convergent.

Power Series:-

A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots + a_n (z - z_0)^n + \cdots$$

is called the power series, where the constants $a_0, a_1, a_2, \ldots, a_n$ are independent of z.

If $z - z_0 = t$, then the series reduces to $\sum_{n=0}^{\infty} a_n t^n$

Hence, it is sufficient to consider a power series of the form $\sum_{n=0}^{\infty} a_n z^n$.

Examples:-

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} + \cdots$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

are examples of power series.

Circle and Radius of convergence:-

If a power series is convergent at any point within the circle, Then the circle is said to be the circle of convergence and the radius is said to be the radius of convergence

Suppose the series $\sum_{n=0}^{\infty} a_n z^n$

$$U_n = a_n z^n$$

$$U_{n+1} = a_{n+1} z^n$$

Ratio Test :-

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| < 1$$

then series is convergent.

$$\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| > 1$$ then series is

If divergent. If $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = 0$, then series has no information (result).

For convergence :-

$$\lim_{n\to\infty}\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\frac{z^{n+1}}{z}\right| < 1$$

$$|z| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

$$|z| < \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Ex:-

Find the radius of convergence of the power series

$$az + \frac{a-b}{2!}z^2 + \frac{(a-b)(a-2b)}{3!}z^3 + \cdots$$

Sol:-

Given series : $az + \frac{a-b}{2!}z^2 + \frac{(a-b)(a-2b)}{3!}z^3 + \cdots$

on comparing with

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

then $a_0 = 0$, $a_1 = a$, $a_2 = \frac{a-b}{2!}$

$a_3 = \frac{(a-b)(a-2b)}{3!}$

$$a_n = \frac{(a-b)(a-2b)\cdots(a-(n-1)b)}{n!}$$

$$a_{n+1} = \frac{(a-b)(a-2b)\cdots(a-nb)}{(n+1)!}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$R = \lim_{n\to\infty}\left|\frac{\frac{\cancel{(a-b)}\cancel{(a-2b)}\cdots\cancel{(a-(n-1)b)}}{n!}}{\cancel{(a-b)}\cancel{(a-2b)}\cdots(a-nb)/(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n+1}{a-nb}\right| \qquad = \frac{n!}{(n+1)(n+1-1)!}$$

$$= \lim_{n\to\infty}\left|\frac{n\left(1+\frac{1}{n}\right)}{n\left(\frac{a}{n}-b\right)}\right| \qquad = \frac{\cancel{n!}}{(n+1)(\cancel{n!})}$$

$$\Rightarrow = \frac{1}{n+1}$$

$$= \lim_{n\to\infty}\frac{1+\frac{1}{n}}{\frac{a}{n}-b}$$

$$R = \frac{1}{|-b|} = \frac{1}{|b|}$$

$$R = \frac{1}{b}$$

Root Test :-

The radius of convergence R of a power series is given by

$$R = \frac{1}{\lim_{n\to\infty}|a_n|^{1/n}}$$

Sol :-

Given power series

$$\sum_{n=0}^{\infty} a_n z^n$$

$$U_n = a_n z^n$$

$$\lim_{n\to\infty} |U_n|^{1/n} = \lim_{n\to\infty} |(a_n z^n)^{1/n}|$$

$$= \lim_{n\to\infty} |(a_n)^{1/n} (z^n)^{1/n}|$$

$$= \lim_{n\to\infty} |(a_n)^{1/n} z| < 1$$

$$= |z| \lim_{n\to\infty} |(a_n)^{1/n}| < 1$$

$$= |z| < \frac{1}{\lim_{n\to\infty} |(a_n)^{1/n}|}$$

$$R = \frac{1}{\lim_{n\to\infty} |(a_n)^{1/n}|}$$

Ex:-

Find the radius of convergence R of series $\sum_{n=1}^{\infty} n^{\log n} z^n$

Sol:-

By Root Test

$$R = \frac{1}{\lim_{n\to\infty} |(a_n)^{1/n}|}$$

$$R = \frac{1}{R'}$$

$$R' = \lim_{n\to\infty} |(a_n)^{1/n}| \qquad \because a_n = n^{\log n}$$

$$R' = \lim_{n\to\infty} |(n^{\log n})^{1/n}|$$

$$\log R' = \lim_{n\to\infty} |\log (n^{\log n})^{1/n}|$$

$$= \lim_{n\to\infty} \left| \frac{1}{n} \log n \log n \right|$$

$$= \lim_{n\to\infty} \left|\frac{(\log n)^2}{n}\right| = \frac{\infty}{\infty}$$

By Del Hospital's theorem.

$$\log R' = \lim_{n\to\infty} \left|\frac{2\log n \cdot \frac{1}{n}}{1}\right| = \frac{\infty}{\infty}$$

$$\log R' = \lim_{n\to\infty} \left|\frac{2}{1}\frac{\log n}{n}\right| = \frac{\infty}{\infty}$$

$$\log R' = \lim_{n\to\infty} \left|\frac{2\frac{1}{n}}{1}\right| = \left|\lim_{n\to\infty} \left|\frac{2}{n}\right|\right| = \frac{2}{\infty} = 0$$

$$\log R' = 0$$

$$R' = e^0 = 1$$

Therefore, $R = \frac{1}{R'}$

$$R = 1$$

Assignments:-

Find the radius of convergence of the series.

(i) $\sum_{n=0}^{\infty} a_n z^n = 1 + 4z + 9z^2 + 16z^3 + 25z^4 + \cdots$

(ii) $a^2 + \frac{a(a-b)z}{1!} + \frac{a(a-2b)}{2!} z^2 + \frac{a(a-3b)}{3!} z^3 + \cdots$

(iii) $a + \frac{a(a-b)z}{} + \frac{a(a-2b)^2}{2!} z^2 + \frac{a(a-3b)^3}{3!} z^3 + \cdots$

(iv) $\sum_{n=0}^{\infty} a_n z^n = 1 - kz + \frac{k(k-3)}{2!} z^2 - \frac{k(k-4)(k-5)}{3!} z^3$

$$+ \frac{k(k-5)(k-6)(k-7)}{4!} z^4 - \cdots$$

(v) $\sum_{k=0}^{\infty} (k+1)^k z^k$

(vi) $\sum_{k=1}^{\infty} \frac{z^k}{k}$

Sol:-

(i) $\sum_{n=0}^{\infty} a_n z^n = 1 + 4z + 9z^2 + 16z^3 + 25z^4 + \cdots$

on comparing with $a_0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots$

$a_0 = 1$, $a_1 = 4$, $a_2 = 9$, $a_3 = 16$

$- - - - - a_n = (n+1)^2$

$a_0 = 1^2$, $a_1 = 2^2$, $a_2 = 3^2$, $a_3 = 4^2$

$- - - - a_n = (n+1)^2$

$a_{n+1} = (n+2)^2$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(n+2)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2 \left(1+\frac{1}{n}\right)^2}{n^2 \left(1+\frac{2}{n}\right)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(1+1/n)^2}{\left(1+\frac{2}{n}\right)^2} \right|$$

$$= \left| \frac{\left(1+\frac{1}{\infty}\right)^2}{\left(1+\frac{2}{\infty}\right)^2} \right|$$

$$= \left| \frac{(1+0)^2}{(1+0)^2} \right|$$

$R = 1$

(ii)

$$\sum_{n=1}^{\infty} a_n z^n = a^2 + \frac{a(a-b)}{1!} z + \frac{a(a-2b)}{2!} z^2 + \frac{a(a-3b)}{3!} z^3 + - - -$$

on comparing with

$a_0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$

$a_0 = a^2$, $a_1 = \dfrac{a(a-b)}{1!}$, $a_2 = \dfrac{a(a-2b)}{2!}$

$a_3 = \dfrac{a(a-3b)}{3!}$ - - - - -

$$a_n = \frac{a(a-nb)}{n!}$$

$$a_{n+1} = \frac{a(a-(n+1)b)}{(n+1)!}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{a(a-nb)}{n!}}{\frac{a(a-(n+1)b)}{(n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{a-nb}{n!}}{\frac{a-(n+1)b}{(n+1)n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)n(\frac{a}{n}-b)}{n(\frac{a}{n}-(1+\frac{1}{n})b)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n(1+\frac{1}{n})(\frac{a}{n}-b)}{\frac{a}{n}-(1+\frac{1}{n})b}\right|$$

$$R = \left|\frac{\infty(1+\frac{1}{\infty})(\frac{a}{\infty}-b)}{\frac{a}{\infty}-(1+\frac{1}{\infty})b}\right|$$

$$R = \left|\frac{\infty}{-1}\right|$$

$$R = \infty$$

(iii)

$$a + a(a-b)z + \frac{a(a-2b)^2}{2!}z^2 + \cdots$$

on comparing with

$$a_0 + a_1z^1 + a_2z^2 + \cdots + a_nz^n + \cdots$$

$$a_0 = a, \quad a_1 = a(a-b), \quad a_2 = \frac{a(a-2b)^2}{2!}$$

$$+ \cdots + a_n = \frac{a(a-nb)^n}{n!}$$

$$a_{n+1} = \frac{a(a-(n+1)b)^{n+1}}{(n+1)!}$$

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{a(a-nb)^n}{n!}}{\frac{a(a-(n+1)b)^{n+1}}{(n+1)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{(a-nb)^n}{n!}}{\frac{(a-(n+1)b)^{n+1}}{(n+1)n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)(a-nb)^n}{(a-(n+1)b)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{n(1+\frac{1}{n})n^n(\frac{a}{n}-b)^n}{n^{n+1}(\frac{a}{n}-(1+\frac{1}{n})b)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{n^{n+1}}}{\cancel{n^{n+1}}}\,\frac{(1+\frac{1}{n})(\frac{a}{n}-b)^n}{(\frac{a}{n}-(1+\frac{1}{n})b)^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(1+\frac{1}{n})(\frac{a}{n}-b)^n}{(\frac{a}{n}-(1+\frac{1}{n})b)^{n+1}}\right|$$

$$= \left| \frac{(1+\frac{1}{\infty})(\frac{a}{\infty}-b)^n}{(\frac{a}{\infty}-(1+\frac{1}{\infty})b)^{n+1}} \right|$$

$$= \left| \frac{(1+0)(0-b)^n}{(0-(1+0)b)^{n+1}} \right|$$

$$= \left| \frac{-b^n}{-b^{n+1}} \right|$$

$$= |b^{n-n-1}| = |b^{-1}|$$

$$= \left| \frac{1}{b} \right|$$

$$R = \frac{1}{b}$$

(iv)
$$\sum_{n=0}^{\infty} a_n z^n = 1 - Kz + \frac{K(K-3)}{2!} z^2 + \frac{K(K-4)(K-5)}{3!} z^3$$

$$+ \frac{K(K-5)(K-6)(K-7)}{4!} z^4 - - - - -$$

on comparing with

$$a_0 + a_1 z^1 + a_2 z^2 + a_3 z^3 + - - - -$$

$$a_0 = 1, \quad a_1 = -K, \quad a_2 = \frac{K(K-3)}{2!}$$

$$a_3 = \frac{K(K-4)(K-5)}{3!} \quad a_4 = \frac{K(K-5)(K-6)(K-7)}{4!}$$

$$- - - - - - - -$$

$$a_n = \frac{K(K+(n+1))(K-(n+2))(K-(n+3))}{n!}$$

$$a_{n+1} = \frac{K(K-(n+1+1))(K-(n+1+2))(K-(n+1+3))}{(n+1)!}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

So

$$R = \lim_{n\to\infty} \left| \frac{\frac{k(k-(n+1))(k-(n+2))(k-(n+3))}{n!}}{\frac{k(k-(n+2))(k-(n+3))(k-(n+4))}{(n+1)!}} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{(n+1)k(k-(n+1))(k-(n+2))(k-(n+3))}{k(k-(n+2))(k-(n+3))(k-(n+4))} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{\cancel{n}(1+\frac{1}{n})\cdot nk(\frac{k}{n}-(1+\frac{1}{n}))\cancel{n}(\frac{k}{n}-(1+\frac{2}{n}))\cancel{n}(\frac{k}{n}-(1+\frac{3}{n}))}{\cancel{n}\cancel{k}(\frac{k}{n}-(1+\frac{2}{n}))(\frac{k}{n}-(1+\frac{3}{n}))\cancel{n}\cdot\cancel{n}(\frac{k}{n}-(1+\frac{4}{n}))} \right|$$

$$R = \lim_{n\to\infty} \left| \frac{(n+1)(\frac{k}{\infty}-(1+\frac{1}{\infty}))(\frac{k}{\infty}-(1+\frac{2}{\infty}))(\frac{k}{\infty}-(1+\frac{3}{\infty}))}{(\frac{k}{\infty}-(1+\frac{2}{\infty}))(\frac{k}{\infty}-(1+\frac{3}{\infty}))(\frac{k}{\infty}-(1+\frac{4}{\infty}))} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(0-(1+0))(0-(1+0))(0-(1+0))}{(0-(1+0))(0-(1+0))(0-(1+0))} \right|$$

$$= \left| \frac{(\infty+1)(-1)(-1)(-1)}{(-1)(-1)(-1)} \right|$$

$$R = \infty$$

(V) $\sum_{k=0}^{\infty} (k+1)^k z^k = 1 + 2z + 3^2 z^2 + 4^3 z^3 + \cdots$

$$= 1 + 2z + 9z^2 + 64z^3 + \cdots$$

$a_0 = 1,\ a_1 = 2,\ a_3 = 3^2,\ a_4 = 4^3 \ldots\ a_n = (n+1)^n$

$R' = \lim_{n\to\infty} |(a_n)^{1/n}| \qquad \therefore R = \frac{1}{R'}$

$R' = \lim_{n\to\infty} |((n+1)^n)^{1/n}| = \lim_{n\to\infty} |n+1| = \infty$

$R = \frac{1}{R'} = \frac{1}{\infty} = 0$

(VI)

$\sum_{k=1}^{\infty} \frac{z^k}{k} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$ on comparing with $a_0 + a_1 z + a_2 z^2 + \cdots$

$a_0 = 0,\ a_1 = 1,\ a_2 = \frac{1}{2},\ a_3 = \frac{1}{3} \ldots\ a_n = \frac{1}{n},\ a_{n+1} = \frac{1}{n+1}$

$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n}}{\frac{1}{n+1}} \right| = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{\cancel{n}(1+\frac{1}{n})}}$

$\lim_{n\to\infty} |1+\frac{1}{n}| = |1+\frac{1}{\infty}| = |1+0| = 1$

Derived Series :-

A derived series can be obtained by either differentiating the original power series term by term or integrating it term by term.

Consider the power series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_n z^n + \cdots \rightarrow (1)$$

Differentiate (1) by term by term

$$a_1 + 2a_2 z + 3a_3 z^2 + \cdots + (n+1) a_{n+1} z^n + \cdots \rightarrow (2)$$

Integrate (1) term by term

$$a_0 z + a_1 \frac{z^2}{2} + a_2 \frac{z^3}{3} + \cdots + a_n \frac{z^{n+1}}{n+1} + \cdots \rightarrow (3)$$

Equation (2) and (3) are Derived series.

Theorem :-

Prove that the radius of convergence of the derived series remains the same as the radius of convergence of the original series.

Proof :-

The original power series

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_n z^n + \cdots$$

The radius of convergence of series is

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Case I :-

The derived series is obtained by differentiating term by term the original series.

$$a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1} + (n+1) a_{n+1} z^n + \cdots$$

$A_{n} = n a_n$ $\qquad \because \frac{d}{dz}(a_0) = 0$ ("1" term decrease)

$A_{n+1} = (n+1) a_{n+1}$

$$R' = \lim_{n\to\infty} \left| \frac{A_n}{A_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n a_n}{(n+1) a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n a_n}{n(1+\frac{1}{n}) a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_n}{(1+\frac{1}{n}) a_{n+1}} \right|$$

$\because 1 + \frac{1}{\infty} = 1 + 0 = 1$

$$= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Case II :-

The derived series is obtained by integrating the original power series term by term.

$$a_0 z + a_1 \frac{z^2}{2} + a_2 \frac{z^3}{3} + \cdots + a_n \frac{z^{n+1}}{n+1} + \cdots$$

$\because \int a_0 \, dz = a_0 z$ ([illegible])

$$A_{n+1} = \frac{a_n}{n+1}$$

$$A_{n+2} = \frac{a_{n+1}}{n+2}$$

$$R' = \lim_{n\to\infty} \left| \frac{A_{n+1}}{A_{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\frac{a_n}{n+1}}{\frac{a_{n+1}}{n+2}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_n(n+2)}{a_{n+1}(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n(1+\frac{2}{n})a_n}{n(1+\frac{1}{n})a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(1+0)\,a_n}{(1+0)\,a_{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

$$= R$$

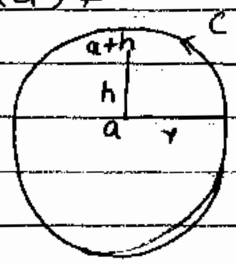
Taylor's theorem:-

Statement:-

If $f(z)$ is analytic on and within closed contour $C : |z-a| = r$ and $a+h$ be an interior point of C, then prove that $f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \cdots$

put $z = a+h$.

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$



Proof:-

Step I:-

Given, $f(z)$ is analytic on and within the closed contour $C : |z-a| = r$; $a+h$ is the interior point

Therefore, by Cauchy's integral formula

$$f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a-h}\, dz \rightarrow (A)$$

Consider

$$\frac{1}{z-a-h} = \frac{1}{z-a} + \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^3} + \cdots + \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)}$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \left(\frac{1}{z-a} + \frac{h}{(z-a)^2} + \frac{h^2}{(z-a)^3} + \cdots + \frac{h^n}{(z-a)^{n+1}} + \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)} \right) f(z)\,dz$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} + \frac{h}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^2} + \cdots + \frac{h^n}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz + \frac{h^{n+1}}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}(z-a-h)}\,dz \to (B)$$

Supposing,

$$R_n = \frac{h^{n+1}}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}(z-a-h)}\,dz$$

As we shall prove $R_n \to 0$ as $n \to \infty$

Step II :-

To prove $R_n \to 0$ as $n \to \infty$

$$|R_n| = \left| \frac{h^{n+1}}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}(z-a-h)}\,dz \right|$$

$$= \frac{h^{n+1}}{2\pi i}\left| \int_C \frac{f(z)}{(z-a)^{n+1}(z-a-h)}\,dz \right|$$

Since $f(z)$ is analytic at all points in C and $z = a+h$ is an interior point

$$\left| \frac{f(z)}{z-a-h} \right| \le M$$

$$|R_n| = \frac{h^{n+1}}{2\pi} \left| \frac{|f(z)|}{(|z-a|)^{n+1}} \right| \left| \int_c dz \right|$$

$\because |z-a| = r \qquad \left| \int_c dz \right| = 2\pi r$

$$|R_n| \le \frac{h^{n+1}}{2\pi} M \cdot \frac{1}{r^{n+1}} 2\pi r$$

$$= h(h/r)^n M$$

Since $h < r$

$n(\frac{h}{r}) < 1$

Hence, $(\frac{h}{r})^n \to 0$ as $n \to \infty$

$\therefore R_n \to 0$ as $n \to \infty$

$$f(a+h) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz + \frac{h}{2\pi i} \int_c \frac{f(z)}{(z-a)^2} dz + \cdots$$

$$+ \frac{h^n}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz + \cdots \to (C).$$

we know that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}} dz$$

$$f(a+h) \Rightarrow \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{f^{(n)}(a)}{n!} \quad ; ; 0 \le n \le \infty$$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \cdots$$

put $z = a+h$

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots$$

$$+ \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Remark:-

In Taylor's series put $a=0$, $h=z$

then $f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots$

$+ \dfrac{z^n}{n!} f^{(n)}(0) + - - - - -$ $C: |z| = r$

is known as Maculaurin's series.

Difference b/w Taylor and Maculaurin's Series:-

In Taylor's series, $f(z)$ is analytic on and within a contour $C: |z-a| = r$ which is a circle, centre at "a" and radius is "r".

In Maculaurin's series $f(z)$ is analytic on and within a contour $C: |z| = r$ which is a circle, centre at origin $(0,0)$ and radius is "r".

(DR. M. IQBAL'S BOOK : PAGE NO 128 – 148)

Ex:-

Expand $f(z) = e^z$ in the form of series. Find the region of convergence also.

Sol:-

Step I:-

$$f(z) = e^z \qquad f(0) = e^0 = 1$$
$$f'(z) = e^z \qquad f'(0) = e^0 = 1$$
$$f''(z) = e^z \qquad f''(0) = e^0 = 1$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(n)}(z) = e^z \qquad f^{(n)}(0) = e^0 = 1$$
$$\vdots \qquad\qquad \vdots$$

Using Maculaurin's series.

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) - - - + \frac{z^n}{n!} f^{(n)}(0) + \dots$$

$$f(z) = e^z = 1 + z + \frac{z^2}{2!} + - - - + \frac{z^n}{n!} + - - -$$

$$a_n = \frac{1}{n!}$$

$$a_{n+1} = \frac{1}{(n+1)!}$$

Radius of convergence $= \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$$= \lim_{n\to\infty} \left|\frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}\right|$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\,1}{(n+1)\cancel{n!}} = \lim_{n\to\infty} (n+1)$$

$$= \infty$$

Region of convergence of the series

$$|z| < \infty$$

Ex:-

Expand $f(z) = \sin z$ in the form of series. Find region of convergence also.

Sol:-

Step I:-

$f(z) = \sin z$ $\qquad$ $f(0) = \sin(0) = 0$

$f'(z) = \cos z$ $\qquad$ $f'(0) = \cos(0) = 1$

$f''(z) = -\sin z$ $\qquad$ $f''(0) = -\sin(0) = 0$

$f'''(z) = -\cos z$ $\qquad$ $f'''(0) = -\cos(0) = -1$

$f^{(4)}(z) = \sin z$ $\qquad$ $f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(z) = \cos z$ $\qquad$ $f^{(5)}(0) = \cos(0) = 1$

By using Maclaurin's series, we have

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \frac{z^3}{3!} f'''(0) + \frac{z^4}{4!} f^{(4)}(0)$$

$$+ \frac{z^5}{5!} f^{(5)}(0) + \cdots + \frac{z^n}{n!} f^{(n)}(0) + \cdots$$

$$\sin z = 0 + z(1) + \frac{z^2}{2!}(0) + \frac{z^3}{3!}(-1) + \frac{z^4}{4!}(0) +$$

$$\frac{z^5}{5!}(1) + \cdots + \frac{z^n}{n!} f^{(n)}(0) + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!}(-1)^n + \cdots$$

Now, we find radius of convergence

$$a_n = \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$a_{n+1} = \frac{(-1)^{n+1} z^{2n+3}}{(2n+3)!}$$

Radius of convergence $= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(-1)^n z^{2n+1}}{(2n+1)!} \times \frac{(2n+3)!}{(-1)^{n+1} z^{2n+3}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{z^{2n+1} (2n+3)(2n+2)(2n+1)!}{(2n+1)! \; z^{2n+2} \cdot z} \right|$$

$$= \infty$$

Region of convergence if this series is

$$|z| < \infty$$

Ex:-

Expand $f(z) = \cos z$ in form of

Sol:-

$f(z) = \cos z$ $\qquad f(0) = \cos(0) = 1$

$f'(z) = -\sin z$ $\qquad f'(0) = -\sin(0) = 0$

$f''(z) = -\cos z$ $\qquad f''(0) = -\cos(0) = -1$

$f'''(z) = \sin z$ $\qquad f'''(0) = \sin(0) = 0$

$f^{(4)}(z) = \cos z$ $\qquad f^{(4)}(0) = \cos(0) = 1$

$f^{(5)}(z) = -\sin z$ $\qquad f^{(5)}(0) = -\sin(0) = 0$

$f^{(6)}(z) = -\cos z$ $\qquad f^{(6)}(0) = -\cos(0) = -1$

By using Maclaurin's series

$$f(z) = f(0) + z f'(0) + \frac{z^2 f''(0)}{2!} + \frac{z^3 f'''(0)}{3!} + \frac{z^4 f^{(4)}(0)}{4!} + \frac{z^5 f^{(5)}(0)}{5!} + \frac{z^6 f^{(6)}(0)}{6!} + \cdots + \frac{z^n f^{(n)}(0)}{n!} + \cdots$$

$$\cos z = 1 + z(0) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(0) + \frac{z^4}{4!}(1) + \frac{z^5}{5!}(0) + \frac{z^6}{6!}(-1) + \cdots + \frac{z^n f^{(n)}(0)}{n!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + \frac{(-1)^n z^{2n}}{2n!} + \cdots$$

$$a_n = \frac{(-1)^n z^{2n}}{(2n)!}$$

$a_{n+1} = \dfrac{(-1)^{n+1} z^{2n+2}}{(2n+2)!}$ Radius of convergence is

$$\lim_{n\to\infty} \left| \frac{(-1)^n z^{2n}}{(2n)!} \times \frac{(2n+2)!}{(-1)^{n+1} z^{2n+2}} \right| = \infty$$

Region of convergence of this series is $|z| < \infty$

Ex:-

Expand $f(z) = \sin z$ at $z = \pi/4$

Sol:-

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n f^{(n)}(a)}{n!} + \cdots$$

Step I:-

$f(z) = \sin z \qquad a = \pi/4$

$f(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4})$

$= \dfrac{1}{\sqrt{2}}$

$f'(z) = \cos z \qquad f'(\tfrac{\pi}{4}) = \cos(\tfrac{\pi}{4}) = \dfrac{1}{\sqrt{2}}$

$f''(z) = -\sin z \qquad f''(\tfrac{\pi}{4}) = -\sin(\tfrac{\pi}{4}) = -\dfrac{1}{\sqrt{2}}$

$f'''(z) = -\cos z \qquad f'''(\tfrac{\pi}{4}) = -\cos\tfrac{\pi}{4} = -\dfrac{1}{\sqrt{2}}$

$f^{(4)}(z) = \sin z \qquad f^{(4)}(\tfrac{\pi}{4}) = \sin(\tfrac{\pi}{4}) = \dfrac{1}{\sqrt{2}}$

Step II:-

Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-\pi/4)^n}{n!} f^{(n)}(\tfrac{\pi}{4})$$

$$f(z) = f(\tfrac{\pi}{4}) + (z-\tfrac{\pi}{4})f'(\tfrac{\pi}{4}) + (z-\tfrac{\pi}{4})^2/2!\, f''(\tfrac{\pi}{4})$$

$$+ \frac{(z-\pi/4)^3}{3!} f'''(\tfrac{\pi}{4}) + \frac{(z-\pi/4)^4}{4!} f^{(4)}(\tfrac{\pi}{4}) + \cdots$$

$$f(z) = \sin z = \frac{1}{\sqrt{2}} + (z-\tfrac{\pi}{4})\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^2}{2!}\frac{1}{\sqrt{2}} -$$

$$\frac{(z-\pi/4)^3}{3!}\frac{1}{\sqrt{2}} + \cdots$$

Ex:-

Expand $f(z) = \cos z$ at $z = \pi/4$

Sol:-

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^{(n)}(a) + \cdots$$

Step I:-

$$f(z) = \cos z \qquad , \qquad a = \pi/4$$

$$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f'(z) = -\sin z$$

$$f'\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4}$$

$$f'\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f''(z) = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = +\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$f^{(4)}(z) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Step II:-

Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}\left(\frac{\pi}{4}\right)$$

$$f(z) = f\left(\frac{\pi}{4}\right) + (z-a) f'\left(\frac{\pi}{4}\right) + \frac{(z-a)^2}{2!} f^{(2)}\left(\frac{\pi}{4}\right) + \frac{(z-a)^3}{3!} f^{(3)}\left(\frac{\pi}{4}\right) + \frac{(z-a)^4}{4!} f^{(4)}\left(\frac{\pi}{4}\right) + \cdots$$

$$f(z) = \cos z = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} - \frac{(z-\pi/4)^2}{2!}\frac{1}{\sqrt{2}} + \frac{(z-\pi/4)^3}{3!}\frac{1}{\sqrt{2}} + \frac{(z-\pi/4)^4}{4!}\frac{1}{\sqrt{2}} + \cdots$$

Ex:-

Expand $f(z) = \log(1+z)$ in form of series. Find region of convergence also or $f(z) = \log(1-z)$.

Soln:-

$$f(z) = \log(1+z) \Rightarrow f(0) = \log(1) = 0$$

$$f'(z) = \frac{1}{1+z} = (1+z)^{-1} \Rightarrow f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = -1(1+z)^{-2} \Rightarrow f''(0) = \frac{-1}{(1+0)^2} = -1, \quad f''(z) = \frac{(-1)^{n-1}}{(1+z)^2}$$

$$f'''(z) = (-1)(-2)(1+z)^{-3} \Rightarrow f'''(0) = \frac{2}{(1+0)^3} = 2, \quad f'''(z) = \frac{(-1)^{n-1}\, 2!}{(1+z)^3}$$

$$f^{(iv)}(z) = 2\cdot(-3)(1+z)^{-4}$$

$$= \frac{-3\cdot 2}{(1+z)^4} = \frac{(-1)^{n-1}\, 3!}{(1+z)^4} \Rightarrow f^{(4)}(0) = -3!$$

$$f^{(n)}(z) = \frac{(-1)^{n-1}(n-1)!}{(1+z)^n} \Rightarrow f^{(n)}(0) = (-1)^n (n-1)!$$

using Maclaurin's series.

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2} f''(0) + \frac{z^3}{3!} f'''(0) + \cdots + \frac{z^n}{n!} f^{(n)}(0) + \cdots$$

$$\log(1+z) = 0 + z(1) + \frac{z^2}{2!}(-1) + \frac{z^3}{3!}(2) + \cdots + \frac{z^n}{n!}(-1)^{n-1}(n-1)!$$

Radius of convergence:-

$$a_n = \frac{(-1)^{n-1}(n-1)!}{n!}$$

$$\Rightarrow a_{n+1} = \frac{(-1)^n\, n!}{(n+1)!}$$

$$\text{Radius of convergence} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n-1}(n-1)!\,(n+1)!}{n!\,(-1)^{n}\, n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n-1}\cancel{(n-1)!}}{\cancel{n!}} \times \frac{(n+1)\, n!}{(-1)^n\, n\, \cancel{(n-1)!}} \right|$$

Radius of Convergence $= \lim_{n\to\infty} \left| -\frac{n+1}{n} \right| = \lim_{n\to\infty} \left[\frac{n(1+\frac{1}{n})}{n} \right] = \left[1 + \frac{1}{\infty}\right] = 1+0 = 1$

is $|z| < 1$

Ex:-

Expand in the power series.

$$f(z) = \frac{1}{z-2}$$

Sol:-

$$f(z) = \frac{1}{z-2}$$

$$f(z) = \frac{1}{z\left[1-\frac{2}{z}\right]}$$

$$f(z) = \frac{1}{z}\left\{\left(1-\frac{2}{z}\right)^{-1}\right\}$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \cdots\right]$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{2^2}{z^3} + \frac{2^3}{z^4} + \cdots$$

Ex:-

Expand $\log(1+z^2)$ as a Taylor's series at $z=1$ and prove that it can be written as $f(z) = a_0 + \sum_{n=1}^{\infty} a_n (z-1)^n$

Also find a_0 and a_n

Sol:-

$$f(z) = \log(1+z^2)$$

$$= \log((z+i)(z-i))$$

$$f(z) = \log(z+i) + \log(z-i)$$

$$f'(z) = \frac{1}{z+i} + \frac{1}{z-i} = (z+i)^{-1} + (z-i)^{-1}$$

$$f''(z) = -1(z+i)^{-2} - (z-i)^{-2}$$

$$f'''(z) = (-1)(-2)(z+i)^{-3} - 1 \times -2(z-i)^{-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f^{(n)}(z) = (-1)(-2)\cdots(-(n-1))(z+i)^{-n} + (-1)(-2)\cdots(-(n-1))(z-i)^{-n}$$

$$f^{(n)}(z) = (-1)^{n-1}(n-1)!\left[\frac{1}{(z+i)^n} + \frac{1}{(z-i)^n}\right]$$

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!\left[\frac{1}{(1+i)^n} + \frac{1}{(1-i)^n}\right] \rightarrow (1)$$

Let $a = 1+i$

$1+i = r(\cos\theta + i\sin\theta)$

$x = r\cos\theta$, $y = r\sin\theta$; $\vec{r} = x+iy$

$r = \sqrt{1+1} = \sqrt{2}$ $\qquad r = 1+iy$

$x = r\cos\theta \Rightarrow 1 = \sqrt{2}\cos\theta$ $\qquad \Rightarrow x=1, y=1$

$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$y = r\sin\theta \Rightarrow 1 = \sqrt{2}\sin\theta$

$\sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(1+i)^n = \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^n$

$(1+i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

Similarly,

$(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$

putting all value's in (1)

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!\left[\frac{1}{2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)} + \frac{1}{2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)}\right]$$

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!\left[\frac{\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4} + \cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}}{2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)}\right]$$

$\because \cos^2\frac{n\pi}{4} + \sin^2\frac{n\pi}{4} = 1$

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!\; 2^{-n/2}\left(2\cos\left(\frac{n\pi}{4}\right)\right)/1$$

$$= (-1)^{n-1}(n-1)!\; 2^{-\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right)$$

$$f^{(n)}(1) = (-1)^{n-1}(n-1)!\; 2^{1-\frac{n}{2}}\cos\left(\frac{n\pi}{4}\right)$$

$$f(z) = \log(1+z^2)$$
$$f(1) = \log(1+1)$$
$$f(1) = \log(2)$$

Taylor's series.

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \frac{(z-a)^3}{3!}f'''(a) + \cdots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \cdots$$

$$= f(a) + \sum_{n=1}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$$

$$= f(1) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} f^{(n)}(1)$$

$$f(z) = \log(2) + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!} \left((-1)^{n-1}(n-1)!\, 2^{-\frac{n}{2}+1}\cos\left(\frac{n\pi}{4}\right)\right)$$

$$a_0 = \log(2) \quad , \quad a_n = \frac{(-1)^{n-1}\, 2^{1-\frac{n}{2}} \cos\left(\frac{n\pi}{4}\right)}{n} \quad \because \frac{(n-1)!}{n(n-1)!} = \frac{1}{n}$$

Laurants Theorem:-

Statement:-

Let C and C' be two closed contour which are concentric circles centre at $z = a$ with radii R and r. C' is the inner circle and C is the outer circle. Let $f(z)$ be a complex valued function analytic on C and C' and in the annular region. Let $z = a+h$ be a point in the annular region then prove that

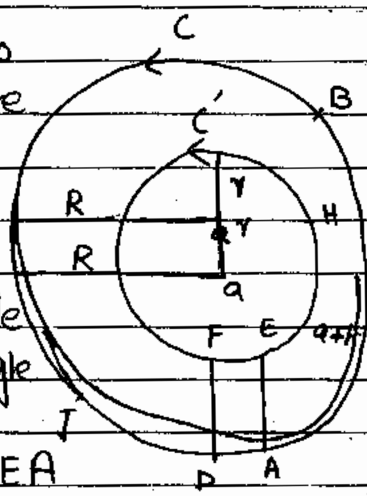
$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n + \cdots + \frac{b_1}{h} + \frac{b_2}{h^2} + \cdots + \frac{b_n}{h^n} + \cdots$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \; ; n = 0, 1, 2, \cdots$

and $b_n = \frac{1}{2\pi i}\int_{c} \frac{f(z)}{(z-a)^{1-n}}\, dz$

$= \frac{1}{2\pi i}\int_{c} f(z)(z-a)^{n-1}\, dz \quad ; n = 1, 2, \ldots$

Proof:-



Given C and C' are two concentric circles with centre at $z = a$ radius R and r respectively.

It is doubly connected domain which can be made simply connected by a single cross cut. So, we have the closed contour ABJDFHEA

Since, $f(z)$ is analytic on C and C' and in the annular region $z = a+h$ is the interior point. Then, by cauchy's integral formula

$$f(a+h) = \frac{1}{2\pi i}\int_{ABJDFHEA} \frac{f(z)}{z-a-h}\, dz \rightarrow (1)$$

$$= \frac{1}{2\pi i}\left[\int_{ABJD} \frac{f(z)}{z-a-h}\, dz + \int_{DF} \frac{f(z)}{z-a-h}\, dz\right.$$

$$\left. + \int_{FHE} \frac{f(z)}{z-a-h}\, dz + \int_{EA} \frac{f(z)}{z-a-h}\, dz\right]$$

$\because$ DF is inverse of EA

$$= \frac{1}{2\pi i}\int_{ABJD} \frac{f(z)}{z-a-h}\, dz + \frac{1}{2\pi i}\int_{FHE} \frac{f(z)}{z-a-h}\, dz$$

$$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a-h} - \frac{1}{2\pi i}\int_{C'} \frac{f(z)\,dz}{z-a-h}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-(a+h)} + \frac{1}{2\pi i}\int_{C'} \frac{f(z)\,dz}{h-(z-a)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a} + \frac{1}{2\pi i}\int_C \frac{h f(z)\,dz}{(z-a)^2} + \cdots$$

$$+ \frac{1}{2\pi i}\int_C \frac{h^n f(z)\,dz}{(z-a)^{n+1}} + \frac{1}{2\pi i}\int_C \frac{h^{n+1} f(z)\,dz}{(z-a)^{n+1}(z-a-h)}$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(z)\,dz}{h} + \frac{1}{2\pi i}\int_{C'} \frac{z-a}{h^2} f(z)\,dz + \cdots$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{(z-a)^{n-1}}{h^n} f(z)\,dz + \frac{1}{2\pi i}\int_{C'} \frac{(z-a)^n f(z)\,dz}{h^{n+1}}$$

Take $R_n = \frac{1}{2\pi i}\int_C \frac{h^{n+1} f(z)\,dz}{(z-a)^{n+1}(z-a-h)}$

and $R'_n = \frac{1}{2\pi i}\int_{C'} \frac{(z-a)^n f(z)\,dz}{h^{n+1}(h-z+a)}$

Now, we show that

$R_n \to 0$ as $n \to \infty$

$R'_n \to 0$ as $n \to \infty$

$$|R_n| = \left| \frac{1}{2\pi i} \int_C \frac{h^{n+1}}{(z-a)^{n+1}(z-a-h)} f(z)\,dz \right|$$

$$\therefore \left| \frac{f(z)}{z-a-h} \right| \leq M \quad , \quad |z-a| = R$$

$$\leq \frac{1}{2\pi} \frac{h^{n+1}}{\cancel{(z-a)}^{n+1}} 2\pi R \quad M^{z-a}$$

R

$$|R_n| = M h \left(\frac{h}{R}\right)^n \qquad \because h < R$$

$$\therefore \frac{h}{R} < 1$$

and $\int_C |dz| = 2\pi R$

$$\left(\frac{h}{R}\right)^n \to 0 \quad \text{as } n \to \infty$$

So, $R_n \to 0$ as $n \to \infty$

$$|R'_n| = \left| \frac{1}{2\pi i} \int_{C'} \frac{(z-a)^n}{h^{n+1}(h-z+a)} f(z)\,dz \right.$$

$$\because z - a = r$$

$$\leq \frac{1}{2\pi} M \frac{r^n}{h^{n+1}} \times 2\pi r \qquad \int_{C'} \left|\frac{f(z)}{h-z+a}\right| \leq M$$

$$|R'_n| = M \left(\frac{r}{h}\right)^{n+1} \qquad \int_{C'} |dz| = 2\pi r$$

Since $r < h$

$$\frac{r}{h} < 1$$

$$\left(\frac{r}{h}\right)^n \to 0 \quad \text{as } n \to \infty$$

Therefore, $R'_n \to 0$ as $n \to \infty$

Therefore,

$$f(a+h) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz + \frac{h}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz$$

$$+ \cdots + \frac{h^n}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz + \cdots + \frac{1}{2\pi i} \times \frac{1}{h}$$

$$\int_{C'} f(z)\,dz + \frac{1}{h^2 \times 2\pi i} \int_{C'} (z-a) f(z)\,dz + \cdots +$$

$$\frac{1}{2\pi i h^n}\int_{c'} (z-a)^{n-1} f(z)\,dz + \cdots$$

$$f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n + \cdots + \frac{b_1}{h} + \frac{b_2}{h^2} + \cdots + \frac{b_n}{h^n} + \cdots \rightarrow (A)$$

where

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a-h)^{n+1}}\,dz \quad ; \; n = 0, 1, 2, \ldots \infty$$

$$b_n = \frac{1}{2\pi i}\int_c (z-a)^{n-1} f(z)\,dz. \quad n = 1, 2, 3, \ldots$$

Remark :-

(i) In Eq (A), $a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n$ is analytic part and $\frac{b'}{h} + \frac{b^2}{h^2} + \cdots + \frac{b^n}{h^n} + \cdots$ is principal part.

(ii) If function is analytic, then its principal part is zero.

(iii) Each Laurent series is Taylor series if its principal part is zero but each taylor series is not Laurent series.

(iv) If function is in the denominator form then series is laurent series, (if) (not) or otherwise the series is Taylor series.

(i-e) $f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n + \cdots + \frac{b_1}{h} + \frac{b_2}{h^2} + \cdots + \frac{b_n}{h^n} + \cdots$ is Laurent series

and $f(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n + \cdots$ is Taylor's seri

Ex:-

Prove that $f(z) = \cos\left(z + \frac{1}{z}\right)$ can be expended as a Laurent's series

$$a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right) \text{ where } a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(2\cos\theta)\cos n\theta \, d\theta$$

Sol:-

Step I:-

Given function $f(z) = \cos\left(z + \frac{1}{z}\right)$ which remains same if we replace z by $\frac{1}{z}$ therefore $b_n = a_n$. Also $f(z)$ ceases (-ve) to be analytic function at $z = 0$ (i-e). we shall expand it for $|z| > 0$.

So $f(z)$ can be written as

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

$\because |z| > 0$ is equation of circle with centre at "0" and radius > 0

$$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \frac{1}{z^n}$$

$$a_0 + \sum_{n=1}^{\infty}\left(a_n z^n + \frac{b_n}{z^n}\right)$$

As $a_n = b_n$

$$a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

Step II:-

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$$a = 0$$

$$f(z) = \cos\left(z + \frac{1}{z}\right)$$

$$z = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$$

$$dz = ie^{i\theta}\, d\theta$$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\cos\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right)}{(e^{i\theta})^{n+1}}\, ie^{i\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos\left(e^{i\theta} + e^{-i\theta}\right)}{(e^{i\theta})^n}\, d\theta$$

$$\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\cos\theta)\, e^{-in\theta}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\cos\theta)(\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(2\cos\theta)\cos n\theta\, d\theta - \frac{i}{2\pi}\int_0^{2\pi} \cos($$

$2\cos\theta)\sin n\theta\, d\theta$

Consider the integral

$$I = \int_0^{2\pi} \cos(2\cos\theta)\sin n\theta\, d\theta$$

put $\theta = 2\pi - \phi$

$d\theta = -d\phi$

$I =$ $\quad \phi = 2\pi - \theta$

at $\theta = 0$, $\phi = 2\pi - 0 = 2\pi$

at $\theta = 2\pi$, $\phi = 2\pi - 2\pi = 0$

$$I = \int_{2\pi}^{0} \cos(2\cos(2\pi-\phi))\sin n(2\pi-\phi)(-d\phi)$$

$$I = \int_0^{2\pi} \cos(2\cos(\phi))(-\sin n\phi)\, d\phi$$

$$I = -\int_0^{2\pi} \cos(2\cos\phi)\sin n\phi\, d\phi$$

$\Rightarrow I = -I$

$I + I = 0$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

$$\Rightarrow \int_0^{2\pi} \cos(2\cos\theta)\sin n\theta\, d\theta = 0$$

Therefore, $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \cos(2\cos\theta)\cos n\theta\, d\theta$.

Ex:-

$\sinh(z + \frac{1}{z})$ in the form of Laurent's series

$f(z) = a_0 + \sum_{n=1}^{\infty} a_n(z^n + \frac{1}{z^n})$ where

$a_n = b_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sinh(2\cos\theta)\cos n\theta\, d\theta$

Sol:-

Step I:

Given function $f(z) = \sinh(z + \frac{1}{z})$ which remains same if we replace z by $\frac{1}{z}$, therefore $b_n = a_n$. Also $f(z)$ ceases

to be analytic function at $z=0$. i-e, we shall expand it for $|z|>0$.
So, $f(z)$ can be written as

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

Step II :-

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$$

As $|z|>0$ then $a=0$

$$f(z) = \sinh\left(z+\frac{1}{z}\right)$$

$$z = e^{i\theta} \; ; \; 0 \le \theta \le 2\pi$$

$$dz = ie^{i\theta}d\theta$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{\sinh(z+\frac{1}{z})\,dz}{(z-0)^{n+1}}$$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sinh(e^{i\theta}+\frac{1}{e^{i\theta}})\,ie^{i\theta}d\theta}{(e^{i\theta})^{n+1}}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sinh(e^{i\theta}+e^{-i\theta})}{e^{in\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sinh(e^{i\theta}+e^{-i\theta})\,e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sinh(2\cos\theta)(\cos n\theta - i\sin n\theta)\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sinh(2\cos\theta)\cos n\theta\,d\theta - \frac{i}{2\pi}\int_0^{2\pi} \sinh(2\cos\theta)\sin n\theta\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sinh(2\cos\theta)\cos n\theta\,d\theta - \frac{i}{2\pi}\int_0^{2\pi} \sinh(2\cos\theta)\sin n\theta\,d\theta$$

Consider, the integral

$$I = \int_0^{2\pi} \sinh(2\cos\theta)\sin n\theta\,d\theta$$

put $\theta = 2\pi - \phi$, $d\theta = -d\phi$

$$\phi = 2\pi - \theta$$

at $\theta = 0$, $\phi = 2\pi$

at $\theta = 2\pi$, $\phi = 0$

$$I = \int_{2\pi}^{0} \text{Sinh}(2\cos(2\pi-\phi))\, \text{Sin}\, n(2\pi-\phi)(-d\phi)$$

Ex:- $I = \int_{2\pi}^{0} \text{Sinh}(2\cos\phi)(-\text{Sin}\, n\phi)(-d\phi)$

(i) Expand $\int_{2\pi}^{0} \text{Sinh}(z + \frac{1}{z})$ in form of laurent's series

(ii) Expand $\int_{0}^{2\pi} e^{2}\text{Sinh}(2\cos\phi)\, \text{Sin}\, n\phi\, d\phi$ in laurent's series

$$I = -I$$

$$I + I = 0$$

$$2I = 0$$

$$\Rightarrow I = 0$$

$$\Rightarrow \int_{0}^{2\pi} \text{Sinh}(2\cos\phi)\, \text{Sin}\, n\phi\, d\phi = 0$$

Therefore, $a_n = \frac{1}{2\pi}\int_{0}^{2\pi} \text{Sinh}(2\cos\theta)\cos n\theta\, d\theta$

Ex:-

Expand $\sin(z + \frac{1}{z})$ in laurent's series.

Sol:-

Step I:-

Given function $f(z) = \sin(z + \frac{1}{z})$ which remains same if we replace z by $\frac{1}{z}$, therefore $b_n = a_n$. Also $f(z)$ ceases to be analytic function at $z = 0$ i-e, we shall expand it for $|z| > 0$

So, $f(z)$ can be written as

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$$

Step II:-

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

As $|z - 0| > 0$, $\Rightarrow a = 0$

$$f(z) = \sin(z + \frac{1}{z})$$

$$z = e^{i\theta} \quad ; \quad 0 \le \theta \le 2\pi$$

$$dz = ie^{i\theta} d\theta$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{\sin\left(z + \frac{1}{z}\right)}{(z-0)^{n+1}} dz$$

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\sin\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right)}{(e^{i\theta})^{n+1}} ie^{i\theta} d\theta$$

$$a_n = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\sin(e^{i\theta} + e^{-i\theta})}{(e^{i\theta})^n} d\theta$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \sin(e^{i\theta} + e^{-i\theta}) e^{-in\theta} d\theta$$

$$\because e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \sin(2\cos\theta)(\cos n\theta - i \sin n\theta) d\theta$$

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \sin(2\cos\theta)\cos n\theta \, d\theta - \frac{i}{2\pi} \int_0^{2\pi} \sin(2\cos\theta) \sin n\theta \, d\theta$$

Consider the integral

$$I = \int_0^{2\pi} \sin(2\cos\theta) \sin n\theta \, d\theta$$

put $\theta = 2\pi - \phi$

$$d\theta = -d\phi$$

$$\phi = 2\pi - \theta$$

at $\theta = 0 \Rightarrow \phi = 2\pi$

at $\theta = 2\pi \Rightarrow \phi = 0$

$$I = \int_{2\pi}^{0} \sin(2\cos(2\pi - \phi)) \sin n(2\pi - \phi)(-d\phi)$$

$$I = \int_{2\pi}^{0} \sin(2\cos\phi)(-\sin n\phi)(-d\phi)$$

$$I = -\int_0^{2\pi} \sin(2\cos\phi) \sin n\phi \, d\phi$$

$$I = -I$$

$$I + I = 0$$

$$2I = 0$$

$$I = 0$$

$$\Rightarrow \int_0^{2\pi} \sin(2\cos\theta)\sin n\theta \, d\theta = 0$$

So

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin(2\cos\theta)\cos n\theta \, d\theta$$

Ex:-

Expand $e^{a(z+\frac{1}{z})}$ or $e^{c(z+\frac{1}{z})}$ in Laurent's series.

Sol:-

Step I:-

Given function $f(z) = e^{c(z+\frac{1}{z})}$ is not analytic at $z=0$, So we can expand the function $f(z)$ as a Laurent's series for all $|z| > 0$, the series is

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots + \frac{b_n}{z^n} + \cdots$$

and $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}} dz \quad ; a = 0$

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-0)^{n+1}} dz$$

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{z^{n+1}} dz$$

Let C be the unit circle i-e $z = e^{i\theta}$

and $f(z) = e^{c(z+\frac{1}{z})} \quad ; \quad 0 \le \theta \le 2\pi.$

$$a_n = \frac{1}{2\pi i}\int_c \frac{e^{c(z+\frac{1}{z})}}{z^{n+1}} dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{c(e^{i\theta} + \frac{1}{e^{i\theta}})}}{(e^{i\theta})^{n+1}} \, i e^{i\theta} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{c(e^{i\theta}+e^{-i\theta})}}{e^{in\theta}} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{c(2\cos\theta)} e^{-n\theta i} d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2c\cos\theta} e^{-n\theta i} d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2c\cos\theta}[\cos n\theta - i\sin n\theta]\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2c\cos\theta}\cos n\theta\,d\theta - \frac{i}{2\pi}\int_0^{2\pi} e^{2c\cos\theta}\sin n\theta\,d\theta$$

Step II :-

Consider the integral

$$I_1 = \int_0^{2\pi} e^{2c\cos\theta}\sin n\theta\,d\theta$$

put $\theta = 2\pi - \phi$, $d\theta = -d\phi$

$\phi = 2\pi - \theta$

at $\theta = 0 \Rightarrow \phi = 2\pi$

at $\theta = 2\pi \Rightarrow \phi = 0$

$$I_1 = \int_{2\pi}^{0} e^{2c\cos(2\pi-\phi)}\sin n(2\pi-\phi)(-d\phi)$$

$$I_1 = -\int_0^{2\pi} e^{2c\cos\phi}(-\sin n\phi)(-d\phi)$$

$$I_1 = -\int_0^{2\pi} e^{2c\cos\phi}\sin n\phi\,d\phi$$

$I_1 = -I_1$

$2I_1 = 0$

$\Rightarrow I_1 = 0$

$$\int_0^{2\pi} e^{2c\cos\phi}\sin n\phi\,d\phi = 0$$

So, $$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2c\cos\theta}\cos n\theta\,d\theta$$

Remarks:-

(i) Laurent's series is used for trignometric functions ($\sin z$, $\cos z$), when its principal part is zero

(ii) Laurent's series is used for expanding the exponential functions.

(i-e) $e^{z+\frac{1}{z}}$, $\sin(z+\frac{1}{2})$

Ex :-

Expand the following functions in the Laurent's series or in form of Laurent's series

(i) $e^{uz+\frac{v}{z}}$

(ii) $e^{a(z+\frac{1}{z})}$

(iii) $f(z) = \dfrac{2z-1}{(z+1)(z-4)}$

Sol :-

(i) $f(z) = e^{uz+\frac{v}{z}}$

Step I :-

Given function $f(z) = e^{uz+\frac{v}{z}}$ remains unaltered (unchange) if we replace z by $\frac{1}{z}$ and u by v, therefore, b_n can be obtained by putting u by v in the expansion of a_n. Also $f(z)$ is not analytic at $z=0$. So, we shall expand it for $|z|>0$ i-e $a=0$

So, $f(z)$ can be written as

$$f(z) = a_0 + \sum_{n=1}^{\infty}\left(a_n z^n + \frac{b_n}{z^n}\right)$$

where

a_n and b_n are to be determind.

Step II :-

Let us determine a_n

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad \text{As } a=0$$

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z^{n+1}}$$

Let C be a unit circle and $f(z) = e^{uz+\frac{v}{z}}$

i-e $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$; $0 \le \theta \le 2\pi$

$$e^{uz+\frac{v}{z}} = e^{uz+vz^{-1}}$$

$$= e^{ue^{i\theta}+ve^{-i\theta}}$$

$$= e^{u(\cos\theta+i\sin\theta)+v(\cos\theta-i\sin\theta)}$$

$$e^{uz+\frac{v}{z}} = e^{u\cos\theta + iu\sin\theta + v\cos\theta - iv\sin\theta}$$

$$= e^{(u+v)\cos\theta + i(u-v)\sin\theta}$$

$$= e^{(u+v)\cos\theta} \cdot e^{i(u-v)\sin\theta}$$

$$e^{uz+\frac{v}{z}} = e^{(u+v)\cos\theta}\left(\cos((u-v)\sin\theta) + i\sin((u-v)\sin\theta)\right)$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz \qquad a=0$$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{(u+v)\cos\theta}\left(\cos((u-v)\sin\theta) + i\sin((u-v)\sin\theta)\right)}{(e^{i\theta})^{n+1}}\, i e^{i\theta}\,d\theta$$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{(u+v)\cos\theta}\, e^{i(u-v)\sin\theta}\, e^{-n\theta i}\, i\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\left[\cos((u-v)\sin\theta) + i\sin((u-v)\sin\theta)\right)(\cos n\theta - i\sin n\theta)]\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\{[\cos[[(u-v)\sin\theta] - n\theta] + i\sin[[(u-v)\sin\theta] - n\theta]]\,d\theta\} \rightarrow (1)$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\cos[(u-v)\sin\theta - n\theta]\,d\theta + i\int_0^{2\pi} e^{(u+v)\cos\theta}\sin[(u-v)\sin\theta - n\theta]\,d\theta \rightarrow (2)$$

Consider the integral $I_1 = \int_0^{2\pi} e^{(u+v)\cos\theta}\sin[(u-v)\sin\theta - n\theta]\,d\theta \rightarrow (3)$

put $\theta = 2\pi - \phi$, $d\theta = -d\phi$

$\phi = 2\pi - \theta$

at $\theta = 0 \Rightarrow \phi = 2\pi$.

at $\theta = 2\pi \Rightarrow \phi = 0$ Eq (3) becomes

$$I_1 = \int_{2\pi}^{0} e^{(u+v)\cos(2\pi-\phi)}\sin[(u-v)\sin(2\pi-\phi) - n(2\pi-\phi)(-d\phi)$$

$$= -\int_0^{2\pi} e^{(u+v)\cos\phi}\sin[-(u-v)\sin\phi - 2n\pi + n\phi](-d\phi)$$

$$= -\int_0^{2\pi} e^{(u+v)\cos\phi}(-\sin[2n\pi + \{(u-v)\sin\phi - n\phi\}\,d\phi$$

$$= -\int_0^{2\pi} e^{(u+v)\cos\phi}\sin[(u-v)\sin\phi - n\phi]\,d\phi$$

$$I_1 = -I_1$$

$$I_1 + I_1 = 0$$
$$2I_1 = 0$$
$$\Rightarrow I_1 = 0$$
$$\Rightarrow I_1 = \int_0^{2\pi} e^{(u+v)\cos\phi} \sin[(u-v)\sin\phi - n\phi)]\,d\phi$$

Eq (3) becomes

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\phi} \cos[(u-v)\sin\phi - n\phi]\,d\phi \quad \text{and}$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\phi} \cos[(u-v)\sin\phi - n\phi]\,d\phi$$

(ii)

$$f(z) = e^{z+\frac{1}{z}}$$

Step I :-

Given function $f(z) = e^{z+\frac{1}{z}}$ remains same, $f(z)$ is not analytic at $z=0$, So we can expand the function $f(z)$ as a laurent's series for all $|z|>0$, the series is

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots + \frac{b_n}{z^n} + \cdots$$

and $a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz \quad ; a=0$

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-0)^{n+1}}\,dz$$

$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{z^{n+1}}\,dz$$

Let C be unit circle i-e $z = e^{i\theta}$

$$f(z) = e^{z+\frac{1}{z}} \qquad ; 0 \le \theta \le 2\pi$$

$$a_n = \frac{1}{2\pi i}\int_c \frac{e^{z+\frac{1}{z}}}{(e^{i\theta})^{n+1}}\, i e^{i\theta}\,d\theta$$

$$a_n = \frac{1}{2\pi}\int \frac{e^{e^{i\theta}+e^{-i\theta}}}{e^{ni\theta} e^{i\theta}}\, e^{i\theta}\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{(2\cos\theta)} \cdot e^{-n\theta i}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta - in\theta} (\cos n\theta - i\sin n\theta) d\theta$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} e^{2\cos\theta}\cos n\theta\, d\theta - i\int_0^{2\pi} e^{2\cos\theta}\sin n\theta\, d\theta\right]$$

$$a_n = \frac{1}{2\pi}\left[\int_0^{2\pi} e^{2\cos\theta}\cos n\theta\, d\theta\right] - \frac{i}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\sin n\theta\, d\theta \rightarrow (1)$$

Step II :-

Consider the integral

$$I = \int_0^{2\pi} e^{2\cos\theta}\sin n\theta\, d\theta$$

put $\theta = 2\pi - \phi$, $d\theta = -d\phi$

$\phi = 2\pi - \theta$ at $\theta = 0 \Rightarrow \phi = 2\pi$

at $\theta = 2\pi$ $\phi = 0$

$$I = \int_{2\pi}^{0} e^{2\cos(2\pi-\phi)} \sin n(2\pi - \phi)(-d\phi)$$

$$= \int_0^{2\pi} e^{2\cos(\phi)} (\sin n(2\pi - \phi))\, d\phi$$

$$I = -\int_0^{2\pi} e^{2\cos\phi}\sin n\phi\, d\phi$$

$I = -I$

$\Rightarrow I + I = 0$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

$$\Rightarrow \int_0^{2\pi} e^{2\cos\theta}\sin n\theta\, d\theta = 0$$

Eq (1) becomes.

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta\, d\theta$$

(iii) $f(z) = \dfrac{2z-1}{(z+1)(z-4)}$

By partial fraction

$$\frac{2z-1}{(z+1)(z-4)} = \frac{A}{z+1} + \frac{B}{z-4} \rightarrow (a)$$

Multiplying both sides by $(z+1)(z-4)$

$$2z-1 = A(z-4) + B(z+1) \rightarrow (1)$$

put $z+1=0$ or $z=-1$ in (1)

$$2(-1)-1 = A(-1-4) + B(0)$$

$$-2-1 = A(-5)$$

$$-3 = A(-5)$$

$$A = \frac{3}{5}$$

put $z-4=0$ or $z=4$ in (1)

$$2(4)-1 = A(0) + B(4+1)$$

$$8-1 = 5B$$

$$7 = 5B$$

$$B = 7/5$$

put A, B values in (a)

$$\frac{2z-1}{(z+1)(z-4)} = \frac{3}{5}\left(\frac{1}{z+1}\right) + \frac{7}{5}\left(\frac{1}{z-4}\right)$$

$$f(z) = \frac{3}{5}\left(\frac{1}{z+1}\right) + \frac{7}{5}\left(\frac{1}{z-4}\right)$$

$$= \frac{3}{5}\,\frac{1}{z\left(1+\frac{1}{z}\right)} + \frac{7}{5}\,\frac{1}{-4\left(1-\frac{z}{4}\right)}$$

$$f(z) = \frac{3}{5} z^{-1}\left(1+\frac{1}{z}\right)^{-1} + \frac{7}{5}(-4)^{-1}\left(1-\frac{z}{4}\right)^{-1}$$

$$f(z) = \frac{3}{5}\,\frac{1}{z}\left(1-\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{7}{20}\left(1+\frac{z}{4}+\frac{z^2}{16}+\cdots\right)$$

is the form of Laurent's series

Ex :-

Prove that Laurent's series is the power of $(z+1)$ which represent the function

$f(z) = \frac{z^2+1}{z(z^2-3z+2)}$ in the region $|z+1| > 3$ is given by $\frac{1}{2}\sum_{n=0}^{\infty}(1-2^{n+2}+5\cdot 3^n)(z+1)^{-(n+1)}$

Sol:-

$$f(z) = \frac{z^2+1}{z(z^2-3z+2)}$$

By partial fraction.

$$\frac{z^2+1}{z(z^2-3z+2)} = \frac{z^2+1}{z(z-2)(z-1)}$$

$\because z^2-3z+2$
$\Rightarrow z^2-2z-z+2$
$\Rightarrow z(z-2)-1(z-2)$
$\Rightarrow (z-2)(z-1)$

$$\frac{z^2+1}{z(z^2-3z+2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2} \rightarrow (a)$$

Multiplying both sides by $z(z-1)(z-2)$

$$z^2+1 = A(z-1)(z-2) + Bz(z-2) + Cz(z-1) \rightarrow (1)$$

put $z = 0$ in (1)

$$0+1 = A(0-1)(0-2) + B(0) + C(0)$$
$$1 = A(2)$$
$$A = 1/2$$

put $z-1 = 0$ or $z = 1$ in (1)

$$1+1 = A(0) + B(1)(1-2) + C(0)$$
$$2 = B(1)(-1)$$
$$2 = -B$$
$$B = -2$$

put $z-2 = 0$ or $z = 2$ in (1)

$$2^2+1 = A(0) + B(0) + C(2)(2-1)$$
$$4+1 = C(2)(1)$$
$$5 = 2C$$
$$C = 5/2$$

$$f(z) = \frac{1}{2z} - \frac{2}{z-1} + \frac{5}{2(z-2)}$$

Take $z+1 = t$

$$z = t-1$$
$$z-1 = t-2$$

$$z-2=t-3$$

$$f(z)=f(t-1)=\frac{1}{2(t-1)}-\frac{2}{t-2}+\frac{5}{2(t-3)}$$

$$=\frac{1}{2}(t-1)^{-1}-2(t-2)^{-1}+\frac{5}{2}(t-3)^{-1}$$

$$=\frac{1}{2}\left[(t-1)^{-1}-4(t-2)^{-1}+5(t-3)^{-1}\right]$$

$$=\frac{1}{2}\left[t^{-1}\left(1-\frac{1}{t}\right)^{-1}-2^2t^{-1}\left(1-\frac{2}{t}\right)^{-1}+5t^{-1}\left(1-\frac{3}{t}\right)^{-1}\right]$$

$$=\frac{1}{2}\left[t^{-1}\left(1+\frac{1}{t}+\frac{1}{t^2}+\cdots\right)-2^2t^{-1}\left(1+\frac{2}{t}+\frac{2^2}{t^2}+\cdots\right)\right.$$

$$\left.+5t^{-1}\left(1+\frac{3}{t}+\frac{3^2}{t^2}+\frac{3^3}{t^3}+\cdots\right)\right]$$

$$=\frac{1}{2}\left[\sum_{n=0}^{\infty}t^{-1}t^{-n}-2^2\sum_{n=0}^{\infty}t^{-1}t^{-n}2^n+5\sum_{n=0}^{\infty}t^{-1}t^{-n}3^n\right]$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}t^{-1}t^{-n}\left[1-2^2.2^n+5.3^n\right]$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}t^{-(n+1)}\left[1-2^{n+2}+5.3^n\right]$$

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left[1-2^{n+2}+5.3^n\right]t^{-(n+1)}$$

Put $t=z+1$ or $z=t-1$

$$f(z)=\frac{1}{2}\sum_{n=0}^{\infty}\left[1-2^{n+2}+5.3^n\right](z+1)^{-(n+1)}$$

Ex:-

Expand $\sin u\left(z+\frac{1}{z}\right)$ in form of Laurent's Series.

Sol:-

Step I:-

Given function $f(z)=\sin u\left(z+\frac{1}{z}\right)$ remains same if we replace z by $\frac{1}{z}$. Therefore, $a_n=b_n$ and $f(z)$ ceases to be analytic function at $z=0$, i-e we shall expand it for $|z|>0$

So, the function $f(z)$ becomes

$$f(z)=a_0+\sum_{n=1}^{\infty} a_n\left(z^n+\frac{1}{z^n}\right)$$

Step II :-

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

AS $|z|>0$, $a=0$

$z=e^{i\theta}$, $dz=ie^{i\theta}d\theta$; $0\le\theta\le 2\pi$

$$a_n = \frac{1}{2\pi i}\int_C \frac{\sin u\left(z+\frac{1}{z}\right)dz}{(z-0)^{n+1}}$$

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sin u\left(e^{i\theta}+e^{-i\theta}\right)ie^{i\theta}d\theta}{(e^{i\theta})^{n+1}}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin u(2\cos\theta)\,d\theta}{(e^{i\theta})^n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin u(2\cos\theta)\,e^{-in\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin u(2\cos\theta)(\cos n\theta - i\sin n\theta)\,d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin u(2\cos\theta)\cos n\theta\,d\theta - \frac{i}{2\pi}\int_0^{2\pi} \sin u(2\cos\theta)\sin n\theta\,d\theta \to (1)$$

Step III :-

Consider the integral

$$I = \int_0^{2\pi} \sin u(2\cos\theta)\sin n\theta\,d\theta$$

put $\theta = 2\pi - \phi$, $d\theta = -d\phi$

$\phi = 2\pi - \theta$ at $\theta = 0 \Rightarrow \phi = 2\pi$.

at $\theta = 2\pi$, $\phi = 0$.

$$\int_{2\pi}^{0} \sin u(2\cos(2\pi-\phi))\sin n(2\pi-\phi)(-d\phi)$$

$$= \int_0^{2\pi} \sin u(2\cos\phi)(-\sin n(\phi))\,d\phi$$

$$I = -\int_0^{2\pi} \sin u(2\cos\phi)\sin n\phi\, d\phi$$

$I = -I$

$I + I = 0$

$2I = 0$

$I = 0$

$$\Rightarrow \int_0^{2\pi} \sin u(2\cos\theta)\sin n\theta\, d\theta = 0$$

Then, eq (1) becomes

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin u(2\cos\theta)\cos n\theta\, d\theta$$

As $a_n = b_n$

So, $$b_n = \frac{1}{2\pi}\int_0^{2\pi} \sin u(2\cos\theta)\cos n\theta\, d\theta$$

$\sin \pi i = \frac{e^{\pi i} - e^{-\pi i}}{2} = \frac{1}{2}\{\cos\pi + i\sin\pi - (\cos\pi + i\sin\pi)\}$

$= 2\cos\pi$

$= \frac{2i\sin\pi}{2}$

$= i\sin\pi$

CHAPTER # 07

CALCULAS OF RESIDUES

Zero of a function:-

Let $f(z)$ be a complex valued function defined in D_f and $z = a$ is also a point in D_f such that $f(a) = 0$ then $z = a$ is known as zero of $f(z)$.

Further, if $f(a) = 0$ and $f'(a) \neq 0$ then the zero is called zero of order "1" or Simple zero.

If $f(a) = 0$ and $f'(a) = 0$ but $f''(a) \neq 0$, then the zero is called zero of order "2".

In general, if $f(a) = 0$, $f'(a) = 0$, $f''(a) = 0 \ldots$ $f^{(n-1)}(a) = 0$, but $f^{(n)}(a) \neq 0$ then, the zero is called zero of order "n" at $z = a$

Ex:-

$$f(z) = z - 2$$
$$f(2) = 2 - 2 = 0$$
$$f'(z) = 1$$

$\therefore f(z) = z - 2$ has a zero at $z = 2$ of order "1" or first order

Ex:-

$$f(z) = (z-2)^3$$
$$f(2) = 0$$
$$f'(z) = 3(z-2)^2$$
$$f'(2) = 0$$
$$= 6(z-2)$$
$$f''(2) = 0$$
$$f'''(z) = 6$$

$f(z)$ has a zeros of order "3" at $z = 2$

Ex :- $f(z) = \sin z$

$$f(0) = \sin(0) = 0$$
$$f'(z) = \cos z$$
$$f'(0) = \cos(0) = 1$$

Then function $f(z) = \sin z$ has simple zero at $z = 0$. or The function $f(z)$ has zeros of order "1" at $z = 0$.

Ex :-

$$f(z) = 1 - \cos z$$
$$f(2n\pi) = 0$$
$$f'(z) = \sin z$$
$$f'(2n\pi) = 0$$
$$f''(z) = \cos z$$
$$f''(2n\pi) = 1$$

The function $f(z)$ has zeros of order "2" at $z = 2n\pi$.

Ex :-

Find the zeros of the following function

(i) $f(z) = \sinh z$

(ii) $f(z) = \cosh z$

(iii) $F(z) = z \sinh z$

Sol :-

(i) $f(z) = \sinh z$

$$f(z) = \frac{e^{z} - e^{-z}}{2}$$

$$f(0) = \frac{e^{0} - e^{-0}}{2} = \frac{1-1}{2} = 0$$

$$f'(z) = \cosh z$$
$$= \frac{e^{z} + e^{-z}}{2}$$

$$f'(0) = \frac{e^{0} + e^{-0}}{2} = \frac{1+1}{2} = \frac{2}{2}$$

$$f'(0) = 1$$

Then function f(z) has zeros of order "1" at $z=0$

(iii)

$$f(z) = z \sinh z$$

$$f(z) = z\left(\frac{e^{z}+e^{-z}}{2}\right)$$

$$f(0) = (0)\left(\frac{e^{0}-e^{-0}}{2}\right)$$

$$f(0) = 0$$

$$f'(z) = z \cdot \cosh z + 1 \sinh z$$

$$f'(z) = z\left(\frac{e^{z}+e^{-z}}{2}\right) + \frac{e^{z}-e^{-z}}{2}$$

$$f'(0) = 0 + \frac{e^{0}-e^{-0}}{2}$$

$$f'(0) = 0 + 0$$

$$f'(0) = 0$$

$$f''(z) = z(\sinh z) + 1 \cdot \cosh z + \cosh z$$

$$f''(z) = z\,\frac{e^{z}-e^{-z}}{2} + \frac{e^{z}+e^{-z}}{2} + \frac{e^{z}+e^{-z}}{2}$$

$$f''(z) = 2\left(e^{z}+e^{-z}\right)$$

$$f''(z) = e^{z}+e^{-z}\,2$$

$$f''(0) = e^{0}+e^{-0} = 1+1 = 2$$

The function f(z) has zeros of order "2" at $z=0$

(ii)

$$f(z) = \cosh z$$

Singularities :-

Given a function $f(z)$. If $f(z)$ ceases to be analytic at a centre (i.e.) point $z=a$, then the function is said to have singularity at $z=a$

(i) ceases means negative or not

(ii) The point at which the function $f(z)$ is non-analytic is singular point).

Further, if $f(z)$ has a singularity at a certain point and in the neighbourhood of that point there is no other singular point that singularity is called isolated singularity, otherwise it is called non-isolated singularity.

Isolated singularity can be further classified

Isolated singularity
- Poles
- Essential Isolated Singularity

Pole :-

If the principal part of $f(z)$ at "a" contains at least one non-zero term but the number of such term is finite, there exists an integer m such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \dots = 0$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \frac{b_n}{z^n} \rightarrow (1)$$

That is expansion (1) takes the form.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \dots + \frac{b_m}{(z-a)^m}$$

In this case, the isolated singular

point "a" is called a pole of order "m".

Ex :-

Find the pole of $f(z) = \frac{\sinh z}{z^6}$

Sol :-

$$f(z) = \sinh z / z^6$$

$$f(z) = \frac{1}{z^6}\left\{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right\}$$

$$f(z) = \frac{1}{z^5} + \frac{1}{3!\,z^3} + \frac{1}{5!\,z} + \frac{z}{7!} + \frac{z^3}{9!} + \cdots$$

It has a pole at $z=0$ of order 5.

Ex :-

$$f(z) = \frac{2}{z} + \frac{4}{z^2} + \frac{1}{z^7}$$

$f(z)$ has a pole at $z=0$ of order 7. because in denominator $(z-0)^7$, $z=0$,

Ex :-

$$f(z) = 3 + 5/z$$

it has a simple pole at $z=0$

Remark :-

(i) In Laurent's series the principal part of series is finite and has at least non-zero term. And the point at which the function is non-analytic is called pole.

(ii) Every pole is singular point but a singular point is not a pole.

(iii) In Laurent's expansion, if no of terms of principal part is finite and function is not analytic at $z=0$ then $z=0$ is called pole.

(iv) Entire function is also called integral function (analytic at $z=a$).

Entire function :-

A function $f(z)$ is said to be a entire function, if it has no singularies in the finite part of plane. or, the function which is non analytic at a certain point $z = a$ is called entire function.

Ex :-

e^z, $\sin z$ are analytic at $z = 0$.

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

$$\sin z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots$$

are entire functions because th

Meromorphic function :-

A complex valued function $f(z)$ is said to be meromorphic if it is analytic everywhere except at a finite number of poles.

Ex :- (The function which is non-analytic at poles and analytic at everywhere except pole)

Ex :-

$$f(z) = \frac{1}{z} + \frac{1}{z^2}$$

is meromorphic because it is non-analytic at $z = 0$.

Ex :-

$$f(z) = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

is not meromorphic.

Ex :-

$$f(z) = z^2 + 1 / z(z+1)(z^2+4)$$

it is meromorphic function because except $z = 0, -1, \pm 2i$, the function is analytic everywhere.

where $z = 0$, $z = -1$ and $z = \pm 2i$ are finite poles of $f(z)$.

Essential Singularity:-

When the principal part of f at 'a' has an infinite number of non-zero terms, then the point is called an essential singular point.

Examples:-

(i) If $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \cdots$ because it has infinite non-zero terms in principal part.

(ii) If $f(z) = \sin(z + \frac{1}{z})$

$$= (z + \tfrac{1}{z}) - \frac{(z + 1/z)^3}{3!} + \frac{(z + 1/z)^5}{5!} + \cdots$$

(iii) $f(z) = \frac{1}{z-1} + \frac{1}{z}$ in this function $z=1$ is pole

but $\frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots$

then $z=1$ is essential singular point.

Remark:-

(i) In Laurent's expantion, when the principal part of f at 'a' has an in-finite number of non-zero terms then the point 'a' is called essential in singular point.

(ii) In Laurent's expantion, when the principal part of function at "a" has finite number of non-zero terms, then the point "a" is called pole.

Removable singularity:-

When all the co-efficients b_n in the principal part of f at an isolated singular point "a" are zero, the point "a" is called a removable singular point of f. In this case, Laurent's expantion has only analytic part Because, when the co-efficients of principal parts are zero, then or the principal part becomes zero.

Ex:- $f(z) = \dfrac{e^{z}-1}{z}$ $\quad z=0$ is singular point.

$$= \frac{\left\{\cancel{1} + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots - \cancel{1}\right\}}{z}$$

$f(z) = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$ is not singularity

So, f(z) is removable singularity.

Note:-

If we simplify laurent's series and obtain power series at point at which function is not analytic.

So, it is removable singularity.

Theorem:-

If a complex valued function f(z) has certain zeros, Then, Prove that those zeros of f(z) are the poles of

$$F(z) = \frac{1}{f(z)}$$

Proof:-

Since, f(z) is a complex valued function, Let $a_1, a_2, \dots, a_n$ be the zeros of f(z) of order $r_1, r_2, \dots, r_n$

then $f(z) = \prod_{i=1}^{n} (z - a_i)^{r_i} g(z)$, $g(z) \neq 0$ at $z = a_i$

$f(z) = (z-1)^2 (z+2)$ $\quad \because z+2 = 1+2 = 3 \neq 0$ at $z=1$

$f(z) = z^3 - 3z + 2$

$f(1) = (1)^3 - 3(1) + 2 = -2 + 2 = 0$

$f'(z) =$

$f'(1) = 3-3 = 0$

$f''(z) = 6z$

$f''(1) = 6$

The function $f(z)$ has zeros of order "2" at $z = 1$

Now $F(z) = \dfrac{1}{f(z)}$

$$= \frac{1}{\prod_{i=1}^{n} (z-a_i)^{r_i}} G(z) \quad ; G(z) = \frac{1}{g(z)}$$

This relation shows that $z = a_1, a_2, \ldots, a_n$ are the poles of $F(z)$.

Note:-

Poles of any function $f(z)$ are also its zeros in denominator.

Ex:-

Find the poles of the following function
(i) $f(z) = \cot z$ (ii) $f(z) = z \operatorname{Cosec} z$

Sol:-

(i) $f(z) = \cot z$

$f(z) = \dfrac{\cos z}{\sin z}$

zeros of $\sin z$ are at $n\pi$; $n = 0, \pm 1, \pm 2, \ldots$
Therefore, the poles of $\cot z$ are at $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

(ii)

$f(z) = z \operatorname{Cosec} z$

$f(z) = \dfrac{z}{\sin z}$

zeros of $\sin z$ are at $n\pi$; $n = 0, \pm 1, \pm 2, \ldots$ Therefore, the poles of $z \operatorname{Cosec} z$ are at $z = n\pi$; $n = 0, \pm 1, \pm 2, \ldots$

Behaviour of the function $f(z)$ at the point $z=\infty$

Procedure :-

Given a function $f(z)$. If we desired to discuss its zeros, poles and singularities

Then put $w = \frac{1}{z}$, $z = \frac{1}{w}$

The given function $f(z)$ reduces to $R(w)$

The zeros, poles and singularities of $R(\omega)$ at $w=0$ will be the zeros, poles and singularities of $f(z)$ at $z=\infty$

Ex :-

Find the zeros, poles, and singularities of the following functions at $z=\infty$

Sol :-

(i) $f(z) = z^2$ (ii) $f(z) = e^{2z}$ (iii) $f(z) = z^2+1$

(i) $f(z) = z^2$

put $z = \frac{1}{w}$

$$R(\omega) = \frac{1}{\omega^2} = \frac{1}{(\omega-0)^2}$$

It has a pole of order 2 at $\omega=0$

Therefore, $f(z)$ has a pole of order 2 at $z=\infty$

(ii) $f(z) = e^{2z}$

put $z = \frac{1}{w}$

$$R(\omega) = e^{2/w}$$

$$= 1 + \frac{2}{\omega} + \left(\frac{2}{\omega}\right)^2 \cdot \frac{1}{2!} + \cdots$$

It has an essential isolated singularity at $\omega=0$.

Therefore, $f(z) = e^{2z}$ has an essential singularity at $z = \infty$

(iii) $f(z) = z^2 + 1$

put $z = 1/w$

$$R(w) = \frac{1}{w^2} + 1$$

$$= 1 + \frac{1}{w^2}$$

It has pole of order 2 at $w = 0$
Therefore, $f(z)$ has a pole of order 2 at $z = \infty$

Ex :-

Investigate whether the following functions are Entire or Meromorphic

(i) $z^3 e^{4z}$ (ii) $\dfrac{\sin z}{(z+2)(z^2+a)^2}$ (iii) $\sin 1/z$

Sol :-

(i) $f(z) = z^3 e^{4z}$

$f(z) = z^3 \{1 + 4z + (4z)^2/2! + \cdots\}$

Since, the function $f(z)$ has no singular point. So, $f(z) = z^3 e^{4z}$ is an entire function.

(ii)

$f(z) = \sin z / (z+2i)(z^2+a)^2$

$f(z)$ is analytic everywhere in the finite part of plane except at finite number of poles.

i-e $z = -2i$ or $z = \pm a$

Therefore, $f(z)$ is meromorphic function

(iii)

$f(z) = \sin(1/z)$

It has singular point and is analytic at pole. So, $\sin z$ is neither an entire nor meromorphic.

Ex :-

Discuss the nature of singularities of the functions.

(i) $f(z) = \dfrac{z+1}{z^3(z^2+1)} = \dfrac{z+1}{(z-0)^3(z^2+1)}$

$f(z)$ has 3 isolated singular points at $z = 0, \pm i$

(ii) $f(z) = \dfrac{z^2-2z+3}{z-2}$ $\quad \because z^2-2z+3$

$= z^2-3z+z+3$

$f(z) = \dfrac{(z-3)(z+1)}{z-2}$ $\quad = z(z-3)+1(z-3)$

$= (z-3)(z+1)$

$f(z)$ has a simple pole of order "1" at $z = 2$.

(iii) $f(z) = \dfrac{z-\sin z}{z^3}$

$$f(z) = \frac{z-\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right]}{z^3}$$

$$= \frac{1}{z^3}\left[\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right]$$

$$f(z) = \frac{1}{3!}-\frac{z^2}{5!}+$$

It has removable singularity at $z=0$

(iv) $f(z) = e^{1/2z}$

$$f(z) = 1+\frac{1}{2z}+\frac{1}{2!}\left(\frac{1}{2z}\right)^2+\cdots$$

It has an isolated singularity at $z=0$

Pages (Module for singularities zeros and poles)

Residue of f(z) at a pole:-

Residue of a function $f(z)$ at $z=a$, is the co-efficient of $\frac{1}{z-a}$ (i-e) b_1.

Also, we know that

$$b_n = \frac{1}{2\pi i}\int_C f(z)(z-a)^{n-1}\,dz$$

$$b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i b_1 = 2\pi i R$$

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Example:-

E Find the residue of $\frac{\sin z}{z^6}$

Sol:-

$$f(z) = \frac{\sin z}{z^6}$$

$$= \frac{1}{z^6}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \frac{1}{z^5} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z} - \frac{z}{7!} + \cdots$$

$$\text{Residue} = \frac{1}{5!} = \frac{1}{120}$$

So,

$$R(f, 0) = \frac{1}{120}$$

Ex:-

Find the residue of $\frac{z^2-2z+3}{z-2}$

Sol:-

$$f(z) = \frac{z^2-2z+3}{z-2}$$

$$f(z) = z + \frac{3}{z-2} = 2+(z-2)+\frac{3}{z-2}$$

It has a pole at $z=2$ of order 1. So, $R(f,2) = 3$

Ex:-

Find the residue of $f(z) = \frac{\operatorname{cosec} z \cdot \operatorname{cosech} z}{z^3}$

Sol :-

$$f(z) = \frac{\text{Cosec}\, z \ \text{Cosech}\, z}{z^3}$$

$$= \frac{1}{z^3}\left[\frac{1}{\sin z} \cdot \frac{1}{\sinh z}\right]$$

$$= \frac{1}{z^3}\left[\frac{1}{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots} \quad \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots}\right]$$

$$= \frac{1}{z^5}\left[\frac{1}{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots} \quad \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots}\right]$$

$$f(z) = \frac{1}{z^5}\left[\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots\right)\right]$$

$$f(z) = \frac{1}{z^3 \cdot z^2}\left[\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \frac{z^6}{7!} + \cdots\right)\right]$$

$$f(z) = \left[\left(\frac{1}{z^3} - \frac{1}{3!}\frac{1}{(z-0)} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots\right)\left(\frac{1}{z^2} + \frac{1}{3!} + \frac{z^2}{5!} + \cdots\right)\right]$$

Residue = Co-efficient of $\frac{1}{z-0}$ which $-\frac{1}{3!}$

$$b = -\frac{1}{3!} = \frac{-1}{6}$$

$$R(f, 0) = -\frac{1}{6}$$

Formula :-

$$R(f, a) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)_{z=a}$$

where m is the order of pole

Ex :-

Find the Residue of $f(z) = \frac{2z-5}{(z-1)^2(z+2)}$

Sol :-

$$f(z) = \frac{2z-5}{(z-1)^2(z+2)}$$

It has a pole of order 2 at $z=1$ and a simple pole at $z=-2$

$$R_1(f,1) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z-1)^2 \times \frac{2z-5}{(z-1)^2(z+2)}\right)_{z=1}$$

$$R_1(f,1) = \frac{1}{(2-1)!}\frac{d}{dz}\left(\frac{2z-5}{z+2}\right)_{z=1}$$

$$R_1(f,1) = \left(\frac{(z+2)(2-0)-(2z-5)(1)}{(z+2)^2}\right)_{z=1}$$

$$= \left(\frac{2z+4-2z+5}{(z+2)^2}\right)_{z=1}$$

$$= \left(\frac{9}{(z+2)^2}\right)_{z=1} = \frac{9}{(1+2)^2} = \frac{9}{3^2} = \frac{9}{9}$$

$$R_1(f,1) = 1$$

$$R_2(f,-2) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z+2)^2 \times \frac{2z-5}{(z-1)^2(z+2)}\right)_{z=-2}$$

$$= \frac{d}{dz}\left((z+2) \times \frac{(2z-5)}{(z-1)^2(z+2)}\right)_{z=-2}$$

$$= \frac{d}{dz}\left(\frac{(z+2)(2z-5)}{(z-1)^2}\right)_{z=-2} = \frac{d}{dz}\left(\frac{2z^2+4z-10}{(z-1)^2}\right)_{z=-2}$$

$$= \left(\frac{(z-1)^2(4z-1)-(2z^2-z-10)2(z-1)(1)}{((z-1)^2)^2}\right)_{z=-2}$$

$$= \left((4z-1)(z^2-2z+1) - 2(2z^3-2z^2-z^2+z-10z+10)\right)/(z-1)^4$$

$$= (4z^3-8z^2+4z-z^2+2z-1-4z^3+4z^2+2z^2-2z+20z-20)\ z=-2$$

$$= \left(-3z^2+24z-21/(z-1)^4\right)_{z=-2} = \left(\frac{-3(z^2-8z+7)}{(z-1)^4}\right)_{z=-2} = \frac{-3(4+16+7)}{(-2-1)^4}$$

$$R_2(f,-2) = \frac{-8}{9} = -1$$

Ex:-

Find the residue of $f(z) = \frac{\sin z\, e^z}{(z^2+a^2)^2}$

Sol:-

It has a pole of order 2 at $z = \pm ai$

$$R(f,ai) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left((z-ai)^2 f(z)\right)_{z=ai}$$

$$= \frac{d}{dz}\left((z-ai)^2 \frac{\sin z\, e^z}{(z^2+a^2)^2}\right)_{z=ai}$$

$$= \frac{d}{dz}\left((z-ai)^2 \times \frac{\sin z\, e^z}{(z-ai)^2(z+ai)^2}\right)_{z=ai}$$

$$R_1(f, ai) = \frac{d}{dz}\left(\frac{\sin z\, e^z}{(z+ai)^2}\right)_{z=ai}$$

$$= \left(\frac{(z+ai)^2(e^z\cos z + e^z\sin z) - (2(z+ai)e^z\sin z)}{((z+ai)^2)^2}\right)_{z=ai}$$

$$= \left(\frac{(z+ai)e^z[(z+ai)(\cos z+\sin z) - 2\sin z]}{(z+ai)^4}\right)_{z=ai}$$

$$= \left(\frac{e^z[(z+ai)(\cos z+\sin z) - 2\sin z]}{(z+ai)^3}\right)_{z=ai}$$

$$= \left(\frac{e^{ai}[(ai+ai)(\cos(ai) + i\sin(ai)) - 2i\sin(ai)]}{(ai+ai)^3}\right)$$

$$= \frac{e^{ai}[(2ai)(\cosh a + i\sinh a) - 2i\sinh a]}{(2ai)^3}$$

$$= \frac{2e^{ai}[ai(\cosh a + i\sinh a) - i\sinh a]}{8a^3i^3} \qquad i^3 = i^2 \cdot i$$

$$= \frac{-e^{ai}i}{4a^3i}[ai\cosh a + ai\sinh a - \sinh a] \qquad = -i$$

$$= \frac{-1}{4a^3i}e^{ai}[ai\cosh a + (ai-1)\sinh a]$$

$$R_1(f, ai) = \frac{-1}{4a^3i}e^{ai}[ai\cosh(a) + (ai-1)\sinh(a)]$$

$$R_2(f, -ai) = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\left((z+ai)^2 f(z)\right)_{z=-ai}$$

$$R_2(f,-ai) = \frac{1}{4}\frac{d}{dz}\left((z+ai)^2 \frac{\sin z\, e^z}{(z-ai)^2(z+ai)^2}\right)_{z=-ai}$$

$$R_2(f,-ai) = \frac{d}{dz}\left(\frac{e^z \sin z}{(z-ai)^2}\right)_{z=-ai}$$

$$= \left(\frac{(z-ai)^2(e^z\cos z + e^z \sin z) - (2(z-ai)e^z \sin z)}{((z-ai)^2)^2}\right)_{z=-ai}$$

$$= \left(\frac{(z-ai)e^z[(z-ai)(\cos z+\sin z) - 2\sin z]}{(z-ai)^4}\right)_{z=-ai}$$

$$= \left[\frac{e^z[(z-ai)(\cos z+\sin z) - 2\sin z]}{(z-ai)^3}\right]_{z=-ai}$$

$$= \frac{e^{-ai}[(-ai-ai)(\cos(-ai)+\sin(-ai)) - 2\sin(-ai))]}{(-ai-ai)^3}$$

$$= \frac{e^{-ai}[(-2ai)(\cosh(a) - i\sinh(a)) - 2i(-\sinh(a))]}{(-2ai)^3}$$

$$= \frac{e^{-ai}(-2)[ai(\cosh(a) - i\sinh(a)) + i\sinh(a)]}{-8a^3 i^3} \qquad \because i^3 = -i$$

$$= \frac{e^{-ai}\, i\, [ai\cosh(a) - ai\sinh(a) - \sinh(a)]}{4a^3 i^3}$$

$$= \frac{e^{-ai}\, i\, [ai\cosh(a) - (ai+1)\sinh(a)]}{4a^3(-i)}$$

$$= \frac{e^{-ai}}{-4a^3 i}[ai\cosh(a) + (ai+1)\sinh(a)]$$

$$R_2(f,-ai) = \frac{-1}{4a^3 i} e^{-ai}[ai\cosh a - (ai+1)\sinh a]$$

$$R_2(f,-ai) = \frac{-1}{4a^3} e^{-ai}[a\cosh a - (ai+1)\sinh a]$$

Ex :-

Find the residue of $\frac{ze^z}{(z^2+a^2)^2}$, $\frac{z^3-z}{\sin\pi z}$.

Sol :-

(i) $f(z) = \frac{z^3-z}{\sin\pi z}$

It has pole of order "1" at $z = n$

$$R(f,0) = \left((z-n)\frac{z^3-z}{\sin\pi z}\right)_{z=n}$$

$$R(f,0) = \left((n-n)\left(\frac{n^3-n}{\sin n\pi}\right)\right) = \frac{0}{0}$$

By Del Hospital's theorem.

$$R(f,0) = \frac{(z-n)(3z^2-1)+(z^3-z)}{\pi\cos\pi z}\bigg|_{z=n}$$

$$= \frac{(n-n)(3n^2-1)+(n^3-n)}{\pi\cos(n\pi)} \qquad \because \cos(n\pi) = (-1)^n$$

$$= 0 + \frac{n^3-n}{\pi(-1)^n}$$

$$R(f,0) = \frac{n^3-n}{(-1)^n\pi}$$

(ii)

$$f(z) = \frac{e^z z}{(z^2+a^2)^2}$$

It has pole of order "2" at $z = \pm ai$

$$R(f,ai) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z-ai)^2\frac{e^z z}{(z+ai)^2(z-ai)^2}\right)_{z=ai}$$

$$= \frac{d}{dz}\left(\frac{e^z z}{(z+ai)^2}\right)_{z=ai}$$

$$= \left(\frac{(z+ai)^2(e^z+ze^z)-(2(z+ai)e^z z)}{((z+ai)^2)^2}\right)_{z=ai}$$

$$= \left(\frac{e^z(z+ai)[(z+ai)(1+z)-2z]}{(z+ai)^4}\right)_{z=ai}$$

$$= \left(\frac{e^z \left[(z+ai)(1+z) - 2z \right]}{(z+ai)^3} \right)_{z=ai}$$

$$= \frac{e^{ai} \left[(ai+ai)(1+ai) - 2ai \right]}{(ai+ai)^3}$$

$$= \frac{e^{ai} \left[(2ai)(1+ai) - 2ai \right]}{(2ai)^3}$$

$$= \frac{e^{ai} \left[2ai + 2(ai)^2 - 2ai \right]}{8(ai)^3}$$

$$= e^{ai} \left[\frac{2a^2 i^2}{8a^3 i^3} \right] \qquad i^3 = i^2 \cdot i = -i$$

$$R_1(f, ai) = +e^{ai} \frac{1}{4ai}$$

$$R_2(f, -ai) = \frac{1}{(2-1)!} \frac{d}{dz} \left((z+ai)^2 \frac{e^z z}{(z+ai)^2 (z-ai)^2} \right)_{z=-ai}$$

$$= \frac{d}{dz} \left(\frac{e^z z}{(z-ai)^2} \right)_{z=-ai}$$

$$= \left(\frac{(z-ai)^2 (e^z + z e^z) - (2(z-ai) e^z z)}{((z-ai)^2)^2} \right)_{z=-ai}$$

$$= \left(\frac{e^z (z-ai) \left[(z-ai)(1+z) - 2z \right]}{(z-ai)^4} \right)_{z=-ai}$$

$$= \left[\frac{e^z \left[(z-ai)(1+z) - 2z \right]}{(z-ai)^3} \right]_{z=-ai}$$

$$= \frac{e^{-ai} \left[(-ai-ai)(1-ai) - 2(-ai) \right]}{(-ai-ai)^3}$$

$$= \frac{e^{-ai} \left[(-2ai)(1-ai) + 2ai \right]}{(-2ai)^3}$$

$$= \frac{e^{-ai} \left[-2ai + 2(ai)^2 + 2ai \right]}{-8a^3 i^3}$$

$$= \frac{e^{-ai} \left[2a^2 i^2 \right]}{-8a^3 i^3} \qquad i^3 = -i$$

$$R_2(f, -ai) = +e^{-ai} \frac{1}{4ai}$$

Ex :-

Find Residue of the following function

(i) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

(ii) $f(z) = e^z \operatorname{cosec}^2 z$

(iii) $f(z) = \dfrac{\cot z \times \coth z}{z^3}$

Sol :-

(i) $f(z) = \dfrac{z^2 - 2z}{(z+1)^2 (z^2+4)}$

It has a pole of order 2 at $z = -1$ and a simple pole at $z = \pm 2i$

$$R_1(f, a) = \frac{1}{(m-1)!} \cdot \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z) \right)_{z=a}$$

$$R_1(f, -1) = \frac{1}{(2-1)!} \cdot \frac{d^{2-1}}{dz^{2-1}} \left((z+1)^2 f(z) \right)_{z=-1}$$

$$= \frac{1}{1!} \cdot \frac{d}{dz} \left((z+1)^2 \frac{z^2 - 2z}{(z+1)^2 (z^2+4)} \right)_{z=-1}$$

$$= \frac{d}{dz} \left(\frac{z^2 - 2z}{z^2+4} \right)_{z=-1}$$

$$= \left(\frac{(z^2+4)(2z-2) - (z^2-2z)(2z)}{(z^2+4)^2} \right)_{z=-1}$$

$$= \left(\frac{\cancel{2z^3} + 8z - 2z^2 + 8 - \cancel{2z^3} + 4z^2}{(z^2+4)^2} \right)_{z=-1}$$

$$= \left(\frac{2z^2 + 8z - 8}{(z^2+4)^2} \right)_{z=-1} = \frac{2(-1)^2 + 8(-1) - 8}{((-1)^2+4)^2}$$

$$= \frac{2(1) - 8 - 8}{(1+4)^2}$$

$$= \frac{2 - 16}{(5)^2} = \frac{-14}{25}$$

$$R_1(f, -1) = \frac{-14}{25}$$

$$R_2(f, 2i) = \frac{1}{(1-1)!} \frac{d^0}{dz^0}\left((z-2i)^2 \frac{(z^2-2z)}{(z+1)^2(z-2i)(z+2i)}\right)_{z=2i}$$

$$= \frac{d}{dz}\left(\frac{z^2-2z}{(z+1)^2(z+2i)}\right)_{z=2i}$$

$$(2i+1)^2(2i(2i)^2 \ldots 2(2i) - 2) + (2i-2i)(2i \ldots)$$

$$= 4i^2 - 4i + 1 \qquad i^2 = -1$$

$$(1+4i+4i^2)(4i)$$

$$(4i(-4i) - 4i \ldots (4i+4-4i) \qquad = -4(1+i)$$

$$(1+4i+4(-1))(4i) \qquad 4i+16i^2-16i \qquad -12i-16$$

$$\frac{-4i(1+i)}{-4i \quad 4i(1+2i)^2}$$

$$4i(1+2i)^2$$

$$4i(1+2i)^2$$

$$4i(1+2i) \qquad 4i$$

ON PAPER

(ii) $f(z) = e^z \operatorname{Cosec}^2 z$

It has double pole at $z = 0, \pm\pi, \pm 2\pi, \ldots$

(i.e) $z = n\pi$; $n = 0, \pm 1, \pm 2, \ldots$

∴ Residue at $z = n\pi$

$$\lim_{z\to n\pi} \frac{1}{(2-1)!} \frac{d}{dz}\left\{(z-n\pi)^2 \frac{e^z}{\sin^2 z}\right\}$$

$$= \lim_{z\to n\pi} e^z \left[\frac{(z-n\pi)^2 \sin z + 2(z-n\pi)\sin z - 2(z-n\pi)\cos z}{\sin^3 z}\right]$$

Let $z - n\pi = u \Rightarrow z = u + n\pi$; $\sin(u+n\pi) = \sin u$

then $\lim_{u\to 0} e^{u+n\pi} \left[\frac{u^2 \sin u + 2u\sin u - 2u^2 \cos u}{\sin^3 u}\right]$

$$= e^{0+n\pi}\left[\lim_{u\to 0} \frac{u^2 \sin u + 2u\sin u - 2u^2\cos u}{\sin^3 u}\right] = \frac{0}{0}$$

By Del' Hospital theorem

$$e^{n\pi} \lim_{u\to 0} \frac{u^3}{\sin^3 u} = \left(\lim_{u\to 0} \frac{u}{\sin u}\right)^3 = 1$$

then

$$= e^{n\pi}\left[\lim_{u\to 0}\left(\frac{u^2\sin u + 2u\sin u - 2u^2\cos u}{u^3}\right) \lim_{u\to 0} \frac{u^3}{\sin^3 u}\right]$$

$$= e^{n\pi} \lim_{u\to 0} \frac{u^2 \sin u + 2u\sin u - 2u^2\cos u}{u^3} = e^{n\pi}$$

Using L' Hospital theorem (rule) several times in evaluating this limit, we can instead use the series expensions $\sin u = u - \frac{u^3}{3!} + \ldots$, $\cos u = 1 - \frac{u^2}{2!} + \ldots$

(iii)

$$f(z) = \frac{\cot z \coth z}{z^3} \quad \text{at } z = 0$$

$$f(z) = \frac{1}{z^3}\left\{\frac{\cos z}{\sin z} \cdot \frac{\cosh z}{\sinh z}\right\} = \frac{1}{z^3}\left[\frac{1 - \frac{z^2}{2!} + \ldots}{z - \frac{z^3}{3!} + \ldots} \cdot \frac{1 + \frac{z^2}{2!} + \ldots}{z + \frac{z^3}{3!} + \ldots}\right]$$

$$= \frac{1}{z^3}\left[\frac{1 + \frac{z^2}{2!} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^4}{4!} + \frac{z^4}{4!} - \ldots}{z \cdot z\left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots\right]\left[1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \ldots\right]}\right]$$

$$= \frac{(1 - z^4/6 + \ldots)}{z^5\left(1 - \frac{z^2}{6} + \frac{z^4}{120} - \frac{z^2}{6}\right.}$$

on pages.

Ex :-

$$f(z) = \sinh z + \cos^3 z / z^4$$

Sol :-

$$f(z) = \frac{\sinh z + \cos^3 z}{z^4}$$

$$f(z) = \left[\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) + \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right)^3\right]\frac{1}{z^4}$$

$$f(z) = \left[\left(\frac{1}{z^3} + \frac{1}{3!\,z} + \frac{z^5}{5!} + \dots\right) + \left(\frac{1}{z^4} - \frac{1}{z^2\,2!} + \frac{1}{4!} - \frac{z^2}{6!} + \dots\right)\right]$$

$$R(f,0) = \frac{1}{3!} = \frac{1}{6}$$

Therefore $\therefore$ $\displaystyle\int_c \frac{\sinh z + \cos^3 z}{z^4} = 2\pi i R$

$$= 2\pi i \times \frac{1}{6}$$

$$\int_c \frac{\sinh z + \cos^3 z}{z^4} = \frac{1}{3}\pi i$$

Note :-

(i) The function $f(z)$ which has pole $= -i$, of order $m = 7$ and Residue $R = -1$ is

$$f(z) = \frac{1}{(z+i)^7} - \frac{1}{z+i}$$

(ii) The function which has pole $= -i$ of order $m = 1$ or simple pole and Residue $R = -1$ is $f(z) = \dfrac{-1}{z+i}$

(iii) Pole ~~can~~ cannot be ∞, because pole is any finite value.

Imp

Theorem :-

State and prove Cauchy's Residue theorem.

Statement :-

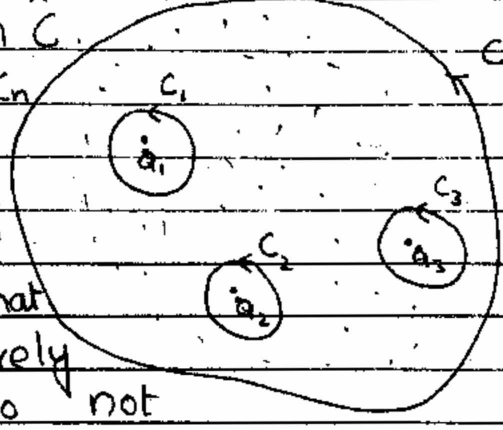
If $f(z)$ is analytic except

at a finite number of poles within a closed contour 'C' and continuous on the boundary of C, then Prove that

$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} R_j$ where R_j is the sum of the Residues of $f(z)$ at its poles within C.

Proof :-

Step I :- Let $a_1, a_2, a_3, \ldots, a_n$ be the n poles within C. Further, Let $C_1, C_2, C_3, \ldots, C_n$ be circles with centres $a_1, a_2, a_3, \ldots, a_n$ respectively and each of radius ε. Now, ε is so small that all the circles lie entirely within C and they do not overlap.



So, f(z) is analytic in the region between C and circles. So that we have, by Cauchy's fundamental theorem.

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz \rightarrow (1)$$

Step II :-

Now, if f(z) has a pole of order m_1 at $z = a_1$ then by Laurent's theorem, we have

$$f(z) = \phi(z) + \sum_{j=1}^{m_1} \frac{b_j}{(z-a_1)^j} \rightarrow (2)$$

where $\phi(z)$ is analytic on and within

C_1. Taking integral of both sides of equation (2) along C_1

$$\int_{C_1} f(z)dz = \int_{C_1} \phi(z)dz + \int_{C_1} \frac{b_1}{(z-a_1)}dz + \int_{C_2} \frac{b_2}{(z-a_1)^2}dz$$

$$+ \int_{C_3} \frac{b_3}{(z-a_1)^3}dz + \cdots + \int_{C_1} \frac{b_{m_1}}{(z-a_1)^{m_1}}dz$$

Since, $\phi(z)$ is analytic on and within C_1, therefore by Cauchy's fundamental theorem.

$$\int_{C_1} \phi(z)dz = 0$$

$$\int_{C_1} \frac{b_{m_1}}{(z-a_1)^{m_1}}dz = b_{m_1}\int_{C_1} \frac{dz}{(z-a_1)^{m_1}}$$

put $z - a_1 = \varepsilon e^{i\theta}$

$dz = i\varepsilon e^{i\theta} d\theta$

then $$b_{m_1}\int_{C_1} \frac{dz}{(z-a_1)^{m_1}} = b_{m_1}\int_{0}^{2\pi} \frac{i\varepsilon e^{i\theta}d\theta}{(\varepsilon e^{i\theta})^{m_1}}$$

$$= \frac{i b_{m_1}}{\varepsilon^{m_1-1}}\int_0^{2\pi} e^{-i(m_1-1)\theta} d\theta$$

$$= 0 \quad \text{if } m_1 \neq 1 \text{ and } < 0$$

when $m_1 = 1$ then

$$\int_{C_1} \frac{b_1}{z-a_1}dz = b_1\int_0^{2\pi} \frac{i\varepsilon e^{i\theta}d\theta}{(\varepsilon e^{i\theta})^1}$$

$$= b_1 i \int_0^{2\pi} d\theta$$

$$= i b_1 \theta \Big|_0^{2\pi}$$

$$= i b_1 (2\pi - 0)$$

$$= 2\pi i b_1$$

$$= 2\pi i R_1$$

$$\therefore \int_{C_1} f(z)\,dz = 2\pi i R_1$$

Likewise $\int_{C_2} f(z)\,dz = 2\pi i R_2$

$\int_{C_2} f(z)\,dz = 2\pi i R_3$

$\int_{C_n} f(z)\,dz = 2\pi i R_n$

$$\therefore \int_C f(z)\,dz = 2\pi i R_1 + 2\pi i R_2 + 2\pi i R_3 + \dots + 2\pi i R_n$$
$$= 2\pi i \sum_{j=1}^{n} R_j$$

Rough work :-

$$\int_{C_2} \frac{b_{m_2}}{(z-a_1)^{m_2}}\,dz = b_{m_2}\int_{C_2} \frac{dz}{(z-a)^{m_2}}$$

put $z - a_1 = \varepsilon e^{i\theta}$

$dz = i\varepsilon e^{i\theta}\,d\theta$

then
$$\int_{C_2} \frac{b_{m_2}}{(z-a_1)^{m_2}}\,dz = b_{m_2}\int_0^{2\pi} \frac{i\varepsilon e^{i\theta}\,d\theta}{(\varepsilon e^{i\theta})^{m_2}}$$
$$= \frac{i\,b_{m_2}}{\varepsilon^{m_2-1}}\int_0^{2\pi} e^{i\theta}\cdot e^{-m_2 i\theta}\,d\theta$$
$$= \frac{i\,b_{m_2}}{\varepsilon^{m_2-1}}\int_0^{2\pi} e^{-i(m_2-1)\theta}\,d\theta$$

when $m_2 = 2$ then
$$\int_{C_2} \frac{b_2}{(z-a_1)^2}\,dz = \frac{i\,b_2}{\varepsilon^{2-1}}\int_0^{2\pi} e^{-i(2-1)\theta}\,d\theta$$
$$= \frac{i\,b_2}{\varepsilon}\int_0^{2\pi} e^{-i\theta}\,d\theta$$
$$= \frac{i\,b_2}{\varepsilon}\,\frac{e^{-i\theta}}{-i}\Big|_0^{2\pi}$$
$$= -\frac{b_2}{\varepsilon}\left(e^{-2\pi i} - e^{0}\right)$$
$$= -\frac{b_2}{\varepsilon}\left(\cos(2\pi) - i\sin 2\pi\right) + \frac{b_2}{\varepsilon}$$
$$= -\frac{b_2}{\varepsilon} + \frac{b_2}{\varepsilon} = 0$$

$$\Rightarrow \int_{C_2} \frac{b_2}{(z-a_1)^2}\,dz = 0$$

Ex:-

Evaluate $\int_c \frac{ze^z}{(z^2+a^2)^2}\,dz$ where c: encloses the point $z = ia$ or $\pm ai$

Sol:-

Given function $f(z) = \frac{ze^z}{(z^2+a^2)^2}$

It has poles at $z = \pm ai$ of order 2

The only pole which lie inside C is $z = ai$

$$R(f, a) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\Big((z-a)^m f(z)\Big)\Big|_{z=a}$$

$$R(f_1, ai) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z-ai)^2 \frac{ze^z}{(z+ai)^2(z-ai)^2}\right)\Bigg|_{z=ai}$$

$$= \frac{d}{dz}\left(\frac{ze^z}{(z+ai)^2}\right)\Bigg|_{z=ai}$$

$$= \left(\frac{(z+ai)^2(1\cdot e^z+ze^z)-(ze^z)(2(z+ai))}{((z+ai)^2)^2}\right)\Bigg|_{z=ai}$$

$$= \frac{(z+ai)e^z\big((z+ai)(1+z)-2z\big)}{(z+ai)^4}$$

$$= \frac{e^z\big((z+ai)(1+z)-2z\big)}{(z+ai)^3}\Bigg|_{z=ai}$$

$$= \frac{e^{ai}\big(2ai(1+ai)-2ai\big)}{(2ai)^3}$$

$$R(f_1, ai) = \frac{e^{ai}}{4ai}$$

$\therefore$ by Cauchy's Residue theorem

$$\int_c \frac{ze^z}{(z^2+a^2)^2}\,dz = \frac{2\pi i \times e^{ai}}{4ai}$$

$$= \frac{\pi}{2a}e^{ai}$$

Now

$$R(f_2, -ai) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z+ai)^2\frac{ze^z}{(z+ai)^2(z-ai)^2}\right)\Bigg|_{z=-ai}$$

$$= \frac{d}{dz}\left(\frac{ze^z}{(z-ai)^2}\right)\Bigg|_{z=-ai}$$

$$= \left(\frac{(z-ai)^2(1e^z+ze^z)-(ze^z)(2(z-ai))}{(z-ai)^4}\right)\Bigg|_{z=-ai}$$

$$= \left(\frac{e^{z}[(z-ai)(1+z) - 2z]}{(z-ai)^3} \right)_{z=-ai}$$

$$= \frac{e^{-ai}[(-2ai)(1-ai) + 2ai]}{(-2ai)^3} = \frac{e^{-ai}\cdot[-2ai+2a^2i^2+2ai]}{-8a^3i^3}$$

$$= -e^{-ai}/4ai$$

$$\int_c f(z)dz = 2\pi i \sum_{j=1}^{n} R_j$$

$$\int_c \frac{ze^z}{(z^2+a^2)^2} dz = 2\pi i R_1 + 2\pi i R_2$$

$$= 2\pi i \frac{e^{ai}}{4ai} + 2\pi i \left(-\frac{e^{-ai}}{4ai}\right)$$

$$= \frac{\pi}{2a} e^{ai} + \frac{\pi}{2a} e^{-ai} = \frac{\pi}{2a}[e^{ai} + e^{-ai}]$$

Ex:-

Evaluate $\int_c \frac{\sin z}{z^4} dz$ and $\int_c \frac{\sin z}{z^5} dz$,

C: encloses at point $z=0$.

Sol:-

(i) $\int_c \frac{\sin z}{z^4} dz = 2\pi i R_1 \to (1)$

$$\frac{\sin z}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} + \cdots$$

Residue $R(f,0) = \frac{1}{3!} = \frac{1}{6}$ put in (1)

(i) $\int_c \frac{\sin z}{z^4} dz = 2\pi i \frac{1}{6} = \frac{\pi}{3} i$

(ii) $\int_c \frac{\sin z}{z^5} dz = 2\pi i R_2 \to (2)$

$$\frac{\sin z}{z^5} = \frac{1}{z^5}\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right\} = \frac{1}{z^4} - \frac{1}{3!\,z^2} + \frac{1}{5!} + \cdots$$

Residue $R_2(f,0) = 0$

Put in (2)

$$\int_c \frac{\sin z}{z^5} dz = 0$$

Ex:-

Evaluate $\int_c \frac{5z-2}{z(z-1)}dz, \quad c: |z| = 2$

Sol:-

$$\int_c \frac{5z-2}{z(z-1)}dz = 2\pi i \sum_{j=1}^{2} R_j$$

$\int_c \frac{5z-2}{z(z-1)}dz$, it has 2 poles of order 1 or simple poles at $z=0$ and $z=1$.

So, $R(f,a) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right)_{z=a}$

$$R(f,0) = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left((z-0)^1 \frac{5z-2}{z(z-1)}\right)_{z=0}$$

$$= z\left(\frac{5z-2}{z(z-1)}\right)_{z=0} = \left(\frac{5z-2}{z-1}\right)_{z=0}$$

$$R(f,0) = \frac{5(0)-2}{0-1} = \frac{-2}{-1} = 2$$

Now,

$$R(f,1) = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left((z-1)\frac{5z-2}{z(z-1)}\right)_{z=1}$$

$$R(f,1) = \left(\frac{(z-1)(5z-2)}{z(z-1)}\right)_{z=1}$$

$$= \left(\frac{5z-2}{z}\right)_{z=1}$$

$$R(f,1) = \frac{5(1)-2}{1}$$

$$R(f,1) = 5-2$$

$$R(f,1) = 3$$

So,

$$\int_c \frac{5z-2}{z(z-1)}dz = 2\pi i(2+3) = 2\pi i(5) = 10\pi i$$

Ex:-

Show that $\int_c \frac{(1+z^5)\sinh z}{z^6} dz = \frac{\pi i}{60}$

where C is unit circle ($z=0$ (pole)) described in the positive direction

Sol:-

$$\int_c \frac{(1+z^5)\sinh z}{z^6} = 2\pi i R$$

$$= \frac{1+z^5}{z^6}\sinh z = \left(\frac{1}{z^6}+\frac{1}{z}\right)\left[z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right]$$

$$\frac{1+z^5}{z^6}\sinh z = \left\{\frac{1}{z^5}+\frac{1}{3!z^3}+\frac{1}{5!z}+\cdots\right\}\left[1+\frac{z^2}{3!}+\frac{z^4}{4!}+\cdots\right]$$

$$R(f_1,0) = \frac{1}{5!} = \frac{1}{120}$$

$$R(f_2,0) = 0$$

$$\Rightarrow R(f_1,0)+R(f_2,0) = \frac{1}{120}+0 = \frac{1}{120}$$

$$\int_c \frac{(1+z^5)\sinh z}{z^6} = 2\pi i \frac{1}{120} = \frac{\pi i}{60}$$

Ex:-

If $f(z)$ is a complex valued function having pole of order 2 at $z=0$ with residue 2, and also it has pole of order 1 at $z=1$ with residue 2, Further, $f(z)$ is analytic except function finite poles $z=0$ and $z=1$ and is bounded $|z| \to \infty$.

Find $f(z)$ when $f(2)=5$, $f(-1)=2$

Sol:-

$$f(z) = \frac{2}{z}+\frac{b_2}{z^2}+\frac{2}{z-1}$$

Since $f(z)$ is bounded as $|z| \to \infty$, then by Liouville's theorem.

$f(z) - \frac{2}{z} - \frac{b_2}{z^2} + \frac{2}{z-1} = \text{Constant (say } a)$

$f(z) = a + \frac{2}{z} + \frac{b_2}{z^2} + \frac{2}{z-1}$

$f(2) = 5$

$5 = a + \frac{2}{2} + \frac{b_2}{4} + \frac{2}{1} = \frac{4a+4+b_2+8}{4}$

$20 = 4a + b_2 + 12$

$4a + b_2 = 8 \rightarrow (1)$

$f(-1) = 2$

$2 = a - 2 + b_2 + \frac{2}{-2}$

$2+2 = a + b_2 - 1$

$4+1 = a + b_2$

$a + b_2 = 5 \rightarrow (2)$

Solve (1) and (2) Simultaneously

$4Eq(2) - Eq(1)$

$4a + 4b_2 = 20$

$\pm 4a \pm b_2 = \pm 8$

$3b_2 = 12$

$b_2 = 4$

put in (2)

$a + 4 = 5$

$\Rightarrow a = 1, \quad b_2 = 4$

$f(z) = 1 + \frac{2}{z} + \frac{4}{z^2} + \frac{2}{z-1}$

Check: $f(2) = 1 + \frac{2}{2} + \frac{4}{4} + \frac{2}{1}$

$= 1+1+1+2 = 5$

$f(-1) = 1 - 2 + 4 + \frac{2}{-2}$

$= -1 + 4 - 1$

$f(-1) = 2$

Formula:-

Statement:-

Let $f(z)$ be analytic inside and on a simple closed curve C except at a pole of order m inside C. Prove that the residue of $f(z)$ at "a" is given by $a_{-1} = \lim\limits_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$

or

$$R(f, a) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

Proof:-

If $f(z)$ has a pole of order m, then by Laurent's series of $f(z)$ is

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-(m-1)}}{(z-a)^{m-1}} + \cdots + \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots \rightarrow (1)$$

Multiplying both sides by $(z-a)^m$

$$(z-a)^m f(z) = a_{-m} + a_{-(m-1)}(z-a) + \cdots + (a_{-1})(z-a)^{m-1} + a_0(z-a)^m + a_1(z-a)^{m+1} + \cdots \rightarrow (2)$$

This represents the Tylor's series about $z = a$ of analytic function on the left. Differentiate both sides $(m-1)$ times with respect to z.

$$\frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] = (m-1)!(a_{-1}) + m(m-1)\cdots(z-a) \cdots + 2a_0(z-a) + \cdots$$

Thus, on letting $z \to a$

$$\lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] = (m-1)!(a_{-1})$$

$$\Rightarrow \quad a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)] \quad \text{(proved)}.$$

Contour Integration
(Applications of Cauchy's Residue Theorem)

Type I:-

Form of the integral is

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$$

where f is a rational function of $\sin\theta, \cos\theta$

Working Rule:-

Step I:-

Make the substitution $z = e^{i\theta}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ $\qquad dz = ie^{i\theta}d\theta = iz\,d\theta$

$d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

and the integral $\int_0^{2\pi} f(\theta)\,d\theta = \int_c f(z)\,dz$

Step II:-

Calculate the poles of $f(z)$, select those poles which lie in the unit circle $|z| = 1$. Then, finding the residues at the selected poles.

Step III:-

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta = \int_c f(z)\,dz$$

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta = 2\pi i \sum_{j=1}^{n} R_j$$

Ex:-

Prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2} = \frac{2\pi}{1-p^2}$; $0 < p < 1$

Sol:-

Step I:-

put $z = e^{i\theta}$ $\qquad$; $0 \le \theta \le 2\pi$; $C: |z| = 1$

$$z+\frac{1}{z} = 2\cos\theta$$

$$dz = ie^{i\theta}d\theta$$
$$dz = iz\,d\theta$$
$$d\theta = \frac{dz}{iz}$$

The integral becomes $\int_c \frac{\frac{dz}{iz}}{1-p(z+\frac{1}{z})+p^2}$

$$= \int_c \frac{1}{iz}\frac{dz}{\left(1-p\left(\frac{z^2+1}{z}\right)+p^2\right)} = \int_c \frac{i\,dz}{i^2(z-p(z^2+1)+p^2z)} = \frac{i^2}{-i^2p}\int_c \frac{dz}{\left(-\frac{1}{p}z+z^2+1+pz\right)}$$

$$= \frac{i}{p}\int_c \frac{dz}{z^2-(p+\frac{1}{p})z+1}$$

$$\int_c \frac{dz/iz}{1-p(z+\frac{1}{z})+p^2} = \frac{i}{p}\int_c \frac{dz}{z^2-(p+\frac{1}{p})z+1}$$

Step II :-

Take $z^2-(p+\frac{1}{p})z+1 = 0$

$z = p, \frac{1}{p}$

The poles of $f(z) = \frac{1}{z^2-(p+\frac{1}{p})z+1}$ are at

$z = p, \frac{1}{p}$

The only pole which lie in the unit circle is $z = p$

Residue $= R(f, p) = \frac{1}{(1-1)!}\frac{d^{1-1}}{dz^{1-1}}\left((z-p)^1 \frac{1}{(z^2-(p+\frac{1}{p})z+1)}\right)_{z=p}$

$\because z^2-(p+\frac{1}{p})z+1=0$

$\Rightarrow z^2-pz-\frac{1}{p}z+1=0$ $\qquad = \left((z-p)\frac{1}{(z-p)(z-\frac{1}{p})}\right)_{z=p}$

$\Rightarrow z(z-p)-\frac{1}{p}(z-p)$

$\Rightarrow (z-p)(z-\frac{1}{p})=0$ $\qquad = \left(\frac{1}{z-\frac{1}{p}}\right)_{z=p}$

$\Rightarrow z=p,\ z=\frac{1}{p}$

$$R(f, p) = \frac{1}{p-\frac{1}{p}} = \frac{1}{\frac{p^2-1}{p}}$$

$$= \frac{p}{p^2-1}$$

Step III :-

$$\frac{i}{p}\int_C \frac{dz}{z^2-(p+\frac{1}{p})z+1} = \frac{i}{p}\left(2\pi i \times \frac{p}{p^2-1}\right)$$

$$= \frac{-2\pi}{p^2-1}$$

$$= \frac{2\pi}{1-p^2}$$

So, $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2p\cos\theta+p^2} = \frac{2\pi}{1-p^2}$

Ex :-

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$; $a>b$

and use it to calculate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$

and Also show that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}} \quad ; \quad a>b>0$$

Sol :-

Step I :-

Put $z = e^{i\theta}$; $0 \le \theta \le 2\pi$, $C : |z| = 1$

$$z + \frac{1}{z} = 2\cos\theta$$

$$dz = ie^{i\theta}d\theta$$

$$d\theta = \frac{dz}{iz}$$

The given integral becomes

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_C \frac{\frac{dz}{iz}}{a+b\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}$$

$$= \frac{1}{i}\int \frac{dz}{z\left[\frac{2az+b(z^2+1)}{2z}\right]}$$

$$= \frac{2}{i}\int_c \frac{dz}{2az+bz^2+b}$$

$$= \frac{2}{ib}\int_c \frac{dz}{\frac{2a}{b}z+z^2+1}$$

$$= \frac{2}{ib}\int_c \frac{dz}{z^2+\frac{2a}{b}z+1}$$

Step II :-

Take $z^2+\frac{2a}{b}z+1=0$

$a=1,\ b=\frac{2a}{b},\ c=1$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\left(\frac{2a}{b}\right)^2-4(1)(1)}}{2(1)}$$

$$= \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$= \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$= \frac{-\frac{2a}{b} \pm \frac{2}{b}\sqrt{a^2-b^2}}{2}$$

$$= \frac{2}{b}\left[\frac{-a \pm \sqrt{a^2-b^2}}{2}\right]$$

$$z = \frac{-a+\sqrt{a^2-b^2}}{b}$$

The poles of $f(z) = \frac{1}{z^2+\frac{2a}{b}z+1}$ are

at $z = \alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$

and $z = \beta = \frac{-a - \sqrt{a^2-b^2}}{b}$

The only pole which lie in the unit circle is $z = \alpha$

$$R(f, \alpha) = \frac{1}{(1-1)!} \frac{d^{1-1}}{dz^{1-1}}\left((z-\alpha)^1 \frac{1}{(z-\alpha)(z-\beta)}\right)_{z=\alpha}$$

$$= \left(\frac{(z-\alpha)}{(z-\alpha)(z-\beta)}\right)_{z=\alpha}$$

$$= \left(\frac{1}{z-\beta}\right)_{z=\alpha}$$

$$= \frac{1}{\alpha-\beta}$$

$$R(f, \alpha) = \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} + \frac{a+\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1}{\frac{-\cancel{a}+\sqrt{a^2-b^2}+\cancel{a}+\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}}$$

$$R(f, \alpha) = \frac{b}{2\sqrt{a^2-b^2}}$$

Step III :-

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{ib} \int_C \frac{dz}{z^2 + \frac{2a}{b}z + 1}$$

$$= \frac{2}{ib}\left(2\pi i \times R\right)$$

$$= \frac{2}{ib}\left(2\pi i \times \frac{b}{2\sqrt{a^2-b^2}}\right)$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Now $$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{2\pi}{\sqrt{5^2-3^2}}$$

$$= \frac{2\pi}{\sqrt{25-9}} = \frac{2\pi}{\sqrt{16}} = \frac{2\pi}{4} = \frac{\pi}{2}$$

Now $$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Differentiate it w.r.t "a" on both sides.

$$\int_0^{2\pi} \frac{-1\, d\theta}{(a+b\cos\theta)^2} = \frac{-\cancel{1}\cancel{2}\pi(2a)}{\cancel{2}(a^2-b^2)^{3/2}}$$

$$-\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = -\frac{2\pi a}{(a^2-b^2)^{3/2}}$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{(a+b\cos\theta)^2}\, d\theta = \frac{2\pi a}{(a^2-b^2)^{3/2}}$$

Assignment :-

Ex :-

Prove that $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta}$; $a>0$

Sol :-

$$\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} \; ; \; a>0$$

put $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$$\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \int_0^{\pi} \frac{a\,d\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta}$$

put $2\theta = \phi$

$2d\theta = d\phi$

$$2\theta = \phi$$

at $\theta = 0$

$\phi = 0$ and at $\theta = \pi$, $\phi = 2\pi$.

$$\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \int_0^{2\pi} \frac{1}{2}\left(\frac{a\,d\phi}{2a^2+1-\cos\phi}\right)$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi}$$

Step I :-

put $z = e^{i\phi/2}$ ... $e^{i\phi}$; $0 \le \phi \le 2\pi$ $(\because |z|=1$

$$dz = i\phi e^{i\phi} d\phi$$

$$dz = iz\,d\phi$$

$$d\phi = \frac{dz}{iz}, \quad \cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\frac{1}{2}\int_0^{2\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi} = \frac{1}{2}\int_0^{2\pi} \frac{a\frac{dz}{iz}}{2a^2+1-\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{a\,dz}{z\left[2a^2+1-\left(\frac{z^2+1}{2z}\right)\right]}$$

$$= \frac{1}{2i}\int_0^{2\pi} \frac{a\,dz}{z\left[\frac{4a^2z+2z-z^2-1}{2z}\right]}$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{a\,dz}{4a^2z+2z-z^2-1}$$

$$= \frac{2}{-i}\int_0^{2\pi} \frac{a\,dz}{z^2-(4a^2+2)z+1}$$

$$= \frac{-2a}{i}\int_0^{2\pi} \frac{dz}{z^2-(4a^2+2)z+1}$$

Step II :-

Take $z^2-(4a^2+2)z+1=0$

$$z^2 - 2(2a^2+1)z + 1 = 0$$

$$a = 1, \quad b = -2(2a^2+1), \quad c = 1$$

$$z = \frac{2(2a^2+1) \pm \sqrt{4(4a^4+4a^2+1) - 4}}{2}$$

$$= \frac{2(2a^2+1) \pm 2\sqrt{4a^4+4a^2+1-1}}{2}$$

$$= \frac{2\left((2a^2+1) \pm 2\sqrt{a^4+a^2}\right)}{2}$$

$$z = (2a^2+1) \pm 2\sqrt{a^4+a^2}$$

$$z = \alpha = (2a^2+1) + 2\sqrt{a^4+a^2} \quad \therefore |\alpha| > 1$$

$$z = \beta = (2a^2+1) - 2\sqrt{a^4+a^2} \quad \therefore |\beta| < 1$$

The only pole which lie in the unit circle is β.

So, $$R(f, \beta) = \frac{1}{(1-1)!} \frac{d^0}{dz^0}\left((z-\beta)^1 \frac{1}{(z-\alpha)(z-\beta)}\right)_{z=\beta}$$

$$= \left(\frac{1}{z-\alpha}\right)_{z=\beta}$$

$$= \frac{1}{\beta - \alpha}$$

$$= \frac{1}{(\cancel{2a^2+1}) - 2\sqrt{a^4+a^2} - (\cancel{2a^2+1}) - 2\sqrt{a^4+a^2}}$$

$$R(f, \beta) = \frac{1}{-4\sqrt{a^4+a^2}}$$

Step III :-

$$-\frac{2a}{i}\int_0^{2\pi} \frac{dz}{z^2 - 2(2a^2+1)z + 1} = -\frac{2a}{i}(2\pi i \times R)$$

$$= -4a\pi \times \frac{1}{-4\sqrt{a^4+a^2}}$$

$$\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{a\pi}{\sqrt{a^4+a^2}} = \frac{\pi}{\sqrt{a^2+1}}$$

Ex :-

$$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p} \quad ; 0<p<1$$

Sol :-

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2} d\theta$$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{2\cos^2 3\theta}{1-2p\cos 2\theta + p^2} d\theta \qquad \because \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$2\cos^2\theta = 1+\cos 2\theta$$

$$2\cos^2(3\theta) = 1+\cos(6\theta)$$

$$I = \frac{1}{2}\int_0^{2\pi} \frac{(1+\cos 6\theta)\, d\theta}{1-2p\cos 2\theta + p^2} \qquad e^{i6\theta} = \cos 6\theta + i\sin 6\theta$$

$I =$ Real part of $\frac{1}{2}\int_0^{2\pi} \frac{(1+e^{i6\theta})\, d\theta}{1-2p\cos 2\theta + p^2}$

Step I :-

put $z = e^{i\theta}$ $\quad d\theta = \frac{dz}{iz}$

$\Rightarrow z^6 = e^{i6\theta}$

and $\cos 2\theta = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right)$

$$I = \frac{1}{2}\int_c \frac{1+z^6}{1-2p\left[\frac{1}{2}\left(z^2+\frac{1}{z^2}\right)\right]+p^2} \frac{dz}{iz}$$

$$= \frac{1}{2i}\int_c \frac{(1+z^6)\, dz}{1-p\left(z^2+\frac{1}{z^2}\right)+p^2}$$

Step II :-

Take $1-p\left(z^2+\frac{1}{z^2}\right)+p^2 = 0$

$$1-p\left(\frac{z^4+1}{z^2}\right)+p^2 = 0$$

$$z^2 - p(1+z^4) + p^2z^2 = 0$$

$$z^2 - p - pz^4 + p^2z^2 = 0$$

$$z^2 - pz^4 - p + p^2z^2 = 0$$

$$z^2(1-pz^2) - p(1-pz^2) = 0$$

$$(z^2-p)(1-pz^2)=0$$

$$z^2-p=0, \qquad 1-pz^2=0$$

$$z^2=p, \qquad 1=pz^2$$

$$z=\pm\sqrt{p}, \qquad \frac{1}{p}=z^2$$

$$z=\pm\sqrt{p}, \qquad z=\pm\sqrt{\frac{1}{p}}$$

The only pole which lie in unit circle is $z=\pm\sqrt{p}$

$$R_1(f,\sqrt{p})=\frac{1}{(1-1)!}\frac{d^0}{dz^0}\left(\frac{(z-\sqrt{p})z(z^6+1)}{(z\pm\sqrt{p})(1-pz^2)}\right)_{z=\sqrt{p}}$$

$$=\left(\frac{z(z^6+1)}{(z+\sqrt{p})(1-pz^2)}\right)_{z=\sqrt{p}}$$

$$R(f,\sqrt{p})=\frac{\sqrt{p}(p^3+1)}{(2\sqrt{p})(1-p^2)}=\frac{\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)}$$

$$R_2(f,-\sqrt{p})=\frac{-\sqrt{p}(1+p^3)}{-2\sqrt{p}(1-p^2)}=\frac{\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)}$$

$R=R_1+R_2$

$$R=\frac{\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)}+\frac{\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)}$$

$$=\frac{2\sqrt{p}(1+p^3)}{2\sqrt{p}(1-p^2)}=\frac{1+p^3}{1-p^2}$$

$$=\frac{(1+p)(p^2-p+1)}{(1+p)(1-p)}=\frac{p^2-p+1}{1-p}$$

Step III :-

$$\therefore \frac{1}{2i}\int_c \frac{1+z^6}{p^2-p(z^2+\frac{1}{z^2})+1}dz=\frac{1}{2i}(2\pi i\times R)$$

$$=\pi\frac{(p^2-p+1)}{1-p}$$

$$\int_0^{2\pi}\frac{\cos^2 3\theta\, d\theta}{1-2p\cos 2\theta+p^2}=\frac{\pi(1-p+p^2)}{1-p}$$

Ex :-

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{\pi}{12}$$

Sol :-

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta \qquad e^{i3\theta} = \cos 3\theta + i\sin 3\theta.$$

$$= \text{Real part of } \int_0^{2\pi} \frac{e^{i3\theta}}{5-4\cos\theta}\, d\theta$$

Step I :- put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$$\Rightarrow z^3 = e^{i3\theta}$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$\int_0^{2\pi} \frac{e^{i3\theta}}{5-4\cos\theta}\, d\theta = \int_0^{2\pi} \frac{z^3}{5-4\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)} \frac{dz}{iz}.$$

$$= \frac{1}{i}\int_c \frac{z^3}{5-2\left(\frac{z^2+1}{z}\right)} \frac{dz}{z}$$

$$= \frac{1}{i}\int_c \frac{z^2\, dz}{\frac{5z-2(z^2+1)}{z}}$$

$$= \frac{1}{i}\int_c \frac{z^3\, dz}{5z-2z^2-2}$$

$$= \frac{1}{-2i}\int_c \frac{z^3\, dz}{z^2-\frac{5}{2}z+1}$$

Step II :-

Take $z^2 - \frac{5}{2}z + 1 = 0$.

$a = 1$, $b = -\frac{5}{2}$, $c = 1$

$$z = \frac{\frac{5}{2} \pm \sqrt{\frac{25}{4}-4}}{2} = \frac{\frac{5}{2} \pm \sqrt{\frac{25-16}{4}}}{2}$$

$$z = \frac{5}{2} \pm \sqrt{9/4}\Big/2$$

$$z = \frac{\frac{5}{2} \pm \frac{3}{2}}{2}$$

$$z = \frac{\frac{1}{2}(5 \pm 3)}{2}$$

$$z = \frac{5 \pm 3}{4}$$

$$z = \frac{5+3}{4}, \qquad z = \frac{5-3}{4}.$$

$$z = 2, \qquad z = \frac{1}{2}$$

The only pole, which lie in unit circle is $z = \frac{1}{2}$

$$R(f, \tfrac{1}{2}) = \frac{1}{(1-1)!} \frac{d^0}{dz^0}\left((z-\tfrac{1}{2})^1 \frac{z^3}{(z-2)(z-\frac{1}{2})} \right)_{z=\frac{1}{2}}$$

$$= \left(\frac{z^3}{(z-2)} \right)_{z=\frac{1}{2}}$$

$$= \frac{(\frac{1}{2})^3}{\frac{1}{2}-2} = \frac{\frac{1}{8}4}{\frac{1-4}{2}} = \frac{1}{(1-4)4}$$

$$R(f, \tfrac{1}{2}) = -\frac{1}{18}$$

Step III :-

$$\therefore -\frac{1}{2i} \int_C \frac{z\,dz}{z^2 - \frac{5}{2}z + 1} = -\frac{1}{2i}(2\pi i \times R)$$

$$= -\pi \times -\frac{1}{18}$$

$$\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5 - 4\cos\theta} \, d\theta = \frac{\pi}{18}$$

Ex :-

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos^2\theta)^2} = \frac{\pi(2a+b)}{a^{3/2}(a+b)^{3/2}} \; ; \; a > b > 0$$

Sol :-

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2}$$

put $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$a+b\cos^2\theta = a+b\left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{1}{2}\left[2a+b(1+\cos 2\theta)\right]$$

$$a+b\cos^2\theta = \frac{1}{2}\left[2a+b+b\cos 2\theta\right]$$

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2} = \int_0^{2\pi} \frac{d\theta}{\left(\frac{2a+b+b\cos 2\theta}{2}\right)^2}$$

$$= \int_0^{2\pi} \frac{2\,d\theta}{2a+b+b\cos 2\theta}$$

put $2\theta = \phi$

$2\,d\theta = d\phi$

at $\theta = 0 \Rightarrow \phi = 0$

at $\theta = 2\pi \Rightarrow \phi = 4\pi$.

$$\int_0^{2\pi} \frac{2\,d\theta}{2a+b+b\cos 2\theta} = \int_0^{4\pi} \frac{d\phi}{2a+b+b\cos\phi}$$

is even function. So,

$$\int_0^{4\pi} \frac{d\phi}{2a+b+b\cos\phi} = 2\int_0^{2\pi} \frac{d\phi}{2a+b+b\cos\phi}$$

Step I :-

put $z = e^{i\phi}$ $\quad d\phi = \frac{dz}{iz}$

$\cos\phi = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$2\int_0^{2\pi} \frac{d\phi}{2a+b+b\cos\phi} = 2\int_0^{2\pi} \frac{\frac{dz}{iz}}{2a+b+b\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)}$$

$$= 2\int_0^{2\pi} \frac{1}{iz}\,\frac{dz}{\left[2a+b+b\left(\frac{z^2+1}{2z}\right)\right]}$$

$$= \frac{2}{i}\int_0^{2\pi} \frac{1}{z}\frac{dz}{\frac{4az+2bz+bz^2+b}{2z}}$$

$$= \frac{4}{i}\int_0^{2\pi} \frac{dz}{bz^2+(4a+2b)z+b}$$

$$= \frac{4}{ib}\int_0^{2\pi} \frac{dz}{z^2+\frac{2}{b}(2a+b)z+1}$$

$$= \frac{4}{ib}\int_0^{2\pi} \frac{dz}{z^2+2(2\frac{a}{b}+1)z+1}$$

Step II :-

Take $z^2+2(2\frac{a}{b}+1)z+1=0$.

$$z = \frac{-2(2\frac{a}{b}+1) \pm \sqrt{(-2(2\frac{a}{b}+1))^2-4}}{2}$$

$$= \frac{-2(2\frac{a}{b}+1) \pm \sqrt{4(\frac{4a^2}{b^2}+\frac{4a}{b}+1)-4}}{2}$$

$$= \frac{-2(2\frac{a}{b}+1) \pm 2\sqrt{4(a^2/b^2+a/b)}}{2}$$

$$= -(2\frac{a}{b}+1) \pm 2\sqrt{\frac{a^2}{b^2}+a/b}$$

$$= \frac{-(2a+b)}{b} \pm 2\sqrt{\frac{a^2+ab}{b^2}}$$

$$z = \frac{-(2a+b) \pm 2\sqrt{a(a+b)}}{b}$$

$$z = \alpha = \frac{-(2a+b)+2\sqrt{a(a+b)}}{b}$$

$$z = \beta = \frac{-(2a+b)-2\sqrt{a(a+b)}}{b}$$

$\because |\alpha| < 1, \quad |\beta| > 1$

The only pole which lie inside unit circle is $z=\alpha$

$$R(f,\alpha) = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left((z-\alpha)\frac{1}{(z-\alpha)(z-\beta)}\right)_{z=\alpha}$$

$$= \left(\frac{1}{z-\beta}\right)_{z=\alpha}$$

$$= \frac{1}{\alpha-\beta}$$

$$= \frac{1}{-(2a+b)+2\sqrt{a(a+b)}+(2a+b)+2\sqrt{a(a+b)}}$$

$$R(f,\alpha) = \frac{b}{4\sqrt{a(a+b)}} \qquad \frac{b}{}$$

Step III :-

$$\frac{4}{ib}\int_0^{2\pi}\frac{dz}{z^2+2(2\frac{a}{b}+1)z+1} = \frac{4}{ib}\,2\pi i\times R$$

$$= \frac{8\pi}{b}\,\frac{b}{4\sqrt{a(a+b)}}$$

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos^2\theta} = \frac{2\pi}{\sqrt{a(a+b)}}$$

we have to calculate

$$\int_0^{2\pi}\frac{d\theta}{(a+b\cos^2\theta)^2}$$

Differentiate w.r.t "a" on both sides.

$$-\int_0^{2\pi}\frac{d\theta}{(a+b\cos^2\theta)^2} = \frac{-1}{2}\frac{2\pi}{(a(a+b))^{3/2}}\frac{d}{da}(a(a+b))$$

$$= \frac{-\pi}{(a(a+b))^{3/2}}\frac{d}{da}(a^2+ab)$$

$$-\int_0^{2\pi}\frac{d\theta}{(a+b\cos^2\theta)^2} = \frac{-\pi}{a^{3/2}(a+b)^{3/2}}(2a+1.b)$$

$$\int_0^{2\pi}\frac{d\theta}{(a+b\cos^2\theta)^2} = \frac{\pi(2a+b)}{a^{3/2}(a+b)^{3/2}}$$

Ex:-

$$\int_0^{2\pi} \frac{d\theta}{(a-b\sin\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}, \quad a>b>0$$

Sol:-

$$\int_0^{2\pi} \frac{d\theta}{(a-b\sin\theta)^2}$$

$$\int_0^{2\pi} \frac{d\theta}{a-b\sin\theta}$$

Step I:- put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$\sin\theta = \frac{1}{2}\left(z-\frac{1}{z}\right)$

$$\int_0^{2\pi} \frac{d\theta}{a-b\sin\theta} = \int_0^{2\pi} \frac{\frac{dz}{iz}}{a-b\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}$$

$$= \frac{1}{i}\int_0^{2\pi} \frac{dz}{z\left[a-\frac{b}{2i}\left(\frac{z^2-1}{z}\right)\right]}$$

$$= \frac{1}{i}\int_0^{2\pi} \frac{dz}{z\left[\frac{2aiz-bz^2+b}{2iz}\right]}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{dz}{-bz^2+2aiz+b}$$

$$\int_0^{2\pi} \frac{d\theta}{a-b\sin\theta} = \frac{-2}{ib}\int_0^{2\pi} \frac{dz}{z^2-\frac{2a}{b}iz-1}$$

Step II:-

Take $z^2 - \frac{2a}{b}iz - 1 = 0$

$a = 1, \quad b = -\frac{2a}{b}i, \quad c = -1$

$$z = \frac{+\frac{2a}{b}i \pm \sqrt{-\frac{4a^2}{b^2}+4}}{2} \qquad \because i^2 = -1$$

$$z = \frac{-\frac{2ai}{b} \pm \sqrt{\frac{-4a^2+4b^2}{b^2}}}{2}$$

$$z = \frac{-\frac{2ai}{b} \pm \sqrt{\frac{4b^2-4a^2}{b^2}}}{2}$$

$$z = \frac{-2ai \pm 2\sqrt{b^2-a^2}}{2b}$$

$$z = \frac{-ai \pm \sqrt{b^2-a^2}}{b}$$

$$z = \alpha = \frac{-ai + \sqrt{b^2-a^2}}{b}$$

$$z = \beta = \frac{-ai - \sqrt{b^2-a^2}}{b}$$

The only pole which lie in unit circle is

$$z = \alpha = \frac{ai + \sqrt{b^2-a^2}}{b}$$

$$R(f, \alpha) = \left((z-\alpha) \frac{1}{(z-\alpha)(z-\beta)} \right)_{z=\alpha}$$

$$= \left(\frac{1}{z-\beta} \right)_{z=\alpha}$$

$$= \frac{1}{\alpha-\beta}$$

$$= \frac{1}{\frac{ai+\sqrt{b^2-a^2}}{b} - \left(\frac{ai - \sqrt{b^2-a^2}}{b} \right)}$$

$$= \frac{b}{ai + \sqrt{b^2-a^2} - ai + \sqrt{b^2-a^2}}$$

$$R(f, \alpha) = \frac{b}{2\sqrt{b^2-a^2}}$$

By cauchy Residue theorem.

$$\int_0^{2\pi} \frac{d\theta}{(a-b\sin\theta)} = -\frac{2}{b}\int_c f(z)\,dz$$

$$= -\frac{2}{b} \times 2\pi i \times \frac{b}{2\sqrt{b^2-a^2}}$$

$$\int_0^{2\pi} \frac{d\theta}{a-b\sin\theta} = \frac{-2\pi i}{\sqrt{b^2-a^2}}$$

Differentiate w.r.t "a"

$$\int_0^{2\pi} \frac{-d\theta}{(a-b\sin\theta)^2} = -\left(\frac{-2\pi i(-2a)}{2(b^2-a^2)^{3/2}}\right)$$

$$-\int_0^{2\pi} \frac{d\theta}{(a-b\sin\theta)^2} = \frac{-2\pi a i}{(b^2-a^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{d\theta}{(a-b\sin\theta)^2} = 2\pi a i/(b^2-a^2)^{3/2}$$

Ex:-

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{2\pi}{n!}$$

Sol:-

$$\int_0^{2\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\,d\theta$$

$I =$ Real part of $\int_0^{2\pi} e^{\cos\theta}\, e^{-i(n\theta - \sin\theta)}\,d\theta$

$$I = \int_0^{2\pi} e^{\cos\theta - in\theta + i\sin\theta}\,d\theta$$

$$= \int_0^{2\pi} e^{\cos\theta + i\sin\theta}\cdot e^{-in\theta}\,d\theta$$

Step I:-

put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} e^{\cos\theta + i\sin\theta}\cdot e^{-in\theta}\,d\theta = \int_0^{2\pi} e^{e^{i\theta}}\cdot e^{-in\theta}\,d\theta$$

$$\therefore e^{i\theta} = \cos\theta + i\sin\theta.$$

$$\int_0^{2\pi} \frac{e^{i\theta}}{e^{in\theta}} \, d\theta = \int_0^{2\pi} \frac{e^z}{z^n} \frac{dz}{iz}.$$

$\because e^{in\theta} = z^n$

$$= \frac{1}{i} \int_0^{2\pi} \frac{e^z}{z^{n+1}} \, dz.$$

Step II :-

$f(z)$ has a pole of order $n+1$ at $z=0$.

$$R(f,0) = \frac{1}{(n+1-1)!} \frac{d^{n+1-1}}{dz^{n+1-1}} \left((z-0)^{n+1} \frac{e^z}{z^{n+1}} \right)_{z=0}.$$

$$= \frac{1}{n!} \frac{d^n}{dz^n} \left(z^{n+1} \frac{e^z}{z^{n+1}} \right)_{z=0}$$

$$= \frac{1}{n!} \frac{d^n}{dz^n} \left(e^z \right)_{z=0}.$$

$$= \frac{1}{n!} \left(e^z \right)_{z=0}$$

$$= \frac{1}{n!} e^0$$

$$= \frac{1}{n!}$$

Step III :-

$$\frac{1}{i} \int_0^{2\pi} \frac{e^z}{z^{n+1}} \, dz = \frac{1}{i} \left(2\pi i \times R \right)$$

$$\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) \, d\theta = \frac{2\pi}{n!}$$

Type II :-

Form the integral will be either $\int_{-\infty}^{\infty} f(x)dx$ or $\int_0^{\infty} f(x)dx$

Working Rule:-

Step I:-

Replace x by z in the integrand and test whether $zf(z) \to 0$ as $|z| \to \infty$

Step II :-

Find the poles of $f(z)$. Locate those poles which lie in the upper half plane. Find the residue at the located poles.

Step III :-

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{n} R_j$$

Ex :-

Prove that $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+a^2}\,dx = \frac{\pi}{2a}$; $a>0$

Sol :-

$$f(x) = \frac{1}{x^2+a^2}$$

Step I :- Replace "x" by "z"

$$f(z) = \frac{1}{z^2+a^2}$$

$$z(f(z)) = \frac{z}{z^2+a^2}$$

$$= \frac{z}{z^2\left[1+\frac{a^2}{z^2}\right]}$$

$$= \frac{1}{z\left[1+\frac{a^2}{z^2}\right]}$$

$$= \frac{\frac{1}{z}}{1+\frac{a^2}{z^2}} \to 0 \text{ as } |z| \to \infty$$

Step II :-

The poles of $f(z) = \dfrac{1}{z^2+a^2}$ are at $z = \pm ai$. The only pole which lie in the upper half plane is $z = ai$

$$R(f, ai) = \frac{1}{(1-1)!} \frac{d^0}{dz^0}\left((z-ai)\frac{1}{(z-ai)(z+ai)}\right)_{z=ai}$$

$$= \left(\frac{1}{z+ai}\right)_{z=ai}$$

$$= \frac{1}{ai+ai} = \frac{1}{2ai}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx = 2\pi i \times \frac{1}{2ai}$$

$$= \frac{\pi}{a}$$

$$2\int_{0}^{\infty} \frac{1}{x^2+a^2}\,dx = \frac{\pi}{a}$$

$$\int_{0}^{\infty} \frac{1}{x^2+a^2}\,dx = \frac{\pi}{2a} \quad ; \quad a>0$$

Ex :-

Evaluate $\int_{0}^{\infty} \frac{1}{x^4+a^4}\,dx \quad , \quad a>0$

Sol :-

$$f(x) = \frac{1}{x^4+a^4}$$

Step I :- Replace "x" by z

$$f(z) = \frac{1}{z^4+a^4}$$

$$z\,f(z) = \frac{z}{z^2\left[z^2+\frac{a^4}{z^2}\right]}$$

$$= \frac{1}{z\left[z^2+\frac{a^4}{z^2}\right]}$$

$$= \frac{\frac{1}{z}}{z^2+\frac{a^4}{z^2}} \to 0 \quad , \quad |z| \to \infty$$

Step II :-

The poles of $f(z) = \frac{1}{z^4+a^4}$

are all $z^4 + a^4 = 0$, $z^4 = -a^4 = a^4[\cos(2k+1)\pi + i\sin(2k+1)\pi]$

$z = a[\cos(2k+1)\pi + i\sin(2k+1)\pi]^{1/4} = a[\cos\frac{1}{4}(2k+1)\pi + i\sin\frac{1}{4}(2k+1)\pi]$

$z_0 = \alpha = a[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}] = a(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$ $\quad k = 0, 1, 2, 3$

$z_1 = \beta = a(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}) = a(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}})$, $z_2 = a[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i] = \gamma$

$z_3 = \delta = a[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}]$

The only poles which lie in upper half plane are α and β

$$R(f,\alpha) = \left((z-\alpha)\frac{1}{(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)}\right)_{z=\alpha} = \frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}$$

$$= \frac{1}{a^3[\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}][\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}][\frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}]}$$

$$= \frac{1}{a^3(\frac{2}{\sqrt{2}})(\frac{2}{\sqrt{2}} + \frac{2i}{\sqrt{2}})(\frac{2}{\sqrt{2}}i)} = \frac{1}{a^3(\sqrt{2}(\sqrt{2}+\sqrt{2}i)(\sqrt{2}i)}$$

$$= \frac{1}{a^3 i\, 2\cdot\sqrt{2}(1+i)} = \frac{1}{2\sqrt{2}a^3 i(1+i)} = \frac{i(1-i)}{-2\sqrt{2}a^3(1+i)(1-i)}$$

Step III $= \frac{i(1-i)}{-4\sqrt{2}a^3} = \frac{1+i}{-4\sqrt{2}a^3}$ Similarly, $R(f,\beta) = \frac{-1+i}{-4\sqrt{2}a^3}$

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\pi i \sum_j R_j$$

$$= 2\pi i\left[\frac{1+i}{-4\sqrt{2}a^3} + \frac{-1+i}{-4\sqrt{2}a^3}\right]$$

$$= 2\pi i\left[\frac{+1+i-1+i}{-4\sqrt{2}a^3}\right]$$

$$= 2\pi i\left[\frac{2i}{-4\sqrt{2}a^3}\right]$$

$$= \frac{\pi i^2}{-\sqrt{2}a^3}$$

$$2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{\sqrt{2}a^3}$$

$$\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3}$$

Ex:-

Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+1}{(x^2+1)(x^2+9)}\,dx$

Sol:-

$$f(x) = \frac{x^2+x+1}{(x^2+1)(x^2+9)}$$

Step I:-

Replace "x" by "z"

$$f(z) = \frac{z^2+z+1}{(z^2+1)(z^2+9)}$$

$$z\,f(z) = \frac{z(z^2+z+1)}{(z^2+1)(z^2+9)}$$

$$= \frac{z \cdot \cancel{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}\right)}{z^2\left(1+\frac{1}{z^2}\right)\cdot z^2\left(1+\frac{9}{z^2}\right)}$$

$$= \frac{\left(1+\frac{1}{z}+\frac{1}{z^2}\right)}{z\left(1+\frac{1}{z^2}\right)\left(1+\frac{9}{z^2}\right)}$$

$$z\,f(z) = \frac{\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}\right)}{\left(1+\frac{1}{z^2}\right)\left(1+\frac{9}{z^2}\right)}$$

$z\,f(z) \to 0$ as $|z| \to \infty$

Step II:-

The poles of $f(z) = \frac{1}{(z^2+1)(z^2+9)}$

are $z = \pm i$ and $z = \pm 3i$

The only pole which lie in the upper half plane are $z = i, 3i$

then $R(f, i) = \frac{1}{(1-1)!} \frac{d^0}{dz^0}\left((z-i)\left(\frac{z^2+z+1}{(z^2+1)(z^2+9)}\right)\right)_{z=i}$

$$R_1(f,i) = \left(\cancel{(z-i)}\,\frac{z^2+z+1}{\cancel{(z-i)}(z+i)(z^2+9)}\right)_{z=i}$$

$$= \left(\frac{z^2+z+1}{(z+i)(z^2+9)}\right)_{z=i}$$

$$= \frac{i^2+i+1}{(i+i)(i^2+9)}$$

$$= \frac{-1+i+1}{(2i)(-1+9)} = \frac{i}{2i(8)} = \frac{i}{16i} = \frac{1}{16}$$

$$R_2(f,3i) = \left(\cancel{(z-3i)}\,\frac{z^2+z+1}{(z^2+1)\cancel{(z-3i)}(z+3i)}\right)_{z=3i}$$

$$= \left(\frac{z^2+z+1}{(z^2+1)(z+3i)}\right)_{z=3i}$$

$$= \frac{(3i)^2+3i+1}{((3i)^2+1)(3i+3i)}$$

$$= \frac{9i^2+3i+1}{(9i^2+1)(6i)} = \frac{-9+3i+1}{(-9+1)(6i)}$$

$$= \frac{-8+3i}{-8(6i)}$$

$$= \frac{-8+3i}{-48i}$$

$$R_2(f,3i) = \frac{(8-3i)}{48i}$$

$$R = R_1 + R_2$$

$$R = \frac{1}{16} + \frac{8-3i}{48i}$$

$$= \frac{\cancel{3i}+8-\cancel{3i}}{48i}$$

$$R = \frac{8}{48i} = \frac{1}{6i}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{x^2+x+1}{(x^2+1)(x^2+9)}\,dx = 2\pi i \times R$$

$$= 2\pi i \times \frac{1}{6i}$$

$$\int_{-\infty}^{\infty} \frac{x^2+x+1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{3}$$

Assignment :-

(i) $\displaystyle\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{3\pi}{16a^5}, \quad a>0.$

Sol :-

$$f(x) = \frac{1}{(x^2+a^2)^3}$$

Step I :-

Replace "x" by "z"

$$f(z) = \frac{1}{(z^2+a^2)^3}$$

$$z\,f(z) = \frac{1\cdot z}{(z^2+a^2)^3}.$$

$$= \frac{z}{(z^2)^3\left[1+\frac{a^2}{z^2}\right]^3}$$

$$= \frac{z}{z^6\left[1+\frac{a^2}{z^2}\right]^3}$$

$$= \frac{1}{z^5\left[1+\frac{a^2}{z^2}\right]^3}$$

$$z\,f(z) = \frac{\frac{1}{z^5}}{\left[1+\frac{a^2}{z^2}\right]^3}$$

$z f(z) \to 0 \quad as \quad |z| \to \infty$

Step II :-

The function $f(z) = \dfrac{1}{(z^2+a^2)^3}$

has pole of order $m=3$ at $z = \pm ai$
The only pole which lie in the upper half is $z = ai$

$$R(f, ai) = \frac{1}{(3-1)!} \frac{d^{3-1}}{dz^{3-1}} \left((z-ai)^3 \frac{1}{(z^2+a^2)^3} \right)_{z=ai}$$

$$= \frac{1}{2!} \frac{d^2}{dz^2} \left((z-ai)^3 \frac{1}{(z+ai)^3 (z-ai)^3} \right)_{z=ai}$$

$$= \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{1}{(z+ai)^3} \right)_{z=ai}$$

$$= \frac{1}{2} \frac{d}{dz} \left(\frac{0 + 3(z+ai)^2}{(z+ai)^6} \right)_{z=ai}$$

$$= \frac{1}{2} \frac{d}{dz} \left(\frac{3(z+ai)^2}{(z+ai)^6} \right)_{z=ai}$$

$$= \frac{3}{2} \frac{d}{dz} \left(\frac{1}{(z+ai)^4} \right)_{z=ai}$$

$$= \frac{3}{2} \left(\frac{0 + 4(z+ai)^3}{((z+ai)^4)^2} \right)_{z=ai}$$

$$= \frac{3}{\cancel{2}} \left(\frac{\overset{2}{\cancel{4}}(z+ai)^3}{(z+ai)^8} \right)$$

$$R(f, ai) = 3 \left(\frac{2}{(z+ai)^5} \right)$$

$$R(f, ai) = \left(\frac{6}{(z+ai)^5}\right)_{z=ai} = \frac{6}{(2ai)^5}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^3} = 2\pi i \times \frac{6}{2^5 a^5 i^5}$$

$$= 2\pi i \times \frac{6}{32 a^5 i}$$

$i^5 = i^2 \times i^2 \cdot i$

$i^5 = (-1)(-1)\cdot i$

$i^5 = i$

$$2\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{6\pi}{16a^5}$$

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^3} = \frac{1}{2}\left(\frac{6\pi}{16a^5}\right)$$

$$= \frac{3\pi}{16a^5}, \quad a > 0$$

(ii)

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(a+bx^2)^4} = \frac{\pi}{16 a^{3/2} b^{5/2}}; \quad a > b > 0$$

Sol :-

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(a+bx^2)^4}$$

$$f(x) = \frac{x^4}{(a+bx^2)^4}$$

Replace x by z

$$f(z) = \frac{z^4}{(a+bz^2)^4}$$

$$z f(z) = \frac{z \cdot z^4}{(z^2)^4\left(b + \frac{a}{z^2}\right)^4}$$

$$zf(z) = \frac{z^{9}}{z^{8}\left(b+\frac{a}{z^{4}}\right)^{4}}$$

$$zf(z) = \frac{1}{z^{9}\left(b+\frac{a}{z^{4}}\right)^{4}}$$

$$zf(z) = \frac{\frac{1}{z^{9}}}{\left(b+\frac{a}{z^{2}}\right)^{4}}$$

$$zf(z) \to 0 \quad \text{as} \quad |z| \to \infty$$

Step II :-

Consider the integral $\int_{-\infty}^{\infty} \frac{1}{a+bz^2}\,dz$

$bz^2 = -a$

$$\int_{-\infty}^{\infty} \frac{dz}{a+bz^2} = \int_{-\infty}^{\infty} \frac{dz}{b\left(\frac{a}{b}+z^2\right)} \qquad -\frac{a}{b}$$

The integral $\frac{1}{b}\int_{-\infty}^{\infty} \frac{dz}{z^2+\frac{a}{b}}$ which lie in the upper half is ... of order $m=4$

The poles of $f(z)$ are the roots of

$$(a+bz^2) = 0$$

$$bz^2 = -a$$

$$z^2 = -\frac{a}{b}$$

$$z = \pm\sqrt{\frac{a}{b}}\,i$$

The only pole which lie in the upper half plane is $z = \sqrt{\frac{a}{b}}\,i$

$$R\left(f, \sqrt{\frac{a}{b}}\,i\right) = \left(\left(z-\sqrt{\frac{a}{b}}\,i\right)\frac{1}{\left(z-\sqrt{\frac{a}{b}}\,i\right)\left(z+\sqrt{\frac{a}{b}}\,i\right)}\right)_{z=\sqrt{\frac{a}{b}}\,i}$$

$$= \left(\frac{1}{z+\sqrt{\frac{a}{b}}\,i}\right)_{z=\sqrt{\frac{a}{b}}\,i}$$

$$R\left(f, \sqrt{\tfrac{a}{b}}\, i\right) = \frac{1}{2\sqrt{\frac{a}{b}}\, i}$$

$$R\left(f, \sqrt{\tfrac{a}{b}}\, i\right) = \frac{1}{2i\sqrt{\frac{a}{b}}}$$

Step III

$$\int_{-\infty}^{\infty} \frac{dx}{a+bx^2} = \frac{1}{b}\int_{-\infty}^{\infty} \frac{dz}{z^2+\frac{a}{b}}$$

$$= \frac{1}{b}\left[2\pi i \times \frac{1}{2i\sqrt{\frac{a}{b}}}\right]$$

$$\int_{-\infty}^{\infty} \frac{dx}{a+bx^2} = \frac{\pi}{b\sqrt{\frac{a}{b}}} = \frac{\pi}{\sqrt{ab}} \rightarrow (1)$$

Differentiate w.r.t b

$$\int_{-\infty}^{\infty} \frac{dx}{(a+bx^2)^2}(-x^2) = \frac{\pi}{\sqrt{a}}\left(\frac{-1}{2(b)^{3/2}}\right)$$

$$\int_{-\infty}^{\infty} \frac{-x^2\,dx}{(a+bx^2)^2} = -\frac{\pi}{2\sqrt{a}\,b^{3/2}}$$

Again Differentiate it w.r.t b

$$\int_{-\infty}^{\infty} \frac{(a+bx^2)^2(0) - x^2(2(a+bx^2))(x^2)}{(a+bx^2)^4}\,dx = \frac{\pi}{2\sqrt{a}}\left(\frac{-3}{2}\right) b^{-5/2}$$

$$\int_{-\infty}^{\infty} \frac{-2x^4}{(a+bx^2)^3}\,dx = \frac{\pi}{\sqrt{a}}\left(-\frac{3}{4}\right) b^{-5/2}$$

$$\frac{2}{3b}\int_{-\infty}^{\infty}\frac{2x^4(z+\frac{a}{b}i)^5}{(a+bx^2)^3}\,dx \left[\frac{3\sqrt{\pi}}{4\sqrt{a}}\,3\frac{a}{b}^{-5/2}(2-2z)+4z^2(0-2)\right]$$

$$(z+\frac{a}{b}i)^{12}$$

$$\int_{-\infty}\frac{12x^4(z+\frac{a}{b}i)^5}{(a+bx^2)^3}\left(4\frac{3\sqrt{\pi}}{4\sqrt{a}}\,\frac{3a}{b^{5/2}}i-2z\right)\Bigg]_{z=\frac{a}{b}i}$$

$$=\frac{2}{3}\frac{a}{b}\left[\text{Differentiate w.r.t. } a \text{ at } z = \frac{a}{b}\ -16z^2-3z^2\right]+6$$

$$(z+\frac{a}{b}i)^5(12\frac{a}{b}iz^2-2z^3)$$

$$\int_{-\infty}^{\infty}\frac{(a+bx^2)^3(0)b-2x^4\cdot 3(a+bx^2)\,dx}{\ }=\frac{3}{4}\frac{\pi}{b^{5/2}}\,a^{-3/2}$$

$$\frac{2}{3}\frac{a}{b}i\left[(z+\frac{a}{b}i)^6\right]\left[24\frac{a}{b}iz-3z^2\right]^2+$$

$$\int_{-\infty}\frac{b-6x^4(a+bx^2)^2\,dx}{(a+bx^2)^6(12\frac{a}{b}iz^2-3z^3)}=\frac{b\,3\pi}{8\,a^{3/2}b^{5/2}}\Bigg|_{z=\frac{a}{b}i}$$

$$=\frac{2}{3}\frac{a}{b}\int_{-\infty}^{\infty}\frac{-6x^4\,dx}{(a+bx^2)^{6+3}}\left(\frac{a}{b}i\right)^2=\left[\left(\frac{3\pi}{8a^{3/2}b^{5/2}}\frac{a}{b}i\right)\left(24\frac{a}{b}i\frac{a}{b}i\right.\right.$$

$$\left.-24(\frac{a}{b}i)^2\right]+6\left(12\frac{a}{b}i\left(\frac{a}{b}i\right)^2-8\left(\frac{a}{b}i\right)^3\right]$$

$$\Rightarrow\frac{2}{3}\frac{a}{b}i\int\frac{6x^4\,dx}{(a+bx^2)^4}=\frac{3\pi}{8a^{3/2}b^{5/2}}\left(24\frac{a^2}{b^2}i^2-24\frac{a^2}{b^2}i^2\right)$$

$$\Rightarrow+6\int_{-\infty}^{\infty}\frac{x^4\,dx}{(a+bx^2)^4}\,\frac{a^2}{b^2}\,\frac{3\pi}{48a^{3/2}b^{5/2}}-2\frac{a^3}{b^3}i^3\Bigg]\Bigg|_{i^2=-1}$$

$$=\frac{2}{3}\frac{a}{b}i\left[\ \ \right]+6\left(12\frac{a^3}{b^3}i(-1)-8\frac{a^3}{b^3}(-i)\right]$$

$$=\frac{2}{3}\frac{a}{b}i\int_{-\infty}^{\infty}\left[\frac{x^4\,dx}{(a+bx^2)^4}i\right]=\frac{3\pi}{16a^{3/2}ib^{5/2}}$$

$$=\frac{2}{3}\frac{a}{b}i\int_{-\infty}^{\infty}\frac{x^4\,dx}{(a+bx^2)^4}=\frac{\pi}{16\,a^{3/2}b^{5/2}};\ a>b>0.$$

Q:- $12\frac{a^4}{b^4}i^2$

$$\int_{-\infty}^{\infty}\frac{x^4\,dx}{(a+bx^2)^4}\,b^4\,dx=\frac{\pi}{48a^{3/2}b^{3/2}};\ a>b>0$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{(a+bx^2)^4} = 2\pi i \times R$$

$$= 2\pi i \times \frac{128}{3}\frac{a^4}{b^4}$$

(iii)

$$\int_{0}^{\infty} \frac{dx}{(x^4+a^4)^2} = \frac{3\pi}{8\sqrt{2}\,a^7} \; ; \quad a>0$$

Sol:-

$$f(x) = \frac{1}{(x^4+a^4)^2}$$

Step I :- Replace x by z

$$f(z) = \frac{1}{(z^4+a^4)^2}$$

$$z\,f(z) = \frac{z}{(z^4)^2\left[1+\frac{a^4}{z^4}\right]^2}$$

$$z\,f(z) = \frac{z}{z^8\left[1+\frac{a^4}{z^4}\right]^2}$$

$$z\,f(z) = \frac{1}{z^7\left[1+\frac{a^4}{z^4}\right]^2}$$

$$z\,f(z) = \frac{1/z^7}{\left[1+\frac{a^4}{z^4}\right]^2}$$

$z\,f(z) \to 0$ as $|z| \to \infty$

Step II :- Consider the integral $\int_0^{\infty} \frac{dx}{x^4+a^4}$

Take $z^4 + a^4 = 0$

$\Rightarrow z^4 = -a^4$

$z^4 = a^4 [\cos(2k+1)\pi + i\sin(2k+1)\pi]$

$z = a[\cos(2k+1)\frac{\pi}{4} + i\sin(2k+1)\frac{\pi}{4}]$

where $k = 0, 1, 2, 3$.

when $k = 0$ poles of z are

$z = a[\cos(2(0)+1)\frac{\pi}{4} + i\sin(2(0)+1)\frac{\pi}{4}]$

$z = a[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}] = ae^{i\pi/4}$

when $k = 1$

$z = a[\cos\frac{\pi}{4}(2+1) + i\sin(2+1)\frac{\pi}{4}]$

$z = a[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}] = ae^{i3\pi/4}$

when $k = 2$

$z = a[\cos\frac{\pi}{4}(2(2)+1) + i\sin\frac{\pi}{4}(2(2)+1)]$

$z = a[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}] = ae^{i5\pi/4}$

when $k = 3$

$z = a[\cos\frac{\pi}{4}(2(3)+1) + i\sin\frac{\pi}{4}(2(3)+1)]$

$z = a[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}] = ae^{i7\pi/4}$.

The only poles which lie in the upper half plane are $ae^{i\pi/4}$ and $ae^{i\frac{3\pi}{4}}$

let $z = \beta$ denote any one of these poles. such that in $z^4 + a^4 = 0$

we have $z^4 = \beta^4 = -a^4$

$$R_1(f, \beta) = \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-\beta)^2 \frac{1}{(z^2+a^4)^2}\right]_{z=\beta}$$

$$= \frac{d}{dz}\left[\frac{1}{(z+\beta)^2}\right]_{z=\beta}$$

$$= \left[\frac{(z+\beta)^2(0) - 2(z+\beta)}{(z+\beta)^4}\right]_{z=\beta}$$

$$= \left[\frac{2(z+\beta)}{(z+\beta)^4}\right]_{z=\beta^2}$$

$$= \left[\frac{2}{(z+\beta)^3}\right]_{z=\beta}$$

$$= \frac{2}{(\beta+\beta)^3} = \frac{2}{(2\beta)^3} = \frac{2}{8\beta^3} = \frac{1}{4\beta^3}$$

$$R_1(f,\beta) = \frac{\beta}{4\beta^4} = \frac{\beta}{-4a^4}$$

$$\Rightarrow R_1[f, e^{i\pi/4}] = -\frac{e^{i\pi/4}}{4a^4}$$

$$R_2[f, e^{3\pi/4 i}] = -\frac{e^{3\pi/4 i}}{4a^4}$$

$$R = R_1 + R_2 = -\frac{e^{\pi/4 i}}{4a^4} - \frac{e^{3\pi/4 i}}{4a^4}$$

$$= -\frac{1}{4a^4}\left[e^{\pi/4 i} + e^{3/4\pi i}\right]$$

$$= -\frac{1}{4a^4}\left[\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right]$$

$$= -\frac{1}{4a^4}\left[\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\right]$$

$$= -\frac{1}{4a^4}\left[\cancel{\frac{1}{\sqrt{2}}} + \frac{i}{\sqrt{2}} - \cancel{\frac{1}{\sqrt{2}}} + \frac{i}{\sqrt{2}}\right]$$

$$= -\frac{1}{4a^4}\left\{\frac{2i}{\sqrt{2}}\right\}$$

$$= -\frac{i}{2\sqrt{2}\,a^4}$$

Step III :— $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4} = 2\pi i \times -\frac{i}{2\sqrt{2}a^4}$

we already calculated

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}\,a^3}$$

Differentiate it w.r.t "a"

$$-\int_0^\infty \frac{dx}{(x^4+a^4)^2}\,4a^3 = \frac{\pi}{2\sqrt{2}}\left(-\frac{3}{a^4}\right)$$

$$\therefore \int_0^\infty \frac{dx}{(x^4+a^4)^2} = \frac{3\pi}{8a^3a^4\sqrt{2}}$$

$$= \frac{3\pi}{8\sqrt{2}\,a^7}$$

$$\int_0^\infty \frac{dx}{(x^4+a^4)^2} = \frac{3\pi}{8\sqrt{2}\,a^7}$$

Q(v)

$$\int_{-\infty}^{\infty} \frac{x^2+2x+1}{x^4+8x^2+16}\,dx = \frac{5\pi}{16}$$

Sol:-

$$f(x) = \frac{x^2+2x+1}{x^4+8x^2+16}$$

Step I:-

Replace "x" by z.

$$f(z) = \frac{z^2+2z+1}{z^4+8z^2+16}$$

$$zf(z) = \frac{z\cdot z^2\left[1+\frac{2}{z}+\frac{1}{z^2}\right]}{z^4\left[1+\frac{8}{z^2}+\frac{16}{z^4}\right]}$$

$$= \frac{\left[\frac{1}{z^2}+\frac{2}{z}+1\right]}{z\left[\frac{16}{z^4}+\frac{8}{z^2}+1\right]}$$

$$zf(z) = \frac{\frac{1}{z}\left[\frac{1}{z^2}+\frac{2}{z}+1\right]}{\left[\frac{16}{z^4}+\frac{8}{z^2}+1\right]}$$

$zf(z) \to 0$ as $|z| \to \infty$.

Step II :-

$(z^2+4)^2 = z^4 + 8z^2 + 16 = 0$

$z^4 + 4z^2 + 4z^2 + 16 = 0$

$z^2(z^2+4) + 4(z^2+4) = 0$

$(z^2+4)(z^2+4) = 0$

$\Rightarrow z^2+4 = 0, \qquad z^2+4 = 0$

$\Rightarrow z = \pm 2i, \qquad z = \pm 2i$

The only poles which lie in the upper half plane is $z = 2i$, i.e. of order 2.

$$R(f, 2i) = \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} \left((z-2i)^2 \frac{z^2+2z+1}{(z+2i)^2(z-2i)^2} \right)_{z=2i}$$

$$= \left(\frac{d}{dz} \frac{(z^2+2z+1)}{(z+2i)^2} \right)_{z=2i}$$

$$= \left(\frac{(z+2i)^2(2z+2) - (z^2+2z+1)(2(z+2i))}{(z+2i)^4} \right)_{z=2i}$$

$$= \left[\frac{(z+2i)[2(z+2i)(z+1) - 2(z^2+2z+1)]}{(z+2i)^4} \right]_{z=2i}$$

$$= \frac{2[(z+2i)(z+1) - (z^2+2z+1)]}{(z+2i)^3}\Bigg]_{z=2i}$$

$$= \frac{2[(2i+2i)(2i+1) - (4i^2+4i+1)]}{(2i+2i)^3}$$

$$= \frac{2[4i(1+2i) - (-4+4i+1)]}{(4i)^3} \qquad i^3 = i^2 \cdot i = -i$$

$$= \frac{2[4i + 8i^2 + 4 - 4i - 1]}{-64i} = \frac{4i - 8 + 8 - 8i - 2}{-32i}$$

$$= \frac{4\cancel{3} - 8}{2\cancel{3}2i}$$

$$= \frac{-4i\,5 - 2i}{2\cancel{3}2i \cdot 2i} =$$

$$R(f, 2i) = -\frac{5}{32i} \quad \frac{-2i - 1}{i}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{x^2+2x+1}{x^4+8x^2+16}\,dx = 2\pi i \times R$$

$$= 2\pi i \times \frac{(5\,4i - 2)}{32i\,i}$$

$$\int_{-\infty}^{\infty} \frac{x^2+2x+1}{x^4+8x^2+16}\,dx = \frac{(-5\pi}{16} - 2\,\frac{16}{\pi}.$$

(V)

$$\int_{0}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,dx = \frac{\pi}{16a^3} \quad ; \quad a>0 .$$

Sol:-

$$f(x) = \frac{x^2}{(x^2+a^2)^3}$$

Step I :-

Replace x by z.

$$f(z) = \frac{z^2}{(z^2+a^2)^3}$$

$$z\,f(z) = \frac{z^3}{(z^2)^3\left[1+\frac{a^2}{z^2}\right]^3}$$

$$= \frac{1}{z^3\left[1+\frac{a^2}{z^2}\right]^3}$$

$$z f(z) = \frac{\frac{1}{z^3}}{\left[1+\frac{a^2}{z^2}\right]^3} \quad z f(z) \to 0 \text{ as } |z| \to \infty$$

Step II :-

Consider the integral $\int_0^\infty \frac{x^2\,dx}{x^2+a^2}$

or $\int_0^\infty \frac{z^2\,dz}{z^2+a^2}$

The poles of $f(z)$ are $z = \pm ai$

The only pole which lie in upper half plane is $z = ai$.

$$R(f, ai) = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left((z-ai)\left(\frac{z^2}{z^2+a^2}\right)\right)_{z=ai}$$

$$= \left(\cancel{(z-ai)}\left(\frac{z^2}{\cancel{(z-ai)}(z+ai)}\right)\right)_{z=ai}$$

$$= \left(\frac{z^2}{z+ai}\right)_{z=ai}$$

$$= \frac{(ai)^2}{ai+ai}$$

$$= \frac{a^2 i^2}{2ai}$$

$$R(f, ai) = \frac{-a^2}{2ai} = -\frac{a}{2i}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^2+a^2} = \cancel{2}\pi\cancel{i} \times -\frac{a}{\cancel{2i}}$$

$$= -a\pi$$

Differentiatiat e it w.r.t "a".

$$\int_{-\infty}^{\infty} \frac{(x^2+a^2)(0)+(0\mp 2a)x^2}{(x^2+a^2)^2}\,dx = -\pi$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{-2ax^2}{(x^2+a^2)^2}\,dx = -\pi$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^2} = \frac{\pm\pi}{2a}$$

Again Differentiate w.r.t "a"

$$\int_{-\infty}^{\infty} \frac{(x^2+a^2)^2(0) \mp x^2(2(x^2+a^2)(0+2a))}{(x^2+a^2)^4}dx$$

$$= \frac{\pi}{2}\left(\frac{-1}{a^2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{-4ax^2(x^2+a^2)}{(x^2+a^2)^4}\,dx = \frac{-\pi}{2a^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,dx = \frac{-\pi}{2a^2} \times \frac{1}{-4a}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = \frac{\pi}{8a^3}$$

$$2\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = \frac{\pi}{8a^3}$$

$$\int_{0}^{\infty} \frac{x^2\,dx}{(x^2+a^2)^3} = \frac{\pi}{16a^3}$$

(vi)

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2(x^2+b^2)} = \frac{\pi(2a+b)}{2a^3b(a+b)^2}$$

Sol:-

$$f(x) = \frac{1}{(x^2+a^2)^2(x^2+b^2)}$$

Replace "x" by "z"

$$f(z) = \frac{1}{(z^2+a^2)^2(z^2+b^2)}$$

$$z\,f(z) = \frac{z}{(z^2)^2\left[1+\frac{a^2}{z^2}\right]^2 z^2\left[1+\frac{b^2}{z^2}\right]}$$

$$z f(z) = \frac{z}{z^6\left[1+\frac{a^2}{z^2}\right]\left[1+\frac{b^2}{z^2}\right]}$$

$$z f(z) = \frac{\frac{1}{z^5}}{\left[1+\frac{a^2}{z^2}\right]\left[1+\frac{b^2}{z^2}\right]}$$

$$z f(z) \to 0 \quad \text{as} \quad |z| \to \infty$$

<u>Step II :-</u>

The pole of $f(z)$ are $x = \pm ai, \pm bi$
The only pole which lie in upper half plane is $z = ai$ of order "2" and is $z = bi$ of order "1"

$$R(f, ai) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2(z^2+b^2)}\right)_{z=ai}$$

$$= \frac{d}{dz}\left((z+ai)^{-2}(z^2+b^2)^{-1}\right]_{z=ai}$$

$$= \left\{\frac{(z+ai)^{-2}(-1)(z^2+b^2)^{-2}(2z)+(z^2+b^2)^{-1}(-2)(z+ai)^{-3}}{(z+ai)^4}\right]_{z=ai}$$

$$= \left[-(2ai)^{-2}(-a^2+b^2)^{-2}(2ai)-2(-a^2+b^2)(2ai)^{-3}\right]$$

$$= \frac{-2ai}{(2ai)^2(b^2-a^2)^2} - \frac{2}{(2ai)^3(b^2-a^2)}$$

$$= \frac{-1}{2ai(b^2-a^2)^2} - \frac{2}{(b^2-a^2)(2ai)^3}$$

$$= \frac{-1}{2ai(b^2-a^2)}\left[\frac{1}{b^2-a^2} + \frac{2}{-4a^2}\right]$$

$$= \frac{-1}{2ai(b^2-a^2)}\left[\frac{4a^2-2(b^2-a^2)}{4a^2(b^2-a^2)}\right]$$

$$= \frac{-1}{2ai(b^2-a^2)}\left[\frac{4a^2-2b^2+2a^2}{4a^2(b^2-a^2)}\right]$$

$$= \frac{-1}{2ai(b^2-a^2)}\left[\frac{6a^2-2b^2}{4a^2(b^2-a^2)}\right] = \frac{-(3a^2-b^2)}{4ia^3(b^2-a^2)^2}$$

$$R(f, bi) = \left[(z-bi) \times \frac{1}{(x^2+a^2)^2(z+bi)(z-bi)}\right]_{z=bi}$$

$$= \frac{1}{2bi(-b^2+a^2)^2} = \frac{1}{2bi(a^2-b^2)^2}$$

Step III :-

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+a^2)^2} = 2\pi i\left[\frac{-(3a^2-b^2)}{4a^3 i(b^2-a^2)^2} + \frac{1}{2bi(a^2-b^2)^2}\right]$$

$$= \frac{\cancel{2}\pi \cancel{i}}{\cancel{2}\cancel{i}}\left[\frac{-3a^2+b^2}{2a^3(b^2-a^2)^2} + \frac{1}{b(a^2-b^2)^2}\right]$$

$$= \frac{\pi}{(a^2-b^2)^2}\left[\frac{-3a^2+b^2}{2a^3} + \frac{1}{b}\right]$$

$$= \frac{\pi}{(a^2-b^2)^2}\left[\frac{-3a^2b+b^3+2a^3}{2a^3b}\right]$$

$$= \frac{\pi}{2a^3b(a+b)^2}\left[\frac{2a^3+b^3-3a^2b}{(a-b)^2}\right]$$

$$= \frac{\pi}{2a^3b(a+b)^2}\left[\frac{(2a+b)(a^2+b^2-2ab)}{(a^2+b^2-2ab)}\right]$$

$$\because \quad 2a^3+b^3-3a^2b = (2a+b)(a^2+b^2-2ab)$$

$$= 2a(a^2+b^2-2ab)+b(a^2+b^2-2ab)$$

$$= 2a^3+\cancel{2ab^2}-4a^2b+a^2b+b^3-\cancel{2ab^2}$$

$$2a^3+b^3-3a^2b = 2a^3+b^3-3a^2b$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+a^2)^2} = \frac{\pi}{2a^3b(a+b)^2}\,\frac{(2a+b)\cancel{(a-b)^2}}{\cancel{(a-b)^2}}$$

$$\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+a^2)^2} = \frac{\pi(2a+b)}{2a^3b(a+b)^2}$$

108

(vii) $\int_{-\infty}^{\infty} \frac{x^4\,dx}{(a+bx^2)^4} = \frac{\pi}{48}\,\frac{1}{a^{3/2}b^{5/2}}$, $a>b>0$.

Sol:-

$$f(x) = \frac{x^4}{(a+bx^2)^4}$$

Step I:-

Replace "x" by "z"

$$f(z) = \frac{z^4}{(a+bz^2)^4}$$

$$z\,f(z) = \frac{z^5}{(a+bz^2)^4}$$

$$= \frac{z^5}{(z^2)^4\left[b+\frac{a}{z^2}\right]}$$

$$= \frac{z^5}{z^8\left[b+\frac{a}{z^2}\right]}$$

$$= \frac{1}{z^3\left[b+\frac{a}{z^2}\right]}$$

$$z\,f(z) = \frac{1/z^3}{\left[b+\frac{a}{z^2}\right]}$$

$$z\,f(z) \to 0 \quad \text{as} \quad |z| \to \infty$$

Step II:-

Consider the integral $\int_{-\infty}^{\infty} \frac{z^4\,dz}{(a+bz^2)^4}$

The pole of $f(z) = \frac{z^4}{a+bz^2}$ are

$$a+bz^2 = 0 \Rightarrow z^2 = -\frac{a}{b} \Rightarrow z = \pm\sqrt{\frac{a}{b}}\,i$$

The only pole which lie in the upper half plane is $z = \sqrt{\frac{a}{b}}\, i$

$$R\left(f, \sqrt{\frac{a}{b}}\, i\right) = \frac{1}{(1-1)!} \frac{d^0}{dz^0}\left(\left(z - \sqrt{\frac{a}{b}}\, i\right) \frac{z^4}{\left(z - \sqrt{\frac{a}{b}}\, i\right)\left(z + \sqrt{\frac{a}{b}}\, i\right)} \right)_{z = \sqrt{\frac{a}{b}} i}$$

$$R\left(f, \sqrt{\frac{a}{b}}\, i\right) = \left[\frac{z^4}{z + \sqrt{\frac{a}{b}}\, i} \right]_{z = \sqrt{\frac{a}{b}}\, i}$$

$$= \frac{\left(\sqrt{\frac{a}{b}}\, i\right)^4}{\sqrt{\frac{a}{b}}\, i + \sqrt{\frac{a}{b}}\, i} = \frac{\left(\frac{a}{b} i^2\right)^2}{2\sqrt{\frac{a}{b}}\, i}$$

$$= \frac{\frac{a^2}{b^2}(-1)^2}{2\sqrt{\frac{a}{b}}\, i} = \frac{\left(\sqrt{\frac{a}{b}}\right)^4}{2\sqrt{\frac{a}{b}}\, i}$$

$$R\left(f, \sqrt{\frac{a}{b}}\, i\right) = \frac{\left(\sqrt{a/b}\right)^3}{2i}$$

$$= \frac{\left(\sqrt{\frac{a}{b}}\right)^2 \left(\sqrt{\frac{a}{b}}\right)}{2i} = \frac{\frac{a}{b}\sqrt{\frac{a}{b}}}{2i}$$

$$R\left(f, \sqrt{\frac{a}{b}}\, i\right) = \frac{a}{2bi}\sqrt{\frac{a}{b}} = \frac{a^{1+\frac{1}{2}} / b^{1+\frac{1}{2}}}{2i}$$

$$R\left(f, \sqrt{\frac{a}{b}}\, i\right) = \frac{a^{3/2}}{2b^{3/2} i}$$

$$\int_{-\infty}^{\infty} \frac{x^4\, dx}{(a+bx^2)^4} = 2\pi i \times \frac{a^{3/2}}{2ib^{3/2}}$$

$$= \frac{\pi a^{3/2}}{b^{3/2}}$$

Differentiate it w.r.t "b"

$$-\int_{-\infty}^{\infty} \frac{(a+bx^2)(0) - x^4(0+bx^2)}{(a+bx^2)^2}\, dx = \frac{\pi}{b^{3/2}}\left(\frac{3}{2} a^{\frac{3}{2}-1}\right)$$

$$-\int_{-\infty}^{\infty} \frac{-x^6}{(a+bx^2)^2}\, dx = \frac{\pi}{b^{3/2}}\left[\frac{3}{2} a^{\frac{3-2}{2}}\right]$$

$$= \frac{3\pi}{2b^{3/2}}\, a^{1/2}$$

$$\int_{-\infty}^{\infty} \frac{x^6\, dx}{(a+bx^2)^2} = \frac{3}{2}\pi\, a^{1/2} b^{-3/2}$$

Differentiate it w.r.t "a"

$$\int_{-\infty}^{\infty} \frac{(a+bx^2)^2(6x^5) - x^6(2(a+bx^2)(1))}{(a+bx^2)^4}\, dx = \frac{3\pi}{2}\frac{1}{2} a^{-1/2} b^{-3/2}$$

$$\int_{-\infty}^{\infty} \frac{6x^5(a+bx^2)^2 - 2x^6(a+bx^2)}{(a+bx^2)^4}\, dx = \frac{3}{4}\pi a^{-1/2} b^{-3/2}$$

(viii) $$\int_{-\infty}^{\infty} \frac{\ln(1-x^2)}{1+x^2}\,dx = \pi \ln(2)$$

Sol:-

$$f(x) = \frac{\ln(1-x^2)}{1+x^2}$$

Step I:-

Replace "x" by "z"

$$f(z) = \frac{\ln(1-z^2)}{1+z^2}$$

$$f(z) = \frac{\ln(1+z)(1-z)}{1+z^2}$$

$$f(z) = \frac{\ln(1+z)+\ln(1-z)}{1+z^2}$$

$$zf(z) = \frac{z[\ln(1+z)+\ln(1-z)]}{z^2[1+\frac{1}{z^2}]}$$

$$= \frac{[\ln(1+z)+\ln(1-z)]}{z[1+\frac{1}{z^2}]}$$

$$zf(z) = \frac{\frac{1}{z}[\ln(1+z)+\ln(1-z)]}{[1+\frac{1}{z^2}]}$$

$zf(z) \to 0$ as $|z| \to \infty$.

Step II:-

The poles of $f(z)$ are $z = \pm i$

The only pole which lie in the upper half plane is $z = i$

$$R(f, i) = \left((z-i)\frac{\ln(1-z^2)}{(z-i)(z+i)}\right)_{z=i}$$

$$R(f, i) = \left(\frac{\ln(1-z^2)}{z+i}\right)_{z=i}$$

$$= \frac{\ln(1-(i^2))}{i+i}$$

$$= \frac{\ln(1-(-1))}{2i}$$

$$= \frac{\ln(1+1)}{2i}$$

$$R(f, i) = \frac{\ln(2)}{2i}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{\ln(1-x^2)}{1+x^2} dx = 2\pi i \times R.$$

$$= 2\pi i \times \frac{\ln(2)}{2i}$$

$$\int_{-\infty}^{\infty} \frac{\ln(1-x^2)}{1+x^2} dx = \pi \ln(2)$$

TYPE III :-

Form of integral is either

$$\int_{-\infty}^{\infty} f(x) \, [\sin mx \text{ or } \cos mx] \qquad m \text{ is constt.}$$

or $\int_{0}^{\infty} f(x) \, [\sin mx \text{ or } \cos mx]$

Working Rule :-

Step I :- Replace x by z and $\cos mz$ and $\sin mz$ by e^{imz}

Step II :-

Find the poles of $e^{imz} f(z)$. Locate the poles which lie in the upper half plane and find residue at choosen poles.

Step III :-

$$\int_{-\infty}^{\infty} f(x)\sin mx\,dx = \mathrm{Img}\left[2\pi i \times \sum_{j=1}^{n} R_j\right]$$

$$\int_{-\infty}^{\infty} f(x)\cos mx\,dx = \mathrm{Real}\left[2\pi i \times \sum_{j=1}^{n} R_j\right]$$

Ex :-

Evaluate $\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx$, m, a are constants.

Sol :-

$$f(x) = \frac{x}{x^2+a^2}$$

Step I :- Replace x by z and $\sin mx = e^{imz}$

Then, the given integral becomes

$$\int_{C} \frac{z e^{imz}}{z^2+a^2}\,dz.$$

Step II :-

The poles of $f(z)e^{imz}$ are at $z = \pm ai$

The only poles which lie in the upper half plane is $z = ai$

$$R(f, ai) = \left(\cancel{(z-ai)}\,\frac{z e^{imz}}{\cancel{(z-ai)}(z+ai)}\right)_{z=ai}$$

$$= \left(\frac{ai\, e^{im(ai)}}{ai+ai}\right)$$

$$= \frac{\cancel{ai}\, e^{-ma}}{2\cancel{ai}} = \frac{e^{-ma}}{2}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{x\sin mx}{x^2+a^2}\,dx = \mathrm{Img}\left[2\pi i \times R\right]$$

$$= \mathrm{Img}\left[\cancel{2}\pi i \times \frac{e^{-ma}}{\cancel{2}}\right]$$

$$\int_{-\infty}^{\infty} \frac{x\sin mx\,dx}{x^2+a^2} = \pi e^{-ma}$$

Assignment :-

i) $\int_{-\infty}^{\infty} \frac{\cos^2 x}{(x^2+1)^2} dx = \frac{\pi}{4}(1+3e^{-2})$

Sol:-

$$f(x) = \frac{1}{(x^2+1)^2}$$

$$\cos^2 x = \frac{1+2\cos 2x}{2}$$

Replace x by z and $\cos 2x = e^{i2z}$ then, the integral becomes

$$\int_{-\infty}^{\infty} \frac{1}{2}\left(\frac{1+2\cos 2x}{(z^2+1)^2}\right)dx = \frac{1}{2}\int_c \frac{1+e^{i2z}}{(z^2+1)^2} dz.$$

The pole of $f(z)e^{i2z}$ are at $z = \pm i$. The only pole which lie in upper half plane is $z = i$ of order "2".

$$R(f,i) = \left(\frac{1}{(2-1)!}\frac{d}{dz}\frac{\cancel{(z-i)^2}(1+e^{2iz})}{(z-i)(z+i)\cancel{(z-i)^2}(z+i)^2}\right)_{z=i}$$

$$= \frac{d}{dz}\left(\frac{1+e^{2iz}}{e^{2i}(z+i)^2}\right)_{z=i}$$

$$= \left(\frac{(z+i)^2(0+2ie^{2iz}) - (1+e^{2iz})2(z+i)}{1+e^{-2}(z+i)^4}\right)_{z=i}$$

$$= \left[\frac{2\cancel{(z+i)}[e^{2iz}[i(z+i) - (\frac{1}{e^{2iz}})]]}{1+e^{-2}(z+i)^4}\right]_{z=i}$$

$$R(f,i) = \left[\frac{2e^{2iz}[i(z+i) - (1+\frac{1}{e^{2iz}})]}{(z+i)^3}\right]_{z=i}$$

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{1+2e^{-2}[i(i+i)+((1+\frac{1}{e^{-2}}))]+e}{x^2+1} \quad \frac{}{(i+i)^3} \quad \frac{}{-i}$$

$$\int_{-\infty}^{\infty} \frac{\cos^2 x}{(x^2+1)} dx = \frac{2e^{-2}[i(2i)-(1+e^2)]^{-2}}{(2i)^3 2}$$

$$= 2e^{-2}\left[-2-1-e^{2}\right]/8i^{3}$$

$$= -2e^{-2}\left[-3-e^{2}\right] \qquad i^{3}=i^{2}\cdot i=-i$$

$$= -\frac{1}{4i}\left[+3e^{-2}8i+1\right] = -\frac{1}{4i}\left[1+3e^{-2}\right]$$

(ii) $$\int_{-\infty}^{\infty}\frac{\cos^{2}x}{(1+x^{2})^{2}}dx = \frac{1}{2}\text{Real}\left[2\pi i\times\frac{1}{4i}(1+3e^{-2})\right] = \frac{\pi}{4}(1+3e^{-2})$$

$$\int_{-\infty}^{\infty}\frac{\sin^{2}2x}{(1+x^{2})^{2}}dx = \frac{\pi}{4}(1-5e^{-4})$$

Sol:-

$$f(x) = \frac{1}{(1+x^{2})^{2}}$$

$$\sin^{2}2x = \frac{1-\cos 4x}{2}$$

Replace x by z and $\cos 4x$ by e^{4iz}.
Then, the integral becomes

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1-\cos 4x}{(1+x^{2})^{2}}dx = \frac{1}{2}\int_{c}\frac{1-e^{4iz}}{(1+z^{2})^{2}}dz$$

The pole of $f(z)$. are at $z=\pm i$ of order '2'. The only pole which lie in upper half plane is $z=i$

$$R(f,i) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z-i)^{2}\frac{(1-e^{4iz})}{(z-i)^{2}(z+i)^{2}}\right)_{z=i}$$

$$= \frac{d}{dz}\left(\frac{1-e^{4iz}}{(z+i)^{2}}\right)_{z=i}$$

$$= \left.\frac{(z+i)^{2}(0-4ze^{4iz})-(1-e^{4iz})(2(z+i))}{(z+i)^{4}}\right|_{z=i}$$

$$= \left(\frac{-4ze^{4iz}(z+i)^{2}-2(1-e^{4iz})(z+i)}{(z+i)^{4}}\right)_{z=i}$$

$$= \left.\frac{-2(z+i)\left[2ze^{4iz}(z+i)+(1-e^{4iz})\right]}{(z+i)^{4}}\right|_{z=i}$$

$$= -2e^{4iz}\left[\frac{2z(z+i)+\frac{1}{e^{4iz}}-1}{(z+i)^3}\right]_{z=i}$$

$$= -2e^{-4}\left[\frac{2i(i+i)-1+\frac{1}{e^{-4}}}{(i+i)^3}\right]$$

$$= -2e^{-4}\left[\frac{2i(2i)-1+e^4}{(2i)^3}\right]$$

$$= -2e^{-4}\left[\frac{4i^2-1+e^4}{8i^3}\right] \qquad i^3 = i^2 \cdot i = -i$$

$$= -2e^{-4}\left[\frac{-4-1+e^4}{-8i}\right]$$

$$= \frac{e^{-4}}{4i}\left[-5+e^4\right]$$

$$= \frac{1}{4i}\left[-5e^{-4}+e^{4-4}\right]$$

$$= \frac{1}{4i}\left[-5e^{-4}+e^0\right]$$

$$\Rightarrow \frac{1}{4i}\left[1-5e^{-4}\right]$$

$$\int_{-\infty}^{\infty}\frac{\sin^2 2x}{(x^2+1)^2}\,dx = \frac{1}{2}\left(\text{Img}\left[2\pi i \times \frac{1}{4i}(1-5e^{-4})\right]\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4}(1-5e^{-4})\right)$$

(iii)

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right]$$

Sol:-

$$f(x) = \frac{1}{(x^2+a^2)(x^2+b^2)}$$

Replace x by z and $\cos x$ by e^{iz}

Then, the integral becomes.

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)} = \int_c \frac{e^{iz}\, dz}{(z^2+a^2)(z^2+b^2)}$$

The pole are at $z = \pm ai$, $z = \pm bi$. The only the pole which lie in the upper half plane is $z = ai$, $z = bi$.

$$+R_1(f, ai) = \left((z-ai) \frac{e^{iz}}{(z-ai)(z+ai)(z^2+b^2)} \right)_{z=ai}$$

$$R_1(f, ai) = \left(\frac{e^{iz}}{(z+ai)(z^2+b^2)} \right)_{z=ai}$$

$$= \frac{e^{i(ai)}}{(ai+ai)((ai)^2+b^2)}$$

$$= \frac{e^{-a}}{(2ai)(a^2i^2+b^2)} = \frac{e^{-a}}{2ai(-a^2+b^2)}$$

$$R_1(f, ai) = \frac{e^{-a}}{2ai(b^2-a^2)}$$

$$R_2(f, bi) = \left(\frac{e^{iz}\cancel{(z-bi)}}{\cancel{(z-bi)}(z+bi)(z^2+a^2)} \right)_{z=bi}$$

$$= \left(\frac{e^{-b}}{(bi+bi)((bi)^2+a^2)} \right)_{z=bi}$$

$$= \frac{e^{-b}}{(2bi)(-b^2+a^2)} = \frac{e^{-b}}{2bi(a^2-b^2)}$$

$$R = R_1 + R_2 = \frac{e^{-a}}{2ai(b^2-a^2)} + \frac{e^{-b}}{2bi(a^2-b^2)}$$

$$= \frac{1}{2i}\left[\frac{-e^{-a}}{a(a^2-b^2)} + \frac{e^{-b}}{b(a^2-b^2)} \right]$$

$$= \frac{1}{2i}\left[\frac{e^{-b}}{b(a^2-b^2)} - \frac{e^{-a}}{a(a^2-b^2)}\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \text{Real}\left\{2\pi i \times \frac{1}{2i}\left(\frac{e^{-b}}{b(a^2-b^2)} - \frac{e^{-a}}{a(a^2-b^2)}\right)\right.$$

$$= \pi\left[\frac{ae^{-b} - be^{-a}}{ab(a^2-b^2)}\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

(iv)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+5}\,dx = \frac{\pi \sin(1)}{2e^2}$$

Sol:-

$$f(x) = \frac{1}{x^2-2x+5}$$

Replace x by z and $\sin x = e^{iz}$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+5}\,dx = \int_C \frac{e^{iz}}{z^2-2z+5}\,dz$$

$z^2-2z+5=0$

$z = \frac{2 \pm \sqrt{4-4(5)}}{2}$

$= \frac{2 \pm \sqrt{-16}}{2}$

$= \frac{2 \pm 4i}{2}$

$z = 1 \pm 2i$

The pole of $f(z)$ are at $z = 1+2i$

The only pole which lie in the upper half plane is $z = 1+2i$.

$$R(f, 1+2i) = \left((z-1-2i)\frac{e^{iz}}{(z-1-2i)(z-1+2i)}\right)_{z=1+2i}$$

$$= \left(\frac{e^{iz}}{z+1+2i}\right)_{z=1+2i}$$

$$= \frac{e^{i(1+2i)}}{1+2i-1+2i} = \frac{e^{i-2}}{2i+2i}$$

$$= \frac{e^{i}e^{-2}}{4i e^{2}} = \frac{e^{i}}{4ie^{2}}$$

$$R(f, 1+2i) = \frac{1}{e^2 4i}(\cos 1 + i\sin 1)$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+5}\,dx = \text{Img}\left(2\pi i \times \frac{1}{4ie^2}(\cos 1 + i\sin 1)\right)$$

$$= \frac{\pi}{2e^2}\sin 1 = \frac{\pi \sin(1)}{2e^2}$$

Type IV

Form of the integral is either

$$\int_{-\infty}^{\infty} f(x)\,[\sin mx \text{ or } \cos mx]\,dx$$

or $\int_{0}^{\infty} f(x)\,[\sin mx \text{ or } \cos mx]\,dx$; m is integer.

Working Rule :-

Step I :- Replace 'x' by 'z' and $\sin mx$ or $\cos mx$ by e^{imz}

Step II :-

Find the poles of the integrand. Locate those poles which lie in the upper half plane and on the real axis.

Find the residue of the poles in the upper half plane say R_p and residues at the poles on the real axis says R_x.

Step III :-

$$\int_{-\infty}^{\infty} f(x)\sin mx\,dx = \operatorname{Img}[2\pi i R_p + \pi i R_x]$$

$$\int_{-\infty}^{\infty} f(x)\cos mx\,dx = \operatorname{Real}[2\pi i R_p + \pi i R_x]$$

Ex :-

Prove that $\int_{0}^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$

Sol :-

$$f(x) = \frac{1-\cos x}{x^2}$$

Step I :-

Replace x by z and $\cos x$ by e^{iz}

$$f(z) = \frac{1-\cancel{\cos z}\; e^{iz}}{z^2}$$

$$\int_C \frac{1-e^{iz}}{z^2}\,dz$$

The poles of $\frac{1-e^{iz}}{z^2}$ is at $z=0$ of order 2.

$$R_x(f,0) = \frac{1}{(2-1)!}\frac{d}{dz}\left[(z-0)^2 \frac{1-e^{iz}}{z^2}\right]_{z=0}$$

$$R_x(f,0) = \frac{d}{dz}\left(1-e^{iz}\right)_{z=0} = \left(0-ie^{iz}\right)_{z=0}$$

$$R_x(f,0) = -i$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \text{Real}\left[\pi i \times -i\right]$$

$$2\int_{0}^{\infty} \frac{1-\cos x}{x^2}\,dx = \pi$$

$$\Rightarrow \int_{0}^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$$

Assignment of Type II :-

(V)
$$\int_{0}^{\infty} \frac{x \sin ax}{x^4+x^2+1}\,dx = \frac{\pi}{\sqrt{3}}\, e^{-\frac{\sqrt{3}}{2}a} \sin(a/2)$$

Sol:-

$$f(x) = \frac{x}{x^4+x^2+1}$$

Replace x by z and $\sin ax = e^{iaz}$

Then, the integral becomes

$$\int_C \frac{z\, e^{iaz}}{z^4+z^2+1}$$

$$z^2 = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

The pole of $f(z)$ are at $z^2 = \frac{-1 \pm \sqrt{3}i}{2}$

The only pole which lie in the upper half plane is $z = e^{i\pi/3}$ $e^{i\frac{2\pi}{3}}$

$$z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

Let $z = \alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\beta = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$$R(f,\alpha) = \left((z-\alpha) \frac{z e^{iaz}}{(z-\alpha)(z-\beta)} \right)_{z=\alpha}$$

$$= \left(\frac{z e^{iaz}}{z-\beta} \right)_{z=\alpha}$$

$$R(f,\alpha) = \frac{\alpha e^{ia\alpha}}{\alpha-\beta}$$

put $\alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$R(f,\alpha) = \frac{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) e^{ia\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)}}{-\frac{1}{2} + i\frac{\sqrt{3}}{2} + \frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

$$= \frac{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) e^{\left(-\frac{ai}{2} - \frac{\sqrt{3}}{2}a\right)}}{\cancel{2} i \frac{\sqrt{3}}{\cancel{2}}}$$

(vi)
$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = 2\pi e^{-a}$$

Sol:-
$$\int_{-\infty}^{\infty} \left(\frac{a\cos x + x\sin x}{x^2+a^2}\right)dx = \int_{-\infty}^{\infty} \left(\frac{a\cos x}{x^2+a^2} + \frac{x\sin x}{x^2+a^2}\right)dx$$

$$= \int_{-\infty}^{\infty} \frac{a\cos x}{x^2+a^2}\,dx + \int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx \rightarrow (1)$$

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{a\cos x}{x^2+a^2}\,dx$

$$f(x) = \frac{1}{x^2+a^2}$$

Replace "x" by "z" and $\cos x = e^{iz}$

$$\int_{-\infty}^{\infty} \frac{a\cos x}{x^2+a^2}\,dx = \int \frac{ae^{iz}}{z^2+a^2}\,dz.$$

The pole of $f(z)$ are at $z = \pm ai$. The only pole which lie in upper half plane is $z = ai$

$$R(f, ai) = \left((z-ai)\frac{(ae^{iz})}{(z-ai)(z+ai)}\right)_{z=ai}$$

$$= \left(\frac{ae^{iz}}{z+ai}\right)_{z=ai}$$

$$R(f, ai) = \frac{ae^{-a}}{2ai} = \frac{e^{-a}}{2i}$$

$$\int_{-\infty}^{\infty} \frac{a\cos x}{x^2+a^2}\,dx = \text{Real}\left[2\pi i \times \frac{e^{-a}}{2i}\right]$$

$$\int_{-\infty}^{\infty} \frac{a\cos x}{x^2+a^2}\,dx = \pi e^{-a}$$

(b) $$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx$$

Replace x by z and $\sin x = e^{iz}$.

$$\int_c \frac{z e^{iz}}{z^2+a^2}\,dz.$$

The only pole which lie in upper half plane is $z = ai$

$$R(f, ai) = \left((z-ai)\frac{z e^{iz}}{(z-ai)(z+ai)}\right)_{z=ai}$$

$$= \left(\frac{z e^{iz}}{z+ai}\right)_{z=ai}$$

$$= \frac{ai e^{-a}}{2ai} = \frac{e^{-a}}{2}.$$

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}\,dx = \text{Img}\left[2\pi i \times \frac{e^{-a}}{2}\right]$$

$$= \pi e^{-a}$$

putting these in (1).

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = \pi e^{-a} + \pi e^{-a}$$

$$= 2\pi e^{-a}$$

(vii)

$$\int_{-\infty}^{\infty} \frac{x\cos \pi x}{x^2+2x+5}\,dx = \frac{\pi}{2} e^{-2\pi}$$

Sol:-

$$f(x) = \frac{x}{x^2+2x+5}$$

Replace "x" by "z" and $\cos\pi x = e^{i\pi z}$

Then, the integral becomes.

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+2x+5}\,dx = \int_C \frac{z\,e^{i\pi z}}{z^2+2z+5}\,dz$$

The pole of $f(z)$ are at $z = -1 \pm 2i$ - The only pole which lie in upper half plane is $-1+2i = \alpha$, $-1-2i = \beta$

$$R(f,\alpha) = \left(\cancel{(z-\alpha)} \frac{z\,e^{i\pi z}}{\cancel{(z-\alpha)}(z-\beta)} \right)_{z=\alpha}$$

$$= \left(\frac{z e^{i\pi z}}{z-\beta} \right)_{z=\alpha}$$

$$= \frac{\alpha\, e^{i\pi\alpha}}{\alpha-\beta}$$

put $\alpha = 2i-1$

$$= \frac{(2i-1)\, e^{i\pi(2i-1)}}{(2i-1)-(-2i-1)}$$

$$= \frac{(2i-1)\, e^{-2\pi - \pi i}}{(2i-1)+(2i+1)}$$

$$= \frac{(2i-1)\, e^{-2\pi} \cdot e^{-\pi i}}{2i \cancel{-1} + 2i \cancel{+1}}$$

$$= \frac{(2i-1)\, e^{-2\pi} \cdot (\cos\pi - i\sin\pi)}{4i}$$

$$= \frac{2i e^{-2\pi}(\cos\pi - i\sin\pi) - e^{-2\pi}(\cos\pi - i\sin\pi)}{4i}$$

$$R(f,\alpha) = \frac{\cancel{2i}\, e^{-2\pi}(\cos\pi - i\sin\pi)}{4i} - \frac{e^{-2\pi}(\cos\pi - i\sin\pi)}{4i}$$

$$R(f,\alpha) = \frac{e^{-2\pi}}{2}(\cos\pi - i\sin\pi) - \frac{e^{-2\pi}}{4i}(\cos\pi - i\sin\pi)$$

$$= \frac{e^{-2\pi}}{2}(-1-0) - \frac{e^{-2\pi}}{4i}(-1-0) = \frac{e^{-2\pi}}{4i}(1-2i)$$

$$R(f,\alpha) = e^{-2\pi}(1-2i)/4i$$

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+2x+5}\,dx = \text{Real}\,[2\pi i \times R]$$

$$= \text{Real}\left[2\pi i \times \frac{e^{-2\pi}((1-2i))}{4i}\right]$$

$$\text{Real}\left[\pi i \cdot e^{-2\pi}(1-0) - \frac{e^{-2\pi}}{2i}(1-0i)\right]$$

$$\pi e^{-2\pi} = \frac{\pi e^{-2\pi}}{2}((1-2i))$$

$$= \pi e^{-2\pi}(1)(\cos\pi) \qquad \because \cos\pi = 1$$

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x\,dx}{x^2+2x+5} = \frac{\pi e^{-2\pi}}{2} = \frac{\pi}{2}e^{-2\pi}$$

(viii)

$$\int_0^{\infty} \frac{x\sin mx\,dx}{x^4+a^4} = \frac{\pi}{2a^3}\, e^{-\frac{ma}{\sqrt{2}}}\sin\left(\frac{ma}{\sqrt{2}}\right)$$

Sol:-

$$\int_0^{\infty} \frac{x\sin mx}{x^4+a^4}\,dx$$

Replace 'x' by 'z' and $\sin mx = e^{imz}$

$$\int_{-\infty}^{\infty} \frac{z e^{imz}}{z^4+a^4}$$

The poles of $f(z)$ are at $z^4+a^4=0$

$$z^4 = (-1)a^4 \Rightarrow z^4 = a^4 e^{i(2n+1)\pi}$$

$$z_n = a\cdot e^{\frac{i(2n+1)\pi}{4}}, \quad n=0,1,2,3$$

The only poles which lie in the upper half plane are $z_0 = a e^{\frac{\pi}{4}i} = a\left(\cos\frac{\pi}{4} + i\sin\left(\frac{\pi}{4}\right)\right)$

$$\alpha = a\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \frac{a}{\sqrt{2}}(1+i)$$

$$z_1 = ae^{\frac{3\pi i}{4}} = \frac{a}{\sqrt{2}}(-1+i) = \beta$$

$$R(f,\alpha) = \left(\frac{(z-\alpha)\,z\cdot e^{imz}}{(z-\alpha)(z-\beta)(z-\gamma)(z-\delta)}\right)_{z=\alpha} = \frac{0}{0}\text{ form}$$

$$= \frac{\alpha\, e^{im\alpha}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}$$

$$R(f,\alpha) = \frac{a\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)e^{ima\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{-4\sqrt{2}\,a^3} \times (i+1)$$

$$= \frac{\frac{1}{\sqrt{2}}(1+i)^2 e^{\frac{a}{\sqrt{2}} im(1+i)}}{-4\sqrt{2}\,a^2} = \frac{(1-1+2i)e^{\frac{a}{\sqrt{2}}m(i-1)}}{-4(2)a^2}$$

$$= \frac{i e^{\frac{a}{\sqrt{2}}mi - \frac{a}{\sqrt{2}}m}}{-4a^2} = \frac{e^{\frac{a}{\sqrt{2}}mi - \frac{a}{\sqrt{2}}m}}{4ia^2}$$

$$R(f,\beta) = \frac{\beta e^{im\beta}}{-2\sqrt{2}a^3}\cdot i = \frac{\frac{a}{\sqrt{2}}(i-1)e^{iam/\sqrt{2}(i-1)}}{-4\sqrt{2}a^3}\cdot i(i+1)$$

$$= \frac{(i-1)e^{-\frac{am}{\sqrt{2}}-\frac{aim}{\sqrt{2}}}(-1+i)}{-4(2)a^2}$$

$$= \frac{(i-1)^2 e^{-\frac{am}{\sqrt{2}}-\frac{am}{\sqrt{2}}i}}{-8a^2} = \frac{(-1+1-2i)e^{-\frac{am}{\sqrt{2}}-\frac{ami}{\sqrt{2}}}}{-8a^2}$$

$$R(f,\beta) = \frac{ie^{-\frac{am}{\sqrt{2}}-\frac{am}{\sqrt{2}}i}}{4a^2}$$

Step III :-

$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^4+a^4}\,dx = \mathrm{Img}\left[2\pi i\left[\frac{ie^{-\frac{am}{\sqrt{2}}-\frac{am}{\sqrt{2}}i}}{4a^2} + \frac{e^{\frac{ami}{\sqrt{2}}-\frac{am}{\sqrt{2}}}}{4ia^2}\right]\right]$$

$$= \mathrm{Img}\left[\frac{-\pi e^{-\frac{am}{\sqrt{2}}-\frac{am}{\sqrt{2}}i}}{2a^2} + \frac{\pi e^{\frac{am}{\sqrt{2}}i-\frac{am}{\sqrt{2}}}}{2a^2}\right]$$

$$= \frac{\pi}{2a^2}\,\mathrm{Img}\left[-e^{-\frac{am}{\sqrt{2}}-\frac{am}{\sqrt{2}}i} + e^{\frac{am}{\sqrt{2}}i-\frac{am}{\sqrt{2}}}\right]$$

$$= \frac{\pi}{2a^2}\,\mathrm{Img}\, e^{-am/\sqrt{2}}\left[\frac{e^{-\frac{am}{\sqrt{2}}i} - e^{\frac{am}{\sqrt{2}}i}}{2i}\times 2i\right]$$

$$= \frac{\pi}{2a^2}\,\mathrm{Img}\, e^{-\frac{am}{\sqrt{2}}}\left[\sin\left(\frac{am}{\sqrt{2}}\right)\times 2i\right]$$

$$= \frac{\pi}{\cancel{2}a^2}\, e^{-\frac{am}{\sqrt{2}}}\,\cancel{2}\sin\left(\frac{am}{\sqrt{2}}\right)$$

$$\int_{-\infty}^{\infty}\frac{x\sin mx\,dx}{x^4+a^4} = \frac{\pi}{a^2}\,e^{-\frac{am}{\sqrt{2}}}\sin\left(\frac{am}{\sqrt{2}}\right)$$

$$2\int_{0}^{\infty}\frac{x\sin mx\,dx}{x^4+a^4} = \frac{\pi}{a^2}\,e^{-\frac{am}{\sqrt{2}}}\sin\left(\frac{am}{\sqrt{2}}\right)$$

$$\int_{0}^{\infty}\frac{x\sin mx\,dx}{x^4+a^4} = \frac{\pi}{2a^2}\,e^{-\frac{am}{\sqrt{2}}}\sin\left(\frac{am}{\sqrt{2}}\right)$$

(ix) $\int_0^\infty \frac{(3x^2-a^2)\cos mx \, dx}{(x^2+b^2)^2}$

Sol:-

Replace "x" by "z" and $\cos mx = e^{imz}$

$$\int_0^\infty \frac{3x^2-a^2}{(x^2+b^2)^2}\cos mx \, dx = \int_c \frac{(3z^2-a^2)}{(z^2+b^2)^2} e^{imz} dz$$

The pole of $f(z)$ are at $z = \pm bi$

The only pole which lie in the upper half plane is $z = bi$ of order "2"

$$R(f, bi) = \frac{d}{dz}\left[(z-bi)^2 \frac{(3z^2-a^2)e^{imz}}{(z-bi)^2(z+bi)^2}\right]_{z=bi}$$

$$= \frac{d}{dz}\left[\frac{(3z^2-a^2)e^{imz}}{(z+bi)^2}\right]_{z=bi}$$

$$= \frac{(z+bi)^2[(3z^2-a^2)e^{imz}(im)+e^{imz}(6z)]-2e^{imz}(z+bi)(3z^2-a^2)}{(z+bi)^4}\Bigg|_{z=bi}$$

$$= \left.\frac{(z+bi)[(3z^2-a^2)e^{imz}(im)+6ze^{imz}]-2(3z^2-a^2)e^{imz}}{(z+bi)^3}\right]_{z=bi}$$

$$= e^{imz}\left[\frac{6z+3imz^2-ia^2m(z+bi)-2(3z^2-a^2)}{(z+bi)^3}\right]_{z=bi}$$

$$= \frac{e^{imz}}{(z+bi)^3}\left[(6z+6bzi+3imz^3-3bmz^2-ia^2mz+a^2bm-6z^2 + 2a^2\right]_{z=bi}$$

$$= \frac{e^{im(bi)}}{(bi+bi)^3}\left[6b(bi)i+3imb^3i^3-3bmb^2i^2-ia^2mb(i)+ a^2bm+2a^2\right]$$

$$= \frac{e^{-mb}}{(2bi)^3}\left[+6b^2+3b^3m+3b^3m+a^2mb+a^2mb+2a^2\right]$$

$$\frac{e^{-mb}}{8b^3i^3}\left[-6b^2+6b^3m+2a^2mb+2a^2\right]$$

$$= \frac{e^{-mb}}{-8b^3 i}\left[-3(2b^2-2b^3m)+2a^2(mb+1)\right]$$

$$= \frac{e^{-mb}}{-8b^3 i}\left[-6b^2(1-bm)+2a^2(1+bm)\right]$$

$$= \frac{e^{-mb}\,i}{8b^3}\left[-6b^2(1-bm)+2a^2(1+bm)\right]$$

$$= \frac{e^{-mb}\,i\,2}{8b^3}\left[-3b^2(1-bm)+a^2(1+bm)\right]$$

$$= \frac{e^{-mb}}{4b^3}\,i\left[-3b^2+3b^3m+a^2+a^2bm\right]$$

Step III :-

$$\int_{-\infty}^{\infty}\frac{(3x^2-a^2)\cos mx}{(x^2+b^2)^2}\,dx = \text{Real}$$

Assignment of Type IV :-

i) Show that $\displaystyle\int_0^\infty \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{-\pi}{2a^2}e^{-ma} + \frac{\pi}{2a^2}$

Deduce the results for $\displaystyle\int_0^\infty \frac{\sin mx}{x(x^2+a^2)^2}\,dx$

Sol:-

$$\int_0^\infty \frac{\sin mx}{x(x^2+a^2)}\,dx$$

$$f(x) = \frac{1}{x(x^2+a^2)}$$

Replace "x" by "z" and $\sin mx = e^{imz}$

$$\int_0^\infty \frac{\sin mx}{x(x^2+a^2)}\,dx = \int_c \frac{e^{imz}}{z(z^2+a^2)}\,dz$$

The pole of $f(z)$ are at $z=0$ and $z=\pm ai$

The only pole which lie in upper half plane is $z=ai$ and at real is $z=0$

$$R(f,0) = \left((z-0)\frac{e^{imz}}{z(z^2+a^2)}\right)_{z=0}$$

$$R(f,0) = \left(\frac{e^{imz}}{z^2+a^2}\right)_{z=0} = \frac{e^0}{0+a^2} = \frac{1}{a^2}$$

$$R(f,ai) = \left((z-ai)\frac{e^{imz}}{z(z-ai)(z+ai)}\right)_{z=ai}$$

$$= \left(\frac{e^{imz}}{z(z+ai)}\right)_{z=ai}$$

$$= \frac{e^{-am}}{ai(ai+ai)} = \frac{e^{-am}}{(2ai)ai} = \frac{e^{-am}}{2a^2i^2}$$

$$R(f,ai) = \frac{e^{-am}}{-2a^2}$$

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \mathrm{Img}\left[\cancel{2}\pi i\left(\frac{e^{-am}}{-\cancel{2}a^2}\right) + \pi i\left(\frac{1}{a^2}\right)\right]$$

$$= \mathrm{Img}\left[-\frac{\pi}{a^2}e^{-am}i + \frac{\pi}{a^2}i\right]$$

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = Img\left[\left(\frac{-\pi e^{-ma}}{a^2} + \frac{\pi}{a^2}\right)i\right]$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{-\pi}{a^2} e^{-ma} + \frac{\pi}{a^2}$$

$$2\int_{0}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = -\frac{\pi}{a^2} e^{-ma} + \frac{\pi}{a^2}$$

$$\int_{0}^{\infty} \frac{\sin mx}{x(x^2+a^2)}\,dx = \frac{-\pi}{2a^2} e^{-ma} + \frac{\pi}{2a^2}$$

Differentiate it w.r.t "a"

$$\int_{0}^{\infty} \frac{x(x^2+a^2)(0) - \sin mx\,(0+2a)x}{(x(x^2+a^2))^2} = \frac{-\pi e^{-ma}(-2a^{-3})}{2} + \frac{\pi}{2}(-2)a^{-3}$$

$$\int_{0}^{\infty} \frac{0 - 2ax\sin mx\,dx}{x^2(x^2+a^2)^2} = -\frac{\pi}{2} e^{-ma}\frac{(-2)}{a^3} + \frac{\pi}{2}\frac{(-2)}{a^3}$$

$$= \frac{\pi e^{-ma}}{a^3} - \frac{\pi}{a^3}$$

$$-2a\int_{0}^{\infty} \frac{x\sin mx\,dx}{x^2(x^2+a^2)^2} = \frac{\pi}{a^3}\left[e^{-ma} - 1\right]$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = -\frac{\pi}{2a^4}\left[e^{-ma} - 1\right]$$

$$\int_{0}^{\infty} \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{2a^4}\left[1 - e^{-ma}\right]$$

(ii)

$$\int_{0}^{\infty} \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi$$

Sol:-

$$f(x) = \frac{1}{x(1-x^2)}$$

Replace "x" by "z" and $\sin \pi x = e^{-i\pi z}$

Then, the integral becomes.

$$\int_{0}^{\infty} \frac{\sin\pi x}{x(1-x^2)}\,dx = \int_{C} \frac{e^{i\pi z}}{z(1-z^2)}\,dz$$

The pole of $f(z)$ are at $z=0$ at real axis and are at $z=\pm 1$

$$R_1(f,0) = \left((z-0)\frac{e^{i\pi z}}{z(1-z^2)}\right)_{z=0}$$

$$= \left(\frac{e^{i\pi z}}{1-z^2}\right)_{z=0}$$

$$R_1(f,0) = \frac{e^{0}}{1-0} = 1$$

$$R_2(f,1) = \left((z-1)\frac{e^{i\pi z}}{z(1-z)(1+z)}\right)_{z=1}$$

$$= \left(\frac{\cancel{(z-1)}\,e^{i\pi z}}{-z\cancel{(z-1)}(1+z)}\right)_{z=1}$$

$$= \left(\frac{e^{i\pi z}}{-z(1+z)}\right)_{z=1}$$

$$R_2(f,1) = \frac{e^{i\pi}}{-1(1+1)} = \frac{e^{i\pi}}{-2}$$

$$R_3(f,-1) = \left(\cancel{(z+1)}\frac{e^{i\pi z}}{z(1-z)\cancel{(1+z)}}\right)_{z=-1}$$

$$= \left(\frac{e^{i\pi z}}{z(1-z)}\right)_{z=-1}$$

$$R_3(f,-1) = \frac{e^{-i\pi}}{-1(1+1)} = \frac{e^{-i\pi}}{-2}$$

$$R = R_1 + R_2 + R_3$$

$$R = 1 + \frac{e^{i\pi}}{-2} + \frac{e^{i\pi}}{-2} = 1 + \cancel{2}\left(\frac{e^{i\pi}}{-\cancel{2}}\right)$$

$$R = 1 - e^{i\pi}$$

$$\int_{-\infty}^{\infty} \frac{\sin\pi x}{x(1-x^2)}\,dx = \mathrm{Img}\left(\pi i\left(1-e^{i\pi}\right)\right)$$

$$= Img(\pi i - \pi i e^{i\pi})$$

$$= Img(\pi i - \pi i(\cos\pi + i\sin\pi))$$

$$= Img(\pi i - \pi(i\cos\pi + i^2\sin\pi))$$

$$= Img(\pi i - \pi[i\cos\pi - 1\cdot\sin\pi])$$

$$= Img(\pi i - \pi[i\cos\pi - \sin\pi])$$

$$\int_{-\infty}^{\infty} \frac{\sin\pi x\,dx}{x(1-x^2)} = Img(\pi i - \pi i\cos\pi - \pi\sin\pi)$$

$$= Img(\pi i - \pi i(-1) - \pi\sin\pi]$$

$$= Img(\pi i + \pi i - 0)$$

$$\int_{-\infty}^{\infty} \frac{\sin\pi x\,dx}{x(1-x^2)} = Img(2\pi i)$$

$$\int_{-\infty}^{\infty} \frac{\sin\pi x\,dx}{x(1-x^2)} = 2\pi i$$

$$2\int_{0}^{\infty} \frac{\sin\pi x\,dx}{x(1-x^2)} = 2\pi$$

$$\int_{0}^{\infty} \frac{\sin\pi x\,dx}{x(1-x^2)} = \pi$$

(iii)

$$\int_{-\infty}^{\infty} \frac{\sin 2(x-a)\,dx}{(x-a)(x^2+b^2)}$$

Sol:-

$$f(x) = \frac{1}{(x-a)(x^2+b^2)}$$

Replace "x" by "z" and $\sin 2(x-a) = e^{2i(z-a)}$

Then, the integral becomes.

$$\int_C \frac{e^{2i(z-a)}}{(z-a)(z^2+b^2)}\,dz.$$

The pole of $f(z)$ are at $z=a$ and are at $z = \pm bi$

The only pole which lie in upper half plane is $z = bi$ and at real $z=a$.

$$R_x(f,a) = \left(\frac{\cancel{(z-a)}\, e^{2i(z-a)}}{\cancel{(z-a)}(z^2+b^2)}\right)_{z=a}$$

$$R_x(f,a) = \left(\frac{e^{2i(a-a)}}{a^2+b^2}\right)$$

$$R_x(f,a) = \frac{e^0}{a^2+b^2} = \frac{1}{a^2+b^2}$$

$$R_{p_1}(f,bi) = \left(\frac{\cancel{(z-bi)}\, e^{2i(z-a)}}{(z-a)\cancel{(z-bi)}(z+bi)}\right)_{z=bi}$$

$$= \left(\frac{e^{2i(z-a)}}{(z-a)(z+bi)}\right)_{z=bi}$$

$$R_{p_1}(f,bi) = \frac{e^{2i(bi-a)}}{(bi-a)(bi+bi)} = \frac{e^{-2b}\cdot e^{-2ai}}{2bi(bi-a)}$$

~~$R = R_x + R_p$~~

$$R_p(f,bi) = \frac{e^{-2b} \cdot e^{-2ai}}{2bi^2 - 2abi} = \frac{e^{-2b}\cdot e^{-2ai}}{-2b^2-2abi}$$

$$\int_{-\infty}^{\infty} \frac{\sin 2(x-a)}{(x-a)(x^2+b^2)}\,dx = \operatorname{Img}\left(2\pi i \frac{e^{-2b}\cdot e^{(-2ai)}}{-2(b^2+iab)} + \pi i \frac{1}{a^2+b^2}\right)$$

$$= \operatorname{Img}\left(\frac{\pi i}{a^2+b^2} - \frac{\pi i e^{-2b}}{b^2+iab}(\cos(2a) - i\sin(2a))\right)$$

$$= \operatorname{Img}\left[\frac{\pi i}{a^2+b^2} - \frac{i\pi e^{-2b}\cos(2a)}{b^2+iab} + \frac{\pi i^2 e^{-2b}\sin(2a)}{b^2+iab}\right)$$

$$= \operatorname{Img}\left[\frac{\pi i}{a^2+b^2} - \frac{i\pi e^{-2b}}{b^2+iab}[\cos 2a - i\sin 2a]\right]$$

$$= \operatorname{Img}\left[\frac{i\pi}{a^2+b^2} - \frac{\pi e^{-2b}}{b^2+iab}(\cos 2a - i\sin 2a)\right]$$

$$\int_{-\infty}^{\infty} \frac{\sin 2(x-a)}{(x-a)(x^2+b^2)}\,dx = \frac{\pi}{a^2+b^2} - \frac{\pi e^{-2b}}{(b^2+ab)}(\cos 2a - \sin 2a)$$

(iv) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(x-a)\sin(x-b)}{(x-a)(x-b)}\,dx = \frac{\pi}{a-b}\sin(a-b)$

Sol:-

$$\int_{-\infty}^{\infty} \frac{\sin(x-a)\sin(x-b)}{(x-a)(x-b)}\,dx \qquad -2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$$

Step I :- $\displaystyle -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(x-a+x-b)-\cos(x-a-x+b)}{(x-a)(x-b)} \Rightarrow \sin\alpha\sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(2x-a-b)\,dx}{(x-a)(x-b)} - \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(b-a)\,dx}{(x-a)(x-b)} \rightarrow (A)$$

Replace "x" by "z" and $\cos(2x-a-b) = e^{i(2z-a-b)}$, $\cos(b-a) = e^{i(b-a)}$

$$= -\frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i(2z-a-b)}}{(z-a)(z-b)}\,dz - \frac{1}{2}\int \frac{e^{i(b-a)}}{(z-a)(z-b)}\,dz$$

II : The poles of $f(z)$ are at $z=a$, $z=b$.

$$R(f,a) = \left.\frac{(z-a)\,e^{i(2z-a-b)}}{(z-a)(z-b)}\right|_{z=a}, \quad R(f,a) = \left.\frac{(z-a)\,e^{i(b-a)}}{(z-a)(z-b)}\right|_{z=a}$$

$$= \frac{e^{i(2a-a-b)}}{a-b}, \quad R(f,a) = \frac{e^{(b-a)i}}{a-b}$$

$$R(f,a) = \frac{e^{i(a-b)}}{a-b}, \quad R(f,a) = \frac{e^{i(b-a)}}{a-b}$$

$$R(f,b) = \left.\frac{(z-b)\,e^{i(2z-a-b)}}{(z-a)(z-b)}\right|_{z=b}, \quad R(f,b) = \left.\frac{(z-b)\,e^{i(b-a)}}{(z-a)(z-b)}\right|_{z=b}$$

$$R(f,b) = \frac{e^{i(2b-a-b)}}{b-a}, \quad R(f,b) = \frac{e^{i(b-a)}}{b-a}$$

$$R(f,b) = \frac{e^{i(b-a)}}{b-a}, \quad R(f,b) = \frac{e^{i(b-a)}}{b-a}$$

Step III :-

$$\int_{-\infty}^{\infty} \frac{\cos(2x-a-b)\,dx}{(x-a)(x-b)} = \text{Real}\left[\pi i \times \frac{e^{i(a-b)}}{a-b} + \frac{e^{i(b-a)}}{b-a}\right]$$

$$= \text{Real}\left[\frac{\pi i}{a-b}\left(e^{i(a-b)} - e^{-i(a-b)}\right)\right]$$

$$= \text{Real}\left[\frac{\pi i}{a-b}\left(2i\sin(a-b)\right)\right] = \frac{-2\pi}{a-b}\sin(a-b)$$

$$\int_{-\infty}^{\infty} \frac{\cos(b-a)\,dx}{(x-a)(x-b)} = \text{Real}\left[\pi i\left[\frac{e^{i(b-a)}}{a-b} - \frac{e^{i(b-a)}}{a-b}\right]\right]$$

$$\int_{-\infty}^{\infty} \frac{\cos(b-a)x\, dx}{(x-a)(x-b)} = 0$$

$$\int_{-\infty}^{\infty} \frac{\sin(x-a)\sin(x-b)\,dx}{(x-a)(x-b)} = \frac{1}{2}\left[\frac{-2\pi}{a-b}\sin(a-b)+0\right]$$

$$= \frac{\pi}{a-b}\left[\sin(a-b)\right]$$

$$\int_{-\infty}^{\infty} \frac{\sin(x-a)\sin(x-b)\,dx}{(x-a)(x-b)} = \frac{\pi}{a-b}\sin(a-b)$$

(V)

$$\int_{-\infty}^{\infty} \frac{(x^2 - ix^2 + x + 2)}{x^4-5x^2+4}\,dx = 0$$

Sol:-

$$f(x) = \frac{(i^2 (x^2+x+2))}{x^4-5x^2+4}$$

Replace 'x' by z

$$\int_C \frac{(z^2 - iz^2 + z + 2)}{z^4-5z^2+4}\,dz$$

The pole of $f(z) \Rightarrow z^4-5z^2+4=0$ are

$$z^4-4z^2-z^2+4=0$$

$$z^2(z^2-4)-1(z^2-4)=0$$

$$\Rightarrow (z^2-4)(z^2-1)=0$$

$$\Rightarrow z^2-4=0, \quad z^2-1=0$$

$$\Rightarrow z^2=4, \quad z^2=1$$

$$\Rightarrow z=\pm 2, \quad z=\pm 1$$

are at real

$$R_{x_1}(f,2) = \left(\frac{\cancel{(z-2)}\,(z^2 - iz^2 + z + 2)}{\cancel{(z-2)}(z+2)(z^2-1)}\right)_{z=2} = \frac{4+2+2}{4(3)}$$

$$R_{x_1}(f,2) = \frac{8}{12} = \frac{2}{3}$$

$$R_{x_2}(f,-2) = \left(\frac{\cancel{(z+2)}\,(z^2 - iz^2 + z + 2)}{(z-2)\cancel{(z+2)}(z^2-1)}\right)_{z=-2}$$

$$R_{x_2}(f,-2) = \left(\frac{z^2+z+2}{(z-1)(z-2)}\right)_{z=-2} = \frac{4}{(3)(-4)} = -\frac{1}{3}$$

$$R_{x_2}(f,-2) = -\frac{1}{3}$$

$$R_{x_3}(f,1) = \left(\frac{(z-1)(z^2-1)(z^2+z+2)}{(z^2-1)(z+2)(z-2)}\right)_{z=1}$$

$$= \left(\frac{z^2+z+2}{(z+1)(z^2-2^2)}\right)_{z=1} = \frac{1+1+2}{(1+1)(1-4)} = -\frac{2}{3}$$

$$R_{x_3}(f,1) = -\frac{2}{3}$$

$$R_{x_4}(f,-1) = \left(\frac{(z+1)(z^2-1)(z^2+z+2)}{(z^2-1)(z+2)(z-2)}\right)_{z=-1}$$

$$= \left(\frac{z^2+z+2}{(z-1)(z^2-4)}\right)_{z=-1} = \frac{1-1+2}{(-1-1)(1-4)} = \frac{2}{(-2)(-3)}$$

$$R_{x_4}(f,-1) = \frac{1}{3}$$

$$R_x = R_{x_1} + R_{x_2} + R_{x_3} + R_{x_4}$$

$$R_x = \frac{2}{3} - \frac{1}{3} - \frac{2}{3} + \frac{1}{3}$$

$$\int_{-\infty}^{\infty} \frac{(x^2-1)(x+2)\,dx}{x^4-5x^2+4} = 0$$

(vi)

$$\int_0^{\infty} \frac{\cos 2\alpha x - \cos 2\beta x}{x^2}\,dx = \pi(\beta-\alpha)$$

$$\beta > \alpha > 0$$

Sol:-

$$\int_0^{\infty} \frac{\cos 2\alpha x - \cos 2\beta x}{x^2}\,dx = \int_0^{\infty} \frac{\cos 2\alpha x}{x^2}\,dx - \int_0^{\infty} \frac{\cos 2\beta x}{x^2}\,dx \rightarrow (1)$$

(a) $$\int_0^{\infty} \frac{\cos 2\alpha x}{x^2}\,dx$$

Replace "x" by z and $\cos 2\alpha x = e^{2i\alpha z}$
Then, the integral becomes.

$$\int_C \frac{e^{2i\alpha z}}{z^2}\,dz.$$

The pole of $f(z)$ is $z=0$ of order "2"

$$R_x(f,0) = \frac{1}{(2-1)!}\frac{d}{dz}\left((z-0)^2\frac{e^{2i\alpha z}}{z^2}\right)_{z=0}$$

$$= \frac{d}{dz}\left(e^{2i\alpha z}\right)_{z=0}$$

$$= \left(2\alpha i\, e^{2i\alpha z}\right)_{z=0} = (2\alpha i e^0)$$

$$= 2i\alpha$$

Now $\int_0^\infty \frac{\cos 2\beta x}{x^2}\,dx$

$$= \int_C \frac{e^{2i\beta z}}{z^2}\,dz.$$

The pole is $z=0$ of order "2".

$$R_x(f,0) = \frac{d}{dz}\left((z-0)^2\frac{e^{2i\beta z}}{z^2}\right)_{z=0}$$

$$= \frac{d}{dz}\left(e^{2i\beta z}\right)_{z=0}$$

$$R_x(f,0) = \left(2i\beta\, e^{2i\beta z}\right)_{z=0} = 2i\beta$$

Putting these in (1).

$$\int_0^\infty \frac{\cos 2\alpha x}{x^2} = \text{Real}\,\frac{1}{2}[\pi i \times 2i\alpha] = -\frac{2}{2}\alpha\pi = -\alpha\pi$$

$$\int_0^\infty \frac{\cos 2\beta x}{x^2}\,dx = \text{Real}\,\frac{1}{2}[\pi i \times 2i\beta] = -\beta\pi$$

$$\int_0^\infty \frac{\cos 2\alpha x - \cos 2\beta x}{x^2}\,dx = -\alpha\pi - (-\beta\pi)$$

$$= -\alpha\pi + \beta\pi$$

$$\int_0^\infty \frac{\cos 2\alpha x - \cos 2\beta x}{x^2}\,dx = \pi(\beta - \alpha)$$

(vii) $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-a)(x-b)} dx = \frac{\pi}{b-a} (\cos mb - \cos ma)$

Sol:-

Consider the integral $\int_{-\infty}^{\infty} \frac{\sin mx}{(x-a)(x-b)} dx$

Step I:

Replace x by z and $\sin mx = e^{imz}$

$$\int_c \frac{e^{imz}}{(z-a)(z-b)} dz$$

Step II:- The poles of $f(z) = \frac{1}{(z-a)(z-b)}$ are at

$z = a$ and $z = b$.

$$R_{x_1}(f, a) = \left((z-a) \frac{1 \cdot e^{imz}}{\cancel{(z-a)}(z-b)} \right)_{z=a}$$

$$= z\left(\frac{e^{imz}}{z-b} \right)_{z=a}$$

$$R_{x_1}(f, a) = \frac{e^{ima}}{a-b}$$

$$R_{x_2}(f, b) = \left((z-\cancel{b}) \frac{1 \cdot e^{imz}}{(z-a)(z-\cancel{b})} \right)_{z=b}$$

$$= \left(\frac{1 e^{imz}}{z-a} \right)_{z=b}$$

$$R_{x_2}(f, b) = b \frac{e^{imb}}{b-a}$$

$$R_x = R_{x_1} + R_{x_2}$$

$$= \frac{e^{ima}}{a-b} + \frac{e^{imb}}{b-a}$$

$$= \frac{1}{a-b} e^{ima} + \frac{1}{b-a} e^{imb}$$

$$R_x = \frac{1}{a-b} \{\cos ma + i \sin ma\} + \frac{1}{b-a} (\cos mb + i \sin mb)$$

Step III:-

$$\int_{-\infty}^{\infty} \frac{\operatorname{Sin} mx}{(x-a)(x-b)}\,dx = \operatorname{Img}(2\pi i R_p + \pi i R_x)$$

$$= \operatorname{Img}\left(\pi i\left(\frac{1}{a-b}(\operatorname{Cos} ma + i\operatorname{Sin} ma)\right.\right.$$

$$\left.\left.-\frac{1}{a-b}(\operatorname{Cos} mb + i\operatorname{Sin} mb)\right)\right)$$

$$= \operatorname{Img}\left[\frac{\pi}{a-b}\left[i\operatorname{Cos} ma + i^2\operatorname{Sin} ma\right.\right.$$

$$\left.\left.-i\operatorname{Cos} mb - i^2\operatorname{Sin} mb\right]\right]$$

$$= \operatorname{Img}\left[\frac{\pi}{a-b}(\operatorname{Cos} ma\, i - \operatorname{Sin} ma\right.$$

$$\left.-\operatorname{Cos} mb\, i + \operatorname{Sin} mb)\right]$$

$$= \operatorname{Img}\left[\frac{\pi}{a-b}(\operatorname{Cos} ma - \operatorname{Cos} mb)i\right.$$

$$\left.-\operatorname{Sin} ma + \operatorname{Sin} mb)\right]$$

$$\int_{-\infty}^{\infty} \frac{\operatorname{Sin} mx}{(x-a)(x-b)}\,dx = \frac{\pi}{a-b}(\operatorname{Cos} ma - \operatorname{Cos} mb)$$

$$\int_{-\infty}^{\infty} \frac{\operatorname{Sin} mx}{(x-a)(x-b)}\,dx = \frac{\pi}{b-a}(\operatorname{Cos} mb - \operatorname{Cos} ma)$$

(viii)

$$\int_{-\infty}^{\infty} \frac{2\operatorname{Sin}\alpha x\operatorname{Cos}\beta x\,dx}{x(x^2+a^2)} = \int_{-\infty} \frac{\operatorname{Sin}(\alpha+\beta)x + \operatorname{Sin}(\alpha-\beta)x}{x(x^2+a^2)}\,dx$$

Separate it and then solve.

Sol:-

Consider the integral $\int_{-\infty}^{\infty} \frac{2\operatorname{Sin}\alpha x\operatorname{Cos}\beta x\,dx}{x(x^2+a^2)}$

Step I:-

Replace "x" by "z" and $\operatorname{Sin}\alpha x = e^{i\alpha x}$

$\operatorname{Cos}\beta x = e^{i\beta z}$

then $\displaystyle\int_{-\infty}^{\infty} \frac{2e^{i\alpha z}e^{i\beta z}}{z(z^2+a^2)}\,dz = \int_{-\infty}^{\infty} \frac{2e^{iz(\alpha+\beta)}}{z(z^2+a^2)}\,dz$

The poles of $f(z)$ are at $z=0$, $z=\pm ai$

The only poles which lie on real axis is $z=0$ and upper half plane is $z=ai$

$$R_x(f,0) = \left(\frac{(z-0)\,2e^{iz(\alpha+\beta)}}{z(z^2+a^2)}\right)_{z=0}$$

$$= \left(\frac{2e^{iz(\alpha+\beta)}}{z^2+a^2}\right)_{z=0}$$

$$R_x(f,0) = \frac{2e^0}{0+a^2} = \frac{2}{a^2}$$

$$R_p(f,ai) = \left(\frac{\cancel{(z-ai)}\cdot 2e^{iz(\alpha+\beta)}}{z(z+ai)\cancel{(z-ai)}}\right)_{z=ai}$$

$$= \frac{2e^{i(ai)(\alpha+\beta)}}{ai(ai+ai)}$$

$$= \frac{2e^{ai^2(\alpha+\beta)}}{ai(2ai)}$$

$$= \frac{2e^{-a(\alpha+\beta)}}{2a^2i^2}$$

$$= \frac{e^{-a(\alpha+\beta)}}{-a^2}$$

$$R_p(f,ai) = -\frac{e^{-a(\alpha+\beta)}}{a^2}$$

$$\int_{-\infty}^{\infty}\frac{2\sin\alpha x\cos\beta x\,dx}{x(x^2+a^2)} = \left[2\pi i\left(-\frac{e^{-a(\alpha+\beta)}}{a^2}\right)+\pi i\frac{2}{a^2}\right]$$

$$= \left[\frac{2\pi i}{a^2} - \frac{2\pi i e^{-a(\alpha+\beta)}}{a^2}\right]$$

$$\int_{-\infty}^{\infty}\frac{2\sin\alpha x\cos\beta x\,dx}{x(x^2+a^2)} = \frac{2\pi}{a^2}\left[1-e^{-a(\alpha+\beta)}\right] \text{ or } \frac{\pi}{a^2}\left[2-(e^{-a(\alpha+\beta)}+e^{-a(\alpha-\beta)}\right]$$

(ix)

$$\int_0^{\infty}\frac{x^4}{x^6-1}\,dx$$

Sol:-

$$\int_0^{\infty}\frac{x^4}{x^6-1}\,dx$$

Step I:- Replace "x" by "z" and

$$\int_c \frac{z^4}{z^6-1}\,dz$$

Step II:-

Poles of $f(z)$ are at $z^6-1=0 \Rightarrow z^6=1$

or $z^6-1=0 \Rightarrow (z^2-1)(z^4+z^2+1)=0$

$\Rightarrow z^2-1=0$ or $z^4+z^2+1=0 \Rightarrow (z^2+z+1)(z^2-z+1)=0$

$z=\pm 1$ $\quad z^2+z+1=0, \quad z^2-z+1=0$

$z=1$ $\quad z=\frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i)$

$z=-1$ $\quad z=\frac{1}{2}(-1-\sqrt{3}i), \frac{1}{2}(1-\sqrt{3}i)$

The only poles which lie on the upper half plane are $z=\frac{1}{2}(1+\sqrt{3}i), \frac{1}{2}(-1+\sqrt{3}i)$

And on real axis $z=1, -1$

$$R(f,1) = (z-1)\frac{z^4}{(z-1)(z+1)(z^4+z^2+1)}\Bigg|_{z=1}$$

$$= \frac{1}{2(3)} = \frac{1}{6}$$

$$R(f,-1) = (z+1)\frac{z^4}{(z-1)(z+1)(z^4+z^2+1)}\Bigg|_{z=-1}$$

$$= \frac{+1}{-6} = \frac{-1}{6}$$

$$R\left(f, \tfrac{1}{2}(1+\sqrt{3}i)\right) = \left(z-\tfrac{1}{2}(1+\sqrt{3}i)\right)\frac{z^4}{(z^2-1)(z-\frac{1}{2}(1+\sqrt{3}i))(z-\frac{1}{2}(-1+\sqrt{3}i))(z-\frac{1}{2}(1-\sqrt{3}i))(z-\frac{1}{2}(-1-\sqrt{3}i))}\Bigg|_{z=\frac{1}{2}(1+}$$

$$= \frac{\left(\frac{1}{2}(1+\sqrt{3}i)\right)^4}{\left(\frac{1}{4}(1+\sqrt{3}i)^2-1\right)\left(\frac{1+\sqrt{3}i+1-\sqrt{3}i}{2}\right)\left(\frac{1+\sqrt{3}i-1+\sqrt{3}i}{2}\right)\left(\frac{1+\sqrt{3}i+1+\sqrt{3}i}{2}\right)}$$

$$= \frac{(\frac{1}{2})^4(1+\sqrt{3}i)^4}{\frac{1-3+2\sqrt{3}i-4}{4}\,(1)\,(\sqrt{3}i)(1+\sqrt{3}i)}$$

$$= \frac{1}{4\cdot 4}(1+\sqrt{3}i)^3 \Big/ \frac{2\sqrt{3}-6}{4}(\sqrt{3}i)$$

$$= \frac{\frac{1}{8}(1+\sqrt{3}i)^3}{8(\sqrt{3}i-3)\sqrt{3}i} = \frac{(\frac{1}{2}+\frac{\sqrt{3}}{2}i)^3}{(\sqrt{3}i-3)\sqrt{3}i} = \frac{(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})^3}{(\sqrt{3}i-3)(\sqrt{3}i)}$$

$$= \frac{\cos\pi + i\sin\pi}{\sqrt{3}i(\sqrt{3}i-3)} = \frac{-1}{(\sqrt{3}i-3)\sqrt{3}i}$$

Now $R(f, \frac{1}{2}(-1+\sqrt{3}i)) = \left(\frac{z^4}{(z-\frac{1}{2}(1+\sqrt{3}i))(z-\frac{1}{2}(-1-\sqrt{3}i))(z-\frac{1}{2}(1-\sqrt{3}i))}\right)_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i}$

$$= \frac{(\frac{1}{2})^4(-1+\sqrt{3}i)^3}{(\frac{1}{4}(-1+\sqrt{3}i)^2-1)(1)(\sqrt{3}i)(-1+\sqrt{3}i)}$$

$$= \frac{\frac{1}{2\cdot 2^3}(-1+\sqrt{3}i)^3}{\frac{1}{4}(\sqrt{3}i)(\cancel{2}(3+\sqrt{3}i)}$$

$(+1+3i^2-2\sqrt{3}i)(-1+\sqrt{3}i)$

$\Rightarrow [-1+\sqrt{3}i+3-3\sqrt{3}i+2\sqrt{3}i-2(3)i$

$\Rightarrow [2-6i]$

$$= \frac{\frac{1}{2}\left[-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right]^3}{\frac{1}{\cancel{4}}\sqrt{3}i(\cancel{2}(3+\sqrt{3}i))} = \frac{\frac{1}{2}\left[\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right]^3}{\frac{1}{2}\sqrt{3}i(3+\sqrt{3}i)}$$

$$= \frac{\cos(2\pi)+i\sin(2\pi)}{3(\sqrt{3}i-1)} = \frac{1}{3(\sqrt{3}i-1)}$$

Step III :-

$$\int_{-\infty}^{\infty}\frac{x^4\,dx}{x^6-1} = \text{Real}\left[\pi i\left(\frac{1}{6}-\frac{1}{6}\right)+2\pi i\left[\frac{1}{3(\sqrt{3}i+1)}+\frac{1}{3(\sqrt{3}i-1)}\right]\right]$$

$$= \text{Real}\left[\frac{2\pi i}{3}\left[\frac{\cancel{1}-\sqrt{3}i-\cancel{1}-\sqrt{3}i}{(\sqrt{3}i)^2-(1)^2}\right]\right]$$

$$= \text{Real}\left[\frac{2\pi i}{3}\left(\frac{+2\sqrt{3}i}{-4}\right)\right]$$

$$= \text{Real}\left[\frac{\pi}{3}i^2\sqrt{3}\right]$$

$$= -\frac{\pi}{3}\sqrt{3} = -\frac{\pi}{\sqrt{3}}$$

$$\int_{-\infty}^{\infty}\frac{x^4\,dx}{x^6-1} = -\frac{\pi}{\sqrt{3}}$$

$$\int_{0}^{\infty}\frac{x^4\,dx}{x^6-1} = -\frac{\pi}{2\sqrt{3}}$$

Type V

Form of the integral is

$$\int_0^\infty x^{\alpha-1} f(x)\, dx \quad \text{where } \alpha \text{ is fraction}$$

where $x^{-}\phi(x) - x^{\alpha-1} f(x)$

or $\int^{\infty} x^{\infty} f(x)\, dx$

Working Rule:-

Step I:-

Replace x by z and

$\phi(z) = z^{\alpha-1} f(z)\, z^{\alpha-1} = e^{\alpha-1}[\log(-z) + \pi i]$

$z^{\alpha-1} = e^{(\alpha-1)} f(z)$

Step II:-

Find the poles of $f(z)$, calculate all the poles which lie in the whole complex plane. Calculate the residue at these poles. (Poles will not lie in the real axis)

Step III:-

$$\int_0^\infty x^{\alpha-1} f(x)\, dx = \frac{-\pi}{\sin \alpha\pi} e^{-\alpha\pi i} \sum_{j=1}^{n} R_j$$

Ex:-

Prove that $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$, $0<p<1$

Sol:-

Consider the integral $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx$

Step I:-

Replace "x" by "z" and

$x^{p-1} = z^{p-1} = e^{(p-1)[\log(-z) + \pi i]}$ $\quad \phi(z) = e^{(p-1)[\log(-z)+\pi i]} f(z)$

Step II:-

The pole of $f(z) = \frac{1}{z+1}$ is at $z=-1$

$$R(q,-1) = \left((z+1) \frac{e^{(p-1)[\log(-z)+\pi i]}}{z+1} \right)_{z=-1}$$

$= \left(e^{p-1\,(\log(-1)+\pi i)} \cancel{[\log(-z))+\pi i]}\right)_{z=-1}$ $\quad \log(z) = \ln|z| + i\arg z$

$\arg z = \tan^{-1}\frac{y}{x}$

$= e^{(p-1)\pi i}[\log(1) + \pi i]$

$= e^{p-1} . e^{\pi i} + e^{-\pi i}]$ $\quad e^{(p-1)(-\pi i)}$

$= e^{-p\pi i} . e^{\pi i}$

$= e^{p\pi i} . e^{-\pi i}$ $\quad e^{-\pi i} = \cos\pi - i\sin\pi$

$= -e^{p\pi i}$ $\quad e^{-\pi i} = -1.$

Step III :-

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{-\pi}{\sin p\pi}\, e^{-p\pi i} \times R$$

$$= \frac{-\pi}{\sin p\pi}\, e^{-p\pi i} . -e^{p\pi i}$$

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx = \frac{\pi}{\sin p\pi}$$

Ex :-

Prove that $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{x^2+1}\,dx = \frac{\pi}{2}\operatorname{Cosec}\left(\frac{\pi\alpha}{2}\right)$; $0<\alpha<1$

Sol:-

Consider the integral $\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{x^2+1}\,dx$

Step I :-

Replace "x" by "z"

and $z^{\alpha-1} = e^{(\alpha-1)(\log(-z)+\pi i)} \cancel{[\log(-z)+\pi i]}$

$$\int_c \frac{e^{(\alpha-1)(\log(-z)+\pi i)} \cancel{[\log(-z)+\pi i]}}{z^2+1}\,dz$$

$\phi(z) = e^{(\alpha-1)[\log(-z)+\pi i]}$

$f(z)$

Step II :-

The poles of $f(z) = \dfrac{1}{z^2+1}$ are at $z = \pm i$

$$R(\phi, i) = \left(\cancel{(z-i)}\, \frac{e^{(\alpha-1)(\log(-z)+\pi i)} \cancel{[\log(-z)+\pi i]}}{\cancel{(z-i)}(z+i)}\right)_{z=i}$$

$$= \left(\frac{e^{(\alpha-1)[\log(-z)+\pi i]} \cancel{[\log(-z)+\pi i]}}{z+i}\right)_{z=i}$$

$$= \frac{e^{(\alpha-1)[\log(-i)+\pi i]}}{i+i}$$

$$R_1(\varphi, i) = \frac{e^{(\alpha-1)[\log(-i)+\pi i]}}{2i}$$

we know $\log z = \ln|z| + i\arg(z)$

$$\log(-i) = \ln|(-i)| + i\tan^{-1}\left(\frac{-1}{0}\right) \qquad \left(\arg z = \tan^{-1}\frac{y}{x}\right)$$

$$-i = 0 - i$$

$$= \ln\left(\sqrt{0^2+(-1)^2}\right) + i$$

$$= \ln(1)$$

$$\log(-i) = \log e^{-\frac{\pi}{2}i}$$

$$\therefore e^{-\frac{\pi}{2}i} = \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) = -i$$

$$\log(-i) = -\frac{\pi}{2}i$$

$$R_1(\varphi, i) = \frac{e^{(\alpha-1)[(-\frac{\pi}{2}i)+\pi i]}}{2i}$$

$$= \frac{e^{(\alpha-1)[(-\frac{\pi}{2}+\pi)i]}}{2i}$$

$$= \frac{e^{(\alpha-1)[(\frac{-\pi+2\pi}{2})i]}}{2i}$$

$$= \frac{e^{(\alpha-1)[\frac{\pi}{2}i]}}{2i}$$

$$= \frac{e^{(\alpha-1)\frac{\pi}{2}i}}{2i}$$

$$R_2(\varphi, -i) = \left(\frac{\cancel{(z+i)}\, e^{(\alpha-1)[\log(-z)+\pi i]}}{\cancel{(z+i)}(z-i)} \right)_{z=-i}$$

$$= \left(\frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z-i} \right)_{z=-i}$$

$$= \frac{e^{(\alpha-1)[\log(i)+\pi i]}}{-i-i}$$

$$= \frac{e^{(\alpha-1)[\log(i)+\pi i]}}{-2i}$$

$$\log(i) = \ln|i| + i\arg(i)$$

$$\log(i) = \ln\left(\sqrt{(0)^2+(1)^2}\right) + i\left(\frac{1}{0}\right)$$

$$\log(i) = \ln e^{\frac{\pi}{2}i}$$

$$\log(i) = \frac{\pi}{2} i$$

$$R_2(4, -i) = \text{Re}^{(\alpha-1)(\frac{\pi}{2}i+2\pi i)}$$

$$= \log e^{\frac{\pi}{2}+2i} + \log e^{\frac{\pi}{2}2i}$$

$$= e^{(\alpha-1)((\frac{\pi}{2}+\pi)i)}$$

$$R = 2\log 2i^{\frac{\pi}{2}2i}$$

$$R_2(a, -i) = \frac{e^{(\alpha-1)(\frac{3\pi}{2}i)}}{-2i}$$

$$R = R_1 + R_2$$

$$R = R_1 + R_2$$

$$R = \frac{e^{(\alpha-1)(\frac{\pi}{2}i)}}{4\ 2i} - \frac{e^{(\alpha-1)(\frac{3\pi}{2}i)}}{4\ 2i}$$

$$= \frac{e^{(\alpha-1)\frac{\pi}{2}i}}{4} + \frac{e^{(\alpha-1)(\frac{3\pi}{2}i)}}{2i}$$

$$= \frac{e^{\alpha\frac{\pi}{2}i} \cdot e^{-\frac{\pi}{2}i}}{4} - \frac{e^{\alpha\frac{3\pi}{2}i} \cdot e^{-\frac{3\pi}{2}i}}{2i}$$

$$R = \frac{e^{(\alpha\frac{\pi}{2}i)}(-i) - e^{\alpha(\frac{3\pi}{2}i)}(i)}{2i}$$

$$e^{-\frac{\pi}{2}i} = \cos\left(+\frac{\pi}{2}\right) - i\sin\left(\frac{\pi}{2}\right)$$

$$e^{-\frac{\pi}{2}i} = -i$$

$$e^{-\frac{3\pi}{2}i} = \cos\left(\frac{3\pi}{2}\right) - i\sin\left(\frac{3\pi}{2}\right)$$

$$= 0 - i(-1)$$

$$= i$$

~~Step III :-~~

$$R = \frac{\left(-e^{\alpha\frac{\pi}{2}i} \cdot e^{\alpha\frac{3\pi}{2}i}\right)i}{2} \qquad = \frac{-\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i} \cdot R$$

$$R = \frac{-e^{\alpha\pi i}\left[e^{\alpha\frac{\pi}{2}i} + e^{-\alpha\frac{\pi}{2}i}\right]}{2} \qquad \frac{-\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i} \cdot \pi e^{\alpha-1\cdot\frac{\pi}{2}}$$

$$\because \cos\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}$$

$$R = -e^{\alpha\pi i}\cos\left(\frac{\alpha\pi}{2}\right)$$

Step III :-

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^2}\,dx = \frac{-\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i} \cdot R$$

$$= \frac{-\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i}\left[-e^{\alpha\pi i}\cos\frac{\alpha\pi}{2}\right]$$

$$= \frac{\pi}{2\sin\frac{\alpha\pi}{2}\cos\frac{\alpha\pi}{2}}\left(\cos\frac{\alpha\pi}{2}\right)$$

$$\because \sin 2a = 2\sin\frac{a}{2}\cos\frac{a}{2}$$

$$= \frac{\pi}{2\sin\frac{\alpha\pi}{2}}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{Cosec}\left(\frac{\alpha\pi}{2}\right)$$

Assignment :-

(i) $$\int_0^\infty \frac{x^{\alpha-1}}{x+e^{i\beta}}\,dx = \frac{\pi e^{(\alpha-1)\beta i}}{\sin\alpha\pi}$$

Sol :-

$$\int_0^\infty \frac{x^{\alpha-1}}{x+e^{i\beta}}\,dx$$

Replace x by z and

$$z^{\alpha-1} = e^{(\alpha-1)[\log(-z)+\pi i]} \qquad \phi(z) = e^{(\alpha-1)[\log(-z)+\pi i]}\;f(z)$$

$$\int_c \frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z+e^{i\beta}}\,dz$$

The pole of $f(z)$ are at $z = -e^{i\beta}$

$$R(\phi, -e^{i\beta}) = \left((z+e^{i\beta})\left(\frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z+e^{i\beta}}\right)\right)_{z=-e^{i\beta}}$$

$$= \left(e^{(\alpha-1)[\log(-z)+\pi i]}\right)_{z=-e^{i\beta}}$$

$$= e^{(\alpha-1)[\log(+e^{i\beta})+\pi i]} \qquad \log e^{i\beta} = i\beta$$

$$= e^{(\alpha-1)[i\beta+\pi i]}$$

$$= e^{(\alpha-1)[(\beta+\pi)i]} \qquad \because e^{-(\beta+\pi)i} = \cos(\beta+\pi) + i\sin(\beta+\pi)$$

$$= e^{(\alpha\beta+\alpha\pi)i}\cdot e^{-(\beta+\pi)i}$$

$$\int_0^\infty \frac{x^{\alpha-1}}{x+e^{i\beta}}\,dx = \frac{-\pi}{\sin\alpha\pi}\,e^{-\alpha\pi i}\cdot R$$

$$= \frac{-\pi}{\sin\alpha\pi}\cdot e^{-\alpha\pi i}\,e^{\alpha\beta i}\cdot e^{\alpha\pi i}\,e^{-\beta i}\cdot e^{-\pi i} \qquad \because e^{-\pi i} = \cos\pi - i\sin\pi$$

$$= \frac{-\pi}{\sin\alpha\pi}\,e^{\alpha\beta i}\cdot e^{-\beta i}(-1) \qquad e^{-\pi i} = -1+0$$

$$= \frac{\pi}{\sin\alpha\pi}\,e^{(\alpha-1)\beta i}$$

(ii) $$\int_0^\infty \frac{x^\alpha}{(x+a)(x+b)}dx = \frac{\pi}{\sin\alpha\pi}\left[\frac{a^\alpha-b^\alpha}{a-b}\right]$$

Sol:-

$$\int_0^\infty \frac{x^\alpha}{(x+a)(x+b)}dx$$

Replace "x" by z and

$$z^{\alpha-1} = e^{(\alpha-1)[\log(-z)+\pi i]}$$

$$\int_c \frac{z^{(\alpha-1)+1}}{(z+a)(z+b)}dz$$

$$= \int_c \frac{e^{(\alpha)[\log(-z)+\pi i]}}{(z+a)(z+b)}dz \quad ; \phi(z) = \frac{e^{(\alpha)[\log(-z)+\pi i]}}{f(z)}$$

The poles of $f(z)$ are at $z=-a$ and $z=-b$.

$$R_1(\varphi,-a) = \left(\frac{(z+a)\,e^{(\alpha)[\log(-z)+\pi i]}}{(z+a)(z+b)}\right)_{z=-a}$$

$$= \left[\frac{e^{(\alpha)[\log(-z)+\pi i]}}{z+b}\right]_{z=-a} \qquad a = a+0\cdot i$$

$$= \frac{e^{(\alpha)[\log(a)+\pi i]}}{-a+b} \qquad \log(a) = \log|a| + i\arg(a)$$

$$= \frac{e^{(\alpha)[\log a+\pi i]}}{-a+b} \qquad = \sqrt{a^2} + i(0)$$

$$\log(a) = a$$

$$= -\left[\frac{e^{(\alpha)(\log a+\pi i)}}{a-b}\right] \qquad e^{-i\pi} = \cos\pi - i\sin\pi$$

$$= -\frac{(e^{\alpha\log a}\cdot e^{\alpha\pi i})}{a-b} \qquad = 1$$

$$= \frac{-(e^{\log a^\alpha}\cdot e^{\alpha\pi i}\cdot 1)}{a-b}$$

$$R_1(\varphi,-a) = \frac{-a^\alpha\cdot e^{\alpha\pi i}}{a-b}$$

$$R_2(\varphi,-b) = \left(\frac{(z+b)\, e^{(\alpha)[\log(-z)+\pi i]}}{(z+a)(z+b)}\right)_{z=-b}$$

$$= \left[\frac{e^{(\alpha)[\log(-z)+\pi i]}}{(z+a)}\right]_{z=-b}$$

$$= \frac{e^{(\alpha)[\log(b)+\pi i]}}{-b+a}$$

$$= \frac{e^{(\alpha)(\log b+\pi i)}}{a-b}$$

$$R_2(\varphi,-b) = \frac{e^{\alpha\log b}\, e^{\alpha\pi i}}{a-b} = \frac{e^{\log b^\alpha}\, e^{\alpha\pi i}}{a-b} = \frac{b^\alpha\cdot e^{\alpha\pi i}}{a-b}$$

$$R = R_1 + R_2$$

$$R = -\frac{a^\alpha e^{\alpha\pi i}}{a-b} + \frac{b^\alpha e^{\alpha\pi i}}{a-b}$$

$$R = \frac{-e^{\alpha\pi i}\left[a^\alpha - b^\alpha\right]}{a-b}$$

$$\int_0^\infty \frac{x^\alpha}{(x+a)(x+b)}\,dx = \frac{-\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i}\cdot R$$

$$= \frac{-\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i}\left\{-e^{\alpha\pi i}\left[\frac{a^\alpha-b^\alpha}{a-b}\right]\right\}$$

$$= \frac{\pi}{\sin\alpha\pi}\left[\frac{a^\alpha-b^\alpha}{a-b}\right]$$

(iii) $\displaystyle\int_0^\infty \frac{x^\alpha}{(x^2+b^2)^2}\,dx = \frac{\pi b^{\alpha-3}(1-\alpha)}{4\cos(\frac{\alpha\pi}{2})}$, $-1<\alpha<3$, $b>0$

Sol :-

Consider the integral $\displaystyle\int_0^\infty \frac{x^\alpha}{(x^2+b^2)^2}\,dx$

Replace "x" by z and $z^{\alpha-1} = e^{(\alpha-1)[\log(-z)+\pi i]}$

$$\int_c \frac{z^{(\alpha-1)+1}}{z^2+b^2}\,dz \qquad \phi(z) =$$

$$\Rightarrow \phi(z) = e^{(\alpha)[\log(-z)+\pi i]}\cdot f(z) \quad \text{where } f(z) = \frac{1}{z^2+b^2}$$

The poles of $f(z)$ are at $z = \pm bi$

$$R_1(\phi, bi) = \left((z-bi)\frac{e^{(\alpha)[\log(-z)+\pi i]}}{(z-bi)(z+bi)}\right)_{z=bi}$$

$$= \left(\frac{e^{(\alpha)[\log(-z)+\pi i]}}{z+bi}\right)_{z=bi}$$

$$= \frac{e^{(\alpha)[\log(-bi)+\pi i]}}{bi+bi}$$

$\log(-bi) = \log b + \log(-i)$
$= \log b + \log e^{-\pi/2}$
$\log(-bi) = \log b - \frac{\pi}{2} i$

$$= \frac{1}{2bi}\, e^{(\alpha)[\log(b) - \frac{\pi}{2}i + \pi i]}$$

$$= \frac{1}{2bi}\left[e^{\alpha[\log(b)+\frac{\pi}{2}i]}\right]$$

$$= \frac{1}{2bi}\left[e^{\alpha\log(b)}\cdot e^{\frac{\pi}{2}\alpha i}\right]$$

$$= \frac{1}{2bi}\left[e^{\log b^\alpha}\cdot e^{\pi/2\,\alpha i}\right]$$

$$= \frac{1}{2bi}\left[b^\alpha\cdot e^{\alpha\frac{\pi}{2}i}\right]$$

Similarly, $R_2(\phi, -bi) = -\frac{1}{2bi}\left[b^\alpha\cdot e^{\alpha\frac{3\pi}{2}i}\right]$

$R = R_1 + R_2$

$$R = \frac{1}{2bi}\, b^\alpha\cdot e^{\alpha\frac{\pi}{2}i} - \frac{1}{2bi}\, b^\alpha\cdot e^{\alpha\frac{3\pi}{2}i}$$

$$R = \frac{b^{\alpha}}{2bi}\left[e^{\alpha\frac{\pi}{2}i} - e^{\alpha\frac{3\pi}{2}i}\right]$$

$$= \frac{b^{\alpha}}{b}\, e^{\alpha\pi i}\left[\frac{e^{-\alpha\frac{\pi}{2}i} - e^{\alpha\frac{\pi}{2}i}}{2i}\right]$$

$$R = \frac{b^{\alpha}}{b}\, e^{\alpha\pi i}\left(-\sin\alpha\frac{\pi}{2}\right)$$

$$\int_0^\infty \frac{x^{(\alpha+1)-1}}{x^2+b^2}\,dx = \frac{-\pi}{\sin\alpha\pi}\, e^{-(\alpha+1)\pi i}\,.\,R$$

$\sin(\alpha+1)\pi = \sin\alpha\pi + \sin\pi = \sin\alpha\pi + 0$

$$= \frac{-\pi}{\sin\alpha\pi}\, e^{-(\alpha+1)\pi i}\,\frac{b^{\alpha}}{b}\, e^{\alpha\pi i}\left(-\sin\frac{\alpha\pi}{2}\right)$$

$$= \frac{\pi}{\sin\alpha\pi}\, e^{-\alpha\pi i}\cdot e^{+\alpha\pi i}\cdot e^{-\pi i}\,\frac{b^{\alpha}}{b}\sin\frac{\alpha\pi}{2}$$

$$= \frac{\pi}{2\sin\frac{\alpha\pi}{2}\cos\frac{\alpha\pi}{2}}\, e^{0}\cdot(1)\, b^{\alpha-1}\sin\left(\frac{\alpha\pi}{2}\right)$$

$$= \frac{\pi}{2}\, b^{\alpha-1}\,\frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)}$$

Differentiate it w.r.t "b"

$$\int_0^\infty \frac{(x^2+b^2)(0) - (2b)\,x^{\alpha}}{(x^2+b^2)^2}\,dx = \frac{\pi}{2}(\alpha-1)\, b^{\alpha-2}\,\frac{1}{\cos\left(\frac{\alpha\pi}{2}\right)}$$

$$\int_0^\infty \frac{-2bx^{\alpha}}{(x^2+b^2)^2}\,dx = \frac{\pi}{2}\,\frac{(-(1-\alpha))\, b^{\alpha-2}}{\cos\left(\frac{\alpha\pi}{2}\right)}$$

$$\int_0^\infty \frac{x^{\alpha}}{(x^2+b^2)^2}\,dx = -\frac{1}{2b}\left[\frac{\pi}{2}\,\frac{(-(1-\alpha))\, b^{\alpha-2}}{\cos\left(\frac{\alpha\pi}{2}\right)}\right]$$

$$= \frac{\pi}{4}\,\frac{(1-\alpha)\, b^{\alpha-3}}{\cos\left(\frac{\alpha\pi}{2}\right)}$$

(iv)

$$\int_0^\infty \frac{x^{\alpha-1}}{x^2+b^2}\,dx = \frac{\pi}{2}\,\frac{b^{\alpha-2}}{\sin\left(\frac{\alpha\pi}{2}\right)}, \quad 0<\alpha<2.$$

202.

Sol:-

Consider the integral $\int_0^\infty \frac{x^{\alpha-1}}{x^2+b^2}\,dx$

Replace "x" by "z" and

$$z^{\alpha-1} = e^{(\alpha-1)[\log(-z)+\pi i]}$$

$$\int_c \frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z^2+b^2}\,dz \qquad \phi(z) = e^{(\alpha-1)[\log(-z)+\pi i]} \; f(z)$$

The poles of $f(z)$ are at $z = \pm bi$.

$$\cdot\; R_1(\varphi, bi) = \left((z-bi) \frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{(z-bi)(z+bi)} \right)_{z=bi}$$

$$= \left(\frac{e^{(\alpha-1)[\log(-z)+\pi i]}}{z+bi} \right)_{z=bi} \qquad bi -$$

$$= e^{(\alpha-1)[\log(-bi)+\pi i]} \Big/ bi + bi$$

$$\log(-bi) = \log(b) + \log(-i)$$

$$= \log b - \frac{\pi}{2} i$$

$$R_1(\varphi, bi) = \frac{e^{(\alpha-1)[\log b - \frac{\pi}{2} i + \pi i]}}{2bi}$$

$$= \frac{e^{(\alpha-1)[\log b + \frac{\pi}{2} i]}}{2bi}$$

$$= \frac{e^{\alpha \log b} \cdot e^{-\log b} \cdot e^{-\frac{\pi}{2} i} \cdot e^{+\frac{\alpha\pi}{2} i}}{2bi} \qquad \cos\left(\tfrac{\pi}{2}\right) - i\sin\left(\tfrac{\pi}{2}\right),\; e^{-\frac{\pi}{2} i} = -i$$

$$= \frac{e^{\log b^\alpha} \cdot e^{\log b^{-1}} (-i) \cdot e^{+\frac{\alpha\pi}{2} i}}{2bi}$$

$$= -\frac{b^\alpha \cdot b^{-1}\, e^{+\frac{\alpha\pi}{2} i}}{2b} \qquad R_2(\varphi, -bi) = e^{(\alpha-1)[\log(bi)+\pi i]}$$

$$= e^{(\alpha-1)(\log b + \frac{\pi}{2} i + \pi i)}$$

$$= -\frac{b^{\alpha-1-1}\, e^{+\frac{\alpha\pi}{2} i}}{2} \qquad = e^{(\alpha-1)(\log b + \frac{3\pi}{2} i)}$$

$$= e^{\alpha \log b} \cdot e^{-\frac{3\pi}{2} i} \cdot e^{-\log b} \cdot e^{\frac{3\alpha\pi}{2}}$$

$$R_1(\varphi, bi) = -\frac{b^{\alpha-2}\, e^{+\frac{\alpha\pi}{2} i}}{2} \qquad = \frac{b^\alpha b^{-1} \cdot e^{\frac{3\alpha\pi i}{2}} (-i)}{2bi}$$

Similarly, $R_2(\varphi, -bi) = -\dfrac{b^{\alpha-2}\, e^{+\frac{3\alpha\pi}{2} i}}{2}$

$$R = R_1 + R_2$$

$$R = -\frac{b^{\alpha-2}}{2} e^{+\frac{\pi\alpha}{2}i} + \frac{b^{\alpha-2}}{2} e^{+\frac{3\pi}{2}\alpha i}$$

$$= \frac{b^{\alpha-2}}{2}\left[+e^{+\frac{\pi}{2}\alpha i} + e^{+\frac{3\pi}{2}\alpha i}\right]$$

$$= \frac{b^{\alpha-2}}{2} e^{-\alpha\pi i}\left[-e^{-\pi/2\,\alpha i} + e^{+\pi/2\,\alpha i}\right]$$

$$R = -b^{\alpha-2} e^{-\alpha\pi i}\left[\frac{e^{\pi/2\,\alpha i} + e^{-\pi/2\,\alpha i}}{2}\right]$$

$$R = -b^{\alpha-2} e^{-\alpha\pi i} \cos\left(\frac{\pi\alpha}{2}\right)$$

$\pi - \frac{\pi}{2} = \frac{2\pi-\pi}{2} = \frac{\pi}{2}$

$\pi - \frac{3\pi}{2} = \frac{2\pi-3\pi}{2} = -\frac{\pi}{2}$

$$\int_0^\infty \frac{x^{\alpha-1}}{x^2+b^2}\,dx = \frac{-\pi}{\sin\alpha\pi} e^{-\alpha\pi i} \cdot R$$

$$= \frac{-\pi}{\sin\alpha\pi} e^{-\alpha\pi i}\left(-b^{\alpha-2} e^{\alpha\pi i}\cos\left(\frac{\pi\alpha}{2}\right)\right)$$

$$= \frac{+\pi\, b^{\alpha-2}\, e^{-\alpha\pi i+\alpha\pi i}}{2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)} \cos\left(\frac{\pi\alpha}{2}\right)$$

$$\int_0^\infty \frac{x^{\alpha-1}}{x^2+b^2}\,dx = \frac{\pi\, b^{\alpha-2}}{2\sin\left(\frac{\alpha\pi}{2}\right)}$$

(V)

$$\int_0^\infty \frac{x^\alpha}{1+2x\cos\theta + x^2}\,dx = \frac{\pi\sin\alpha\theta}{\sin\alpha\pi\sin\theta}$$

Sol:-

Consider the integral $\int_0^\infty \frac{x^\alpha}{1+2x\cos\theta+x^2}\,dx$

Replace "x" by "z" and $z^{\alpha-1} = e^{(\alpha-1)[\log(-z)+\pi i]}$

$$\int \frac{z^{(\alpha-1)+1}}{1+2z\cos\theta+z^2}\,dz \qquad \phi(z) = e^{\alpha[\log(-z)+\pi i]} f(z)$$

The poles of $f(z)$ are at

$$1+2z\cos\theta+z^2=0$$

$$z^2+2z\cos\theta+1=0$$

$$z=\frac{-2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$=\frac{-2\cos\theta\pm 2i\sin\theta}{2}$$

$$z=-\cos\theta\pm i\sin\theta$$

$$z=\alpha=-\cos\theta+i\sin\theta,\quad z=\beta=-\cos\theta-i\sin\theta$$

$$R_1(\phi,\alpha)=\left[(z-\alpha)\frac{e^{\alpha[\log(-z)+\pi i]}}{(z-\alpha)(z-\beta)}\right]_{z\to\alpha}$$

$$=\left[\frac{e^{\alpha[\log(-z)+\pi i]}}{z-\beta}\right]_{z=\alpha}$$

$$=\frac{e^{\alpha[\log(-\alpha)+\pi i]}}{\alpha-\beta}=\frac{e^{\alpha[\log(\cos\theta-i\sin\theta)+\pi i]}}{\alpha-\beta}$$

$$=\frac{e^{\alpha[\ln e^{-i\theta}+\pi i]}}{2i\sin\theta}=\frac{e^{\alpha(-i\theta+\pi i)}}{2i\sin\theta}$$

Similarly,

$$R_2(\phi,\beta)=\frac{e^{\alpha(i\theta+\pi i)}}{-2i\sin\theta}$$

$$\int_0^\infty\frac{x^\alpha}{1+2x\cos\theta+x^2}\,dx=\frac{-\pi}{\sin(\alpha+1)\pi}\cdot e^{-(\alpha+1)\pi i}\cdot R$$

$$=\frac{-\pi}{-\sin\alpha\pi}\,e^{-(\alpha+1)\pi i}\left[\frac{e^{\alpha(-i\theta+\pi i)}}{2i\sin\theta}+\frac{e^{\alpha(i\theta+\pi i)}}{-2i\sin\theta}\right]$$

$$=\frac{+\pi}{\sin\alpha\pi}\,\frac{e^{-\alpha\pi i}\cdot e^{-\pi i}}{2i\sin\theta}\left[e^{\alpha(-i\theta+\pi i)}-e^{\alpha(i\theta+\pi i)}\right]$$

$$=\frac{+\pi}{2i\sin\theta\sin\alpha\pi}\,e^{-\alpha\pi i}(-1)e^{\alpha\pi i}\left[e^{-\alpha i\theta}-e^{\alpha i\theta}\right]$$

$$=\frac{\pi}{2i\sin\alpha\pi\sin\theta}\,e^0\left[e^{\alpha i\theta}-e^{-\alpha i\theta}\right]$$

$$=\frac{\pi}{2i\sin\alpha\pi\sin\theta}\left[\cancel{\cos\alpha\theta+i\sin\alpha\theta}\ 2i\sin\alpha\theta\right]$$

$$=\frac{\pi\sin\alpha\theta}{\sin\alpha\pi\sin\theta}$$

Type VI

Form of integral $\int_0^{\infty} \frac{x^{\alpha-1} f(x)}{\phi(x)} dx$ or $\int_0^{\infty} \frac{x^{\alpha} f(x)}{\phi(x)} dx$; α is a fraction

Working Rule:-

Step I:-

Replace "x" by "z" and $x^{\alpha-1} = z^{\alpha-1}$

$z^{\alpha-1} = e^{(\alpha-1)\log z}$

$\therefore \phi(z) = e^{(\alpha-1)\log z} f(z)$

Step II:-

Find poles of $f(z)$ and calculate the residues at these poles. (will lie in +ve real axis)

Step III:-

$$\int_0^{\infty} x^{\alpha-1} f(x) dx = -\pi \cot \alpha\pi \sum_{j=1}^{n} R_j$$

Ex:-

Prove that $\int_0^{\infty} \frac{x^{\alpha-1}}{1-x} dx = \pi \cot \alpha\pi$

Sol:-

Consider the integral $\int_0^{\infty} \frac{x^{\alpha-1}}{1-x} dx$

Step I:-

Replace x by z and $z^{\alpha-1} = e^{(\alpha-1)\log z}$

Then, the given integral becomes

$$\int_c e^{(\alpha-1)\log z} \frac{1}{1-z} dz$$

Step II:- $\phi(z) = e^{(\alpha-1)\log z} f(z)$

The poles of $f(z) = \frac{1}{1-z}$ is at $z = 1$

$$R(\phi, 1) = \left((z-1) \frac{e^{(\alpha-1)\log(z)}}{1-z} \right)_{z=1}$$

$$= \left(-e^{(\alpha-1)\log z} \right)_{z=1}$$

$$= -e^{(\alpha-1)\log(1)}$$

$$= -e^{0} \qquad \because \log(1) = 0$$

$R(\phi, 1) = -1$

Step III:-

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx = -\pi \cot\alpha\pi \times R$$

$$= -\pi \cot\alpha\pi \times -1$$

$$\int_0^\infty \frac{x^{\alpha-1}}{1-x}\,dx = \pi \cot\alpha\pi$$

Ex:-

Prove that $\int_0^\infty \frac{x^\alpha}{(x^2-b^2)(x^2-c^2)}\,dx = \frac{\pi}{2(b^2-c^2)}[b^{\alpha-1} - c^{\alpha-1}]$

$(\operatorname{Cosec}\alpha\pi - \cot\alpha\pi)$

Sol:-

Consider $\int_0^\infty \frac{x^\alpha}{(x^2-b^2)(x^2-c^2)}\,dx$

Replace "x" by z.

$$\int_0^\infty \frac{z^{(\alpha+1)-1}}{(z^2-b^2)(z^2-c^2)}\,dz$$

The poles of $f(z) = \frac{1}{(z^2-b^2)(z^2-c^2)}$ are

at $z = \pm b, \pm c$

$$R(\phi, b) = \left(\frac{(z-b)\,e^{\alpha\log z}}{(z-b)(z+b)(z^2-c^2)}\right)_{z=b}$$

$$= \left(\frac{e^{\alpha\log z}}{(z+b)(z^2-c^2)}\right)_{z=b}$$

$$= \frac{e^{\alpha\log b}}{2b(b^2-c^2)}$$

$$= \frac{e^{\log b^\alpha}}{2b(b^2-c^2)}$$

$$R(\phi, b) = \frac{b^\alpha}{2b(b^2-c^2)}$$

$$R_2(\varphi, c) = \left(\frac{(z-c)\, e^{\alpha \log z}}{(z-c)(z+c)(z^2-b^2)} \right)_{z=c}$$

$$= \left(\frac{e^{\alpha \log z}}{(z+c)(z^2-b^2)} \right)_{z=c}$$

$$= \frac{e^{\alpha \log c}}{2c(c^2-b^2)}$$

$$R_2(\varphi, c) = \frac{c^{\alpha}}{2c(c^2-b^2)}$$

$$R_3(\varphi, -b) = \left[\frac{(z+b)\, e^{\alpha[\log(-z) + \pi i]}}{(z+b)(z-b)(z^2-c^2)} \right]_{z=-b}$$

$$= \left[\frac{e^{\alpha[\log(-z) + \pi i]}}{(z-b)(z^2-c^2)} \right]_{z=-b}$$

$$= \frac{e^{\alpha[\log(b) + \pi i]}}{(-b-b)(b^2-c^2)}$$

$$= \frac{e^{\alpha \log b} \cdot e^{\alpha \pi i}}{-2b(b^2-c^2)}$$

$$= \frac{b^{\alpha} \cdot e^{\alpha \pi i}}{-2b(b^2-c^2)}$$

$$R_4(\varphi, -c) = \left[\frac{\cancel{(z+c)}\, e^{\alpha[\log(-z) + \pi i]}}{(z^2-b^2)\cancel{(z+c)}(z-c)} \right]_{z=-c}$$

$$= \left[\frac{e^{\alpha[\log(-z) + \pi i]}}{(z^2-b^2)(z-c)} \right]_{z=-c}$$

$$= \frac{e^{\alpha[\log(c) + \pi i]}}{(c^2-b^2)(-c-c)}$$

$$= \frac{e^{\alpha \log c} \cdot e^{\alpha \pi i}}{-2c(-(b^2-c^2))}$$

$$= \frac{c^{\alpha} \cdot e^{\alpha\pi i}}{2c(b^2-c^2)}$$

Step III :-

$$\int_0^\infty \frac{x^\alpha}{(x^2-b^2)(x^2-c^2)}\,dx = \frac{-\pi}{\sin(\alpha+1)\pi} e^{-(\alpha+1)\pi i}(R_3+R_4) - \pi\cot(\alpha+1)\pi\,(R_1+R_2)$$

$$= \frac{-\pi}{-\sin\alpha\pi} e^{-\alpha\pi i} \cdot e^{-\pi i}\left[\frac{b^\alpha \cdot e^{\alpha\pi i}}{-2b(b^2-c^2)} + \frac{c^\alpha\, e^{\alpha\pi i}}{2c(b^2-c^2)}\right] - \pi\cot\alpha\pi\left[\frac{b^\alpha}{2b(b^2-c^2)} + \frac{c^\alpha}{2c(c^2-b^2)}\right]$$

$$= \frac{\pi}{\sin\alpha\pi} e^{-\alpha\pi i} \cdot e^{\alpha\pi i}(-1)\left[\frac{1}{-2(b^2-c^2)}(b^{\alpha-1} - c^{\alpha-1})\right] - \pi\cot\alpha\pi\left[\frac{1}{2(b^2-c^2)}(b^{\alpha-1} - c^{\alpha-1})\right]$$

$$= \frac{\pi}{\sin\alpha\pi} \times \frac{1}{2(b^2-c^2)}(b^{\alpha-1} - c^{\alpha-1}) - \pi\cot\alpha\pi\left[\frac{1}{2(b^2-c^2)}(b^{\alpha-1} - c^{\alpha-1})\right]$$

$$= \frac{1}{2(b^2-c^2)}\left[b^{\alpha-1} - c^{\alpha-1}\right]\left[\frac{\pi}{\sin\alpha\pi} - \pi\cot\alpha\pi\right]$$

$$= \frac{1}{2(b^2-c^2)}\left[b^{\alpha-1} - c^{\alpha-1}\right]\left[\pi\operatorname{cosec}\alpha\pi - \pi\cot\alpha\pi\right]$$

Assignment :-

(i) Prove that $\int_0^\infty \frac{x^{1/2}}{x^2+x+1}\,dx = \frac{\pi}{\sqrt{3}}$

(ii) $\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx = \frac{\pi(1-\alpha)}{4\cos\left(\frac{\alpha\pi}{2}\right)}$

Sol :-

(i) Consider the integral $\int_0^\infty \frac{x^{1/2}\,dx}{x^2+x+1}$

$\alpha = \frac{1}{2} = 1 - \frac{1}{2}$; $\alpha = \frac{3}{2} - 1$

Replace 'x' by 'z' and

$z^{\alpha-1} = e^{(\alpha-1)\log(z)}$

$$\int_0^\infty \frac{z^{(\alpha-1+1)}}{z^2+z+1}\,dz = \int_0^\infty \frac{z^{1/2-1+1}}{z^2+z+1}\,dz = \int_0^\infty \frac{z^{3/2-1}}{z^2+z+1}\,dz$$

$z^{3/2-1} = e^{(3/2-1)\log(z)}$

The poles of $f(z) = \dfrac{1}{z^2+z+1}$ are at

$$z^2+z+1=0$$

$$a=1,\ b=1,\ c=1$$

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

$z = \alpha = \dfrac{-1+\sqrt{3}i}{2}$

$z = \beta = \dfrac{-1-\sqrt{3}i}{2}$

$$R(\varphi, \alpha) = \left[\frac{(z-\alpha)\, e^{\alpha[\log(z)+\pi i]}}{(z-\alpha)(z-\beta)}\right]_{z=\alpha}$$

$$= \left[\frac{e^{\alpha[\log(z)+\pi i]}}{z-\beta}\right]_{z=\alpha}$$

$$= \left[\frac{e^{\alpha[\log(\alpha)+\pi i]}}{\alpha-\beta}\right]$$

$$R_1(\varphi, \alpha) = \frac{e^{\alpha[\log(\frac{1-\sqrt{3}i}{2}) + \pi i]}}{\alpha-\beta} \quad \frac{-1+\sqrt{3}i+1+\sqrt{3}i}{}$$

$$R_2(\varphi, \beta) = \left(\frac{(z-\beta)\, e^{\alpha[\log(1) + i\tan^{-1}(\frac{\sqrt{3}}{2}) + \pi i]}}{(z-\alpha)(z-\beta)}\right)_{z=\beta} \qquad \therefore \tan^{-1}\frac{\sqrt{3}}{2} = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

$$\sqrt{3}i$$

$$= \left(\frac{e^{\alpha[\log(1) + i\frac{\pi}{3} + \pi i]}}{(z-\sqrt{3}i)}\right)_{z=\beta}$$

$$= \frac{e^{\alpha[\log\beta + \frac{3\pi i}{3}]}}{\beta-\sqrt{3}i} \qquad \therefore \alpha = 1/2$$

$$= \frac{e^{\log\beta(\frac{4\pi}{3}i)}}{\beta-\sqrt{3}i}$$

$$R(\varphi, \beta) = \frac{\beta^{1/2} e^{1/2(\frac{2\pi}{3}i)}}{\beta-\alpha\ \sqrt{3}i} = \frac{1}{\sqrt{3}i}\, e^{\pi/3\, i}$$

$$R(\phi, \beta) = R\left[(z-\beta)\, e^{\alpha[\log(-z)+\pi i]}\right]$$

$$R = \frac{\alpha^{1/2}}{\alpha-\beta} = \frac{\beta^{1/2}(z-\alpha)(z-\beta)}{e^{\alpha[\log(-\beta)+\pi i]}\,(\alpha-\beta)(\beta-\alpha)} \quad z=\beta$$

$$R = \frac{\alpha^{1/2}}{} \frac{e^{\alpha[\log\left(\frac{1+\sqrt{3}i}{2}\right)+\pi i]}}{\alpha-\beta} = \frac{e^{\alpha[\log(1)+i\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)+\pi i]}}{\frac{-1-\sqrt{3}i}{2} - \frac{(-1+\sqrt{3}i)}{2}} \quad \frac{1}{2}-\frac{\sqrt{3}}{2}i-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$= \frac{e^{\alpha[\log(1)+i\frac{\pi}{3}+\pi i]}}{-2\sqrt{3}/2\, i} = \frac{e^{\alpha[\frac{\pi}{3}i+\pi i]}}{-\sqrt{3}i}$$

$$= \frac{e^{\alpha[0+(\frac{\pi}{3}+\pi)i]}}{-\sqrt{3}i} = \frac{e^{\alpha[\frac{i\pi+3\pi i}{3}]}}{-\sqrt{3}i} = \frac{e^{\alpha(\frac{4\pi i}{3})}}{-\sqrt{3}i} = \frac{e^{\frac{2\pi}{3}i}}{-\sqrt{3}i} \quad \because \alpha = 1/2$$

$$R(\phi, \beta) = \frac{e^{\frac{2\pi}{3}i}}{-\sqrt{3}i}$$

Step III :-

$$\int_0^\infty \frac{x^{1/2}\,dx}{x^2+x+1} = \frac{-\pi}{\sin(\alpha+1)\pi}$$

$$\int_0^\infty \frac{x^{1/2}\,dx}{x^2+x+1} = \frac{-\pi}{\sin(\frac{1}{2}+1)\pi}\,\frac{e^{-3/2\,\pi i}}{\sqrt{3}i}\left[e^{\pi/3\, i} - e^{\frac{2\pi}{3}i}\right]$$

$$= \frac{-\pi}{\sin(\frac{3\pi}{2})}\,\frac{e^{-3/2\,\pi i}}{\sqrt{3}i}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} - \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right]$$

$$= \frac{-\pi}{(-1)}\,\frac{1}{\sqrt{3}i}\, e^{-3/2\,\pi i}\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]$$

$$= \frac{\pi}{\sqrt{3}i}\left[\cos(\frac{3\pi}{2}) - i\sin(\frac{3\pi}{2})\right]\left[\frac{2}{2}\right]$$

$$= \frac{\pi}{\sqrt{3}i}\left[0 - i(-1)\right]$$

$$= \frac{\pi}{\sqrt{3}i}(0+i)$$

$$= \frac{\pi}{\sqrt{3}i}\, i$$

$$\int_0^\infty \frac{x^{1/2}\,dx}{x^2+x+1} = \frac{\pi}{\sqrt{3}}$$

Ex :-

$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx = \frac{\pi(1-\alpha)}{4\cos(\frac{\alpha\pi}{2})}$$

Sol :-

$$\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}$$

Step I :-

Replace "x" by "z"

$$\int_0^\infty \frac{z^\alpha}{(z^2+1)^2}\,dz \qquad \phi(z) = z^\alpha \cdot f(z)$$

The poles of $f(z) = \frac{1}{(z^2+1)^2}$ are at $z = +i$ of order "2"

$$R(\phi, i) = \frac{d}{dz}\left(\frac{(z-i)^2 e^{\alpha[\log(-z)+\pi i]}}{(z-i)^2(z+i)^2}\right)_{z=i}$$

$$= \left[\frac{(z+i)^2 e^{\alpha[\log(-z)+\pi i]} \cdot \alpha(-\frac{1}{z})(-1) - e^{\alpha[\log(-z)+\pi i]} \cdot 2(z+i)}{(z+i)^4}\right]_{z=i}$$

$$= \left[\frac{\alpha(z+i)e^{\alpha[\log(-z)+\pi i]} - 2z e^{\alpha[\log(-z)+\pi i]}}{z(z+i)^3}\right]_{z=i}$$

$$= \frac{\alpha(2i)e^{\alpha[\log(-i)+\pi i]} - 2i\, e^{\alpha[\log(-i)+\pi i]}}{i(2i)^3}$$

$$= \frac{\alpha e^{\alpha[-\frac{\pi}{2}i+\pi i]} - e^{\alpha[\log(1)-\frac{\pi}{2}i+\pi i]}}{i(2i)^2}$$

$$= \frac{(\alpha-1)e^{\alpha(\frac{\pi}{2}i)}}{-4i} = \frac{(1-\alpha)\, e^{\frac{\alpha\pi}{2}i}}{4i}$$

2.12

$$R(\phi,-i)=\frac{d}{dz}\left(\frac{(z+i)^2 e^{\alpha[\log(-z)+\pi i]}}{(z-i)^2(z+i)^2}\right)_{z=i}$$

$$=\left[\frac{(z-i)^2 e^{\alpha[\log(-z)+\pi i]}\cdot\alpha\left(\frac{1}{z}\right)-e^{\alpha[\log(-z)+\pi i]}\cdot 2(z-i)}{(z-i)^4}\right]_{z=-i}$$

$$=\left\{\frac{\alpha(z-i)e^{\alpha[\log(-z)+\pi i]}-2ze^{\alpha[\log(-z)+\pi i]}}{z(z-i)^3}\right\}_{z=-i}$$

$$=\frac{\alpha(-2i)e^{\alpha[\log(i)+\pi i]}+2ie^{\alpha[\log(i)+\pi i]}}{-i(-2i)^3}$$

$$=\frac{2i(-\alpha+1)\,e^{\alpha[0+\frac{\pi}{2}i+\pi i]}}{i\,8i^2}$$

$$=\frac{(1-\alpha)e^{\frac{3\pi\alpha}{2}i}}{-4i}$$

$$R(\phi,-i)=\frac{(1-\alpha)e^{\alpha\frac{3\pi}{2}i}}{-4i}$$

$$\int_0^\infty\frac{x^\alpha\,dx}{(x^2+1)^2}=\frac{-\pi}{\sin(\alpha+1)\pi}\;e^{-(\alpha+1)\pi i}$$

$$=\frac{-\pi}{\sin(\alpha+1)\pi}\;e^{-(\alpha+1)\pi i}\;\frac{(1-\alpha)}{4i}\left[e^{-\alpha\frac{3\pi}{2}i}+e^{\frac{\alpha\pi i}{2}}\right]$$

$$=\frac{-\pi}{-\sin\alpha\pi}\;e^{-\alpha\pi i}\cdot e^{-\pi i}\cdot e^{\alpha\pi i}\;\frac{(1-\alpha)}{4i}\left[e^{\frac{\alpha\pi}{2}i}+e^{-\frac{\alpha\pi}{2}i}\right]$$

$$=\frac{\pi}{\sin\alpha\pi}\;e^0\,(+1)\;\frac{(1-\alpha)}{2\cancel{4}i}\left[\frac{e^{\frac{\alpha\pi}{2}i}-e^{-\frac{\alpha\pi}{2}i}}{2i}\right]$$

$$=\frac{\pi}{\sin\alpha\pi}\;\frac{(1-\alpha)}{2}\;\sin\left(\frac{\alpha\pi}{2}\right)$$

$$=\frac{\pi}{2\sin\frac{\alpha\pi}{2}\cos\frac{\alpha\pi}{2}}\;\frac{(1-\alpha)}{2}\;\sin\left(\frac{\alpha\pi}{2}\right)$$