Notes written by:

Shaniza Ghafoor

(Bs mathematics)

Dedicated

To

My

Worthy class teacher

Prof. Fazal Abbas sajid

03014695644

Chapter No. 06

Curvilinear Coordinates

Question No. 01:-

Consider the curvilinear coordinat system defined by

$u_1 = x+y$, $u_2 = x-y$, $u_3 = 2z$

i. Solve for x, y & z in term of u_1, u_2 & u_3.

$u_1 = x+y \longrightarrow (1)$

$u_2 = x-y \longrightarrow (2)$

$u_3 = 2z \longrightarrow (3)$

Add (1) & (2)

$u_1 + u_2 = x + y - y + x$

$u_1 + u_2 = 2x$

$$\boxed{x = \frac{u_1+u_2}{2}}$$

Subtract (1) & (2)

$u_1 - u_2 = x + y - x + y$

$u_1 - u_2 = 2y$

$$\boxed{y = \frac{u_1 - u_2}{2}}$$

from (3) $\Rightarrow u_3 = 2z$

$$\boxed{z = \frac{u_3}{2}.}$$

(ii) Show that the system is orthogonal and left-handed. Compute the scalar factors.

Sol:-

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\left(\frac{U_1+U_2}{2}\right)\hat{i} + \left(\frac{U_1-U_2}{2}\right)\hat{j} + \left(\frac{U_3}{2}\right)\hat{k}$$

$$\frac{\partial \vec{r}}{\partial u_1} = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$$h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \Rightarrow \boxed{1/\sqrt{2}}$$

$$\hat{e}_1 = \frac{\partial \vec{r}/\partial u_1}{|\partial \vec{r}/\partial u_1|}$$

$$= \frac{1/2\,\hat{i} + 1/2\,\hat{j}}{1/\sqrt{2}} \Rightarrow \frac{\sqrt{2}}{2}\hat{i} + \frac{\sqrt{2}}{2}\hat{j}$$

$$\boxed{\hat{e}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}}$$

Now,

$$\frac{\partial \vec{r}}{\partial u_2} = \frac{1}{2}\hat{i} - \frac{1}{2}\hat{j}$$

$$h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right| = \sqrt{\left(\frac{1}{2}\right)^2 + (-1/2)^2}$$

$$\boxed{h_2 = 1/\sqrt{2}}$$

written by shaniza credit goes to prof fazal

$\hat{e_1} \times \hat{e_2} = \hat{e_3}$ (Right handed)
$\hat{e_1} \times \hat{e_2} = -\hat{e_3}$ (left handed).

written by shaniza credit goes to prof fazal

$$\hat{e}_2 = \frac{\partial \vec{r}/\partial u_2}{|\partial \vec{r}/\partial u_2|} \quad 03014695644$$

$$\Rightarrow \boxed{\hat{e}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}}$$

Now,

$$\frac{\partial \vec{r}}{\partial u_3} = \frac{1}{2}\hat{k}$$

$$h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| \Rightarrow \sqrt{\left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{4}}$$

$$\boxed{h_3 = \frac{1}{2}}$$

$$\hat{e}_3 = \frac{\partial \vec{r}/\partial u_3}{|\partial \vec{r}/\partial u_3|} = \frac{1/2\,\hat{k}}{1/2} \Rightarrow \boxed{\hat{k} = \hat{e}_3}$$

Orthogonal:-

$\rightarrow$ $\hat{e}_1 \cdot \hat{e}_2 = 0$

$$\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right) = 0$$

$$1/2 - 1/2 = \boxed{0} \quad \text{proved}$$

$\rightarrow$ $\hat{e}_2 \cdot \hat{e}_3 = 0$

$$\left(\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}\right)(\hat{k}) = 0$$

$$0 = 0 \quad \text{proved.}$$

$\rightarrow$ $\hat{e}_3 \cdot \hat{e}_1 = 0$

$$(\hat{k})\left(\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$$

$$0 = 0 \quad \text{proved.}$$

Left handed:-

$$\hat{e}_1 \times \hat{e}_2 \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{vmatrix}$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(-1/2 - 1/2)$$

$$\Rightarrow -\hat{k} \Rightarrow -\hat{e}_3$$

$$\boxed{\hat{e}_1 \times \hat{e}_2 = -\hat{e}_3}$$

$$\hat{e}_2 \times \hat{e}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \hat{i}(-1/\sqrt{2}) - \hat{j}(1/\sqrt{2}) + \hat{k}$$

$$-1(1/\sqrt{2}\,\hat{i} + 1/\sqrt{2}\,\hat{j}) \Rightarrow -\hat{e}_1$$

$$\boxed{\hat{e}_2 \times \hat{e}_3 = -\hat{e}_1}$$

$$\hat{e}_3 \times \hat{e}_1 = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{vmatrix}$$

$$= \hat{i}(0 - 1/\sqrt{2}) - \hat{j}(0 - 1/\sqrt{2}) + \hat{k}(0)$$

$$= -1/\sqrt{2}\,\hat{i} + 1/\sqrt{2}\,\hat{j}$$

$$= -(1/\sqrt{2}\,\hat{i} - 1/\sqrt{2}\,j) \Rightarrow -\hat{e}_2$$

$$\boxed{\hat{e}_3 \times \hat{e}_1 = -\hat{e}_2}$$ proved.

(iii) Find the expression for $(ds)^2$ in this system.

$$x = \frac{u_1 + u_2}{2}, \quad y = \frac{u_1 - u_2}{2}, \quad z = u_3/2$$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= \left(\frac{1}{2}\left[\frac{1}{2}(du_1 + du_2)\right]\right)^2 + \left(\frac{1}{2}\left[du_1 - du_2\right]\right)^2 +$$

$$\left(\frac{1}{2}\left[du_3\right]\right)^2$$

$$= \frac{1}{4}\left[du_1^2 + du_2^2 + 2du_1du_2 + du_1^2 + du_2^2 - 2du_1du_2 + du_3^2\right]$$

$$(ds)^2 = \frac{1}{2\cdot 2}\left[2du_1^2 + 2du_2^2 + du_3^2\right]$$

$$\boxed{(ds)^2 = \frac{1}{2}du_1^2 + \frac{1}{2}du_2^2 + \frac{1}{4}du_3^2}$$

$h_1^2 = 1/2$, $h_2^2 = 1/2$, $h_3^2 = 1/4$.

iv- Find $\Delta\psi$ for $\psi(u_1, u_2, u_3) = u_1 + u_2 + 2u_3$

$$h_1 = \frac{1}{\sqrt{2}}, \quad h_2 = \frac{1}{\sqrt{2}}, \quad h_3 = \frac{1}{2}$$

$$\hat{e}_1 = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}, \quad \hat{e}_2 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}, \quad \hat{e}_3 = \hat{k}$$

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{e}_3$$

$$\nabla\psi = \frac{1}{1/\sqrt{2}}(1)\left[\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right] + \frac{1}{1/\sqrt{2}}\left[\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}j\right] + \frac{1}{2}(2)\hat{k}$$

$$\nabla\psi = \sqrt{2}\left[\frac{1}{\sqrt{2}}i + \cancel{\frac{1}{\sqrt{2}}}j + \frac{1}{\sqrt{2}}i - \cancel{\frac{1}{\sqrt{2}}}j\right] + k$$

$$\nabla\psi = \cancel{\sqrt{2}}\left[\frac{2\hat{i}}{\cancel{\sqrt{2}}}\right] + \hat{k}$$

written by shaniza Ghafoor credit goes to prof fazal

$$\boxed{\nabla\psi = 2\hat{i} + 0\hat{j} + \hat{k}}$$

v. Find the expression for $\nabla^2\psi$ relative to this coordinate system.

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \right.$$

$$\left.\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\cdot\frac{\partial\psi}{\partial u_3}\right)\right].$$

$$= \frac{1}{1/\sqrt{2}\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2}}\left[\frac{\partial}{\partial u_1}\frac{(1/\sqrt{2})(1/2)(1)}{1/\sqrt{2}} + \frac{\partial}{\partial u_2}\frac{(1/\sqrt{2})(1/2)(1)}{1/\sqrt{2}}\right.$$

$$\left. + \frac{\partial}{\partial u_3}\frac{(1/\sqrt{2})(1/\sqrt{2})(2)}{1/2}\right].$$

$$(\sqrt{2})^2 2\,[0+0+0]$$

$$4\,[0+0+0] \Rightarrow \boxed{\nabla^2\psi = 0}$$

Scanned with CamScanner

Problem 2:-

Consider the curvilinear coordinates system defined by

$$x = u_1^2 + u_2^2, \quad y = 2u_1u_2, \quad z = u_3$$

(i) Show that the system is orthogonal and right handed. Compute the scalar factors.

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = (u_1^2 + u_2^2)\hat{i} + (2u_1u_2)\hat{j} + (u_3)\hat{k}$$

$$\frac{\partial \vec{r}}{\partial u_1} = 2u_1\hat{i} + 2u_2\hat{j} + 0\hat{k}$$

Now; $$h_1 = |\partial\vec{r}/\partial u_1|$$

$$\Rightarrow \sqrt{4u_1^2 + 4u_2^2} \Rightarrow \boxed{2\sqrt{u_1^2 + u_2^2} = h_1}$$

$$\hat{e_1} = \frac{\partial\vec{r}/\partial u_1}{|\partial\vec{r}/\partial u_1|}$$

$$= \frac{2(u_1\hat{i} + u_2\hat{j})}{2\sqrt{u_1^2 + u_2^2}} \Rightarrow \boxed{\hat{e_1} = \frac{u_1\hat{i} + u_2\hat{j}}{\sqrt{u_1^2 + u_2^2}}}$$

Now,

$$\frac{\partial\vec{r}}{\partial u_2} = -2u_2\hat{i} + 2u_1\hat{j} = 0$$

$$h_2 = |\partial\vec{r}/\partial u_2| \Rightarrow \sqrt{4u_2^2 + 4u_2^2}$$

$$\boxed{h_2 = 2\sqrt{u_2^2 + u_2^2}}$$

$$\hat{e}_2 = \frac{\partial \vec{r} / \partial u_2}{|\partial \vec{r} / \partial u_2|} \Rightarrow \frac{\cancel{2}(-u_2 \hat{i} + u_1 \hat{j})}{\cancel{2}\sqrt{u_2^2 + u_1^2}}$$

$$\hat{e}_2 = \frac{-u_2 \hat{i} + u_1 \hat{j}}{\sqrt{u_2^2 + u_1^2}}$$

Now,

$$\frac{\partial \vec{r}}{\partial u_3} = \hat{k}$$

$$h_3 = |\partial \vec{r} / \partial u_3| \Rightarrow \boxed{h_3 = 1}$$

$$e_3 = \frac{\partial \vec{r} / \partial u_3}{h_3} \Rightarrow \boxed{e_3 = \hat{k}}$$

Orthogonal:-

written by shaniza credit goes to prof fazal

$\rightarrow$ $\hat{e}_1 \cdot \hat{e}_2 = 0$

$$\left(\frac{u_1 \hat{i} + u_2 \hat{j}}{\sqrt{u_1^2 + u_2^2}} \right) \left(\frac{-u_2 \hat{i} + u_1 \hat{j}}{\sqrt{u_1^2 + u_2^2}} \right)$$

$$\Rightarrow \frac{1}{\sqrt{u_1^2 + u_2^2}} \left[-u_1 u_2 \hat{i} + u_1 u_2 \hat{j} \right] \Rightarrow -u_1 u_2 (i \cdot i) + u_1 u_2 (j \cdot j)$$

$$-u_1 u_2 (1) + u_1 u_2 (1)$$

$$\Rightarrow \boxed{0} \text{ proved}$$

$\rightarrow$ $\hat{e}_2 \cdot \hat{e}_3 = 0$

$$\left(\frac{-u_2 \hat{i} + u_1 \hat{j}}{\sqrt{u_2^2 + u_1^2}} \right) (\hat{k})$$

$$\Rightarrow \boxed{0}$$

$\rightarrow$ $\hat{e}_3 \cdot \hat{e}_1 = 0 \quad \Rightarrow (\hat{k}) \left(\frac{u_1 \hat{i} + u_2 \hat{j}}{\sqrt{u_1^2 + u_2^2}} \right)$

$\Rightarrow \boxed{0}$ proved.

Right handed:-

$$\hat{e}_1 \times \hat{e}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1/\sqrt{u_1^2+u_2^2} & u_2/\sqrt{u_1^2+u_2^2} & 0 \\ -u_2/\sqrt{u_1^2+u_2^2} & u_2/\sqrt{u_1^2+u_2^2} & 0 \end{vmatrix}$$

$$= \hat{k}\left(\frac{u_1^2}{u_2^2+u_1^2} + \frac{u_2^2}{u_1^2+u_2^2}\right)$$

$$= \left(\frac{u_1^2+u_2^2}{u_1^2+u_2^2}\right)\hat{k}$$

$$\boxed{\hat{e}_1 \times \hat{e}_2 = \hat{e}_3 = \hat{k}}$$

$$\hat{e}_2 \times \hat{e}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -u_2/\sqrt{u_1^2+u_2^2} & u_1/\sqrt{u_1^2+u_2^2} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \hat{i}\left(\frac{u_1}{\sqrt{u_1^2+u_2^2}}\right) - \hat{j}\left(\frac{-u_2}{\sqrt{u_1^2+u_2^2}}\right)$$

$$\frac{u_1\hat{i} + u_2\hat{j}}{\sqrt{u_1^2+u_2^2}}$$

$$\boxed{\hat{e}_2 \times \hat{e}_3 = \hat{e}_1}$$ proved

$$\hat{e}_3 \times \hat{e}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ u_1/\sqrt{u_1^2+u_2^2} & u_2/\sqrt{u_1^2+u_2^2} & 0 \end{vmatrix}$$

$$= \hat{i}\left(\frac{-u_2}{\sqrt{u_1^2+u_2^2}}\right) - \hat{j}\left(\frac{-u_1}{\sqrt{u_1^2+u_2^2}}\right) \Rightarrow \frac{-u_2\hat{i} + u_1\hat{j}}{\sqrt{u_1^2+u_2^2}}$$

$$= \boxed{\hat{e}_3 \times \hat{e}_1 = \hat{e}_2}$$ proved.

$$h_1 = h_2 = 2\sqrt{u_1^2+u_2^2}$$
$$h_3 = 1$$

* Find $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$ for the vector field

$$\vec{A} = u_3\hat{e}_1 + u_1\hat{e}_2 + u_2\hat{e}_3$$

$\nabla \cdot \vec{A} = ?$

03014695644

$$\nabla \cdot \vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$= \frac{1}{(2\sqrt{u_1^2+u_2^2})(2\sqrt{u_1^2+u_2^2})(1)}\left[\frac{\partial}{\partial u_1}\left(\frac{1}{2\sqrt{u_1^2+u_2^2}}\right)(u_3) + \frac{\partial}{\partial u_2}\left(\frac{1}{2\sqrt{u_1^2+u_2^2}}(u_2)\right)\right.$$
$$\left. + \frac{\partial}{\partial u_3}\left(\frac{1}{4(u_1^2+u_2^2)}\right)u_2\right]$$

$$= \frac{1}{4(u_1^2+u_2^2)}\left[2\cdot\frac{1}{2}(u_2^2+u_1^2)^{-1/2}(2u_1)(u_3) + 2\cdot\frac{1}{2}(u_1^2+u_2^2)^{-1/2}\, 2u_2u_1 + 0\right]$$

$$\boxed{\nabla \cdot \vec{A} = \frac{1}{4(u_1^2+u_2^2)}\left[\frac{2u_1u_3}{\sqrt{u_1^2+u_2^2}} + \frac{2u_2}{\sqrt{u_1^2+u_2^2}} \cdot u_1\right]}$$

$\nabla \times \vec{A} = ?$

$$\nabla \times \vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e} \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{4(u_1^2+u_2^2)}\begin{vmatrix} 2\sqrt{u_1^2+u_2^2}\cdot\frac{(u_1\hat{i}+u_2\hat{j})}{\sqrt{u_1^2+u_2^2}} & 2\sqrt{u_1^2+u_2^2}\,\frac{(-u_2\hat{i}+u_1\hat{j})}{\sqrt{u_1^2+u_2^2}} & \hat{k} \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ 2\sqrt{u_1^2+u_2^2}\,(u_3) & 2\sqrt{u_1^2+u_2^2}\,(u_1) & 1\,(u_2) \end{vmatrix}$$

written by shaniza credit goes to prof fazal

$$= \frac{1}{4(u_1^2+u_2^2)} \begin{vmatrix} 2(u_1\hat{i}+u_2\hat{j}) & 2(-u_2\hat{i}+u_1\hat{j}) & \hat{k} \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ 2\sqrt{u_1^2+u_2^2}\,(u_3) & 2\sqrt{u_1^2+u_2^2}\,(u_1) & 1(u_2) \end{vmatrix}$$

$$= \frac{1}{4(u_1^2+u_2^2)} \left[(u_1\hat{i}+u_2\hat{j})(1) - (-u_2\hat{i}+u_1\hat{j})(0-2\sqrt{u_1^2+u_2^2}) + \hat{k} \left(2\sqrt{u_1^2+u_2^2} + u_1\left(\frac{2u_1}{\sqrt{u_1^2+u_2^2}}\right)\right)\right]$$

$$= \frac{1}{4(u_1^2+u_2^2)} \left[i\left(u_1 - u_2\sqrt{u_1^2+u_2^2}\right) + j\left(u_2 + 2u_1\sqrt{u_1^2+u_2^2}\right) + \hat{k}\left(\frac{2u_1^2+2u_1^2+u_2^2}{\sqrt{u_1^2+u_2^2}}\right)\right]$$

$$\boxed{= \frac{1}{4(u_1^2+u_2^2)} \left[\hat{i}\left(u_1 - u_2\sqrt{u_1^2+u_2^2}\right) + \hat{j}\left(u_2 + 2u_1\sqrt{u_1^2+u_2^2}\right) + k\left(\frac{3u_1^2+2u_2^2}{\sqrt{u_1^2+u_2^2}}\right)\right]}$$

(iii)

Find the expression for $\nabla^2\psi(u_1,u_2,u_3)$

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\psi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right]$$

$$= \frac{1}{4(u_1^2+u_2^2)}\left(\frac{\partial}{\partial u_1}\frac{2\sqrt{u_1^2+u_2^2}}{2\sqrt{u_1^2+u_2^2}}\frac{\partial\psi}{\partial u_1} + \frac{\partial}{\partial u_2}\frac{2\sqrt{u_1^2+u_2^2}}{2\sqrt{u_1^2+u_2^2}}\frac{\partial\psi}{\partial u_2} + \frac{\partial}{\partial u_3}\frac{4(u_1^2+u_2^2)\partial\psi}{\partial u_3}\right)$$

$$\boxed{\nabla^2\psi = \frac{1}{4(u_1^2+u_2^2)}\left[\frac{\partial^2\psi}{\partial u_1^2} + \frac{\partial^2\psi}{\partial u_2^2} + 4(u_1^2+u_2^2)\frac{\partial^2\psi}{\partial u_3^2}\right]}$$

Question No. 03

Suppose that u_1, u_2, u_3 are orthogonal curvilinear

Coordinates for which
$(ds)^2 = u_2^2 du_1^2 + u_1^2 du_2^2 + du_3^2$

(i)

Calculate $\nabla \cdot \hat{e_1}$ where $\hat{e_1}$ is the unit vector tangent to a u_1-curve.

written by shaniza

Sol:-

$$(ds)^2 = u_2^2 du_1^2 + u_1^2 du_2^2 + du_3^2$$

$$(ds)^2 = h_1^2 (du_1)^2 + h_2 (du_2)^2 + h_3 (du_3)^2 \quad ((\text{compare})$$

$$h_1 = u_2, \quad h_2 = u_1, \quad h_3 = 1$$

03014695644
prof fazal

$$\vec{A} = A_1 \hat{e_1} + A_2 \hat{e_2} + A_3 \hat{e_3}$$

$$\nabla \cdot \hat{e_1} = \vec{A} \; ; \quad A_1 = 1, \quad A_2 = 0, \quad A_3 = 0$$

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} h_2 h_3 A_1 + \frac{\partial}{\partial u_2} h_1 h_3 A_2 + \frac{\partial}{\partial u_3} h_1 h_2 A_3 \right]$$

$$= \frac{1}{u_2 u_1 (1)} \left[\frac{\partial}{\partial u_1} (u_1)(1)(1) + 0 + 0 \right]$$

$$= \frac{1}{u_2 u_1} (1) \Rightarrow \boxed{\nabla \cdot \hat{e_1} = \frac{1}{u_2 u_1}}$$

(ii) Find $\nabla^2 \psi$ if $\psi = (u_1, u_2 u_3)$

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_2} \right) \right].$$

$$= \frac{1}{(U_2)(U_1)(1)}\left[\frac{\partial}{\partial U_1}\left(\frac{U_1(1)}{U_2}(U_2 U_3)\right) + \frac{\partial}{\partial U_2}\left(\frac{U_2(1)}{U_1}(U_1)(U_3)\right)\right.$$

$$\left. + \frac{\partial}{\partial U_3}\left(U_2(U_1)(U_1 U_2)\right)\right]$$

$$= \frac{1}{(U_2)(U_1)}\left[U_3 + U_3 + 0\right]$$

$$\boxed{\nabla^2 \psi = \frac{2U_3}{(U_1)(U_2)}}$$ Answer.

CYLINDERICAL COORDINATE SYSTEM:-

$P(r, \theta, z) = P(x, y, z)$

$r = r\cos\theta \hat{i} + r\sin\theta \hat{j} + z\hat{k}$

$h_r = 1, \quad h_\theta = r, \quad h_z = 1$

$$\begin{bmatrix} \hat{e_r} \\ \hat{e_\theta} \\ \hat{e_z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{e_r} \\ \hat{e_\theta} \\ \hat{e_z} \end{bmatrix}$$

$\vec{A} = A_r\hat{e_r} + A_\theta\hat{e_\theta} + A_z\hat{e_z}$.

written by shaniza Ghafoor
credit goes to prof fazal

Question No. 4:-
Transform $\vec{A} = \frac{x}{y}\hat{i}$ into cylinderical polar coordinates.

Sol:- $\vec{A} = \frac{x}{y}\hat{i}$ put the values

$$\vec{A} = \frac{r\cos\theta}{r\sin\theta}\,(\cos\theta\,\hat{e_r} - \sin\theta\,\hat{e_\theta})$$

$$\vec{A} = \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\,\hat{e_r} - \frac{\cos\theta}{\sin\theta}\cdot\sin\theta\,\hat{e_\theta}$$

$$\boxed{\vec{A} = \cot\theta\cdot\cos\theta\,\hat{e_r} - \cos\theta\,\hat{e_\theta}}$$

Question 05:- 03014695644

In cylinderical polar coordinates $\vec{A} = r\hat{e_r} + r\hat{e_\theta}$. Transform $\vec{A}$ into rectangular carterian coordinates:-

Sol:- $\vec{A} = r\hat{e_r} + r\hat{e_\theta}$ put the values.

$$\vec{A} = \sqrt{x^2+y^2}\,[(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}) + (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})]$$

$$= \sqrt{x^2+y^2}\,[(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) + (\sin\theta\,\hat{j} + \cos\theta\,\hat{j})]$$

$$= \sqrt{x^2+y^2}\,[(\frac{x}{r} - \frac{y}{r})\hat{i} + (\frac{y}{r} + \frac{x}{r})\hat{j}]$$

$$\sqrt{x^2+y^2}\,[(\frac{x-y}{r})\,\hat{i} + (\frac{y+x}{r})\hat{j}]$$

$$\sqrt{x^2+y^2}\,\frac{[(x-y)\hat{i} + (y+x)\hat{j}]}{r} \qquad \because r = \sqrt{x^2+y^2}$$

$\boxed{\vec{A} = (x-y)\hat{i} + (y+x)\hat{j}}$

Question No 6:-

In cylindrical polar coordinates, if $\psi = r^2 z \sin\theta\cos\theta$, find $\nabla\psi$ at the point $r=1$, $\theta = \pi/4$, $z = 2$

Solution:-

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e_1} + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{e_2} + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{e_3}$$

$h_1 = 1$, $h_2 = r$, $h_3 = 1$.

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial u_1}\hat{e_r} + \frac{1}{r}\frac{\partial\psi}{\partial u_2}\hat{e_z} + \frac{1}{1}\frac{\partial\psi}{\partial u_3}\hat{e_3}$$

$$= 1(2rz\sin\theta\cos\theta)\hat{e_r} + \frac{1}{r}\left(rz\cdot\frac{1}{2}(\cos 2\theta)\,2\hat{e_\theta} + 1(r^2\sin\theta\cos\theta)\hat{e_z}\right.$$

$$= rz\sin 2\theta\,\hat{e_r} + rz(\cos 2\theta)\hat{e_\theta} + r^2\sin\theta\cos\theta\,\hat{e_z}$$

put $r=1$, $\theta = \pi/4$, $z=2$

$$\nabla\psi = (1, \tfrac{\pi}{4}, 2) = 2(1)(2)\sin\pi/4\cos\pi/4\,\hat{e_r} + (1)(2)\cos 2(\cos\pi/4)\sin 2\theta\; e_\theta + (1)^2\sin\pi/4\cos\pi/4\,\hat{e_z}$$

$$= 2(\sin 2\pi/4)\hat{e_r} + 0\,\hat{e_\theta} + \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\hat{e_z}$$

$$\boxed{\nabla\psi = 2\hat{e_r} + \frac{1}{2}\hat{e_z}}$$

Question No. 7:-

Verify that in cylindrical polar

Coordinates $\nabla(\ln r) = \nabla \times (\theta \hat{e_z})$

L.H.S

$$\nabla\psi = \nabla(\ln r)$$

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial u_1}\hat{e_1} + \frac{1}{h_2}\frac{\partial\psi}{\partial u_2}\hat{e_2} + \frac{1}{h_3}\frac{\partial\psi}{\partial u_3}\hat{e_3}$$

$$= 1\frac{\partial\psi}{\partial r}\hat{e_r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e_r} + 1\frac{\partial\psi}{\partial z}\hat{e_z}$$

put $\psi = \ln r$

$$\nabla\ln r = 1\frac{\partial \ln r}{\partial r}\hat{e_r} + \frac{1}{r}\frac{\partial \ln r}{\partial\theta}\hat{e_\theta} + \frac{\partial \ln r}{\partial z}\hat{e_z}$$

$$= \frac{1}{r}\hat{e_r} + 0 + 0$$

$$\boxed{\nabla\ln r = \frac{1}{r}\hat{e_r}}$$

written by shaniza Ghafoor
credit goes to prof fazal

R.H.S

$$\nabla\times\vec{A} = \nabla\times(\theta\hat{e_z})$$

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e_1} & h_2\hat{e_2} & h_3\hat{e_3} \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$\nabla\times(\theta\hat{e_z}) = \frac{1}{(1)(r)(1)}\begin{vmatrix} 1\hat{e_r} & r\hat{e_\theta} & 1\hat{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 1(0) & r(0) & 1(\theta) \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{e_r}(1-0) + r\hat{e_\theta}(0-0) + \hat{e_z}(0-0)\right]$$

$$\boxed{\nabla \times (\theta \hat{e_z}) = \frac{1}{r}\hat{e_r}}$$ proved

L.H.S = R.H.S

Question No. 08:-

In cylindrical coordinates $\vec{A} = r\cos\theta\hat{e_r} + r\sin\theta\hat{e_\theta} + 0\hat{e_z}$. Find $\nabla\cdot\vec{A}$.

Sol:-

$$\vec{A} = A_r\hat{e_r} + A_\theta\hat{e_\theta} + A_z\hat{e_z}.$$

$$\nabla\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial u_1}(h_2h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right)$$

$$\nabla\cdot\vec{A} = \frac{1}{(1)(r)(1)}\left(\frac{\partial}{\partial r}(r)(1)A_r + \frac{\partial}{\partial\theta}(1)(1)A_\theta + \frac{\partial}{\partial z}(1)(r)A_z\right).$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos\theta) + \frac{\partial}{\partial\theta}(r\sin\theta) + 0\right].$$

$$= \frac{1}{r}\left[2r\cos\theta + r\cos\theta\right]$$

$$= \frac{1}{r}\cdot r\left[2\cos\theta + \cos\theta\right] \Rightarrow 3\cos\theta$$

$$\boxed{\nabla\cdot\vec{A} = 3\cos\theta}$$

Question No. 09:-

In cylindrical coordinates if $\vec{A}(r,\theta,z) = r\hat{e_r} + z\sin\theta\hat{e_\theta} + rz\hat{e_z}$. Find

(i) $\nabla\cdot\vec{A} = ?$

Sol:- $\vec{A} = r\hat{e_r} + z\sin\theta\hat{e_\theta} + rz\hat{e_r}$.

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial r}(h_2)(h_3)A_r + \frac{\partial}{\partial \theta}(h_1)(h_3)A_\theta + \frac{\partial}{\partial z}(h_1)(h_2)A_z\right]$$

$$= \frac{1}{(1)(r)(1)}\left[\frac{\partial}{\partial r}(r)(1).r + \frac{\partial}{\partial \theta}(1)(1) z\sin\theta + \frac{\partial}{\partial z}(1)(r)(rz)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r} r^2 + \frac{\partial}{\partial \theta} z\sin\theta + \frac{\partial}{\partial z} r^2 z\right]$$

$$= \frac{1}{r}\left[2r + z\cos\theta + r^2\right]$$

$$\boxed{\nabla \cdot \vec{A} = 2 + \frac{z}{r}\cos\theta + r}$$

(ii) $\nabla \times \vec{A} = ?$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_r & h_2\hat{e}_\theta & h_3\hat{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ h_1 A_r & h_2 A_\theta & h_3 A_z \end{vmatrix}$$

$$= \frac{1}{(1)(r)(1)}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ (1)(r) & (r)(z\sin\theta) & (1)(r)z \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{e}_r\left(\frac{\partial}{\partial\theta}(rz) - \frac{\partial}{\partial z}(rz\sin\theta)\right) - r\hat{e}_\theta\left(\frac{\partial}{\partial r}(rz) - \frac{\partial}{\partial z}(r)\right)\right.$$
$$\left. + \hat{e}_z\left(\frac{\partial}{\partial r}(rz\sin\theta) - \frac{\partial}{\partial\theta} r\right)\right]$$

$$= \frac{1}{r}\left[\hat{e}_r(0 - r\sin\theta) - r\hat{e}_\theta(z - 0) + \hat{e}_z(z\sin\theta - 0)\right]$$

$$\boxed{\nabla \times \vec{A} = \frac{1}{r}\left(-r\sin\theta\,\hat{e}_r - rz\,\hat{e}_\theta + z\sin\theta\,\hat{e}_z\right)}$$

Question No. 10

Express the following vector fields in cylindrical polar coordinates and find $\nabla \cdot \vec{A}$ & $\nabla \times \vec{A}$.

1. $\vec{A} = \dfrac{x\hat{i} + y\hat{j}}{x^2+y^2}$

$$\vec{A} = \frac{r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j}}{r^2}$$ put the values of $\hat{i}$ & $\hat{j}$.

$$\vec{A} = \frac{r\cos\theta(\cos\theta\,\hat{e_r} - \sin\theta\,\hat{e_\theta})}{r^2} + \frac{r\sin\theta(\sin\theta\,\hat{e_r} + \cos\theta\,\hat{e_\theta})}{r^2}$$

$$\vec{A} = \frac{\cos^2\theta\,\hat{e_r}}{r} - \frac{\cos\theta\sin\theta\,\hat{e_r}}{r} + \frac{\sin^2\theta\,\hat{e_r}}{r} + \frac{\sin\theta\cos\theta\,\hat{e_\theta}}{r}$$

$$\vec{A} = \frac{1}{r}(\cos^2\theta + \sin^2\theta)\,\hat{e_r}$$

$$\boxed{\vec{A} = \frac{1}{r}(\hat{e_r}).}$$

$\nabla \cdot \vec{A} = ?$

03014695644

prof fazal

$$\nabla \cdot \vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial r}(h_2)(h_3)A_r + \frac{\partial}{\partial \theta}(h_1)(h_3)A_\theta + \frac{\partial}{\partial z}(h_1)(h_2)A_z\right]$$

$$= \frac{1}{(1)(r)(1)}\left[\frac{\partial}{\partial r}(r)(1)\frac{1}{r} + \frac{\partial}{\partial \theta}(1)(1)(0) + \frac{\partial}{\partial z}(1)(r)(0)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(1) + 0 + 0\right]$$

$$\boxed{\nabla \cdot \vec{A} = 0}$$

written by shaniza credit goes to prof fazal

$\nabla \times \vec{A} = ?$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & \hat{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$= \frac{1}{r}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & \hat{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 1/r & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{e_r}(0) - r\hat{e_\theta}(0) + \hat{e_z}(0)\right]$$

$$\boxed{\nabla \times \vec{A} = 0}$$

written by shaniza Ghafoor
credit goes to prof fazal
03014695644

2.

$$\vec{A} = \frac{-y\hat{i} + x\hat{j}}{x^2+y^2}$$

put $x = r\cos\theta$, $y = r\sin\theta$, $\hat{i} = \cos\theta\,\hat{e_r} - \sin\theta\,\hat{e_\theta}$
$, \hat{j} = \sin\theta\,\hat{e_r} + \cos\theta\,\hat{e_\theta}$.

$$\vec{A} = \frac{-r\sin\theta(\cos\theta\,\hat{e_r} - \sin\theta\,\hat{e_\theta}) + r\cos\theta(\sin\theta\,\hat{e_r} + \cos\theta\,\hat{e_\theta})}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \frac{-r\sin\theta\cos\theta\,\hat{e_r} + r\sin^2\theta\,\hat{e_\theta} + r\cos\theta\sin\theta\,\hat{e_r} + r\cos^2\theta\,\hat{e_\theta}}{r^2\cos^2\theta + r^2\sin^2\theta}.$$

$$= \frac{r(\sin^2\theta + \cos^2\theta)\,\hat{e_\theta}}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$\boxed{\vec{A} = \frac{1}{r}\hat{e_\theta}}$$

$\nabla . \vec{A} = ?$

$$\nabla . \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial r}(h_2)(h_3)A_r + \frac{\partial}{\partial \theta}(h_1)(h_3)A_\theta + \frac{\partial}{\partial z}(h_1)(h_2)A_z\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r} r A_r^{\nearrow 0} + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\right)\hat{e_\theta} + \frac{\partial}{\partial z}(1)\,0\right]$$

$$\frac{1}{r} \cdot \frac{1}{r} \Rightarrow \boxed{\nabla . \vec{A} = \frac{1}{r^2}}$$

$\nabla \times \vec{A} = ?$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{e_r} & h_2 \hat{e_\theta} & h_3 \hat{e_z} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ h_1 A_r & h_2 A_\theta & h_3 A_z \end{vmatrix}$$

$$= \frac{1}{r}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & \hat{e_z} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ 0 & r(1/r) & 0 \end{vmatrix}$$

$$\frac{1}{r}\left[\hat{e_r}(0-0) - r\hat{e_\theta}(0-0) + \hat{e_z}(0-0)\right]$$

$$\frac{1}{r}(0)$$

$$\boxed{\nabla \times \vec{A} = 0}$$ Ans.

Q4 (page 338 6.2 theorem)

Question No. 11:-

In cylindrical polar coordinates if $\vec{A} = r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta}$, Evaluate $(\vec{A} \cdot \nabla)\vec{A}$

Sol:- $\vec{A} = r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta}$

$$\vec{A}\cdot\nabla = (r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta})\cdot\left(\hat{e_r}\frac{\partial}{\partial r} + \frac{\hat{e_\theta}}{r}\frac{\partial}{\partial\theta} + \hat{e_z}\frac{\partial}{\partial z}\right)$$

$$= \left(r\cos\theta(\hat{e_r}\cdot\hat{e_r})\frac{\partial}{\partial r} + 0 + 0\right) + \left(\frac{\sin\theta}{r}(1)\frac{\partial}{\partial\theta}\right) + 0 +$$

$$= \left(r\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta}).$$

$$= \left(r\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\vec{A}$$

$$= \left(r\cos\theta\frac{\partial}{\partial r} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta}).$$

$\because \frac{\partial}{\partial r}\hat{e_r} = 0$; $\frac{\partial}{\partial\theta}\hat{e_r} = \hat{e_\theta}$; $\frac{\partial}{\partial z}\hat{e_r} = 0$

$\frac{\partial}{\partial r}\hat{e_\theta} = 0$; $\frac{\partial}{\partial\theta}\hat{e_\theta} = -\hat{e_r}$, $\frac{\partial}{\partial z}\hat{e_\theta} = 0$

$\frac{\partial}{\partial r}\hat{e_z} = 0$, $\frac{\partial}{\partial\theta}\hat{e_z} = 0$; $\frac{\partial}{\partial z}\hat{e_z} = 0$

$$= r\cos\theta\frac{\partial}{\partial r}\left[r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta}\right] + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left[r\cos\theta\,\hat{e_r} + \sin\theta\,\hat{e_\theta}\right]$$

$\frac{\partial\hat{e_\theta}}{\partial r} = 0$ product rule.

$$= r\cos^2\theta\frac{\partial}{\partial r}\left[r\hat{e_r}\right] + r\cos\theta\sin\theta(0) + \sin\theta\frac{\partial}{\partial\theta}\cos\theta\,\hat{e_r} +$$

$$\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(\sin\theta\,\hat{e_\theta}).$$

$$= r\cos^2\theta\left[\hat{e_r} + r(0)\right] + 0 + \sin\theta(-\sin\theta\,\hat{e_r} + \cos\theta(\hat{e_\theta})) +$$

$$\frac{\sin\theta}{r}(\cos\theta\,\hat{e_\theta} + \sin\theta(-\hat{e_r})).$$

$$= r\cos^2\theta\,\hat{e_r} - \sin^2\theta\,\hat{e_r} + \sin\theta\cos\theta\,\hat{e_\theta} + \frac{\sin\theta\cos\theta}{r}\hat{e_\theta} - \frac{\sin^2\theta\,\hat{e_r}}{r}$$

$$\boxed{=\left(r\cos^2\theta-\sin^2\theta-\frac{\sin^2\theta}{r}\right)\hat{e_r}+\hat{e_\theta}\left(\sin\theta\cos\theta+\frac{\sin\theta\cos\theta}{r}\right)}$$

written by shaniza Ghafoor

Question 12:-

In cylindrical polar coordinates if

$\vec{A}=(r\sin\theta\cos\theta+z\cos\theta)\hat{e_r}+(r\cos^2\theta-z\sin\theta)\hat{e_\theta}+r\sin\theta\hat{e_z}$

verify that $\nabla\times\nabla\times\vec{A}=0$

Sol:-

$$\nabla\times\vec{A}=\frac{1}{(1)(r)(1)}\begin{vmatrix}(1)\hat{e_r} & r\hat{e_\theta} & (1)\hat{e_z}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z\\ (1)A_r & rA_\theta & (1)A_z\end{vmatrix}$$

$$=\frac{1}{r}\begin{vmatrix}\hat{e_r} & r\hat{e_\theta} & \hat{e_z}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z\\ r\sin\theta\cos\theta+z\cos\theta & r(r\cos^2\theta-z\sin\theta) & r\sin\theta\end{vmatrix}$$

$$=\frac{1}{r}\left[\hat{e_r}\left(\frac{\partial}{\partial\theta}r\sin\theta-\frac{\partial}{\partial z}(r(r\cos^2\theta-z\sin\theta))\right)-r\hat{e_\theta}\left(\frac{\partial}{\partial r}r\sin\theta-\frac{\partial}{\partial z}(r\sin\theta\cos\theta+z\cos\theta)\right)+\hat{e_z}\left(\frac{\partial r}{\partial r}(r\cos^2\theta-z\sin\theta)-\frac{\partial}{\partial\theta}\frac{(r\sin\theta\cos\theta)}{(z\cos\theta)}\right)\right.$$

$$=\frac{1}{r}\left[\hat{e_r}(r\cos\theta-0+r\sin\theta(1))-r\hat{e_\theta}(\sin\theta-0-\cos\theta)+\hat{e_z}\left(2r\cos^2\theta-z\sin\theta-\frac{2r\cos2\theta}{2}+z\sin\theta\right)\right]$$

$$=\frac{1}{r}\left[\hat{e_r}(r\cos\theta+r\sin\theta)-r\hat{e_\theta}(\sin\theta-\cos\theta)+\hat{e_z}(2r\cos^2\theta-r\cos2\theta)\right]$$

$$=\frac{1}{\cancel{r}}\cancel{r}\left[(\hat{e_r}(\cos\theta+\sin\theta)-\hat{e_\theta}(\sin\theta-\cos\theta)+\hat{e_z}(2\cos^2\theta-\cos2\theta)\right]$$

$$\nabla \times \vec{A} = \hat{e_r}(\cos\theta + \sin\theta) - \hat{e_\theta}(\sin\theta - \cos\theta) + \hat{e_z}(2\cos^2\theta - \cos 2\theta)$$

$\downarrow A_r$ $\downarrow A_\theta$ $\downarrow A_z$

Now

$$\nabla \times \nabla \times A = ?$$

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & \hat{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$= \frac{1}{r}\Big[\hat{e_r}\Big(\frac{\partial}{\partial\theta}(2\cos^2\theta - \cos 2\theta) - \frac{\partial}{\partial z} r(\cos\theta - \sin\theta)\Big) - r\hat{e_\theta}$$

$$\frac{\partial}{\partial r}(2\cos^2\theta - \cos 2\theta) - \frac{\partial}{\partial z}(\cos\theta + \sin\theta)\Big) + \hat{e_z}$$

$$\Big(\frac{\partial}{\partial r}(r(\cos\theta - \sin\theta)) - \frac{\partial}{\partial\theta}(\cos\theta + \sin\theta)\Big]$$

$$= \frac{1}{r}\Big[\hat{e_r}(-4\cos\theta\sin\theta + 2\sin 2\theta - 0) - \cancel{A\sin}\, r\hat{e_r}(0-0)$$

$$e^{z}(-\cancel{\sin\theta} + \cancel{\cos\theta} + \sin\theta - \cos\theta)$$

$$= \frac{1}{r}\Big[\hat{e_r}\Big(-\overset{2}{\cancel{4}}\cdot\frac{\sin 2\theta}{\cancel{2}} + 2\sin\theta\Big)\Big].$$

$$= \frac{1}{r}\big[\hat{e_r}(-2\sin 2\theta + 2\sin\theta)\big]$$

$$= \frac{1}{r}(0)$$

$$= \boxed{0} \quad R.H.S$$

written by shaniza Ghafoor credit goes to sir fazal 03014695644

So

$$\boxed{\nabla \times \nabla \times A = 0}$$ proved.

Question No. 13:-

In cylindrical polar coordinates if $\vec{A} = \hat{e_\theta}$ verify the identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

L.H.S $\nabla \times \nabla \times \vec{A}$

$$\nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & \hat{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & r & 0 \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{e_r}(0-0) - r\hat{e_\theta}(0-0) + \hat{e_z}(1-0)\right]$$

$$\Rightarrow \boxed{\nabla \times \vec{A} = \frac{\hat{e_z}}{r}.}$$

written by shaniza Ghafoor
credit goes to prof fazal

Now

$\nabla \times \nabla \times \vec{A} = ?$

$$\nabla \times (\nabla \times \vec{A}) = \frac{1}{r}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & \hat{e_z} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ 0 & 0 & 1/r \end{vmatrix}$$

$$= \frac{1}{r}\left[\hat{e_r}(0-0) - r\hat{e_\theta}\left(\frac{-1}{r^2}-0\right) + \hat{e_z}(0-0)\right]$$

$$\boxed{\nabla \times (\nabla \times \vec{A}) = \frac{1}{r^2}\hat{e_\theta}}$$

R.H.S

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2\vec{A} = ? \longrightarrow (A)$$

$$\nabla \cdot \vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial r}(h_2)(h_3)A_r + \frac{\partial}{\partial\theta}(h_1)(h_3)A_\theta + \frac{\partial}{\partial z}(h_1)(h_2)A_z\right].$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r)(0) + \frac{\partial}{\partial\theta}(1) + \frac{\partial}{\partial z}(0)\right]$$

$$\Rightarrow \boxed{0} \longrightarrow (1)$$

Now

$$\nabla^2 \vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\hat{e_\theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\hat{e_\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial}{\partial z}\hat{e_\theta}\right)\right]$$

$$= \frac{1}{r}\left[0 + \frac{\partial}{\partial \theta}\left(\frac{1}{r}(-\hat{e_r})\right) + \frac{\partial}{\partial z}(0)\right]$$

$\because \frac{\partial}{\partial \theta}\hat{e_\theta} = -\hat{e_r}$

$\frac{\partial}{\partial \theta}(\hat{e_r}) = \hat{e_\theta}$

$$= \boxed{-\frac{1}{r^2}e^{\hat{\theta}}} \rightarrow ②$$ put ① & ② in (A)

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

$$0 - (-1/r^2 e^{\theta})$$

$$\Rightarrow \boxed{1/r^2 e^{\theta}}$$ proved L.H.S = R.H.S

Spherical Coordinate System:-

$P(x, y, z) = P(r, \theta, \phi)$; $h_1 = 1$, $h_2 = r$

$h_3 = r\sin\theta$.

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$.

$$\begin{bmatrix} \hat{e_r} \\ \hat{e_\theta} \\ \hat{e_\phi} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e_r} \\ \hat{e_\theta} \\ e_\phi \end{bmatrix}$$

written by shaniza Ghafoor

Question No. 14:-

Express the vector field $\vec{A} = -y\hat{i} + x\hat{j}$ in spherical polar coordinates.

Sol:-

Since $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$\hat{i} = \sin\theta\cos\phi\,\hat{e_r} + \cos\theta\cos\phi\,\hat{e_\theta} - \sin\phi\,\hat{e_\phi}$

$\hat{j} = \sin\theta\sin\phi\,\hat{e_r} + \cos\theta\sin\phi\,\hat{e_\theta} + \cos\phi\,\hat{e_\phi}$.

put values in

$$\vec{A} = -y\hat{i} + x\hat{j}$$

$$\vec{A} = -(r\sin\theta\sin\phi)(\sin\theta\cos\phi\,\hat{e_r} + \cos\theta\cos\phi\,\hat{e_\theta} - \sin\phi\,\hat{e_\phi})$$
$$+ (r\sin\theta\cos\phi)(\sin\theta\sin\phi\,\hat{e_r} + \cos\theta\sin\phi\,\hat{e_\theta} + \cos\phi\,\hat{e_\phi})$$

$$\vec{A} = [\cancel{-r\sin^2\theta\sin\phi\cos\phi\,\hat{e_r}} - \cancel{r\sin\theta\cos\theta\sin\phi\cos\phi\,\hat{e_\theta}} + r\sin\theta\sin^2\phi\,\hat{e_\phi} + \cancel{r\sin^2\theta\cos\phi\sin\phi\,\hat{e_r}} + \cancel{r\sin\theta\cos\theta\cos\phi\sin\phi\,\hat{e_\theta}}$$
$$+ r\sin\theta\cos^2\phi\,\hat{e_\phi}].$$

$$= r\sin\theta\sin^2\phi\,\hat{e_\phi} + r\sin\theta\cos^2\phi\,\hat{e_\phi}$$

$$r\sin\theta(\sin^2\phi + \cos^2\phi)\,\hat{e_\phi}$$

$$\boxed{\vec{A} = r\sin\theta\,e_\phi.}$$ Answer.

Question No. 15:-

In spherical polar coordinates $\vec{A} = \frac{1}{r}\hat{e_r}$. Transform $\vec{A}$ in rectangular cartisian coordinates.

Sol:-

$$\vec{A} = \frac{1}{r}\hat{e_r}$$

$\because \hat{e_r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$r = \sqrt{x^2+y^2+z^2}$

$\frac{x}{r} \Rightarrow x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

put the values

$$\vec{A} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left[\underbrace{\sin\theta\cos\phi}_{x/r}\,\hat{i} + \underbrace{\sin\theta\sin\phi}_{y/r}\,\hat{j} + \underbrace{\cos\theta}_{z/r}\,\hat{k}\right]$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right]$$

$$\vec{A} = \frac{1}{r^2}\left[x\hat{i} + y\hat{j} + z\hat{k}\right].$$

$$\boxed{\vec{A} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}}$$ Answer.

Question No 16:-

In spherical polar coordinates, $\vec{A} = r\hat{e_\theta} + r\hat{e_\phi}$

Find the spherical polar coordinates of

$\frac{\partial\vec{A}}{\partial r}$, $\frac{\partial\vec{A}}{\partial\theta}$ and $\frac{\partial\vec{A}}{\partial\phi}$.

Sol:- $\vec{A} = 0\hat{e_r} + r\hat{e_\theta} + r\hat{e_\phi}$

$$\frac{\partial\vec{A}}{\partial r} = \frac{\partial}{\partial r}(r\hat{e_\theta} + r\hat{e_\phi}) \quad \text{product Rule}$$

$$= (1)\hat{e_\theta} + r(0) + (1)\hat{e_\phi} + r(0)$$

$$\boxed{\frac{\partial\vec{A}}{\partial r} = 0\cdot\hat{e_r} + \hat{e_\theta} + \hat{e_\phi}} \Rightarrow (0, 1, 1)$$

(Formulas Page 353)

$$\partial \hat{e_r} = 0 \; ; \; \frac{\partial \hat{e_r}}{\partial \theta} = \hat{e_\theta} \; ; \; \frac{\partial \hat{e_r}}{\partial \phi} = \sin\theta \hat{e_\phi}$$

$$\frac{\partial \hat{e_\theta}}{\partial r} = 0 \; ; \; \frac{\partial}{\partial \theta}\hat{e_\theta} = -\hat{e_r} \; ; \; \frac{\partial}{\partial \phi}\hat{e_\theta} = \cos\theta \hat{e_\phi}$$

$$\frac{\partial \hat{e_\phi}}{\partial r} = 0 \; ; \; \frac{\partial}{\partial \theta}\hat{e_\phi} = 0 \; ; \; \frac{\partial}{\partial \phi}\hat{e_\phi} = -\sin\theta \hat{e_r} - \cos\theta \hat{e_\theta}$$

$$\frac{\partial \vec{A}}{\partial \theta} = \frac{\partial}{\partial \theta}(r\hat{e_\theta} + r\hat{e_\phi})$$

$$= r(-\hat{e_r}) + r(0)$$

written by shaniza Ghafoor credit goes to pr

$$\boxed{\frac{\partial \vec{A}}{\partial \theta} = -r\hat{e_r}} \Rightarrow (-r, 0, 0).$$

$$\frac{\partial \vec{A}}{\partial \phi} = \frac{\partial}{\partial \phi}(r\hat{e_\theta} + r\hat{e_\phi})$$

$$= r(\cos\theta \hat{e_\phi}) + r(-\sin\theta \hat{e_r} - \cos\theta \hat{e_\theta})$$

$$= -r\sin\theta \hat{e_r} - r\cos\theta \hat{e_\theta} + r\cos\theta \hat{e_\phi}$$

$$\boxed{\frac{\partial \vec{A}}{\partial \phi} = (-r\sin\theta, -r\cos\theta, r\cos\theta)}$$ Answer

Question No. 17:-

Find $\nabla\psi$ in spherical coordinates if $\psi(r,\theta,\phi) = \frac{\cos\theta}{r^2}$.

Sol:-

$$\nabla\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial r}\hat{e_r} + \frac{1}{h_2}\frac{\partial\psi}{\partial\theta}\hat{e_\theta} + \frac{1}{h_3}\frac{\partial\psi}{\partial\phi}\hat{e_\phi}$$

written by shaniza Ghafoor

OR

$$= \frac{1}{h_1}\frac{\partial \psi}{\partial u_1}\hat{e_1} + \frac{1}{r}\frac{\partial \psi}{\partial u_2}\hat{e_2} + \frac{1}{h_3}\frac{\partial \psi}{\partial u_3}\hat{e_3}$$

$$\nabla\psi = (1)\frac{\partial \psi}{\partial r}\hat{e_r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{e_\phi}$$

$$\because \nabla\psi = \cos\theta / r^2$$

$$\nabla\left(\frac{\cos\theta}{r^2}\right) = \frac{\partial}{\partial r}\left(\frac{\cos\theta}{r^2}\right)\hat{e_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\cos\theta}{r^2}\right)\hat{e_\theta} +$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{\cos\theta}{r^2}\hat{e_\phi}$$

$$= \frac{-2}{r^3}\cos\theta\hat{e_r} + \frac{1}{r^3}(-\sin\theta)\hat{e_\theta} + 0$$

$$\boxed{\nabla(\cos\theta/r^2) = (-2/r^3\cos\theta, -\sin\theta/r^3, 0)}$$

Question No. 18:- prof fazal 03014695644

Verify that in Spherical polar coordinates

$$\nabla\phi = \nabla \times \left(\frac{r\nabla\theta}{\sin\theta}\right)$$

$P(r,\theta,\phi)$
$h_r = h_1 = 1$
$h_\theta = h_2 = r$
$h_\phi = h_3 = r\sin\theta$

L.H.S:-

$$\nabla\psi = 1\frac{\partial}{\partial r}\psi\hat{e_r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial \phi}\hat{e_\phi}$$

$$\nabla\phi = \frac{1}{1}\frac{\partial \phi}{\partial r}\hat{e_r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \phi}\hat{e_\phi}$$

$$= 0 + 0 + \frac{1}{r\sin\theta}(1)\hat{e_\phi}$$

$$\boxed{\nabla\phi = \hat{e_\phi}/r\sin\theta \Rightarrow \frac{1}{r\sin\theta}\hat{e_\phi}}$$

R.H.S

$$\nabla \times \left(\frac{r \nabla \theta}{\sin\theta} \right)$$

$$\nabla \theta = 1 \frac{\partial \theta}{\partial r} \hat{e_r} + \frac{1}{r} \frac{\partial \theta}{\partial \theta} \hat{e_\theta} + \frac{1}{r \sin\theta} \frac{\partial \theta}{\partial \phi} \hat{e_\phi}$$

$$\nabla \theta = 0 + \frac{1}{r} (1) \hat{e_\theta} + 0$$

$$\boxed{\nabla \theta = 1/r \, \hat{e_\theta}}$$

written by shaniza Ghafoor
credit goes to prof fazal

Now,

$$\frac{r}{\sin\theta} \nabla \theta = \frac{\cancel{r}}{\sin\theta} \left(\frac{1}{\cancel{r}} \hat{e_\theta} \right)$$

$$\Rightarrow \boxed{\frac{\hat{e_\theta}}{\sin\theta}}$$

Now,

$$\nabla \times \left(\frac{r \nabla \theta}{\sin\theta} \right) = ? \qquad \nabla \times \vec{A} = ?$$

$$\nabla \times \left(\frac{1}{\sin\theta} \hat{e_\theta} \right) = \frac{1}{(1)(r)(r\sin\theta)} \begin{vmatrix} (1)\hat{e_r} & r\hat{e_\theta} & r\sin\theta \hat{e_\phi} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ h_r A_r & h_\theta A_\theta & h_\phi A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & r\sin\theta \hat{e_\phi} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ 0 & r(1/\sin\theta) & 0 \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \left(\hat{e_r}(0) + r\hat{e_\theta}(0-0) + \hat{e_\phi} r \sin\theta \left(\frac{1}{\sin\theta} - 0 \right) \right)$$

$$\Rightarrow 1/r^2 \sin\theta \, (r \hat{e_\phi})$$

$$\Rightarrow \boxed{\frac{1}{r\sin\theta}\,\hat{e_\phi}}$$ proved

written by shaniza Ghafoor
credit goes to prof fazal

L.H.S = R.H.S

Question No. 19:-

In spherical polar coordinates if $\vec{A}(r,\theta,\phi) = \hat{e_r} + r\hat{e_\theta} + r\cos\phi\,\hat{e_\phi}$. Find

(i) $\nabla\cdot\vec{A}$.

$$\nabla\cdot\vec{A} = \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}(r)(r\sin\theta)(1) + \frac{\partial}{\partial\theta}(1)(r\sin\theta)(r)\right.$$

$$\left. + \frac{\partial}{\partial\phi}(1)(r)(r\cos\phi)\right]$$

$$\nabla\cdot\vec{A} = \frac{1}{r^2\sin\theta}\left[2r\sin\theta + r^2\cos\theta - r^2\sin\phi\right]$$

$$= \frac{1}{r^2\sin\phi}\left[r2\sin\theta + r\cos\theta - r^2\sin\phi)\right]$$

$$\boxed{\nabla\cdot\vec{A} = \frac{1}{r\sin\phi}(2\sin\theta + r\cos\theta - r\sin\phi).}$$

(ii)

$\nabla\times\vec{A} = ?$

$$\nabla\times\vec{A} = \frac{1}{(h_1)(h_2)(h_3)}\begin{vmatrix} h_1\hat{e_r} & h_2\hat{e_\theta} & h_3\hat{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

$$= \frac{1}{(1)(r)(r\sin\theta)}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & r\sin\theta\,\hat{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ (1)(1) & (r)(r) & r\sin\theta\,(r\cos\phi) \end{vmatrix}$$

written by Shaniza Ghafoor credit goes to prof fazal

$$= \frac{1}{r^2\sin\theta}\Big[e\hat{r}\Big(\frac{\partial}{\partial\theta}(r^2\sin\theta\cos\phi) - \frac{\partial}{\partial\phi}(r^2)\Big] - re\hat{\theta}\Big(\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi)$$

$$- \partial/\partial\phi(1)\Big) + r\sin\theta e\hat{\phi}\Big(\partial/\partial r(r^2) - \partial/\partial\theta(1)\Big)\Big].$$

$$= \frac{1}{r^2\sin\theta}\Big[e\hat{r}(r^2\cos\theta\cos\phi) - 0 - re\hat{\theta}(2r\sin\theta\cos\phi) - 0$$

$$+ r\sin\theta e\hat{\phi}(2r - 0)\Big].$$

$$= \frac{1}{r^2\sin\theta}\Big[(r^2\cos\theta\cos\phi)e\hat{r} - (2r^2\sin\theta\cos\phi)e\hat{\theta} + (2r^2\sin\theta)e\hat{\phi}\Big]$$

$$= \frac{1}{\cancel{r^2}\sin\theta}\cancel{r^2}\Big[(1\cos\theta\cos\phi)e\hat{r} - (2\sin\theta\cos\phi)e\hat{\theta} + (2\sin\theta)e\hat{\phi}\Big].$$

$$\boxed{\nabla\times\vec{A} = \frac{1}{\sin\theta}\Big[(\cos\theta\cos\phi)e\hat{r} - (2\sin\theta\cos\phi)e\hat{\theta} + (2\sin\theta)e\hat{\phi}\Big]}.$$

Question 20:- 03014695644

In spherical polar coordinates if $A = e\hat{\theta}$, find the complement of $\nabla(\nabla\cdot\vec{A})$.

Sol:-

$$\vec{A} = 0e\hat{r} + 1e\hat{\theta} + 0e\hat{\phi}.$$

$$\nabla\cdot\vec{A} = \frac{1}{(1)(r)(r\sin\theta)}\Big[\frac{\partial}{\partial r}(r\cdot r\sin\theta)(0) + \frac{\partial}{\partial\theta}(1)(r\sin\theta)(1) + \frac{\partial}{\partial\phi}(r)(1)(0)\Big]$$

$$= \frac{1}{r^2\sin\theta}\Big[0 + r\cos\theta + 0\Big]$$

$$\Rightarrow \boxed{\frac{1}{r}\cot\theta} \Rightarrow \nabla\cdot\vec{A}.$$

Now $\nabla(\nabla \cdot \vec{A}) = ?$

$$\nabla\psi = \frac{1}{1}\frac{\partial\psi}{\partial r}\hat{e_r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{e_\phi}$$

$$\nabla\left(\frac{\cot\theta}{r}\right) = 1\frac{\partial}{\partial r}(\cot\theta/r)\hat{e_r} + \frac{1}{r}\frac{\partial}{\partial r}(\cot\theta/r)\hat{e_\theta}$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cot\theta/r)\hat{e_\phi}$$

$$= -\frac{1}{r^2}\cot\theta\hat{e_r} - \frac{\operatorname{cosec}^2\theta}{r^2}\hat{e_\theta} + 0$$

$$\boxed{\nabla\left(\frac{\cot\theta}{r}\right) = -1/r^2\cot\theta\hat{e_r} - \frac{\operatorname{cosec}^2\theta}{r^2}\hat{e_\theta}}$$ Answer

written by shaniza Ghafoor
credit goes to prof fazal
03414695644

Question 21:-

In spherical polar coordinates of $\vec{A} = e\hat{\theta}$ find the component of $\nabla(\nabla \cdot \vec{A})$.

$\vec{A} = A_1\hat{e_r} + A_\theta\hat{e_\theta}$ where A_r, A_θ are independent of ϕ. prove that $\nabla \times \nabla \times \vec{A}$ is of similar form to $\vec{A}$.

Sol:-

$$\vec{A} = A_r\hat{e_r} + A_\theta\hat{e_\theta} + 0\hat{e_\phi}$$

$$\nabla \times \vec{A} = \frac{1}{(1)(r)(r\sin\theta)} \begin{vmatrix} (1)\hat{e_r} & r\hat{e_\theta} & r\sin\theta\,\hat{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ (1)A_r & rA_\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{e_r}(0-0) + r\hat{e_\theta}(0-0) + r\sin\theta\, e_\phi\left(r\frac{\partial}{\partial r}A_\theta + (1)A_\theta - \partial/\partial\phi\, A_r\right]\right.$$

$$\frac{1}{r^2\sin\theta}\left[0-0+r\sin\theta\, \hat{e_\phi}\left(r\frac{\partial}{\partial r}A_\theta+A_\theta-\frac{\partial}{\partial\theta}A_r\right)\right]$$

$$\boxed{\nabla\times\vec{A}=1/r\left[r\frac{\partial}{\partial r}A_\theta+A_\theta-\frac{\partial}{\partial\theta}A_r\right]\hat{e_\phi}}$$

$$\nabla\times\nabla\times\vec{A}=\frac{1}{(1)(r)(r\sin\theta)}\begin{vmatrix}(1)\hat{e_r} & r\hat{e_\theta} & r\sin\theta\,\hat{e_\phi}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ 0 & 0 & \frac{\partial}{\partial r}A_\theta+\frac{1}{r}A_\theta-\frac{1}{r}A_r\end{vmatrix}$$

if

$$=\frac{1}{r^2\sin\theta}\left[\hat{e_r}\left(\frac{\partial^2}{\partial\theta}\frac{A_\theta}{\partial r}+\frac{1}{r}\frac{\partial}{\partial\theta}A_\theta-\frac{1}{r}\frac{\partial^2A_r}{\partial\theta^2}-0\right)\right.$$

$$-r\hat{e_\theta}\left(\frac{\partial^2A_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial A_\theta}{\partial r}-\frac{1}{r^2}\frac{\partial A_r}{\partial\theta}-\right.$$

written by shaniza Ghafoor

$$\left.\underbrace{\frac{1}{r}\frac{\partial^2A_r}{\partial r\partial\theta}-0}\right]$$

$$\boxed{\frac{1}{r^2\sin\theta}\left[\hat{e_r}\left(\frac{\partial^2}{\partial\theta}\frac{A_\theta}{\partial r}+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}-\frac{1}{r}\frac{\partial^2A_r}{\partial\theta}\right)\right]-\left[r\hat{e_\theta}\left(\frac{\partial^2A_\theta}{\partial r^2}+1/r\frac{\partial A_\theta}{\partial r}-1/r^2\frac{\partial A_r}{\partial\theta}-1/r\frac{\partial^2A_r}{\partial r\partial\theta}\right)\right]}$$

$=0$

$\frac{\partial u}{\partial\theta}$

Question No. 22

In spherical polar coordinates, a vector field $\vec{A}=A_r\hat{e_r}+A_\theta\hat{e_\theta}+A_\phi\hat{e_\phi}$. Show that if $A_r=A_\theta=\frac{\partial A_\phi}{\partial\phi}=0$ and $\nabla\times\nabla\times\vec{A}=\bar{0}$ then

$$r^2\frac{\partial^2A_\phi}{\partial r^2}+2r\frac{\partial A_\phi}{\partial r}+\frac{\partial}{\partial\theta}\left[\frac{\partial A_\phi}{\partial\theta}A_\phi\cot\theta\right]=0$$

Sol:-

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & r\sin\theta\,\hat{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{e_r}\left(r\cos\theta A_\phi + r\sin\theta\frac{\partial A_\phi}{\partial\theta}\right) + r\hat{e_\theta}\left(0 - \sin\theta A_\phi\right.\right.$$

$$\left.\left. - r\sin\theta\frac{\partial A_\phi}{\partial r}\right)\right].$$

$$\nabla\times\vec{A} = \hat{e_r}\left(\frac{\cos A_\phi}{r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right) + \hat{e_\theta}\left(-\frac{1}{r}A_\phi - \frac{\partial A_\phi}{\partial r}\right)$$

$$\nabla\times(\nabla\times\vec{A}) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{e_r} & r\hat{e_\theta} & r\sin\theta\,\hat{e_\phi} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{1}{r}\cos A_\phi + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta} & -\frac{r\partial A_\phi}{\partial r} - A_\phi & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left\{0 - 0 + r\sin\theta\,\hat{e_\phi}\left(-\frac{\partial}{\partial r}A_\phi - \frac{\partial A_\phi}{\partial r} - r\frac{\partial^2 A_\phi}{\partial r^2}\right.\right.$$

written by Shaniza Ghafoor $\left.\left. - \frac{\partial}{\partial\theta}\left(\frac{1}{r}\cos\theta A_\phi + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta}\right)\right)\right\}.$

$$0\hat{e_r} + 0\hat{e_\theta} + 0\hat{e_\phi} = \frac{1}{r}\left\{-2\frac{\partial A_\phi}{\partial r} - r\frac{\partial^2 A_\phi}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial A_\phi}{\partial\theta} + A_\phi\cos\theta\right)\right.$$

Comparing $\hat{e_\phi}$ and $(-r)$

$$\boxed{\frac{r^2\partial^2 A_\phi}{\partial r^2} + \frac{2r\partial A_\phi}{\partial r} + \frac{\partial}{\partial\theta}\left(\frac{\partial A_\phi}{\partial\theta} + A_\phi\cos\theta\right) = 0}$$

proved.

Question No 23:-

Prove that

written by shaniza Ghafoor
credit goes to sir fazal
03014695644

(i) $\nabla \times \nabla \psi = 0$

(ii) $\nabla \cdot \nabla \times \vec{A} = 0$ in cylindrical and spherical coordinate systems.

Sol:-

(i)

$\nabla \times \nabla \psi = 0$

$P(r, \theta, z)$

$h_1 = h_r = 1$

$h_2 = h_\theta = r$

$h_3 = h_z = 1$

In cylindrical

$$\nabla \psi = \frac{1}{1}\frac{\partial}{\partial r}\psi \hat{e_r} + \frac{1}{r}\frac{\partial \psi}{\partial \theta}\hat{e_\theta} + \frac{1}{1}\frac{\partial \psi}{\partial z}\hat{e_z}$$

$$\nabla \times \nabla \psi = \frac{1}{(1)(r)(1)} \begin{vmatrix} (1)\hat{e_r} & r\hat{e_\theta} & (1)\hat{e_z} \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ 1\,\partial \psi/\partial r & r/r\,\partial \psi/\partial \theta & \partial \psi/\partial z \end{vmatrix}$$

$$\frac{1}{r}\left[\hat{e_r}\left(\frac{\partial^2\psi}{\partial\theta\partial z}-\frac{\partial^2\psi}{\partial\theta\partial z}\right)-r\hat{e_\theta}\left(\frac{\partial^2\psi}{\partial r\partial z}-\frac{\partial^2\psi}{\partial r\partial z}\right)+\right.$$

$$\left.\hat{e_z}\left(\frac{\partial^2\psi}{\partial r\partial\theta}-\frac{\partial^2\psi}{\partial r\partial\theta}\right)\right]$$

$\boxed{\nabla\times\nabla\psi=0}$ proved.

In spherical:-

$h_1=h_r=1$

$h_2=h_\theta=r$

$h_3=h_\phi=r\sin\theta$

$$\nabla\psi=\frac{1}{1}\frac{\partial}{\partial r}\psi\hat{e_r}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{e_\phi}$$

$$\nabla\times\nabla\psi=\frac{1}{(1)(r)r\sin\theta}\begin{vmatrix}(1)\hat{e_r} & r\hat{e_\theta} & r\sin\theta\hat{e_\phi}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi\\ 1\,\partial\psi/\partial r & \frac{r}{r}\partial\psi/\partial\theta & \frac{r\sin\theta}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\end{vmatrix}$$

$$=\frac{1}{r^2\sin\theta}\left[\hat{e_r}\left(\frac{\partial^2\psi}{\partial\phi\partial\theta}-\frac{\partial^2\psi}{\partial\theta\partial\phi}\right)-r\hat{e_\theta}\left(\frac{\partial^2\psi}{\partial r\partial\phi}-\frac{\partial^2\psi}{\partial r\partial\phi}\right)\right.$$

$$\left.+r\sin\theta\hat{e_\phi}\left(\frac{\partial^2\psi}{\partial r\partial\theta}-\frac{\partial^2\psi}{\partial r\partial\theta}\right)\right].$$

$\boxed{\nabla\times\nabla\psi=0}$ proved

(ii) $\nabla\cdot\nabla\times\vec{A}$

In cylindrical:-

$$\nabla\times\vec{A}=\frac{1}{(1)(r)(1)}\begin{vmatrix}1\hat{e_r} & r\hat{e_\theta} & 1\hat{e_z}\\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z\\ (1)A_r & rA_\theta & 1A_z\end{vmatrix}$$

$$=\frac{1}{r}\left[\hat{e_r}\left(\frac{\partial}{\partial\theta}A_z-rA_\theta\frac{\partial}{\partial z}\right)-r\hat{e_\theta}\left(\frac{\partial}{\partial r}A_z-\frac{\partial}{\partial z}A_r\right)\right.$$
$$\left.+\hat{e_z}\left(\frac{\partial}{\partial r}rA_\theta-\frac{\partial}{\partial\theta}A_r\right)\right]$$

Question No. 24

Express the heat equation
$\frac{\partial U}{\partial t} = c^2 \nabla^2 U$ in spherical coordinate if is independent of
(i) ϕ (ii) ϕ and θ (iii) r and t
(iv) ϕ, θ and t.

Sol:-

$$\partial U/\partial t = c^2 \nabla^2 U \quad \text{—(1)}$$

As we know that

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial r}\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial r} + \frac{\partial}{\partial \theta}\frac{h_1 h_3}{h_2}\frac{\partial \psi}{\partial \theta} + \frac{\partial}{\partial \phi}\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial \phi}\right]$$

put $\psi = u$, $h_1 = 1$, $h_2 = r$ and $h_3 = r\sin\theta$

$$\nabla^2 U = \frac{1}{(1)(r)(r\sin\theta)}\left[\frac{\partial}{\partial r}\frac{(r)(r\sin\theta)}{1}\frac{\partial U}{\partial r} + \frac{\partial}{\partial \theta}\frac{(1)(r\sin\theta)}{r}\frac{\partial U}{\partial \theta} + \frac{\partial}{\partial \phi}\frac{(1)(r)}{(r\sin\theta)}\frac{\partial U}{\partial \phi}\right]$$

$$\nabla^2 U = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial U}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\frac{1}{\sin\theta}\frac{\partial U}{\partial \phi}\right]$$

(i) U is independent of "ϕ"

By putting the value of $\nabla^2 U$

$$① \to \frac{\partial U}{\partial t} = \frac{c^2}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)\right]$$

+ 0 (bcz ϕ is independent ...)

$$\boxed{\frac{\partial U}{\partial t} = \frac{c^2}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)\right]}$$

(ii)

U is independent "ϕ" & "θ".

$$\frac{\partial U}{\partial t} = \frac{c^2}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial U}{\partial r}\right)\right] + 0 + 0.$$

$$\boxed{\partial U/\partial t = c^2/r^2\sin\theta\left[\partial/\partial r\left(r^2\sin\theta\,\partial U/\partial r\right)\right]}$$

(iii)

U is independent "r" & "t"

$$\because \frac{\partial U}{\partial t} = 0 \qquad (t = 0)$$

So

$$0 = \frac{c^2}{r^2\sin\theta}\left[0 + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}\right)\right]$$

$$\frac{r^2\sin\theta}{c^2}\cdot 0 = \left[\frac{\partial}{\partial\theta}\sin\theta\frac{\partial U}{\partial\theta} + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial U}{\partial\phi}\right)\right].$$

$$\boxed{0 = \left[\partial/\partial\theta\left(\sin\theta\,\partial U/\partial\theta\right) + \partial/\partial\phi\left(1/\sin\theta\;\partial U/\partial\phi\right)\right]}$$

(iv) U is independent of ϕ, θ, t

$$0 = \frac{\partial}{\partial r}\left[\left(\frac{r^2 \sin\theta}{t}\frac{\partial u}{\partial r}\right) + 0 + 0\right]$$

$$\boxed{0 = \left[\partial/\partial r\left(r^2 \sin\theta/t \;\; \partial u/\partial r\right)\right]}$$

Question 25:- Evaluation of Integral:-

using cylindrical polar coordinates

$$\int_0^2 \int_0^{4-\sqrt{x^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dz\,dy\,dx$$

Sol:-

In cylindrical coordinates system

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$0 \le r \le 2, \quad 0 \le \theta \le \pi/2$

$dz\,dy\,dx = r\,dz\,d\theta\,dr$

$\because -x^2 - y^2 = -(x^2+y^2)$

$r^2\cos^2\theta + r^2\sin^2\theta$

$r^2(\cos^2\theta + \sin^2\theta)$

$r^2(1) = r^2$

$$\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} \frac{r\cos\theta}{r}\, r\,dz\,dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^2 z\Big|_0^{\sqrt{4-r^2}} \cos\theta\, dr\,d\theta$$

$$\int_0^{\pi/2}\int_0^2 (\sqrt{4-r^2} - 0)\cos\theta\, r\,dr\,d\theta$$

$$\int_0^2 \sin\theta\Big|_0^{\pi/2} \sqrt{4-r^2}\,(r)\,dr$$

$$\int_0^2 (\sin \pi/2 - \sin 0)\, r\sqrt{4-r^2}\, dr$$

$$\int_0^2 1\,(r\sqrt{4-r^2})\, dr$$

$\times$ & $\div$ by -2

$$\int_0^2 \frac{-2r\,(4-r^2)^{1/2}}{-2r}\, dr$$

$$\frac{(4-r^2)^{1/2+1}}{(1/2+1)(-2)}\Big|_0^2 \Rightarrow \frac{(4-r^2)^{3/2}}{3/2\,(-2)}\Big|_0^2$$

$$+\frac{(4-r^2)^{3/2}}{3}\Big|_0^2 \Rightarrow \frac{(4-2^2)^{3/2}}{-3} - \frac{(4-0)^{3/2}}{-3}$$

$$0+\frac{(2^2)^{3/2}}{3}$$

$\boxed{+8/3.}$ Answer

Question No. 26:-

Using cylindrical polar coordinates. Evaluate $\iiint_R x\,dV$, where R is the region in the first octant bounded by the cylinder $x^2+y^2=8$ & the planes $z=0$ and $z=6$.

Sol:-

$$\int_0^{\pi/2}\int_0^{\sqrt{8}}\int_0^{6} r\cos\theta\, r\, dz\, dr\, d\theta$$

$$\int_0^{\pi/2}\cos\theta\, d\theta \cdot \int_0^{\sqrt{8}} r\, dr \cdot \int_0^{6} dz \Rightarrow +\sin\theta\Big|_0^{\pi/2} \cdot \frac{r^2}{2}\Big|_0^{\sqrt{8}}$$

$(\sin \pi/2 - \sin 0) \cdot \left(\frac{(\sqrt{8})^2}{2} - \frac{0}{2} \right) \cdot (6-0)$

$(1-0) \cdot \left(\frac{8}{2}\right) \cdot (6)$

$\boxed{24}$ Answer

Question No. 27.

Using cylindrical coordinates evaluate $\iiint_R (x^2+y^2)\,dv$ where R in the region R bounded by two cylinder $r=1$ and $r=2$. for $0 \le z \le 2$.

Sol:-

$$\int_0^{2\pi}\int_1^2\int_0^2 (x^2+y^2)\,dz\,dy\,dx$$

In cylindrical $dz\,dy\,dx = dz\,dr\,d\theta$

$$\int_0^{2\pi} d\theta \cdot \int_1^2 dr(x^2+y^2) \cdot \int_0^2 dz$$

written by shaniza Ghafoor

$$\theta\Big|_0^{2\pi} \cdot \int_1^2 r^2\,dr \int_0^2 dz$$

03014695644

$$2\pi - 0 \cdot \frac{r^3}{3}\Big|_1^2 \cdot z\Big|_0^2 \Rightarrow 2\pi \cdot \left(\frac{2^3}{3} - \frac{1^3}{3}\right) \cdot (2-0)$$

$$2\pi\left(\frac{8}{3} - \frac{1}{3}\right) \cdot 2 \Rightarrow 4\pi\left(\frac{8-1}{3}\right)$$

$\boxed{\frac{28\pi}{3}}$ Ans

Question No. 28.

Using spherical polar coordinates, evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{1}{1+x^2+y^2+z^2}\, dz\, dy\, dx$$

In spherical coordinates

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$0 \le r < 2$, $0 < \theta < \pi/2$

$0 \le \phi < \pi/2$

In spherical

$\because x^2+y^2+z^2 = r^2$

$dv = r^2 \sin\theta\, dr\, d\theta\, dz$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-r^2}} 1$$

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \frac{r^2}{1+r^2} \sin\theta\, dr\, d\theta\, dz$$

$$\int_0^{\pi/2} \sin\theta\, d\theta \cdot \int_0^{\pi/2} dz \cdot \int_0^2 \frac{r^2}{1+r^2}\, dr$$

$\because \frac{r^2}{1+r^2}$ $+1-1$

$\frac{r^2+1-1}{1+r^2} \Rightarrow \frac{1+r^2}{1+r^2} - \frac{1}{1+r^2}$

$1 - \frac{1}{1+r^2}$

$$\cos\theta \Big|_0^{\pi/2} \cdot z \Big|_0^{\pi/2} \cdot \int_0^2 1 - \frac{1}{1+r^2}\, dr$$

$$(\cos\pi/2 - \cos 0) \cdot (\pi/2 - 0) \cdot (r - \tan^{-1} r)\Big|_0^2$$

$$(1-0)\,(\pi/2)\,(2 - \tan^{-1}2 - 0 + \tan^{-1}0)$$

$$\pi/2\,(2 - \tan^{-1}2 + 0)$$

$$\pi/2\,(2 - \tan^{-1}2)$$

$$\boxed{\frac{\pi}{2}[2 - \tan^{-1}2]}$$

Question No. 29:-

Using spherical polar coordinates, evaluate $\iiint_R \sqrt{x^2+y^2+z^2}\, dv$, where R is the region bound...

by the xy-plane & the hemisphere $x^2+y^2+z^2=9,\ z\geq 0$

Sol:-

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} r\cdot(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{3} r^3\sin\theta\,dr\,d\theta\,d\phi \Rightarrow \int_0^{\pi/2}\sin\theta\,d\theta\cdot\int_0^{2\pi}d\phi\cdot\int_0^{3} r^3$$

$$= \cos\theta\Big|_0^{\pi/2}\cdot\phi\Big|_0^{2\pi}\cdot\frac{r^4}{4}\Big|_0^{3}$$

$$= (\cos\pi/2-\cos 0)\cdot(2\pi-0)\frac{1}{4}(3^4-0^4)$$

$$= \frac{1}{4_2}2\pi(0-1)\ 81$$

$$\boxed{\frac{81\pi}{2}}$$ Answer.

Question No. 30:-

Using spherical polar coordinates, evaluate $\iiint_R x^2 dV$, where R is the region bounded by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$.

Sol:- As we know that / since

Equation of cone

$$z=\sqrt{x^2+y^2}$$

Equation of sphere

$x^2+y^2+z^2=1$, $r=1^2$ $\boxed{1=r}$

$x^2+y^2=z^2$

So $z^2+z^2=1 \Rightarrow 2z^2=1$

$z^2=1/2 \Rightarrow \boxed{z=\pm\frac{1}{\sqrt{2}}}$

For a sphere

$x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$, $z=r\cos\theta$

$\Rightarrow z=r\cos\theta=\frac{1}{\sqrt{2}}$

$\theta=\cos^{-1}(1/\sqrt{2}) \Rightarrow \boxed{\theta=\pi/4}$

$$\iiint_R x^2\,dV=\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^1 r^2\sin^2\theta\cos^2\phi\,(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$=\int_0^{2\pi}\int_{\pi/4}^{\pi}\int_0^1 r^4\sin^3\theta\cos^2\phi\,dr\,d\theta\,d\phi.$$

$$=\int_0^{2\pi}d\phi\int_{\pi/4}^{\pi}\sin^3\theta \ldots$$

$$=\int_0^{2\pi}\cos^2\phi\,d\phi\cdot\int_0^1 r^4\,dr\cdot\int_{\pi/4}^{\pi}\sin\theta\sin^2\theta\,d\theta$$

$$=\int_0^{2\pi}\frac{1+\cos2\phi}{2}\,d\phi\cdot\frac{r^5}{5}\Big|_0^1\cdot\int_{\pi/4}^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta$$

$$\frac{1}{2}\left[\theta+\frac{\sin2\theta}{2}\right]_0^{2\pi}\cdot\frac{1}{5}[1-0]\cdot\int_{\pi/4}^{\pi}\sin\theta\,d\theta-\cos^2\theta\sin\theta\,d\theta$$

$$\frac{1}{5}\cdot\frac{1}{2}\left[2\pi+\sin2(2\pi)-\left(0+\frac{\sin2(0)}{2}\right)\right]\cdot\left(-\cos\theta-\frac{\cos^3\theta}{3}\right)_{\pi/4}^{\pi}$$

$\sin 4\pi = 0$

$\frac{1}{10}([2\pi + 0] - 0) \cdot (1 - \cos\pi - \frac{\cos^3\pi}{3} + \cos\pi/4 + \frac{\cos^3\pi/4}{3})$

$\frac{\pi}{5}\left(1 - \frac{1}{3} + \frac{1}{\sqrt{2}} - \frac{\frac{1}{2\sqrt{2}}}{3}\right)$

$\frac{\pi}{5}\left(\frac{2}{3} + \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) \Rightarrow \frac{\pi}{5}\left(\frac{4\sqrt{2} + 6 - 1}{6\sqrt{2}}\right)$

$\frac{\pi/5 \; (4\sqrt{2} + 5)}{6\sqrt{2}}$

By rationalizing

$\frac{\pi}{6 \cdot 5}\left[\frac{4\sqrt{2}+5}{\cancel{6}\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}\right] \Rightarrow \frac{\pi}{30}\left[\frac{4(2) + 5\sqrt{2}}{2}\right]$

$\frac{\pi}{60}(8 + 5\sqrt{2})$

$\Rightarrow \boxed{\frac{(8 + 5\sqrt{2})\pi}{60}}$ Answer