Topology: Handwritten Notes
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Partial Contents

These are the handwritten notes. These notes are lecture delivered by Mr. Tahir Mehmood.

METRIC SPACE.

DEFINITION:

Let X be a non-empty set. A function $d: X \times X \to \mathbb{R}$ is said to be metric on X, if for all $x, y, z \in X$, it satisfies the following axioms:

M_1): $d(x,y) \geq 0$

M_2): $d(x,y) = d(y,x)$

M_3): $d(x,y) = 0 \iff x = y$

M_4): $d(x,z) \leq d(x,y) + d(y,z)$

EXAMPLE 1:

Let $X = \mathbb{R}$ then $d(x,y) = |x-y|$

SOLUTION:

i) As $|x-y| \geq 0$

$\therefore d(x,y) \geq 0$

ii) $d(x,y) = |x-y|$

$= |y-x|$

$= d(y,x)$

iii) $d(x,y) = 0 \iff |x-y| = 0$

$\iff x-y = 0$

$\iff x = y$

iv) $d(x,z) = |x-z|$

$= |x-y+y-z|$

$\leq |x-y| + |y-z|$

$d(x,z) \leq d(x,y) + d(y,z)$

Then (X,d) is a metric space.

EXAMPLE 2:

For set $R^2 = \{(x_1, x_2), (y_1, y_2)\}$

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

SOLUTION:

i) As $|x_1 - y_1| + |x_2 - y_2| \geq 0$

$\therefore d((x_1, x_2), (y_1, y_2)) \geq 0$

ii) $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$

$= |y_1 - x_1| + |y_2 - x_2|$

$= d((y_1, y_2), (x_1, x_2))$

iii) $d((x_1, x_2), (y_1, y_2)) = 0$

$\Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0$

$\Leftrightarrow |x_1 - y_1| = 0, |x_2 - y_2| = 0$

$\Leftrightarrow x_1 - y_1 = 0, x_2 - y_2 = 0$

$\Leftrightarrow x_1 = y_1, x_2 = y_2$

$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$

iv) $d((x_1, x_2), (z_1, z_2)) = |x_1 - z_1| + |x_2 - z_2|$

$= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$

$\leq (|x_1 - y_1| + |y_1 - z_1|) + (|x_2 - y_2| + |y_2 - z_2|)$

$= (|x_1 - y_1| + |x_2 - y_2|) + (|y_1 - z_1| + |y_2 - z_2|)$

$$d((x_1,x_2),(z_1,z_2)) \leq d((x_1,x_2),(y_1,y_2)) + d((y_1,y_2),(z_1,z_2))$$

As all the axioms of the metric are satisfied. So (X,d) is a metric space

EXAMPLE 3:

Let (X,d_1) and (X,d_2) be two metric spaces defined as:

$$d'((x_1,x_2),(y_1,y_2)) = \sum_{i=1}^{2} d_i(x_i,y_i).$$

Is d' metric on X?

EXAMPLE 4:

Let $X = C$ (complex no.) and $d : X \times X \Rightarrow R$ be defined as:

$$d(z_1,z_2) = |z_1 - z_2|.$$

SOLUTION:

i) As $|z_1 - z_2| \geq 0 \Rightarrow d(z_1,z_2) \geq 0.$

ii) $d(z_1,z_2) = |z_1 - z_2|$
$= |-(z_2 - z_1)|$
$= |z_2 - z_1| = d(z_2,z_1).$

iii) $d(z_1,z_2) = 0 \Leftrightarrow |z_1 - z_2| = 0$
$\Leftrightarrow z_1 - z_2 = 0$
$\Leftrightarrow z_1 = z_2$

iv) Let $z_1, z_2, z_3 \in C$.

$$d(z_1, z_3) = |z_1 - z_3|$$
$$= |z_1 - z_2 + z_2 - z_3|$$
$$\leq |z_1 - z_2| + |z_2 - z_3|$$

$$\therefore d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$$

$\Rightarrow (z, d)$ is a metric space.

EXAMPLE 5: (DISCRETE METRIC SPACE).

Let X be a non empty set and $d: X \times X \to R$ is defined as:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Then show that d is metric on X.

SOLUTION:

i) As $d(x, y) = 0$ if $x = y$
and $d(x, y) = 1$ if $x \neq y$

$$\Rightarrow d(x, y) \geq 0$$

ii) $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$$= \begin{cases} 0 & \text{if } y = x \\ 1 & \text{if } y \neq x \end{cases}$$

$$= d(y, x)$$

iii) $d(x,y)=0$ if $x=y$ (By definition)

iv) $d(x,z) \leq d(x,y)+d(y,z)$

Case I: $x \neq y \neq z$

$d(x,y)=1,\ d(y,z)=1,\ d(z,x)=1.$

$1 < 1+1$

$\Rightarrow d(x,z) < d(x,y)+d(y,z) \rightarrow (1)$

Case II: $x \neq y = z$

$d(x,y)=1,\ d(y,z)=0,\ d(z,x)=1$

$1 = 1+0$

$d(x,z) = d(x,y)+d(y,z) \rightarrow (2)$

Case III: $x=y=z$

$d(x,y)=0,\ d(y,z)=0,\ d(z,x)=0$

$0 = 0+0$

$d(x,z) = d(x,y)+d(y,z) \rightarrow (3)$

From (1), (2) and (3), we conclude

$d(x,z) \leq d(x,y)+d(y,z)$

Thus, d is a metric on X.

MINKOWSKI'S INEQUALITY:

$$\left\{\sum_{i=1}^{\infty} d(x_i - y_i)^p\right\}^{1/p} \leq \sum_{i=1}^{\infty} (x_i^{\,p})^{1/p} + \sum_{i=1}^{\infty} (y_i^{\,p})^{1/p}$$

OPEN BALL: (DEF).

Let (X,d) be a metric space and $x_0 \in X$. Then, for $r \in R^+$ we define an open ball with centre at 'x_0' and radius 'r' as a set consisting of all those points of X whose distance from 'x_0' is less than 'r'. Mathematically, it is defined and denoted as:

$$B(x_0, r) = S_r(x_0) = \{x : x \in X \text{ and } d(x, x_0) < r\}.$$

EXAMPLE:

An open ball in a usual metric is an open interval.

SOLUTION:

Let (R,d) be the usual metric. Let us consider the open ball

$$S_r(a) = \{x : x \in R \wedge d(x,a) < r\}$$

$$= \{x : x \in R \wedge |x-a| < r\}$$

$$= \{x : x \in R \wedge a-r < x < a+r\}$$

$= \,]a-r, a+r[$ Which is an open interval.

OPEN SET: (DEF).

Let (X,d) be the metric space and $A \subseteq X$. Then, A is said to be open set if for each $x \in A$. There exist some open ball $S_r(x)$ such that $x \in S_r(x) \subseteq A$.

THEOREM:

Prove That an open ball is an open set.

PROOF: Let us consider,

$S_r(x_0) = \{x : x \in X \wedge d(x, x_0) < r\}$.

Let $x \in S_r(x_0) \Rightarrow d(x, x_0) < r$.

$\Rightarrow r > d(x, x_0)$.

$\Rightarrow r - d(x, x_0) > 0$.

Put $r_1 = r - d(x, x_0) > 0$.

Consider $S_{r_1}(x) = \{y \in X : d(y, x) < r_1\}$.

We now show That $S_{r_1}(x) \subseteq S_r(x_0)$.

Let $y \in S_{r_1}(x) \Rightarrow d(y, x) < r_1 = r - d(x, x_0)$

$d(y, x) + d(x, x_0) < r$.

Now $d(y, x_0) \leq d(y, x) + d(x, x_0)$

$< r$

$\Rightarrow d(y, x_0) < r$.

$y \in S_r(x_0)$.

$\Rightarrow S_{r_1}(x) \subseteq S_r(x_0)$.

Hence, $S_r(x_0)$ is open set.

THEOREM: Let (X, d) be a metric space. Then:

i) ϕ and X are open sets.

ii) Union of any number of open sets is open.

iii) Intersection of finite number of open sets is open.

PROOF:

i) To prove: ϕ is open.

For this, we have to prove for each $x \in \phi$, there exists open ball $B(x, r)$ such that $x \in B(x, r) \subseteq \phi$. But since ϕ contains no element so automatically it is proved that ϕ is open set.

Next to prove, X is open. Let $x \in X$ and for any $r \in R^+$, define $B(x, r) = \{y \in X : d(y, x) < r\} \subseteq X$.
Hence, X is open.

ii) To prove: Union of any number of open sets is open.

Let $\{U_\alpha : \alpha \in I\}$ be collection of open sets.
To prove: $\bigcup_{\alpha \in I} U_\alpha$ is open.

Let $x \in \bigcup_{\alpha \in I} U_\alpha \Rightarrow x \in U_\alpha$ for some $\alpha \in I$.

As $x \in U_\alpha$ and U_α is open. Then by definition of open set there exist an open ball $B(x, r)$ such that:

$x \in B(x, r) \subseteq U_\alpha$

$\Rightarrow x \in B(x, r) \subseteq \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is open set.

iii) Let $\{U_1, U_2, \dots U_n\}$ be the finite collection of open sets.

To prove: $\bigcap_{i=1}^{n} U_i$ is open.

Let $x \in \bigcap_{i=1}^{n} U_i \Rightarrow x \in U_i$ for each i, $1 \le i \le n$

$\Rightarrow$ There exist open ball $B(x, r_i)$ such that $x \in B(x, r_i) \subseteq U_i$ $\because$ U_i is open set.

Let $r = \min\{r_1, r_2, \dots, r_n\}$.

Then for each i, $1 \le i \le n$.

$B(x, r) \subseteq B(x, r_i)$.

$\Rightarrow$ for each i, $1 \le i \le n$.

$x \in B(x, r) \subseteq B(x, r_i) \subseteq U_i$

$\Rightarrow x \in B(x, r) \subseteq U_i$ for each i

$\Rightarrow x \in B(x, r) \subseteq \bigcap_{i=1}^{n} U_i$

Hence, $\bigcap_{i=1}^{n} U_i$ is an open set

CLOSED BALL: (DEF).

Let (X, d) be a metric space, then for $x_0 \in X$ and $r \in \mathbb{R}^+$, we denote and defined closed ball as:

$\bar{B}(x_0, r) = S_r[x_0] = \{x \in X : d(x, x_0) \le r\}$

CLOSED SET (DEF).

Let (X,d) be a metric space and $A \subseteq X$. Then, A is said to be closed set if and only if A' is open.

REMARK:

We have just proved that intersection of finite number of open sets is open. This is not valid for the case of intersection of infinite number of open sets.

For example:

Let (R,d) be the usual metric and $\{I_n =]-1/n, 1/n[: n \in N\}$ be an infinite collection of open sets. Then, $\bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty}]-1/n, 1/n[= \{0\}$

Which is not open set.

DISTANCE OF A POINT FROM A SET: (DEF).

Let (X,d) be a metric space with $A \subseteq X$ and $x \in X$. Then, the distance of x from A is denoted and defined as:

$$d(x,A) = \inf\{d(x,y) : y \in A\}$$

DIAMETER OF A SET: (DEF).

Let (X,d) be a metric space and $A \subseteq X$. Then, diameter of "A" is denoted and defined as:

$$d(A) = \delta(A) = \sup\{d(x,y) : x, y \in A\}$$

BOUNDED SET: (DEF).

Let (X,d) be the metric space and $A \subseteq X$. Then, A is said to be bounded set if and only if diameter of A is finite.

REMARK:

Diameter of an empty set is considered as $-\infty$.

THEOREM:

Diameter of a closed ball is less than or equal to two times of its radius.

PROOF:

Let $A = \bar{B}(x_0, r) = \{y \in X : d(x_0, y) \leq r\}$ be a closed ball in metric space (X,d).

Now, $\delta(A) = \sup\{d(x,y) : x, y \in A\}$
$\leq 2r$

THEOREM:

Prove that union of two bounded sets is bounded:

PROOF:

Let A and B be the two bounded sets in metric space (X,d)

To prove: $A \cup B$ is bounded.
Since A and B are bounded.
So $\delta(A)$ and $\delta(B)$ are finite.
Let $x, y \in A \cup B$

Then, there arises the three cases.

Case I: If $x, y \in A$

Then $d(x,y) \leq \delta(A)$

$\Rightarrow \sup_{x,y \in A\cup B} d(x,y) \leq \delta(A)$

$\Rightarrow \delta(A\cup B) \leq \delta(A)$

Since $\delta(A)$ is finite

So $\delta(A\cup B)$ is also finite.

Hence $A\cup B$ is bounded.

Case II: If $x, y \in B$

Then by the similar argument as in Case I, $A\cup B$ is again bounded.

Case III: If $x \in A$ and $y \in B$.

Let $a \in A$ and $b \in B$.

Now $d(x,y) \leq d(x,a) + d(a,y)$

$\leq d(x,a) + d(a,b) + d(b,y)$

$\sup_{x,y \in A\cup B} d(x,y) \leq \sup_{x,a \in A} d(x,a) + d(a,b) + \sup_{b,y \in B} d(b,y)$

$\Rightarrow \delta(A\cup B) \leq \delta(A) + d(a,b) + \delta(B)$

Since, R.H.S is finite

So L.H.S is also finite.

Hence, $A\cup B$ is bounded.

LIMIT POINT (DEF):

Let (X,d) be a metric space and $A \subseteq X$. Then an element $x \in X$ is said to be limit point of A if and only if for every open ball $B(x,r)$.

$$B(x,r) \cap A \setminus \{x\} \neq \phi$$

In other words each open ball $B(x,r)$ contains a point of A different from x.

The set of all limit points of A is said to be derived set and is denoted by $d(A)$ or $D(A)$ or A^d or A'.

EXAMPLE: Let (R,d) be the usual metric and $A =]2,3]$, $B = [1,2,3,4]$

SOLUTION:

Consider $A =]2,3]$.

Let $x \in A$ i.e. $2 < x \leq 3$

then obviously for any $r > 0$, however small, $B(x,r) \cap A \setminus \{x\} \neq \phi$.

So x is then limit point of A.

Let $x = 2$, then again for any $r > 0$, however small, $B(x,r) \cap A \setminus \{x\} \neq \phi$.

$\Rightarrow x = 2$ is limit point of A.

If $x < 2$ or $x > 3$, then x is not limit point of A e.g. if $x = 1.8$ then $B(x, 0.1) \cap A \setminus \{x\} = \phi \Rightarrow D(A) = [2,3]$.

Also $D(B) = \phi$, because B is finite set.

DEF: CLOSURE OF A SET:

Let (X,d) be a metric space and $A \subseteq X$. Then, closure of A is denoted and defined as:

$$\bar{A} = A \cup D(A)$$

THEOREM:

Let (X,d) be the metric space and $A \subseteq X$, Then:

i) $\bar{A}$ is the intersection of all the closed supersets of A.

ii) $\bar{A}$ is the smallest closed superset of A.

iii) $\bar{A}$ is closed.

PROOF:

Let us define:

$\gamma = \{F : F$ is closed set of $A\}$

To prove: $\bar{A} = \cap\gamma$

Let $x \in \bar{A} \Rightarrow x \in A \cup D(A)$

$\Rightarrow x \in A$ or $x \in D(A)$.

If $x \in A \Rightarrow x \in \cap\gamma \Rightarrow A \subseteq \cap\gamma$.

If $x \notin A \Rightarrow x \in D(A)$ and we have to prove $x \in \cap\gamma$.

Suppose $x \notin \cap\gamma \Rightarrow x \notin F$ for some $F \in \gamma$.

$\Rightarrow x \in F'$

As F is closed $\Rightarrow F'$ is open

Then, by the definition of open set there exist some open ball $B(x,r)$ such that:

$x \in B(x,r) \subseteq F'$

Now as $F \cap F' = \phi$

and $B(x,r) \subseteq F'$ and $A \subseteq F$

$\Rightarrow B(x,r) \cap A = \phi$

$\Rightarrow B(x,r) \cap A - \{x\} = \phi$

$\Rightarrow x$ is not the limit point of A

$\Rightarrow x \notin D(A)$

Which is a contradiction

$\because x \in D(A)$

So, our supposition is wrong.

Hence, $x \in \cap\gamma \Rightarrow \bar{A} \subseteq \cap\gamma \rightarrow (1)$

Now let $x \in \cap\gamma$

To prove: $x \in \bar{A} = A \cup D(A)$

If $x \in A \Rightarrow x \in A \cup D(A) \Rightarrow x \in \bar{A}$

If $x \notin A$ then to prove $x \in D(A)$

Suppose $x \notin D(A)$

$\Rightarrow x$ is not the limit point of A.

Then, There exists some open ball $B(x,r)$

such that $B(x,\gamma) \cap A - \{x\} = \phi$

$\Rightarrow B(x,\gamma) \cap A = \phi$

$\Rightarrow A \subseteq (B(x,\gamma))'$

Since $B(x,\gamma)$ is an open set

So, $(B(x,\gamma))'$ is a closed set

$\Rightarrow (B(x,\gamma))'$ is the closed super set of A with

$x \notin (B(x,\gamma))' \Rightarrow x \notin \cap\gamma$

Which is a contradiction.

So, our supposition is wrong.

And hence, $x \in D(A) \Rightarrow x \in A \cup D(A)$

$\Rightarrow x \in \bar{A} \Rightarrow \cap\gamma \subseteq \bar{A} \rightarrow (2)$

① and ② $\Rightarrow \bar{A} = \cap \gamma$.

ii)

To prove: $\bar{A}$ is the smallest closed superset of A.

Since $\bar{A}$ is the intersection of all closed supersets of A.

As intersection of any number of closed sets is closed. So, $\bar{A}$ is the closed superset of A.

Now, we prove $\bar{A}$ is the smallest such set, let B be another closed superset of A.

$\Rightarrow B \in \gamma$

$\Rightarrow \cap \gamma \subseteq B$

$\Rightarrow \bar{A} \subseteq B$

$\Rightarrow \bar{A}$ is the smallest closed superset of A.

iii) Since $\bar{A}$ is the intersection of closed sets.

Hence A is closed.

THEOREM:

Let (X,d) be a metric space and $A \subseteq X$. Then, A is open if and only if A is the union of open balls/spheres.

PROOF:

Let (X,d) be a metric space and $A \subseteq X$.

To prove: A is the union of open spheres.

Let x be an arbitrary point of A.

As A is open, so there exists some open ball $S_{r_x}(x)$ such that:

$$x \in S_{r_x}(x) \subseteq A .$$

Then, $\bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} S_{r_x}(x) \subseteq A$.

$$\Rightarrow A \subseteq \bigcup_{x \in A} S_{r_x}(x) \subseteq A .$$

$$\Rightarrow A = \bigcup_{x \in A} S_{r_x}(x) .$$

Conversely suppose that, A is the union of open spheres.

To prove: A is open.

Since, each open sphere is a open set in a metric space and the union of any number of open sets is also open.

So, A being the union of open sets, is open.

THEOREM:

Let (X, d) be a metric space and $A \subseteq X$. Then a point $x \in X$ and $x \notin A$ is called limit point of 'A' if every open ball containing 'x' contains infinite number of points of A.

PROOF:

Suppose $B(x, r)$ containing 'x' contains finite number of points of A, then:

$$B(x_0, r) \cap A = \{x_1, x_2, \ldots, x_n\} .$$

As $x \notin A \Rightarrow x \notin \{x_1, x_2, \dots, x_n\}$.

$\Rightarrow x \neq x_i \ \forall i, 1 \leq i \leq n$.

$\Rightarrow d(x, x_i) > 0 \ \forall \ i = 1, 2, \dots, n$.

Put $d(x, x_i) = r_i$.

and let $r^* = \min\{r_1, r_2, \dots, r_n\}$.

$\Rightarrow r^* \leq r_i \ \forall \ i = 1, 2, \dots, n$.

Now consider,

$B(x, r^*) = \{y \in X : d(x, y) < r^*\}$.

For $y \in B(x, r^*) \Rightarrow d(x, y) < r^* < r_i, \ i = 1, 2, \dots, n$

$\Rightarrow d(x, y) < r_i = d(x, x_i)$

$\Rightarrow d(x, y) < d(x, x_i)$

$\Rightarrow \ y \neq x_i \quad i = 1, 2, \dots, n$.

$\Rightarrow B(x, r^*)$ doesnot contain any point of A different from x.

So, x is not the limit point of A.

Which is a contradiction.

So our supposition is wrong.

And hence, every open ball containing x contains infinite number of points of 'A'.

"CONVERGENCE IN METRIC SPACES AND COMPLETE METRIC SPACES"

DEFINITION:

Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to a point 'x' [written as $x_n \to x$] of X. If for every $\varepsilon > 0$ There exists a positive integer

n_0 such that $d(x_n, x) < \varepsilon$ whenever $n \geq n_0$ or in other words:

$$\lim_{n \to \infty} d(x_n, x) = 0$$

Then, we write $\lim_{n \to \infty} x_n = x$

Then, x is called the limit point of $\{x_n\}$.

CAUCHY SEQUENCE : (DEF).

Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer n_0, such that:

$$d(x_m, x_n) < \varepsilon \text{ whenever } m, n \geq n_0$$

THEOREM:

Prove that every convergent sequence is cauchy.

PROOF:

Let $\{x_n\}$ be a convergent sequence in metric space (X, d).

To prove: $\{x_n\}$ is cauchy in X

Since $\{x_n\}$ is convergent in X. So, there exist some $x \in X$ such that $x_n \to x$. Then, for every $\varepsilon > 0$, there exist some positive integer n_0 such that:

$$d(x_n, x) < \varepsilon/2 \text{ whenever } n \geq n_0$$

Now consider $m, n \geq n_0$.
Then, $d(x_m, x) < \varepsilon/2$ and $d(x_n, x) < \varepsilon/2$.
Now $d(x_m, x_n) \leq d(x_m, x) + d(x, x_n)$.

$\Rightarrow d(x_m, x_n) \leq \varepsilon/2 + \varepsilon/2$

$\Rightarrow d(x_m, x_n) \leq \varepsilon \quad (m, n \geq n_0)$.

Hence sequence is Cauchy in X.

REMARK:
Converse of the above theorem is not true in general i.e. there may be a sequence which is Cauchy in X but is not convergent in X.
e.g. Let $X =]0, 1]$ and define $d: X \times X \to R$ by $d(x, y) = |x - y|$. Then (X, d) is metric space. Consider $x_n = 1/n$.
Then $\{x\}$ is Cauchy in X.
But $\{x\}$ is not convergent in X
$\because 1/n \to 0 \notin X$

BOUNDED SEQUENCE: (DEF).
A sequence $\{x_n\}$ in metric space (X, d) is said to be bounded if there exists some positive real number λ, however large, such that:
$d(x_n, x) \leq \lambda$ for all n.
Here $x \in X$ is some fixed element.

THEOREM:

Let (X,d) be a metric space. Then,-

i). Every convergent sequence is bounded and limit of the convergent sequence is unique.

ii) If $x_n \to x$ and $y_n \to y$. Then,
$d(x_n, y_n) \to d(x, y)$.

PROOF:

i) a): Since $\{x_n\}$ is convergent, so say, $x_n \to x \in X$. Then, for every $\varepsilon > 0$; there exist some positive integer n_0 such that:

$$d(x_n, x) < \varepsilon, \text{ whenever } n \geq n_0.$$

Let $\lambda_1 = \max\{d(x_1, x), d(x_2, x), \ldots, d(x_{n-1}, x)\}$.

Then, $d(x_n, x) < \lambda_1 + \varepsilon \quad \forall \, n \geq 1$.

$\Rightarrow d(x_n, x) < \lambda \quad \forall \, n \geq 1$

$\Rightarrow \{x_n\}$ is bounded.

b): Now, we prove the uniqueness of the limit. On the contrary, say $x_n \to x$ and $x_n \to y$.

Then; $\lim\limits_{n \to \infty} d(x_n, x) \to 0$.

and $\lim\limits_{n \to \infty} d(x_n, y) \to 0$

Now, $d(x, y) \leq d(x, x_n) + d(x_n, y) \to 0 + 0$ When $n \to \infty$

$\Rightarrow d(x, y) \leq 0$

$\Rightarrow d(x, y) = 0 \Leftrightarrow x = y$

Hence, limit of convergent sequence is unique.

ii) Given $x_n \to x$ and $y_n \to y$.
To prove: $d(x_n, y_n) \to d(x, y)$
Since, $x_n \to x$ and $y_n \to y$.

So, $\lim_{n\to\infty} d(x_n, x) \to 0$ and $\lim_{n\to\infty} d(y_n, y) \to 0$.

Now, $|d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(x, y_n) - d(x, y)|$.

$\Rightarrow |d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(x, y) + d(y, y_n) - d(x, y)|$.

$\Rightarrow |d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(y, y_n)| \to 0$
when $n \to \infty$

$\Rightarrow d(x_n, y_n) \to d(x, y)$.

★ V-Imp
THEOREM:
Let (X, d) be a metric space and $M \subseteq X$, then:

i) $x \in \overline{M}$ if and only if there exist a sequence $\{x_n\}$ in M such that $x_n \to x$.

ii) M is closed if and only if the situation sequence $\{x_n\}$ in M such that $x_n \to x$. Then, $x \in M$.

PROOF:
i) Suppose $x \in \overline{M} \Rightarrow x \in M \cup D(M)$
$\Rightarrow x \in M$ or $x \in D(M)$

If $x \in M$, then we have the constant sequence $(x, x, x, \text{-----}) \to x$.

If $x \notin M$, then $x \in D(M)$:

Then, for each positive integer n, open ball $B(x, 1/n)$ contains a point of M different from x.

$\Rightarrow$ This is a sequence $\{x_n\}$ in M and $x_n \in B(x, 1/n)$

$\Rightarrow d(x_n, x) < 1/n \to 0$ when $n \to \infty$.

$\Rightarrow x_n \to x$.

Conversely suppose that there is a sequence $\{x_n\}$ in M such that $x_n \to x$.

To prove: $x \in \bar{M}$:

As $x_n \to x$. Then each open ball $B(x, r)$ contains all the points of the sequence $\{x_n\}$ except a finite number of points.

$\Rightarrow B(x, r)$ contains the points of M different from x

$\Rightarrow x \in D(M)$

$\Rightarrow x \in M \cup D(M)$

$\Rightarrow x \in \bar{M}$

ii). Let us consider M is closed and $\{x_n\}$ be a sequence in M such that $x_n \to x$.

To prove: $x \in M$.

By (i) $x \in \bar{M}$.

Since M is closed. So $M = \bar{M}$
$\Rightarrow x \in M$.

Conversely, let $\{x_n\}$ be a sequence in M, such that $x_n \to x$ and $x \in M$.

To prove: M is closed.
$M \subseteq \bar{M} \to (1)$

Let $x \in \bar{M}$, then by part (i) there exist a sequence $\{x_n\}$ in M, such that $x_n \to x$.

Then, by given $x \in M$.
$\Rightarrow \bar{M} \subseteq M \to (2)$
(1) and (2) $\Rightarrow M = \bar{M}$

$\Rightarrow M$ is closed.

THEOREM:
If a convergent sequence in a metric space has infinitely many distinct points then its limit is a limit point of the set of the points of the sequence.

PROOF:
Let (X, d) be a metric space and $\{x_n\}$ be a convergent sequence in X with infinitely many distinct points. Let A be the set of points of the sequence and $x_n \to x$.

To prove:

x is the limit point of A.

Let us consider any open ball $B(x,r)$. Then, obviously by given condition $B(x,r)$ contains the infinite many distinct points of A.

$\Rightarrow x \in D(A)$.

COMPLETE METRIC SPACE: (DEF).

Let (X,d) be a metric space then, X is said to be complete metric space if and only if every cauchy sequence in X converges to a point in X.

THEOREM:

A subspace M of a complete metric space X is complete if and only if M is closed in X.

PROOF:

Suppose M is complete.

To prove: M is closed in X.

Let $\{x_n\}$ be a sequence in M such that $x_n \to x$.

As every convergent sequence is Cauchy so $\{x_n\}$ is Cauchy in M.

As M is complete, so, $x_n \to x \in M$. But $x_n \to x$, so $x \in M$. So, M is closed. ($\because$ M is closed if and only if the situation sequence $\{x_n\}$ in M such that $x_n \to x$. Then $x \in M$).

Conversely, suppose M is closed in X.
To prove: M is complete.

Let $\{x_n\}$ be a Cauchy sequence in M.
Since, $\{x_n\}$ is a sequence in M and $M \subseteq X$.
So $\{x_n\}$ is also a Cauchy sequence in X.
As X is complete so $x_n \to x \in X$

Hence $\{x_n\}$ is a Cauchy sequence in M and $x_n \to x$.
But M is closed.
Then, by above theorem $*$, $x \in M$.

$\Rightarrow M$ is complete

NESTED SEQUENCE: (DEF).

Let (X, d) be a metric space and $\{A_n\}$ be a sequence of non empty subsets of X. Then, this sequence is called nested sequence if:

i) $A_n \supseteq A_{n+1} \quad \forall\, n \geq 1$
ii) $\delta(A_n) \to 0$ when $n \to \infty$

Nested Interval Property
OR
CANTOR'S INTERSECTION THEOREM:

STATEMENT:

Let (X, d) be a complete metric space and $\{F_n\}$ be a decreasing sequence of closed subsets of X, such that $\delta(F_n) \to 0$ as $n \to \infty$. Then, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

PROOF:

First we show that $\bigcap_{n=1}^{\infty} F_n \neq \phi$

As F_n is non empty for all n.
So, let $x_n \in F_n$ and $x_m \in F_m$.
Put $m = \max(m, n)$.
Then as $\{F_n\}$ is decreasing so $F_m \subseteq F_n$.
$\Rightarrow x_m \in F_m \subseteq F_n$.
$\Rightarrow x_m \in F_n$.
$\Rightarrow$ Both x_m and $x_n \in F_n$.
$\Rightarrow d(x_m, x_n) \leq \delta(F_n) \to 0$ when $n \to \infty$.

$\Rightarrow d(x_m, x_n) \to 0$ when $n \to \infty$
$\Rightarrow \{x_n\}$ is cauchy sequence in X.
But X is complete
As X is complete, so $\{x_n\}$ converges to a point in X; say $x_n \to x \in X$.

Now, here arises two cases.

Case I:

If the sequence $\{x_n\}$ contain finite number of distinct points, Then, x is that point which repeats infinitely many times. Then, There exists a positive integer n_0 such that:

$$x \in F_n \quad \forall\, n \geq n_0$$

But as $\{F_n\}$ is decreasing.
So, $x \in F_n \;\forall\, n$.
$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n \Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \phi$.

Case II:

If $\{x_n\}$ contains infinite many distinct points. Then, x is the limit point of the set of the points of the sequence. Then, using the facts that $\{F_n\}$ is decreasing and its elements are closed sets we have:

$$x \in F_n, \ \forall n$$
$$\Rightarrow x \in \bigcap_{n=1}^{\infty} F_n \Rightarrow \bigcap_{n=1}^{\infty} F_n \neq \phi.$$

Now we show that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Consider $x, y \in \bigcap_{n=1}^{\infty} F_n$

$$\Rightarrow x, y \in F_n \ \forall n$$
$$\Rightarrow d(x,y) \leq \delta(F_n) \to 0 \text{ when } n \to \infty$$
$$\Rightarrow d(x,y) \to 0 \Rightarrow x = y.$$

Hence, $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

CONTINUOUS FUNCTION:

Let (X, d_X) and (Y, d_Y) be two metric spaces then a function $f: X \to Y$ is said to be continuous at a point $x_0 \in X$, if for every $\varepsilon > 0$ there must exist some $\delta > 0$ such that:

$$d_Y(f(x), f(x_0)) < \varepsilon \text{ whenever } d_X(x, x_0) < \delta.$$

THEOREM:

Let (X,d) be a metric space, a $f: X \to R$ is defined as $f(x) = d(x,z)$, where z is a fixed point of X, then show that f is continuous on X.

PROOF:

Let $y \in X$ and $\varepsilon > 0$

Choose, $\delta = \varepsilon$ such that $d(x,y) < \delta$.

Now, $|f(x) - f(y)| = |d(x,z) - d(y,z)|$

$\leq d(x,y) < \delta = \varepsilon$

$\therefore |f(x) - f(y)| < \varepsilon$

Hence, f is continuous at y. Since, y is an arbitrary point of X. So, f is continuous on X.

Imp.

THEOREM:

Let (X,d_X) and (Y,d_Y) be two metric spaces and $f: X \to Y$ is a function then, f is said to be continuous at $x_0 \in X$ if and only if there exist a sequence $\{x_n\}$ in X such that $f(x_n) \to f(x_0)$ when $x_n \to x_0$.

PROOF:

Suppose $f: X \to Y$ is continuous at $x_0 \in X$. And let $\{x_n\}$ be a sequence in X such that $x_n \to x_0$.

To prove: $f(x_n) \to f(x_0)$.

Since 'f' is continuous at x_0. So, for every $\varepsilon > 0$, there exist a $\delta > 0$ such that:

$d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d(x, x_0) < \delta \rightarrow$ ①.

Now as $x_n \to x_0$. So, for $\delta > 0$, there exist some positive integer n_0 such that:

$d_X(x_n, x_0) < \delta$ whenever $n \geq n_0$.
So by ①, whenever $n \geq n_0$:
$d_Y(f(x_n), f(x_0)) < \varepsilon$
$\Rightarrow f(x_n) \to f(x_0)$

Conversely, suppose if $x_n \to x_0$, then, $f(x_n) \to f(x_0)$. To prove: f is continuous at x_0.
Suppose on the contrary, f is discontinuous at x_0.
Then, for every $\delta > 0$, there exist $\varepsilon > 0$, such that:
$d_X(x, x_0) < \delta$ but $d_Y(f(x), f(x_0)) \not< \varepsilon$

$\Rightarrow d_Y(f(x), f(x_0)) \geq \varepsilon$
Let $\delta = 1/n$

Then there exist $x_n \in X$ such that $d_X(x_n, x_0) < 1/n$ but $d_Y(f(x_n), f(x_0)) \geq \varepsilon$.

Now as $d_X(x_n, x_0) < 1/n$.
So when $n \to \infty$. Then $d_X(x_n, x_0) \to 0$.
$\Rightarrow x_n \to x_0$
Then, by the hypothesis, $f(x_n) \to f(x_0)$.

But as $d_Y(f(x_n), f(x_0)) \geq \varepsilon$
$\Rightarrow f(x_n) \not\to f(x_0)$.

Which is a contradiction.
So, our supposition is wrong.
Hence, f is continuous at x_0.

THEOREM:
Prove that every singleton set in a metric space is closed:

PROOF:
Let $F = \{x\}$ be a singleton subset of a metric space X.

Let $y \in F^c \Rightarrow y \notin F$.
So $x \neq y \Rightarrow d(x,y) \neq 0$.

Let $d(x,y) = r$.
then, $x \notin S_r(y) \Rightarrow S_r(y) \subset F^c$.

$\Rightarrow F^c$ is open.
$\Rightarrow F$ is closed.

THEOREM:
Prove that every finite set in a metric space is closed:

PROOF:
Let 'F' be a finite subset of a metric space X, then, we have to show that F is closed in X.

Suppose on the contrary that F is not closed in X, then, there must exist at least one limit point of F which is not in F. Let that limit point of F be x, then each open ball centred at x must contain infinite number of points of F. But F is finite.

So this condition cannot be satisfied. This shows that x is not a limit point of F.

Which is a contradiction to our assumption that 'x' is not a limit point of F.

So, F is closed.

THEOREM:

Let (X, d) be a metric space and $A \subseteq X$. Then, A is closed if and only if $D(A) \subseteq A$.

PROOF:

Suppose A is closed.

To prove: $D(A) \subseteq A$

Since, A is closed. So, A' is open.

Let $x \in D(A)$

Then, x is the limit point of A.

Then, for each open ball $B(x, r)$,

$$B(x, r) \cap A - \{x\} \neq \phi$$

Now if $x \in A$, then there is nothing to prove.

Suppose $x \notin A \Rightarrow A - \{x\} = A$

$\Rightarrow B(x, r) \cap A - \{x\} \neq \phi$

$\Rightarrow B(x, r) \cap A \neq \phi$

$\Rightarrow B(x, r) \nsubseteq A'$

$\Rightarrow A'$ is not open set

$\Rightarrow A$ is not closed.

Which is a contradiction.

So our supposition is wrong.

Hence $x \in A \Rightarrow D(A) \subseteq A$

Conversely, let us suppose $D(A) \subseteq A$

To prove: A is closed set. For this, we prove A' is open.

Let $x \in A' \Rightarrow x \notin A \Rightarrow x \notin D(A)$ $(\because D(A) \subseteq A)$

$\Rightarrow x$ is not the limit point of A.

Then there exist an open ball $B(x,r)$ such that:

$$B(x,r) \cap A - \{x\} = \phi$$

$$\Rightarrow B(x,r) \cap A = \phi$$

$$\Rightarrow B(x,r) \subseteq A'$$

$\Rightarrow x \in B(x,r) \subseteq A' \Rightarrow A'$ is open

$\Rightarrow A$ is closed.

THEOREM:

Prove that closed ball in usual metric space is closed interval.

PROOF:

Let (R,d) be the usual metric space.

Let us consider the closed ball

$$S_r[a] = \{x : x \in R \wedge d(x,a) \leq r\}$$
$$= \{x : x \in R \wedge |x - a| \leq r\}$$
$$= \{x : x \in R \wedge a - r \leq x \leq a + r\}$$
$$= [a - r \quad a + r]$$

Which is closed interval.

THEOREM:

Prove that in metric space (X,d) closed ball is a closed set:

PROOF:

Let us consider,

$S_r[x_0] = \{x \in X : d(x, x_0) \leq r\}$.

Then we have to show that closed ball i.e, $S_r[x_0]$ is a closed set.

Let $x \in S_r^c[x_0]$.

$\Rightarrow x \notin S_r[x_0]$.

So, $d(x, x_0) > r \Rightarrow d(x, x_0) - r > 0$.

Put $r_1 = d(x, x_0) - r \rightarrow (1)$.

Consider $S_{r_1}[x] = \{y \in X : d(y, x) < r_1\}$.

We now show that: $S_{r_1}[x] \subseteq S_r^c[x_0]$.

Let $y \in S_{r_1}[x] \Rightarrow d(y, x) < r_1$

$d(x, x_0) \leq d(x, y) + d(y, x_0)$.

$d(x, x_0) < r_1 + d(y, x_0)$.

$d(y, x_0) > d(x, x_0) - r_1 \rightarrow (2)$

$d(y, x_0) > d(x, x_0) - d(x, x_0) + r$ (using (1)).

$\Rightarrow d(y, x_0) > r$.

So $y \notin S_r[x_0]$.

$\Rightarrow y \in S_r^c[x_0]$.

$\Rightarrow S_{r_1}[x] \subseteq S_r^c[x_0]$.

$\Rightarrow S_r^c[x_0]$ is open.

$\Rightarrow S_r[x_0]$ is closed.
Hence closed balls are closed sets.

THEOREM:
Prove that in metric space (X,d):

i) ϕ and X are closed sets.

ii) Intersection of any number of closed sets is closed.

iii) Union of finite number of closed sets is closed.

PROOF:

i) As ϕ is open. So ϕ' is closed.
But $\phi' = X \Rightarrow X$ is closed.
As X is open. So X' is closed.
As $X' = \phi \Rightarrow \phi$ is closed.

ii) Let $\{A_\alpha : \alpha \in I\}$ be any collection of closed sets.
$\Rightarrow \{A'_\alpha : \alpha \in I\}$ is the collection of open sets in X. $\Rightarrow \bigcup_{\alpha\in I} A'_\alpha$ is open set in X.

$$\Rightarrow \left(\bigcup_{\alpha\in I} A'_\alpha\right)' \text{ is closed.}$$

As, $\left(\bigcup_{\alpha\in I} A'_\alpha\right)' = \bigcap_{\alpha\in I} (A'_\alpha)' = \bigcap_{\alpha\in I} A_\alpha$

$$\Rightarrow \bigcap_{\alpha\in I} A_\alpha \text{ is closed set.}$$

iii) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of closed sets.
So, $\{A'_1, A'_2, \ldots, A'_n\}$ is the collection of open sets. As intersection of finite number

of open sets is open. So,

$\bigcap_{i=1}^{n} A_i'$ is open $\Rightarrow (\bigcap_{i=1}^{n} A_i')'$ is closed.

$\Rightarrow \bigcup_{i=1}^{n} A_i$ is closed.

THEOREM:

Prove that $C(X,\mathbb{R})$ is the closed.

PROOF:

To prove: $C(X,\mathbb{R}) = \overline{C(X,\mathbb{R})}$

As., $C(X,\mathbb{R}) \subseteq \overline{C(X,\mathbb{R})} \rightarrow (1)$

Now let $f \in \overline{C(X,\mathbb{R})}$. If $f \in C(X,\mathbb{R})$. Then, there is nothing to prove.

If $f \notin C(X,\mathbb{R})$. Then, f is limit point of $C(X,\mathbb{R})$ $(\because \bar{A} = A \cup d(A))$.

Then for each open ball $B(f, \varepsilon/3)$

$B(f, \varepsilon/3) \cap C(X,\mathbb{R}) \setminus \{f\} \neq \phi$

$\Rightarrow B(f, \varepsilon/3) \cap C(X,\mathbb{R}) \neq \phi$

$\Rightarrow f_1 \in B(f, \varepsilon/3) \cap C(X,\mathbb{R})$

$\Rightarrow f_1 \in B(f, \varepsilon/3)$ and $f_1 \in C(X,\mathbb{R})$

As $f_1 \in B(f, \varepsilon/3)$ and $d(f_1, f) = \sup_{x \in \mathbb{R}} |(f_1 - f)(x)| < \varepsilon/3$

$\Rightarrow |f_1(x) - f(x)| < \varepsilon/3 \;\; \forall x \in R.$

As $f_1 \in C(X,R)$, so f_1 is continuous at $x_0 \in X$. Then, for $\varepsilon > 0$ there exist $\delta > 0$ such that:

$$|f_1(x) - f_1(x_0)| < \varepsilon/3 \text{ whenever } d(x, x_0) < \delta.$$

Now,

$$|f(x) - f(x_0)| = |f(x) - f_1(x) + f_1(x_0) - f(x_0) + f_1(x) - f_1(x_0)|$$
$$\leq |f(x) - f_1(x)| + |f_1(x_0) - f(x_0)| + |f_1(x) - f_1(x_0)|$$

$\Rightarrow |f(x) - f(x_0)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3$

$\Rightarrow |f(x) - f(x_0)| \leq \varepsilon$

$\Rightarrow f$ is continuous on X

$\Rightarrow f \in C(X,R)$

$\Rightarrow \overline{C(X,R)} \subseteq C(X,R) \rightarrow (2)$

(1) and (2) $\Rightarrow C(X,R) = \overline{C(X,R)}$

$\Rightarrow C(X,R)$ is closed.

Topological Spaces

Topology is the generalization of the Metric Space. The word Topology is composed of two words.

- **Top** means twisting instruments.
- **Logy** a Latin word means Analysis.

Definition 1: Topological spaces

Suppose that X be a non-empty set and τ be the collection of subsets of X, then τ is called a topology on X if the following axioms are satisfied.

1. ϕ and X are in τ.
2. The union of the elements of any sub collection of τ is in τ.
3. The intersection of the elements of any finite sub collection of τ is in τ.

We call the set X together with topology τ is a topological space and denote it (X, τ).

The subset A of X is an open subset of X if $A \in \tau$, so we can say that a topological space together with its subsets are all open, such that X and ϕ are both open and also the infinite union and finite intersection of open sets is also open.

Examples:

(1) Let $X = \{a, b, c\}$, and consider the collection

$$\tau = \{X, \phi, \{a\}, \{b, c\}\}$$

- X and ϕ belongs to τ.
- The union of any sub collection of τ belongs to τ.
- The intersection of finite sub collection of τ belongs to τ.

All the three axioms are satisfied, hence τ is a topology on X.

(2) Let X be any set, and $P(X)$ called the power set of X consisting of all subsets of X is a topology on X. It is called discrete topology.

(3) The collection consisting of the set X and empty set only is also a topology on X, it is called indiscrete topology or trivial topology.

METRIC TOPOLOGY: (DEF).

Let (X,d) be a metric space and $\mathcal{T} = \{U \subseteq X : U \text{ is open in } (X,d)\}$. Then, $\mathcal{T}$ is said to be topology on X and is called metric topology.

EXAMPLE:

Let $X = \{1, 2, 3\}$, then make all the possible topologies on X.

COFINITE TOPOLOGY: (DEF).

Let X be a non empty set and $\mathcal{T}_c$ be the collection containing ϕ and all those subsets of X whose compliments are finite. Then $\mathcal{T}_c$ is a topology on X, it is called cofinite topology.

SOLUTION:

i) $\phi \in \mathcal{T}_c$ (Given)

Also as $X' = \phi$, which is finite.

So $X \in \mathcal{T}_c$

ii) Let γ be the collection of elements of $\mathcal{T}_c$. To prove: $\cup\gamma \in \mathcal{T}_c$

Here arises two cases:

Case I: $\phi \notin \gamma$

Now $(\cup\gamma)' = (\bigcup_{F\in\gamma} F)'$

$$= \bigcap_{F\in\gamma} F'$$

Now here for each $F \in \gamma \subseteq \mathcal{F}_c$:

$\Rightarrow F'$ is finite.

$\Rightarrow \bigcap_{F\in\gamma} F'$ is finite.

$\Rightarrow (\cup\gamma)'$ is finite.

$\Rightarrow \cup\gamma \in \mathcal{F}_c$.

Case II: If $\phi \in \gamma$

Then, we have γ_1 such that:

$\gamma_1 = \gamma - \{\phi\}$.

Then $\phi \notin \gamma_1$

Then, by Case I, $\cup\gamma_1 \in \mathcal{F}_c$.

As $\cup\gamma = \phi \cup (\cup\gamma_1)$.

$= \cup\gamma_1 \in \mathcal{F}_c$.

$\Rightarrow \cup\gamma \in \mathcal{F}_c$.

iii) Let $\alpha = \{F_1, F_2, \ldots, F_n\}$ be a finite collection of elements of $\mathcal{F}_c$

To prove: $\cap\alpha = \bigcap_{i=1}^{n} F_i \in \mathcal{F}_c$

Here arises two cases:

Case I: If $\phi \notin \alpha$

Then, $(\cap\alpha)' = (\bigcap_{i=1}^{n} F_i)'$

$= \bigcup_{i=1}^{n} F_i'$

As $F_i \in \alpha \subseteq \mathcal{F}_c \Rightarrow$ for each i, $1 \leq i \leq n$.

F_i' is finite.

As finite union of finite sets is finite.

So $\bigcup_{i=1}^{n} F_i'$ is finite.

$\Rightarrow (\cap\alpha)'$ is finite $\Rightarrow \cap\alpha \in \mathcal{F}_c$

Case II: If $\phi \in \alpha$.
Then, we can find $\beta = \alpha - \{\phi\}$.
Then $\phi \notin \beta$. Then by Case I: $\cap\beta \in \mathcal{F}_c$.

Now, $\cap\alpha = \phi \cap (\cap\beta)$
$= \phi \in \mathcal{F}_c$

Hence, $\mathcal{F}_c$ is a topology on X.

REMARKS.
(i) If X is finite, then $\mathcal{F}_D = \mathcal{F}_c$
(ii) If X is singelton set, then $\mathcal{F}_I = \mathcal{F}_D = \mathcal{F}_c$

OPEN SET: (DEF):
Let $(X, \mathcal{F})$ be a topological space and $A \subseteq X$. Then, A is said to be open set if $A \in \mathcal{F}$.

EXAMPLE:
If $X = \{1, 2, 3\}$.
$\mathcal{F} = \{\phi, X, \{1\}, \{2\}, \{1,2\}\}$ is a Topology on X.
Then $\{1,2\}$ is open set in $(X, \mathcal{F})$.

But $\{1,3\}, \{2,3\}, \{3\}$ are not open sets in $(X, \mathcal{F})$.

THEOREM:
Let $(X, \mathcal{T})$ be a topological space. Then,

i) ϕ and X are open sets.
ii) Union of any number of open sets is open.
iii) Intersection of finite number of open sets is open.

PROOF:
i). As $\phi \in \mathcal{T} \Rightarrow \phi$ is open.
As $X \in \mathcal{T} \Rightarrow X$ is open.

ii) Let $\{A_\alpha : \alpha \in I\}$ be a collection of any number of open sets.
To prove: $\bigcup_{\alpha \in I} A_\alpha$ is open.

As A_α is collection of open sets.
So A_α is open.
$\Rightarrow A_\alpha \in \mathcal{T}$
$\Rightarrow \bigcup A_\alpha \in \mathcal{T}$ ($\because \mathcal{T}$ is topology on X).
$\Rightarrow \bigcup A_\alpha$ is open.

iii) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of open sets.

To prove: $\bigcap_{i=1}^{n} A_i$ is open.

As A_i is open $\Rightarrow A_i \in \mathcal{T}$
$\Rightarrow \bigcap_{i=1}^{n} A_i \in \mathcal{T}$ ($\because \mathcal{T}$ is topology on X)
$\Rightarrow \bigcap_{i=1}^{n} A_i$ is open.

Hence proved.

CLOSED SET: (DEF).

Let $(X,\mathcal{T})$ be a topological space and $A \subseteq X$, then A is said to be closed if and only if A' is open.

THEOREM:

Let $(X,\mathcal{T})$ be a topological space, then:

i) ϕ and X are closed sets.

ii) Intersection of any number of closed sets is closed.

iii) Union of finite number of closed sets is closed.

PROOF:

i) As ϕ is open so ϕ' is closed.

As $\phi' = X \Rightarrow X$ is closed.

As X is open so X' is closed.

As $X' = \phi \Rightarrow \phi$ is closed.

ii) Let $\{A_\alpha : \alpha \in I\}$ be any collection of closed sets.

$\Rightarrow \{A'_\alpha : \alpha \in I\}$ is the collection of open sets.

$\Rightarrow \bigcup_{\alpha \in I} A'_\alpha$ is open set in X.

$\Rightarrow \left(\bigcup_{\alpha \in I} A'_\alpha\right)'$ is closed.

As, $\left(\bigcup_{\alpha \in I} A'_\alpha\right)' = \bigcap_{\alpha \in I} (A'_\alpha)'$

$= \bigcap_{\alpha \in I} A_\alpha \Rightarrow \bigcap_{\alpha \in I} A_\alpha$ is closed set.

iii) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite collection of closed sets.

So, $\{A_1', A_2', \ldots, A_n'\}$ is the finite collection of open sets.

As intersection of finite number of open sets is open. So,

$\bigcap_{i=1}^{n} A_i'$ is open.

$\Rightarrow \left(\bigcap_{i=1}^{n} A_i'\right)'$ is closed.

$\Rightarrow \bigcup_{i=1}^{n} A_i$ is closed.

CLOSURE OF A SET: (DEF).

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, closure of A, denoted by $\bar{A}$ is the intersection of all the closed supersets of A.

EXAMPLE:

Let $X = \{a, b, c, d\}$, $\mathcal{T} = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$
$A = \{b, c\}$ Then find $\bar{A}$

SOLUTION:

Closed sets are: $X, \phi, \{b, c, d\}, \{c, d\}, \{d\}$.

Closed supersets of A are: $X, \{b, c, d\}$.

$$\bar{A} = X \cap \{b, c, d\}$$
$$= \{b, c, d\}$$

THEOREM:

Let $(X, \mathcal{T})$ be a topological space $A, B \subseteq X$.
Then:

i) $\bar{A}$ is closed. ii) A is closed iff $A = \bar{A}$

iii) $\bar{\phi} = \phi$ and $\bar{X} = X$. iv) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$

v) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

vi) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Also show by example $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$

PROOF:

i) $\bar{A}$ is closed:

As $\bar{A}$ is the intersection of closed sets (which are also supersets of A) and intersection of closed sets is closed. So $\bar{A}$ is closed.

ii) A is closed iff $A = \bar{A}$:

Suppose A is closed:
To prove: $A = \bar{A}$.

As $\bar{A}$ is the intersection of all the closed supersets of A.
$\Rightarrow A \subseteq \bar{A} \rightarrow (1)$

Now let $x \in \bar{A}$.
Then, x belongs to each closed superset of A.
As $A \subseteq A$ and A is closed
$\Rightarrow A$ is the closed superset of A.

So, $x \in A \Rightarrow \bar{A} \subseteq A \rightarrow$ (2)

(1) and (2) $\Rightarrow \bar{A} = A$.

Conversely, let $A = \bar{A}$
To prove: A is closed.

Since $\bar{A}$ is closed.
$\Rightarrow A$ is closed $(\because A = \bar{A})$.

iii) $\bar{\phi} = \phi$ and $\bar{X} = X$.

As $\bar{\phi}$ and $\bar{X}$ are the intersection of all the closed supersets of ϕ and X respectively. Since ϕ and X are closed.

So, $\bar{\phi} \subseteq \phi \subseteq \bar{\phi}$
and $\bar{X} \subseteq X \subseteq \bar{X}$.

then, $\bar{\phi} = \phi$ and $\bar{X} = X$.

iv) If $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

Given: $A \subseteq B$
To prove: $\bar{A} \subseteq \bar{B}$.

As $A \subseteq B$ and $B \subseteq \bar{B}$
$\Rightarrow A \subseteq \bar{B}$

Also, as $\bar{B}$ is closed. So $\bar{B}$ is the closed superset of A.

But $\bar{A}$ is the smallest closed superset of A.

$\Rightarrow \bar{A} \subseteq \bar{B}$.

v). $\overline{A\cup B} = \bar{A} \cup \bar{B}$.

As $A \subseteq A\cup B$ and $B \subseteq A\cup B$.
$\Rightarrow \bar{A} \subseteq \overline{A\cup B}$ and $\bar{B} \subseteq \overline{A\cup B}$.
$\Rightarrow \bar{A}\cup\bar{B} \subseteq \overline{A\cup B} \rightarrow ①$

Now as, $A \subseteq \bar{A}$ and $B \subseteq \bar{B}$.
$\Rightarrow A\cup B \subseteq \bar{A}\cup\bar{B}$.

As $\bar{A}$ and $\bar{B}$ are closed.
$\Rightarrow \bar{A}\cup\bar{B}$ is closed.

So $\bar{A}\cup\bar{B}$ is the closed superset of $A\cup B$.
But $\overline{A\cup B}$ is the smallest closed superset of $A\cup B$.

$\Rightarrow \overline{A\cup B} \subseteq \bar{A}\cup\bar{B} \rightarrow ②$

① and ② $\Rightarrow \overline{A\cup B} = \bar{A}\cup\bar{B}$.

vi). $\overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$.

As $A\cap B \subseteq A$ and $A\cap B \subseteq B$.

$\Rightarrow \overline{A\cap B} \subseteq \bar{A}$ and $\overline{A\cap B} \subseteq \bar{B}$.

$\Rightarrow \overline{A\cap B} \subseteq \bar{A}\cap\bar{B}$.

EXAMPLE:

Show that $\overline{A\cap B} \neq \bar{A}\cap\bar{B}$

Let $A=\{1\}$; $B=\{2,3\}$.

$X=\{1,2,3\}$; $\mathfrak{F}=\{\phi, X\}$.

Then, $\bar{A}=X$, $\bar{B}=X$

$\bar{A}\cap\bar{B} = X\cap X = X \to (1)$

As $A\cap B=\phi$

And $\overline{A\cap B} = \bar{\phi}$

$= \phi \to (2)$

(1) and (2) $\Rightarrow \overline{A\cap B} \neq \bar{A}\cap\bar{B}$.

Hence Proved.

NEIGHBOURHOOD: (DEF)

Let $(X,\mathfrak{F})$ be a topological space and $x\in X$. Then, a subset N of X is said to be neighbourhood of x if there exists some open set U in X such that $x\in U\subseteq N$.

EXAMPLE:

Let $X=\{1,2,3,4\}$, $\mathfrak{F}=\{\phi, X, \{1\}, \{4\}, \{2,3\}, \{1,2,3\}, \{1,4\}, \{2,3,4\}\}$. $N=\{1,2,4\}$

SOLUTION:

As $1\in\{1\}\subseteq N$.

N is not the neighbourhood of 2.

As 3 doesnot belong to N

$\therefore N$ is not the neighbourhood of 3.

As $4\in\{1,4\}\subseteq N$

$\therefore N$ is the neighbourhood of 1 and 4.

REMARK:

i) If N is an open set. Then, as for each $x \in N$ $\Rightarrow x \in N \subseteq N$. So it means N is then neighbourhood of each of its element. It is called open neighbourhood.

ii) If $x \in X$, then the set of all neighbourhoods of x is denoted by N_x and is called neighbourhood system. It is i.e, N_x is always non ~~negative~~ empty because X is always in N_x.

INTERIOR POINT: (DEF). $x \in X$

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, an element $\underline{x \in A}$ is said ? to be an interior point of A if there exists some open set U in X such that $x \in U \subseteq A$. In other words, x is an interior point of A if and only if A is neighbourhood of x.

OR:

x is an interior point of A if there exist an open set U containing x such that $U \cap A' = \phi$.

EXTERIOR POINT: (DEF).

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then $x \in X$ is said to be an exterior point of A if x is an interior point of A' i.e, x is said to be exterior point of A if there exist some open set

U such that $x \in U \subseteq A'$.

OR:
x is exterior to A if there exist open set U containing x such that $U \cap A = \phi$

BOUNDARY POINT: (DEF).

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, $x \in X$ is said to be boundary point or frontier point of A if x is neither the interior point of A nor the exterior point of A. In other words $x \in X$ is said to be boundary point of $A \subseteq X$ if for every open set U containing x,

$$U \cap A \neq \phi \text{ and } U \cap A' \neq \phi$$

REMARK:

A° or $\text{Int}(A)$ denotes the set of all interior points of A, $\text{Ext}(A)$ denotes the set of all exterior points of A, $b(A)$ or $Fr(A)$ denotes the set of all boundary points of A.

EXAMPLE:

Let $X = \{2, 1, 3, 4\}$
$\mathcal{T} = \{\phi, X, \{1, 2\}, \{2, 3\}, \{2\}, \{1, 2, 3\}\}$
and $A = \{2, 3, 4\}$, Then find A°, $\text{Ext}(A)$, $b(A)$.

SOLUTION:

If $A = \{2, 3, 4\}$ then $A' = \{1\}$.

$1 \in b(A)$
$2 \in Int(A)$
$3 \in Int(A)$
$4 \in b(A)$

$\therefore Int(A) = \{2, 3\}$
$Ext(A) = \phi$
$b(A) = \{1, 4\}$

i) $Int(A) \cup Ext(A) \cup b(A) = X$
ii) $Int(A) \cap Ext(A) = \phi$
$Int(A) \cap b(A) = \phi$
$b(A) \cap Ext(A) = \phi$

THEOREM:
Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$, then $\bar{A} = \{x : x \in X, \text{ each open neighbourhood of } x \text{ intersects } A\}$ or $\bar{A} = \{x \in X : \text{for each open set } U \text{ containing } x,\ U \cap A \neq \phi\}$.

PROOF:
Let $B = \{x \in X : \text{for each open set } U \text{ containing } x,\ U \cap A \neq \phi\}$.

To show: $\bar{A} = B$
Let $x \in \bar{A}$. To show: $x \in B$
Suppose $x \notin B$. Then, there exist some open set U containing x such that $U \cap A = \phi$

$\Rightarrow A \subseteq U'$
As U is open. So, U' is closed.
$\Rightarrow U'$ is the closed superset of A

As $x \in U$, so $x \notin U'$.

$\Rightarrow x \notin \bar{A}$ ($\because \bar{A}$ is the intersection of all the closed supersets of A)

Which is a contradiction.

$\therefore x \in \bar{A}$

So our supposition is wrong.

Hence, $x \in B \Rightarrow \bar{A} \subseteq B \rightarrow$ (1)

Now let $x \in B$. To prove: $x \in \bar{A}$.

Suppose $x \notin \bar{A}$. Then there exists some closed superset F of A such that $x \notin F$.

$\Rightarrow x \in F' = U$.

Since, F is closed. So $F' = U$ is open set with $x \in U$. As $A \subseteq F \Rightarrow A \cap F' = \phi \Rightarrow A \cap U = \phi$.

$\Rightarrow$ There is an open set U containing x such that $U \cap A = \phi \Rightarrow x \notin B$.

Which is a contradiction.

$\therefore x \in B$

So, our supposition is wrong.

Hence, $x \in \bar{A} \Rightarrow B \subseteq \bar{A} \rightarrow$ (2).

(1) and (2) $\Rightarrow B = \bar{A}$.

LIMIT POINT: (DEF).

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, a point $x \in X$ is said to be limit point of A if for every open set U

containing x, $U \cap A \setminus \{x\} \neq \phi$. i.e, each open set U containing x, contains atleast one point of A different from 'x'.

The set of all limit points of A is denoted by $d(A)$ or $D(A)$ or A^d or A' and is called derived set of A.

EXAMPLE:

Let $X = \{1, 2, 3, 4\}$
$\mathcal{F} = \{\phi, X, \{1\}, \{2,3\}, \{1,2,3\}, \{3\}, \{1,3\}\}$
$A = \{2, 3, 4\}$. Find the limit points of A.

SOLUTION:

$x=1$ is not the limit point of A
because for open set $\{1\}$, $\{1\} \cap A \setminus \{1\} = \phi$

$x=2$: open sets containing 2 are:
X, $\{2,3\}$, $\{1,2,3\}$
and $X \cap A - \{2\} \neq \phi$
$\{2,3\} \cap A - \{2\} \neq \phi$
$\{1,2,3\} \cap A - \{2\} \neq \phi$
$\therefore$ 2 is the limit point of A

$x=3$ is not the limit point of A
because for open set $\{3\}$, $\{3\} \cap A - \{3\} = \phi$.

$x=4$: open sets containing 4 are: X
and $X \cap A - \{4\} \neq \phi$.
$\therefore$ 4 is the limit point of A

$\Rightarrow D(A) = \{2, 4\}$.

REMARK:

For any topological space $(X, \mathcal{T})$ and $A \subseteq X$. If $\{x\}$ is an open set then x is not the limit point of A because

$\{x\} \cap A - \{x\} = \phi$

THEOREM:

Let $(X, \mathcal{T})$ be a topological space and $A, B \subseteq X$. Then,

i) A° is the union of all open sets that are contained in A or, A° is the union of all open subsets of A.

Proof:

Let $\gamma = \{V : V \text{ is an open subset of } A\}$

To prove: $A^\circ = \cup\gamma = \bigcup_{V \in \gamma} V$

Let $x \in \bigcup_{V \in \gamma} V$, To prove: $x \in A^\circ$

Now $x \in \bigcup_{V \in \gamma} V \Rightarrow x \in V$ for some $V \in \gamma$.

$\Rightarrow x \in V \subseteq A$ (By definition of γ)

$\Rightarrow x \in A^\circ$

$\Rightarrow \bigcup_{V \in \gamma} V \subseteq A^\circ \rightarrow$ ①

Now let $x \in A^\circ$, then there exist some open set V in X such that $x \in V \subseteq A$.

$\Rightarrow x \in \bigcup_{V \in \gamma} V$

$\Rightarrow A^\circ \subseteq \bigcup_{V \in \gamma} V \rightarrow$ ②

① and ② $\Rightarrow A^\circ = \bigcup_{V \in \gamma} V$

ii). A° is the largest open subset of A.

Proof: To prove, A° is the largest open subset of A.

Since, A° is the union of all open subsets of A. And union of any number of opensets is open and union of subsets of anyset is again subset of that set.

So A° is an open subset of A.

Now, let D be another open subset of A. Then, by part i) $D \in \gamma$.

$\Rightarrow D \subseteq \bigcup \gamma$.

$\Rightarrow D \subseteq A^\circ$ $(\because A^\circ = \bigcup \gamma)$.

$\Rightarrow A^\circ$ is the largest open subset of A.

iii). A° is an open set. $A^\circ \subseteq A$.

Proof: Proved above in (ii).

iv). A is open if and only if $A = A^\circ$.

Proof:- Let $\gamma = \{V \subseteq X : V$ is open subset of $A\}$.

Then, $A^\circ = \bigcup \gamma$.

Let A be an open set. To prove $A = A^\circ$.

As A is open and $A \subseteq A$.

$\Rightarrow A \in \gamma$

$\Rightarrow A \subseteq \bigcup \gamma \Rightarrow A \subseteq A^\circ \to (1)$ $(\because A^\circ = \bigcup \gamma)$

But $A^\circ \subseteq A \to (2)$

(1) and (2) $\Rightarrow A = A^\circ$

Conversely, suppose $A = A^\circ$

To prove: A is open.

As, we have already proved that A° is an open set and here as $A^\circ = A$

So, A is open.

v) $(A^\circ)^\circ = A^\circ$

Proof: As A° is an open set (Proved above) so by part (iv), $(A^\circ)^\circ = A^\circ$

vi) $\phi^\circ = \phi$, $X^\circ = X$

Proof: As ϕ and X are open sets.

So $\phi^\circ = \phi$ and $X^\circ = X$

vii) If $A \subseteq B$ then $A^\circ \subseteq B^\circ$

Proof: Given $A \subseteq B$. To prove: $A^\circ \subseteq B^\circ$

Let $x \in A^\circ$, then there exists an open set U such that $x \in U \subseteq A$

$\Rightarrow x \in U \subseteq A \subseteq B$

$\Rightarrow x \in U \subseteq B$

$\Rightarrow x \in B^\circ$

$\Rightarrow A^\circ \subseteq B^\circ$

viii) $(A\cap B)^\circ = A^\circ \cap B^\circ$

Proof: As $A\cap B \subseteq A$ and $A\cap B \subseteq B$

$\Rightarrow (A\cap B)^\circ \subseteq A^\circ$ and $(A\cap B)^\circ \subseteq B^\circ$

$\Rightarrow (A\cap B)^\circ \subseteq A^\circ \cap B^\circ \rightarrow$ ①

Now as, $A^\circ \subseteq A$ and $B^\circ \subseteq B$

$\Rightarrow A^\circ \cap B^\circ \subseteq A\cap B$

$\Rightarrow A^\circ \cap B^\circ$ is an open subset of $A\cap B$

But $(A\cap B)^\circ$ is the largest open subset of $A\cap B$

$\Rightarrow A^\circ \cap B^\circ \subseteq (A\cap B)^\circ \rightarrow$ ②

① and ② $\Rightarrow (A\cap B)^\circ = A^\circ \cap B^\circ$

ix) $(A\cup B)^\circ \supseteq (A^\circ \cup B^\circ)$, Give an example to show:

$(A\cup B)^\circ \neq A^\circ \cup B^\circ$

Proof: As $A \subseteq A\cup B$ and $B \subseteq A\cup B$

$\Rightarrow A^\circ \subseteq (A\cup B)^\circ$ and $B^\circ \subseteq (A\cup B)^\circ$

$\Rightarrow A^\circ \cup B^\circ \subseteq (A\cup B)^\circ$

Example:

Let $X = \{a,b,c\}$; $A = \{a\}$, $B = \{b,c\}$

$\mathcal{T} = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$

Solution: $A\cup B = \{a,b,c\} = X$

$(A\cup B)^\circ = X^\circ = X$

$A^\circ = A$, $B^\circ = \{b\}$

$A^\circ \cup B^\circ = \{a,b\} \neq (A\cup B)^\circ$

HENCE PROVED.

THEOREM:
Let $(X, \mathcal{T})$ be a topological space and A be a closed subset of X. Then, A is the disjoint union of A° and $b(A)$.

PROOF:
To prove: i) $A = A^\circ \cup b(A)$
ii) $A^\circ \cap b(A) = \phi$

i) Let $x \in A$. To prove: $x \in A^\circ \cup b(A)$
If $x \in A^\circ$ then $x \in A^\circ \cup b(A)$ and there is nothing to prove.
If $x \notin A^\circ$, then to prove $x \in b(A)$.
Suppose $x \notin b(A)$.
$\Rightarrow$ Either $x \in A^\circ$ or $x \in Ext(A)$.
But $x \notin A^\circ$, so $x \in Ext(A)$. Then, there exist some open set U such that $x \in U \subseteq A'$.
$\Rightarrow x \in A'$

Which is a contradiction.
$\because x \in A$
So, our supposition is wrong.
Hence, $x \in b(A)$
$\Rightarrow x \in A^\circ \cup b(A) \Rightarrow A \subseteq A^\circ \cup b(A) \rightarrow (1)$

Now, let $x \in A^\circ \cup b(A)$. To prove: $x \in A$
$\Rightarrow x \in A^\circ$ or $x \in b(A)$. (Given)
If $x \in A^\circ$ then as $A^\circ \subseteq A$, so $x \in A$ and there is nothing to prove.

If $x \notin A^\circ$ then $x \in b(A)$ and to prove $x \in A$.

Suppose $x \notin A \Rightarrow x \in A'$
Since A is closed, so A' is open
$\Rightarrow x \in A' \subseteq A'$
$\Rightarrow x \in Ext(A) \Rightarrow x \notin b(A)$.

Which is a contradiction.
$\because x \in b(A)$
So, our supposition is wrong.
Hence, $x \in A \Rightarrow A^\circ \cup b(A) \subseteq A \rightarrow$ (2)

(1) and (2) $\Rightarrow A = A^\circ \cup b(A)$.

ii) To prove: $A^\circ \cap b(A) = \phi$
Let $A^\circ \cap b(A) \neq \phi$. Then, there exist an element $x \in A^\circ \cap b(A)$
$\Rightarrow x \in A^\circ$ and $x \in b(A)$.
If $x \in A^\circ$, then there exist an open set U containing x such that $x \in U \subseteq A \Rightarrow U \cap A' = \phi$

$\Rightarrow x \notin b(A)$ $(\because U \cap A' \neq \phi)$
Which is a contradiction
$\because x \in b(A)$.
So, our supposition is wrong.
Hence, $A^\circ \cap b(A) = \phi$.

Proved.

THEOREM:
Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then; i) $\bar{A} = A \cup D(A)$.
ii) A is closed iff $D(A) \subseteq A$.

PROOF:

i) $\bar{A} = A \cup D(A)$

Let $x \in \bar{A}$. To prove: $x \in A \cup D(A)$

If $x \in A$, then $x \in A \cup D(A)$. Then, there is nothing to prove.

If $x \notin A$. Then, to prove $x \in D(A)$

Suppose $x \notin D(A)$

$\Rightarrow x$ is not the limit point of A.

Then, there exist at least one open set U containing x such that $U \cap A \setminus \{x\} = \phi$

$\Rightarrow U \cap A = \phi \Rightarrow A \subseteq U'$

As $x \in U \Rightarrow x \notin U'$.

As U is open. So, U' is closed.

$\Rightarrow U'$ is the closed superset of A.

As $x \in U$ and $x \notin U'$

$\Rightarrow x \notin \bar{A}$ ($\because \bar{A}$ is the intersection of all closed supersets of A)

Which is a contradiction

$\because x \in \bar{A}$

So, our supposition is wrong.

Hence, $x \in D(A) \Rightarrow x \in A \cup D(A)$

$\Rightarrow \bar{A} \subseteq A \cup D(A) \rightarrow ①$

Now let, $x \in A \cup D(A)$. To prove: $x \in \bar{A}$.

As $x \in A \cup D(A)$

$\Rightarrow x \in A$ or $x \in D(A)$

If $x \in A \Rightarrow x \in \bar{A}$ ($\because A \subseteq \bar{A}$)
If $x \notin A$ then $x \in D(A)$. To prove: $x \in \bar{A}$
Suppose, $x \notin \bar{A}$

Then, there exist atleast one closed superset F of A such that $x \notin F \Rightarrow x \in F'$

Now, $A \subseteq F \Rightarrow A \cap F' = \phi$
$\Rightarrow F' \cap A = \phi$
$\Rightarrow F' \cap A \setminus \{x\} = \phi$ ($\because x \notin A$)
$\Rightarrow x \notin D(A)$

Which is a contradiction $\because x \in D(A)$
So, our supposition is wrong.
Hence, $x \in \bar{A} \Rightarrow A \cup D(A) \subseteq \bar{A} \rightarrow (2)$

(1) and (2) $\Rightarrow \bar{A} = A \cup D(A)$

ii) A is closed iff $D(A) \subseteq A$

Suppose A is closed.
To prove: $D(A) \subseteq A$
Let $x \in D(A)$
$\Rightarrow x \in A \cup D(A) = \bar{A}$
$\Rightarrow x \in \bar{A} = A$
As A is closed $\therefore A = \bar{A}$
$\Rightarrow x \in A$
$\Rightarrow D(A) \subseteq A$

Conversely, suppose that $D(A) \subseteq A$

To prove: A is closed.
For this we have to show that $A=\bar{A}$
As $A \subseteq \bar{A} \rightarrow (1)$
Let $x \in \bar{A}$
$\Rightarrow x \in A \cup D(A)$
If $x \in A$, then there is nothing to prove.

If $x \in D(A) \subseteq A$
$\Rightarrow x \in A$
$\Rightarrow \bar{A} \subseteq A \rightarrow (2)$

(1) and (2) $\Rightarrow A=\bar{A} \Rightarrow A$ is closed.

ISOLATED POINT: (DEF).

Let $(X,\mathcal{T})$ be a topological space and ? $A \subseteq X$, then a point $x \in A$ is said to be isolated point of A, if x is not the limit point of A. i.e, there exist an open set U containing x such that $U \cap A \setminus \{x\} = \phi$. The set of all isolated points of A is denoted by A^*.

THEOREM:

Let $(X,\mathcal{T})$ be a topological space, then, any closed subset A of X is the disjoint union of A^* and $D(A)$.

PROOF: It is obvious $A^* \cap D(A) = \phi$.
Now, we prove that $A = A^* \cup D(A)$.
Let $x \in A$, to prove: $x \in A^* \cup D(A)$.
If $x \in A^*$, then $x \in A^* \cup D(A)$ and there is nothing to prove.

If $x \notin A^*$, then to prove $x \in D(A)$.
Suppose, $x \notin D(A)$.
then $x \in A^*$
Which is a contradiction
$\because x \notin A^*$
So, our supposition is wrong.
Hence, $x \in D(A) \Rightarrow x \in A^* \cup D(A)$
$\Rightarrow A \subseteq A^* \cup D(A) \rightarrow$ ①

Conversely, let $x \in A^* \cup D(A)$. To prove: $x \in A$
As $x \in A^* \cup D(A) \Rightarrow x \in A^*$ or $x \in D(A)$.
If $x \in A^*$, then $x \in A$ (by definition) and
There is nothing to prove.
If $x \notin A^*$, then $x \in D(A)$. To prove: $x \in A$

Suppose, $x \notin A$, then $x \in A'$
Since, A is closed so, A' is open. And $x \in A'$.
Now, $A' \cap A = \phi$.
$\Rightarrow A' \cap A / \{x\} = \phi \Rightarrow x \notin D(A)$.
Which is a contradiction
$\because x \in D(A)$.
So, our supposition is wrong.
Hence, $x \in A \Rightarrow A^* \cup D(A) \subseteq A \rightarrow$ ②

① and ② $\Rightarrow A = A^* \cup D(A)$.

HENCE PROVED.

DENSE: (DEF).
Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. Then, A is called Dense (every where dense) in X if $\bar{A} = X$.

EXAMPLE:

Let $X = \{1, 2, 3, 4\}$.

$\mathcal{T} = \{\phi, X, \{1\}, \{2,3\}, \{1,2,3\}, \{4\}, \{1,4\}, \{2,3,4\}\}$.

Then show that A and B are dense in X.

SOLUTION: Let $A = \{1, 3, 4\}$, $B = \{1, 4\}$

Closed sets are $\{X, \phi, \{2,3,4\}, \{1,4\}, \{4\}, \{1,2,3\}, \{2,3\}, \{1\}\}$.

Closed supersets of A are : X.

$\Rightarrow \bar{A} = X$

Closed superset of B are : $X, \{1,4\}$

$\Rightarrow \bar{B} = X \cap \{1,4\} = \{1,4\} \neq X$.

This shows that A is dense in X and B is not dense in X.

SEPARABLE: (DEF).

Let $(X, \mathcal{T})$ be a topological space then it is said to be separable if it has countable dense set.

COUNTABLE (DEF):

Any set A is said to be countable if: i) It is finite.

ii) It has one-one correspondence with the set of natural numbers.

e.g. $\{1, 2, 3, \ldots, 10\}, \{a, b, c, \ldots, l\}, N, Z, Q, W$ all are countable.

EXAMPLE:

If: i) X is itself countable then for any Topology $\mathcal{T}$, $(X,\mathcal{T})$ is separable. (X is itself dense of X).

ii). Let $X = \mathbb{R}$ (Uncountable set) and $\mathcal{T} = \mathcal{T}_D = P(X)$. Then $(X,\mathcal{T})$ is not separable.

SOLUTION:

$X = \mathbb{R}$ and $\mathcal{T} = P(X)$.

Then $\mathbb{R}$ is the only subset of $\mathbb{R}$ such that with respect to this $\mathcal{T}$, $\bar{\mathbb{R}} = \mathbb{R}$. Because for any proper subset A of $\mathbb{R}$ i.e $A \subseteq \mathbb{R}$ but $A \neq \mathbb{R}$. Then, A is closed.

Then $\bar{A} = A$ ($\because A$ is closed iff $A = \bar{A}$)

$\Rightarrow \bar{A} \neq \mathbb{R}$ ($\because A \neq \mathbb{R}$).

$\Rightarrow \mathbb{R}$ is the only dense set in $\mathbb{R}$.

Since, $\mathbb{R}$ is uncountable.

So $(\mathbb{R},\mathcal{T})$ is not separable.

BASE: (DEF).

Let $(X,\mathcal{T})$ be a Topological space. Then, a collection $\mathcal{B}$ of subsets of X is said to be base for X if:

i) $\mathcal{B} \subseteq \mathcal{T}$.

ii) For every $U \in \mathcal{T}$, there is a subfamily γ of $\mathcal{B}$ such that $U = \cup\gamma$. (The second condition can also be stated as every $U \in \mathcal{T}$ can be expressed as the union of some members of $\mathcal{B}$).

EXAMPLE: Let $X=\{1,2,3,4,5\}$,
$\mathcal{T}=\{\phi, X, \{1,2\}, \{3\}, \{4\}, \{1,2,3\}, \{1,2,4\}, \{3,4\}\}$
$\mathcal{B}=\{\phi, \{1,2\}, \{3\}, \{4\}\}$. Show that $\mathcal{B}$ is base for X.

SOLUTION: As both conditions of $\mathcal{B}$ as a base are satisfied. $\therefore$ $\mathcal{B}$ is a base for X.

REMARK: It is not always necessary that $\phi\in\mathcal{B}$ because if $\phi\notin\mathcal{B}$ then for every $\phi\in\mathcal{T}$, we consider the empty subfamily say γ of $\mathcal{B}$ such that $\phi=\cup\gamma$.

THEOREM: Let $(X,\mathcal{T})$ be a topological space and $\mathcal{B}$ be a family of some subsets of X, then $\mathcal{B}$ is base for X if and only if:

i) $\mathcal{B}\subseteq\mathcal{T}$. ii) for every $U\in\mathcal{T}$ and $x\in U$ then there is $V\in\mathcal{B}$ such that $x\in V\subseteq U$.

Proof: Let $\mathcal{B}$ be the base for X. Then,
① $\mathcal{B}\subseteq\mathcal{T}$. ② For every open set $U\in\mathcal{T}$, there is a subfamily γ of $\mathcal{B}$ such that $U=\cup\gamma$.

Now, by ① and ②, we have to prove i) and ii).
i) is same as ①. Now we prove ii)
Let $U\in\mathcal{T}$ and $x\in U$, then by ② there is a subfamily γ of $\mathcal{B}$ such that $U=\cup\gamma$.
Now $x\in U=\cup\gamma \Rightarrow x\in\cup\gamma \Rightarrow x\in V$ for some $V\in\gamma$.
$\Rightarrow x\in V,\ V\in\gamma\subseteq\mathcal{B} \Rightarrow V\in\mathcal{B}$.

Also $V \in \gamma \Rightarrow V \subseteq \cup\gamma = U \Rightarrow V \subseteq U$.

Hence, we have $V \in \mathcal{B}$ such that $x \in V \subseteq U$.

Conversely, let (i) and ii) holds.

To prove: $\mathcal{B}$ is base for X. For this, we have to prove:

① $\mathcal{B} \subseteq \mathcal{T}$. ② for every $U \in \mathcal{T}$, there is a subfamily γ of $\mathcal{B}$ such that $U = \cup\gamma$.

① is same as (i). To prove②

Let $U \in \mathcal{T}$. If $U = \phi$. Then, we can consider the empty subfamily γ of $\mathcal{B}$ such that $U = \cup\gamma$. But if $U \neq \phi$. Then let $x \in U$. Then, by given ii) there is $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U$.

Then consider $\gamma = \{V_x \in \mathcal{B} : x \in U \text{ and } x \in V_x \subseteq U\}$.

Then, $x \in V_x \subseteq U \Rightarrow \{x\} \subseteq V_x \subseteq U$.

$\Rightarrow \bigcup_{x \in U} \{x\} \subseteq \bigcup_{x \in U} V_x \subseteq U$.

$\Rightarrow U \subseteq \cup\gamma \subseteq U \Rightarrow U = \cup\gamma \Rightarrow$ We have a subfamily γ of $\mathcal{B}$ such that $U = \cup\gamma$.

$\Rightarrow \mathcal{B}$ is base for X.

THEOREM:

Let X be a non empty set. A family $\mathcal{B}$ of subsets of X is base for some topology $\mathcal{T}$ on X if and only if:

i) $X = \bigcup_{\alpha \in I} B_\alpha$, $B_\alpha \in \mathcal{B}$.

ii) For $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, Then there exist $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

PROOF: Given $\mathcal{B}$ is base for some topology $\mathfrak{T}$ on X.
To prove condition (i) and (ii) described in the Theorem.

i). Let $x \in X$ (note that $X \in \mathfrak{T}$). Then there exist some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X$.

$$\Rightarrow \{x\} \subseteq B_x \subseteq X$$

$$\Rightarrow \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} B_x \subseteq X$$

$$\Rightarrow X \subseteq \bigcup_{x \in X} B_x \subseteq X$$

$$\Rightarrow X = \bigcup_{x \in X} B_x$$

So, condition (i) is proved.

ii). Now let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$.
Since $\mathcal{B}$ is base for $\mathfrak{T}$. So $\mathcal{B} \subseteq \mathfrak{T}$

$$\Rightarrow B_1, B_2 \in \mathfrak{T}$$

Since $\mathfrak{T}$ is topology. So $B_1 \cap B_2 \in \mathfrak{T}$.
Hence, we have $B_1 \cap B_2 \in \mathfrak{T}$, $x \in B_1 \cap B_2$ and $\mathcal{B}$ is base for $\mathfrak{T}$.
So there exists $B_3 \in \mathcal{B}$ such that,

$$x \in B_3 \subseteq B_1 \cap B_2$$

So condition (ii) is proved.

Conversely, let the collection $\mathcal{B}$ of subsets of X satisfies the two conditions described in the statement of theorem.

We have to prove that $\mathcal{B}$ is base for some topology $\mathfrak{T}$ on X.
Let $\mathfrak{T}$ be the collection of all possible

unions of all the subfamilies of $\mathcal{B}$. Now we prove that $\mathcal{J}$ is topology on X and $\mathcal{B}$ is the base for this topology.

1) Let γ be the empty subfamily of $\mathcal{B}$. Then $U\gamma = \phi$. Then by defination of $\mathcal{J}$, $U\gamma = \phi \in \mathcal{J}$. Now, by given condition (i), there is a subfamily $\{B_\alpha : \alpha \in I\}$ of $\mathcal{B}$ such that:

$$X = \bigcup_{\alpha \in I} B_\alpha$$

Then, by the construction of $\mathcal{J}$,

$$\bigcup_{\alpha \in I} B_\alpha \in \mathcal{J} \text{ i.e } X \in \mathcal{J}.$$

2). Union of any number of elements of $\mathcal{J}$ is in $\mathcal{J}$:

Let $\{U_\alpha : \alpha \in I\}$ be a collection of elements of $\mathcal{J}$.

To prove: $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{J}$.

Since for each $\alpha \in I$, $U_\alpha \in \mathcal{J}$. So, by the construction of $\mathcal{J}$, there exists some subfamily $\{B_\beta^\alpha : \beta \in I_1\}$ such that: $U_\alpha = \bigcup_{\beta \in I_1} B_\beta^\alpha$.

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I}\Big(\bigcup_{\beta \in I_1} B_\beta^\alpha\Big) \in \mathcal{J} \text{ (By construction of } \mathcal{J}\text{)}.$$

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \in \mathcal{J}$$

3). Intersection of finite number of elements of $\mathcal{J}$ is in $\mathcal{J}$:

Let $U_1, U_2 \in \mathcal{F}$. To prove: $U_1 \cap U_2 \in \mathcal{F}$.
As, $U_1 \in \mathcal{F}$, so there exists a subfamily $\{B_\alpha : \alpha \in I\}$ of $\mathcal{B}$ such that $U_1 = \bigcup_{\alpha \in I} B_\alpha$.

As, $U_2 \in \mathcal{F}$, so there exists a subfamily $\{B_{\alpha'} : \alpha' \in I'\}$ of $\mathcal{B}$ such that $U_2 = \bigcup_{\alpha' \in I'} B_{\alpha'}$.

$$U_1 \cap U_2 = \left(\bigcup_{\alpha \in I} B_\alpha\right) \cap \left(\bigcup_{\alpha' \in I'} B_{\alpha'}\right)$$

$$= \bigcup (B_\alpha \cap B_{\alpha'}) \quad \text{(Distributive law)}$$

Here arises two cases:

a) If $B_\alpha \cap B_{\alpha'} = \phi \quad \forall\ \alpha, \alpha'$

$\Rightarrow \bigcup (B_\alpha \cap B_{\alpha'}) = \phi$

$\Rightarrow U_1 \cap U_2 = \phi \Rightarrow U_1 \cap U_2 \in \mathcal{F}$ ($\because \phi \in \mathcal{F}$)

b) If $B_\alpha \cap B_{\alpha'} \neq \phi \Rightarrow x \in B_\alpha \cap B_{\alpha'}$.
Then, by given conditions, there exists some $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq B_\alpha \cap B_{\alpha'}$.

$\Rightarrow \{x\} \subseteq B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\Rightarrow \bigcup_{x \in B_\alpha \cap B_{\alpha'}} \{x\} \subseteq \bigcup_{x \in B_\alpha \cap B_{\alpha'}} B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\Rightarrow B_\alpha \cap B_{\alpha'} \subseteq \bigcup_{x \in B_\alpha \cap B_{\alpha'}} B_x \subseteq B_\alpha \cap B_{\alpha'}$

$\Rightarrow B_\alpha \cap B_{\alpha'} = \bigcup_{x \in B_\alpha \cap B_{\alpha'}} B_x$

Thus, $U_1 \cap U_2 = \bigcup \left(\bigcup_{x \in B_\alpha \cap B_{\alpha'}} B_x \right) \in \mathcal{F}$

$\Rightarrow U_1 \cap U_2 \in \mathcal{F}$

$\Rightarrow \mathcal{F}$ is topology for X.

Further, by the construction of $\mathcal{T}$, $\mathcal{B}$ is base for $\mathcal{T}$.

NEIGHBOURHOOD BASE OR LOCAL BASE OR BASE AT A POINT. (DEF).

Let $(X,\mathcal{T})$ be a topological space and $x\in X$. Then, a collection B_x of subsets of X is said to be local base for X at x if:

i) $B_x \subseteq \mathcal{T}$.

ii) For every $U\in\mathcal{T}$ such that $x\in U$, there exists some $V\in B_x$ such that $x\in V\subseteq U$.

EXAMPLE: Let $X=\{1,2,3\}$, $\mathcal{T}=\{\phi, X, \{1\}, \{2,3\}\}$.

SOLUTION: $x=1$. $B_x=\{\{1\}\}$.

Then B_x is the local base at $x=1$.

FIRST COUNTABLE SPACE: (DEF).

A topological space $(X,\mathcal{T})$ is said to be first countable space if it has countable local base at each of its point.

SECOND COUNTABLE SPACE: (DEF).

A topological space $(X,\mathcal{T})$ is said to be second countable space (or it satisfies the 2^{nd} axiom of countability) if it has a countable base.

EXAMPLES:

1) For any set X with any topology $\mathcal{T}$ on X, if X itself is countable then $(X,\mathcal{T})$ is both

first countable as well as 2^{nd} countable.

2) If X is uncountable and $\mathcal{T} = \mathcal{T}_I = \{\phi, X\}$. Then, $(X, \mathcal{T}_I)$ is both 1^{st} and 2^{nd} countable.

3). If $X = \mathbb{R}$ (i.e uncountable) and $\mathcal{T} = \mathcal{T}_D$. Then space $(X, \mathcal{T})$ is first countable but it is not the 2^{nd} countable space because the base for $(X, \mathcal{T})$ with minimum number of elements is $\mathcal{B} = \{\{x\} : x \in X\}$ is also uncountable.

THEOREM: Prove that every second countable space is first countable.

PROOF: Let $(X, \mathcal{T})$ be a 2^{nd} countable space.
To prove: X is first countable.
For this, let $x \in X$ be an arbitrary point of X. We have to prove that this X has a countable local base for this x.
Since X is second countable space, so X has a countable base.
Let $\mathcal{B} = \{B_n\}$ be a countable base for X.
Put $B_x = \{B_{nx} : x \in B_n$ and $B_{nx} \in \mathcal{B}\}$
Then, $B_x \subseteq \mathcal{B}$
$\Rightarrow B_x$ is countable ($\because \mathcal{B}$ is countable)
Further, also $B_x \subseteq \mathcal{T}$ ($\because B_x \subseteq \mathcal{B} \subseteq \mathcal{T}$).
Now let $U \in \mathcal{T}$ such that $x \in U$. Then, as $\mathcal{B}$ is base so there exist $V \in \mathcal{B}$ such that $x \in V \subseteq U$. Then, by the construction of B_x, $V \in B_x$. So we have $V \in B_x$ such that $x \in V \subseteq U$.
Hence, B_x is the local base at x.
$\Rightarrow X$ is first countable space.

REMARK: Converse of the above theorem is not true in general, i.e., a first countable space need not to be 2^{nd} countable necessarily. e.g., If X is an uncountable set and $\mathcal{T}_D$ is discrete topology on X. Then $(X, \mathcal{T}_D)$ is 1^{st} countable space because for every $x \in X$ we have countable local base $B_x = \{\{x\}\}$ for $x \in X$. But $(X, \mathcal{T}_D)$ is not 2^{nd} countable space because the smallest base for X that can be considered is $\mathcal{B} = \{\{x\} : x \in X\}$ which is not countable.

OPEN COVER: (DEF).

Let $(X, \mathcal{T})$ be a topological space then a collection $\{U_\alpha : \alpha \in I\}$ of open sets in X is said to be open cover for X if $X \subseteq \bigcup_{\alpha \in I} U_\alpha$.
Since, for each $\alpha \in I$, $U_\alpha \subseteq X$.
So $\bigcup_{\alpha \in I} U_\alpha \subseteq X$. Hence if $\{U_\alpha : \alpha \in I\}$ is open cover for X. Then $\bigcup_{\alpha \in I} U_\alpha = X$.

EXAMPLE: Let $X = \{1, 2, 3, 4, 5\}$,
$\mathcal{T} = \{\phi, X, \{1\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}, \{1, 4, 5\}, \{2, 3, 4, 5\}\}$.

SOLUTION:
$\gamma = \{\phi, \{1\}, \{2, 3\}, \{4, 5\}, \{1, 2, 3\}\}$.
As $\bigcup \gamma = X$. So γ is open cover for X.
As $\beta = \{\{1\}, \{2, 3\}, \{4, 5\}\} \subseteq \gamma$.
Also $\bigcup \beta = X$. Then β is open subcover for X.
Every open cover has atleast one open subcover which is open cover itself.

LINDELOF SPACE: (DEF).

A topological space $(X, \mathcal{T})$ is said to be Lindelof space if every open cover for X has a countable subcover.

LINDELOF THEOREM:

Let X be a second countable space. If a non-empty set G is represented as a union of open sets then G can be represented as a countable union of open sets.

PROOF:

Let $\{G_i\}$ be a collection of open sets in X such that $G = \cup G_i$.

Since, X is 2^{nd} countable space, so X has a countable base. Let $\mathcal{B} = \{B_n\}$ be a countable base for X.

Now let $x \in G \Rightarrow x \in \cup G_i$ ($\because G = \cup G_i$)

$\Rightarrow x \in G_i$ for some i.

Now as $x \in G_i$, G_i is an open set and $\mathcal{B} = \{B_n\}$ is a countable base for X, then, there exists $B_{n_x} \in \mathcal{B}$ such that $x \in B_{n_x} \subseteq G_i$.

$\Rightarrow \{x\} \subseteq B_{n_x} \subseteq G_i$

$\Rightarrow \bigcup_{x \in G} \{x\} \subseteq \bigcup_{x \in G} B_{n_x} \subseteq \cup G_i$

$\Rightarrow G \subseteq \bigcup_{x \in G} B_{n_x} \subseteq G \Rightarrow G = \bigcup_{x \in G} B_{n_x}$.

Now as $\mathcal{B} = \{B_n\}$ is countable. So, $\{B_{n_x} : B_{n_x} \in \mathcal{B}$ and $\bigcup_{x \in G} B_{n_x} = G\}$ is countable.

Thus for each basic open set in $\{B_{n_x}\}$ we can chose a set G_i which contains it. (i.e, $B_{n_x} \subseteq G_i$), then the class $\{G_i\}$ which arise in this way is countable and its union is G.

THEOREM: Every separable metric space is second countable.

PROOF: Let (X, d) be a separable metric space.
To prove: X is 2^{nd} countable.
As X is separable. So, X has countable dense set in X. Let A be countable dense set in X i.e., A is countable subset of X and $\bar{A} = X$.

Put $\gamma = \{B(x, r) : x \in A \text{ and } r \in Q\}$
then it is clear that γ is countable.
Now, we prove that γ is base for X.
Clearly, $\gamma \subseteq \mathcal{T}$.
Further let G be an open set in X and $y \in G$. Since G is an open set. So, there exist an open ball $B(y, r)$ such that:
$y \in B(y, r) \subseteq G$.
Now consider $B(y, r/3)$ as the concentric open ball of $B(y, r)$ then as A is dense in X. So, $B(y, r/3) \cap A \neq \phi$.

Let $a \in B(y, r/3) \cap A$
$\Rightarrow a \in B(y, r/3)$ and $a \in A$.
Let $r_1 \in Q$ such that $r/3 < r_1 < 2r/3$.
We know claim that, $B(a, r_1) \subseteq B(y, r)$.
Let $z \in B(a, r_1) \Rightarrow d(z, a) < r_1$.
Now as, $d(z, y) \leq d(z, a) + d(a, y)$
$< r_1 + r/3$ ($\because a \in B(y, r/3) \Rightarrow d(a, y) < r/3$).
$< 2r/3 + r/3 = r$.

$\Rightarrow z \in B(y, r)$

$\Rightarrow B(a, r_1) \subseteq B(y, r) \subseteq G$

Further as, $d(a, y) < r/3 < r_1$

$\Rightarrow d(a, y) < r_1 \Rightarrow y \in B(a, r_1)$.

Since $a \in A$ and $r_1 \in \mathbb{Q}$ so $B(a, r_1) \in \gamma$.

Hence, we have proved that for any $G \in \mathcal{T}$ and $y \in G$, There is an element $B(a, r_1) \in \gamma$ such that $x \in B(a, r_1) \subseteq G$. Hence γ is base for $\mathcal{T}$. So X is 2nd countable.

THEOREM: Let X is an uncountable set with cofinite topology then X is neither first countable nor second countable.

PROOF: Suppose X is 1st countable. Then, for each $x \in X$, there exist a countable local base for $x \in X$. Let $\mathcal{B} = \{B_n\}$ be countable local base at $x \in X$. Since $\mathcal{B} \subseteq \mathcal{T}_c$ (By definition of local base). So, for each $B_n \in \mathcal{B} \Rightarrow B_n \in \mathcal{T}_c$. So for all n, B_n' is finite $\Rightarrow \bigcup_n B_n'$ is countable.

Put $G = \bigcup_n B_n' \Rightarrow G$ is countable. As G is countable and X is uncountable (Given)

So $G' = X/G$ is uncountable.

Then, there exist $y \in G'$, such that $x \neq y$.

As $y \in G'$ so $y \notin G \Rightarrow y \notin \bigcup_n B_n'$.

$\Rightarrow y \in (\bigcup_n B_n')' \Rightarrow y \in \bigcap_n B_n \Rightarrow y \in B_n \ \forall n$.

Put, $U = X - \{y\}$. Then $U' = \{y\} \Rightarrow U \in \mathcal{T}_c$.

Hence, U is an open set and $x \in U$.

$(\because U = X - \{y\}$ and $x \neq y)$.

and $\mathcal{B}$ is local base at x. Then, there exist

$B_n \in \mathcal{B}$, such that $x \in B_n \subseteq U$.
Now as, $y \notin U$ and $B_n \subseteq U \Rightarrow y \notin B_n$.

Which is a contradiction $\because y \in B_n \ \forall n$.
So our supposition is wrong.
Hence, X is not 1^{st} countable.

Now, suppose X is 2^{nd} countable. Then, by a well known theorem X is first countable.
Which is a contradiction because X is not 1^{st} countable (Proved above).
So our supposition is wrong.
Hence, X is not 2^{nd} countable.
PROVED.

RELATIVE TOPOLOGY OR SUBSPACE: (DEF).
Let $(X, \mathcal{T})$ be a topological space and Y be a non empty subset of X. Then, the collection $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y and is called subspace of $(X, \mathcal{T})$.

EXAMPLE:
Let $X = \{1, 2, 3, 4, 5\}$.
$\mathcal{T} = \{\phi, X, \{1,2\}, \{3,4\}, \{5\}, \{1,2,3,4\}, \{1,2,5\}, \{3,4,5\}\}$.
Then $(X, \mathcal{T})$ be a topological space.
Let $Y = \{1, 3, 5\}$.
Then $\mathcal{T}_Y = \{\phi, Y, \{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}\}$.

THEOREM: Every subspace of first countable space is first countable.

PROOF: Let X be 1^{st} countable space and Y be a subspace of X.

To prove: Y is first countable.

Let $x \in Y$. To prove: there exist a countable local base at x for Y.

Now, as $x \in Y$ and $Y \subseteq X$. So $x \in X$ and X is first countable. So, X has a countable local base at x for X. Let B_x be the countable local base at x for X.

Put $B_x^* = \{V \cap Y : V \in B_x\}$.

Then, B_x^* is countable ($\because B_x$ is countable).

Now, let U_Y be an open set in Y and $x \in U_Y$. Since U_Y is an open set in Y and Y is subspace of X, then there exist an open set U_X in X such that $U_Y = U_X \cap Y$.

As $x \in U_Y \Rightarrow x \in U_X \cap Y$.

$\Rightarrow x \in U_X$ and $x \in Y$.

Now as, $x \in U_X$, U_X is open in X and B_x is a local base at x for X. Then there exists $V \in B_x$ such that $x \in V \subseteq U_X$.

As $x \in Y \Rightarrow x \in V \cap Y \subseteq U_X \cap Y = U_Y$

$\Rightarrow x \in V \cap Y \subseteq U_Y$. Where $V \cap Y \in B_x^*$.

$\Rightarrow B_x^*$ is local base at x for Y.

Hence, Y is first countable.

THEOREM: Every closed subspace of Lindelof space is Lindelof.

PROOF: Let X be a Lindelof space and Y be a closed subspace of X.

To prove: Y is Lindelof.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for

Y i.e $Y = \bigcup_{\alpha\in I} U_\alpha$. Since for each $\alpha \in I$, U_α is an open set in Y, so for each $\alpha \in I$, there exists an open set V_α in X such that $U_\alpha = V_\alpha \cap Y$.

Further as Y is closed in X, so Y' is open in X.

Now as $U_\alpha = V_\alpha \cap Y \Rightarrow U_\alpha \subseteq V_\alpha$

$$\bigcup_{\alpha\in I} U_\alpha \subseteq \bigcup_{\alpha\in I} V_\alpha \Rightarrow Y \subseteq \bigcup_{\alpha\in I} V_\alpha$$

$$\Rightarrow Y \cup Y' \subseteq \left(\bigcup_{\alpha\in I} V_\alpha\right) \cup Y'$$

$$\Rightarrow X \subseteq \left(\bigcup_{\alpha\in I} V_\alpha\right) \cup Y' \subseteq X \Rightarrow X = \left(\bigcup_{\alpha\in I} V_\alpha\right) \cup Y'$$

$\Rightarrow \{Y', V_\alpha : \alpha \in I\}$ is an open cover for X. Since X is Lindelof. So this open cover for X has a countable subcover. Let $\{Y', V_n\}$ be a countable subcover for X.

$\Rightarrow X = (\bigcup_n V_n) \cup Y'$

$\Rightarrow Y \subseteq X = (\bigcup_n V_n) \cup Y' \Rightarrow Y \subseteq (\bigcup_n V_n) \cup Y'$

$$\Rightarrow Y \subseteq \bigcup_n V_n \Rightarrow Y = \left(\bigcup_n V_n\right) \cap Y$$

$$\Rightarrow Y = \bigcup_n (V_n \cap Y) \Rightarrow Y = \bigcup_n U_n$$

$\Rightarrow \{U_n\}$ is a countable subcover for Y. Hence, Y is Lindelof.

THEOREM: Every 2^{nd} countable space is Lindelof.

PROOF: Let X be a 2^{nd} countable space.

To prove: X is Lindelof.

As X is 2^{nd} countable. So, X has a countable base. Let $B=\{B_n\}$ be a countable base for X and let $\gamma=\{U_\alpha ; \alpha\in I\}$ be an open cover for X, then $X=\bigcup_{\alpha\in I} U_\alpha$.

Let $x\in X \Rightarrow x\in \bigcup_{\alpha\in I} U_\alpha \Rightarrow x\in U_\alpha$ for some $\alpha\in I$.

Since for some $\alpha\in I$, $x\in U_\alpha$ and $B=\{B_n\}$ is a base for X, then there exists some $B_{n_x}\in B$ such that $x\in B_{n_x}\subseteq U_\alpha$.

$\Rightarrow \{x\}\subseteq B_{n_x}\subseteq U_\alpha$.

$\Rightarrow \bigcup_{x\in X}\{x\}\subseteq \bigcup_{x\in X} B_{n_x}\subseteq \bigcup_{\alpha\in I} U_\alpha$

$\Rightarrow X\subseteq \bigcup_{x\in X} B_{n_x}\subseteq X \Rightarrow X=\bigcup_{x\in X} B_{n_x}$

Now, for each $B_{n_x}\in B$, we can find a $U_{\alpha_x}\in\gamma$ such that $B_{n_x}\subseteq U_{\alpha_x}$ and $X=\bigcup U_{\alpha_x}$. As $\{B_{n_x}\}$ is countable. So, $\{U_{\alpha_x}\}$ is also countable.

$\Rightarrow \{U_{\alpha_x}\}$ is a countable subcover for X.

Hence, X is Lindelof.

THEOREM: Let $(X,\mathcal{T})$ be a topological space and $A\subseteq X$, then A has empty boundary if and only if A is both open and closed.

PROOF: Suppose A is both open and closed.

To prove: $b(A)=\phi$.

Since A is open then $A=A^\circ$.

Since A is also closed $\Rightarrow A'$ is open.

$\Rightarrow (A')^\circ=A' \Rightarrow A'=Ext(A)$ ($\because x$ is an exterior point of A iff x is an interior point of A').

$\Rightarrow A^\circ \cup Ext(A) = X \quad (\because A \cup A' = X)$

Now $b(A) = X - \{A^\circ \cup Ext(A)\}$

$= X - X = \phi$

Conversely suppose $b(A) = \phi$.

To prove: A is both open and closed.

Since $b(A) = \phi$, so $X = A^\circ \cup Ext(A)$.

Now, let $x \in A \Rightarrow x \in X \; (\because A \subseteq X)$

$\Rightarrow x \in A^\circ \cup Ext(A)$

$\Rightarrow x \in A^\circ$ or $x \in Ext(A)$.

But $x \in A$ so $x \notin Ext(A)$.

So $x \in A^\circ \Rightarrow A \subseteq A^\circ$, but $A^\circ \subseteq A$

$\Rightarrow A = A^\circ$. Hence A is open.

~~Now to show, A is also closed.~~

~~Let x ∈ A' ⇒ x ∉ A ⇒ x ∉ Int(A)~~

~~⇒ x ∈ Ext(A) ⇒ A' ⊆ Ext(A)~~

~~But Ext(A) ⊆ A'~~

~~⇒ Ext(A) = A' ⇒ [Int(A)]' = A'~~

~~⇒ A' is closed ⇒ A is open.~~

Now to show, A is closed

As $b(A) = \phi$

and $\bar{A} = A \cup b(A)$.

So $\bar{A} = A \cup \phi \Rightarrow \bar{A} = A$

$\Rightarrow A$ is closed.

Hence PROVED.

THEOREM: Prove that every second countable space is separable.

PROOF:

Let X be a second countable space.
To prove: X is separable.
As X is 2^{nd} countable. So X has countable base. Let $\mathcal{B} = \{B_n\}$ be a countable base for X.
Put $A = \{x_n : x_n \in B_n\}$.
then $\bar{A} \subseteq X \rightarrow (1)$.
Now let $x \in X$. Further let $U \in \mathcal{T}$ such that $x \in U$, there exists $B_n \in \mathcal{B}$ such that:
$x \in B_n \subseteq U$. ($\because \mathcal{B}$ is base for X).
As, $B_n \subseteq U$ and $B_n \cap A \neq \phi$.
$\Rightarrow U \cap A \neq \phi$.
$\Rightarrow$ For each open set U containing x, $U \cap A \neq \phi$.
$\Rightarrow x \in \bar{A} \Rightarrow$ For each $x \in X \Rightarrow x \in \bar{A}$
$\Rightarrow X \subseteq \bar{A} \rightarrow (2)$.
(1) and (2) $\Rightarrow \bar{A} = X$.
Further as $\mathcal{B}$ is countable.
So, A is countable.
$\Rightarrow A$ is countable dense in X.
$\Rightarrow X$ is separable.

THEOREM: Every subspace of second countable space is second countable.

PROOF: Let X be a second countable space and Y be a subspace of X.
To prove: Y be second countable.

As X is second countable so X has a countable base $\mathcal{B} = \{B_n : n \in N\}$
Let $\mathcal{B}^* = \{B_n \cap Y : B_n \in \mathcal{B}\}$

As $\mathcal{B}$ is countable. So $\mathcal{B}^*$ is also countable.
Now let U be open in Y. Then, there exist open set V in X such that: $U = V \cap Y$.
Let $x \in Y$ such that $x \in U$.
As $x \in Y \subseteq X \Rightarrow x \in X$.
Now $x \in U = V \cap Y$.
$\Rightarrow x \in V$ and $x \in Y$.
As $x \in V$ and $\mathcal{B}$ is base for X.
So, there exist $B_n \in \mathcal{B}$ such that $x \in B_n \subseteq V$.
$\Rightarrow x \in B_n \cap Y \subseteq V \cap Y$.

$\Rightarrow x \in B_n \cap Y \subseteq U$. and $B_n \cap Y \in \mathcal{B}^*$.
$\Rightarrow \mathcal{B}^*$ is base for Y.

$\Rightarrow Y$ is second countable.

HENCE PROVED.

Theorem:- Let S be a non-empty collection of subsets of X with $\cup S = X$. Then S is sub-base for some topology on X.

Proof:- Let $\mathcal{B}$ be the collection of all possible finite intersections of members (subfamilies) of S i.e.
$\mathcal{B} = \{ B : B = \bigcap_{i=1}^{n} S_i ,\ S_i \in S,\ 1 \leq i \leq n \}$.
Then $X = \cup S \subseteq \cup \mathcal{B} \subseteq X$
$\Rightarrow X = \cup \mathcal{B}$

Further let $B_1, B_2 \in \mathcal{B}$ & $x \in B_1 \cap B_2$. Since B_1 & B_2 are finite intersections of subfamilies of S, so $B_1 \cap B_2$ is finite intersection of some subfamily of $S \Rightarrow B_1 \cap B_2 = B_3 \in \mathcal{B}$. Then $x \in B_3 \subseteq B_1 \cap B_2$

Then, by ~~2nd~~ Theorem of base $\mathcal{B}$ is base for some topology on X.

Separation Axiom:

T_0 - Space: (Def).
A topological space $(X, \mathcal{J})$ is said to be T_0-space if for each $x, y \in X$ such that $x \neq y$, either there exists an open set U such that $x \in U$ and $y \notin U$ or there exists an open set V such that $y \in V$ and $x \notin V$.

Examples:
1) Let $X = \{1, 2\}$ and $\mathcal{J} = \{\phi, X, \{1\}\}$, then $\{X, \mathcal{J}\}$ is T_0 space.
2) $(X, \mathcal{J}_D)$ is also T_0-space.
3) $(X, \mathcal{J}_I)$ is not T_0-space.

Theorem: Every subspace of T_0 space is T_0.

Proof: Let $(X, \mathcal{J})$ be a T_0-space and Y be a subspace of X. To prove: Y is T_0.
Let $x, y \in Y$ such that $x \neq y$.
As $x, y \in Y$ and $Y \subseteq X$ so $x, y \in X$.
Since, X is T_0 space, so either there exist an open set U such that $x \in U$ and $y \notin U$ or ~~their~~ there exist an open set V such that $y \in V$ and $x \notin V$.
With out any loss of generality, suppose there exist an open set U such that $x \in U$ and $y \notin U$.
Now as U is open in X, so $U \cap Y$ is open in Y.
As $x \in U$ and $x \in Y \Rightarrow x \in U \cap Y = U_1$.
As $y \notin U \Rightarrow y \notin U \cap Y = U_1$.
Hence, we have found open set U_1 in Y

such that $x \in U_1$ and $y \notin U_1$.

$\Rightarrow Y$ is T_0 space.

THEOREM: A topological space $(X, \mathcal{T})$ is T_0 iff for every $a, b \in X$ such that $a \neq b$ then $\overline{\{a\}} \neq \overline{\{b\}}$.

PROOF: Suppose X is T_0-space and $a, b \in X$ such that $a \neq b$. To prove: $\overline{\{a\}} \neq \overline{\{b\}}$.

Since X is T_0 and $a, b \in X$ with $a \neq b$, so say, there exists an open set U in X such that $a \in U$ and $b \notin U$.

As, $a \in U$ and $b \notin U \Rightarrow U$ is an open set containing 'a' such that $U \cap \{b\} = \phi$.

$\Rightarrow a \notin \overline{\{b\}}$ $\quad$ ($\overline{A} = \{x :$ for each open set U containing x, $U \cap A \neq \phi\}$).

$\Rightarrow \{a\} \not\subseteq \overline{\{b\}}$.

Further as, $\{a\} \subseteq \overline{\{a\}}$

$\Rightarrow \overline{\{a\}} \not\subseteq \overline{\{b\}}$.

$\Rightarrow \overline{\{a\}} \neq \overline{\{b\}}$.

Conversely suppose that for every $a, b \in X$ such that $a \neq b$, then $\overline{\{a\}} \neq \overline{\{b\}}$.

To prove: X is T_0-space.

Suppose on the contrary that X is not T_0-space. Then, there is a pair $a, b \in X$ such that $a \neq b$ and for every open set U containing 'a' also contain 'b' and for every open set V containing 'b' contains 'a'.

Hence, now for every open set U containing

a, $U \cap \{b\} \neq \phi \Rightarrow a \in \overline{\{b\}} \Rightarrow \{a\} \subseteq \overline{\{b\}}$

$\Rightarrow \overline{\{a\}} \subseteq \overline{\overline{\{b\}}}$ ($\because$ If $A \subseteq B$ then $\bar{A} \subseteq \bar{B}$).

$\Rightarrow \overline{\{a\}} \subseteq \overline{\{b\}}$ ($\because A = \bar{A} \Rightarrow \bar{\bar{A}} = \bar{A}$)

Similarly, $\overline{\{b\}} \subseteq \overline{\{a\}}$.

$\Rightarrow \overline{\{a\}} = \overline{\{b\}}$.

Which is a contradiction.

$\because \overline{\{a\}} \neq \overline{\{b\}}$.

So our supposition is wrong.

Hence, X is T_0 space.

T_1 – SPACE: (DEF).

A topological space $(X, \mathcal{F})$ is said to be T_1-space if for every $x, y \in X$ such that $x \neq y$, there exists two open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

EXAMPLE: Let $X = \{1, 2\}$

$\mathcal{F} = \{\phi, X, \{1\}, \{2\}\}$.

THEOREM:

Every subspace of T_1-space is T_1.

PROOF:

Let $(X, \mathcal{F})$ be a T_1-space and Y be a subspace of X. To prove: Y is T_1.

Let $x, y \in Y$ such that $x \neq y$.

Now, $x, y \in Y$ and $Y \subseteq X$, so $x, y \in X$.

As X is T_1 space, so there exists two open sets U and V in X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Now as U and V are open sets in X, so $U_1 = U \cap Y$ and $V_1 = V \cap Y$ are open in Y.

As $x \in U$ and $x \in Y \Rightarrow x \in U_1$ and as $y \notin U \Rightarrow y \notin U_1$.

As $y \in V$ and $y \in Y \Rightarrow y \in V \cap Y = V_1$ and as $x \notin V \Rightarrow x \notin V \cap Y = V_1$

Hence, we have found two open sets U_1 and V_1 in Y such that:
$x \in U_1, y \notin U_1$ and $y \in V_1, x \notin V_1$
Hence Y is T_1- space.

THEOREM: Prove that every T_1-space is T_0-space.

PROOF: Let X be T_1-space. To prove: X is T_0-space.
Let $x, y \in X$ such that $x \neq y$.
Since, X is T_1, so there exists two open sets U and V in X such that $x \in U$, $y \notin U$ and $y \in V, x \notin V$.
Since here we have an open set U in X with $x \in U$, $y \notin U$. $\Rightarrow X$ is also T_0-space.

REMARK: Converse of the above theorem is not true in general i.e a T_0-space need not necessarily to be T_1-space.

EXAMPLE: Let $X = \{1, 2\}$, $\mathcal{T} = \{\phi, X, \{1\}\}$.
Hence X is T_0-space. But it is not T_1.

THEOREM: A topological space $(X, \mathcal{T})$ is T_1-space iff each singelton subset of X is closed in X.

PROOF: Let us suppose X is T_1-space.
To prove: Each singleton subset of X is closed.
Let $\{x\}$ be the singleton subset of X.
To prove: $\{x\}$ is closed.
For this, we prove $\{x\}'$ is open in X.
Let $y \in \{x\}' \Rightarrow y \notin \{x\} \Rightarrow x \neq y$.
Hence, we have $x, y \in X$ such that $x \neq y$ and X

is T_1-space. So there exists two open sets U_x and V_y in X such that $x \in U_x$, $y \notin U_x$ and $y \in V_y$, $x \notin V_y$.

Now as $V_y \subseteq X$ and $x \notin V_y \Rightarrow V_y \subseteq X - \{x\}$.
As $y \in V_y \subseteq X - \{x\} \Rightarrow \{y\} \subseteq V_y \subseteq \{x\}'$

$$\Rightarrow \bigcup_{y \in \{x\}'} \{y\} \subseteq \bigcup_{y \in \{x\}'} V_y \subseteq \{x\}'.$$

$$\Rightarrow \{x\}' \subseteq \bigcup_{y \in \{x\}'} V_y \subseteq \{x\}'.$$

$$\Rightarrow \{x\}' = \bigcup_{y \in \{x\}'} V_y.$$

Since, V_y is open set and union of any number of open sets is open. So $\bigcup_{y \in \{x\}'} V_y$ is open.

$\Rightarrow \{x\}'$ is open $\Rightarrow \{x\}$ is closed.

Conversely, suppose each singleton subset of X is closed. To prove: X is T_1-space.
For this, let $x, y \in X$ such that $x \neq y$.
Then by supposition $\{x\}$ and $\{y\}$ are closed.
$\Rightarrow \{x\}'$ and $\{y\}'$ are open.
Let $U = \{y\}'$ and $V = \{x\}'$
Then $x \in U$, $y \notin U$, $y \in V$, $x \notin V$.
$\Rightarrow X$ is T_1-space.

THEOREM: Every finite T_1-space is discrete.

PROOF: Let X be a T_1-space. To prove: X is discrete. For this, we will have to show that each subset of X is closed. Let $A \subseteq X$.

If $A=\phi$, then A is closed.
If $A\neq\phi$, then A contains some elements.
Since X is finite. So A is also finite.
Let $A=\{x_1, x_2, \cdots, x_n\} \Rightarrow A=\bigcup_{i=1}^{n}\{x_i\}$

Since X is T_1. So each singleton subset of X is closed $\Rightarrow$ for each i, $1\leq i\leq n$, $\{x_i\}$ is closed.
Since union of finite number of closed sets is closed. So $\bigcup_{i=1}^{n}\{x_i\}$ is closed $\Rightarrow A$ is closed.

$\Rightarrow X$ is discrete.

THEOREM: A topological space $(X,\mathcal{T})$ is T_1 space iff each subset of X is the intersection of its open supersets.

PROOF: Let X be a T_1-space and $A\subseteq X$.
To prove: A is the intersection of its open supersets.
Let $y\in X$ such that $y\notin A \Rightarrow y\in A'$
Now, as X is T_1-space. So, each singleton subset of X is closed $\Rightarrow \{y\}$ is closed
$\Rightarrow \{y\}'$ is open. Now as $A\subseteq X$ and $y\notin A$.

$\Rightarrow A\subseteq X-\{y\} \Rightarrow A\subseteq \{y\}'$.
$\Rightarrow \{y\}'$ is an open superset of A.
Now, we prove $A=\bigcap_{y\in A'}\{y\}'$

Let $x\in A \Rightarrow x\in\{y\}'$ for all $y\in A'$.
$\Rightarrow x\in \bigcap_{y\in A'} y' \Rightarrow A\subseteq \bigcap_{y\in A'}\{y\}' \rightarrow (1)$

Now, let $x \in \bigcap_{y\in A'} \{y\}' \Rightarrow x \in \{y\}'$ for all $y \in A'$

$\Rightarrow x \notin \{y\}, \forall\, y \in A' \Rightarrow x \neq y, \forall\, y \in A'$

$\Rightarrow x \notin A' \Rightarrow x \in A \Rightarrow \bigcap_{y\in A'} \{y\}' \subseteq A \rightarrow (2)$

(1) and (2) $\Rightarrow A = \bigcap_{y \in A'} \{y\}'$.

Hence, A is the intersection of its supersets.

Conversely, suppose that in topological space $(X, \mathcal{T})$ each subset of X is the intersection of its open supersets. To prove: X is T_1.

Suppose X is not T_1.
Then, there exists $x, y \in X$ such that $x \neq y$.
And either each open set containing x also contains y or each open set containing y also contains x.

Say, each open set containing x also contains y, so by given condition $\{x\}$ is the intersection of its open supersets.

$\Rightarrow \{x\}$ is open $\Rightarrow y \in \{x\} \Rightarrow y = x$.

Which is a contradiction $\because y \neq x$.
So our supposition is wrong.
Hence, X is T_1.

THEOREM: Let X be T_1 space and $A \subseteq X$ and $x \in X$ is the limit point of A. Then every open set containing x contains infinite number of distinct points of A.

PROOF:
Suppose given is not true i.e each open set

containing x doesnot contain infinite number of distinct points of A. Then, there exists an open set U containing x which contains finite number of distinct points of A. i.e.,

$U \cap A = \{x_1, x_2, x_3, \dots, x_n\} = B$

As X is T_1 space and B is finite subset of X. So, B is closed.

$\Rightarrow B'$ is open, as $x \notin B$ so $x \in B'$

$\Rightarrow B'$ is an open set containing x.

Also $B' \cap A = \phi \Rightarrow B' \cap A \setminus \{x\} = \phi \Rightarrow x \notin D(A)$

Which is a contradiction $\because x \in D(A)$

So our supposition is wrong.

And hence, each open set in X containing x contains infinite number of distinct points of A.

SUBBASE: (DEF).

If $(X, \mathcal{T})$ is a topological space. Then, a subcollection S of $\mathcal{T}$ is called subbase for $\mathcal{T}$ if and only if finite intersection of members of S form a base for $\mathcal{T}$.

EXAMPLE: Let $X = \{a, b, c, d, e\}$

$\mathcal{T} = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}, \{b, c, d, e\}, \{a, d, e\}, \{d, e\}\}$

and $S = \{\{a\}, \{a, b, c\}, \{b, c, d, e\}, \{a, d, e\}\}$

Then S is subbase for $\mathcal{T}$.

Because $B = \{\{a\}, \{a, b, c\}, \{b, c, d, e\}, \{a, d, e\}, \phi, \{b, c\}, \{d, e\}, X\}$.

GENERATION OF TOPOLOGIES:

Let X be a non empty set and S

be the collection of some subsets of X including X itself. Then, S is the subbase for some topology $\mathcal{T}$ on X. i.e:

$S \xrightarrow[\text{of elements of } S]{\text{All possible finite intersections}} B \xrightarrow[\text{elements of } B]{\text{All possible unions of}} \mathcal{T}$

This procedure is called Generation of topologies.

REMARK: X may not be included in S because in this case, we take an empty subcollection γ of S such that $\cap\gamma = X$.

e.g. Let $X = \{1,2,3,4,5\}$

$S = \{\{1\}, \{1,2,3\}, \{3,4\}\}$

$B = \{\{1\}, \{1,2,3\}, \{3,4\}, \{3\}, \phi, X\}$

$\mathcal{T} = \{\phi, X, \{1\}, \{3\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,3\}, \{1,2,3,4\}\}$.

T_2 SPACE: (DEF).

Let $(X,\mathcal{T})$ be a topological space, then it is said to be T_2-space or Hausdorff space if for every $x,y \in X$ such that $x \neq y$, then there exists two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

EXAMPLE: Let $X = \{1,2\}$, $\mathcal{T} = \{\phi, X, \{1\}, \{2\}\}$

$1 \in \{1\}$, $2 \in \{2\}$, $\{1\} \cap \{2\} = \phi$.

$\Rightarrow (X,\mathcal{T})$ is a T_2-space.

THEOREM: Every T_2 space is T_1-space.

PROOF:

Let X be T_2 space. To prove: X is T_1-space.

Let $x,y \in X$ such that $x \neq y$.

As X is T_2, so there exists two open sets U and V such that: $x \in U$, $y \in V$ and $U \cap V = \phi$:

Now as $x \in U$ and $U \cap V = \phi \Rightarrow x \notin V$.
As $y \in V$ and $U \cap V = \phi \Rightarrow y \notin U$.
$\Rightarrow X$ is T_1-space.

REMARK: Converse of the above theorem is not true in general i.e, a T_1-space is not necessarily T_2-space.

EXAMPLE: Let $X = \mathbb{N}$ and $\mathcal{F} = \mathcal{F}_c$.
Now, for every $x, y \in X$ such that $x \neq y$. We have:
$x \in X - \{y\}$, $y \notin X - \{y\}$ and $y \in X - \{x\}$, $x \notin X - \{x\}$.
$X - \{x\}$ and $X - \{y\}$ are open in X.
Hence, X is T_1-space.
But X is not T_2-space.
Because, on the contrary if we suppose that X is T_2-space, then there exists two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.
Now $U \cap V = \phi \Rightarrow (U \cap V)' = \phi'$
$\Rightarrow U' \cup V' = X \Rightarrow U' \cup V' = \mathbb{N}$.
Now as $(X, \mathcal{F}_c)$ is cofinite and U, V are open in X.
$\Rightarrow U'$ and V' are finite.
$\Rightarrow U' \cup V'$ is finite $\Rightarrow X = \mathbb{N}$ is finite.
Which is a contradiction.
So our supposition is wrong.
Hence, X is not T_2-space.

THEOREM: Every subspace of T_2-space is T_2-space.

PROOF: Let X be a T_2-space and Y be a subspace of X. To prove: Y is T_2.

Let $x, y \in Y$ such that $x \neq y$.
As $x, y \in Y$ and $Y \subseteq X \Rightarrow x, y \in X$ such that $x \neq y$.
As X is T_2. So, there exist two open sets U and V in X such that: $x \in U$, $y \in V$ and $U \cap V = \phi$.
Put $U_1 = U \cap Y$ and $V_1 = V \cap Y$.
then, U_1 and V_1 are open in Y.
As $x \in U$, $x \in Y \Rightarrow x \in U \cap Y \Rightarrow x \in U_1$
As $y \in V$, $y \in Y \Rightarrow y \in V \cap Y \Rightarrow y \in V_1$.
Now, $U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$.
$= (U \cap V) \cap Y = \phi \cap Y = \phi$
$\Rightarrow Y$ is T_2.

THEOREM: In T_1-space, no finite subset has the limit point.

PROOF:

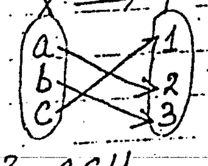
Let X be a T_1-space and A be a finite subset of X. Suppose $x \in D(A)$.
Then, each open set U containing x contains infinite number of distinct points of A.
Which is a contradiction.
$\because$ A itself is finite.
So our supposition is wrong.
Hence, in T_1-space a finite set has no limit points.

CONTINUOUS FUNCTION: (DEF).

Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be the two topological spaces then a function $f: X \to Y$ is said to be continuous at a point $x \in X$ if for every open set V in Y containing $f(x)$, there exist an open set U in X containing x such that $f(U) \subseteq V$.

f is said to be continuous on X if it is continuous at each point on X.

EXAMPLE: Let $X=\{a,b,c\}$, $\mathcal{T}_1=\{\phi, X, \{a,b\}, \{c\}\}$.
$Y=\{1,2,3\}$, $\mathcal{T}_2=\{\phi, Y, \{1\}, \{2,3\}\}$.
We define function $f: X\to Y$ as:



$x=a$, $f(a)=\{2\}$.
$V=\{2,3\}$, $V=Y \Rightarrow f(a)\subseteq V$.
$U=X$, $a\in X$
$f(U)=f(X)=Y\subseteq V$.
$V=\{2,3\}$, $f(a)\subseteq V$, $U=\{a,b\}$, $a\in U$,
$f(U)=\{2,3\}\subseteq V$.
Hence, f is continuous at $x=a$, similarly we can check f is continuous at $x=b,c$.
Hence f is continuous on X.

THEOREM: Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two topological spaces. A function $f: X\to Y$ is said to be continuous iff inverse image of each open set is open.

PROOF: Suppose f is continuous on X.
To prove: Inverse image of each open set in Y is open in X. Let G be an open set in Y. To prove: $f^{-1}(G)$ is open in X.
If $G=\phi$ then $f^{-1}(G)=\phi$, which is open in X and there is nothing to prove.
If $G\neq\phi$ then $f^{-1}(G)\neq\phi \Rightarrow x\in f^{-1}(G)\Rightarrow f(x)\in G$.

As f is continuous on X, so f is continuous at $x\in X$. Then, by the definition of continuity, there exist an open set U_x in X such that $x\in U_x$ and $f(U_x)\subseteq G$.
$\Rightarrow x\in U_x$ and $U_x\subseteq f^{-1}(G)$.

$\Rightarrow x \in U_x \subseteq f^{-1}(G)$.

$\Rightarrow \{x\} \subseteq U_x \subseteq f^{-1}(G)$.

$\Rightarrow \bigcup_{x \in f^{-1}(G)} \{x\} \subseteq \bigcup_{x \in f^{-1}(G)} U_x \subseteq f^{-1}(G)$.

$\Rightarrow f^{-1}(G) \subseteq \bigcup_{x \in f^{-1}(G)} U_x \subseteq f^{-1}(G) \Rightarrow f^{-1}(G) = \bigcup_{x \in f^{-1}(G)} U_x$.

Since U_x is open in X and union of open sets is open. So $\bigcup U_x$ is open in X.

$\Rightarrow f^{-1}(G)$ is open in X.

Conversely, suppose inverse image of each open set in Y is open in X.

To prove: f is continuous on X.

Let $x \in X$, to show f is continuous at x.

Let V be an open set in Y containing $f(x)$

i.e, $f(x) \in V \Rightarrow x \in f^{-1}(V) = U$.

Since, by assumption, inverse image of each open set is open and V is open in Y.

So $U = f^{-1}(V)$ is open in X and $x \in U$.

Now as $U = f^{-1}(V) \Rightarrow f(U) \subseteq V$.

$\Rightarrow f$ is continuous at $x \in X$.

Since, x is arbitrary. So f is continuous at each point of X. Hence, f is continuous on X.

THEOREM: Every metric space is T_2-space.

PROOF: Let (X, d) be the metric space.

To prove: X is T_2-space.

Let $x, y \in X$ such that $x \neq y$.

$\Rightarrow d(x, y) > 0$

Let $d(x, y) = r > 0$.

Now consider $U = B(x, r/2)$ and $V = B(y, r/2)$.
$\Rightarrow U$ and V are open sets in X ($\because$ Open balls are open sets).

and $x \in U$, $y \in V$.
Now to show $U \cap V = \phi$.
Suppose, on the contrary that $U \cap V \neq \phi$.
$\Rightarrow z \in U \cap V \Rightarrow z \in U$ and $z \in V$.
$\Rightarrow z \in B(x, r/2)$ and $z \in B(y, r/2)$
$\Rightarrow d(z, x) < r/2$ and $d(z, y) < r/2$.
Now $d(x, y) \leq d(x, z) + d(z, y)$.
$r < r/2 + r/2 = r$
$\Rightarrow r < r$
A contradiction.
So our supposition is wrong.
Hence $U \cap V = \phi$
$\Rightarrow X$ is T_2-space.

PRODUCT TOPOLOGY: (DEF).

Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ be two topological spaces and $X \times Y$ be the cartesian product of X and Y. Define a subset $U \times V$ of $X \times Y$ to be open in $X \times Y$. if $U \in \mathcal{T}_1$ and $V \in \mathcal{T}_2$, then the class of all subsets $U \times V$ of $X \times Y$ is the base for the topology $\mathcal{T}$ on $X \times Y$, called product topology on $X \times Y$.

EXAMPLE: Let $X = \{1, 2, 3\}$, $\mathcal{T}_1 = \{\phi, X, \{1\}, \{2, 3\}\}$
$Y = \{a, b, c, d\}$, $\mathcal{T}_2 = \{\phi, Y, \{a, b\}\}$.
$\mathcal{B} = \{\phi, X \times Y, \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\},$
$\{(1,a), (1,b), (1,c), (1,d)\}, \{(1,a), (1,b)\}, \{(2,a), (2,b),$
$(2,c), (2,d), (3,a), (3,b), (3,c), (3,d)\}, \{(2,a), (2,b),$
$(3,a), (3,b)\}\}$.

$\mathcal{T} = \{U\gamma : \gamma$ is a subfamily of $B\}$.

THEOREM: The following statements about the topological space are equivalent:
i) X is T_2- space. ii) The diagonal $D = \{(x,x) : x \in X\}$ is closed in $X \times X$.

PROOF: i) $\Rightarrow$ ii) i.e, we assume that X is T_2 space and prove that D is closed in $X \times X$.
For this, we prove that D' is open in $X \times X$.
Let $(x,y) \in D' \Rightarrow x \neq y$.
Hence, we have $x, y \in X$ such that $x \neq y$ and X is T_2 space. So, there exists two open sets U_x and V_y such that $x \in U_x$ and $y \in V_y$ and $U_x \cap V_y = \phi$.

Now let $(u,v) \in U_x \times V_y$.
$\Rightarrow u \in U_x$ and $v \in V_y$.
As $U_x \cap V_y = \phi \Rightarrow u \neq v \Rightarrow (u,v) \in D'$.
$\Rightarrow U_x \times V_y \subseteq D' \Rightarrow (x,y) \in U_x \times V_y \subseteq D'$.

$\Rightarrow \{(x,y)\} \subseteq U_x \times V_y \subseteq D'$.

$$\Rightarrow \bigcup_{(x,y)\in D'} \{(x,y)\} \subseteq \bigcup_{(x,y)\in D'} U_x \times V_y \subseteq D'.$$

$$\Rightarrow D' \subseteq \bigcup_{(x,y)\in D'} U_x \times V_y \subseteq D'$$

$$\Rightarrow D' = \bigcup_{(x,y)\in D'} U_x \times V_y.$$

Now as U_x and V_y are open in X. So, $U_x \times V_y$ is open in $X \times X$ and as union of any number of open sets is open. So D' is open.
$\Rightarrow D$ is closed.

ii) $\Rightarrow$ 1) i.e., here we assume that D is closed in $X \times X$ and we have to prove that X is T_2-space.

Let $x, y \in X$ such that $x \neq y \Rightarrow (x, y) \in D'$.

Now as D' is an open set in $X \times X$.

So, there exists open set $U_x \times V_y$ in $X \times X$ such that $(x, y) \in U_x \times V_y \subseteq D'$.

$\Rightarrow (x, y) \in U_x \times V_y$.

$\Rightarrow x \in U_x$ and $y \in V_y$.

Now to prove $U_x \cap V_y = \phi$.

Suppose on the contrary $U_x \cap V_y \neq \phi$.

Let $z \in U_x \cap V_y \Rightarrow z \in U_x$ and $z \in V_y$.

$\Rightarrow (z, z) \in U_x \times V_y$.

$\Rightarrow (z, z) \in D'$ ($\because U_x \times V_y \subseteq D'$).

$\Rightarrow z \neq z$.

Which is a contradiction.

$\because z = z$

So our supposition is wrong.

Hence, $U_x \cap V_y = \phi$.

$\Rightarrow X$ is T_2-space.

THEOREM: Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be the two topological spaces then a function $f: X \to Y$ is said to be continuous on X if and only if inverse image of each closed set is closed.

PROOF:

Suppose f is continuous and A is closed in $Y \Rightarrow A'$ is open in Y.

$\Rightarrow f^{-1}(A')$ is open in X.

Now, $[f^{-1}(A')]' = [f^{-1}(Y - A)]'$

$= [f^{-1}(Y) - f^{-1}(A)]' = [X - f^{-1}(A)]'$

$= f^{-1}(A)$.

$\Rightarrow f^{-1}(A)$ is closed in X.

Conversely, suppose inverse image of each closed set in Y is closed in X.

To prove: f is continuous.

Let A be open in Y.

$\Rightarrow A'$ is closed in $Y \Rightarrow f^{-1}(A')$ is closed in X.

$\Rightarrow [f^{-1}(A')]'$ is open in X.

And $[f^{-1}(A')]' = f^{-1}(A)$

$\Rightarrow f^{-1}(A)$ is open in X.

$\Rightarrow f$ is continuous.

CONVERGENCE: (DEF).

Let $(X, \mathcal{T})$ be a topological space then a sequence $\{x_n\}$ in X is said to converge to a point $x \in X$ if for every open set U containing x there is a natural number n_0 such that:

$x_n \in U$, for all $n \geq n_0$.

THEOREM: Let X be a T_2-space. Then, any sequence in X can converge to atmost one point. i.e, in T_2 space, limit of the sequence is unique.

PROOF: Suppose $\{x_n\}$ is a sequence in X and $x_n \to x$ and $x_n \to y$ and suppose $x \neq y$.

As X is T_2-space so, then there exists two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Now as $x_n \to x \in U$, so then, there exists some positive integer n_0, such that $x_n \in U \ \forall n \geq n_0$.

Also as $x_n \to y \in V$, so then there exist some

positive integer n_1 such that $x_n \in V$, $\forall\ n \geq n_1$.
Let $m = \max(n_0, n_1)$.
Then $\forall\ n \geq m$, $x_n \in U$ and $x_n \in V$.
$\Rightarrow U \cap V \neq \phi$
Which is a contradiction.
$\because U \cap V = \phi$.
So, our supposition is wrong.
Hence $x = y$.
Hence, limit of the sequence is unique.

THEOREM: Let $(X, \mathcal{T})$ be a topological space and Y be a T_2-space and $f: X \to Y$ is a continuous function, then the graph $G = \{(x,y) : y = f(x)\}$ is closed in $X \times Y$.

PROOF: We prove G' is open in $X \times Y$.
Let $(x,y) \in G' \Rightarrow y \neq f(x)$.
As $y, f(x) \in Y$, $y \neq f(x)$ and Y is T_2, so then there exists two open sets V and V_1 in Y such that: $y \in V$, $f(x) \in V_1$ and $V \cap V_1 = \phi$

Let $U = f^{-1}(V_1)$
As V is open in Y and f is continuous, so inverse image $f^{-1}(V_1) = U$ is open in X.
Now, $f(x) \in V_1$
$\Rightarrow x \in f^{-1}(V_1) \Rightarrow x \in U$
$\Rightarrow x \in U$, $y \in V \Rightarrow (x,y) \in U \times V$
Hence, $(x,y) \in U \times V \subseteq G'$

But $U \times V$ is open in $X \times Y$.
$\Rightarrow G'$ is open in $X \times Y$.
$\Rightarrow G$ is closed in $X \times Y$

THEOREM: Let X be a topological space and Y be a T_2-space and $f, g: X \to Y$ be two continuous functions, then prove that $A = \{x : x \in X \wedge f(x) = g(x)\}$ is closed in X.

PROOF: We prove, A' is open in X.

Let $a \in A' \Rightarrow f(a) \neq g(a)$.

As $a \in X \Rightarrow f(a), g(a) \in Y$ and $f(a) \neq g(a)$ and Y is T_2, so then there exists two open sets V and V_1 in Y such that:

$f(a) \in V$, $g(a) \in V_1$ and $V \cap V_1 = \phi$.

As f and g are continuous so, $f^{-1}(V)$ and $g^{-1}(V_1)$ are open in X

As $f(a) \in V \Rightarrow a \in f^{-1}(V)$

$g(a) \in V_1 \Rightarrow a \in g^{-1}(V_1)$

$\Rightarrow a \in f^{-1}(V) \cap g^{-1}(V_1) \subseteq A'$

As $f^{-1}(V) \cap g^{-1}(V_1)$ is open in X.

$\Rightarrow A'$ is open in X

$\Rightarrow A$ is closed in X.

THEOREM: Let $f: X \to Y$ and $g: X \to Y$ be two continuous functions from a topological space X to a T_2 space Y and $f(x) = g(x)$ for all $x \in D$, where D is dense in X. Then, $f(x) = g(x)$ $\forall$ $x \in X$.

PROOF: If $D = X$, then theorem is trivially proved.

If $D \neq X$, then $D \subset X$, then there is $z \in X$ such that $z \notin D$. To prove: $f(z) = g(z)$.

Suppose $f(z) \neq g(z)$. As $f(z), g(z) \in Y$, $f(z) \neq g(z)$ and Y is a T_2-space. Then, there exists two open sets U and V such that $f(x) \in U$, $g(x) \in V$ and $U \cap V = \phi$

Put $U_1 = f^{-1}(U)$ and $V_1 = g^{-1}(V)$

Since U and V are open in Y and f, g are continuous functions so then U_1 and V_1 are open in X.

Further $f(x) \in U$ and $g(x) \in V$.

$\Rightarrow x \in f^{-1}(U)$ and $x \in g^{-1}(V)$.

$\Rightarrow x \in U_1$ and $x \in V_1 \Rightarrow x \in U_1 \cap V_1$

Now as D is dense in X. So $\overline{D} = X$.

$\Rightarrow x \in \overline{D}$

$\Rightarrow (U_1 \cap V_1) \cap D \neq \phi$

Let $d \in (U_1 \cap V_1) \cap D$

$\Rightarrow d \in U_1$, $d \in V_1$ and $d \in D$.

$\Rightarrow d \in f^{-1}(U)$ and $d \in g^{-1}(V)$ and $d \in D$.

Now $d \in D \Rightarrow f(d) = g(d)$ $(\because \forall x \in D, f(x) = g(x))$

Also, $d \in f^{-1}(U)$ and $d \in g^{-1}(V)$

$\Rightarrow f(d) \in U$ and $g(d) \in V$

$\Rightarrow f(d) \neq g(d)$ $(\because U \cap V = \phi)$

Which is a contradiction.

So, our supposition is wrong.

Hence, $f(x) = g(x)$, $\forall x \in X$.

THEOREM: A topological space X is T_2-space iff for any two distinct points $a, b \in X$, there are closed sets C_1 and C_2 such that $a \in C_1$, $b \notin C_1$ and $b \in C_2$, $a \notin C_2$ and $C_1 \cup C_2 = X$.

PROOF:

Suppose X is T_2-space.

As $a, b \in X$ and $a \neq b$. As X is T_2-space. So there exist two open sets U and V such that:

$a \in U$, $b \in V$ and $U \cap V = \phi$.

Put $C_1 = V'$ and $C_2 = U'$.

As U and V are open. So C_1 and C_2 are closed in X.

As $a \in U$ and $U \cap V = \phi \Rightarrow a \notin V$.

$b \in V$ and $U \cap V = \phi \Rightarrow b \notin U$.

Now, $a \in U \Rightarrow a \notin U' \Rightarrow a \notin C_2$.

$a \notin V \Rightarrow a \in V' \Rightarrow a \in C_1$.

$b \in V \Rightarrow b \notin V' \Rightarrow b \notin C_1$

$b \notin U \Rightarrow b \in U' \Rightarrow b \in C_2$.

Further, $U \cap V = \phi \Rightarrow (U \cap V)' = \phi'$

$\Rightarrow U' \cup V' = X \Rightarrow C_1 \cup C_2 = X$.

Conversely suppose $C_1 \cup C_2 = X$.

To prove: X is T_2

Let $U = C_2'$ and $V = C_1'$

As C_1 and C_2 are closed.

So U and V are open in X.

As $a \notin C_2 \Rightarrow a \in C_2' \Rightarrow a \in U$.

$b \notin C_1 \Rightarrow b \in C_1' \Rightarrow b \in V$.

Now, $U \cap V = C_2' \cap C_1' = (C_2 \cup C_1)'$

$= X' = \phi$.

$\Rightarrow X$ is T_2-space.

THEOREM: A topological space X is T_2 space iff for every point $a \in X$, $\{a\} = \bigcap_{\alpha \in I} C_\alpha$ where each C_α is a closed set containing an open set U such that $a \in U$.

PROOF: Suppose X is T_2-space.

To prove: $\{a\} = \bigcap_{\alpha \in I} C_\alpha$

Let $b \in X$ such that $a \neq b$, then there exists two open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \phi$

Put $V' = C_\alpha$. As V is open so V' is closed.

Now as $U \cap V = \phi$. So $U \subseteq V'$.

$\Rightarrow a \in U \subseteq C_\alpha$. Now as $b \in V \Rightarrow b \notin V'$

$\Rightarrow b \notin C_\alpha$.

Now as for every point $b \in X$ distinct from a, we have a closed set C_α such that $a \in C_\alpha$ and $b \notin C_\alpha$.

$\Rightarrow a \in \bigcap C_\alpha$ and $b \notin \bigcap C_\alpha$

$\Rightarrow \{a\} = \bigcap_{\alpha \in I} C_\alpha$.

Conversely, suppose in a topological space X, for every point $a \in X$, $\{a\} = \bigcap_{\alpha \in I} C_\alpha$, where C_α is a closed set containing an open set U such that $a \in U$.

To prove: X is T_2-space.

Let $b \in X$ such that $a \neq b$

$\Rightarrow b \notin \bigcap_{\alpha \in I} C_\alpha$

$\Rightarrow b \notin C_\alpha$ for some α.

Put $V = C_\alpha' \Rightarrow b \in V$

$\Rightarrow a \in U$ and $b \in V$

Now as $U \subseteq C_\alpha$.

$\Rightarrow U \cap C_\alpha' = \phi \Rightarrow U \cap V = \phi$.

$\Rightarrow X$ is T_2-space.

THEOREM: A 1^{st} countable space X is T_2-space if and only if every convergent sequence has a unique limit.

PROOF: Suppose X is 1^{st} countable space which is T_2. To prove: Every convergent sequence has unique limit.

Suppose on the contrary that $x_n \to x$ and $x_n \to y$ and $x \neq y$.

Since, X is T_2, so there exists two open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$ $\to$ (1)

Now $x_n \to x \in U$, so there exist $n_1 \in N$ such that $x_n \in U$, $\forall\ n \geq n_1$.

As $x_n \to y \in V$, so there exist $n_2 \in N$ such that $x_n \in V$, $\forall\ n \geq n_2$.

Put $n_0 = \max[n_1, n_2]$

$\Rightarrow x_n \in U \quad \forall\ n \geq n_0$

and $x_n \in V \quad \forall\ n \geq n_0$

$\Rightarrow U \cap V \neq \phi \to$ (2)

(1) and (2) gives the contradiction

So our supposition is wrong.

Hence $x = y$.

$\Rightarrow \{x_n\}$ has unique limit.

Conversely suppose in a first countable space X, every convergent sequence has a unique limit. To prove: X is T_2 space.

Let $a, b \in X$ such that $a \neq b$.

To prove : X is T_2 - space.

We suppose X is not T_2 - space.
Then, every open set containing 'a' has a non-empty intersection with every open set which contains 'b'.

Let $\{U_n\}$ and $\{V_n\}$ be countable nested bases at 'a' and 'b' respectively.

Then $U_n \cap V_n \neq \phi$.

$\Rightarrow a_n \in U_n \cap V_n, \forall n$

Then $a_n \to a$ and $a_n \to b$.

Which is a contradiction.

$\because$ Every convergent sequence in X has unique limit.

So our supposition is wrong.

Hence, X is T_2 space.

THEOREM:

Every T_2 space is T_0 - space.

REGULAR SPACE: (DEF).

A topological space $(X, \mathcal{T})$ is said to be regular space if for every $x \in X$ and for any closed subset A of X with $x \notin A$, there exists two open sets U and V such that $x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

EXAMPLE: Let $X = \{a, b\}$

$\mathcal{T} = \{\phi, X, \{a\}, \{b\}\}$

Then $(X, \mathcal{T})$ is regular.

THEOREM: The following statements about a topological space are equivalent.

1) X is regular.
2) For any open set U in X and $x \in U$, there is an open set V containing x such that $x \in \bar{V} \subseteq U$.
3) Each element of X has a local base containing closed sets.

PROOF: 1) $\Rightarrow$ 2) ie, here it is given that X is regular and to prove ②

Let U be an open set in X with $x \in U$.
To prove: There exist an open set V in X containing x such that $x \in \bar{V} \subseteq U$.

Now as $x \in U$ and U is open set.
$\Rightarrow x \notin U'$ and U' is closed.
Then, by the definition of regular

space, there exist two open sets V and V_1 such that $x \in V$, $U' \subseteq V_1$ and $V \cap V_1 = \phi$.

Now, $U' \subseteq V_1 \Rightarrow V_1' \subseteq U$.
Also, $V \cap V_1 = \phi \Rightarrow V \subseteq V_1'$.

$$\Rightarrow x \in V \subseteq V_1' \subseteq U.$$

Now as V_1 is an open set, so V_1' is closed set. So V_1' is closed superset of V. But $\bar{V}$ is the smallest closed superset of $V \Rightarrow x \in V \subseteq \bar{V} \subseteq V_1' \subseteq U$.

$$\Rightarrow x \in \bar{V} \subseteq U.$$

2) ⇒ 3): Let $x \in X$. To prove: X has a local base containing closed sets.

Let U be an open set such that $x \in U$. Then, by condition ②, there exist an open set V such that $x \in \bar{V} \subseteq U$.

This shows that local base at x contains sets of the form $\bar{V}$ which is of course closed set.

3) ⇒ 1): Let $x \in X$ and A be closed subset of X such that $x \notin A$.

$\Rightarrow x \in A'$. Further as A is closed. So, A' is open set. Then by ③, there is a closed set B in the local base at x such that $x \in B \subseteq A'$. Now $B \subseteq A' \Rightarrow A \subseteq B'$.

Let $U = B$ and $V = B'$.
Then, U is open as U is in local base.
V is open because $V = B'$ and B is closed.

Further $x \in U$, $A \subseteq V$ and $U \cap V = \phi$ ($\because B \cap B' = \phi$).

Hence, it shows that ①, ② and ③ are equivalent.

COMPLETELY REGULAR SPACE: (DEF).
A topological space $(X, \mathcal{T})$ is said to be completely regular space if for any closed set A in X and $x \in X$ such that $x \notin A$, there exist a continuous function $f: X \to [0\ 1]$ such that $f(x) = 0$ and $f(A) = 1$.

EXAMPLE:

Every metric space is completely Regular.

THEOREM:
Every completely regular space is regular.

PROOF:
Let X be a completely regular space.
To prove: X is regular.

Let $x \in X$ and A be closed subset of X such that $x \notin A$. Then, as X is completely regular so there exist a continuous function $f: X \to [0\ 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Let $U = [0\ 1/2[$ and $V =]1/2\ \ 1]$

Then U and V are open in $[0\ 1]$.

As f is continuous. So $f^{-1}(U)$ and $f^{-1}(V)$ are open in X.

And $x \in f^{-1}(U)$, $A \subseteq f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$.

So, X is regular.

HENCE PROVED.

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Theorem: Every subspace of the completely regular space is completely regular.

Proof: Let X be a completely regular space and Y be a subspace of X.

To prove: Y is completely regular.

Let $x \in Y$ and A be a closed subset of Y such that $x \notin A$. As $x \in Y$ and $Y \subseteq X$ so $x \in X$.

Further as, A is closed in Y and Y is subspace of X. So, then there exists a closed subset B in X such that $A = B \cap Y$.

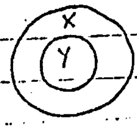
As $x \notin A$ and $x \in Y \Rightarrow x \notin B$.

As X is completely regular, so there exist a countinuous function $f: X \to [0\ 1]$ such that:

$f(x) = 0$ and $f(B) = 1$

Now define $g: Y \to [0\ 1]$ by $g(x) = f(x)\ \forall\, x \in Y$.

Then, $x \in Y \Rightarrow g(x) = f(x) = 0$



$f: A \to B$
$E \subseteq A$
$g: E \to B$
$g(x) = f(x)\ \forall\, x \in E$

$f(E) = 1$
$A = \{x_1, x_3\}$
$B = \{x_1, x_2, x_3\}$
$Y = \{x_1, x_3, x_4, x_5\}$

$g(A) = f(A) \quad \because A \subseteq Y$

$= f(B \cap Y)$

$= f(B) \cap f(Y) = 1$

g is restriction of f and Then g is restriction of f. f is continuous so g is also continuous. Hence Y is completely regular.

Definition: T_3 - Space:
A regular T_1-space is called T_3-space.

Definition: $T_{3\frac{1}{2}}$-Space:
A completely regular T_1-space is called $T_{3\frac{1}{2}}$-space or Tychonoff space.

Theorem. A completely regular T_1-space is T_2-space.

Proof: Let X be a completely regular T_1-space.
To prove: X is T_2-space.
Let $x, y \in X$ such that $x \neq y$.
As X is T_1 space. So each singelton subset in X is closed.
So $A = \{y\}$ is closed set in X and $x \notin A$.
As X is completely regular so then there exists a continuous function $f: X \to [0\ 1]$ such that:

$$f(x) = 0, \quad f(A) = 1 \Rightarrow f(y) = 1.$$

Now let $U = [0\ \tfrac{1}{2}[$ and $V =]\tfrac{1}{2}\ 1]$ be two open sets in $[0\ 1]$, Then as f is continuous so $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Further as:

$$f(x) = 0 \in U \Rightarrow x \in f^{-1}(U)$$

$$f(y) = 1 \in V \Rightarrow y \in f^{-1}(V).$$

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$$
$$= f^{-1}(\phi) = \phi$$

$\Rightarrow X$ is T_2 space.

Theorem: A subspace of $T_{3\frac{1}{2}}$-space is $T_{3\frac{1}{2}}$-space.

Proof: Let X be a $T_{3\frac{1}{2}}$ space and Y be a subspace of X.
To prove: Y is $T_{3\frac{1}{2}}$.
As X is $T_{3\frac{1}{2}}$ space so X is completely regular and X is T_1 space.
A subspace of completely regular space is completely regular. As subspace of T_1-space is T_1. So Y is completely regular and T_1-space.
$\Rightarrow Y$ is $T_{3\frac{1}{2}}$.

Hence Proved

Set of all continuous functions from $X \to R$.

Theorem: For any topological space X if $C(X,R)$ separates the points of X, then X is T_2-space.

Proof: Let $x, y \in X$ such that $x \neq y$.
And let $f \in C(X,R)$.
i.e f is a continuous function from $X \to R$.
Now by the given condition $f(x) \neq f(y)$.
Say $f(x) < f(y)$.
As $f(x), f(y) \in R$ and $f(x) < f(y)$.
So then there exist $r \in R$ s.t.
(there is a real no. b/w two real nos.) $f(x) < r < f(y)$.
Put $U = \{u : u \in X$ and $f(u) < r\}$. or $U < f^{-1}(r)$
$V = \{v : v \in X$ and $f(v) > r\}$. or $V > f^{-1}(r)$
then $x \in U$ and $y \in V$. $X = \{x_1, x_2, x_3\}$ $U = \{x_1, x_2\}$ $V = \{x_3\}$
Further, $U = f^{-1}(]-\infty, r[)$. r
and $V = f^{-1}(]r, \infty[)$. $-\infty$ $f(U) < r < f(V)$ ∞
Now as $]-\infty, r[$ and $]r, \infty[$ are open sets in R and f is continuous. So inverse images of open sets is also open. So U and V are open in X. Also by definition of U and V.
$$U \cap V = \phi$$
Hence, we have found two open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.
$\Rightarrow X$ is T_2.

Definition: NORMAL SPACE:
A topological space $(X, \mathcal{T})$ is said to be normal space if for any two closed disjoints subsets A and B of X there are open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Theorem. Every discrete space with atleast two points is normal.

Proof: Let X be a discrete space with atleast two points. To prove: X is normal.

As X is discrete, so, each subset of X is open as well as closed.

Let A and B be two disjoint closed sets in X. Let $U = A$ and $V = B$.

Then U and V are open and $U \cap V = \phi$.

$A \subseteq U$, $B \subseteq V$.

$\Rightarrow X$ is normal.

Theorem: Every subspace of a regular space is regular.

Proof: Let X be a regular space and Y be a subspace of X.

To prove: Y is regular.

Let $x \in Y$ and A be a closed set in Y such that $x \notin A$.

Now as A is closed in Y and Y is subspace of X, so then there exists a closed set B in X, such that $A = B \cap Y$.

Further $x \notin A \Rightarrow x \notin B \cap Y$.

$\Rightarrow x \notin B$ $(\because x \in Y)$.

Further $x \in Y \subseteq X \Rightarrow x \in X$.

So $x \in X$ and B is closed set in X such that $x \notin B$ and X is regular so then there exists two open sets U and V in X such that:

$x \in U$, $B \subseteq V$ and $U \cap V = \phi$.

Put $U_1 = U \cap Y$ and $V_1 = V \cap Y$.

As U and V are open in X so, U_1 and V_1 are open in Y.

Now as $x \in U$, $x \in Y \Rightarrow x \in U \cap Y \Rightarrow x \in U_1$.

Also $B \subseteq V \Rightarrow B \cap Y \subseteq V \cap Y \Rightarrow A \subseteq V_1$.

And $U_1 \cap V_1 = (U \cap Y) \cap (V \cap Y)$
$= (U \cap V) \cap Y$
$= \phi \cap Y = \phi$
$\Rightarrow Y$ is regular.

Theorem:- Every metric space is normal space.
Proof:- Let (X,d) be a metric space.
To prove: X is normal space.
Let A and B be the two disjoint non-empty closed sets in X. Let $a \in A$, Then
$d(a,B) = \inf_{b \in B} d(a,b)$
definition of a distance of a point.
As, $A \cap B = \phi$ and $a \in A$ so $a \notin B$.
Further as B is closed. So, $d(a,B) > 0$.
Let $d(a,B) = r_a$.
Similarly let $b \in B$ and $d(b,A) = r_b$.
Now consider the open balls,
$B(a, r_a/3)$ and $B(b, r_b/3)$.
Put $U = \bigcup_{a \in A} B(a, r_a/3)$, $V = \bigcup_{b \in B} B(b, r_b/3)$.

Then U and V are open, ($\because$ Union of open balls is open)

with $A \subseteq U$ and $B \subseteq V$.
Now we prove that $U \cap V = \phi$.
Suppose on the contrary that $U \cap V \neq \phi$
$\Rightarrow x \in U \cap V \Rightarrow x \in U$ and $x \in V$.
$\Rightarrow x \in \bigcup_{a \in A} B(a, r_a/3)$ and $x \in \bigcup_{b \in B} B(b, r_b/3)$

$\Rightarrow x \in B(a_1, r_{a_1}/3)$ and $x \in B(b_1, r_{b_1}/3)$.
$\Rightarrow d(x, a_1) < r_{a_1}/3$ and $d(x, b_1) < r_{b_1}/3$.
As $a_1 \in A$, $b_1 \in B$ and $A \cap B = \phi \Rightarrow a_1 \neq b_1$
$\Rightarrow d(a_1, b_1) > 0$.

اگر closed set ہو تو اسکے باہر کے ہر element کا greater dist ہونا چاہیے۔
یا greater ہی ہو گا ٫٫ ٫٫ ٫٫ ٫٫ ٫٫ ٫٫ تو ہو ٫٫ ٫٫ ٫٫ ٫٫ equal to Zero ہو گا۔

Let $d(a_1, b_1) = r$

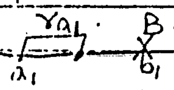
Clearly, $r_{a_1} \leq r$ and $r_{b_1} \leq r$ r_{a_1} and r_{b_1} are min dist from each other.

arbitrary

Now $r = d(a_1, b_1) \leq d(a_1, x) + d(x, b_1)$

$< r_{a_1}/3 + r_{b_1}/3$

$\leq r/3 + r/3 = 2r/3$ $r_{a_1} = d(a, B)$



$\Rightarrow r < 2r/3$

$\Rightarrow 3r < 2r$.

Which is a contradiction.

So our supposition is wrong

Hence $U \cap V = \phi$

$\Rightarrow X$ is normal.

DEFINITION: T_4 SPACE.

A normal T_1-space is called T_4-space.

Theorem: T_4 space is regular space.

Proof: Let X be a T_4 space.

To prove: X is regular.

As X is T_4, so X is normal and X is T_1 as well.

Let $x \in X$ and A be a closed set in X s.t. $x \notin A$

As X is T_1 so $\{x\}$ is closed.

Hence $\{x\}$ and A are two disjoint closed sets in X.

As X is normal so then there exists two open sets U and V in X such that

$\{x\} \subseteq U$, $A \subseteq V$ and $U \cap V = \phi$.

$\Rightarrow x \in U$, $A \subseteq V$ and $U \cap V = \phi$.

$\Rightarrow X$ is regular.

V-Imp

Theorem. A topological space $(X, \mathcal{T})$ is normal iff for any closed set A and open set U containing A, there is atleast one open set V containing 'A' such that $A \subseteq V \subseteq \bar{V} \subseteq U$.

Proof: Let X be a normal space. And let 'A' be a closed set in X, U be an open set in X with $A \subseteq U$. To prove: There is atleast one open set 'V' in X with $A \subseteq V \subseteq \bar{V} \subseteq U$.

Now, $A \subseteq U \Rightarrow A \cap U' = \phi$.

As U is open so U' is closed in X. So, we have A and U' two closed disjoint sets in X. As, X is normal so then there exists two open sets V and V_1 in X such that $A \subseteq V$, $U' \subseteq V_1$ and $V \cap V_1 = \phi$.

Now, $U' \subseteq V_1 \Rightarrow V_1' \subseteq U$. Also as $V \cap V_1 = \phi \Rightarrow V \subseteq V_1'$.

$\Rightarrow A \subseteq V \subseteq V_1' \subseteq U$.

Now, as V_1 is open, so V_1' is closed. So it means V_1' is the closed superset of V.

But as $\bar{V}$ is the smallest closed superset of V so $V \subseteq \bar{V} \subseteq V_1'$.

$\Rightarrow A \subseteq V \subseteq \bar{V} \subseteq V_1' \subseteq U$.

$\Rightarrow A \subseteq V \subseteq \bar{V} \subseteq U$.

Conversely, let it is given that in a topological space $(X, \mathfrak{J})$ for any closed set A in X, U is an open set in X with $A \subseteq U$, there is atleast one open set V in X such that $A \subseteq V \subseteq \bar{V} \subseteq U$.

To prove: X is normal. Let A and B be the two closed disjoint sets in X. As $A \cap B = \phi \Rightarrow A \subseteq B'$

As B is closed so B' is open.

Now, by given condition, there is an open set V in X such that $A \subseteq V \subseteq \bar{V} \subseteq B'$.

Now, $A \subseteq V$ and $\bar{V} \subseteq B' \Rightarrow A \subseteq V$ and $B \subseteq \bar{V}'$.

Now as $\bar{V}$ is closed so $\bar{V}'$ is open.

Further as $V \subseteq \bar{V} \Rightarrow V \cap (\bar{V})' = \phi$

Hence, we have found two open sets V and $\bar{V}'$

in X such that $A \subseteq V$, $B \subseteq \overline{V}'$ and $V \cap \overline{V}' = \phi$.

$\Rightarrow X$ is normal.

Theorem: Every closed subspace of a normal space is normal.

Proof: Let X be a normal space and Y be a closed subspace of X.

To prove: Y is normal.

Let A_1 and A_2 be the two disjoint closed sets in Y. As Y is subspace. So then there are two disjoint closed sets B_1 and B_2 in X such that:

$A_1 = B_1 \cap Y$, $A_2 = B_2 \cap Y$.

Now as B_1 and B_2 are two disjoint closed sets in X and X is normal. So then, there are two open sets U_1 and U_2 such that $B_1 \subseteq U_1$, $B_2 \subseteq U_2$ and $U_1 \cap U_2 = \phi$.

Put $V_1 = U_1 \cap Y$ and $V_2 = U_2 \cap Y$.

$\Rightarrow V_1$ and V_2 are open in Y.

As $B_1 \subseteq U_1 \Rightarrow B_1 \cap Y \subseteq U_1 \cap Y \Rightarrow A_1 \subseteq V_1$

Also $B_2 \subseteq U_2 \Rightarrow B_2 \cap Y \subseteq U_2 \cap Y \Rightarrow A_2 \subseteq V_2$.

$$V_1 \cap V_2 = (U_1 \cap Y) \cap (U_2 \cap Y)$$
$$= (U_1 \cap U_2) \cap Y$$
$$= \phi \cap Y = \phi$$

$\Rightarrow Y$ is normal.

Theorem: Every metric space is completely regular.

Proof: Let (X,d) be a metric space.

To prove: X is completely regular.

Let A be a closed subset of X and $x \in X$ such that $x \notin A$. And we have to find out a continuous function $f: X \to [0\ 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Define $g: X \to [0\ 1]$ by $g(y) = d(y, B)$ where B is any other closed set in X with $A \cap B = \phi$ and $x \in B$.

Then, i) $g(x) = d(x, B) = 0$.

ii) $g(A) = d(A, B) > 0$ ($\because$ A and B are closed).

Let $d(A, B) = K$.

iii) Now for any $\varepsilon > 0$, we chose $\delta = \varepsilon$ such that whenever $d(y, y') < \delta$.

Then $|g(y) - g(y')| = |d(y, B) - d(y', B)|$

$\leq d(y, y')$

$< \delta = \varepsilon$.

$\Rightarrow |g(y) - g(y')| < \varepsilon$. $(|d(x, y) - d(x, z)| \leq d(y, z))$

$\Rightarrow g$ is continuous on X.

Now $f: X \to [0\ 1]$ by $f(y) = \frac{1}{K} g(y)$.

As g is continuous, so f is continuous with

$f(x) = \frac{1}{K} g(x) = \frac{1}{K}(0) = 0$.

$f(A) = \frac{1}{K} g(A) = \frac{1}{K} d(A, B) = \frac{1}{K} \cdot K = 1$.

$\Rightarrow X$ is completely regular.

Open Function:

A function 'f' is said to be open function if image of each open set is open.

Example:

Let $X = \{1, 2, 3, 4\}$, $\mathcal{T}_X = \{\phi, X, \{1, 2\}, \{3, 4\}\}$

$Y = \{a, b, c, d\}$, $\mathcal{T}_Y = \{\phi, Y, \{b, c\}, \{a, d\}\}$.

Define $f: X \to Y$ as $f(1) = b$, $f(2) = c$, $f(3) = a$, $f(4) = d$. Then f is open.

Closed function:
A function 'f' is said to be closed if image of each closed set is closed.

Theorem: A closed and continuous image of a normal space is normal.

Proof: Let X be a normal space and $Y = f(X)$ is its closed continuous image.
To prove: $Y = f(X)$ is normal.
Let A_1 and B_1 be the two disjoint closed in $Y = f(X)$.
As f is continuous so inverse image of each closed set is closed.
So $f^{-1}(A_1)$ and $f^{-1}(B_1)$ are closed in X.
Let $A = f^{-1}(A_1)$ and $B = f^{-1}(B_1)$
Then, A and B are closed in X and

$$A \cap B = f^{-1}(A_1) \cap f^{-1}(B_1)$$
$$= f^{-1}(A_1 \cap B_1)$$
$$= f^{-1}(\phi) = \phi$$

$\Rightarrow$ A and B are two disjoint closed sets in normal space X. So, then there exists two open sets U and V in X such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.
As U and V are open in X. So, U' and V' are closed in X.
Further as f is closed:
So $f(U')$ and $f(V')$ are closed
As $Y = f(X)$
Put $U_1 = Y \setminus f(U')$
$V_1 = Y \setminus f(V')$
$\Rightarrow U_1$ and V_1 are open in $Y = f(X)$.
Now we show $A_1 \subseteq U_1$.

Let $x \in A_1$

$\Rightarrow x \in f(A)$

$\Rightarrow f^{-1}(x) \in A \subseteq U$

$\Rightarrow f^{-1}(x) \in U$

$\Rightarrow f^{-1}(x) \notin U'$ $\quad \because A = f^{-1}(A_1)$

$\Rightarrow x \notin f(U')$ $\quad \Rightarrow f(A) = A_1$

$\Rightarrow x \in Y \setminus f(U')$

$\Rightarrow x \in U_1$

$\Rightarrow A_1 \subseteq U_1$

Similarly $B_1 \subseteq V_1$

Now, $U_1 \cap V_1 = (f(U'))' \cap (f(V'))'$

$= [f(U') \cup f(V')]'$

$= [f(U' \cup V')]'$

$= [f(X)]'$

$= Y' = \phi$

$\Rightarrow f(X)$ is normal.

"COMPACTNESS IN TOPOLOGICAL SPACES".

DEFINITION: Compactness:

A topological space $(X,\mathcal{T})$ is said to be compact if every open cover for X has a finite subcover.

Examples:

1) If X is any set with indiscrete topology Then X is compact.
2) If X is finite set Then for any topology $\mathcal{T}$ on X, $(X,\mathcal{T})$ is compact.
3) If X is any set with $A \subseteq X$, then $\mathcal{T} = \{\phi, X, A, A'\}$ then, $(X,\mathcal{T})$ is compact.

REMARK:

If $(X,\mathcal{T})$ is compact space then it is Lindelof. But converse is not true because e.g. If $X = \mathbb{N}$ and $\mathcal{T} = \mathcal{T}_D$

Then $(X,\mathcal{T})$ is Lindelof space but $(X,\mathcal{T})$ is not compact because $\{\{x\} : x \in \mathbb{N}\}$ is an open cover for X; which has no finite subcover. $\{\bigcup\{x\} = X$ open cover all are open$\}$.

THEOREM: Let X be an infinite set with cofinite topology then X is compact.

PROOF: Let $\gamma = \{U_\alpha : \alpha \in I\}$ be an open cover for X. We have to find a finite subcover of γ for X. Since γ is an opencover for X.

So $X = \bigcup\gamma = \bigcup_{\alpha\in I} U_\alpha$.

Now, for any $U_\alpha \in \gamma$, $\Rightarrow U_\alpha$ is an open set.

$\Rightarrow U'_\alpha$ is finite.

$\Rightarrow U'_\alpha = \{x_1, x_2, x_3, \ldots, x_n\}$

Now as γ is an open cover for X i.e,

$X = \bigcup_{\alpha\in I} U_\alpha$.

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›› ERROR

So, for any $x_i \in U'_\alpha$, $1 \le i \le n$.

$\Rightarrow x_i \in \bigcup_{\alpha \in I} U_\alpha$

$\Rightarrow x_i \in U_{\alpha_i}$ for some $\alpha_i \in I$

$\Rightarrow \{x_i\} \subseteq U_{\alpha_i}$

$\Rightarrow \bigcup_{i=1}^{n} \{x_i\} \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\Rightarrow U'_\alpha \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$

$\Rightarrow U_\alpha \cup U'_\alpha \subseteq U_\alpha \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$

$\Rightarrow X \subseteq U_\alpha \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right) \subseteq X$

$\Rightarrow X = U_\alpha \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$

$\Rightarrow \{U_\alpha, U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ is a finite open subcover for X.

Hence, X is compact space.

THEOREM: The real line $\mathbb{R}$ is not compact with usual topology.

topology [illegible] respect [illegible] metric space [illegible] Real line [illegible] Usual topology [illegible] define

PROOF: Let $\gamma = \{U_n =]-n\ n[: n \in \mathbb{N}\}$ be an open cover for X.

To prove: $\mathbb{R}$ is not compact.

Suppose on the contrary that $\mathbb{R}$ is compact. Then by the definition of compact space, γ has a finite subcover for $\mathbb{R}$.

Let $\{U_{n_1}, U_{n_2}, U_{n_3}, \ldots U_{n_r}\}$ be the finite subcover for $\mathbb{R} \Rightarrow \mathbb{R} = \bigcup_{i=1}^{r} U_{n_i}$

$U_1 =]-1\ 1[$, $U_2 =]-2\ 2[$

$U_1 \cup U_2 = U_2 =]-2\ 2[$

Let $m = \max(n_1, n_2, n_3, \ldots, n_r)$

$U_1 \cup U_2 \cup U_3 = U_3$

then $m \in \mathbb{N}$ and $\bigcup_{i=1}^{r} U_{n_i} =]-m\ m[$

$\bigcup_{n=1}^{\infty} U_n =]-\infty\ \infty[= \mathbb{R}$

$\Rightarrow \mathbb{R} =]-m \ \ m[$ which is a contradiction.
So, our supposition is wrong.
Hence, $\mathbb{R}$ is not compact.

FINITE INTERSECTION PROPERTY:
Let $(X, \mathcal{T})$ be a topological space and $\gamma = \{U_\alpha : \alpha \in I\}$ be a collection of some subsets of X, then γ is said to have finite intersection property if each finite subcollection of γ has non-empty intersection e.g. Let $X = \mathbb{N}$, $\mathcal{T} = \mathcal{T}_c$ and $\gamma = \{\{1\}, \{2\}, \{1,2,3\}, \{1,2,3,4\}, \ldots\}$ then γ satisfies finite intersection property.

THEOREM: A topological space $(X, \mathcal{T})$ is compact if and only if every collection of closed sets in X which satisfy finite intersection property has non-empty intersection.

PROOF: Let X be compact and $\{U_\alpha : \alpha \in I\}$ be the collection of closed sets which satisfy finite intersection property.
To prove: $\bigcap_{\alpha \in I} U_\alpha \neq \phi$.

Suppose on the contrary that $\bigcap_{\alpha \in I} U_\alpha = \phi$.

$\Rightarrow \left(\bigcap_{\alpha \in I} U_\alpha\right)' = \phi' \Rightarrow \bigcup_{\alpha \in I} U'_\alpha = X$.

Now as $\{U_\alpha : \alpha \in I\}$ is the collection of closed sets so $\{U'_\alpha : \alpha \in I\}$ is the collection of open sets with $\bigcup_{\alpha \in I} U'_\alpha = X$.

$\Rightarrow \{U'_\alpha : \alpha \in I\}$ is an open cover of X, where X

is compact space.

$\Rightarrow \{U'_{\alpha_1}, U'_{\alpha_2}, U'_{\alpha_3}, \dots, U'_{\alpha_n}\}$ is a finite open subcover for X.

$\Rightarrow \bigcup_{i=1}^{n} U'_{\alpha_i} = X$.

$\Rightarrow (\bigcup_{i=1}^{n} U'_{\alpha_i})' = X' \Rightarrow \bigcap_{i=1}^{n} U_{\alpha_i} = \phi$.

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite subcollection of $\{U_\alpha : \alpha \in I\}$ with empty intersection.

$\Rightarrow \{U_\alpha : \alpha \in I\}$ does not satisfy finite intersection property. Which is a contradiction.

$\because \{U_\alpha : \alpha \in I\}$ satisfies finite intersection property.

So our supposition is wrong.

And hence, $\bigcap_{\alpha \in I} U_\alpha \neq \phi$

Conversely, let in a topological space $(X, \mathcal{F})$ each collection of closed sets in X which satisfies finite intersection property has non-empty intersection.

To prove: X is compact.

Let $\{O_\alpha : \alpha \in I\}$ be an open cover for X.

i.e., $\bigcup_{\alpha \in I} O_\alpha = X \Rightarrow (\bigcup_{\alpha \in I} O_\alpha)' = X' \Rightarrow \bigcap_{\alpha \in I} O'_\alpha = \phi$.

$\Rightarrow \{O'_\alpha : \alpha \in I\}$ is a collection of closed sets with empty intersection.

Then, by given hypothesis $\{O'_\alpha : \alpha \in I\}$ does not satisfy finite intersection property. Then, there exists a finite subcollection $\{O'_{\alpha_1}, O'_{\alpha_2}, \dots, O'_{\alpha_n}\}$ with empty intersection.

$\Rightarrow \bigcap_{i=1}^{n} O'_{\alpha_i} = \phi \Rightarrow (\bigcap_{i=1}^{n} O'_{\alpha_i})' = \phi' \Rightarrow \bigcup_{i=1}^{n} O_{\alpha_i} = X$.

$\Rightarrow \{O_{\alpha_1}, O_{\alpha_2}, \dots, O_{\alpha_n}\}$ is a finite open subcover for $X \Rightarrow X$ is compact.

THEOREM: Every closed subspace of a compact space is compact.

PROOF: Let X be compact and Y be a closed subspace of X.

To prove: Y is compact.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover for Y.

As U_α, $\alpha \in I$, is an open set in Y and Y is subspace of X, so then there is an open set V_α in X such that:

$$U_\alpha = V_\alpha \cap Y$$

$$\Rightarrow U_\alpha \subseteq V_\alpha$$

$$\Rightarrow \bigcup_{\alpha\in I} U_\alpha \subseteq \bigcup_{\alpha \in I} V_\alpha$$

$$\Rightarrow Y \subseteq \bigcup_{\alpha \in I} V_\alpha$$

Now, $X = Y \cup Y' \subseteq \left(\bigcup_{\alpha\in I} V_\alpha\right) \cup Y' \subseteq X$

$$\Rightarrow X = \left(\bigcup_{\alpha \in I} V_\alpha\right) \cup Y'$$

As Y is closed so Y' is open in X.

$\Rightarrow \{Y', V_\alpha : \alpha \in I\}$ is an open cover for X.

As X is compact so this open cover has a finite subcover $\{Y', V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}\}$

i.e. $X = \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup Y'$

$A \subseteq B \cup C \not\Rightarrow A \subseteq B$ or $A \subseteq C$

$A \subseteq B \cup A' \Rightarrow A \subseteq B$

$A \subseteq B \Rightarrow A = B \cap A$

Now, $Y \subseteq X = \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cup Y'$

$$\Rightarrow Y \subseteq \bigcup_{i=1}^{n} V_{\alpha_i} \quad \because Y \cap Y' = \phi$$

$$\Rightarrow Y = \left(\bigcup_{i=1}^{n} V_{\alpha_i}\right) \cap Y$$

$$\Rightarrow Y = \bigcup_{i=1}^{n} (V_{\alpha_i} \cap Y) \Rightarrow Y = \bigcup_{i=1}^{n} U_{\alpha_i}$$

$\Rightarrow \{U_{\alpha_1}, U_{\alpha_2}, \dots, U_{\alpha_n}\}$ is a finite subcover for Y.
$\Rightarrow Y$ is compact.

V-Imp.
THEOREM: Continuous image of a compact space is compact.

PROOF: Let $f: X \to Y$ be a continuous function from a compact space X to a topological space Y.
To prove: $f(X)$ is compact.
Let $\{U_\alpha : \alpha \in I\}$ be an open cover for $f(X)$, where $f(X)$ is the subspace of Y.
As, $\{U_\alpha : \alpha \in I\}$ is an open set in $f(X)$ and $f(X)$ is subspace of Y.
So, then, there exists an open set V_α in Y such that
$U_\alpha = V_\alpha \cap f(X)$.

$\Rightarrow U_\alpha \subseteq V_\alpha \Rightarrow \bigcup_{\alpha \in I} U_\alpha \subseteq \bigcup_{\alpha \in I} V_\alpha$

$\Rightarrow f(X) \subseteq \bigcup_{\alpha \in I} V_\alpha$

$\Rightarrow X \subseteq f^{-1}\left(\bigcup_{\alpha \in I} V_\alpha\right)$.

$\Rightarrow X \subseteq \bigcup_{\alpha \in I} f^{-1}(V_\alpha) \subseteq X$ — $V_\alpha \subseteq Y$, $f^{-1}(V_\alpha) \subseteq X$.

$\Rightarrow X = \bigcup_{\alpha \in I} f^{-1}(V_\alpha)$

As $V_\alpha, \alpha \in I$ is open in Y and $f: X \to Y$ is continuous.
$\Rightarrow f^{-1}(V_\alpha), \alpha \in I$, is open in X ($\because$ Inverse image of open set is open).
$\Rightarrow \{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover for X.
Since, X is compact.
So, this open cover has a finite subcover, $\{f^{-1}(V_{\alpha_1}), f^{-1}(V_{\alpha_2}), \dots, f^{-1}(V_{\alpha_n})\}$ for X.
$\Rightarrow \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) = X$.

$\Rightarrow f^{-1}(\bigcup_{i=1}^{n} V_{\alpha i}) = X$

$\Rightarrow X = f^{-1}(\bigcup_{i=1}^{n} V_{\alpha i})$

$\Rightarrow f(X) \subseteq \bigcup_{i=1}^{n} V_{\alpha i}$

$\Rightarrow f(X) = (\bigcup_{i=1}^{n} V_{\alpha i}) \cap f(X)$.

$\Rightarrow f(X) = \bigcup_{i=1}^{n} (V_{\alpha i} \cap f(X))$ (Distributive property).

$= \bigcup_{i=1}^{n} U_{\alpha i}$

$\Rightarrow \{U_{\alpha 1}, U_{\alpha 2}, \ldots, U_{\alpha n}\}$ is a finite subcover for $f(X)$.

Hence, $f(X)$ is compact.

THEOREM: Prove that in a T_2 space any point and disjoint compact subspace of X can be separated by open sets, in the sence they have disjoint neighbourhoods.

PROOF: Let $x \in X$ and 'C' be a compact subspace of X such that $x \notin C$.

To prove: x and 'C' can be separated by open sets.

Let $y \in C \Rightarrow x \neq y$, then as $x, y \in X$, $x \neq y$, X is T_2 space, so then there exists two open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \phi$.

Now as, $y \in V_y \Rightarrow \{y\} \subseteq V_y$

$\Rightarrow \bigcup_{y \in C} \{y\} \subseteq \bigcup_{y \in C} V_y$

$\Rightarrow C \subseteq \bigcup_{y \in C} V_y \Rightarrow C = (\bigcup_{y \in C} V_y) \cap C$

$\Rightarrow C = \bigcup_{y \in C} (V_y \cap C)$

As for every $y \in C$, V_y is open in X, so $V_y \cap C$ is open in C.

$\Rightarrow \{V_y \cap C : y \in C\}$ is an open cover for C.

As C is compact. So, this open cover has a finite subcover $\{V_1 \cap C, V_2 \cap C, V_3 \cap C, \dots, V_n \cap C\}$

$\Rightarrow C = \bigcup_{i=1}^{n} (V_i \cap C) = (\bigcup_{i=1}^{n} V_i) \cap C$

$\Rightarrow C \subseteq \bigcup_{i=1}^{n} V_i$

Put $U = \bigcap_{i=1}^{n} U_i$ and $V = \bigcup_{i=1}^{n} V_i$

$\Rightarrow x \in U,\ C \subseteq V$

Now to prove $U \cap V = \phi$.

Suppose on the contrary, $U \cap V \neq \phi$.

$\Rightarrow z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$

$\Rightarrow z \in \bigcap_{i=1}^{n} U_i$ and $z \in \bigcup_{i=1}^{n} V_i$

Then, there is an i, $1 \leq i \leq n$ such that $z \in U_i$ and $z \in V_i$.

$\Rightarrow U_i \cap V_i \neq \phi$.

Which is a contradiction.

So our supposition is wrong.

Hence $U \cap V = \phi$.

THEOREM: Compact subspace of a T_2-space is closed.

PROOF: Let X be a T_2 space and C be a compact subspace ~~sub~~ of X.

To prove: C is closed.

We prove C' is open.

If $C' = \phi$, then C' is open.

If $C' \neq \phi$, then $x \in C' \Rightarrow x \notin C$.

Then, by a well known theorem, in a T_2-space any point and a disjoint subspace of X can be separated by open sets in the sense they have disjoint neighbourhoods, there exists two open sets U_x and V_x such that $x \in U_x$, $C \subseteq V_x$ and $U_x \cap V_x = \phi$.

Now, $x \in U_x$

$\Rightarrow \{x\} \subseteq U_x \subseteq V_x'$ $(\because U_x \cap V_x = \phi)$

Also, $C \subseteq V_x \Rightarrow V_x' \subseteq C'$

$\Rightarrow \{x\} \subseteq U_x \subseteq V_x' \subseteq C'$

$\Rightarrow \{x\} \subseteq U_x \subseteq C'$

$\Rightarrow \bigcup_{x \in C'} \{x\} \subseteq \bigcup_{x \in C'} U_x \subseteq C'$

$\Rightarrow C' \subseteq \bigcup_{x \in C'} U_x \subseteq C'$

$\Rightarrow C' = \bigcup_{x \in C'} U_x$

As, U_x is open so $\bigcup_{x \in C'} U_x$ is open.

$\Rightarrow C'$ is open.

$\Rightarrow C$ is closed.

DEFINITION: HOMEOMORPHISM.

A function $f: X \to Y$ is said to be homeomorphism if:

i) f is continuous ii) f is open iii) f is bijective.

THEOREM: A 1-1 continuous mapping from a compact space X onto a T_2-space Y is Homeomorphism.

PROOF: As given 'f' is continuous and bijective so, we have just to prove that 'f' is open.

Let G be an open set in X.

$\Rightarrow G'$ is closed set in X.

As closed subspace of a compact space is compact. So, G' is compact.

Further as, continuous image of a compact space is compact. So $f(G')$ is compact.

$\Rightarrow f(G')$ is a compact subspace of Y.

As compact subspace of T_2 space is closed.

And $f(G')$ is a compact subspace of T_2-space.

$\Rightarrow f(G')$ is closed in Y.

Now,
$$
\begin{aligned}
f(G') &= f(X-G) \\
&= f(X) - f(G) \\
&= Y - f(G) \quad (\because f \text{ is onto}) \\
&= [f(G)]'
\end{aligned}
$$

$\Rightarrow [f(G)]'$ is closed in Y.

$\Rightarrow f(G)$ is open in Y.

$\Rightarrow f$ is open function.

Hence, f is homeomorphism.

THEOREM: A topological space X is compact if and only if every class of closed sets with empty intersection has a finite sub class with empty intersection.

PROOF: Given X is compact and $\{C_\alpha : \alpha \in I\}$ be a class of closed sets in X with $\bigcap_{\alpha\in I} C_\alpha = \phi$. To prove: There is a finite subclass of $\{C_\alpha : \alpha \in I\}$ with empty intersection.

Now as $\bigcap_{\alpha\in I} C_\alpha = \phi$.

$$\Rightarrow \left(\bigcap_{\alpha\in I} C_\alpha\right)' = \phi' \Rightarrow \bigcup_{\alpha\in I} C'_\alpha = X$$

As C_α is closed so, C'_α is open.

$\Rightarrow \{C'_\alpha : \alpha \in I\}$ is an open cover for X.

As X is compact so this open cover has a finite subcover $\{C'_{\alpha 1}, C'_{\alpha 2}, \ldots, C'_{\alpha n}\}$.

$$\Rightarrow \bigcup_{i=1}^{n} C'_{\alpha i} = X$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} C'_{\alpha i}\right)' = X'$$

$$\Rightarrow \bigcap_{i=1}^{n} C_{\alpha i} = \phi$$

$\Rightarrow \{C_{\alpha 1}, C_{\alpha 2}, \ldots, C_{\alpha n}\}$ is a finite subclass $\{C_\alpha : \alpha \in I\}$ with empty intersection.

Conversely, suppose in a Topological space X each class $\{C_\alpha : \alpha \in I\}$ of closed sets with empty intersection has a finite subclass with empty intersection.

To prove: X is compact.

Now, let $\{U_\alpha : \alpha \in I\}$ be an open cover for X.

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha = X \Rightarrow \left(\bigcup_{\alpha \in I} U_\alpha\right)' = X' \Rightarrow \bigcap_{\alpha \in I} U'_\alpha = \phi.$$

$\Rightarrow \{U'_\alpha : \alpha \in I\}$ is a class of closed sets with empty intersection. Then, by given condition there is a finite subclass $\{U'_{\alpha 1}, U'_{\alpha 2}, \ldots, U'_{\alpha n}\}$ with,

$$\bigcap_{i=1}^{n} U'_{\alpha i} = \phi$$

$$\Rightarrow \left(\bigcap_{i=1}^{n} U'_{\alpha i}\right)' = \phi' \Rightarrow \bigcup_{i=1}^{n} U_{\alpha i} = X$$

$\Rightarrow \{U_{\alpha 1}, U_{\alpha 2}, \ldots, U_{\alpha n}\}$ is a finite open subcover for X.

$\Rightarrow X$ is compact.

Hence Proved.

THEOREM: Every compact T_2-space is normal.

PROOF: Let X be a compact T_2-space.

To prove: X is normal.

Let A and B be the two closed disjoint subsets of X. We have to prove there exists two open set U and V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$.

Let $x \in A$. As A and B are disjoint i.e $A \cap B = \phi$. So $x \notin B$. Then, as X is compact and B is closed in X so, B is also compact.

Further as X is also T_2 and in a T_2 space a point and a disjoint compact subspace can be separated by open sets, so then there exists two open sets U_x and V_x such that:

$x \in U_x$, $B \subseteq V_x$ and $U_x \cap V_x = \phi$.

Now, $x \in U_x \Rightarrow \{x\} \subseteq U_x \Rightarrow \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x$

$$\Rightarrow A \subseteq \bigcup_{x \in A} U_x$$

$$\Rightarrow A = \left(\bigcup_{x \in A} U_x\right) \cap A \Rightarrow A = \bigcup_{x \in A} (U_x \cap A)$$

$\Rightarrow \{U_x \cap A : x \in A\}$ is an open cover for A.

Again as A is a closed subspace of compact space X, so A is compact.

So, the open cover $\{U_x \cap A : x \in A\}$ for A has a finite subcover $\{U_{x_1} \cap A, U_{x_2} \cap A, \ldots, U_{x_n} \cap A\}$ for A.

$$\Rightarrow A = \bigcup_{i=1}^{n} (U_{x_i} \cap A)$$

$$= \left(\bigcup_{i=1}^{n} U_{x_i}\right) \cap A$$

$$\Rightarrow A \subseteq \bigcup_{i=1}^{n} U_{x_i}$$

Put $U = \bigcup_{i=1}^{n} U_{x_i}$ and $V = \bigcap_{i=1}^{n} V_{x_i}$.

Then U and V are open.

$A \subseteq U$, $B \subseteq V$

Now to prove only $U \cap V = \phi$.

Suppose on the contrary $U \cap V \neq \phi$

$\Rightarrow z \in U \cap V$

$\Rightarrow z \in U$ and $z \in V$

$\Rightarrow z \in \bigcup_{i=1}^{n} U_{x_i}$ and $z \in \bigcap_{i=1}^{n} V_{x_i}$

$\Rightarrow z \in U_{x_i}$ for some i.

$z \in V_{x_i}$ for all i.

$\Rightarrow U_{x_i} \cap V_{x_i} \neq \phi$. A contradiction. So our supposition is wrong and hence $U \cap V = \phi$.

$\Rightarrow X$ is normal.

HEINE BOREL THEOREM: V-Imp

Statement: Every closed and bounded subspace of a real line $\mathbb{R}$ is compact.

PROOF: Closed and bounded subspace of $\mathbb{R}$ is some closed interval $[a\ b]$.

Case I: If $a = b$, then $[a, b] = \{a\}$, which is compact.

Case II: If $a < b$, then the class of all intervals $[a\ c[\,;\,]c\ b]$ is an open subbase for $[a\ b]$.
Similarly, the class of all intervals $[a\ c], [d\ b]$ is a closed subbase for $[a\ b]$.

Let $S = \{[a\ c_i], [d_j\ b]\}$ be the class of those subbasic closed sets which satisfy finite intersection property.

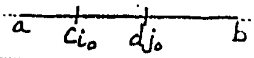
Now here arises the following cases.

i) If S contains only the interval of the form $[a\ c_i]$ then always $a \in \cap S$.

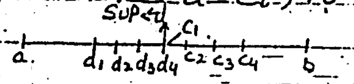
$\Rightarrow \cap S \neq \phi$. So closed interval $[a\ b]$ is compact.

ii) If S contains only the intervals of the form $[d_j, b]$, Then always $b \in \cap S \Rightarrow \cap S \neq \phi$. So $[a\ b]$ is compact.

iii) If S contains the intervals of both forms. Then, put
$d = \sup\{d_j\}$
To prove, $d \leq c_i,\ \forall i$.
Suppose on the contrary that it is not true.
Then for some i_0, $d > c_{i_0}$
$\Rightarrow c_{i_0} < d = \sup\{d_j\}$.
Then, there exists some d_{j_0} such that $c_{i_0} < d_{j_0}$.
Then, $[a\ c_{i_0}] \cap [d_{j_0}\ b] = \phi$.



$\Rightarrow S$ doesnot satisfy finite intersection property.
Which is a contradiction.
So, our supposition is wrong and hence $d \leq c_i, \forall i$.
$\Rightarrow \cap S \neq \phi$
$\Rightarrow [a\ b]$ is compact.



Real Analysis

THEOREM: Every compact subspace of real line $\mathbb{R}$ is closed and bounded.

PROOF: Let 'C' be a compact subspace of $\mathbb{R}$.
To prove: C is closed and bounded.
As $\mathbb{R}$ is T_2-space and compact subspace of a T_2-space is closed. So, C is closed.
Now it remains only to prove that C is bounded.
Let $U_k = B(0,k)$, $k \in \mathbb{N}$, Then, $\{U_k : k \in \mathbb{N}\}$ is an open cover for $\mathbb{R}$. $B(0,k) =]-k\ k[$. $]-1\ 1[\cup]-2\ 2[\cup \ldots]-\infty\ \infty[= \mathbb{R}$.
Now as U_k is open in $\mathbb{R}$ and C is subspace of $\mathbb{R}$. So, $U_k \cap C$ is a open set in C.
Let $\{U_k \cap C : k \in \mathbb{N}\}$ be an open cover for C.
As C is compact. So this open cover has a finite

subcover, $\{U_{k_1}\cap C, U_{k_2}\cap C, \dots, U_{k_n}\cap C\}$.

$\Rightarrow C = \bigcup_{i=1}^{n} (U_{k_i}\cap C)$

$= (\bigcup_{i=1}^{n} U_{k_i})\cap C$

$\Rightarrow C \subseteq \bigcup_{i=1}^{n} U_{k_i}$.

Put $m = \max(k_1, k_2, \dots, k_n)$.
Then $\bigcup_{i=1}^{n} U_{k_i} = U_m =]-m, m[$.

$\Rightarrow C \subseteq]-m, m[$.
$\Rightarrow C$ is bounded.

Real Analysis

THEOREM: Prove that compact subspace of R^n is closed and bounded.

PROOF: Let C be a compact subspace of R^n.
To prove: C is closed and bounded.
As $\mathbb{R}^n$ is T_2 and C is a compact subspace of T_2 space $\mathbb{R}^n$. So C is closed.
Now, let $U_k = B(0, k)$ (where $0 = (0, 0, \dots, 0)$ & $k \in N$).
Then $\{U_k : k \in N\}$ is an open cover for R^n.
As U_k is an open set in $\mathbb{R}^n$.
So, $U_k \cap C$ is open set in C.
Let $\{U_k \cap C : k \in N\}$ be an open cover for C.
As 'C' is compact. So this open cover has a finite subcover $\{U_{k_1}\cap C, U_{k_2}\cap C, \dots, U_{k_n}\cap C\}$.

$\Rightarrow C = \bigcup_{i=1}^{n} (U_{k_i}\cap C)$.

$\Rightarrow C = (\bigcup_{i=1}^{n} U_{k_i})\cap C$.

$\Rightarrow C \subseteq \bigcup_{i=1}^{n} U_{k_i} \subseteq U_{n^*}$ where $n^* = \max(k_1, k_2, \dots, k_n)$.

$\Rightarrow C \subseteq U_n^*$.

As U_n^* is bounded so C is bounded.

Real Analysis

THEOREM: A continuous real valued function defined on a compact space is bounded and attains its bounds.

PROOF: Let $f: X \to \mathbb{R}$ be a continuous function from a compact space X.

To prove: f is bounded.

For this, we prove $f(X)$ is bounded.

As X is compact and f is continuous and continuous image of a compact space is compact. So; $f(X)$ is compact. Hence $f(X)$ is a compact subspace of $\mathbb{R}$.

As a compact subspace of $\mathbb{R}$ is closed and bounded, so $f(X)$ is closed and bounded.

Let $M = \operatorname{Sup} f(X)$ and $m = \operatorname{Inf} f(X)$.

As $f(X)$ is bounded so M and m exists.

As sup and Inf of a set are its limit points. So M and m are the limit points of $f(X)$.

As $f(X)$ is closed.

So $M, m \in f(X)$.

DEFINITION: COUNTABLY COMPACT SPACE:

A topological space X is said to be countably compact space if every countable open cover for X has a finite subcover for X.

REMARK:

Every compact space is also countably compact space.

THEOREM: A topological space X is countably compact if and only if every countable collection of closed sets in X which satisfy finite intersection property has non-empty intersection.

PROOF: Suppose that X is countably compact space and $\{U_n : n \in N\}$ be the countable collection of closed sets which satisfy finite intersection property.

To prove: $\bigcap_{n \in N} U_n$ is non empty.

Suppose on the contrary, $\bigcap_{n \in N} U_n = \phi$

$\Rightarrow (\bigcap_{n \in N} U_n)' = \phi' \Rightarrow \bigcup_{n \in N} U_n' = X$.

As U_n is closed so U_n' is open for all n.

$\Rightarrow \{U_n' : n \in N\}$ is a countable open cover for X.

As X is countably compact so this countable open cover has a finite subcover $\{U_1', U_2', --- U_n'\}$.

$\Rightarrow \bigcup_{i=1}^{n} U_i' = X$

$\Rightarrow (\bigcup_{i=1}^{n} U_i')' = X' \Rightarrow \bigcap_{i=1}^{n} U_i = \phi$

$\Rightarrow$ The class $\{U_n : n \in N\}$ does not satisfy finite intersection property.

Which is a contradiction.

$\because \{U_n : n \in N\}$ satisfy finite intersection property.

So our supposition is wrong.

Hence, $\bigcap_{n \in N} U_n \neq \phi$.

Conversely, suppose that for every countable collection of closed sets $\{U_n : n \in N\}$ which satisfy finite intersection property has non empty intersection.

To prove: X is countably compact.

Suppose, X is not countably compact.
Let $\{U_n : n \in N\}$ be a countably open cover for X.
As X is not countably compact space so for every finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\{U_n : n \in N\}$.

$$\bigcup_{i=1}^{n} U_i \neq X.$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} U_i\right)' \neq X'.$$

$$\Rightarrow \bigcap_{i=1}^{n} U_i' \neq \phi.$$

As $\{U_n : n \in N\}$ be collection of open sets with

$$\bigcup_{n \in N} U_n = X$$

$\Rightarrow \{U_n' : n \in N\}$ is the collection of closed sets with $\bigcap_{n \in N} U_n' = \phi$.

$\Rightarrow \{U_n' : n \in N\}$ is the class of closed sets

THEOREM: Let X be a topological space, then any infinite subset of X has a limit point if and only if every countably infinite subset of X has a limit point.

PROOF: Let us suppose in topological space X every infinite subset of X has limit point. Then trivially every countably infinite subset of X also has a limit point.

Conversely, suppose every countably infinite subset of X has the limit point.
To prove: Every infinite subset of X has a limit point.

$\therefore \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \ldots$ are irrational nos.
$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13}, \ldots \subseteq$ irrational nos. and 1-1 correspondence with prime no. and prime no. $\subseteq \mathbb{N}$ and $\mathbb{N}$ is countably.

Let 'A' be an infinite subset of X. Then, by a well known result of a set theory, A has a countable infinite subset B.

Then, by the hypothesis B has the limit point say x. Then, for every open set U in X containing x,

$$U \cap B \setminus \{x\} \neq \phi$$
$$\Rightarrow U \cap A \setminus \{x\} \neq \phi \quad \because B \subseteq A$$
$\Rightarrow x$ is also the limit point of A.

THEOREM: Let X be a countably compact space then every infinite subset of X has a limit point.

PROOF: Let A be an infinite subset of a countably compact space X. To prove A has a limit point.

Suppose, A has no limit point.

Let $B = \{x_1, x_2, \ldots\}$ be a countably infinite subset of A. Then B has no limit point. Now, consider the subset $C_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$, $n \in \mathbb{N}$.

Then, as for every C_n, $D(C_n) = \phi \subseteq C_n$. (If $D(A) \subseteq A$ then A is closed.)

Hence, C_n is closed for all n.

$\Rightarrow \{C_n : n \in \mathbb{N}\}$ is a class of closed sets which satisfy finite intersection property.

Because, for every finite subcollection $\{C_{n_1}, C_{n_2}, \ldots, C_{n_r}\}$, $\bigcap_{i=1}^{r} C_{n_i} = C_{n'} \neq \phi$

where, $n' = \max(n_1, n_2, \ldots, n_r)$.

Hence, $\{C_n : n \in \mathbb{N}\}$ is a class of closed sets which satisfy finite intersection property and $\bigcap_{i=1}^{\infty} C_n = \phi \Rightarrow X$ is not countably compact.

Which is a contradiction.

So, our supposition is wrong.

Hence, A has a limit point.

BOLZANO WEIERSTRASS PROPERTY:

A space X is said to satisfy, B.W property if and only if every infinite subset of X has a limit point in X.

COROLLARY: Every countably compact space satisfies B.W Property.

PROOF: Let X be a countably compact space and A be an infinite subset of X. Then, by a well known theorem (Previous), A has limit point. So, X satisfies B.W property.

SEQUENTIALLY COMPACT SPACE:

A space 'X' is said to be sequentially compact space if and only if every sequence in X has a convergent subsequence.

THEOREM: A metric space is sequentially compact if and only if it satisfies B.W Property.

PROOF: A metric space is sequentially compact.
To prove: X satisfies B.W property.
Let A be an infinite subset of X.
To prove: A has limit point in X.
Let $\{x_n\}$ be a sequence in A. As $A \subseteq X$, so $\{x_n\}$ is also a sequence in X.
As, X is sequentially compact, so this sequence $\{x_n\}$ has a convergent subsequence.
Let $\{x_{n_k}\}$ be a convergent subsequence of $\{x_n\}$

such that $x_{n_k} \to x \in X$.

Let B be the set of the point of $\{x_{n_k}\}$.

Then, x is the limit point of B.

As $B \subseteq A$. So, x is the limit point of A.

$\Rightarrow X$ satisfy B.W. Property.

Conversely, suppose X satisfies B.W. property.

To prove: X is sequentially compact.

Let $\{x_n\}$ be a sequence in X.

If a point x in $\{x_n\}$ repeated infinitely many times then $(x, x, x, \cdots) \to x$ is convergent subsequence of $\{x_n\}$.

If no point repeated infinitely many times then set A be the set of the points of sequence $\{x_n\}$. As X satisfies B.W. property. So, A has limit point 'x'. Then we can chose a sequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$.

$\Rightarrow X$ is sequentially compact.

THEOREM: Every compact metric space is sequentially compact.

PROOF: Let X be a compact metric space.

To prove: X is sequentially compact.

For this, we prove X satisfies B.W. property.

"Because a metric space is sequentially compact iff it satisfies B.W. property".

Let 'A' be an infinite subset of X.

To prove: A has limit point.

Suppose, A has no limit point.

Then, there exists an open ~~set~~ ball $B(x, r_x)$ which doesnot contain any point of A different from x.

$\Rightarrow \{B(x, r_x) : x \in X\}$ is an open cover for X.

As, X is compact. So this open cover has a finite subcover $\{B(x_1, r_{x_1}), B(x_2, r_{x_2}), \dots, B(x_n, r_{x_n})\}$.

As, A may contains only centres of these open balls.

$\Rightarrow A$ is finite.

Which is a contradiction.

$\because A$ is an infinite set.

So our supposition is wrong.

Hence, A has a limit point.

$\Rightarrow X$ satisfies B.W property.

$\Rightarrow X$ is sequentially compact.

LEBESGUE NUMBER:

Let (X, d) be a metric space and $\{G_i\}$ be an open cover for X. A real number $a > 0$ is called Lebesgue number for this given open cover $\{G_i\}$ if each subset of X whose diameter is less than 'a' is contained in atleast one G_i.

BIG SET: A subset of X is called big set if it is not contained in any G_i.

V-Imp

LEBESGUE COVERY LEMMA: In a sequentially compact metric space every open cover has a Lebesgue number.

PROOF: Let X be a sequentially compact metric space and $\{G_i\}$ be an open cover for X.

To prove: open cover $\{G_i\}$ has a Lebesgue number.

Here, arises two cases.

Case-I: If Big set does not exists, Then for $a>0$, as big set doesnot exist. So, for every $A \subseteq X$ such that diam $d(A) < a$, there is an open set $G_{i'}$ such that $A \subseteq G_{i'}$.

$\Rightarrow a$ is the Lebesgue number for open cover $\{G_i\}$.

Case-II: If big sets exist then let a' be the greatest lower bound of diameters of these big sets, then $0 < a' \leq \infty$. Then,

i) If $a' = \infty$, then any positive real number 'a' is the Lebesgue number for the open cover $\{G_i\}$ ($\because$ for $a>0$, let $A \subseteq X$ such that $d(A) < a < a'$
$\Rightarrow A$ is not big set because $d(A) < a'$ and a' itself is infimum and infimum is less than all the other diameters).
So, for atleast one G_i, $A \subseteq G_i \Rightarrow a$ is Lebesgue number.*

ii) If $0 < a' < \infty$. Then for any a, $0 < a < a'$, then a is Lebesgue number for open cover $\{G_i\}$ (By: *).

iii) If $a' = 0$. Then, there exists a big set B_n such that $d(B_n) < 1/n$. Now choose $x_n \in B_n$
$\Rightarrow \{x_n\}$ is a sequence in X.
As X is sequentially compact. So this sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ and $x_{n_k} \to x \in X$. As $\{G_i\}$ is an open cover for X so $X = \cup G_i \Rightarrow x \in X \Rightarrow x \in \cup G_i$
$\Rightarrow x \in G_{i_0}$ for some $i = i_0$.
As $x \in G_{i_0}$ and G_{i_0} is an open set.
So, there exist an open ball $B(x, r)$ such that
$x \in B(x, r) \subseteq G_{i_0}$.

Let $B(x, r/2)$ be the concentric open ball to $B(x,r)$.
Then, as $x_{n_k} \Rightarrow x \in B(x, r/2)$, so $B(x, r/2)$ contains all the points of x_{n_k} and hence of x_n, except a finite number of points.

Let n_0 be the positive integer such that $1/n_0 < r/2$ and $x_{n_0} \in B(x, r/2)$.

$n_0 > \frac{2}{r}$

$1/n_0 < \frac{r}{2}$

We claim that $B_{n_0} \subseteq B(x,r) \subseteq G_0$

Let $y \in B_{n_0}$, Now by triangular inequality.

$d(y,x) < d(y, x_{n_0}) + d(x_{n_0}, x)$.

As $y, x_{n_0} \in B_{n_0} \Rightarrow d(y, x_{n_0}) < d(B_{n_0}) < 1/n_0 < r/2$.

$\Rightarrow d(y, x_0) < r/2$.

Also, as $x_{n_0} \in B(x, r/2) \Rightarrow d(x_{n_0}, x) < r/2$

So, $d(y,x) < d(y, x_{n_0}) + d(x_{n_0}, x)$.

$d(y,x) < r/2 + r/2$.

$\Rightarrow d(y,x) < r$.

$\Rightarrow y \in B(x,r)$

$\Rightarrow B_{n_0} \subseteq B(x,r) \subseteq G_{i_0}$

$\Rightarrow B_{n_0} \subseteq G_{i_0}$

$\Rightarrow B_{n_0}$ is not a big set.

Which is a contradiction.

$a' \neq 0$. Hence $0 < a' < \infty$.

Then, by (i) and (ii) Lebesgue number exists.

$x_{n_0} \in B_{n_0}$

$d(B_{n_0}) < 1/n_0 < r/2$

diameter $d(B_{n_0}) < r/2$

distance $d(x_{n_0}, x) < r/2$

Hence Proved.

DEFINITION: (ε-NET):

Let (X,d) be a metric space and $\varepsilon > 0$, a subset A of X is called ε-net if:

1) A is finite. 2) $X = \bigcup_{a \in A} B(a, \varepsilon)$.

DEFINITION: TOTALLY BOUNDED:

A metric space (X,d) is said to be

totally bounded if it has an ε-net for each $\varepsilon>0$.

THEOREM. Every sequentially compact metric space is totally bounded.

PROOF: Let (X,d) be a sequentially compact metric space.

To prove: X is totally bounded.

Let $\varepsilon>0$ and $a_1\in X$.

If $B(a_1,\varepsilon)=X$

Then, $\{a_1\}$ is an ε-net.

If $B(a_1,\varepsilon)\neq X$.

$\Rightarrow B(a_1,\varepsilon)\subsetneq X$.

Let $a_2\in X$ such that $a_2\notin B(a_1,\varepsilon)$

Now if $B(a_1,\varepsilon)\cup B(a_2,\varepsilon)=X$.

Then $\{a_1,a_2\}$ is an ε-net for X.

If $B(a_1,\varepsilon)\cup B(a_2,\varepsilon)\neq X$.

Then, $a_3\in X$ such that $a_3\notin B(a_1,\varepsilon)\cup B(a_2,\varepsilon)$.

Now, if $B(a_1,\varepsilon)\cup B(a_2,\varepsilon)\cup B(a_3,\varepsilon)=X$.

Then, $\{a_1,a_2,a_3\}$ is an ε-net for X.

If $B(a_1,\varepsilon)\cup B(a_2,\varepsilon)\cup B(a_3,\varepsilon)\neq X$

Then, continuing this way we have.

$B(a_1,\varepsilon)\cup B(a_2,\varepsilon)\cup\cdots\cup B(a_n,\varepsilon)=X$.

If not, then the sequence $\{a_1,a_2,a_3,\ldots,a_n,\ldots\}$ has no convergent subsequence.

$\Rightarrow X$ is not sequentially compact.

Which is a contradiction.

So our supposition is wrong.

And hence $\{a_1,a_2,\ldots,a_n\}$ is an ε-net.

$\Rightarrow X$ is totally bounded.

THEOREM: Every sequentially compact metric space is compact.

PROOF: Let X be a sequentially compact metric space. To prove: X is compact.

Let $\{G_i\}$ be an open cover for X. As X is sequentially compact so by lebesgue covery lemma $\{G_i\}$ has a lebesgue number 'a' put $\varepsilon = a/3 > 0$. As every sequentially compact metric space is totally bounded, then for $\varepsilon > 0$, it has an ε-net.

$A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$.

$\Rightarrow X = \bigcup_{k=1}^{n} B(a_k, \varepsilon)$, $a_k \in A$

Now $d(B(a_k, \varepsilon)) < 2\varepsilon < 2(a/3) < a$.

$\Rightarrow d(B(a_k, \varepsilon)) < a$

Then, by definition of Lebesgue number, $B(a_k, \varepsilon) \subseteq G_{ik}$ for some $G_{ik} \in \{G_i\}$.

$\Rightarrow \bigcup_{k=1}^{n} B(a_k, \varepsilon) \subseteq \bigcup_{k=1}^{n} G_{ik}$.

$\Rightarrow X = \bigcup_{k=1}^{n} B(a_k, \varepsilon) \subseteq \bigcup_{k=1}^{n} G_{ik} \subseteq X$.

$\Rightarrow X = \bigcup_{k=1}^{n} G_{ik}$

$\Rightarrow \{G_{i1}, G_{i2}, \ldots, G_{in}\}$ is finite subcover for X.

$\Rightarrow X$ is compact.

Real Analysis.

V-V-Imp

THEOREM: A continuous function from a compact metric space to a metric space is uniformly continuous.

PROOF: Let $f: X \to Y$ be a continuous mapping from a compact metric space X to a metric space Y.

To prove: f is uniformly continuous.

Let d_1 and d_2 be the metrices on X and Y respectively.

Let $x \in X$ and $\varepsilon > 0$. Now as $x \in X$
$\Rightarrow f(x) \in Y$.

Now consider the open ball $B(f(x), \varepsilon/2)$, which is an open set in Y.

As $f: X \to Y$ is continuous, so $f^{-1}(B(f(x), \varepsilon/2))$ is an open set in X. (∵ Inverse image of each open set is open)

Then, $\bigcup_{x \in X} f^{-1}(B(f(x), \varepsilon/2)) = X$.

$\Rightarrow \gamma = \{f^{-1}(B(f(x), \varepsilon/2)) : x \in X\}$ is an open cover for X.

As X is compact and every compact metric space is sequentially compact and as every sequentially compact metric space has a Lebesgue number. So, X has a Lebesgue number, say δ, for open cover γ.

Let $x_1, x_2 \in X$ such that $d_1(x_1, x_2) < \delta$.

Now as $d_1(x_1, x_2) < \delta$
$\Rightarrow d\{x_1, x_2\} < \delta$.

Then, by the definition of Lebesgue number $\{x_1, x_2\} \subseteq f^{-1}(B(f(x^*), \varepsilon/2))$

$$\Rightarrow f(\{x_1, x_2\}) \subseteq B(f(x^*), \varepsilon/2)$$

$$\Rightarrow \{f(x_1), f(x_2)\} \subseteq B(f(x^*), \varepsilon/2)$$

$$\Rightarrow f(x_1), f(x_2) \in B(f(x^*), \varepsilon/2)$$

$$\Rightarrow d_2(f(x_1), f(x^*)) < \varepsilon/2$$

$$d_2(f(x_2), f(x^*)) < \varepsilon/2$$

Now, $d_2(f(x_1), f(x_2)) \leq d_2(f(x_1), f(x^*)) + d_2(f(x^*), f(x_2))$

$$< \varepsilon/2 + \varepsilon/2$$

$$\Rightarrow d_2(f(x_1), f(x_2)) < \varepsilon$$

$\Rightarrow f$ is uniformly continuous.

THEOREM: Every sequentially compact metric space is countably compact.

PROOF: Let X be a sequentially compact metric space. To prove: X is countably compact.

Let $C = \{U_1, U_2, \ldots, U_n, \ldots\}$ be a countable open cover for X.

Suppose X is not countably compact. Then, this open cover has no finite subcover i.e., for every subcollection $\{U_1, U_2, U_3, \ldots, U_n\}$ of C,

$$\bigcup_{i=1}^{n} U_i \neq X$$

Then, there is $x_n \in X$ such that:

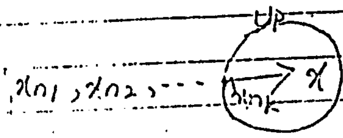
$$x_n \notin \bigcup_{i=1}^{n} U_i$$

So then, we have an infinite sequence $\{x_n\}$ in X.

But as X is sequentially compact. So, this sequence has a convergent subsequence $\{x_{n_k}\}$ and $x_{n_k} \to x \in X$.

Now as $X = \bigcup_{n=1}^{\infty} U_n$

$\Rightarrow x \in \bigcup_{n=1}^{\infty} U_n$



$\Rightarrow x \in U_p$, for some $U_p \in C$.

But, then for sufficiently large k, $x_{n_k} \in U_p$

Which is a contradiction.

So, our supposition is wrong.

Hence, X is countably compact.

DEFINITION: ε-net:

Let (X,d) be a metric space and $\varepsilon > 0$ be any real number and $M \subseteq X$. Then a finite subset A of X is said to be ε-net for M if for every $x \in M$ there is atleast one $a \in A$ such that $d(x,a) < \varepsilon$ i.e, $x \in B(a,\varepsilon)$. If $M = X$, then A is called ε-net for X.

THEOREM: A subset $A = \{a_1, a_2, \ldots, a_n\}$ of X is an ε-net for X iff $X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

PROOF: Suppose A is ε-net for X.

To prove: $X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

$\bigcup_{i=1}^{n} B(a_i, \varepsilon) \subseteq X$.

Let $x \in X$. Then, as A is an ε-net for X, so, there exists a_i, $1 \le i \le n$ in A such that $d(x, a_i) < \varepsilon$

$\Rightarrow x \in B(a_i, \varepsilon)$

$\Rightarrow x \in \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

As $x \in X$ is an arbitrary so,

$X \subseteq \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

$\Rightarrow X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

Conversely, suppose $X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

To prove: A is an ε-net for X:

Let $x \in X$

$\Rightarrow x \in \bigcup_{i=1}^{n} B(a_i, \varepsilon)$.

$\Rightarrow x \in B(a_i, \varepsilon)$, for some $a_i \in A$.

$\Rightarrow d(x, a_i) < \varepsilon$ for some $a_i \in A$

$\Rightarrow A$ is ε-net for X.

THEOREM: Every totally bounded metric space is bounded.

PROOF: Let (X, d) be a totally bounded metric space. To prove: X is bounded.

As X is totally bounded. Then, for $\varepsilon > 0$, X has ε-net.

$F = \{a_1, a_2, \dots, a_n\}$.
i.e, $X = \bigcup_{i=1}^{n} B(a_i, \varepsilon)$. So Then for every $x \in X$, there is $a_i \in F$ such that $d(a, x_i) < \varepsilon$.

Similarly, for $y \in X$, there is $a_j \in F$ such that $d(y, a_j) < \varepsilon$

Now $d(x,y) < d(x,a_i) + d(a_i,y)$
$< d(x,a_i) + d(a_i,a_j) + d(a_j,y)$
$< \varepsilon + d(a_i,a_j) + \varepsilon$
$\Rightarrow d(x,y) < d(a_i,a_j) + 2\varepsilon$.
$\sup d(x,y) < \sup d(a_i,a_j) + 2\varepsilon$
$\delta(X) < \delta(F) + 2\varepsilon$

As F is finite. So, diameter of F is finite.
So R.H.S is finite.
$\Rightarrow \delta(X)$ is finite
$\Rightarrow \delta(X)$ is bounded.

These Notes are the lectures delivered by

Tahir Mahmood

"CONNECTED SPACES"

DEFINITION: DISCONNECTED:

A topological space $(X, \mathcal{T})$ is said to be disconnected if there exists two non-empty open (or closed) sets A and B in X such that $A \cup B = X$ and $A \cap B = \phi$.

e.g. If $X = \{1, 2, 3, 4\}$, $\mathcal{T} = \{\phi, X, \{1, 3\}, \{2, 4\}\}$

Then, $(X, \mathcal{T})$ is disconnected because we have $A = \{1, 3\}$, $B = \{2, 4\}$. A and B are open, $A \cup B = X$, $A \cap B = \phi$.

DEFINITION: CONNECTED SPACES:

A topological space $(X, \mathcal{T})$ is said to be connected if it is not disconnected.

e.g. If $X = \{1, 2, 3\}$, $\mathcal{T} = \{\phi, X, \{2\}\}$. Then $(X, \mathcal{T})$ is connected.

REMARK: For any set X if $\mathcal{T} = \mathcal{T}_I$, then $(X, \mathcal{T})$ is connected.

THEOREM: For any X with more than two points $(X, \mathcal{T}_D)$ is disconnected.

PROOF: Since $\mathcal{T} = \mathcal{T}_D$, so every subset of X is open (as well as closed).

Proper subset Let $A \subsetneq X$. Then $A' = X \setminus A \neq \phi$.

Also A' is open ($\because A$ is also closed)

So we have two open sets A and $B = A'$ such that $A \cup B = X$ and $A \cap B = \phi$.

So $(X, \mathcal{T}_D)$ is disconnected.

Hence proved.

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THEOREM: If X is infinite. Then $(X, \mathcal{F}_c)$ is connected.

PROOF: On the contrary suppose $(X, \mathcal{F}_c)$ is disconnected. Then, there exists two open sets (or closed sets) A and B such that $A \cup B = X$ and $A \cap B = \phi$

Now as A and B are open and $\mathcal{F} = \mathcal{F}_c$.
So A' and B' are finite. Now $A \cap B = \phi$
$\Rightarrow (A \cap B)' = \phi'$

$\Rightarrow A' \cup B' = X$.
$\Rightarrow X$ is finite ($\because$ Union of two finite sets is finite).
Which is a contradiction.
$\because X$ is infinite.
So, our supposition is wrong.
Hence $(X, \mathcal{F}_c)$ is connected.

THEOREM: Continuous image of a connected space is connected.

PROOF: Let X be a connected space and $f: X \to Y$ be a continuous function.
To prove: $f(X)$ is connected.
Suppose on the contrary that $f(X)$ is disconnected. Then, there exists two non-empty open sets A and B in $f(X)$ such that $A \cup B = f(X)$ and $A \cap B = \phi$.

Now $A \cup B = f(X)$
Now as A and B are open in $f(X)$ and f is continuous function.

So, $f^{-1}(A)$ and $f^{-1}(B)$ are open in X.
Further $f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(f(X)) = X$
And, $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$

$\Rightarrow X$ is disconnected.
Which is a contradiction.
$\because X$ is connected.
So, our supposition is wrong.
Hence, $f(X)$ is connected.

THEOREM: The space $\mathbb{Q}$ as subspace of $\mathbb{R}$ is disconnected.

PROOF: Let r be any irrational number.
Then, $]-\infty\ r[\,,\ ]r\ \infty[$ are open in $\mathbb{R}$.

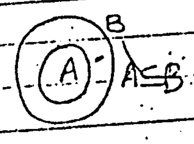
$\Rightarrow\ ]-\infty\ r[\ \cap \mathbb{Q}\ ,\ ]r\ \infty[\ \cap \mathbb{Q}$ are open sets in $\mathbb{Q}$ with $(]-\infty\ r[\ \cap \mathbb{Q}) \cup (]r\ \infty[\ \cap \mathbb{Q})$

$= (]-\infty\ r[\ \cup\ ]r\ \infty[) \cap \mathbb{Q}$ (By distributive law)

$= (\mathbb{R} \setminus \{r\}) \cap \mathbb{Q}$

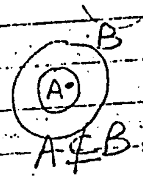
$A \subseteq B \cdot A \cap B = A$

$= \mathbb{Q}$



And $(]-\infty\ r[\ \cap \mathbb{Q}) \cap (]r\ \infty[\ \cap \mathbb{Q})$
$= (]-\infty\ r[\ \cap\ ]r\ \infty[) \cap \mathbb{Q}$

$= \phi \cap \mathbb{Q} = \phi$



$\Rightarrow \mathbb{Q}$ is disconnected.

THEOREM: A topological space X is disconnected if and only if X contains a non-empty subset A which is both open and closed.

PROOF: Suppose X is disconnected and $A \neq \phi$ be a subset of X.

To prove: A is both open and closed.

As X is disconnected then there exists two open sets A and B such that $A \cup B = X$ and $A \cap B = \phi$.

Now A is open.

Now as $A \cup B = X$ and $A \cap B = \phi$. So, by law of complements $B = A'$.

Also B is open $\Rightarrow A'$ is open $\Rightarrow A$ is closed.

$\Rightarrow A$ is both open and closed.

Conversely, suppose that for topological space X, there is non-empty subset A of X which is both open and closed.

To prove: X is disconnected.

Let $B = A'$. So A is closed.

$\Rightarrow A'$ is open $\Rightarrow B$ is open.

$\Rightarrow A$ and B are open in X with $A \cup B = A \cup A' = X$

And $A \cap B = A \cap A' = \phi$.

$\Rightarrow X$ is disconnected.

THEOREM: A space X is connected if and only if there does not exists a continuous surjective function from X to discrete Two point space.

PROOF. Let X be connected:
To prove: There does not exists a continuous surjective function from X to discrete Two point space $Y = \{a, b\}$.

Suppose on the contrary that there exists a function $f: X \to Y = \{a, b\}$; Y is discrete, which is continuous and onto. As Y is discrete so $\phi, \{a\}, \{b\}, \{a, b\}$ are open sets.

As f is continuous so $f^{-1}(\phi), f^{-1}(\{a\}), f^{-1}(\{b\})$ and $f^{-1}(\{a, b\})$ all are open in X.

Now as f is onto. So $f(X) = Y \Rightarrow X = f^{-1}(Y)$

$$\Rightarrow X = f^{-1}(\{a, b\})$$
$$= f^{-1}(\{a\} \cup \{b\})$$
$$= f^{-1}(\{a\}) \cup f^{-1}(\{b\}).$$

Further $f^{-1}(\{a\}) \cap f^{-1}(\{b\})$
$= f^{-1}(\{a\} \cap \{b\}) = f^{-1}(\phi) = \phi$
$\Rightarrow X$ is disconnected.
Which is a contradiction.
$\because X$ is connected.
So, our supposition is wrong.
Hence, There does not exist a continuous function from X onto discrete Two point space Y.

Conversely suppose there doesnot exist a continuous function from X onto discrete two point space Y.

To prove: X is connected.

Suppose on the contrary that X is not connected. Then X is disconnected. So, there exists two non-empty open (or closed) sets A and B in X such that $A \cup B = X$ and $A \cap B = \phi$.

Now define a function $f: X \to Y = \{a, b\}$ by $f(A) = a$ and $f(B) = b$

$$\Rightarrow A = f^{-1}(\{a\}) \text{ , } B = f^{-1}(\{b\})$$

Now as $Y = \{a, b\}$ is discrete so, $\phi, \{a\}, \{b\}, \{a, b\}$ are open in Y.

$$\text{Now } f^{-1}(\phi) = f^{-1}(\{a\} \cap \{b\})$$
$$= f^{-1}(\{a\}) \cap f^{-1}(\{b\}) = A \cap B = \phi$$

$$f^{-1}(Y) = f^{-1}(\{a\} \cup \{b\})$$
$$= f^{-1}(\{a\}) \cup f^{-1}(\{b\})$$
$$= A \cup B = X$$

So f is continuous. ($\because$ Inverse image of each open set is open and here all four open sets of Y have open inverse images).

Which is a contradiction.

So our supposition is wrong.
Hence X is connected.
Hence Proved

THEOREM: A topological space X is disconnected iff there exist a continuous function from X onto discrete two points space.

THEOREM: A topological space X is said to be connected iff every continuous function from X to discrete space Y reduces to a constant function.

PROOF: Suppose X is connected.
Then, X has no its proper subset which is both open and closed.
Let $a \in Y$. As Y is discrete. So $\{a\}$ is both open and closed.

$\Rightarrow f^{-1}(\{a\})$ is open and closed in X.
$\Rightarrow f^{-1}(\{a\})$ is not a proper subset of X.
$\Rightarrow f^{-1}(\{a\}) = \phi$ or $f^{-1}(\{a\}) = X$.

But $f^{-1}(\{a\}) \neq \phi$.
So, $f^{-1}(\{a\}) = X \Rightarrow f(X) = \{a\}$.
$\Rightarrow f$ is constant function.

Conversely suppose every continuous function f from X to discrete space Y reduces to a constant function.

To prove: X is connected.
Suppose, X is disconnected. Then, There exists a continuous function $f: X \to Y = \{a, b\}$. Which is continuous and is onto and Y is discrete $\Rightarrow f(X) = Y \Rightarrow f$ is not constant.

Which is a contradiction.
So, our supposition is wrong.
And hence, X is connected.

THEOREM:

Let X be disconnected space with disconnection $\{A, B\}$ and C is a connected subspace of X. Then, either $C \subseteq A$ or $C \subseteq B$.

PROOF:

Suppose on the contrary that $C \nsubseteq A$ and $C \nsubseteq B$. Then, $C \cap A$ and $C \cap B$ are non empty open sets in C, ($\because C$ is subspace so $C \cap A$ is open in C)

with $(C \cap A) \cup (C \cap B) = C \cap (A \cup B)$
$= C \cap X = C$

and $(C \cap A) \cap (C \cap B) = C \cap (A \cap B)$.
$= C \cap \phi = \phi$

$\Rightarrow C$ is disconnected.
A contradiction.
$\because C$ is connected.
So our supposition is wrong.
Hence, $C \subseteq A$ or $C \subseteq B$.

THEOREM:

Let $X = \bigcup X_\alpha$ where each X_α is connected and $\bigcap\limits_{\alpha \in I} X_\alpha \neq \phi$. Then, X is connected.

PROOF: Suppose X is disconnected. Then there exists two non-empty open sets A and B in X such that $A \cup B = X$ and $A \cap B = \phi$.

Now as $X = \bigcup\limits_{\alpha \in I} X_\alpha \Rightarrow$ for each $\alpha \in I$, $X_\alpha \subseteq X$.

As for each $\alpha \in I$, X_α is connected. So, by well known theorem, for each $\alpha \in I$,

either $X_\alpha \subseteq A$ or $X_\alpha \subseteq B$

But as $\bigcap_{\alpha \in I} X_\alpha \neq \phi$,

So $\bigcup_{\alpha \in I} X_\alpha \subseteq A$ or $\bigcup_{\alpha \in I} X_\alpha \subseteq B$.

$\Rightarrow X \subseteq A$ or $X \subseteq B$.

If $X \subseteq A \Rightarrow A = X \Rightarrow B = \phi$.

If $X \subseteq B \Rightarrow B = X \Rightarrow A = \phi$.

Which is a contradiction

$\because$ Both A and B are non empty.

So, our supposition is wrong.

And, hence X is connected.

THEOREM: A topological space X is connected iff for every pair of points in X there is some connected subspace of X which contains both.

PROOF: Suppose X is connected and $x, y \in X$ such that $x \neq y$. (To prove: there is some connected subspace of X which contains both x and y.) Then X itself is the connected subspace of X which contains both x and y.

Conversely, suppose in a topological space X, for every pair of points $x, y \in X$ such that $x \neq y$, there is some connected subspace of X which contains both x and y.

To prove: X is connected:

Now let, $a \in X$ be some fixed point such that for $x \in X$, $a \neq x$. Then, by the hypothesis, there is a connected subspace $C_{a,x}$ of X such that $a, x \in C_{a,x}$.

Then we have a collection $\{C_{a,x} : x \in X\}$ of connected subspace of X such that,

$$\bigcap_{x \in X} C_{a,x} \neq \phi \text{ and } \bigcup_{x \in X} C_{a,x} = X$$

Then, by a well known theorem, X is connected.

THEOREM: Let C be a connected subspace of X and for some subset A of X, $C \subseteq A \subseteq \overline{C}$. Then, A is connected: In particular $\overline{C}$ is connected.

Proof: Suppose on the contrary that A is disconnected. Then, there exists two non-empty open sets U_1 and V_1 of A such that:

$$U_1 \cup V_1 = A \text{ and } U_1 \cap V_1 = \phi.$$

As U_1 and V_1 are open in A and A is subspace of X. So, then there exists two disjoint open sets U and V in X such that:

$$U_1 = U \cap A \text{ and } V_1 = V \cap A.$$

Now, $C \subseteq A = U_1 \cup V_1 \subseteq U \cup V$

$\Rightarrow C \subseteq U \cup V$ and C is connected and $U \cap V = \phi$. Then, by a well known theorem either $C \subseteq U$ or $C \subseteq V$.

With out any loss of generality, suppose

$C \subseteq U$.

As $U \cap V = \phi \Rightarrow U \subseteq V'$

$\Rightarrow C \subseteq U \subseteq V' \Rightarrow C \subseteq V'$.

As V is open $\Rightarrow V'$ is closed.

So V' is the closed superset of C.

But $\overline{C}$ is the smallest closed superset of C. So $\overline{C} \subseteq V'$.

$\Rightarrow C \subseteq A \subseteq \overline{C}$ (given)

$\Rightarrow C \subseteq A \subseteq \overline{C} \subseteq V'$

$\Rightarrow A \subseteq V' \Rightarrow A \cap V = \phi \Rightarrow V_1 = \phi$

Which is a contradiction.

$\therefore V_1 \neq \phi$.

So, our supposition is wrong.

Hence, A is connected.

Now to prove $\overline{C}$ is connected.

As $C \subseteq \overline{C} \subseteq \overline{C}$ So, by the above argument $\overline{C}$ is connected.

THEOREM: A subspace X of a real line R is connected if and only if X is an interval.

PROOF: Suppose, X is connected. To prove: X is an interval. Suppose X is not an interval, then there exists x, y, z such that $x < y < z$ and $x, z \in X$ but $y \notin X$.

Now $]-\infty, y[$ and $]y\ \infty[$ are open in R

$\Rightarrow]-\infty, y[\cap X$ and $]y\ \infty[\cap X$ are open in X with

$$(]-\infty\ y[\cap X) \cup (]y\ \infty[\cap X) = (]-\infty\ y[\cup]y\ \infty[) \cap X$$
$$= (R \setminus \{y\}) \cap X = X.$$

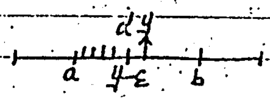
and $(]-\infty, y[\cap X) \cap (]y, \infty[\cap X)$
$= (]-\infty, y[\cap]y, \infty[) \cap X$
$= \phi \cap X = \phi$

$\Rightarrow X$ is disconnected.
Which is a contradiction
$\because X$ is connected.
So, our supposition is wrong.
Hence, X is an interval.

Conversely suppose X is an interval
To prove: X is connected. On the contrary suppose X is disconnected. Then there exists two non empty open disjoint subsets A and B of X such that $A \cup B = X$ and $A \cap B = \phi$.

Let $a \in A$ and $b \in B$.
As $A \cap B = \phi \Rightarrow a \neq b$.
Let $a < b$. Put $y = \sup([a, b] \cap A)$.
Then, by the definition of supremum for every $\varepsilon > 0$, there is some point a' in A such that $y - \varepsilon < a'$

$\Rightarrow y - a' < \varepsilon$



$\Rightarrow d(y, a') < \varepsilon$
$\Rightarrow a' \in B(y, \varepsilon)$
So, every open ball with centre at y contains a point of A different from y.
$\Rightarrow y$ is the limit point of A.

As A is also closed.
So, $y \in A$. Similarly $y \in B \Rightarrow A \cap B \neq \phi$.

Which is a contradiction.
$\because A \cap B = \phi$.
So, our supposition is wrong.
Hence, X is connected.

COMPONENT: (DEF).

The maximal connected subspace of topological space X is called component of X. i.e, a connected subspace of topological space X is called component of X if it is not contained in any other connected subspace of X.

THEOREM: Let X be a topological space, then:

i) Each $x \in X$ is contained in exactly one component of X.

ii) Each connected subspace of X is contained in exactly one component of X.

iii) Each connected subspace of X which is both open and closed is component of X.

iv) Every component of X is closed in X.

PROOF: i) Let $\gamma = \{C_\alpha : \alpha \in I$ and $x \in C_\alpha\}$ be a collection of all connected subspace of X which contains x.

Then, $\bigcap_{\alpha \in I} C_\alpha \neq \phi$.

Then, by a well known theorem,
$C = \bigcup \gamma = \bigcup_{\alpha \in I} C_\alpha$ is connected

subspace of X and $x \in C$ and for every $\alpha \in I$, $C_\alpha \subseteq C$. This shows that C is component of X. Now we show that C is the only component of X containing x. On the contrary let C^* be another component of X containing x.

Now as, C^* is the component of X containing x and C is connected subspace of X. So, $C \subseteq C^*$. Also as C^* is connected subspace of X containing x. So $C^* \in \gamma$

$\Rightarrow C^* \subseteq \cup\gamma = C$

$\Rightarrow C^* \subseteq C \Rightarrow C = C^*$

This shows that C is the only component of X containing x.

ii) Let A be a connected subspace of X and to prove: A is contained only in one component of X. Let $\{C_\alpha : \alpha \in I\}$ be a collection of all connected subspaces of X containing A. Then, $\bigcap_{\alpha \in I} C_\alpha \neq \phi$ and $\bigcup_{\alpha \in I} C_\alpha = C$, which is connected subspace of X. Also, $A \subseteq C$.

$\Rightarrow C$ is connected subspace of X containing A. Also $C = \bigcup_{\alpha \in I} C_\alpha$ so C is such maximal connected subspace of X.

$\Rightarrow C$ is component of X containing A.

Now we show C is the only component of X containing A.

For this, let C^* be another component of X containing A. Now, as C^* is maximal connected subspace of X containing A and C is connected subspace of X containing A so $C \subseteq C^*$. Further also as C^* is connected subspace of X containing A. So, $C^* \in \{C_\alpha : \alpha \in I\}$

$$\Rightarrow C^* \subseteq \bigcup_{\alpha \in I} C_\alpha = C$$

$$\Rightarrow C^* \subseteq C \Rightarrow C = C^*$$

Hence, C is only component of X containing A.

iii) Let A be a connected subspace of X which is both open and closed.

To prove: A is component of X.

Suppose, A is not component of X. Then, A is contained in exactly one component of X, say C. As C is component of X, $A \subseteq C$ and A is not component of X.

$\Rightarrow$ A is proper subset of C.

Then $A \cap C$ and $A' \cap C$ are both non-empty.

Now, as A is open in X, so $A \cap C$ is open in C. Also, as A is closed in $X \Rightarrow A'$ is open in X.

$\Rightarrow A' \cap C$ is open in $C \Rightarrow A \cap C$ and $A' \cap C$ are ~~with~~ open in C with:

$$(A \cap C) \cap (A' \cap C) = (A \cap A') \cap C = \phi \cap C = \phi$$

And $$(A \cap C) \cup (A' \cap C) = (A \cup A') \cap C = X \cap C = C$$

$\Rightarrow$ C is disconnected.

Which is a contradiction.

$\because$ C is component of X.

So our supposition is wrong.

Hence, A is component of X.

iv). Let C be a component of X.

To prove: C is closed.

For this, we prove $C = \bar{C}$.

Suppose $C \neq \bar{C}$.

Now as $C \subseteq \bar{C}$ and $C \neq \bar{C} \Rightarrow C \subset \bar{C}$.

Now as C is connected, then by a well known theorem, $\bar{C}$ is connected $\Rightarrow \bar{C}$ is connected subspace of X containing $C \Rightarrow C$ is not component of X.

A contradiction.

$\because C$ is component of X.

So our supposition is wrong and hence $C = \bar{C}$.

$\Rightarrow C$ is closed.

TOTALLY DISCONNECTED: (DEF).

A topological space X is called totally disconnected if for each pair of points $x, y \in X$ we can form a disconnection $\{A, B\}$ of X such that $x \in A$ and $y \in B$.

THEOREM: Every discrete space is totally disconnected.

PROOF: Let X be a discrete space.

To prove: X is totally disconnected.

Let $x, y \in X$ such that $x \neq y$.

Let $U = \{x\}$ and $V = X - \{x\}$.

As X is discrete so U and V are open in X. Also clearly,

$x \in X$, $y \in V$, $U \cap V = \phi$, $U \cup V = X$.

$\Rightarrow X$ is totally disconnected.

THEOREM: Every totally disconnected is T_2 space.

PROOF: Let X be a totally disconnected.
To prove: X is T_2 space.
Let $x, y \in X$ such that $x \neq y$.
As X is totally disconnected so then there exist two open sets U and V in X such that $x \in U$, $y \in V$, $U \cap V = \phi$ and $U \cup V = X$.
ie, we have two open sets U and V in X such that $x \in U$; $y \in V$ and $U \cap V = \phi$.
$\Rightarrow X$ is T_2 space.

THEOREM: A subspace Q of rationals in real line R is totally disconnected.

PROOF: To prove: Q is totally disconnected in R.
Let $r_1, r_2 \in Q$ such that $r_1 \neq r_2$. Without any loss of generality, suppose $r_1 < r_2$.

Now as by a well known theorem of calculus here is an irrational number between every two rational numbers. There is an irrational number 't' such that $r_1 < t < r_2$.
Now $]-\infty\ t[$ and $]t\ \infty[$ are two open sets in R. Now as Q is subspace of R. So,
$U = Q \cap]-\infty\ t[$ and $V = Q \cap]t\ \infty[$ are open in Q.
Also $r_1 \in U$, $r_2 \in V$, $U \cap V = \phi$, $U \cup V = Q$.

$\Rightarrow Q$ is totally disconnected.

Hence Proved.

THEOREM: The components of totally disconnected space are its singleton subsets.

Proof: Let X be a totally disconnected space.
To prove: Components of X are its singleton subset.
For this, we show that no two points subspace of X is connected.

Let $x, y \in X$ such that $x \neq y$ and $C = \{x, y\}$ be a subspace of X. As X is totally disconnected and $x, y \in X$ such that $x \neq y$, so then there exist two open set U and V in X such that:

$x \in U$, $y \in V$, $U \cup V = X$, $U \cap V = \phi$.

Now as U and V are open in X and C is subspace of X so,

$U \cap C$ and $V \cap C$ are open in C.

Also, $x \in U$ and $x \in C \Rightarrow x \in U \cap C$.

$y \in V$ and $y \in C \Rightarrow y \in V \cap C$.

$$(U \cap C) \cup (V \cap C) = (U \cup V) \cap C$$
$$= X \cap C = C$$
$$(U \cap C) \cap (V \cap C) = (U \cap V) \cap C$$
$$= \phi \cap C = \phi$$

$\Rightarrow C$ is disconnected.

HENCE PROVED.

THEOREM: If a T_2-space has an open base whose sets are also closed. Then, X is totally disconnected.

PROOF:

Let $x, y \in X$ such that $x \neq y$. As X is T_2 space. So then, there exists two open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

Let $\mathcal{B}$ be an open base for X whose elements are also closed. As $x \in U$, U is an open set, $\mathcal{B}$ is base, so then there is $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

Now as $B \subseteq U$ and $U \cap V = \phi$

So $B \cap V = \phi \Rightarrow V \subseteq B' = W$

As $B \in \mathcal{B}$, so B is closed.

$\Rightarrow B'$ is open. Now B and W are two open sets in X with $x \in B$. As $y \notin B$

$\Rightarrow y \in B' = W \Rightarrow y \in W$.

Also $B \cap W = B \cap B' = \phi$

$B \cup W = B \cup B' = X$.

$\Rightarrow X$ is totally disconnected.

THEOREM: Let X be a compact housdorff space then X is totally disconnected iff it has an open base whose sets are also closed.

Proof: Let the compact T_2 space has an open base whose sets are closed. To prove: X is totally disconnected.

Let $x, y \in X$ such that $x \neq y$. As X is T_2 so then there is an open set U such that $x \in U$ and $y \notin U$. Now as $x \in U$ and U is open in X with X has base $\mathcal{B}$. Then, there is $G \in \mathcal{B}$ such that $x \in G \subseteq U$.

As $y \notin U$ and $G \subseteq U \Rightarrow y \notin G \Rightarrow y \in G'$

As $G \in \mathcal{B}$. So G is also closed.

$\Rightarrow G'$ is open. Put $G' = H$.

Hence, we have two open sets G and H in X such that: $x \in G$, $y \in H$

$G \cup H = G \cup G' = X$ and $G \cap H = G \cap G' = \phi$.

$\Rightarrow X$ is totally disconnected.

Conversely, suppose X is totally disconnected. (Where X is also compact and T_2).
To prove: X has an open base whose sets are also closed.
Let $\mathcal{B}$ be an open base for X.
To prove: elements of $\mathcal{B}$ are also closed.
Let $x \in X$ and G be an open set in X such that $x \in G$.

Case I: If $G = X$, then $B_x = X \in \mathcal{B}$ such that $x \in B_x \subseteq G$. Clearly B_x is both open and closed.

Case II: If $G \neq X \Rightarrow G \subset X$. Now as G is an open set so G' is closed. As $x \in G$ so $x \notin G'$. Now as, G' is closed subspace of X and X is compact. So G' is also compact ($\because$ Closed subspace of a compact space is compact). As X is totally disconnected so $\forall\ y \in G'$ such that $x \neq y$, there is subset H_y of X which is both open and closed, such that: $y \in H_y$ and $x \notin H_y$. Then, the set $\{H_y : y \in X\}$ is an open cover for G'.

As G' is compact, so this open cover has a finite subcover $\{H_1, H_2, \ldots, H_n\}$
$$\Rightarrow G' = \bigcup_{i=1}^{n} H_i = H \Rightarrow G' \subseteq H$$

Clearly H is both open and closed.
Further as $x \notin H_y \Rightarrow x \notin \bigcup_{i=1}^{n} H_i = H \Rightarrow x \notin H$.
$\Rightarrow x \in H' = B_x$.

Here B_x is both open and closed.

Now let $z \in B_x$.

$\Rightarrow z \in H' \Rightarrow z \notin H \Rightarrow z \notin G' \Rightarrow z \in G$

$\Rightarrow B_x \subseteq G \Rightarrow x \in B_x \subseteq G$.

$\therefore$ The collection of all such B_x form an open base whose elements are also closed.

HENCE PROVED.

THEOREM: Let X and Y be two topological space, then a function $f: X \to Y$ is continuous iff for every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.

PROOF: Given: f is continuous and $A \subseteq X$.

To prove: $f(\overline{A}) \subseteq \overline{f(A)}$.

As $f(A) \subseteq \overline{f(A)} \Rightarrow A \subseteq f^{-1}(\overline{f(A)})$.

As f is continuous and $\overline{f(A)}$ is closed in Y. Then, $f^{-1}(\overline{f(A)})$ is closed in X. ($\because f$ is continuous iff inverse image of each closed set is closed).

$\Rightarrow A \subseteq f^{-1}(\overline{f(A)})$ and $f^{-1}(\overline{f(A)})$ is closed subset of X.

$\Rightarrow f^{-1}(\overline{f(A)})$ is the closed superset of A.

But $\overline{A}$ is the smallest closed superset of A.

So, $\overline{A} \subseteq f^{-1}(\overline{f(A)})$.

$\Rightarrow f(\overline{A}) \subseteq \overline{f(A)}$.

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Conversely, suppose for any subset A of X;

$f(\bar{A}) \subseteq \overline{f(A)}$.

To prove: f is continuous.

Let C be a closed set in Y.

To prove: f is continuous, we have to prove $f^{-1}(C)$ is closed in X.

Let $A = f^{-1}(C)$

$\bar{A} = \overline{f^{-1}(C)} \Rightarrow f(\bar{A}) = f(\overline{f^{-1}(C)})$

$\subseteq \overline{f(f^{-1}(C))}$ (By given condition).

$f(\bar{A}) \subseteq \overline{(ff^{-1})(C)} = \overline{I(C)}$

$= \bar{C} = C$ ($\because C$ is closed).

$\Rightarrow f(\bar{A}) \subseteq C \Rightarrow f^{-1}(f(\bar{A})) \subseteq f^{-1}(C)$

$\Rightarrow \bar{A} \subseteq A$ ($\because f^{-1}(C) = A$).

But $A \subseteq \bar{A}$

$\Rightarrow A = \bar{A} \Rightarrow A$ is closed in X.

$\Rightarrow f^{-1}(C)$ is closed in X.

$\Rightarrow f$ is continuous.

DEFINITION:

Let X be a topological space and A and B are subsets of X. Then, A and B are said to be separated if and only if $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$.

THEOREM: Let X be a topological space and A, B are the subsets of X if A and B are separated in X then $A \cup B$ is disconnected.

PROOF: Let $Y = A \cup B$.

Now as, A and B are separated in X,

So, $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$.

Now let $G = \bar{B}'$ and $H = \bar{A}'$

Then as $\bar{A}$ and $\bar{B}$ are closed.

$\Rightarrow \bar{A}'$ and $\bar{B}'$ are open.

$\Rightarrow H$ and G are open in X

$\Rightarrow Y \cap G$ and $Y \cap H$ are open in Y ($\because Y$ is subspace).

Now $A \cap \bar{B} = \phi \Rightarrow A \subseteq \bar{B}' \Rightarrow A \subseteq G$.

Further as, $B \subseteq \bar{B} \Rightarrow B \cap \bar{B}' = \phi$.

Now, $Y \cap G = (A \cup B) \cap G = (A \cap G) \cup (B \cap G)$.

$\Rightarrow Y \cap G = A \cup (B \cap \bar{B}')$

$= A \cup \phi = A$.

Similarly, $Y \cap H = B$.

Now, $Y \cap G$ and $Y \cap H$ are open in Y with $(Y \cap G) \cup (Y \cap H) = A \cup B = Y$.

and $(Y \cap G) \cap (Y \cap H) = A \cap B = \phi$.

$\Rightarrow Y$ is disconnected.

THEOREM: Let G and H be the disconnection of a subset A of a topological space X then, show that $A \cap G$ and $A \cap H$ are separated.

PROOF: To prove: $A\cap G$ and $A\cap H$ are separated.

i.e, $(A\cap G)\cap(\overline{A\cap H})=\phi$ and $(\overline{A\cap G})\cap(A\cap H)=\phi$.

First we prove, if $x\in D(A\cap G)$ then; $x\notin A\cap H$.

Suppose on the contrary; $x\in D(A\cap G)$

$\Rightarrow x\in A\cap H$

$\Rightarrow x\in A$ and $x\in H$.

Now, $A\cap G\subseteq G$ and $A\cap G\subseteq A$.

$\Rightarrow D(A\cap G)\subseteq D(G)$ and $D(A\cap G)\subseteq D(A)$

So $x\in D(A\cap G)\Rightarrow x\in D(G)\Rightarrow x\in G$
($\because G$ is closed)

Now $x\in H$ and $x\in G\Rightarrow x\in G\cap H$

$\Rightarrow G\cap H\neq\phi$.

Which is a contradiction.

So, our supposition is wrong.

Hence for $x\in D(A\cap G)\Rightarrow x\notin A\cap H$.

$\Rightarrow D(A\cap G)\cap(A\cap H)=\phi$

Also, $(A\cap G)\cap(A\cap H)=A\cap(G\cap H)$
$=A\cap\phi=\phi$.

$\Rightarrow [D(A\cap G)\cap(A\cap H)]\cup[(A\cap G)\cap(A\cap H)]=\phi$.

$\Rightarrow [D(A\cap G)\cup(A\cap G)]\cap(A\cap H)=\phi$. $((X\cap Z)\cup(Y\cap Z)=(X\cup Y)\cap Z)$.

$\Rightarrow (\overline{A\cap G})\cap(A\cap H)=\phi$.

Similarly, $(A\cap G)\cap(\overline{A\cap H})=\phi$.

$\Rightarrow (A\cap G)$ and $(A\cap H)$ are separated.

THEOREM: Show that a topological space X is connected if and only if every non empty proper subspace has a non-empty boundary.

PROOF: We know that:

i) A topological space X is disconnected iff it has a subset A which is both open and closed.

ii) If $(X, \mathcal{T})$ is topological space and $A \subseteq X$ then boundary of A is empty iff A is both open and closed.

Now given X is connected and A is non empty proper subspace of X.

To prove: boundary of A is non-empty.

Suppose, boundary of A is empty.

i.e, $b(A) = \phi$ then, by (ii), A is both open and closed, but by (i) X is disconnected.

Which is a contradiction.

So our supposition is wrong.

And hence, $b(A) \neq \phi$.

Conversely suppose in a topological space X every non empty proper subset of X has non-empty boundary.

To prove: X is connected.

Suppose X is disconnected then by (i), there is subset A of X, which is both open and closed.

then by (ii), $b(A) = \phi$.
A contradiction.
$\therefore b(A) \neq \phi$.
So, our supposition is wrong.
And Hence, X is connected.

THEOREM: If X and Y are connected topological spaces then, $X \times Y$ is also connected.

PROOF: Let $x \in X$ and $y \in Y$.
Then, $\{x\} \times Y$ and $X \times \{y\}$ are two topological spaces with $\{x\} \times Y \cong Y$ and $X \times \{y\} \cong X$.

$\Rightarrow$ As X and Y are connected.
So $\{x\} \times Y$ and $X \times \{y\}$ are connected for all $x \in X$ and $y \in Y$.

Also $(x,y) \in (\{x\} \times Y) \cap (X \times \{y\})$.
$\Rightarrow (\{x\} \times Y) \cap (X \times \{y\}) \neq \phi$.

$\Rightarrow (\{x\} \times Y) \cup (X \times \{y\})$ is connected.
($\because$ The union of T.S is connected provided their intersection $\neq \phi$)

Furthermore,
$$\bigcap_{x \in X} T_x \neq \phi \text{ and } \bigcup_{x \in X} T_x = X \times Y.$$
where $T_x = [(\{x\} \times Y) \cup (X \times \{y\})]$

$\Rightarrow X \times Y$ is connected.

"Section II"

Normed Spaces.

Definition: Normed Spaces:

Let X be a vector space (Linear space) over field F. Then, a mapping $\|\cdot\|: X \to F$ is said to be norm on X if:

1) $\|x\| \geq 0 \quad \forall x \in X$.
2) $\|x\| = 0$ iff $x = 0$
3) $\|\alpha x\| = |\alpha| \, \|x\| \quad \forall \alpha \in F$ and $x \in X$.
4) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$.

Then, the pair $(X, \|\cdot\|)$ is called normed space.

Examples:

1. If $X = \mathbb{R}$, $\|\cdot\|: \mathbb{R} \to \mathbb{R}$ defined by $\|x\| = |x|$.
2. If $X = \mathbb{C}$, $\|\cdot\|: \mathbb{C} \to \mathbb{R}$ defined by $\|z\| = |z|$
3. If $X = \mathbb{R}^n$, $\|\cdot\|: X \to \mathbb{R}$ defined by $\|x\| = \left[\sum_{i=1}^{n} (x_i)^2\right]^{1/2}$
4. If $X = \mathbb{C}^n$, $\|\cdot\|: X \to \mathbb{R}$ defined by $\|z\| = \left[\sum_{i=1}^{n} |z_i|^2\right]^{1/2}$
5. If $X = l^\infty$, then $\|x\| = \sup_{n=1}^{\infty} |x_n|$.
6. If $X = l^p$, then $\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p}$

Theorem: Every normed space is metric space:

Proof: Let $(X, \|\cdot\|)$ be the normed space.
To prove: It is also a metric space.
Let $d: X \times X \to \mathbb{R}$ be defined by:

$d(x,y) = \|x-y\|$, $\forall\, x,y \in X$ then:

i) As $\|x-y\| \geq 0 \Rightarrow d(x,y) \geq 0$.

ii) $d(x,y) = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$

So $d(x,y) = 0$ iff $x = y$.

iii)
$$
\begin{aligned}
d(x,y) &= \|x-y\| \\
&= \|(-1)(y-x)\| \\
&= |-1|\, \|y-x\| \\
&= \|y-x\| \\
&= d(y,x)
\end{aligned}
$$
$\Rightarrow d(x,y) = d(y,x)$.

iv) Let $x,y,z \in X$:
$$
\begin{aligned}
d(x,y) = \|x-y\| &= \|(x-z)+(z-y)\| \\
&\leq \|x-z\| + \|z-y\|
\end{aligned}
$$
$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$.

Since, all the conditions are satisfied
So, (X,d) is metric space.

REMARK: Converse of the above theorem is not true in general i.e, a metric space is not necessarily a normed space.

Example: Let $X \neq \phi$ and define $d: X \times X \to R$ by
$$
d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}
$$
Then (X,d) be metric space (Discrete metric) but it is not a normed space.

THEOREM: Show that a metric 'd' induced by norm on X satisfies the following:

i) $d(x+a, y+a) = d(x, y)$.

ii) $d(\alpha x, \alpha y) = |\alpha|\, d(x, y)$ $\forall\, x, y \in X$ and $\alpha \in \mathbb{R}$.

PROOF: i)
$$\begin{aligned} d(x+a, y+a) &= \|(x+a) - (y+a)\| \\ &= \|x+a-y-a\| \\ &= \|x-y\| \\ &= d(x,y) \end{aligned}$$

ii)
$$\begin{aligned} d(\alpha x, \alpha y) &= \|\alpha x - \alpha y\| \\ &= \|\alpha(x-y)\| \\ &= |\alpha|\, \|x-y\| \\ &= |\alpha|\, d(x,y) \end{aligned}$$

THEOREM: Let X be a normed space then $\forall\, x, y \in X$,
$$\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$$

PROOF:
$$\begin{aligned} \|x\| &= \|x-y+y\| \\ &\leq \|x-y\| + \|y\| \end{aligned}$$
$$\Rightarrow \|x\| - \|y\| \leq \|x-y\| \rightarrow ①$$

Interchanging x and y, we get:
$$\|y\| - \|x\| \leq \|y-x\|$$
$$\Rightarrow -[\|x\| - \|y\|] \leq \|-(x-y)\|$$
$$= |-1|\, \|x-y\| = \|x-y\|$$
$$\Rightarrow -(\|x\| - \|y\|) \leq \|x-y\| \rightarrow ②$$
$$\Rightarrow \|x\| - \|y\| \geq -\|x-y\| \rightarrow ②'$$
① and ②' $\Rightarrow$ $-\|x-y\| \leq \|x\| - \|y\| \leq \|x-y\|$
$$\Rightarrow \big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$$

UNIFORMLY CONTINUOUS:

Let $(X, \|\cdot\|)$ be a normed space. Then, $\|\cdot\|$ defined on X is said to be uniformly continuous if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x-y\| < \delta$ whenever:

$$\big|\|x\| - \|y\|\big| < \varepsilon$$

THEOREM: Let X be a normed space and $f: X \to R$ be defined by $f(x) = \|x\|$. Then, f is uniformly continuous.

PROOF: Let $\varepsilon > 0$ and choose $\delta = \varepsilon$.
Then, whenever $\|x-y\| < \delta$.
Then, $\|x-y\| < \varepsilon$
Now,
$$\begin{aligned}|f(x) - f(y)| &= \big|\|x\| - \|y\|\big| \\ &\leq \|x-y\| \\ &< \varepsilon\end{aligned}$$

$$\Rightarrow |f(x) - f(y)| < \varepsilon$$

$\Rightarrow f$ is uniformly continuous.

THEOREM: Let X be a normed space. We defined:
i) $f: X \times X \to X$ by $f(x, y) = x + y$
ii) $g: F \times X \to X$ by $g(\alpha, x) = \alpha x$.

Then, show that f and g are continuous.

PROOF: i) To prove f is continuous.
Let $\{x_n\}$ and $\{y_n\}$ be the two sequences in X such that $x_n \to x$ and $y_n \to y$. Now

$\|f(x_n, y_n) - f(x,y)\| = \|(x_n + y_n) - (x+y)\|$

$= \|(x_n - x) + (y_n - y)\|$

$\le \|x_n - x\| + \|y_n - y\| \to 0 + 0 = 0.$

when $n \to \infty$

$\Rightarrow f(x_n, y_n) \to f(x,y)$ $\quad$ $x_n \to x,\ n \to \infty$

$x_n \to x$

$\Rightarrow f$ is continuous. $\quad$ $f(x_n) \to f(x)$

ii) Let $\{x_n\}$ and $\{\alpha_n\}$ be the two sequences such that $x_n \to x$ and $\alpha_n \to \alpha$.

Now, $\|g(\alpha_n, x_n) - g(\alpha, x)\|$

$= \|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|.$

$= \|\alpha_n (x_n - x) + (\alpha_n - \alpha) x\|.$

$\le \|\alpha_n (x_n - x)\| + \|(\alpha_n - \alpha) x\|.$

$= |\alpha_n|\, \|x_n - x| + |\alpha_n - \alpha|\, \|x\|$

$\to |\alpha_n|(0) + (0)\|x\| = 0$ when $n \to \infty$.

$\Rightarrow g(\alpha_n, x_n) \to g(\alpha, x).$

$\Rightarrow g$ is continuous.

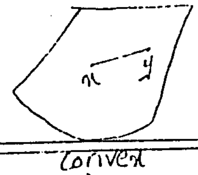
DEFINITION: CLOSED UNIT BALL:

Let X be a normed space. Then, the set denoted and defined by:

$$\overline{B}_{\underset{\text{radius}}{1}}(\underset{\text{centre}}{0}) = \{x : x \in X \wedge \|x\| \le 1\}.$$

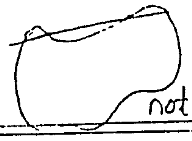
is called closed unit ball.

CONVEX SET: Let X be a normal space and $M \subseteq X$. Then, M is said to be convex if for all $x, y \in M$ and $\alpha \in]0,1[$, $\alpha x + (1-\alpha) y \in M$.



Convex


Convex


not Convex.

THEOREM: Prove that a closed unit ball in a normed space X is convex:

Proof: Let $\bar{B}(0) = \{x \in X : \|x\| \leq 1\}$ be the closed unit ball in a normed space X.

To prove: $\bar{B}(0)$ is convex. Let $x, y \in \bar{B}(0)$ and $\alpha \in]0, 1[$.

As $x, y \in \bar{B}(0)$, so $\|x\| \leq 1$ and $\|y\| \leq 1$.

Now,
$$\begin{aligned}\|\alpha x + (1-\alpha)y\| &\leq \|\alpha x\| + \|(1-\alpha)y\| \\ &= |\alpha|\,\|x\| + |1-\alpha|\,\|y\| \\ &\leq |\alpha|(1) + |1-\alpha|(1) \\ &= \alpha + 1 - \alpha = 1\end{aligned}$$

$\Rightarrow \|\alpha x + (1-\alpha)y\| \leq 1$

$\Rightarrow \alpha x + (1-\alpha)y \in \bar{B}(0)$

$\Rightarrow \bar{B}(0)$ is a convex set.

EXPLANATION:

DEFINITION: VECTOR SPACE:

Let $V \neq \phi$ and F be the field, then:

1) $\forall\ \alpha \in F$ and $x \in V \Rightarrow \alpha x \in V$.
2) $(V, +)$ is abelian group.
3) $\forall\ \alpha, \beta \in F$ and $x \in V$:
   $(\alpha+\beta)x = \alpha x + \beta x$
4) $\forall\ \alpha \in F$ and $x, y \in V$
   $\alpha(x+y) = \alpha x + \alpha y$
5) $\forall\ \alpha, \beta \in F$ and $x \in V$.
   $(\alpha\beta)x = \alpha(\beta x)$
6) $\forall\ x \in V$ and for $1 \in F$
   $1 \cdot x = x$ Then, V is called Vector space.

Example: $V = R^2$, $F = R$.

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$\alpha(x, y) = (\alpha x, 0)$$

$$1(x, y) = (1 \cdot x, 0) = (x, 0) \neq (x, y)$$

LINEAR COMBINATION:

$x_1, x_2, x_3, \ldots, x_n \in V$

$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in F$.

$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_n x_n$ in V is called linear combination of $x_1, x_2, x_3, \ldots, x_n$

DEFINITION: SPANNING SET:

If $S \subseteq V$ over F. Then the set of all linear combinations of finite number of elements of S is called linear span of S or spanning set of S. It is denoted by $\langle S \rangle$ or $L[S]$.

LINEARLY DEPENDENT AND LINEARLY INDEPENDENT:

If $x_1, x_2, x_3, \ldots, x_n \in V(F)$. Then, for any choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$

i) If $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \Rightarrow \alpha_i = 0$ for all i, $1 \le i \le n$, Then $x_1, x_2, \ldots, x_n$ are called linearly independent.

ii) If $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \Rightarrow \alpha_i \neq 0$ for some i, $1 \le i \le n$, Then $x_1, x_2, \ldots, x_n$ are called linearly dependent.

Examples:

Let $V = R^2$ and $F = R$

We define $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

and $\alpha(x, y) = (\alpha x, \alpha y)$. Then,

i) $x = (1, -1)$, $y = (2, 3)$.

and $\alpha x + \beta y = 0$

$\Rightarrow \alpha(1, -1) + \beta(2, 3) = (0, 0)$.

$\Rightarrow (\alpha, -\alpha) + (2\beta, 3\beta) = (0, 0)$

$\Rightarrow (\alpha + 2\beta, -\alpha + 3\beta) = (0, 0)$

$\Rightarrow \alpha + 2\beta = 0 \rightarrow (1)$ and $-\alpha + 3\beta = 0 \rightarrow (2)$

Adding (1) and (2), we get:

$5\beta = 0 \Rightarrow \beta = 0$

Putting value of β in (1):

$\alpha + 2(0) = 0 \Rightarrow \alpha = 0$

$\Rightarrow x, y$ are linearly independent.

ii) Let $x = (1, 2)$, $y = (3, 6)$.

and $\alpha x + \beta y = 0$

$\Rightarrow \alpha(1, 2) + \beta(3, 6) = (0, 0)$

$\Rightarrow (\alpha, 2\alpha) + (3\beta, 6\beta) = (0, 0)$.

$\Rightarrow (\alpha + 3\beta, 2\alpha + 6\beta) = (0, 0)$

$\Rightarrow \alpha + 3\beta = 0 \rightarrow (1)$, $2\alpha + 6\beta = 0 \rightarrow (2)$.

$\alpha + 3\beta = 0$.

Choose $\beta = 1 \Rightarrow \alpha = -3$

$\Rightarrow x, y$ are linearly dependent.

LINEARLY INDEPENDENT LEMMA:

STATEMENT:

Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be linearly independent set of vectors in normal space X, then for any real number $c > 0$ and for every choice of scalar $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq c\,[|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|]$$

This relation is called linearly independent lemma.

PROOF: To prove: $\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq c\,[|\alpha_1| + \ldots + |\alpha_n|] \rightarrow (1)$

Let $S = [|\alpha_1| + |\alpha_2| + |\alpha_3| + \ldots + |\alpha_n|]$.

If $S = 0 \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \ldots, \alpha_n = 0$

Then, (1) is trivially proved.

If $S \neq 0$. Then $S > 0$. Then from (1)

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n\| \geq cS$$

$$\Rightarrow \frac{1}{S} \|\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots + \alpha_n x_n\| \geq c$$

$$\Rightarrow \left\| \left(\frac{\alpha_1}{S}\right) x_1 + \left(\frac{\alpha_2}{S}\right) x_2 + \left(\frac{\alpha_3}{S}\right) x_3 + \ldots + \left(\frac{\alpha_n}{S}\right) x_n \right\| \geq c$$

$$\Rightarrow \|\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots + \beta_n x_n\| \geq c; \text{ where } \beta_j = \alpha_j / S,\ 1 \leq j \leq n.$$

Further, $\displaystyle\sum_{j=1}^{n} |\beta_j| = |\beta_1| + |\beta_2| + \ldots + |\beta_n|$

$$= \left|\frac{\alpha_1}{S}\right| + \left|\frac{\alpha_2}{S}\right| + \ldots + \left|\frac{\alpha_n}{S}\right|$$

$$\sum_{j=1}^{n} |\beta_j| = \frac{1}{S}\left[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\right]$$

$$= \frac{S}{S} = 1.$$

$$\Rightarrow \sum_{j=1}^{n} |\beta_j| = 1$$

Hence, we have:

$$\|\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n\| \geq c \quad \text{②}, \quad \text{with } \sum_{j=1}^{n} |\beta_j| = 1$$

Since ① and ② are equivalent.
So, to prove ①, we prove ②.

On the contrary, suppose ② is wrong.

Then, there exists a sequence $\{y_m\}$ such that $\|y_m\| \to 0$ and $y_m = \beta_1^m x_1 + \beta_2^m x_2 + \cdots + \beta_n^m x_n$ ($m \to$ not power.)

with $\sum_{j=1}^{n} |\beta_j^m| = 1$

Now as, $\sum_{j=1}^{n} |\beta_j^m| = 1 \Rightarrow |\beta_j^m| \leq 1$ for some fix j,

say for $j = 1 \Rightarrow |\beta_1^m| \leq 1$.

$\Rightarrow \{\beta_1^m\}$ is a bounded sequence.

Then, by B.W property, this sequence has a convergent subsequence $\{\gamma_1^m\}$ such that $\gamma_1^m \to \beta_1$.
Then, $\{y_m\}$ has a subsequence $\{y_{(m,1)}\}$ such that

$$y_{(m,1)} = \gamma_1^m x_1 + \beta_2^m x_2 + \beta_3^m x_3 + \cdots + \beta_n^m x_n.$$

Continuing in this way after n- steps, we have a subsequence $\{y(m,n)\}$ of $\{y_m\}$.

and $y(m,n) \to \beta_1\alpha_1 + \beta_2\alpha_2 + \cdots + \beta_n\alpha_n = y$.

and $\sum_{j=1}^{n} |\beta_j| = 1$.

Now as norm is a continuous function so, $y(m,n) \to y \Rightarrow \|y(m,n)\| \to \|y\|$ $(f(x) = \|x\|)$

Now as, $\sum_{j=1}^{n} |\beta_j| = 1$, so not all β_j's' are zero, so $y \neq 0 \to *$

Further as, $\|y_m\| \to 0 \Rightarrow \|y(m,n)\| \to 0$
$\Rightarrow y = 0 \to **$

$*$ and $**$ gives the contradiction.
So our supposition is wrong.

And hence ② is true so, ultimately ① is true.

BASIS OF A VECTOR SPACE:
If $V(F)$ is a vector space and $S \neq \phi$ be subset of V. Then, S is said to be basis for V if.

i) S is linearly independent.
ii) $\langle S \rangle = V$ (i.e, every $v \in V$ is a linear combination of finite number of elements of S).

DEFINITION:

If S is the basis for V then order of S is called the dimension of V. If $O(S)$ is finite, then V is called finite dimensional.

EXAMPLE: Let $V = \mathbb{R}^2$ and $S = \{(1,0), (0,1)\}$

Then, $\alpha(1,0) + \beta(0,1) = 0$

$\Rightarrow (\alpha, 0) + (0, \beta) = (0,0)$

$\Rightarrow (\alpha + 0, 0 + \beta) = (0,0)$

$\Rightarrow \alpha = 0, \beta = 0$

$\Rightarrow S$ is linearly independent.

Now, $\forall$ $v = (x, y) \in V$, then:

$v = (x, y)$

$= (x + 0, 0 + y)$

$= (x, 0) + (0, y)$

$= x(1,0) + y(0,1)$

$\Rightarrow v \in \langle S \rangle$

$\Rightarrow V \subseteq \langle S \rangle$

$\Rightarrow \langle S \rangle = V \Rightarrow S$ is basis for V.

$\Rightarrow \dim(V) = 2$.

DEFINITION: SUBSPACE:

Let V be a vector space over field F and W be a non-empty subset of V then, W is said to be subspace of V if W itself is the vector space over the same field F.

REMARK:

Let V be a vector space over field F and $W \subseteq V$, then W is subspace of V if for all $W_1, W_2 \in W$ and $\alpha, \beta \in F$.

$$\alpha W_1 + \alpha W_2 \in W.$$

THEOREM: Let Y be a finite dimensional subspace of a normed space X. Then, Y is complete.

PROOF. Since Y is finite dimensional, so Y has a finite basis. Let $\{e_1, e_2, \dots, e_n\}$ be a basis for Y. To prove: Y is complete.

Let $\{x^{(m)}\}$ be the Cauchy sequence in Y. Then, for every $\varepsilon > 0$, there exists a natural number $n_0 \in \mathbb{N}$ such that:

$$\|x^{(m)} - x^{(p)}\| < \varepsilon \text{ whenever } m, p \geq n_0.$$
$$\Rightarrow \varepsilon > \|x^{(m)} - x^{(p)}\|, \forall\, m, p \geq n_0.$$

Now as $\{x^{(m)}\}$ is a sequence in Y and $\{e_1, e_2, \dots, e_n\}$ is a basis for Y, so then there exists scalar $\alpha_1^{(m)}, \alpha_2^{(m)}, \dots \alpha_n^{(m)} \in F$ such that $x^{(m)} = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \dots + \alpha_n^{(m)} e_n$.

Similarly, $x^{(p)} = \alpha_1^{(p)} e_1 + \alpha_2^{(p)} e_2 + \dots + \alpha_n^{(p)} e_n$.

Now, $\|x^{(m)} - x^{(p)}\| = \|\alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + \dots + \alpha_n^{(m)} e_n - \alpha_1^{(p)} e_1 - \alpha_2^{(p)} e_2 - \dots - \alpha_n^{(p)} e_n\|$

$$= \|(\alpha_1^{(m)} - \alpha_1^{(p)}) e_1 + (\alpha_2^{(m)} - \alpha_2^{(p)}) e_2 + \dots + (\alpha_n^{(m)} - \alpha_n^{(p)}) e_n\|$$

$$\geq C[|\alpha_1^{(m)} - \alpha_1^{(p)}| + |\alpha_2^{(m)} - \alpha_2^{(p)}| + \dots + |\alpha_n^{(m)} - \alpha_n^{(p)}|],$$

$C > 0$ (By linearly independent Lemma).

$$\Rightarrow \|x^{(m)} - x^{(p)}\| \geq C \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}|$$

$$\Rightarrow \varepsilon > \|x^{(m)} - x^{(p)}\| \geq C \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}| \quad \forall\ m, p \geq n_0.$$

$$\Rightarrow \varepsilon > C \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}| \quad \forall\ m, p \geq n_0$$

$$\Rightarrow \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i^{(p)}| < \frac{\varepsilon}{C} \quad \because C > 0 \quad \forall\ m, p \geq n_0$$

$$\Rightarrow |\alpha_i^{(m)} - \alpha_i^{(p)}| < \varepsilon/C \quad \forall\ m, p \geq n_0.$$

$\Rightarrow \{\alpha_i^{(p)}\}$ is a cauchy sequence in $F = \mathbb{R}$.

Since, $\mathbb{R}$ is complete.
So there exists $\alpha_i \in \mathbb{R}$ such that $\alpha_i^{(p)} \to \alpha_i$

Put $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \dots + \alpha_n e_n \in Y$ ($\because e_1, e_2, \dots, e_n$ are basis of Y and all [illegible] generated by basis $\in Y$)

Now, $\|x^{(m)} - x\| = \|(\alpha_1^{(m)} - \alpha_1)e_1 + (\alpha_2^{(m)} - \alpha_2)e_2 + \dots + (\alpha_n^{(m)} - \alpha_n)e_n\|$

$$\leq |\alpha_1^{(m)} - \alpha_1|\,\|e_1\| + |\alpha_2^{(m)} - \alpha_2|\,\|e_2\| + \dots + |\alpha_n^{(m)} - \alpha_n|\,\|e_n\|$$

$$\leq |\alpha_1^{(m)} - \alpha_1| K + |\alpha_2^{(m)} - \alpha_2| K + \dots + |\alpha_n^{(m)} - \alpha_n| K$$

where $K = \max_{i=1}^{n} \|e_i\|$

$$= K \sum_{i=1}^{n} |\alpha_i^{(m)} - \alpha_i| < K(\varepsilon/C) = \varepsilon'$$

$\Rightarrow \|x^{(m)} - x\| < \varepsilon'$.

$\Rightarrow x^{(m)} \to x \in Y$

$\Rightarrow Y$ is complete.

Hence Proved.

DEFINITION:

Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a linear space X are said to be equivalent norms if there exists two positive real numbers 'a' and 'b' such that:

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \quad \forall x \in X.$$

THEOREM:

Any two norms defined on a finite dimensional normed space X are equivalent.

PROOF: Let X be a finite dimensional normed space with basis $\{e_1, e_2, \ldots, e_n\}$.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the two norms defined on X.

To prove: $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Let $x \in X$, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that: $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$

$\Rightarrow \|x\|_1 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_1$
$\geq c[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|]$, $c > 0$ (By linearly independent lemma)

$\Rightarrow \|x\|_1 \geq cS$ where $S = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$

$\Rightarrow S \leq \frac{1}{c}\|x\|_1 \rightarrow$ ① $(\because c > 0)$.

Also, $\|x\|_2 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_2$
$\leq |\alpha_1|\,\|e_1\| + |\alpha_2|\,\|e_2\| + \cdots + |\alpha_n|\,\|e_n\|$ (By (iii) & (iv) property of norm)
$\leq |\alpha_1|k + |\alpha_2|k + \cdots + |\alpha_n|k$, $k = \max_{i=1}^{n} \|e_i\|$
$= k\sum_{i=1}^{n} |\alpha_i| = kS$, $\left(S = \sum_{i=1}^{n} |\alpha_i|\right)$.

$\Rightarrow \|x\|_2 \leq KS$

$\Rightarrow \frac{1}{K}\|x\|_2 \leq S \leq \frac{1}{C}\|x\|_1$ (By using ①)

$\Rightarrow \frac{1}{K}\|x\|_2 \leq \frac{1}{C}\|x\|_1$

$\Rightarrow \|x\|_2 \leq \frac{K}{C}\|x\|_1 \rightarrow$ ②

Now let $x \in X$ Then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such That, $x = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n$.

$\Rightarrow \|x\|_2 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_2$

$\geq C'[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|]$ $C' > 0$

(By linearly independent lemma)

$\Rightarrow \|x\|_2 \geq C'S$ where $S = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|$

$\Rightarrow S \leq \frac{1}{C'}\|x\|_2 \rightarrow$ ③

Also, $\|x\|_1 = \|\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n\|_1$

$\leq |\alpha_1|\,\|e_1\| + |\alpha_2|\,\|e_2\| + \cdots + |\alpha_n|\,\|e_n\|$

let, $K' = \max_{i=1}^{n} \|e_i\|$

$\Rightarrow \|x\|_1 \leq K'[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|]$.

$\Rightarrow \|x\|_1 \leq K'S$

$\Rightarrow \frac{1}{K'}\|x\|_1 \leq S \leq \frac{1}{C'}\|x\|_2$ (using ③)

$\Rightarrow \frac{C'}{K'}\|x\|_1 \leq \|x\|_2 \rightarrow$ ④

② and ④ $\Rightarrow \frac{C'}{K'}\|x\|_1 \leq \|x\|_2 \leq \frac{K}{C}\|x\|_1$

$\Rightarrow a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1$ with $a = C'/K'$, $b = K/C$

Hence, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

THEOREM: Prove that equivalent norms on a normed space X defines a same Topology on X.

linear space with field over real nos.

PROOF: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the two equivalent norms defined on linear space X.

Then, there exists two positive real nos. a and b such that:

$$a\|x\|_1 \le \|x\|_2 \le b\|x\|_1 \rightarrow ⓐ \quad \forall\ x \in X.$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies defined on X w.r.t $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively.

Let $U_1 \in \mathcal{T}_1$ i.e U_1 is an open set w.r.t $\|\cdot\|_1$. Then, for each $x \in U_1$, there exists an open ball $B_1(x,r)$ such that $x \in B_1(x,r) \subseteq U_1$.

(By Def^n of open set.)

Here $B_1(x,r) = \{y : y \in X \text{ and } \|x-y\|_1 < r\}$

+ve real nos.

Put $r' = ar$

Consider $B_2(x,r') = \{y : y \in X \text{ and } \|x-y\|_2 < r'\}$.

Now, we prove $B_2(x,r') \subseteq B_1(x,r)$.

Let $y \in B_2(x,r') \Rightarrow \|x-y\|_2 < r'$

$\Rightarrow \|x-y\|_2 < ar \Rightarrow \frac{1}{a}\|x-y\|_2 < r$

Now, $a\|x-y\|_1 \le \|x-y\|_2$ (By ⓐ)

$\Rightarrow \|x-y\|_1 \le \frac{1}{a}\|x-y\|_2 < r$

$\Rightarrow \|x-y\|_1 < r \Rightarrow y \in B_1(x,r)$

$\Rightarrow B_2(x,r') \subseteq B_1(x,r)$

$\Rightarrow x \in B_2(x,r') \subseteq B_1(x,r) \subseteq U_1$.

$\Rightarrow x \in B_2(x, r') \subseteq U_1$

$\Rightarrow$ For every $x \in U_1$, we have an open ball $B_2(x, r')$ w.r.t $\|\cdot\|_2$ such that $x \in B_2(x, r') \subseteq U_1$

$\Rightarrow U_1$ is an open set w.r.t $\|\cdot\|_2$.

$\Rightarrow U_1 \in \mathcal{F}_2 \Rightarrow \mathcal{F}_1 \subseteq \mathcal{F}_2 \rightarrow$ ①

Now let $U_2 \in \mathcal{F}_2$

$\Rightarrow U_2$ is an open set w.r.t $\|\cdot\|_2$, then for each $x \in U_2$, there exist an open ball $B_2(x, r)$ w.r.t $\|\cdot\|_2$ such that $x \in B_2(x, r) \subseteq U_2$.

Here $B_2(x, r) = \{y : y \in X \wedge \|x - y\|_2 < r\}$

Let $r' = \frac{1}{b} r$ and consider $B_1(x, r') = \{y : y \in X \wedge \|x - y\|_1 < r'\}$

Now we prove that $B_1(x, r') \subseteq B_2(x, r)$.

Let $y \in B_1(x, r') \Rightarrow \|x - y\|_1 < r'$.

$\Rightarrow \|x - y\|_1 < \frac{1}{b} r$.

$\Rightarrow b\|x - y\|_1 < r$.

Now $\frac{1}{b}\|x - y\|_2 \leq \|x - y\|_1$ (By ⊛).

$\Rightarrow \|x - y\|_2 \leq b\|x - y\|_1 < r$.

$\Rightarrow \|x - y\|_2 < r$.

$\Rightarrow y \in B_2(x, r)$

$\Rightarrow B_1(x, r') \subseteq B_2(x, r)$.

$\Rightarrow B_1(x, r') \subseteq B_2(x, r) \subseteq U_2$.

$\Rightarrow B_1(x, r') \subseteq U_2$.

$\Rightarrow$ For every U_2 is an open set w.r.t $\|\cdot\|_1$.

$\Rightarrow U_2 \in \mathcal{F}_1 = \mathcal{F}_2 \subseteq \mathcal{F}_1 \rightarrow$ ②

① and ② $\Rightarrow \mathcal{F}_1 = \mathcal{F}_2$.

THEOREM: Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be the two equivalent norms defined on linear space X. Then, a Cauchy sequence w.r.t $\|\cdot\|_1$ is also Cauchy sequence w.r.t $\|\cdot\|_2$.

PROOF: Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, so then there exist two positive real numbers a and b such that $a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \rightarrow * \quad \forall x \in X$.

Let $\{x_n\}$ be a cauchy sequence w.r.t $\|\cdot\|_1$. Then, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\|x_m - x_n\|_1 < \varepsilon \quad \forall m,n \geq n_0.$$

Now, $\|x_m - x_n\|_2 \leq b\|x_m - x_n\|_1 < b\varepsilon \quad \forall m,n \geq n_0$.

$$\Rightarrow \|x_m - x_n\|_2 < \varepsilon' \quad , b\varepsilon = \varepsilon', \forall m,n \geq n_0.$$

$\Rightarrow \{x_n\}$ is also cauchy w.r.t $\|\cdot\|_2$.

Conversely let $\{x_n\}$ be cauchy sequence w.r.t $\|\cdot\|_2$. Then for every $\varepsilon > 0$, there exist $n_0 \in N$ such that $\|x_m - x_n\|_2 < \varepsilon \quad \forall m,n \geq n_0$.
Now, $a\|x_m - x_n\|_1 \leq \|x_m - x_n\|_2 < \varepsilon \quad \forall m,n \geq n_0$.

$$\Rightarrow a\|x_m - x_n\|_1 < \varepsilon \quad \forall m,n \geq n_0.$$

$$\Rightarrow \|x_m - x_n\|_1 < \varepsilon/a \quad \forall m,n \geq n_0.$$

$$\Rightarrow \|x_m - x_n\|_1 < \varepsilon', \quad \varepsilon/a = \varepsilon', \forall m,n \geq n_0.$$

$\Rightarrow \{x_n\}$ is Cauchy w.r.t $\|\cdot\|_1$.
Hence Proved.

"BANACH SPACE"

DEFINITION:
A complete normed space is called Banach space.

THEOREM: Prove that $\mathbb{R}$ is banach space:
([illegible] define " Absolute " [illegible] of Real line)

PROOF: Let us define $\|\cdot\|: \mathbb{R} \to \mathbb{R}$ by $\|x\| = |x|$.
First we show that $(\mathbb{R}, \|\cdot\|)$ is normed space.

Let $\alpha \in F$, $x, y \in X$.

i). As $|x| \geq 0$. So $\|x\| \geq 0$.

ii). $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0$.

iii). $\|\alpha x\| = |\alpha x| = |\alpha| |x| = |\alpha| \|x\|$.

iv).
$$\begin{aligned} \|x+y\| &= |x+y| \\ &\leq |x| + |y| \\ &= \|x\| + \|y\|. \end{aligned}$$

$\Rightarrow (\mathbb{R}, \|\cdot\|)$ is normed space.

Now, we prove $\mathbb{R}$ is complete.
Let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$.
Then, for every $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon \quad \forall\, m, n \geq n_0 \rightarrow *$

Let $\varepsilon = 1/2$ and n_0 be the smallest natural number satisfying $*$.
$\varepsilon = 1/2^2$ and n_1 be the smallest natural number satisfying $*$.
$\varepsilon = 1/2^3$ and n_2 be the smallest natural number satisfying $*$.

So on upto:
$\varepsilon = 1/2^{k+1}$ and n_k be the smallest natural number satisfying $*$.

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So, then $n_0 < n_1 \le n_2 < n_3 \le \cdots \le n_k \le \cdots$

So then $\{x_{n_k}\}$ is subsequence of $\{x_n\}$.

Now we show that $x_n \to x \in \mathbb{R}$. Consider the closed interval $I_k = [x_{n_k} - 1/2^k , x_{n_k} + 1/2^k]$,
then $I_{k+1} = [x_{n_{k+1}} - 1/2^{k+1} , x_{n_{k+1}} + 1/2^{k+1}]$

Now we show that $I_{k+1} \subseteq I_k$.
As $\{x_n\}$ is cauchy and $\{x_{n_k}\}$ is subsequence of $\{x_n\}$, so $\{x_{n_k}\}$ is also cauchy so for

$\varepsilon = 1/2^{k+1}$, $n = n_k$, $m = n_{k+1}$, Then from $*$:

$\|x_m - x_n\| < \varepsilon$, $\forall$ $m,n \ge$
$\|x_m - x_n\| \le 1/2^{k+1}$ $\forall$ $m,n \ge n_k$

$\|x_{n_{k+1}} - x_{n_k}\| \le 1/2^{k+1}$

$x_{n_k} - 1/2^{k+1} < x_{n_{k+1}} < x_{n_k} + 1/2^{k+1} \to (1)$

$(\because |x-a| < \delta$ iff $a - \delta < x < a + \delta)$

Now from first half of (1),

$x_{n_k} - 1/2^{k+1} < x_{n_{k+1}}$

$\Rightarrow x_{n_k} - \dfrac{1}{2^k} + \dfrac{1}{2^k} - \dfrac{1}{2} \cdot \dfrac{1}{2^k} < x_{n_{k+1}}$ (Add and Sub $1/2^k$)

$\Rightarrow x_{n_k} - \dfrac{1}{2^k} + \dfrac{1}{2} \cdot \dfrac{1}{2^k} < x_{n_{k+1}}$

$\Rightarrow x_{n_k} - \dfrac{1}{2^k} < x_{n_{k+1}} - \dfrac{1}{2^{k+1}} < x_{n_{k+1}} < x_{n_k} + \dfrac{1}{2^{k+1}} \to (2)$

Now from 2nd part of (1),

$$x_{n_{k+1}} < x_{n_k} + \frac{1}{2^{k+1}} - \frac{1}{2^k} + \frac{1}{2^k}$$

$$\Rightarrow x_{n_{k+1}} < x_{n_k} + \frac{1}{2^k}\cdot\frac{1}{2} - \frac{1}{2^k} + \frac{1}{2^k}$$

$$\Rightarrow x_{n_{k+1}} < x_{n_k} + \frac{1}{2^k}\cdot\frac{1}{2} - \frac{1}{2^k} + \frac{1}{2^k}$$

$$\Rightarrow x_{n_{k+1}} < x_{n_k} + \left(\frac{1}{2} - 1\right)\frac{1}{2^k} + \frac{1}{2^k}$$

$$\Rightarrow x_{n_{k+1}} < x_{n_k} - \frac{1}{2}\cdot\frac{1}{2^k} + \frac{1}{2^k}$$

$$\Rightarrow x_{n_{k+1}} < x_{n_k} - \frac{1}{2^{k+1}} + \frac{1}{2^k}$$

$$\Rightarrow x_{n_{k+1}} + \frac{1}{2^{k+1}} < x_{n_k} + \frac{1}{2^k} \rightarrow (3)$$

Now from (2) and (3), we get:

$$x_{n_k} - \frac{1}{2^k} < x_{n_{k+1}} - \frac{1}{2^{k+1}} < x_{n_{k+1}} + \frac{1}{2^{k+1}} < x_{n_k} + \frac{1}{2^k}$$

$$\Rightarrow [x_{n_{k+1}} - 1/2^{k+1},\ x_{n_{k+1}} + 1/2^{k+1}] \subseteq [x_{n_k} - 1/2^k,\ x_{n_k} + 1/2^k]$$

$$\Rightarrow I_{k+1} \subseteq I_k$$

0 1 2 3 4 5 6

$1 < 2 < 4 < 6$

$[2,\ 4] \subseteq [1\ \ 6]$

$\Rightarrow \{I_k\}$ is a cauchy decreasing sequence of closed sets and $\delta(I_k) \to 0$ when $k \to \infty$. Then, by Cantor's intersection theorem:

$\bigcap_{k=1}^{\infty} I_k$ contains exactly one point, say x

$$\Rightarrow x \in \bigcap_{k=1}^{\infty} I_k \Rightarrow x \in I_k,\ \forall k$$

$$\Rightarrow x \in [x_{n_k} - 1/2^k,\ x_{n_k} + 1/2^k]$$

$\Rightarrow \|x_{n_k} - x\| \leq 1/2^k$.

When $k \to \infty$ $[x-a, x+a]$

$\|x_{n_k} - x\| \to 0$.

$\xleftarrow{\;}\underset{x-a}{|}\overset{a}{\;}\underset{x}{|}\overset{a}{\;}\underset{x+a}{|}\xrightarrow{\;}$

$\Rightarrow x_{n_k} \to x$.

$\Rightarrow x_{n_k} \to x \in \mathbb{R} \Rightarrow \mathbb{R}$ is complete.

$\Rightarrow (\mathbb{R}, \|\cdot\|)$ is banach space.

THEOREM: Prove that $\mathbb{C}$ is Banach space.

PROOF: Define $\|\cdot\| : \mathbb{C} \to \mathbb{R}$ by $\|z\| = |z|$.

First, we show that $(\mathbb{C}, \|\cdot\|)$ is normed space.

Let $\alpha \in F$, $x, y \in z$

i) As $|z| \geq 0$. So $\|z\| \geq 0$.

ii) $\|z\| = 0 \Leftrightarrow |z| = 0 \Leftrightarrow z = 0$.

iii) $\|\alpha z\| = |\alpha z| = |\alpha||z| = |\alpha|\|z\|$.

iv) $\|x+y\| = |x+y|$

$\leq |x| + |y| = \|x\| + \|y\|$.

$\Rightarrow (\mathbb{C}, \|\cdot\|)$ is normed space.

Now, we show that $\mathbb{C}$ is complete.

Let $\{z_m\}$ be a cauchy sequence in $\mathbb{C}$.

Then, for every $\varepsilon > 0$, there exists some positive integer n_0 such that:

$\|z_m - z_n\| < \varepsilon$, $\forall m, n \geq n_0$

$\Rightarrow |z_m - z_n| < \varepsilon$, $\forall m, n \geq n_0$.

Let $z_n = x_n + iy_n$. Then, we have.

$|(x_m + iy_m) - (x_n + iy_n)| < \varepsilon, \forall\ m, n \geq n_0$.

$\Rightarrow |(x_m - x_n) + i(y_m - y_n)| < \varepsilon, \forall\ m, n \geq n_0$.

$\Rightarrow \sqrt{(x_m - x_n)^2 + (y_m - y_n)^2} < \varepsilon, \forall\ m, n \geq n_0$.

$\Rightarrow (x_m - x_n)^2 + (y_m - y_n)^2 < \varepsilon^2, \forall\ m, n \geq n_0$

$\Rightarrow (x_m - x_n)^2 < \varepsilon^2$; $(y_m - y_n)^2 < \varepsilon^2, \forall\ m, n \geq n_0$.

$\Rightarrow |x_m - x_n| < \varepsilon$ and $|y_m - y_n| < \varepsilon, \forall\ m, n \geq n_0$

$\Rightarrow \{x_m\}$ and $\{y_m\}$ are the two cauchy sequences in $\mathbb{R}$. As $\mathbb{R}$ is complete, so $x_m \to x \in \mathbb{R}$ and $y_m \to y \in \mathbb{R}$. Put $z = x + iy \in \mathbb{C}$.

Now as $x_m \to x$ and $y_m \to y$, so for every $\varepsilon > 0$, there exists some positive integers n_1 and n_2 such that $|x_m - x| < \varepsilon/\sqrt{2}, \forall\ m \geq n_1$

and $|y_m - y| < \varepsilon/\sqrt{2}, \forall\ m \geq n_2$.

$\Rightarrow |x_m - x|^2 < \varepsilon^2/(\sqrt{2})^2, \forall\ m \geq n_1$

and $|y_m - y|^2 < \varepsilon^2/(\sqrt{2})^2, \forall\ m \geq n_2$.

$\Rightarrow (x_m - x)^2 + (y_m - y)^2 < \varepsilon^2, \forall\ m \geq n'$ where $n' = \max(n_1, n_2)$.

$\Rightarrow \sqrt{(x_m - x)^2 + (y_m - y)^2} < \varepsilon, \forall\ m \geq n'$

$\Rightarrow |z_m - z| < \varepsilon, \forall\ m \geq n'$

$\Rightarrow \|x_m - x\| < \varepsilon, \ \forall\, m \geq n'$

$\Rightarrow x_m \to x \in C$

$\Rightarrow x_m - x \in C$

$\Rightarrow C$ is Complete

$\Rightarrow (C, \|\cdot\|)$ is Banach space.

THEOREM: Prove that $\mathbb{R}^n$ is Banach space:

PROOF: Let $X = \mathbb{R}^n$ and $\|\cdot\| : X \to F$ be defined by:
$\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$ where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

1) As for all i, $|x_i| \geq 0$

$\Rightarrow \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} \geq 0$

$\Rightarrow \|x\| \geq 0$.

2) $\|x\| = 0 \Leftrightarrow \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} = 0$

$\Leftrightarrow |x_i|^2 = 0 \Leftrightarrow |x_i| = 0$

$\Leftrightarrow x_i = 0 \quad \forall i, 1 \leq i \leq n$

$\Leftrightarrow x = 0$

3) Let $\alpha \in F$, $\|\alpha x\| = \left[\sum_{i=1}^{n} |\alpha x_i|^2\right]^{1/2}$

$= \left[\sum_{i=1}^{n} |\alpha|^2 |x_i|^2\right]^{1/2} = |\alpha| \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$

$= |\alpha| \, \|x\|$.

4) $\|x + y\| = \left[\sum_{i=1}^{n} |x_i + y_i|^2\right]^{1/2}$

$\leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2} + \left[\sum_{i=1}^{n} |y_i|^2\right]^{1/2}$

$= \|x\| + \|y\|$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$.

Hence, $(\mathbb{R}^n, \|\cdot\|)$ is normed space.

Now to prove $\mathbb{R}^n$ is complete.
Let $\{x^{(m)}\}$ be a Cauchy sequence in $\mathbb{R}^n$.

Then, by the definition of cauchy sequence, for every $\varepsilon > 0$ there exist some positive integer n_0 such that.

$$\|x^{(m)} - x^{(P)}\| < \varepsilon \quad \forall\ m, P \geq n_0.$$

$$\Rightarrow \left[\sum_{i=1}^{n} |x_i^{(m)} - x_i^{(P)}|^2\right]^{1/2} < \varepsilon,\ \forall\ m, P \geq n_0$$

$$\Rightarrow \sum_{i=1}^{n} |x_i^{(m)} - x_i^{(P)}|^2 < \varepsilon^2,\ \forall\ m, P \geq n_0$$
$$\forall\ i,\ 1 \leq i \leq n$$

$$\Rightarrow |x_i^{(m)} - x_i^{(P)}|^2 < \varepsilon^2,\ \forall\ m, P \geq n_0 \text{ and } \forall$$

$$\Rightarrow |x_i^{(m)} - x_i^{(P)}| < \varepsilon,\ \forall\ m, P \geq n_0 \text{ and } \forall i,\ 1 \leq i \leq n.$$

$\Rightarrow \{x_i^{(m)}\}$ is a cauchy sequence in $\mathbb{R}$.
Since $\mathbb{R}'$ is complete. So $x_i^{(m)} \to x_i \in \mathbb{R}, \forall i, 1 \leq i \leq n$.

$\Rightarrow$ for every $\varepsilon > 0$, there exists some positive integer n_i such that:

$$|x_i^{(m)} - x_i| < \varepsilon/\sqrt{n} \quad \forall\ m \geq n_i,\ \forall i\ 1 \leq i \leq n.$$

$$\Rightarrow |x_i^{(m)} - x_i|^2 < \varepsilon^2/n \quad \forall\ m \geq n_i,\ \forall i\ 1 \leq i \leq n.$$

$\Rightarrow \sum_{i=1}^{n} |x_i^{(m)} - x_i|^2 < n\,\varepsilon^2/n$, $\forall\, m \geq n'$ where $n' = \max(n_1, n_2, \dots, n_n)$

$\Rightarrow \left[\sum_{i=1}^{n} |x_i^{(m)} - x_i|^2\right]^{1/2} < \varepsilon$, $\forall\, m \geq n'$ where $n' = \max(n_1, n_2, \dots, n_n)$

$\Rightarrow \|x^{(m)} - x\| < \varepsilon$, $\forall\, m \geq n'$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$\Rightarrow x^{(m)} \to x \in \mathbb{R}^n$.

$\Rightarrow \mathbb{R}^n$ is complete.

Hence, $\mathbb{R}^n$ is Banach space.

THEOREM. Prove that l^p is Banach space.

PROOF. Let $X = l^p$ and $\|\cdot\| : X \to F$ be defined by $\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p}$.

i) As $\forall i$, $|x_i| \geq 0$

$\Rightarrow |x_i|^p \geq 0 \Rightarrow \sum |x_i|^p \geq 0$.

$\Rightarrow \left[\sum |x_i|^p\right]^{1/p} \geq 0$.

$\Rightarrow \|x\| \geq 0$.

ii). $\|x\| = 0 \Leftrightarrow |x_i| = 0 \Leftrightarrow x_i = 0 \;\forall i$.

$\Leftrightarrow x = 0$

iii). $\|\alpha x\| = \left[\sum |\alpha x_i|^p\right]^{1/p}$

$= \left[\sum |\alpha|^p |x_i|^p\right]^{1/p}$

$= |\alpha| \left[\sum |x_i|^p\right]^{1/p} = |\alpha|\, \|x\|$

iv). $\cancel{\|x+y\| \leq \|x\| +}$

iv). $\|x+y\| = [\sum |x_i + y_i|^p]^{1/p}$

$\leq [\sum |x_i|^p]^{1/p} + [\sum |y_i|^p]^{1/p}$

$= \|x\| + \|y\|$

$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$

$\Rightarrow (X, \|\cdot\|)$ is normed space.

Now to prove l^p is complete.
Let $\{x^{(m)}\}$ be a cauchy sequence in l^p.
Then for every $\varepsilon > 0$, there exists some positive integer n_0 such that:

$\|x^{(m)} - x^{(n)}\| < \varepsilon \quad , \forall\ m, n \geq n_0$

$\Rightarrow [\sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(n)}|^p]^{1/p} < \varepsilon \ , \forall\ m, n \geq n_0 \rightarrow (1)$

$\Rightarrow |x_i^{(m)} - x_i^{(n)}| < \varepsilon \ , \forall\ m, n \geq n_0$

$\Rightarrow \{x_i^{(n)}\}$ is a cauchy sequence in $\mathbb{R}$.
Since $\mathbb{R}$ is complete. So, $x_i^{(n)} \to x_i \in \mathbb{R}$.
So, when $n \to \infty$ Then from (1):

$[\sum_{i=1}^{\infty} |x_i^{(m)} - x_i|^p]^{1/p} < \varepsilon \quad \forall\ m \geq n_0$

$\Rightarrow \|x^{(m)} - x\| < \varepsilon \quad \forall\ m \geq n_0 , x = (x_1, x_2, \dots x_k)$

$\Rightarrow x^{(m)} \to x$
Now $x = x^{(m)} - (x^{(m)} - x) \in l^p$.

$\Rightarrow x^{(m)} \to x \in l^p \Rightarrow l^p$ is complete.

Hence, l^p is Banach space.

THEOREM: Prove that l^∞ is Banach space.

PROOF: Let us define $\|\cdot\| : l^\infty \to \mathbb{R}$ by $\|x\| = \sup_{n=1}^{\infty} |x_n|$

$x = \{x_n\} \in l^\infty$.

i). As $\forall n \; |x_n| \geq 0 \Rightarrow \sup_{n=1}^{\infty} |x_n| \geq 0$

$\Rightarrow \|x\| \geq 0$.

ii). $\|x\| = 0 \Leftrightarrow \sup_{n=1}^{\infty} |x_n| = 0$

$\Leftrightarrow |x_n| = 0 \quad \forall n$

$\Leftrightarrow x_n = 0 \quad \forall n \Leftrightarrow x = 0$.

iii). Let $\alpha \in F = \mathbb{R}$ and $x \in l^\infty$.

Now $\|\alpha x\| = \sup_{n=1}^{\infty} |\alpha x_n|$

$= |\alpha| \sup_{n=1}^{\infty} |x_n|$

$= |\alpha| \, \|x\|$.

iv). $\|x+y\| = \sup_{n=1}^{\infty} |x_n + y_n|$

$\leq \sup_{n=1}^{\infty} |x_n| + \sup_{n=1}^{\infty} |y_n|$

$= \|x\| + \|y\|$

$\Rightarrow (l^\infty, \|\cdot\|)$ is normed space.

Now to prove $X = l^\infty$ is complete.

 $\{x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \dots)\}$

Let $\{x^{(m)}\}$ be a cauchy sequence in l^∞.
Then, by the definition of cauchy sequence for every $\varepsilon > 0$ there exist some positive integer n_0 such that $\|x^{(m)} - x^{(p)}\| < \varepsilon \;\; \forall\, m, p \geq n_0$.

$\Rightarrow \sup_{n=1}^{\infty} |x_n^{(m)} - x_n^{(p)}| < \varepsilon, \; \forall\, m, p \geq n_0$. $\rightarrow$ ①

$\Rightarrow |x_n^{(m)} - x_n^{(p)}| < \varepsilon, \; \forall\, m, p \geq n_0 \; (\forall n)$.

$\Rightarrow \{x_n^{(p)}\}$ is a cauchy sequence in $\mathbb{R}$.

As $\mathbb{R}$ is complete so there exists some $x_n \in \mathbb{R}$ such that $x_n^{(p)} \rightarrow x_n$ ②

Then, when $p \rightarrow \infty$, then:
$\|x^{(m)} - x\| < \varepsilon$ ③, $\forall\, m \geq n'$ for some $n' \in \mathbb{N}$.
where $x = \{x_n\}$

$\Rightarrow x^{(m)} \rightarrow x$.

Now, $|x_n| = |x_n - x_n^{(m)} + x_n^{(m)}|$
$\leq |x_n - x_n^{(m)}| + |x_n^{(m)}|$.
$< \varepsilon + \lambda$ for some +ve λ

$\Rightarrow |x_n| < \varepsilon + \lambda$
$\Rightarrow x = \{x_n\} \in l^\infty$. Hence $x^m \rightarrow x \in l^\infty$.

$\Rightarrow l^\infty$ is complete.

$\Rightarrow l^\infty$ is Banach Space.

THEOREM: Prove that $C[a\ b]$ is Banach Space.

PROOF: Let $X = C[a\ b]$ and define $\|\cdot\| : X \to \mathbb{R}$ by $\|f\| = \sup_{x \in [a,b]} |f(x)|$. Then:

i) As for all $x \in [a\ b]$, $|f(x)| > 0$

$$\Rightarrow \sup_{x \in [a\ b]} |f(x)| \geq 0$$

$$\Rightarrow \|f\| \geq 0$$

ii) $$\|f\| = 0 \Leftrightarrow \sup_{x \in [a\ b]} |f(x)| = 0$$

$$\Leftrightarrow \underset{x \in [a\ b]}{|f(x)|} = 0$$

$$\Leftrightarrow f(x) = 0 \Leftrightarrow f = 0.$$

iii) $$\|\alpha f\| = \sup_{x \in [a\ b]} |(\alpha f)(x)|$$

$$= \sup_{x \in [a\ b]} |\alpha f(x)| = |\alpha| \sup_{x \in [a\ b]} |f(x)|$$

$$= |\alpha|\ \|f\|$$

iv) $$\|f+g\| = \sup_{x \in [a\ b]} |(f+g)(x)|$$

$$= \sup_{x \in [a\ b]} |f(x) + g(x)|$$

$$\leq \sup_{x \in [a\ b]} |f(x)| + \sup_{x \in [a\ b]} |g(x)|$$

$$= \|f\| + \|g\|$$

$$\Rightarrow \|f+g\| \leq \|f\| + \|g\|.$$

Since, all the conditions are satisfied.

So, $(X, \|\cdot\|)$ is normed space.

Now we prove that X is complete.
Let for this $\{f_m\}$ be a cauchy sequence in X. Then, by the definition of cauchy sequence for every $\varepsilon > 0$, there is some integer n_0, such that:

$\|f_m - f_n\| < \varepsilon \ \forall \ m, n \geq n_0$.

$\Rightarrow \sup_{x \in [a,b]} |(f_m - f_n)(x)| < \varepsilon, \ \forall \ m, n \geq n_0$.

$\Rightarrow \sup_{x \in [a\ b]} |f_m(x) - f_n(x)| < \varepsilon, \ \forall \ m, n \geq n_0 \rightarrow ①$.

$\Rightarrow |f_m(x) - f_n(x)| < \varepsilon, \ \forall \ m, n \geq n_0$.

$\Rightarrow \{f_n(x)\}$ is a cauchy sequence in $\mathbb{R}$.
Since $\mathbb{R}$ is complete. So, then there exists some function f such that $f_n(x) \rightarrow f(x) \rightarrow ②$

Further as for all n, f_n is continuous. So, f being uniform limit of f_n, is also continuous. Then $f \in C[a\ b]$.

From ① and ②,

$\sup_{x \in [a\ b]} |f_m(x) - f(x)| < \varepsilon, \ \forall \ m \geq n'$, for some $n' \in \mathbb{N}$.

$\Rightarrow \|f_m - f\| < \varepsilon, \ \forall \ m \geq n'$.

$\Rightarrow f_m \rightarrow f \in C[a\ b]$.

$\Rightarrow C[a\ b]$ is complete.

$\Rightarrow C[a\ b]$ is Banach space.

THEOREM: A subspace Y of a Banach space X is complete iff Y is closed in X.

PROOF: Suppose Y is complete.
To prove: Y is closed in X.

Let $\{x_n\}$ be a sequence in Y such that $x_n \to x$. Then, $\{x_n\}$ is also a cauchy sequence ($\because$ every convergent sequence is cauchy).

As Y is complete. So, $x_n \to x \in Y$
$\Rightarrow Y$ is closed in X

Conversely, let us suppose Y is closed in X.
To prove: Y is complete.

Let $\{x_n\}$ be a cauchy sequence in Y. As $Y \subseteq X$. So $\{x_n\}$ is also a cauchy sequence in X. As X is Banach space so X is also complete. Then $x_n \to x \in X$.
As Y is closed so $x \in Y$.
Thus, Y is complete.

THEOREM: Prove that every finite dimensional subspace of a Banach space is Banach space.

PROOF: As every finite dimensional subspace of a normed space is closed so if X is

The Banach space and Y be the finite dimensional subspace of X. Then Y is closed in X.

As A subspace Y of a Banach space X is complete iff Y is closed in X.

$\Rightarrow Y$ is also complete.

Hence, Y is Banach space.

THEOREM: Let X be a finite dimensional space, then any $M \subseteq X$ is compact iff M is closed and bounded.

A M.S X is compact if every sequence in X has a convergent subsequence.

PROOF: Suppose M is compact.

To prove: M is closed and bounded.

$M \subseteq \bar{M} \rightarrow ①$

Now let $x \in \bar{M}$. Then there exists a sequence $\{x_n\}$ in M such that $x_n \Rightarrow x$.

As $\{x_n\}$ is a sequence in M and M is compact so then this sequence has a convergent subsequence i.e, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \Rightarrow x \in M$.

Then we have $x_n \rightarrow x \in M \Rightarrow x \in M$

$\Rightarrow \bar{M} \subseteq M \rightarrow ②$

① and ② $\Rightarrow M = \bar{M}$

$\Rightarrow M$ is closed.

Now, we prove that M is bounded.

Suppose M is not bounded and let $b \in M$ be a fixed point. As M is not bounded then for some $n \in N$, there is some $x_n \in M$ such that:
$$d(x_n, b) > n.$$
If it is bounded then it is closed

Then, the sequence $\{x_n\}$ in M has no convergent subsequence. Then, M is not compact.

Which is a contradiction.
So our supposition is wrong.
And hence, M is bounded.

Conversely assume M is closed and bounded.
To prove: M is compact.

Let $\{x_m\}$ be a sequence in M.
Now as X is finite dimensional. So X has the finite basis $\{e_1, e_2, \dots, e_n\}$. Now as $x_m \in M \subseteq X$
$\Rightarrow x_m \in X$. Then, there exists scalars $\alpha_1^m, \alpha_2^m, \dots, \alpha_n^m$ such that $x_m = \alpha_1^m e_1 + \alpha_2^m e_2 + \dots + \alpha_n^m e_n$.

$$\Rightarrow \|x_m\| = \|\alpha_1^m e_1 + \alpha_2^m e_2 + \dots + \alpha_n^m e_n\|$$
$$\geq c\left[|\alpha_1^m| + |\alpha_2^m| + \dots + |\alpha_n^m|\right], \; c > 0 \quad (\text{By L.f.L})$$

Further as M is bounded, so there exists some positive real number k, such that:
$$\|x_m\| \leq k \Rightarrow k \geq \|x_m\| \geq c\left[|\alpha_1^m| + |\alpha_2^m| + \dots + |\alpha_n^m|\right]$$
$$\Rightarrow k \geq c\left[|\alpha_1^m| + |\alpha_2^m| + \dots + |\alpha_n^m|\right]$$
$$\Rightarrow \sum_{i=1}^{n} |\alpha_i^m| \leq k/c$$

$\Rightarrow |\alpha_i^m| < K/c$, for some fixed i.

$\Rightarrow \{\alpha_i^m\}$ is a bounded sequence in $\mathbb{R}$.

Then, by B-W property $\{\alpha_i^m\}$ has a convergent subsequence $\{\beta_i^m\}$ such that $\beta_i^m \to \beta_i$

Put $z_m = \beta_1^m e_1 + \beta_2^m e_2 + \dots + \beta_n^m e_n$

Then $z_m \to \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n = x$ (When $m \to \infty$)

So $\{z_m\}$ is a subsequence of $\{x_m\}$ and $z_m \to x$.

As $z_m \in M$ and $z_m \to x$ and M is closed

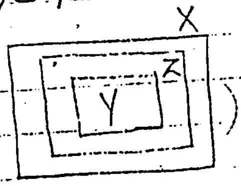
$\Rightarrow x \in M \Rightarrow M$ is compact.

"REIZ LEMMA"

V-Imp

STATEMENT: Let Y and Z be the two subspaces of a normed space X and Y is closed proper subspace of Z, then for any real number θ, $0<\theta<1$, there exist a point $z \in Z$ such that $\|z\| = 1$ and $\|z - y\| > \theta$ $\forall$ $y \in Y$.

Small



PROOF: As Y is closed proper subspace of Z. So Then there exist some $u \in Z$ such that $u \notin Y$.

Put $a = \inf \|u - y\|$ $\forall$ $y \in Y$

Then $a > 0$ and $a \leq \|u - y\| \to (1)$ $\forall$ $y \in Y$.

Also, by the definition of infimum for $\varepsilon > 0$,

There exists some point $y_0 \in Y$ such that:

$\|u - y_0\| < a + \varepsilon$ (By def. of inf)

Let $a + \varepsilon = a/\theta$, $\theta \in]0, 1[$

then, $\|u - y_0\| < a/\theta$

Put $z = \frac{u-y_0}{\|u-y_0\|} \in Z$.

Further, $\|z\| = \left\| \frac{u-y_0}{\|u-y_0\|} \right\|$

$= \frac{\|u-y_0\|}{\|u-y_0\|} = 1$

$\Rightarrow \|z\| = 1$

Now it remains to prove:

$\|z-y\| > 0 \quad \forall \; y \in Y$

Now let $y \in Y$ and $\alpha = \|u-y_0\|$, Then

$\|z-y\| = \left\| \frac{u-y_0}{\alpha} - y \right\|$

$= \frac{1}{|\alpha|} \|u-y_0-\alpha y\| = \frac{1}{\alpha}\|u-(y_0+\alpha y)\|$ $(\because \alpha > 0)$

Now as $y_0, y \in Y$ and Y is subspace so,

$y_0 + \alpha y \in Y$.

Then, $\|z-y\| = \frac{1}{\alpha} \|u-(y_0+\alpha y)\|$

$\geq \frac{1}{\alpha} \cdot a$ (by ①)

$\Rightarrow \|z-y\| \geq a/\alpha$

Now $\|u-y_0\| < a/\theta \Rightarrow \alpha < a/\theta$ $(\because \|u-y_0\| = \alpha)$

$\Rightarrow \theta < a/\alpha \leq \|z-y\| \quad \forall \; y \in Y$

$\Rightarrow \theta < \|z-y\| \quad \forall \; y \in Y$

$\Rightarrow \|z-y\| > \theta \quad \forall \; y \in Y$

HENCE PROVED.

"HILBERT SPACE:"

Inner product space:

Let V be a vector space over the field F ($\mathbb{R}$ or $\mathbb{C}$). Then a mapping $\langle\cdot,\cdot\rangle : V\times V \to F$ is called inner product space if:

i) $\langle x, x\rangle \geq 0 \quad \forall\, x\in X$

ii) $\langle x, x\rangle = 0$ iff $x=0$.

iii) $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

iv) $\langle x, y\rangle = \overline{\langle y, x\rangle}$

v) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$

Then, the pair $(V, \langle\cdot,\cdot\rangle)$ is called inner product space.

EXAMPLE 1:

Let $V=\mathbb{R}^n$ over the field $F=\mathbb{R}$. We define $\langle\cdot,\cdot\rangle : V\times V \to F$ by:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \text{where } x=(x_1, x_2, \dots, x_n),\ y=(y_1, y_2, \dots, y_n)$$

SOLUTION:

i) $\langle x, x\rangle = \sum_{i=1}^{n} x_i^2 \geq 0$

ii) $\langle x, x\rangle = 0 \iff \sum_{i=1}^{n} x_i^2 = 0$

$\iff x_i^2 = 0 \iff x_i = 0$

$\iff (x_1, x_2, \dots, x_n) = 0$

$\iff x = 0$

iii) Let $x=(x_1, x_2, \dots, x_n)$, $y=(y_1, y_2, \dots, y_n)$ and $z=(z_1, z_2, \dots, z_n)\in\mathbb{R}^n$.

Then: $\langle x+y, z\rangle = \sum_{i=1}^{n} (x_i+y_i)z_i$

$= \sum_{i=1}^{n} (x_iz_i + y_iz_i)$

$= \sum_{i=1}^{n} x_iz_i + \sum_{i=1}^{n} y_iz_i$

$= \langle x, z\rangle + \langle y, z\rangle$

iv) Let $\alpha \in F$, then $\langle \alpha x, y\rangle = \sum_{i=1}^{n} (\alpha x_i)y_i$

$\Rightarrow \langle \alpha x, y\rangle = \alpha \sum_{i=1}^{n} x_iy_i$

$= \alpha\langle x, y\rangle$

v) $\langle x, y\rangle = \sum_{i=1}^{n} x_iy_i$

$= \sum_{i=1}^{n} y_ix_i = \sum_{i=1}^{n} \overline{y_ix_i}$

$= \overline{\langle y, x\rangle}$

Since, all the conditions are satisfied, so, $(R^n, \langle\cdot,\cdot\rangle)$ is an inner product space.

"SOME PROPERTIES OF INNER PRODUCT:"

i) $\langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle$

L.H.S $= \langle x, y+z\rangle$

$= \overline{\langle y+z, x\rangle}$

$= \overline{\langle y, x\rangle + \langle z, x\rangle}$

$= \overline{\langle y, x\rangle} + \overline{\langle z, x\rangle}$

$= \langle x, y\rangle + \langle x, z\rangle =$ R.H.S

ii) $\langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle$.

$L.H.S = \langle x, \alpha y\rangle = \overline{\langle \alpha y, x\rangle}$
$= \bar{\alpha}\,\overline{\langle y, x\rangle}$
$= \bar{\alpha}\langle x, y\rangle = R.H.S.$

iii). $\langle \alpha x + \beta y, z\rangle$
$= \langle \alpha x, z\rangle + \langle \beta y, z\rangle$.
$= \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

THEOREM:
Prove that every inner product space is normed space.

PROOF:
Let V be an inner product space define $\|\cdot\| : V \to F$ by $\|x\| = \{\langle x, x\rangle\}^{1/2}$ $\forall\ x \in V$.
Then as:

i) As $\langle x, x\rangle \geq 0$ $\forall\ x \in V$.
$\Rightarrow \{\langle x, x\rangle\}^{1/2} \geq 0$
$\Rightarrow \|x\| \geq 0$ $\forall\ x \in V$.

ii) $\|x\| = 0 \Leftrightarrow \{\langle x, x\rangle\}^{1/2} = 0$
$\Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0$
$\Rightarrow \|x\| = 0$ iff $x = 0$.

iii). Let $\alpha \in F$ and $x \in V$, then
$\|\alpha x\| = \{\langle \alpha x, \alpha x\rangle\}^{1/2}$
$= \{\alpha\bar{\alpha}\langle x, x\rangle\}^{1/2}$.
$= \{|\alpha|^2\langle x, x\rangle\}^{1/2}$.

$$\|\alpha x\| = |\alpha| \{\langle x, x\rangle\}^{1/2}$$
$$= |\alpha| \, \|x\|.$$

iv) $\|x+y\| = \{\langle x+y, x+y\rangle\}^{1/2}$

$$\Rightarrow \|x+y\|^2 = \langle x+y, x+y\rangle.$$
$$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle.$$
$$= \|x\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \|y\|^2.$$
$$= \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2.$$
$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2.$$
$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \text{ (By C.S.I).}$$
$$= (\|x\| + \|y\|)^2.$$
$$\Rightarrow \|x+y\|^2 \leq (\|x\| + \|y\|)^2.$$
$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|.$$

Since all the conditions are satisfied. So V is also normed space.

PARALLELOGRAM LAW:

Let V be an inner product space. Then for any $x, y \in V$.
Then: $\|x+y\|^2 + \|x-y\|^2 = 2[\|x\|^2 + \|y\|^2]$.

PROOF:

$L.H.S = \|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle$
$+ \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2 + \|x\|^2 + \|y\|^2$
$= 2[\|x\|^2 + \|y\|^2] = R.H.S$

Hence Proved.

POLARIZATION IDENTITY:-

i) $Re\langle x,y\rangle = 1/4\ \{\|x+y\|^2 - \|x-y\|^2\}$

ii) $Im\langle x,y\rangle = 1/4\ \{\|x+iy\|^2 - \|x-iy\|^2\}$ is called polarization identity.

PROOF:

i) $R.H.S = 1/4\ \{\|x+y\|^2 - \|x-y\|^2\}$

$= \frac{1}{4}\{\langle x+y, x+y\rangle - \langle x-y, x-y\rangle\}$

$= \frac{1}{4}\{\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle - \langle x,x\rangle$
$+ \langle x,y\rangle + \langle y,x\rangle - \langle y,y\rangle\}$

$= \frac{1}{4}\{\langle x,y\rangle + \langle y,x\rangle + \langle x,y\rangle + \langle y,x\rangle\}$

$= \frac{1}{4}\{2\langle x,y\rangle + 2\langle y,x\rangle\}$

$= \frac{1}{2}\{\langle x,y\rangle + \overline{\langle x,y\rangle}\}$

$= \frac{1}{2}\cdot 2\ Re\langle x,y\rangle = Re\langle x,y\rangle = L.H.S$

ii) $R.H.S = \frac{1}{4}\{\|x+iy\|^2 - \|x-iy\|^2\}$

$= \frac{1}{4}\{\langle x+iy, x+iy\rangle - \langle x-iy, x-iy\rangle\}$

$= \frac{1}{4}\{\langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle + \langle iy,iy\rangle$

$- \langle x,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle - \langle iy,iy\rangle\}$

$= \frac{1}{4}\{\langle x,iy\rangle + \langle iy,x\rangle + \langle x,iy\rangle + \langle iy,x\rangle\}$

$= \frac{1}{4}\{2\langle x,iy\rangle + 2\langle iy,x\rangle\}$

$= \frac{1}{2}\{\bar{i}\langle x,y\rangle + \overline{\bar{i}\langle x,y\rangle}\}$

$= \frac{1}{2}\cdot 2\, Im\langle x,y\rangle$

$= Im\langle x,y\rangle = L.H.S$

Hence Proved.

CAUCHY SCHEWARZ INEQUALITY:

In an inner product space V for all $x,y \in V$, $|\langle x,y\rangle| \le \|x\|\,\|y\|$.

PROOF:

For any $\lambda \in F$.

$\|x-\lambda y\|^2 \ge 0$

$\Rightarrow 0 \le \|x-\lambda y\|^2$

$= \langle x-\lambda y, x-\lambda y\rangle$

$= \langle x,x\rangle - \bar{\lambda}\langle x,y\rangle - \lambda\langle y,x\rangle + \lambda\bar{\lambda}\langle y,y\rangle$

$= \|x\|^2 - \bar{\lambda}\langle x,y\rangle - \lambda\langle y,x\rangle + \lambda\bar{\lambda}\langle y,y\rangle$

$= \|x\|^2 - \bar{\lambda}\langle x,y\rangle - \lambda[\langle y,x\rangle - \bar{\lambda}\|y\|^2]$

Put $\bar{\lambda} = \dfrac{\langle y, x \rangle}{\|y\|^2}$.

$\Rightarrow \langle y, x \rangle - \bar{\lambda}\|y\|^2 = 0$.

Then $0 \leq \|x\|^2 - \dfrac{\langle y, x \rangle}{\|y\|^2}\langle x, y \rangle - \lambda(0)$.

$$0 \leq \|x\|^2\|y\|^2 - \langle x, y \rangle \overline{\langle x, y \rangle}$$

$$0 \leq \|x\|^2\|y\|^2 - |\langle x, y \rangle|^2$$

$$\Rightarrow |\langle x, y \rangle|^2 \leq \|x\|^2\|y\|^2$$

$$\Rightarrow |\langle x, y \rangle| \leq \|x\|\,\|y\|.$$

Hence Proved.

REMARK:
We know that every inner product space is normed space but converse is not true in general i.e., a normed space need not to be an inner product space.

e.g: If we choose $X = l^p$, $p > 2$.
And we define, $\|\cdot\| : X \to \mathbb{R}$ by

$$\|x\| = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p}, \quad x = (x_1, x_2, \dots).$$

Then, $(X, \|\cdot\|)$ is normed space.
But X is not an inner product space. Because
e.g: if we choose.
$x = (1, 1, 0, 0, 0, \dots)$, $y = (1, -1, 0, 0, 0, \dots) \in l^p$.

Then, $\|x\| = 2^{1/p}$, $\|y\| = 2^{1/p}$.

$$\Rightarrow 2[\|x\|^2 + \|y\|^2] = 2[2^{2/p} + 2^{2/p}].$$

$$= 2[2 \cdot 2^{2/p}] = 4 \cdot [2^{2/p}].$$

Now, $x + y = (2, 0, 0, 0, \ldots)$

$\|x + y\| = 2 \Rightarrow \|x + y\|^2 = 4$.

and $x - y = (0, 2, 0, 0, \ldots)$

$\|x - y\| = 2 \Rightarrow \|x - y\|^2 = 4$.

$$\|x + y\|^2 + \|x - y\|^2 = 8 \neq 4(2^{2/p}).$$

$$\Rightarrow \|x + y\|^2 + \|x - y\|^2 \neq 2[\|x\|^2 + \|y\|^2]$$

$\Rightarrow$ $\|$gm law fails to hold.

Hence, X is not an inner product space.

APPALONIUS IDENTITY:

Let V be an inner product space then $\forall\, x, y, z \in V$:

$$\|z - x\|^2 + \|z - y\|^2 = \tfrac{1}{2}\|x - y\|^2 + 2\|z - \tfrac{1}{2}(x + y)\|^2$$

is called Appalonius identity.

PROOF: In the parallelogram law:

$$\|x' + y'\|^2 + \|x' - y'\|^2 = 2\{\|x'\|^2 + \|y'\|^2\}$$

Put $x' = z - x$ and $y' = z - y$.

then,

$$\|z-x+z-y\|^2+\|z-x-z+y\|^2 = 2\|z-x\|^2+2\|z-y\|^2$$

$$\Rightarrow \|2z-(x+y)\|^2+\|-x+y\|^2 = 2\|z-x\|^2+2\|z-y\|^2$$

$$\Rightarrow 4\left\|z-\frac{(x+y)}{2}\right\|^2+\|x-y\|^2 = 2\|z-x\|^2+2\|z-y\|^2$$

$$\Rightarrow \|z-x\|^2+\|z-y\|^2 = \frac{1}{2}\|x-y\|^2+2\|z-\tfrac{1}{2}(x+y)\|^2.$$

Hence Proved.

THEOREM:
Prove that inner product map is a continuous function.

PROOF:
Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space.
To prove: $\langle\cdot,\cdot\rangle$ is a continuous function.

For this, let $x_n \to x$ and $y_n \to y$ in V.
Then, when $n \to \infty$, $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$.

Now, $|\langle x_n, y_n\rangle - \langle x, y\rangle|$

$= |\langle x_n, y_n\rangle - \langle x_n, y\rangle + \langle x_n, y\rangle - \langle x, y\rangle|$

$= |\langle x_n, y_n - y\rangle + \langle x_n - x, y\rangle|$

$\leq |\langle x_n, y_n - y\rangle| + |\langle x_n - x, y\rangle|$

$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$ (By Cauchy Schwarz inequality).

$\rightarrow \|x_n\|(0) + (0)\|y\|$ when $n \to \infty$.
$\rightarrow 0$.
$\Rightarrow \langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$.

$\Rightarrow$ The $\langle \cdot, \cdot \rangle$ is continuous.

HILBERT SPACE: (DEF).
A complete inner product space is called Hilbert space.

e.g: R^n and C^n are Hilbert spaces.

DEFINITION:
Let V be an inner product space. Then, $x, y \in V$ are said to be orthogonal, denoted by $x \perp y$, if $\langle x, y \rangle = 0$.

DEFINITION:
Let V be an inner product space and $A \subseteq V$. Then, an element $x \in V$ is orthogonal to A, $x \perp A$, if: $\langle x, a \rangle = 0$, $\forall\, a \in A$.

DEFINITION:
Let V be an inner product space and $A \subseteq V$. Then, the ~~element~~ orthogonal compliment of A is denoted and defined by:

$$A^{\perp} = \{x \in V : x \perp A\}.$$

PATHAGORIAN THEOREM:
Let V be an inner product space and $x, y \in V$ such that $x \perp y$, Then;

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

PROOF:

$$
\begin{aligned}
L.H.S &= \|x+y\|^2 \\
&= \langle x+y, x+y \rangle \\
&= \langle x,x \rangle + \langle x,y \rangle + \langle y,x \rangle + \langle y,y \rangle \\
&= \|x\|^2 + 0 + 0 + \|y\|^2 \quad (\because x \perp y) \\
&= \|x\|^2 + \|y\|^2 \\
&= R.H.S
\end{aligned}
$$

Hence Proved.

EXAMPLE:

Let V be an inner ~~prod~~ product space and $U, V \in V$ such that:

$$\langle x, U \rangle = \langle x, V \rangle \quad \forall x \in V$$

Then, $U = V$.

SOLUTION:

It is given that $\forall x \in V$, $\langle x, U \rangle = \langle x, V \rangle$

To show, $U = V$

Now, $\langle x, U \rangle = \langle x, V \rangle$

$\Rightarrow \langle x, U \rangle - \langle x, V \rangle = 0$

$\Rightarrow \langle x, U - V \rangle = 0$

Since, this result holds for all $x \in V$

So it holds for $x = U - V \in V$.

$\Rightarrow \langle U-V, U-V \rangle = 0 \Longleftrightarrow U - V = 0$

$\Rightarrow U = V$.

DEFINITION:

Let V be an inner product space and $x \in V$ and $M \subseteq V$. Then, the distance between x and M is denoted and defined as:

$$\delta = \inf \|x - y\|, \ \forall \ y \in M.$$

THEOREM: MINIMIZING VECTOR.

Let X be an inner product space and M be a non empty convex subset of X which is complete. Then, for $x \in X$, there exists a unique $y' \in M$ such that $\delta = \inf_{y \in M} \|x - y\| = \|x - y'\|$.

PROOF:

By definition $\delta = \inf_{y \in M} \|x - y\|$.

Then, by definition of infimum, there exists a sequence $\{y_n\}$ in M such that:

$$\delta = \lim_{n \to \infty} \|x - y_n\|.$$

Now, we prove that $\{y_n\}$ in M is Cauchy sequence. Let $V_n = y_n - x$. Then,

$$\|V_n\| = \|y_n - x\| = \|x - y_n\| = \delta_n \ (\text{Say})$$

Then, $\|V_n + V_m\| = \|y_n - x + y_m - x\|$

$$= \|y_n + y_m - 2x\|.$$

$$= 2 \|1/2 (y_n + y_m) - x\|$$

Now as $y_n, y_m \in M$ and M is convex, so $1/2\, y_n + 1/2\, y_m \in M$.

$\Rightarrow \|\frac{1}{2}(y_n+y_m)-x\| \geq \delta$

$\Rightarrow 2\|\frac{1}{2}(y_n+y_m)-x\| \geq 2\delta$

$\Rightarrow \|v_n+v_m\| \geq 2\delta$

Now, $\|v_n-v_m\| = \|y_n - x - y_m + x\|$
$= \|y_n - y_m\|$

$\Rightarrow \|y_n-y_m\|^2 = \|v_n-v_m\|^2$
$= 2\|v_n\|^2 + 2\|v_m\|^2 - \|v_n+v_m\|^2$ (By ||gm law).
$\leq 2(\delta_n)^2 + 2(\delta_m)^2 - (2\delta)^2$
$\rightarrow 2(\delta)^2 + 2(\delta)^2 - 4\delta^2$ when $m,n\to\infty$
$\rightarrow 0$

$\Rightarrow \|y_n-y_m\| \to 0$ when $m,n\to\infty$

$\Rightarrow \{y_n\}$ is Cauchy sequence.

Now as M is complete. So $y_n \to y' \in M$.
So $\delta = \lim_{n\to\infty} \|x-y_n\| = \|x-y'\|$ $(\because \lim_{n\to\infty} y_n = y')$.

Now we prove the uniqueness of y'. Let us assume $\delta = \|x-y'\|$ and also $\delta = \|x-y''\|$, $y', y'' \in M$.

Now $\|y'-y''\|^2 = \|y'-x+x-y''\|^2$
$= \|(y'-x)+(x-y'')\|^2$
$= 2\|y'-x\|^2 + 2\|x-y''\|^2 - \|(y'-x)-(x-y'')\|^2$ (By || gm law)
$= 2\|y'-x\|^2 + 2\|x-y''\|^2 - 4\|\frac{1}{2}(y'+y'')-x\|^2$

$\leq 2\delta^2 + 2\delta^2 - 4\delta^2$

$\Rightarrow \|y'-y''\|^2 \leq 0$

$\Rightarrow \|y'-y''\|^2 = 0$

$\Rightarrow \|y'-y''\| = 0$ iff $y' = y''$

Hence, the existance of y' is unique.

THEOREM:

Let Y be a complete subspace of an inner product space X, then for $x \in X$, there exist a unique $y \in Y$ such that $z = x - y$ is orthogonal to Y.

PROOF:

Let $x \in X$. Then as Y is complete subspace of an inner product space X, then there exist a unique $y \in Y$ such that $\delta = \|x-y\|$. Let $z = x-y$.

To prove: $z \perp Y$.

Suppose z is not orthogonal to Y. Then, there exist some $y_1 \in Y$ such that:

$$\langle z, y_1 \rangle \neq 0$$

Let $\langle z, y_1 \rangle = \alpha \in F$.

Now let $\beta \in F$. Then,

$$\|z - \beta y_1\|^2 = \langle z - \beta y_1, z - \beta y_1 \rangle$$

$$= \langle z, z \rangle - \beta \langle y_1, z \rangle - \bar{\beta} \langle z, y_1 \rangle + \beta\bar{\beta} \langle y_1, y_1 \rangle$$

$$= \|z\|^2 - \bar{\beta} \langle z, y_1 \rangle - \beta \{ \langle y_1, z \rangle - \bar{\beta} \langle y_1, y_1 \rangle \}$$

Chose $\bar{\beta} = \dfrac{\langle y_1, z \rangle}{\|y_1\|^2}$.

Then, $\|z - \beta y_1\|^2 = \|z\|^2 - \dfrac{\langle y_1, z \rangle}{\|y_1\|^2} \langle z, y_1 \rangle - \beta(0)$

$$\Rightarrow \|z-\beta y_1\|^2 = \|z\|^2 - \frac{\langle y_1, z\rangle}{\|y_1\|^2}\langle z, y_1\rangle.$$

$$= \|z\|^2 - \frac{|\langle z, y_1\rangle|^2}{\|y_1\|^2}$$

$$< \|z\|^2 = \delta^2.$$

$$\Rightarrow \|z-\beta y_1\|^2 < \delta^2.$$

$$\Rightarrow \|z-\beta y_1\| < \delta \rightarrow (1).$$

Also $\|z-\beta y_1\| = \|x-y-\beta y_1\|$.

$$= \|x-(y+\beta y_1)\| \geq \delta.$$

$$\Rightarrow \|z-\beta y_1\| \geq \delta \rightarrow (2).$$

(1) and (2) gives the contradiction.
So, our supposition is wrong.
And hence, $z \perp Y$.

DEFINITION:
A vector space X is said to be the direct sum of its subspace Y and Z, if for every $x \in X$, there exists $y \in Y$ and $z \in Z$ such that $x = y + z$ and this representation is unique. In this case, we write $X = Y \oplus Z$. (In this case, $Y \cap Z = \{0\}$)

THEOREM:
Let Y be a closed subspace of a Hilbert space H. Then $H = Y \oplus Y^{\perp}$.

PROOF:

Let $x \in H$. As Y is closed subspace of a complete space H and closed subspace of a complete space is complete. So, Y is complete. So, then by a well known theorem, there exist a unique $y \in Y$ such that $x - y = z \perp Y$.

$\Rightarrow x - y \in Y^{\perp}$ and $x - y = z \Rightarrow x = y + z$ and $y \in Y$ and $z \in Y^{\perp}$.

Now, we prove the expression $x = y + z$ is unique. Let us assume $x = y_1 + z_1$ and $x = y_2 + z_2$ where $y_1, y_2 \in Y$ and $z_1, z_2 \in Y^{\perp}$.

Now $x = y_1 + z_1$ and $x = y_2 + z_2$

$\Rightarrow y_1 + z_1 = y_2 + z_2$

$\Rightarrow \underbrace{y_1 - y_2}_{\in Y} = \underbrace{z_2 - z_1}_{\in Y^{\perp}} \in Y \cap Y^{\perp} = \{0\}$

$\Rightarrow y_1 - y_2 = 0,\ z_2 - z_1 = 0$

$\Rightarrow y_1 = y_2$ and $z_1 = z_2$

$\Rightarrow$ The representation $x = y + z$ is unique.

Hence, $H = Y \oplus Y^{\perp}$.

$z \in Y \cap Y^{\perp}$

$\Rightarrow z \in Y$ and $z \in Y^{\perp} \Rightarrow z \in Y$ and $z \perp Y$

As $z \perp Y$ so $\langle z, y \rangle = 0 \ \forall y \in Y$.

In particular as $z \in Y$ so $\langle z, z \rangle = 0$

$\Rightarrow z = 0$

$\Rightarrow Y \cap Y^{\perp} = \{0\}$

Hence Proved.

THEOREM:

Let Y be a closed subspace of Hilbert space H. Then, $Y = Y^{\perp\perp}$.

PROOF:

Let $x \in Y$. To prove: $x \in Y^{\perp\perp}$

Now for $y \in Y^{\perp} \Rightarrow y \perp Y$.

Then, as $x \in Y$ and $y \perp Y \Rightarrow \langle x, y \rangle = 0$.

Hence, for any $y \in Y^{\perp} \Rightarrow \langle x, y \rangle = 0$.

$$\Rightarrow x \in (Y^{\perp})^{\perp} \Rightarrow x \in Y^{\perp\perp} \Rightarrow Y \subseteq Y^{\perp\perp} \rightarrow ①$$

Now let $x \in Y^{\perp\perp} \subseteq H$

$\Rightarrow x \in H = Y \oplus Y^{\perp}$

So, then there exists unique $y \in Y$ and $z \in Y^{\perp}$ such that $x = y + z$. Now as $y \in Y \subseteq Y^{\perp\perp}$

$$\Rightarrow y \in Y^{\perp\perp} \Rightarrow x, y \in Y^{\perp\perp} \Rightarrow x - y \in Y^{\perp\perp}$$

$$\Rightarrow x - y \perp Y^{\perp} \Rightarrow x - y = z \perp Y^{\perp}$$

$\Rightarrow$ for all element in $Y^{\perp}$ and so for

$$z \in Y^{\perp} \Rightarrow \langle z, z \rangle = 0 \Rightarrow z = 0 \Rightarrow x = y$$

$$\Rightarrow x \in Y \quad (\because y \in Y)$$

$$\Rightarrow Y^{\perp\perp} \subseteq Y \rightarrow ②$$

① and ② $\Rightarrow Y = Y^{\perp\perp}$.

THEOREM:

For a nonempty subspace M of a Hilbert space H, the span of M is dense in H if and only if $M^{\perp} = \{0\}$

PROOF:

Let $V = \langle M \rangle$. Then to show $\bar{V} = H$ iff $M^{\perp} = \{0\}$.

Suppose V is dense in H i.e $\bar{V} = H$.

Then, to prove, $M^{\perp} = \{0\}$.

Let $x \in M^{\perp} \subseteq H \Rightarrow x \in H \Rightarrow x \in \bar{V}$ $\because \bar{V} = H$.

Then, there exist a sequence $\{x_n\}$ in V such that $x_n \rightarrow x$. As $x \in M^{\perp} \Rightarrow x \perp M$, then for any $v \in V$ as $V = \langle M \rangle$.

So, $v = \alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_r m_r$,

where $m_1, m_2, \dots, m_r \in M$ and $\alpha_1, \dots, \alpha_r \in F$.

$$
\begin{aligned}
\text{Then, } \langle v, x \rangle &= \langle \alpha_1 m_1 + \alpha_2 m_2 + \dots + \alpha_r m_r, x \rangle \\
&= \alpha_1 \langle m_1, x \rangle + \alpha_2 \langle m_2, x \rangle + \dots + \alpha_r \langle m_r, x \rangle \\
&= \alpha_1 (0) + \alpha_2 (0) + \dots + \alpha_r (0) \quad (\because x \perp M) \\
&= 0
\end{aligned}
$$

$\Rightarrow \forall\ v \in V$, $\langle v, x \rangle = 0 \Rightarrow x \perp V$. Further as $x \perp V$ and $\{x_n\}$ is a sequence in V. So, $\forall\ n$, $\langle x_n, x \rangle = 0$.

Further as, $x_n \rightarrow x$ and $x \rightarrow x$.

$\Rightarrow \langle x_n, x \rangle \rightarrow \langle x, x \rangle$

But $\langle x_n, x \rangle = 0$

$\Rightarrow \langle x, x \rangle = 0 \Rightarrow x = 0 \Rightarrow M^{\perp} = \{0\}$.

Conversely, suppose $M^{\perp} = \{0\}$

To prove: $\langle M \rangle = \bar{V} = H$.

Obviously, $\langle \bar{M} \rangle \subseteq H \rightarrow ①$.

Now let $0 \neq x \in H \Rightarrow x \in H = V \oplus V^{\perp}$

$\Rightarrow x \in V \oplus V^{\perp}$
$\Rightarrow x \in V$ or $x \in V^{\perp}$.

If $x \in V \Rightarrow x \in \bar{V}$ $\because V \subseteq \bar{V}$
If $x \in V^{\perp} \Rightarrow x \perp V \Rightarrow x \perp M$ $(\because M \subseteq V)$
$\Rightarrow x \in M^{\perp} \Rightarrow x = 0$
Which is impossible.
So, $x \notin V^{\perp}$
$\Rightarrow x \in V \Rightarrow x \in \bar{V}$
$\Rightarrow H \subseteq \bar{V} \rightarrow ②$

① and ② $\Rightarrow \bar{V} = H$
$\Rightarrow \langle M \rangle = V$ is dense in H.

HENCE PROVED.

ORTHOGONAL SET: (DEF).

Let X be an inner product space and $M \subseteq X$. Then, M is said to be orthogonal if for all $x, y \in M$.

$$\langle x, y \rangle = \begin{cases} 0 & \text{if } x \neq y \\ \text{non zero} & \text{if } x = y \end{cases}$$

ORTHONORMAL SET: (DEF).

Let X be an inner product space and $M \subseteq X$. Then, M is said to be orthonormal if for all $x, y \in M$.

$$\langle x, y \rangle = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

THEOREM:

An orthonormal set $\{e_1, e_2, \ldots, e_n\}$ in an inner product space X is linearly independent.

PROOF: Let $\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$ where, $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

Then $\langle \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n, e_i \rangle = 0$.

$\Rightarrow \alpha_1 \langle e_1, e_i \rangle + \alpha_2 \langle e_2, e_i \rangle + \cdots + \alpha_{i-1} \langle e_{i-1}, e_i \rangle +$
$\alpha_i \langle e_i, e_i \rangle + \alpha_{i+1} \langle e_{i+1}, e_i \rangle + \cdots + \alpha_n \langle e_n, e_i \rangle = 0$

$\Rightarrow \alpha_1(0) + \alpha_2(0) + \cdots + \alpha_{i-1}(0) + \alpha_i(1) + \alpha_{i+1}(0) + \cdots + \alpha_n(0) = 0$

($\because$ They are orthonormal)

$\Rightarrow \alpha_i = 0 \ \forall\, i, 1 \le i \le n$.

$\Rightarrow \{e_1, e_2, \ldots, e_n\}$ is linearly independent.

BESSEL'S INEQUALITY:

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal set in an inner product space X and x be an arbitrary element in X. Then for the scalar $a_1, a_2, \ldots, a_n$, the expression $\|x - \sum_{k=1}^{n} a_k e_k\|^2$ assumes its minimum value for $a_k = \alpha_k = \langle x, e_k \rangle$, $k = 1, 2, \ldots, n$. Then, minimum value ~~d~~ equals $\|x\|^2 - \sum_{k=1}^{n} |\alpha_k|^2$ and also $\sum_{k=1}^{\infty} |\alpha_k|^2 \le \|x\|^2$.

PROOF: Obviously $0 \le \|x - \sum_{k=1}^{n} a_k e_k\|^2$

$= \langle x - \sum_{k=1}^{n} a_k e_k, x - \sum_{k=1}^{n} a_k e_k \rangle$

$= \langle x - \sum_{i=1}^{n} a_i e_i, x - \sum_{k=1}^{n} a_k e_k \rangle$

$= \langle x, x \rangle - \langle x, \sum_{k=1}^{n} a_k e_k \rangle - \langle \sum_{i=1}^{n} a_i e_i, x \rangle +$

$\langle \sum_{i=1}^{n} a_i e_i, \sum_{k=1}^{n} a_k e_k \rangle$.

$$= \|\alpha\|^2 - \sum_{k=1}^{n} \overline{a_k} \langle \alpha, e_k \rangle - \sum_{i=1}^{n} a_i \langle e_i, \alpha \rangle + \sum_{i=1}^{n} a_i \sum_{k=1}^{n} \overline{a_k} \langle e_i, e_k \rangle$$

$$= \|\alpha\|^2 - \sum_{k=1}^{n} \overline{a_k} O_k - \sum_{i=1}^{n} a_i \overline{O_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} a_i \overline{a_k} \langle e_i, e_k \rangle$$

$$= \|\alpha\|^2 - \sum_{k=1}^{n} \overline{a_k} O_k - \sum_{i=1}^{n} a_i \overline{O_i} + \sum_{k=1}^{n} a_k \overline{a_k} \ (1)$$

$$= \|\alpha\|^2 - \sum_{k=1}^{n} \overline{a_k} O_k - \sum_{k=1}^{n} a_k \overline{O_k} + \sum_{k=1}^{n} |a_k|^2 \ (\because z\bar{z} = |z|^2)$$

$$= \|\alpha\|^2 - \sum_{k=1}^{n} |O_k|^2 + \sum_{k=1}^{n} |O_k|^2 - \sum_{k=1}^{n} \overline{a_k} O_k - \sum_{k=1}^{n} a_k \overline{O_k} + \sum_{k=1}^{n} |a_k|^2$$

$$= \|\alpha\|^2 - \sum_{k=1}^{n} |O_k|^2 + \sum_{k=1}^{n} \{ |O_k|^2 - \overline{a_k} O_k - a_k \overline{O_k} + |a_k|^2 \}$$

$$= \|\alpha\|^2 - \sum_{k=1}^{n} |O_k|^2 + \sum_{k=1}^{n} \{ |a_k - O_k|^2 \}$$

$$\Rightarrow 0 \le \|\alpha - \sum_{k=1}^{n} a_k e_k\|^2 = \|\alpha\|^2 - \sum_{k=1}^{n} |O_k|^2 + \sum_{k=1}^{n} |a_k - O_k|^2$$

Obviously, $\|\alpha - \sum_{k=1}^{n} a_k e_k\|^2$ has minimum value when: $\sum_{k=1}^{n} |a_k - O_k|^2 = 0 \Rightarrow |a_k - O_k|^2 = 0$

$$\Rightarrow |a_k - O_k| = 0 \Rightarrow a_k = O_k = \langle \alpha, e_k \rangle$$

HENCE, $\|\alpha - \sum_{k=1}^{n} a_k e_k\|^2$ has minimum value

when $a_k = O_k = \langle \alpha, e_k \rangle$

Further more, minimum value is $\|\alpha\|^2 - \sum_{k=1}^{n} |O_k|^2$. Further $0 \le \|\alpha\|^2 - \sum_{k=1}^{n} |O_k|^2$

$\Rightarrow \sum_{k=1}^{n} |O_k|^2 \leq \|x\|^2$.

Since, R.H.S of inequality is independent of n, so it holds even when $n \to \infty$.

i.e $\sum_{k=1}^{\infty} |O_k|^2 \leq \|x\|^2$

TOTAL ORTHONORMAL SETS: (DEF)

For orthogonal sets of vectors $\{e_1, e_2, \ldots e_n\}$ of an inner product space is said to be total orthonormal or closed if for each $x \in X$ and $O_k = \langle x, e_k \rangle$, $\|x\|^2 = \sum_{k=1}^{\infty} |O_k|^2$.

PARSEVEL'S EQUALITY:

An orthonormal system $\{e_1, e_2, \ldots e_n, \ldots\}$ in an inner product space X is total orthonormal iff for all $x \in X$, $x = \sum_{k=1}^{n} O_k e_k$; $O_k = \langle x, e_k \rangle$.

PROOF:

If $\{e_1, e_2, \ldots, e_n, \ldots\}$ is total orthonormal Then $\|x\|^2 = \sum_{k=1}^{\infty} |O_k|^2$ and it holds

iff $\lim_{n \to \infty} \|x - \sum_{k=1}^{n} O_k e_k\|^2 = 0$.

$\Leftrightarrow x = \sum_{k=1}^{\infty} O_k e_k$

COROLLARY:

Let A be a closed subspace of a Hilbert space H. And $x \in H \setminus A$. Then there is a unique $y' \in A$ such that $\delta = \inf_{y \in A} \|x - y\| = \|x - y'\|$.

PROOF:

As H is Hilbert space. So then as a metric space, H is complete. Now as A is closed subspace of a complete space is convex so, A is convex and hence then, by the minimizing vector, there is a unique $y' \in A$ such that:

$$\delta = \inf_{y \in A} \|x - y\|$$
$$= \|x - y'\|$$

Proved.

THESE NOTES are the Lectures delivered by Tahir Mahmood

"LINEAR OPERATOR"

LINEAR OPERATOR: (DEF).

Let X and Y be the two normed spaces. Then, an operator $T: X \to Y$ is said to be linear operator, if $T(\alpha x+\beta y) = \alpha T(x)+\beta T(y)$.

EXAMPLES: Let us define $I: N \to N$ by $I(x) = x$.

Then $I(\alpha x+\beta y) = \alpha x+\beta y$

$= \alpha I(x) + \beta I(y)$

$\Rightarrow I$ is linear operator.

(2). Let K be the space of all analytic functions over $\mathbb{C}$ and $D: K \to K$ be defined by: $D(f) = f'$, then:

$$D(\alpha f+\beta g) = (\alpha f+\beta g)'$$
$$= \alpha f' + \beta g'$$
$$= \alpha D(f) + \beta D(g)$$

"THE KERNEL OR NULL SPACE OF A LINEAR OPERATOR:" (DEF).

Let N and M be the two normed spaces over field F and $T: N \to M$, be a linear operator. Then, kernel of T is denoted and defined by:

$ker(T) = \{x \in N : T(x) = 0_M\}$

CONTINUOUS LINEAR OPERATOR: (DEF).

A linear operator $T: N \to M$ is said to be continuous at a point $x_0 \in N$ if for every $\varepsilon > 0$, there is a real number, $\delta > 0$ such that:

$\|T(x) - T(x_0)\| < \varepsilon$ whenever $\|x - x_0\| < \delta$.

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BOUNDED LINEAR OPERATOR: (DEF).

A linear operator $T: N \to M$ is said to be bounded linear operator if for all $x \in N$, there is some positive real number k such that:

$$\|T(x)\| \leq k\|x\|.$$

THEOREM: Let $T: N \to M$ be a linear operator from a normed space N to normed space M. Then:

1) T is continuous on N iff T is bounded.
2) T is continuous iff T is continuous at $0 \in N$.
3) If T is continuous on N, then $\ker T$ is closed in N.

PROOF:

1) Suppose T is continuous on N, then T is continuous at each point in N, say also at $x_0 \in N$. Then, by the definition of continuity for every $\varepsilon > 0$ there exists a $\delta > 0$ such that:

$$\|T(x) - T(x_0)\| < \varepsilon \text{ whenever, } \|x - x_0\| < \delta.$$

Now let $y \in N$ and put $x = x_0 + \left(\frac{\delta}{2\|y\|}\right)y$.

Then, $x - x_0 = \left(\frac{\delta}{2\|y\|}\right)y$.

$$\Rightarrow \|x - x_0\| = \left\|\left(\frac{\delta}{2\|y\|}\right)y\right\|$$

$$= \frac{\delta}{2\|y\|}\|y\| = \frac{\delta}{2} < \delta.$$

$\Rightarrow \|x - x_0\| < \delta$, so then by continuity of T, $\|T(x) - T(x_0)\| < \varepsilon \Rightarrow \|T(x - x_0)\| < \varepsilon$

$\Rightarrow \|T((\delta/2\|y\|)\, y)\| < \varepsilon$

$\Rightarrow \frac{\delta}{2\|y\|} \|T(y)\| < \varepsilon \quad (\because T(\alpha x) = \alpha T(x))$.

$\Rightarrow \|T(y)\| < \frac{2\varepsilon}{\delta} \|y\|$,

$\Rightarrow \|T(y)\| < K \|y\| \quad (\text{where } K = 2\varepsilon/\delta > 0)$.

$\Rightarrow T$ is bounded.

Conversely, suppose T is bounded on N. Then, $\forall\, x \in N$, there exist some positive real number K such that $\|T(x)\| \leq K\|x\|$.

Now to prove T is continuous on N: Let x_0 be any arbitrary element of N. Then, for any $x \in N$ and for $\varepsilon > 0$, choose $\delta = \varepsilon/K$. Then, whenever $\|x - x_0\| < \delta \Rightarrow \|x - x_0\| < \varepsilon/K$.

$\Rightarrow K\|x - x_0\| < \varepsilon$

Then, $\|T(x) - T(x_0)\| = \|T(x - x_0)\|$

$\leq K\|x - x_0\| \quad (\because T \text{ is bounded})$

$< \varepsilon$.

$\Rightarrow \|T(x) - T(x_0)\| < \varepsilon$

$\Rightarrow T$ is continuous at x_0.

Since x_0 is arbitrary point. So T is also continuous on N.

2). Suppose T is continuous on N and so T is continuous on each point of N as so ultimately is continuous at $0 \in N$.

Conversely, suppose T is continuous at $0 \in N$. To prove: T is continuous on N.

Let $x_0 \in N$. Since, T is continuous at zero so for every $\varepsilon > 0$, there exist a $\delta > 0$ such that,

$\|T(x) - T(0)\| < \varepsilon$ whenever $\|x - 0\| < \delta$

$\Rightarrow \|T(x-0)\| < \varepsilon$ whenever $\|x\| < \delta$

$\Rightarrow \|T(x)\| < \varepsilon$ whenever $\|x\| < \delta$

So, in particular, when for all $x \in N$,

$\|x - x_0\| < \delta$ then $\|T(x - x_0)\| < \varepsilon$

$\Rightarrow \|T(x) - T(x_0)\| < \varepsilon$

$\Rightarrow T$ is continuous at $x_0 \in N$. Since, x_0 is arbitrary, so T is continuous on N.

3). Given T is continuous.

To prove: $\ker T$ is closed.

Let $x \in \overline{\ker T}$. Then there exists a sequence $\{x_n\}$ in $\ker T$ such that $x_n \to x$.

Since, T is continuous and $x_n \to x$, so $T(x_n) \to T(x)$. As for all n, $x_n \in \ker T$.

$\Rightarrow T(x_n) = 0$

$\Rightarrow \forall\, n,\ T(x_n) = 0$ and $T(x_n) \to T(x)$

$\Rightarrow T(x) = 0 \Rightarrow x \in \ker T$

$\Rightarrow \overline{\ker T} \subseteq \ker T \to (1)$

But $\ker T \subseteq \overline{\ker T} \to (2)$

(1) and (2) $\Rightarrow \ker T = \overline{\ker T}$

$\Rightarrow \ker T$ is closed.

NORM OF A BOUNDED LINEAR OPERATOR:

Let $T: N \to M$ is a bounded linear operator, then, $\|T\|$ is denoted and defined by: $\|T\| = \sup_{\substack{x \in N \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}$.

REMARK:

$$\|T\| = \sup_{\substack{x \in N \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}$$

$$= \sup_{x \neq 0} \left\| \frac{1}{\|x\|} T(x) \right\|$$

$$= \sup_{x \neq 0} \left\| T\left(\frac{x}{\|x\|}\right) \right\| \quad (\because \alpha T(x) = T(\alpha x))$$

$$= \sup_{\|x\|=1} \|T(x)\|$$

THEOREM: Prove that every linear operator on a finite dimensional normed space is bounded.

PROOF: Let 'N' be a finite dimensional normed space with basis $B = \{e_1, e_2, \ldots, e_n\}$ and let $T: N \to M$ be a linear operator on N.

To prove: T is bounded.

Let $x \in N$ and B is a basis for N.

So, $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$. $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$.

$$\Rightarrow T(x) = T(\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n).$$

$$= \alpha_1 T(e_1) + \alpha_2 T(e_2) + \cdots + \alpha_n T(e_n)$$

$$\|T(x)\| = \|\alpha_1 T(e_1) + \alpha_2 T(e_2) + \cdots + \alpha_n T(e_n)\|$$

$$\|T(x)\| \leq |\alpha_1|\|T(e_1)\| + |\alpha_2|\|T(e_2)\| + \cdots + |\alpha_n|\|T(e_n)\|$$
$$\leq \lambda\,[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|] \text{ where } \lambda = \max_{i=1}^{n} \|T(e_i)\|$$
$$= \lambda S,\; S = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|.$$
$$\Rightarrow \|T(x)\| \leq \lambda S \rightarrow (1)$$

Now also,
$$\|x\| = \|\alpha_1 T(e_1) + \alpha_2 T(e_2) + \cdots + \alpha_n T(e_n)\|$$
$$\geq C\,[|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|]$$ (By linearly independent lemma)
$$= CS,\; S = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$$
$$\Rightarrow CS \leq \|x\| \Rightarrow S \leq \frac{1}{C}\|x\|,$$

From (1),
$$\|T(x)\| \leq \lambda S \leq \lambda \cdot \frac{1}{C}\|x\|$$
$$\Rightarrow \|T(x)\| \leq \frac{\lambda}{C}\|x\| \Rightarrow \|T(x)\| \leq K\|x\|$$ (K is some +ve real no.)
$$\Rightarrow T \text{ is bounded.}$$

REMARK:

A linear operator defined on a finite dimensional normed space is continuous.

THEOREM:

Let $T_1 : N \rightarrow M$ and $T_2 : M \rightarrow K$ be bounded linear operators, then T_2T_1 is also bounded and $\|T_2T_1\| \leq \|T_2\|\|T_1\| = \|T_1\|\|T_2\|$.

In particular if $T : N \rightarrow N$ is bounded linear operator, then $\|T^n\| \leq \|T\|^n$.

PROOF: As T_1 and T_2 are bounded so:

$$\|T_1(x)\| \leq \|T_1\|\|x\| \quad \forall x.$$

and $\|T_2(x)\| \leq \|T_2\|\|x\|$

Now, $\|(T_2T_1)(x)\| = \|T_2(T_1(x))\|$

$$\leq \|T_2\|\,\|T_1(x)\| \quad (\because T_2 \text{ is bounded}).$$

$$\leq \|T_2\|\|T_1\|\|x\| \quad (\because T_1 \text{ is bounded}).$$

$$\Rightarrow \|(T_2T_1)(x)\| \leq \|T_2\|\|T_1\|\|x\|.$$

$\Rightarrow T_2T_1$ is bounded.

Further, $\|(T_2T_1)(x)\| \leq \|T_2\|\|T_1\|\ \|x\|.$

$$\Rightarrow \frac{\|(T_2T_1)(x)\|}{\|x\|} \leq \|T_2\|\,\|T_1\|.$$

$$\Rightarrow \sup_{x\neq 0} \frac{\|(T_2T_1)(x)\|}{\|x\|} \leq \|T_2\|\,\|T_1\|.$$

$$\Rightarrow \|T_2T_1\| \leq \|T_2\|\|T_1\|.$$

As, $\|T_1\|\,\|T_2\| = \|T_2\|\|T_1\|$

$\Rightarrow$ Also $\|T_2T_1\| \leq \|T_1\|\,\|T_2\|$

Next, $\|T^n\| = \|T^{n-1}T\|$

$$\leq \|T^{n-1}\|\,\|T\|$$

$$= \|T^{n-2}T\|\,\|T\|.$$

$$\leq \|T^{n-2}\|\|T\|\|T\|.$$

$$\vdots$$

$$\leq \|T\|\,\|T\|\,\|T\| \quad n \text{ factors}.$$

$$= \|T\|^n.$$

$$\Rightarrow \|T^n\| \leq \|T\|^n.$$

HENCE PROVED.

THEOREM: The space $B(N,M)$ of all bounded (and hence continuous) linear operators from normed space N to normed space M is a normed space under the defined norm,

$$\|T\| = \sup_{\|x\|=1} \|T(x)\|$$

PROOF:

First we show that $B(N,M)$ is a linear space.

Let $T_1, T_2 \in B(N,M)$ and $\alpha \in F$.

Define $(T_1+T_2)(x) = T_1(x) + T_2(x)$ and

$(\alpha T_1)(x) = \alpha T_1(x)$.

1) $(T_1+T_2)(\alpha x + \beta y) = T_1(\alpha x+\beta y) + T_2(\alpha x+\beta y)$

(By definition of T_1+T_2)

$= \alpha T_1(x) + \beta T_1(y) + \alpha T_2(x) + \beta T_2(y)$

($\because T_1$ and T_2 are linear)

$= \alpha(T_1(x)+T_2(x)) + \beta(T_1(y)+T_2(y))$

$= \alpha(T_1+T_2)(x) + \beta(T_1+T_2)(y)$

$\Rightarrow T_1+T_2$ is linear.

Further as, $\|T_1+T_2\| = \sup_{\|x\|=1} \|(T_1+T_2)(x)\|$

$= \sup_{\|x\|=1} \|T_1(x)+T_2(x)\|$

$\leq \sup_{\|x\|=1} \|T_1(x)\| + \sup_{\|x\|=1} \|T_2(x)\|$

$= \|T_1\| + \|T_2\|$

$\Rightarrow \|T_1+T_2\| \leq \|T_1\| + \|T_2\|$

$\Rightarrow T_1+T_2$ is bounded.

$\Rightarrow T_1+T_2 \in B(N,M)$

$\Rightarrow (B(N,M), +)$ is closed.

As, sum of the mappings is always associative and commutative and so is associative and commutative in $B(N,M)$.

Now define, $0 : N \to M$ by $0(x) = 0$,
then $0(\alpha x + \beta y) = 0 = \alpha \cdot 0 + \beta \cdot 0$
$= \alpha\, 0(x) + \beta\, 0(x)$.
$\Rightarrow$ 0 is linear.
Also, $\|0(x)\| = 0 \Rightarrow$ 0 is bounded.
$\Rightarrow 0 \in B(N,M)$.
And for all $T \in B(N,M)$.
$(0+T)(x) = 0(x) + T(x) = 0 + T(x) = T(x)$.
$(T+0)(x) = T(x) + 0(x) = T(x) + 0 = T(x)$.
$\Rightarrow 0 + T = T + 0 = T$.
$\Rightarrow$ 0 is additive identity in $B(N,M)$.

Further, for any $T \in B(N,M)$, define:
$(-T) : N \to M$ by $(-T)(x) = -T(x)$.

Then, $(-T)(\alpha x + \beta y) = -T(\alpha x + \beta y)$
$= -[\alpha T(x) + \beta T(y)]$
$= -\alpha T(x) - \beta T(y)$
$= \alpha(-T)(x) + \beta(-T)(y)$.
$\Rightarrow$ $-T$ is linear operator.

and $\|(-T)(x)\| = \|-T(x)\| = \|T(x)\|$
$\leq \|T\|\,\|x\|$.
$\Rightarrow$ $-T$ is bounded
$\Rightarrow -T \in B(N,M)$. And $(T+(-T))(x) = (T)(x) + (-T)(x)$
$= T(x) - T(x) = 0 = 0(x)$.
and $(-T+T)(x) = (-T)(x) + T(x)$
$= -T(x) + T(x) = 0 = 0(x)$.

$\Rightarrow (-T+T) = (T+(-T)) = 0$.
$\Rightarrow -T$ is additive inverse of T in $B(N,M)$.

$\Rightarrow (B(N,M),+)$ is an abelian group.

2). Let $\alpha, \beta \in F$ and $T \in B(N,M)$

Then, $((\alpha+\beta)T)(x) = (\alpha+\beta)T(x)$.

$= \alpha T(x) + \beta T(x)$.

$= (\alpha T)(x) + (\beta T)(x)$.

$= (\alpha T + \beta T) x$.

$\Rightarrow (\alpha+\beta)T = \alpha T + \beta T$.

3). Let $\alpha \in F$ and $T_1, T_2 \in B(N,M)$

$(\alpha(T_1+T_2))(x) = \alpha(T_1+T_2)(x)$.

$= \alpha(T_1(x) + T_2(x))$

$= \alpha T_1(x) + \alpha T_2(x)$.

$= (\alpha T_1)(x) + (\alpha T_2)(x)$

$= (\alpha T_1 + \alpha T_2)(x)$

$\Rightarrow (\alpha(T_1+T_2)) = \alpha T_1 + \alpha T_2$.

4). Let $\alpha, \beta \in F$ and $T \in B(N,M)$

$\Rightarrow ((\alpha\beta)T)(x) = (\alpha\beta)T(x)$

$= \alpha(\beta T(x))$.

$= (\alpha(\beta T))(x)$.

$\Rightarrow (\alpha\beta)T = \alpha(\beta T)$.

5). Let $1 \in F$ and $T \in B(N,M)$.

$(1 \cdot T)(x) = 1 \cdot T(x) = T(x)$.

$\Rightarrow 1 \cdot T = T$.

Since, all the conditions are satisfied.

So $B(N,M)$ is linear space.

Now, we show $B(N,M)$ is normed space.

i) As $\forall\ x \in N$, $\|T(x)\| \geq 0$
$\Rightarrow \sup\limits_{\|x\|=1} \|T(x)\| \geq 0 \Rightarrow \|T\| \geq 0$.

ii) $\|T\| = 0 \Leftrightarrow \sup\limits_{\|x\|=1} \|T(x)\| = 0$.
$\Leftrightarrow \|T(x)\| = 0 \quad \forall\, x$.
$\Leftrightarrow T(x) = 0_M \quad \forall\, x$.
$\Leftrightarrow T = 0$.

iii). $\|\alpha T\| = \sup\limits_{\|x\|=1} \|(\alpha T)(x)\|$
$= \sup\limits_{\|x\|=1} \|\alpha T(x)\|$
$= \sup\limits_{\|x\|=1} |\alpha| \|T(x)\| = |\alpha| \sup\limits_{\|x\|=1} \|T(x)\|$.
$= |\alpha| \|T\|$.

iv) $\|T_1 + T_2\| = \sup\limits_{\|x\|=1} \|(T_1+T_2)(x)\|$
$= \sup\limits_{\|x\|=1} \|T_1(x) + T_2(x)\|$
$\leq \sup\limits_{\|x\|=1} \|T_1(x)\| + \sup\limits_{\|x\|=1} \|T_2(x)\| = \|T_1\| + \|T_2\|$.

Since, all the conditions are satisfied.
So $B(N,M)$ is normed space.

THEOREM: If M is Banach space, then so is $B(N,M)$ under the norm defined by:
$$\|T\| = \sup_{\|x\|=1} \|T(x)\| \ , \ x \in N.$$

PROOF: $B(N,M)$ is a normed space (Already Proved)

Now let $\{T_n\}$ be a Cauchy sequence in $B(N,M)$. Then, for every $\varepsilon > 0$, there is a natural number n_0 such that:
$$\|T_n - T_m\| < \varepsilon \ \forall \ m,n \geq n_0.$$
$$\Rightarrow \sup_{\|x\|=1} \|(T_n - T_m)(x)\| < \varepsilon \ , \forall \ m,n \geq n_0.$$
$$\Rightarrow \sup_{\|x\|=1} \|T_n(x) - T_m(x)\| < \varepsilon \ , \forall \ m,n \geq n_0.$$
$$\Rightarrow \|T_n(x) - T_m(x)\| < \varepsilon \ , \forall \ m,n \geq n_0.$$

$\Rightarrow \{T_m(x)\}$ is a Cauchy sequence in M.
Since M is complete.
So $T_m(x) \to T(x) \in M$.
$$\Rightarrow \lim_{m\to\infty} T_m(x) = T(x).$$

Now,
$$T(\alpha x + \beta y) = \lim_{m\to\infty} T_m(\alpha x + \beta y)$$
$$= \lim_{m\to\infty} [\alpha T_m(x) + \beta T_m(y)]$$
$$= \alpha T(x) + \beta T(y).$$
$\Rightarrow T$ is linear.

Now,
$$\|T\| = \lim_{m\to\infty} \|T_m(x)\|$$
$$\leq \lim_{m\to\infty} \|T_m\| \|x\| \leq k\|x\| \ , \ k = \sup \|T_m\|$$

$\Rightarrow T$ is bounded $\Rightarrow T \in B(N, M)$.

Hence, from $\sup\limits_{\|x\|=1} \|T_n(x) - T_m(x)\| < \varepsilon \quad \forall\, m, n \geq n_0$.

when $m \to \infty$,

$$\sup_{\|x\|=1} \| T_n(x) - T(x) \| \leq \varepsilon \quad \forall\, n \geq n_0.$$

$$\Rightarrow \|T_n - T\| < \varepsilon \quad \forall\, n \geq n_0.$$

$\Rightarrow T_n \to T \in B(N, M)$

$\Rightarrow B(N, M)$ is complete.

$\Rightarrow B(N, M)$ is Banach space.

LINEAR FUNCTIONALS:

Let N be a normed space over the field F. Then, a function $f : N \to F$ is said to be linear functional if:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y). \quad \forall\, \alpha, \beta \in F \text{ and } x, y \in N.$$

These Notes are the lectures delivered by Tahir Mahmood.