SPECIAL FUNCTIONS

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Conversion of Some Hypergeometric Functions $_7F_6$ in Terms of $_2F_1$

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Introduction

Functions which occur often enough to acquire a name are called special functions such as gamma function, beta function, Bessel function, exponential function, logarithmic function and various trigonometric functions etc. The hypergeometric functions introduced by Euler, are the special functions that are of major importance in mathematics, both the pure and applied, and in many branches of science. The hypergeometric function is a solution of Euler's hypergeometric differential equation

$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0,$$

which has three regular singular points 0, 1 and infinity. The generalization of this equation to three arbitrary regular singular points is given by Riemann's differential equation. Any second order differential equation with three regular singular points can be converted to the hypergeometric differential equation by changing of variables. The hypergeometric function plays a central role in the realm of special mathematical functions, as all special functions can be expressed in terms of these functions. For instance, the following is a partial list of functions that have hypergeometric representation [9]

$$e^x = {}_0F_0(-,-,x),$$

$$Cosx = {}_0F_1(-;\frac{1}{2};\frac{-x^2}{4}),$$

$$Sinx = z_0F_1(-;\frac{3}{2};\frac{-x^2}{4}),$$

$$(1-x)^{-a} \;=\; {}_1F_0(a;-;x),$$

and

$$P_n(x) \;=\; {}_2F_1(n+1,-n;1;\frac{1-x}{2}) \quad etc.$$

It is thus frequently encountered in pure mathematics, and its parametric nature provides a powerful tool for the solution of a wide range of applied problems. In [8], the term hypergeometric series was first used by John Wallis in his Arithmetica Infinitorum (1655) to describe infinite series of the form $1+a+a(a+1)+a(a+1)(a+2)+\cdots$. During the next one hundred and fifty years, many other mathematicians studied similar series, notably Euler, Vandermonde, Hidenberg etc.

The term hypergeometric can be applied to three objects: the hypergeometric equation (a linear second order differential equation), the hypergeometric series (a particular solution of the hypergeometric equation), and the hypergeometric function (the sum of the hypergeometric series). This type of function has been extensively studied by many mathematicians, including Johann Friedrich Pfaff, Kummer, Euler and Gauss. Srinisvasa Ramanujan also independently discovered many of the classical theorems of this function. Much of the theory of hypergeometric series follows from the work done by Gauss, who presented some of his early results in [6].

Gauss first introduced and studied hypergeometric series, paying special attention to the cases when a series can be summed into an elementary function. This gives one of the motivations for studying hypergeometric series, i.e., the fact that the elementary functions and several other important functions in mathematics can be expressed in terms of hypergeometric functions. Hypergeometric functions can also be described as solutions of special second order linear differential equations, the hypergeometric differential equations. Riemann was first who exploited this idea and introduced a special symbol to classify hypergeometric functions by singularities and exponents of differential equations.

The Gauss hypergeometric function can be extended to form the generalized hypergeometric function, which can contain any number of numerator and denominator

parameters. This extension was first done by Thomas Clausen in 1828, using three numerator and two denominator parameters. The generalized hypergeometric function is exceptionally useful, as all the special functions of mathematical physics can be expressed in terms of these functions, increasing in complexity as the number of parameters increases. The Bessel functions, Whittaker functions and many orthogonal polynomials can be written as generalized hypergeometric functions [14].
Equations involving hypergeometric functions are of great interest to mathematicians and scientists, and newly proven identities for these functions assist in finding solutions for many differential and integral equations [8]. Hypergeometric functions thus provide a rich field for ongoing research, which continues to produce new results.
The confluent hypergeometric function is the solution of differential equation

$$zf''(z) + (\gamma - z)f'(z) - \alpha f(z) = 0,$$

where α, γ, z may be complex [4]. An exact solution was given by Kummer's series [2]. The confluent hypergeometric function of the first kind ${}_1F_1(a;b;z)$ is a degenerate form of the hypergeometric function ${}_2F_1(a,b;c;z)$ which arises as a solution of the confluent hypergeometric differential equation. It is also known as Kummer's function of the first kind. There are a number of other notations used for the function including $F(\alpha,\beta,x)$ (Kummer 1836), $M(a,b,z)$ (Airey and Webb 1918), $\Phi(a;b;z)$ (Humbert 1920) [1].
In [1, 3], the confluent hypergeometric function has a hypergeometric series given by

$${}_1F_1(a;b;1) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots \;=\; \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r}\frac{z^r}{r!},$$

where $(a)_r = a(a+1)(a+2)\cdots(a+r-1)$ is Pochhammer's symbols. The series is convergent for all finite $z \in \mathbb{C}$.
The confluent hypergeometric functions have proved useful in many branches of physics. They have been used in problems involving both diffusion and sedil:eritition, for example, in isotope separation and protein molecular weight determinations in the ultracentrifuge. The solution of the equation for the velocity distribution of electrons in high frequency gas discharges may frequently be expressed in terms of these

functions. The high frequency breakdown electric field may then be predicted theoretically for gases by the use of such solutions together with kinetic theory [10].
Confluent hypergeometric functions are arised naturally in the study of some random variable. The function also contains other functions as a especial cases, including many that are widely used in mathematical physics. Special cases include the Bessel function, incomplete gamma function, Laguerre polynomials, Hermite polynomials, Coulomb wave function and parabolic cylinder functions. The confluent hypergeometric function represents a limiting special case of Gauss hypergeometric function [11].
This project is composed of four chapters. In the first chapter, we describe some basic concepts and definitions. Second chapter is devoted to the proof of some important results that are used further to prove results related to generalized hypergeometric functions and confluent hypergeometric functions. In the third chapter, we present integral representation of hypergeometric functions by using ledendre's duplication formula and give generalized hypergeometric functions in terms of ${}_2F_1$. In fourth chapter, we focus our concern on some results related to confluent hypergeometric function.

Chapter 1

Preliminaries

In this chapter, we give some definitions and basic results that are used in later chapters. It includes Pochhammer's symbol, gamma function, beta function, hypergeometric function, generalized hypergeometric function, confluent hypergeometric function and exponential function.

1.1 Pochhammer's Symbol

Many results involving special functions can be expressed more concisely through the use of rising(shifted) factorial. In [8], this is denoted by Pochhammer's symbol $(a)_n$, a is any complex number, as defined by German mathematician Leo Pochhammer,

$$(a)_n = \begin{cases} a(a+1)(a+2)...(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0, a \neq 0. \end{cases}$$

It follows that $(1)_n = n!$ for $n \geq 1, a \neq 0$ and $(a)_0 = 1$.
Note that $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. In manipulations with $(a)_n$, it is important to keep in mind that $(a)_n$ is a product of n factors, starting with a and with each factor large by unity than the preceding factor.

1.2 Gamma Function

The gamma function has its roots in attempt to extend the factorial function to non natural arguments. In the study of special function, the gamma function is fundamental. Gamma function has a number of different notations such as series, limit and integral form and each form has their own advantages in different applications. Let $z \in \mathbb{C}$ ($\mathbb{C}$ is a set of complex numbers), then the gamma function is defined by [5, 12, 13],

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad Re(z) > 0. \tag{1.2.1}$$

In another way, it is defined as

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^{z-1}}{(z)_n}. \tag{1.2.2}$$

Also Weierstrass defined the Gemma function as

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right) \exp\left(\frac{-z}{n}\right) \right], \tag{1.2.3}$$

where γ is Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} (H_n - \log(n)) \approx 0.57721566490$$

where $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

We can easily find the relationship between Pochhammer's symbol and gamma function which is given as

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}.$$

1.3 Beta Function

In [13], the beta function is denoted by $\beta(p,q)$ and defined as

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt, \qquad \text{where} \quad Re(p) > 0, Re(q) > 0. \tag{1.3.1}$$

Beta function can be defined in terms of algebraic functions by putting $t = \sin^2 \Phi$, given as

$$\beta(p,q) = 2 \int_0^{\frac{\pi}{2}} \sin^{2p-1} \Phi \cos^{2q-1} \Phi d\Phi, \qquad Re(p) > 0, Re(q) > 0. \tag{1.3.2}$$

Also $\beta(p,q) = \beta(q,p), Re(p) > 0$ and $Re(q) > 0$ in [8]. Although the beta function is a function of two variables while the gamma function is a function of only one variable, there exists a powerful direct relation between the two functions which is frequently used in establishing hypergeometric identities.

1.4 Hypergeometric Function

For $z \in \mathbb{C}, |z| < 1$, a, b and c are real or complex parameters with $c \neq 0, -1, -2, \cdots$, the hypergeometric function is the infinite series denoted by $F(a,b;c;z)$ and given by [8]

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!},$$

where $(a)_n = (a)(a+1)(a+2)\cdots(a+n-1)$ and $(a)_0 = 1$. This series is known as Gauss hypergeometric series.

It can be easily shown, by using the ratio test, that this series converges for $|z| < 1$. The Gauss series is clearly symmetrical with respect to its numerator parameters, so that $F(a,b;c;z) = F(b,a;c;z)$ and reduces to unity if one or more of the numerator parameters is zero. If one or more of the numerator parameters is a negative integer $n, n \in \mathbb{N}$, the series reduces to a hypergeometric polynomial which terminates at the $(n+1)$-th term.

1.5 Generalized Hypergeometric Function

The Gauss hypergeometric function can be extended to form the generalized hypergeometric function, which can contain any number of numerator and denominator

parameters.
In [13], the generalized hypergeometric functions with $p+q$ parameters, p parameters in numerator and q parameters in denominator, is defined by

$$ {}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \beta_1, \beta_2, \cdots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_p)_n z^n}{(\beta_1)_n (\beta_2)_n \cdots (\beta_q)_n n!} $$

for all $\alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, \cdots, \quad |z| < 1$ where $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$.
It is known that
i) if $p \leq q$, the series converges for all finite z;
ii) if $p = q + 1$, the series converges for $|z| < 1$ and diverges for $|z| > 1$;
iii) if $p > q + 1$, the series diverges for $z \neq 0$ unless the series terminates.

1.6 Confluent Hypergeometric Functions

Confluent hypergeometric function is a solution of confluent hypergeometric differential equation which is a degenerate form of a hypergeometric differential equation where two regular singularities out of three merge into an irregular singularity.
We define the confluent hypergeometric function as

$$ {}_1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!} \tag{1.6.1} $$

for $|z| < \infty$, $c \neq 0, -1, -2, \cdots$.

1.7 Legendre's Duplication Formula

The Legendre's duplication formula is defined as [13]

$$ \sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z + \frac{1}{2}). \tag{1.7.1} $$

1.8 Gauss Multiplication Theorem

The Gauss multiplication theorem is stated as [13]

$$\prod_{s=1}^{n}\Gamma(z+\frac{s-1}{n}) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-nz}\Gamma(nz). \tag{1.8.1}$$

1.9 Exponential Function

The exponential function can be defined in formal power series as

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots . \end{aligned}$$

1.10 Binomial Theorem

Binomial theorem is stated as

$$(1+x)^n = \binom{n}{0}x^0 + \binom{n}{1}x^1 + \cdots \binom{n}{n}x^n + \cdots \tag{1.10.1}$$

i.e. ,

$$(1+x)^n = \sum_{m=0}^{\infty}\binom{n}{m}x^m \tag{1.10.2}$$

and

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots . \tag{1.10.3}$$

1.11 Binomial Co-efficient In Factorial Function Notation

Binomial coefficient in factorial function notation is given as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1.11.1}$$

and

$$\frac{n!}{(n-r)!} = (-1)^r(-n)_r. \tag{1.11.2}$$

1.12 Newton's Generalized Binomial Theorem

Newton's generalized binomial theorem is stated as

$$\binom{-a}{n} = \frac{(-1)^n(a)_n}{n!}. \tag{1.12.1}$$

Chapter 2

Some important results

In this chapter, we provide some important results that will be used later to prove some important identities of generalized hypergeometric function and confluent hypergeometric function.

2.1 Indication of Important Results

Lemma 2.1.1. *For all z, the Euler product is given as*

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right]. \tag{2.1.1}$$

Proof. Using equation (1.2.3), we have

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)\exp\left(\frac{-z}{n}\right)\right]$$

$$\Gamma(z) = \frac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)^{-1}\exp\left(\frac{z}{n}\right)\right]$$

$$z\Gamma(z) = e^{-\gamma z}\lim_{n\to\infty}\prod_{m=1}^{n}\left[\left(1+\frac{z}{m}\right)^{-1}\exp\left(\frac{z}{m}\right)\right]. \tag{2.1.2}$$

Since

$$\gamma = \lim_{n\to\infty}\big[H_n - \log n\big],$$

and thus

$$\gamma = \lim_{n\to\infty}\big[H_n - \log(n+1)\big]. \tag{2.1.3}$$

Since

$$\begin{aligned}\sum_{m=1}^{n}\log\left(\frac{m+1}{m}\right) &= \sum_{m=1}^{n}\big(\log(m+1)-\log m\big)\\ &= \log 2-\log 1+\log 3-\log 2+...+\log(n+1)-\log n\\ &= \log(n+1),\end{aligned}$$

therefore equation (2.1.3) becomes

$$\gamma=\lim_{n\to\infty}\Big[H_n-\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)\Big]. \tag{2.1.4}$$

Multiply the whole equation (2.1.4) by $-z$, we obtain

$$\begin{aligned}-\gamma z &= \lim_{n\to\infty}\Big[-zH_n+\sum_{m=1}^{n}z\log\big(\frac{m+1}{m}\big)\Big]\\ &= \lim_{n\to\infty}\Big[-zH_n+\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)^z\Big].\end{aligned}$$

By taking exponential, we get

$$e^{-\gamma z}=\lim_{n\to\infty}\Big[\exp(-zH_n)\exp\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)^z\Big]. \tag{2.1.5}$$

Since $H_n=\sum\limits_{m=1}^{n}\frac{1}{m}$, put in equation (2.1.5), we have

$$e^{-\gamma z} = \lim_{n\to\infty}\Big[\exp(-z\sum_{m=1}^{n}\frac{1}{m})\exp\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)^z\Big] \tag{2.1.6}$$

$$= \lim_{n\to\infty}\Big[\exp\big(\sum_{m=1}^{n}\frac{-z}{m}\big)\exp\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)^z\Big]. \tag{2.1.7}$$

Since

$$\exp\sum_{m=1}^{n}\log(1+a_m)=\prod_{m=1}^{n}(1+a_m), \tag{2.1.8}$$

thus

$$e^{-\gamma z}=\lim_{n\to\infty}\exp\sum_{m=1}^{n}\log\left(\exp\big(\frac{-z}{m}\big)\right)\exp\sum_{m=1}^{n}\log\big(\frac{m+1}{m}\big)^z$$

$$e^{-\gamma z} = \lim_{n\to\infty}\prod_{m=1}^{n}\exp\left(\frac{-z}{m}\right)\left(\frac{m+1^{z}}{m}\right). \tag{2.1.9}$$

Put the result as given in equation (2.1.9) in equation (2.1.2), we get

$$\begin{aligned} z\Gamma(z) &= \lim_{n\to\infty}\prod_{m=1}^{n}\left[\exp\left(\frac{-z}{m}\right)\left(\frac{m+1}{m}\right)^{z}\left(1+\frac{z}{m}\right)^{-1}\exp\left(\frac{z}{m}\right)\right] \\ &= \lim_{n\to\infty}\prod_{m=1}^{n}\left[\left(\frac{m+1}{m}\right)^{z}\left(1+\frac{z}{m}\right)^{-1}\right]. \end{aligned}$$

This implies that

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(\frac{n+1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right].$$

□

Lemma 2.1.2. *For all finite z, the difference equation is given below*

$$\Gamma(z+1) = z\Gamma(z). \tag{2.1.10}$$

Proof. Using equation (2.1.1), we have

$$\begin{aligned} \frac{\Gamma(z+1)}{\Gamma(z)} &= \frac{\frac{1}{z+1}\prod_{n=1}^{\infty}[(\frac{n+1}{n})^{z+1}(1+\frac{z+1}{n})^{-1}]}{\frac{1}{z}\prod_{n=1}^{\infty}[(\frac{n+1}{n})^{z}(1+\frac{z}{n})^{-1}]} \\ &= \frac{z}{z+1}\prod_{n=1}^{\infty}\left[\frac{(\frac{n+1}{n})(1+\frac{z}{n})}{1+(\frac{z+1}{n})}\right] \\ &= \frac{z}{z+1}\lim_{n\to\infty}\prod_{m=1}^{n}\left[\frac{(\frac{m+1}{m})(1+\frac{z}{m})}{1+\frac{z+1}{m}}\right] \\ &= \frac{z}{z+1}\lim_{n\to\infty}\prod_{m=1}^{n}\left[\frac{(\frac{m+1}{m})(\frac{m+z}{m})}{(\frac{m+z+1}{m})}\right] \\ &= \frac{z}{z+1}\lim_{n\to\infty}\prod_{m=1}^{n}\left(\frac{m+1}{m}\frac{m+z}{m+z+1}\right) \\ &= \frac{z}{z+1}\lim_{n\to\infty}\left(\frac{n+1}{1}\frac{1+z}{n+z+1}\right) = z. \end{aligned}$$

Therefore

$$\Gamma(z+1) = z\Gamma(z).$$

□

Lemma 2.1.3. *If α is neither zero nor a negative integer, then*

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{2.1.11}$$

Proof. Consider

$$\Gamma(\alpha+n) = \Gamma(\alpha+n-1+1).$$

Using equation (2.1.10), for n a positive integer, we have

$$\begin{aligned}\Gamma(\alpha+n) &= (\alpha+n-1)\Gamma(\alpha+n-1)\\ &= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2)\\ &= (\alpha+n-1)(\alpha+n-2)...\alpha\Gamma(\alpha)\end{aligned}$$

i.e.,

$$\Gamma(\alpha+n) = \alpha(\alpha+1)...(\alpha+n-2)(\alpha+n-1)\Gamma(\alpha).$$

Since

$$(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-2)(\alpha+n-1),$$

therefore we have

$$\Gamma(\alpha+n) = (\alpha)_n\Gamma(\alpha).$$

This implies that

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

□

Lemma 2.1.4. *If $Re(p) > 0$ and $Re(q) > 0$, then*

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{2.1.12}$$

Proof. Using equation (1.2.1), We have

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-u}u^{p-1}du \int_0^\infty e^{-v}v^{q-1}dv. \tag{2.1.13}$$

In equation (2.1.13), use $u = x^2$ and $v = y^2$. Thus $du = 2xdx$ and $dv = 2ydy$.
Now if $u \to 0$, then $x \to 0$. If $v \to 0$, then $y \to 0$. If $u \to \infty$, then $x \to \infty$. If $v \to \infty$, then $y \to \infty$. We have

$$\Gamma(p)\Gamma(q) = \int_0^\infty e^{-x^2}x^{2p-2}2xdx \int_0^\infty e^{-y^2}y^{2q-2}2ydy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}x^{2p-1}y^{2q-1}dxdy. \tag{2.1.14}$$

Let $x = r\cos\theta$, $y = r\sin\theta$ and $r^2 = x^2 + y^2$, where $0 < r < \infty$ and $0 < \theta < \frac{\pi}{2}$, and $dxdy = rdrd\theta$. Put in the equation (2.1.14), we have

$$\begin{aligned}\Gamma(p)\Gamma(q) &= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}(r\cos\theta)^{2p-1}(r\sin\theta)^{2q-1}rdrd\theta \\ &= 4\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-r^2}r^{2p-1}\cos^{2p-1}\theta r^{2q-1}\sin^{2q-1}\theta rdrd\theta\end{aligned}$$

$$= 2\int_0^\infty e^{-r^2}r^{2p+2q-2}2rdr\int_0^{\frac{\pi}{2}}\cos^{2p-1}\theta\sin^{2q-1}\theta d\theta. \tag{2.1.15}$$

Put the following in equation (2.1.15)

$$\begin{aligned} r^2 &= t & \theta &= \frac{\pi}{2} - \phi \\ 2rdr &= dt & d\theta &= -d\phi \end{aligned}$$

as $r \to 0$, $t \to 0$ and as $r \to \infty$, $t \to \infty$. Also if $\theta = 0$, then $\phi = \frac{\pi}{2}$ and if $\theta = \frac{\pi}{2}$, then $\phi = 0$.
We have

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-t}t^{p+q-1}dt \quad 2\int_0^{\frac{\pi}{2}}\sin^{2p-1}\phi\cos^{2q-1}\phi d\phi \\ &= \Gamma(p+q)\beta(p,q).\end{aligned}$$

by using equation (1.2.1) and equation (1.3.2). This implies that

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

□

Lemma 2.1.5. *Prove that*

$$\sum_{n=0}^\infty\sum_{r=0}^\infty A(r,n) = \sum_{n=0}^\infty\sum_{r=0}^n A(r,n-r). \tag{2.1.16}$$

Proof. Consider the series $\sum\limits_{n=0}^{\infty}\sum\limits_{r=0}^{\infty} A\big(r,n\big)t^{n+r}$.
Put $r = j, n = i - j$.
Since $r \geq 0 \Rightarrow j \geq 0$ and $n \geq 0 \Rightarrow i - j \geq 0$.
Also, $n + r = i - j + j = i$. Since $i - j \geq 0$ i.e., $j \leq i \Rightarrow 0 \leq j \leq i$ and $i \geq 0$. Then

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A\big(r,n\big)t^{n+r} = \sum_{i=0}^{\infty}\sum_{j=0}^{i} A\big(j, i-j\big)t^{i}.$$

Put $t = 1$ in above equation, we have

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A\big(r,n\big) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} A\big(j, i-j\big).$$

Replace i by n and j by r, we get

$$\sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A\big(r,n\big) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} A\big(r, n-r\big).$$

□

Lemma 2.1.6. *Prove that*

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{n!}. \tag{2.1.17}$$

Proof. The binomial theorem states that

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(-a)(-a-1)(-a-2)...(-a-n+1)(-1)^n y^n}{n!},$$

which may be written as

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)(a+1)(a+2)...(a+n-1)y^n}{n!}.$$

Therefore, in factorial function notation,

$$(1-y)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n y^n}{n!}.$$

□

Chapter 3

Some Hypergeometric Functions in terms of ${}_2F_1$

In this chapter, we present generalized hypergeometric functions ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$ in terms of ${}_2F_1$.

3.1 An Integral Representation of Generalized Hypergeometric Function

We give an integral representation of ${}_3F_2$ by using Legendre's duplication formula as in [7].

Theorem 3.1.1. *If $Re(6c-6b-a)>0$, then*

$$
\begin{aligned}
&{}_7F_6\Big(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{5},c+\frac{5}{6};1\Big)\\
&=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6c-a)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(6b)_{j_1}}{(6c-a)_{j_1}}\prod_{r=4}^{6}\Bigg\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{(6b+\sum_{s=1}^{r-3}j_s)_{j_{r-2}}}{(6c-a+\sum_{s=1}^{r-3}j_s)_{j_{r-2}}}\Bigg\}\\
&\qquad\times{}_2F_1(-j_4,6b+j_1+j_2+j_3+j_4;6c-a+j_1+j_2+j_3+j_4;-1). \qquad (3.1.1)
\end{aligned}
$$

Proof. By the definition of generalized hypergeometric function

$$ {}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1) = \sum_{n=0}^{\infty}\frac{(a)_n\prod\limits_{p=1}^{6}(b+\frac{p-1}{6})_n}{\prod\limits_{q=1}^{6}(c+\frac{q-1}{6})_n n!}. $$

Since $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, we have

$$ {}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6};1) = \sum_{n=0}^{\infty}\frac{(a)_n\prod\limits_{p=1}^{6}\Gamma(b+\frac{p-1}{6}+n)\prod\limits_{q=1}^{6}\Gamma(c+\frac{q-1}{6})}{\prod\limits_{p=1}^{6}\Gamma(b+\frac{p-1}{6})\prod\limits_{q=1}^{6}\Gamma(c+\frac{q-1}{6}+n)n!} $$

$$ = \frac{\prod\limits_{q=1}^{6}\Gamma(c+\frac{q-1}{6})}{\prod\limits_{p=1}^{6}\Gamma(b+\frac{p-1}{6})}\sum_{n=0}^{\infty}\frac{(a)_n\prod\limits_{p=1}^{6}\Gamma(b+\frac{p-1}{6}+n)}{\prod\limits_{q=1}^{6}\Gamma(c+\frac{q-1}{6}+n)n!}. \tag{3.1.2} $$

The Gauss multiplication theorem given as

$$ \prod_{r=1}^{m}\Gamma(z+\frac{r-1}{m}) = (2\pi)^{\frac{1}{2}(m-1)}m^{\frac{1}{2}-mz}\Gamma(mz). $$

Put $m=6$ in above equation, we get

$$ \prod_{p=1}^{6}\Gamma(z+\frac{p-1}{6}) = (2\pi)^{\frac{5}{2}}(6)^{\frac{1}{2}-6z}\Gamma(6z). $$

Then (3.1.6) implies that

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{(2\pi)^{\frac{5}{2}}(6)^{\frac{1}{2}-6c}\Gamma(6c)}{(2\pi)^{\frac{5}{2}}(6)^{\frac{1}{2}-6b}\Gamma(6b)}\sum_{n=0}^{\infty}\frac{(a)_n(2\pi)^{\frac{5}{2}}(6)^{\frac{1}{2}-6b-6n}\Gamma(6b+6n)}{(2\pi)^{\frac{5}{2}}(6)^{\frac{1}{2}-6c-6n}\Gamma(6c+6n)n!}
$$

$$
= \frac{\Gamma(6c)}{\Gamma(6b)}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(6b+6n)}{\Gamma(6c+6n)n!}.
$$

Multiplying and dividing by $\Gamma(6c-6b)$, we have

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{n=0}^{\infty}\frac{(a)_n\Gamma(6b+6n)\Gamma(6c-6b)}{\Gamma(6c+6n)n!}.
$$

Now by using equation (2.1.12), we get

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\beta(6b+6n,6c-6b).
$$

By using equation (1.3.1)

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{n=0}^{\infty}\frac{(a)_n}{n!}\int_0^1 t^{6b+6n-1}(1-t)^{6c-6b-1}dt.
$$

Since the series is uniformly convergent, therefore we interchange order of summation and integral as

$$ {}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1) = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-1}\sum_{n=0}^{\infty}\frac{(a)_n(t^6)^n}{n!}dt. $$

Now by using equation (2.1.17), we can write it as

$$ {}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1) = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-1}(1-t^6)^{-a}dt $$

$$ = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-1}[(1-t)(1+t+t^2+t^3+t^4+t^5)]^{-a}dt $$

$$ = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-1}(1-t)^{-a}(1+t+t^2+t^3+t^4+t^5)^{-a}dt $$

$$ = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-a-1}(1+(t+t^2+t^3+t^4+t^5))^{-a}dt. $$

Now by using equation (1.10.2), we get

$$ {}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1) = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-a-1}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}(t+t^2+t^3+t^4+t^5)^{j_1}dt $$

$$ = \frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\int_0^1 t^{6b-1}(1-t)^{6c-6b-a-1}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}t^{j_1}(1+t+t^2+t^3+t^4)^{j_1}dt. $$

Since the series is uniformly convergent, therefore

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{6b+j_1-1}(1-t)^{6c-6b-a-1}(1+t+t^2+t^3+t^4)^{j_1}dt. \tag{3.1.3}
$$

By using again equation (1.10.2) in equation (3.1.7), we have

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{6b+j_1-1}(1-t)^{6c-6b-a-1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}(t+t^2+t^3+t^4)^{j_2}dt
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{6b+j_1-1}(1-t)^{6c-6b-a-1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}t^{j_2}(1+t+t^2+t^3)^{j_2}dt
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\int_0^1 t^{6b+j_1+j_2-1}(1-t)^{6c-6b-a-1}(1+t+t^2+t^3)^{j_2}dt.
$$

Again applying equation (1.10.2) and interchanging the order of summation and integration, we get

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6};1)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\int_0^1 t^{6b+j_1+j_2-1}(1-t)^{6c-6b-a-1}
$$

$$
\times\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}(t+t^2+t^3)^{j_3}dt
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}
$$

$$
\times\int_0^1 t^{6b+j_1+j_2+j_3-1}(1-t)^{6c-6b-a-1}(1+t+t^2)^{j_3}dt.
$$

Again applying equation (1.10.2) and interchanging the order of summation and integration,we get

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\int_0^1 t^{6b+j_1+j_2+j_3-1}(1-t)^{6c-6b-a-1}
$$

$$
\times\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}(t+t^2)^{j_4}dt
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\int_0^1 t^{6b+j_1+j_2+j_3+j_4-1}(1-t)^{6c-6b-a-1}(1+
$$

Now by using equation (2.1.17), we obtain

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\int_0^1 t^{6b+j_1+j_2+j_3+j_4-1}(1-t)^{6c-6b-a-1}
$$

$$
\times\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(-1)^{j_5}t^{j_5}}{j_5!}dt
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}
$$

$$
\times\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(-1)^{j_5}}{j_5!}\int_0^1 t^{6b+j_1+j_2+j_3+j_4+j_5-1}(1-t)^{6c-6b-a-1}dt.
$$

By using equation (1.3.1), we can write it as

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6}.c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(-1)^{j_5}}{j_5!}
\times\beta(6b+j_1+j_2+j_3+j_4+j_5,6c-6b-a)
$$

$$
=\frac{\Gamma(6c)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(-1)^{j_5}}{j_5!}
\times\frac{\Gamma(6b+j_1+j_2+j_3+j_4+j_5)\Gamma(6c-6b-a)}{\Gamma(6c-a+j_1+j_2+j_3+j_4+j_5)}.
$$

Multiplying and dividing by $\frac{\Gamma(6c-a+j_1+j_2+j_3+j_4)}{\Gamma(6b+j_1+j_2+j_3+j_4)}$, we get

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(-1)^{j_5}}{j_5!}
$$

$$
\times\frac{\Gamma(6b+j_1+j_2+j_3+j_4+j_5)\Gamma(6c-a+j_1+j_2+j_3+j_4)}{\Gamma(6c-a+j_1+j_2+j_3+j_4+j_5)\Gamma(6b+j_1+j_2+j_3+j_4)}\frac{\Gamma(6b+j_1+j_2+j_3+j_4)}{\Gamma(6c-a+j_1+j_2+j_3+j_4)}.
$$

Now using equation (2.1.11), we have

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$

$$
=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(-1)^{j_5}}{j_5!}
\times\frac{(6b+j_1+j_2+j_3+j_4)_{j_5}}{(6c-a+j_1+j_2+j_3+j_4)_{j_5}}\frac{\Gamma(6b+j_1+j_2+j_3+j_4)}{\Gamma(6c-a+j_1+j_2+j_3+j_4)}.
$$

Since

$$
{}_2F_1(-j_4,6b+j_1+j_2+j_3+j_4;6c-a+j_1+j_2+j_3+j_4;-1)
=\sum_{j_5=0}^{j_4}\frac{(-j_4)_{j_5}(6b+j_1+j_2+j_3+j_4)_{j_5}(-1)^{j_5}}{(6c-a+j_1+j_2+j_3+j_4)_{j_5}(j_5)!},
$$

therefore we have

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\frac{\Gamma(6b+j_1+j_2+j_3+j_4)}{\Gamma(6c-a+j_1+j_2+j_3+j_4)}
$$
$$
\times\,{}_2F_1(-j_4,6b+j_1+j_2+j_3+j_4;6c-a+j_1+j_2+j_3+j_4;-1).
$$

Multiplying and dividing by $\frac{\Gamma(6c-a+j_1+j_2+j_3)}{\Gamma(6b+j_1+j_2+j_3)}$ and again using equation (2.1.11), we get

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\frac{(6b+j_1+j_2+j_3)_{j_4}}{(6c-a+j_1+j_2+j_3)_{j_4}}
$$
$$
\times\frac{\Gamma(6b+j_1+j_2+j_3)}{\Gamma(6c-a+j_1+j_2+j_3)}{}_2F_1(-j_4,6b+j_1+j_2+j_3+j_4;6c-a+j_1+j_2+j_3+j_4;-1).
$$

Multiplying and dividing by $\frac{\Gamma(6c-a+j_1+j_2)}{\Gamma(6b+j_1+j_2)}$ and again using equation (2.1.11), we get

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\frac{(6b+j_1+j_2+j_3)_{j_4}}{(6c-a+j_1+j_2+j_3)_{j_4}}\frac{(6b+j_1 -}{(6c-a+j}
$$
$$
\times\frac{\Gamma(6b+j_1+j_2)}{\Gamma(6c-a+j_1+j_2)}{}_2F_1(-j_4,6b+j_1+j_2+j_3+j_4;6c-a+j_1+j_2+j_3+j_4;-1).
$$

Multiplying and dividing by $\frac{\Gamma(6c-a+j_1)\Gamma(6c-a)}{\Gamma(6b+j_1)}$ and using equation (2.1.11), we can write it as

$$
{}_7F_6(a,b,b+\frac{1}{6},b+\frac{2}{6},b+\frac{3}{6},b+\frac{4}{6},b+\frac{5}{6};c,c+\frac{1}{6},c+\frac{2}{6},c+\frac{3}{6},c+\frac{4}{6},c+\frac{5}{6};1)
$$
$$
=\frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6c-a)\Gamma(6c-6b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\binom{j_3}{j_4}\frac{\Gamma(6b+j_1)\Gamma(6c-a)}{\Gamma(6c-a+j_1)\Gamma(6b)}
$$
$$
\times\frac{(6b+j_1)_{j_2}}{(6c-a+j_1)_{j_2}}\frac{(6b+j_1+j_2)_{j_3}}{(6c-a+j_1+j_2)_{j_3}}\frac{(6b+j_1+j_2+j_3)_{j_4}}{(6c-a+j_1+j_2+j_3)_{j_4}}{}_2F_1(-j_4,6b+j_1+j_2+j_3+j_4;6c-a+j_1+j_2
$$

$$= \frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-a)\Gamma(6c-6b)} \sum_{j_1=0}^{\infty} \binom{-a}{j_1} \sum_{j_2=0}^{j_1} \binom{j_1}{j_2} \sum_{j_3=0}^{j_2} \binom{j_2}{j_3} \sum_{j_4=0}^{j_3} \binom{j_3}{j_4} \frac{(6b)_{j_1}}{(6c-a)_{j_1}} \frac{(6b+j_1)_{j_2}}{(6c-a+j_1)_{j_2}}$$

$$\times \frac{(6b+j_1+j_2)_{j_3}}{(6c-a+j_1+j_2)_{j_3}} \times \frac{(6b+j_1+j_2+j_3)_{j_4}}{(6c-a+j_1+j_2+j_3)_{j_4}} {}_2F_1(-j_4, 6b+j_1+j_2+j_3+j_4; 5c-a+j_1+j_2+j_3+j_4; -1)$$

$$= \frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6b)\Gamma(6c-a)\Gamma(6c-6b)} \sum_{j_1=0}^{\infty} \binom{-a}{j_1} \frac{(6b)_{j_1}}{(6c-a)_{j_1}} \sum_{j_2=0}^{j_1} \binom{j_1}{j_2} \frac{(6b+j_1)_{j_2}}{(6c-a+j_1)_{j_2}}$$

$$\times \sum_{j_3=0}^{j_2} \binom{j_2}{j_3} \frac{(6b+j_1+j_2)_{j_3}}{(6c-a+j_1+j_2)_{j_3}} \times \sum_{j_4=0}^{j_3} \binom{j_3}{j_4} \frac{(6b+j_1+j_2+j_3)_{j_4}}{(6c-a+j_1+j_2+j_3)_{j_4}} {}_2F_1(-j_4, 6b+j_1+j_2+j_3+j_4; 6c-a+$$

$$= \frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6c-a)\Gamma(6c-6b)} \sum_{j_1=0}^{\infty} \binom{-a}{j_1} \frac{(6b)_{j_1}}{(6c-a)_{j_1}}$$

$$\times \prod_{r=4}^{6} \left\{ \sum_{j_{r-2}=0}^{j_{r-3}} \binom{j_{r-3}}{j_{r-2}} \frac{(6b+\sum_{s=1}^{r-3} j_s)_{j_{r-2}}}{(6c-a+\sum_{s=1}^{r-3} j_s)_{j_{r-2}}} \right\} {}_2F_1(-j_4, 6b+j_1+j_2+j_3+j_4; 6c-a+j_1+j_2+j_3+j_4; -1).$$

□

Corollary 3.1.2. *If* $Re(6c-6b+n) > 0$, $n \in \mathbb{Z}^+$, *then*

$${}_7F_6\big(-n, b, b+\frac{1}{6}, b+\frac{2}{6}, b+\frac{3}{6}, b+\frac{4}{6}, b+\frac{5}{6}; c, c+\frac{1}{6}, c+\frac{2}{6}, c+\frac{3}{6}, c+\frac{4}{6}, c+\frac{5}{6}; 1)\big)$$

$$= \frac{(6c-6b)_n}{(6c)_n} \sum_{j_1=0}^{n} \binom{n}{j_1} \frac{(6b)_{j_1}}{(6c+n)_{j_1}} \prod_{r=4}^{6} \left\{ \sum_{j_{r-2}=0}^{jr-3} \binom{j_{r-3}}{j_{r-2}} \frac{(6b+\sum_{s=1}^{r-3} j_s)_{j_{r-2}}}{(6c+n+\sum_{s=1}^{r-3} j_s)_{j_{r-2}}} \right\}$$

$$\times {}_2F_1(-j_4, 6b+\sum_{k=1}^{4} j_k; 6c+n+\sum_{k=1}^{4} j_k; -1). \quad (3.1.4)$$

Proof. From equation (3.1.5), we have

$$ {}_7F_6(a, b, b+\frac{1}{6}, b+\frac{2}{6}, b+\frac{3}{6}, b+\frac{4}{6}, b+\frac{5}{6}; c, c+\frac{1}{6}, c+\frac{2}{6}, c+\frac{3}{6}, c+\frac{4}{6}, c+\frac{5}{6}; 1) $$

$$ = \frac{\Gamma(6c)\Gamma(6c-6b-a)}{\Gamma(6c-a)\Gamma(6c-5b)} \sum_{j_1=0}^{\infty} \binom{-a}{j_1} \frac{(6b)_{j_1}}{(6c-a)_{j_1}} \prod_{r=4}^{6} \Big\{ \sum_{j_{r-2}=0}^{j_{r-3}} \binom{j_{r-3}}{j_{r-2}} \frac{(6b+\sum\limits_{s=1}^{r-3} j_s)_{j_{r-2}}}{(6c-a+\sum\limits_{s=1}^{r-3} j_s)_{j_{r-2}}} \Big\} $$

$$ \times {}_2F_1(-j_4, 6b+j_1+j_2+j_3+j_4; 6c-a+j_1+j_2+j_3+j_4; -1). $$

Put $a=-n$ in above equation, we get

$$ {}_7F_6(-n, b, b+\frac{1}{6}, b+\frac{2}{6}, b+\frac{3}{6}, b+\frac{4}{6}, b+\frac{5}{6}; c, c+\frac{1}{6}, c+\frac{2}{6}, c+\frac{3}{6}, c+\frac{4}{6}, c+\frac{5}{6}; 1) $$

$$ = \frac{\Gamma(6c)\Gamma(6c-6b+n)}{\Gamma(6c+n)\Gamma(6c-6b)} \sum_{j_1=0}^{n} \binom{n}{j_1} \frac{(6b)_{j_1}}{(6c+n)_{j_1}} \prod_{r=4}^{6} \Big\{ \sum_{j_{r-2}=0}^{j_{r-3}} \binom{j_{r-3}}{j_{r-2}} \frac{(6b+\sum\limits_{s=1}^{r-3} j_s)_{j_{r-2}}}{(6c+n+\sum\limits_{s=1}^{r-3} j_s)_{j_{r-2}}} \Big\} $$

$$ \times {}_2F_1(-j_4, 6b+j_1+j_2+j_3+j_4; 6c+n+j_1+j_2+j_3+j_4; -1). $$

Since $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, therefore we have

$$ {}_7F_6\big(-n, b, b+\frac{1}{6}, b+\frac{2}{6}, b+\frac{3}{6}, b+\frac{4}{6}, b+\frac{5}{6}; c, c+\frac{1}{6}, c+\frac{2}{6}, c+\frac{3}{6}, c+\frac{4}{6}, c+\frac{5}{6}; 1\big) $$

$$ = \frac{(6c-6b)_n}{(6c)_n} \sum_{j_1=0}^{n} \binom{n}{j_1} \frac{(6b)_{j_1}}{(6c+n)_{j_1}} \prod_{r=4}^{6} \Big\{ \sum_{j_{r-2}=0}^{jr-3} \binom{j_{r-3}}{j_{r-2}} \frac{(6b+\sum\limits_{s=1}^{r-3} j_s)_{j_{r-2}}}{(6c+n+\sum\limits_{s=1}^{r-3} j_s)_{j_{r-2}}} \Big\} $$

$$ \times {}_2F_1(-j_4, 6b+\sum_{k=1}^{4} j_k; 6c+n+\sum_{k=1}^{4} j_k; -1). $$

$\square$

Theorem 3.1.3. *If* $Re(5c-5b-a)>0$*, then*

$$
{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)
$$

$$
=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5c-a)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(5b)_{j_1}}{(5c-a)_{j_1}}\prod_{r=4}^{5}\left\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{(5b+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}{(5c-a+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}\right\}
$$

$$
\times\,{}_2F_1(-j_3,4b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1). \quad (3.1.5)
$$

Proof. By the definition of generalized hypergeometric function

$$
{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)
=\sum_{n=0}^{\infty}\frac{(a)_n\prod\limits_{p=1}^{5}(b+\frac{p-1}{5})_n}{\prod\limits_{q=1}^{5}(c+\frac{q-1}{5})_n n!}.
$$

Since $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$, we have

$$
{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)
$$

$$
=\sum_{n=0}^{\infty}\frac{(a)_n\prod\limits_{p=1}^{5}\Gamma(b+\frac{p-1}{5}+n)\prod\limits_{q=1}^{5}\Gamma(c+\frac{q-1}{5})}{\prod\limits_{p=1}^{5}\Gamma(b+\frac{p-1}{5})\prod\limits_{q=1}^{5}\Gamma(c+\frac{q-1}{5}+n)n!}
$$

$$
=\frac{\prod\limits_{q=1}^{5}\Gamma(c+\frac{q-1}{5})}{\prod\limits_{p=1}^{5}\Gamma(b+\frac{p-1}{5})}\sum_{n=0}^{\infty}\frac{(a)_n\prod\limits_{p=1}^{5}\Gamma(b+\frac{p-1}{5}+n)}{\prod\limits_{q=1}^{5}\Gamma(c+\frac{q-1}{5}+n)n!}. \quad (3.1.6)
$$

The Gauss multiplication theorem given as

$$
\prod_{r=1}^{m}\Gamma(z+\frac{r-1}{m})=(2\pi)^{\frac{1}{2}(m-1)}m^{\frac{1}{2}-mz}\Gamma(mz).
$$

Put $m = 5$ in above equation, we get

$$\prod_{p=1}^{5} \Gamma(z + \frac{p-1}{5}) = (2\pi)^2 (5)^{\frac{1}{2}-5z} \Gamma(5z).$$

Then (3.1.6) implies that

$$_6F_5(a, b, b + \frac{1}{5}, b + \frac{2}{5}, b + \frac{3}{5}, b + \frac{4}{5}; c, c + \frac{1}{5}, c + \frac{2}{5}, c + \frac{3}{5}, c + \frac{4}{5}; 1)$$
$$= \frac{(2\pi)^2 (5)^{\frac{1}{2}-5c} \Gamma(5c)}{(2\pi)^2 (5)^{\frac{1}{2}-5z} \Gamma(5b)} \sum_{n=0}^{\infty} \frac{(a)_n (2\pi)^2 (5)^{\frac{1}{2}-5b-5n} \Gamma(5b + 5n)}{(2\pi)^2 (5)^{\frac{1}{2}-5c-5n} \Gamma(5c + 5n) n!}$$

$$= \frac{\Gamma(5c)}{\Gamma(5b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(5b + 5n)}{\Gamma(5c + 5n) n!}.$$

Multiplying and dividing by $\Gamma(5c - 5b)$, we have

$$_6F_5(a, b, b + \frac{1}{5}, b + \frac{2}{5}, b + \frac{3}{5}, b + \frac{4}{5}; c, c + \frac{1}{5}, c + \frac{2}{5}, c + \frac{3}{5}, c + \frac{4}{5}; 1)$$
$$= \frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c - 5b)} \sum_{n=0}^{\infty} \frac{(a)_n \Gamma(5b + 5n) \Gamma(5c - 5b)}{\Gamma(5c + 5n) n!}.$$

Now by using equation (2.1.12), we get

$$_6F_5(a, b, b + \frac{1}{5}, b + \frac{2}{5}, b + \frac{3}{5}, b + \frac{4}{5}; c, c + \frac{1}{5}, c + \frac{2}{5}, c + \frac{3}{5}, c + \frac{4}{5}; 1)$$
$$= \frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c - 5b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \beta(5b + 5n, 5c - 5b).$$

By using equation (1.3.1)

$$_6F_5(a, b, b + \frac{1}{5}, b + \frac{2}{5}, b + \frac{3}{5}, b + \frac{4}{5}; c, c + \frac{1}{5}, c + \frac{2}{5}, c + \frac{3}{5}, c + \frac{4}{5}; 1)$$
$$= \frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c - 5b)} \sum_{n=0}^{\infty} \frac{(a)_n}{n!} \int_0^1 t^{5b+5n-1} (1 - t)^{5c-5b-1} dt.$$

Since the series is uniformly convergent, therefore we interchange order of summation and integral as

$$ {}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1) $$
$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-1}\sum_{n=0}^{\infty}\frac{(a)_n(t^5)^n}{n!}dt. $$

Now by using equation (2.1.17), we can write it as

$$ {}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1) $$
$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-1}(1-t^5)^{-a}dt $$

$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-1}[(1-t)(1+t+t^2+t^3+t^4)]^{-a}dt $$
$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-1}(1-t)^{-a}(1+t+t^2+t^3+t^4)^{-a}dt $$
$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-a-1}(1+(t+t^2+t^3+t^4))^{-a}dt. $$

Now by using equation (1.10.2), we get

$$ {}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1) $$
$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-a-1}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}(t+t^2+t^3+t^4)^{j_1}dt $$

$$ =\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\int_0^1 t^{5b-1}(1-t)^{5c-5b-a-1}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}t^{j_1}(1+t+t^2+t^3)^{j_1}dt. $$

Since the series is uniformly convergent, therefore

$$
{}_6F_5\left(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1\right)
$$

$$
=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{5b+j_1-1}(1-t)^{5c-5b-a-1}(1+t+t^2+t^3)^{j_1}dt. \tag{3.1.7}
$$

By using again equation (1.10.2) in equation (3.1.7), we have

$$
{}_6F_5\left(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1\right)
$$

$$
=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{5b+j_1-1}(1-t)^{5c-5b-a-1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}(t+t^2+t^3)^{j_2}dt
$$

$$
=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\int_0^1 t^{5b+j_1-1}(1-t)^{5c-5b-a-1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}t^{j_2}(1+t+t^2)^{j_2}dt
$$

$$
=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\int_0^1 t^{5b+j_1+j_2-1}(1-t)^{5c-5b-a-1)}(1+t+t^2)^{j_2}dt.
$$

Again applying equation (1.10.2) and interchanging the order of summation and integration, we get

$$
{}_6F_5\left(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1\right)
$$

$$
=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\int_0^1 t^{5b+j_1+j_2-1}(1-t)^{5c-5b-a-1}
$$

$$
\times\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}(t+t^2)^{j_3}dt
$$

$$
=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}
$$

$$
\times\int_0^1 t^{5b+j_1+j_2+j_3-1}(1-t)^{5c-5b-a-1}(1+t)^{j_3}dt.
$$

Now by using equation (2.1.17), we obtain

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\int_0^1 t^{5b+j_1+j_2+j_3-1}(1-t)^{5c-5b-a-1}\\
&\qquad\times\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(-1)^{j_4}t^{j_4}}{j_4!}dt
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\\
&\qquad\times\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(-1)^{j_4}}{j_4!}\int_0^1 t^{5b+j_1+j_2+j_3+j_4-1}(1-t)^{5c-5b-a-1}dt.
\end{aligned}
$$

By using equation (1.3.1), we can write it as

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(-1)^{j_4}}{j_4!}\\
&\qquad\times\beta(5b+j_1+j_2+j_3+j_4,5c-5b-a)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\Gamma(5c)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(-1)^{j_4}}{j_4!}\\
&\qquad\times\frac{\Gamma(5b+j_1+j_2+j_3+j_4)\Gamma(5c-5b-a)}{\Gamma(5c-a+j_1+j_2+j_3+j_4)}.
\end{aligned}
$$

Multiplying and dividing by $\frac{\Gamma(5c-a+j_1+j_2+j_3)}{\Gamma(5b+j_1+j_2+j_3)}$, we get

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(-1)^{j_4}}{j_4!}\\
&\times\frac{\Gamma(5b+j_1+j_2+j_3+j_4)\Gamma(5c-a+j_1+j_2+j_3)}{\Gamma(5c-a+j_1+j_2+j_3+j_4)\Gamma(5b+j_1+j_2+j_3)}\frac{\Gamma(5b+j_1+j_2+j_3)}{\Gamma(5c-a+j_1+j_2+j_3)}.
\end{aligned}
$$

Now using equation (2.1.11), we have

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&\quad=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(-1)^{j_4}}{j_4!}\\
&\qquad\times\frac{(5b+j_1+j_2+j_3)_{j_4}}{(5c-a+j_1+j_2+j_3)_{j_4}}\frac{\Gamma(5b+j_1+j_2+j_3)}{\Gamma(5c-a+j_1+j_2+j_3)}.
\end{aligned}
$$

Since

$$
\begin{aligned}
&{}_2F_1(-j_3,5b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1)\\
&\qquad=\sum_{j_4=0}^{j_3}\frac{(-j_3)_{j_4}(5b+j_1+j_2+j_3)_{j_4}(-1)^{j_4}}{(5c-a+j_1+j_2+j_3)_{j_4}(j_4)!},
\end{aligned}
$$

therefore we have

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&\quad=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\frac{\Gamma(5b+j_1+j_2+j_3)}{\Gamma(5c-a+j_1+j_2+j_3)}\\
&\qquad\times{}_2F_1(-j_3,5b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1).
\end{aligned}
$$

Multiplying and dividing by $\frac{\Gamma(5c-a+j_1+j_2)}{\Gamma(5b+j_1+j_2)}$ and again using equation (2.1.11), we get

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&\quad=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5b)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\frac{(5b+j_1+j_2)_{j_3}}{(5c-a+j_1+j_2)_{j_3}}\\
&\qquad\times\frac{\Gamma(5b+j_1+j_2)}{\Gamma(5c-a+j_1+j_2)}{}_2F_1(-j_3,5b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1).
\end{aligned}
$$

Multiplying and dividing by $\frac{\Gamma(5c-a+j_1)\Gamma(5c-a)}{\Gamma(5b+j_1)}$ and using equation (2.1.11), we can write it as

$$
{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)
$$

$$
=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5c-a)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\frac{\Gamma(5b+j_1)\Gamma(5c-a)}{\Gamma(5c-a+j_1)\Gamma(5b)}
$$

$$
\times\frac{(5b+j_1)_{j_2}}{(5c-a+j_1)_{j_2}}\frac{(5b+j_1+j_2)_{j_3}}{(5c-a+j_1+j_2)_{j_3}}{}_2F_1(-j_3,5b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1)
$$

$$
=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5b)\Gamma(5c-a)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\frac{(5b)_{j_1}}{(5c-a)_{j_1}}\frac{(5b+j_1)_{j_2}}{(5c-a+j_1)_{j_2}}
$$

$$
\times\frac{(5b+j_1+j_2)_{j_3}}{(5c-a+j_1+j_2)_{j_3}}{}_2F_1(-j_3,5b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1)
$$

$$
=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5b)\Gamma(5c-a)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(5b)_{j_1}}{(5c-a)_{j_1}}\sum_{j_2=0}^{j_1}\binom{j_1}{j_2}\frac{(5b+j_1)_{j_2}}{(5c-a+j_1)_{j_2}}
$$

$$
\times\sum_{j_3=0}^{j_2}\binom{j_2}{j_3}\frac{(5b+j_1+j_2)_{j_3}}{(5c-a+j_1+j_2)_{j_3}}{}_2F_1(-j_3,5b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1)
$$

$$
=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5c-a)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(5b)_{j_1}}{(5c-a)_{j_1}}
$$

$$
\times\prod_{r=4}^{5}\left\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{(5b+\sum_{s=1}^{r-3}j_s)_{j_{r-2}}}{(5c-a+\sum_{s=1}^{r-3}j_s)_{j_{r-2}}}\right\}{}_2F_1(-j_3,4b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1).
$$

□

Corollary 3.1.4. *If $Re(5c-5b+n)>0$, $n\in\mathbb{Z}^+$, then*

$$
\begin{aligned}
&{}_6F_5\big(-n,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1\big)\\
&\quad=\frac{(5c-5b)_n}{(5c)_n}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(5b)_{j_1}}{(5c+n)_{j_1}}\prod_{r=4}^{5}\Big\{\sum_{j_{r-2}=0}^{jr-3}\binom{j_{r-3}}{j_{r-2}}\frac{(5b+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}{(5c+n+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}\Big\}\\
&\qquad\times{}_2F_1(-j_3,5b+\sum_{k=1}^{3}j_k;5c+n+\sum_{k=1}^{3}j_k;-1).
\end{aligned}
\tag{3.1.8}
$$

Proof. From equation (3.1.5), we have

$$
\begin{aligned}
&{}_6F_5(a,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&=\frac{\Gamma(5c)\Gamma(5c-5b-a)}{\Gamma(5c-a)\Gamma(5c-5b)}\sum_{j_1=0}^{\infty}\binom{-a}{j_1}\frac{(5b)_{j_1}}{(5c-a)_{j_1}}\prod_{r=4}^{5}\Big\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{(5b+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}{(5c-a+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}\Big\}\\
&\qquad\times{}_2F_1(-j_3,4b+j_1+j_2+j_3;5c-a+j_1+j_2+j_3;-1).
\end{aligned}
$$

Put $a=-n$ in above equation, we get

$$
\begin{aligned}
&{}_6F_5(-n,b,b+\frac{1}{5},b+\frac{2}{5},b+\frac{3}{5},b+\frac{4}{5};c,c+\frac{1}{5},c+\frac{2}{5},c+\frac{3}{5},c+\frac{4}{5};1)\\
&=\frac{\Gamma(5c)\Gamma(5c-5b+n)}{\Gamma(5c+n)\Gamma(5c-5b)}\sum_{j_1=0}^{n}\binom{n}{j_1}\frac{(5b)_{j_1}}{(5c+n)_{j_1}}\prod_{r=4}^{5}\Big\{\sum_{j_{r-2}=0}^{j_{r-3}}\binom{j_{r-3}}{j_{r-2}}\frac{(5b+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}{(5c+n+\sum\limits_{s=1}^{r-3}j_s)_{j_{r-2}}}\Big\}\\
&\qquad\times{}_2F_1(-j_3,4b+j_1+j_2+j_3;5c+n+j_1+j_2+j_3;-1).
\end{aligned}
$$

Since $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$, therefore we have

$$
\begin{aligned}
&{}_6F_5\big(-n, b, b+\frac{1}{5}, b+\frac{2}{5}, b+\frac{3}{5}, b+\frac{4}{5}; c, c+\frac{1}{5}, c+\frac{2}{5}, c+\frac{3}{5}, c+\frac{4}{5}; 1\big) \\
&\quad = \frac{(5c-5b)_n}{(5c)_n} \sum_{j_1=0}^{n} \binom{n}{j_1} \frac{(5b)_{j_1}}{(5c+n)_{j_1}} \prod_{r=4}^{5} \Big\{ \sum_{j_{r-2}=0}^{jr-3} \binom{j_{r-3}}{j_{r-2}} \frac{(5b+\sum_{s=1}^{r-3} j_s)_{j_{r-2}}}{(5c+n+\sum_{s=1}^{r-3} j_s)_{j_{r-2}}} \Big\} \\
&\qquad \times {}_2F_1(-j_3, 5b+\sum_{k=1}^{3} j_k; 5c+n+\sum_{k=1}^{3} j_k; -1).
\end{aligned}
$$

□

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