So (2) gives

$$F(b) - F(a) = \int_a^b f(t)\,dt = \int_a^b f(z)\,dz \quad \rightarrow ③$$

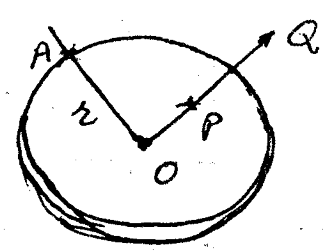
Also $F(z)$ is an indefinite Integral of $f(z)$, we have by definition $F'(z) = f(z)$

Hence (3) ~~gives~~ can be written as

$$F(b) - F(a) = \int_a^b f(z)\,dz = \int_a^b F'(z)\,dz \quad ④$$

DEFINITION: Inverse points w.r.t. a circle

Two pts P and Q are said to be Inverse w.r.t. a circle with centre O and radius r, if the pts O, P and Q are collinear & $OP \cdot OQ = r^2$



§ POISSON'S Integral Formula

Statement: Let $f(z)$ be analytic in the region $|z| < \rho$ and let $(z) = re^{i\theta}$, be any point of this region, then prove that

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})\, d\phi}{R^2 - 2Rr\cos(\phi - \theta) + r^2}$$

where R is any number s.t $r < R < \rho$

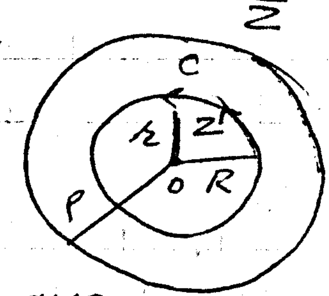
Proof: Let C denotes the circle $|w| = R$ s.t. $r < R < \rho$.

Also $z = re^{i\theta}$ is any point of the region $|z| < \rho$ where $r < R < \rho$. Hence by cauchy's Integral formula, we get

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z}\,dw \quad ①$$

$$(\,|z|^2 = z\bar{z} = R^2)$$

Now the Inverse of the pt z w.r.t. C is $\frac{R^2}{\bar{z}}$ as $z\bar{z} =$
and lies out side C so that
the function $\frac{f(w)}{w - R^2/\bar{z}}$ is analytic



on and within C. Therefore,
by Cauchy's f. Theorem, we have

$$\int_C \frac{f(w)}{w - R^2/\bar{z}}\,dw = 0 \qquad (2)$$

Subtracting (2) from (1), we get

$$f(z) = \frac{1}{2\pi i}\int_C \left| \frac{1}{w-z} - \frac{1}{w - R^2/\bar{z}} \right| f(w)\,dw$$

or $$f(z) = \frac{1}{2\pi i}\int_C \frac{(z - R^2/\bar{z})\, f(w)\,dw}{(w-z)(w - R^2/\bar{z})} \qquad (3)$$

Now $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$ and $w = Re^{i\phi}$
$dw = iRe^{i\theta}d\phi$ putting in (3), we have

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(re^{i\theta} - R^2/re^{-i\theta})\, f(Re^{i\phi})\, Rie^{i\phi}d\phi}{(Re^{i\phi} - re^{i\theta})[Re^{i\phi} - (R^2/re^{-i\theta})]}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{r(r^2 - R^2)e^{i\theta} f(Re^{i\phi})\, Re^{i\phi}d\theta}{r(Re^{i\phi} - re^{i\theta})\, Re^{i\theta}e^{i\phi}(re^{-i\theta} - Re^{-i\phi})}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - R^2)\, f(Re^{i\phi})\,d\phi}{(Re^{i\phi} - re^{i\theta})(re^{-i\theta} - Re^{-i\phi})}$$

$$f(re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{-(r^2 - R^2)\, f(Re^{i\phi})\,d\phi}{R^2 + r^2 - Rr\left(\frac{e^{i(\phi-\theta)} + e^{-i(\phi-\theta)}}{2}\right)2}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\phi})\,d\phi}{R^2 + r^2 - 2rR\cos(\phi - \theta)} \qquad (4)$$

This completes the proof.

③ Ch # 4

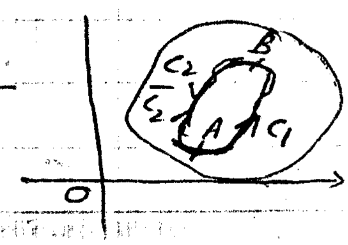
Exercises on Page 125

Q I Prove That $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for all curves C_1 and C_2 contained in a given domain D with the Same initial & final pts iff $\int_C f(z)\,dz = 0$ for all closed curves in D

<u>Solution</u>

To prove $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ for ~~any contour C~~ all curves C_1 & C_2 contained in a given domain D with the Same initial and final pts is equivalent to the Statement That $\int_C f(z)dz = 0$ for any closed contour C in D.

For this, we may write $C = C_1 + \overline{C_2}$ where C_1 and C_2 have initial and Terminal pts and $\overline{C_2}$ is the Inverse of C_2 then



$$\int_C f(z)\,dz = \int_{C_1+\overline{C_2}} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz + \int_{\overline{C_2}} f(z)\,dz$$

$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Conversely, if $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$

$$\Rightarrow \int_C f(z)\,dz = 0$$

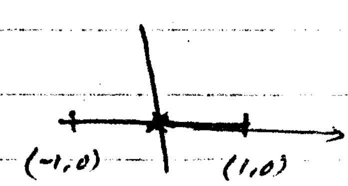
2) $\int_C |z|\,dz$ line segment with initial pt -1 and terminal pt 1.

$|z| = \sqrt{x^2+y^2}$

$y = 0$

$x = t$

$-1 \le t \le 1$



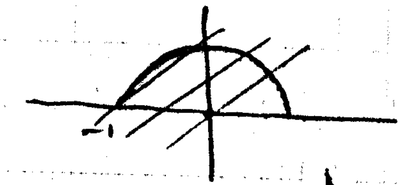
$\int_{-1}^{0} t\,dt$ $\cancel{\int_{0}^{-1} x\,dx}$

$= \left| \frac{t^2}{2} \right|_{-1}^{0} = -\frac{1}{2}$ & $\int_0^1 t\,dt = \frac{t^2}{2} = \frac{1}{2}$

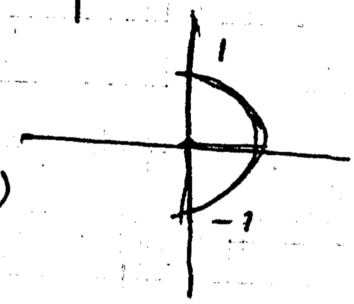
$\Rightarrow \frac{1}{2} - (-\frac{1}{2}) = 1$

2 (ii) $\int_c |z|\, dz$

$x = 0 \quad y \geq 0$ | $\operatorname{Im} z \geq 0$, $y \geq 0$



$y = t \quad -1 \leq t \leq +1$



$\int (1)\, dt = |t|_{-1}^{1} = 1-(-1)$
$= 2$

iii) $|z| = r$

with arbitrary initial & final pts

$z = re^{i\theta}, \quad 0 \leq \theta \leq 2\pi$

$dz = re^{i\theta} \cdot i$

$\int_0^{2\pi} r^2 i e^{i\theta}\, d\theta = r^2 \left[e^{i\theta} \right]_0^{2\pi} = r^2 (1-1)$
$= 0$

Q(3) Evaluate the Integral

$\int_c \frac{dz}{z^2+1}$ | C is the circle $|z+i| = 1$

$(0,-1)$ lies inside C

$|x+iy+i| = 1$

$|x+(y+1)i| = 1$

$x^2 + (y+1)^2 = 1$

$C = (0,-1)$ centre

rad $= 1$

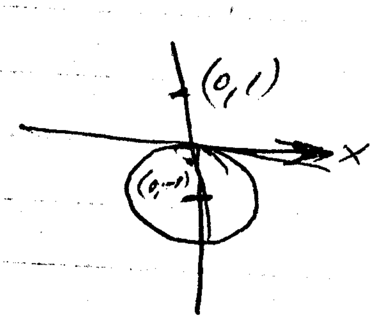
$\int_c \frac{1}{z^2+1}\, dz$

$= \frac{1}{2i}\left(\underbrace{\int \frac{dz}{z-i}}_{I_1} - \underbrace{\int \frac{dz}{z+i}}_{I_2} \right)$

for I_1 $z = i \Rightarrow (0,1)$ lies out side the contour

therefore $\frac{1}{2i}\int \frac{dz}{z-i} = 0$ by Cauchy's Th

& $z = -i = (0,-1)$ lies inside I_2



$\therefore \quad -\frac{1}{2i}(2\pi i) = -\pi$

Q(4) $\int_c \frac{dz}{z^2+1}$ Find all possible values of the Integral where C is a Smooth curve with initial pt. 0 and final pt 1, what restriction be imposed on?

$$\int_c \frac{dz}{z^2+1} = \frac{1}{2i}\int \frac{dz}{z-i} - \frac{1}{2i}\int \frac{dz}{z+i}$$

Not defined at $z=i$ & $z=-i$
C must not passes through $z=\pm i$

$$\cancel{\frac{1}{2i}\ln\left|\frac{z-i}{z+i}\right|_0^1 = \frac{1}{2i}\ln\left|\frac{1-i}{1+i}\right|}$$

$$\int_c \frac{dz}{z^2+1} = \int_0^1 \frac{dz}{z^2+1} = \left|\tan^{-1} z\right|_0^1$$
$$= \tan^{-1}(1) - \tan^{-1}(0)$$
$$= \pi/4 + k\pi$$
$k = \pm 1, \pm 2 \dots$

Q(5)
(i) $\int_c \frac{z^2\,dz}{z+3}$ for the circle $|z|=1$

$z=-3$ is out side the circle $|z|=1$

Hence $\int_c \frac{z^2\,dz}{z+3}$ is zero.

(ii) $\int \frac{(z^2+3z+4)\,dz}{z^4-7z^3+11z^2-28z+28}$

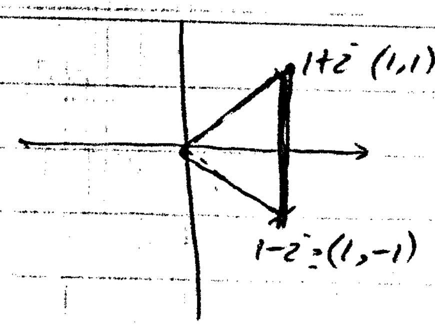
$2i)$	1	-7	11	-28	28
	↓	$2i$	$-4-14i$	$14i+28$	-28
	1	$2i-7$	$7-14i$	$14i$	$\lfloor 0$
$-2i$	↓	$-2i$	$+14i$	$-14i$	
	1	-7	7	$\lfloor 0$	

$z^2-7z+7=0$ $z = \frac{7\pm\sqrt{49-4(7)}}{2} = \frac{7\pm\sqrt{21}}{2}$

$\frac{7\pm 4.58}{2}$
(i) $\frac{11.58}{2} = 5.79$
(ii) $= 1.21$
both out side the circle $|z|=1$
Hence zero

Q(6)

(i) $\displaystyle\int_{1-i}^{1+i} (z^2+1)\,dz$



$x=1,\quad y=t$

$-1\le t\le 1$

$\displaystyle\int_{-1}^{1} \left((1+it)^2+1\right) i\,dt$

$\displaystyle i\int_{-1}^{1} (1-t^2+2it+1)\,dt$

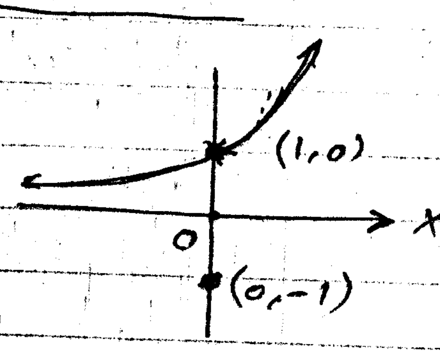
$\displaystyle i\int_{-1}^{1} (2-t^2+2it)\,dt = i\left|2t - \frac{t^3}{3} + 2i\frac{t^2}{2}\right|_{-1}^{1}$

$= i\left\{2(1+1) - \tfrac{1}{3}(1+1) + i(1-1)\right\}$

$= i\left[4 - \tfrac{2}{3}\right] = \dfrac{10i}{3}$

(ii) $\displaystyle\int_{-i}^{i} e^{3z}\,dz$

$x=0$
$y=t$
$z=it$



$\displaystyle\int_{0}^{\pi/2} e^{3it}\, i\,dt$

$= i\left|\dfrac{e^{3it}}{3i}\right|_{0}^{\pi/2} = \tfrac{1}{3}\left(e^{3i\pi/2} - e^{0}\right)$

$= \tfrac{1}{3}\left\{\left[\cos 3\pi/2 + i\sin 3\pi/2\right] - 1\right\}$

$= \tfrac{1}{3}\left(i(-1) - 1\right) = -\tfrac{1}{3} - \tfrac{i}{3}$

& $\tfrac{1}{3}\left|e^{3it}\right|_{0}^{-\pi/2} = \tfrac{1}{3}\left(e^{-3i\pi/2} - e^{0}\right)$

$= \tfrac{1}{3}\left(-i\sin 3\pi/2 - 1\right)$

$= \tfrac{1}{3}\left(\tfrac{i}{3} - 1\right)$

$= \tfrac{1}{3}\left(\tfrac{i}{3} - 1\right) - \left(-\tfrac{1}{3} - \tfrac{i}{3}\right)$

$= \tfrac{2i}{3} + 0$

$\tfrac{1}{3}\left(e^{3i\pi/2} - e^{-3i\pi/2}\right)$

$\tfrac{1}{3}\left(i\sin 3\pi/2 - (-i\sin 3\pi/2)\right)$

$= \dfrac{2i}{3}\sin(3\pi/2)$

$= -2i/3$

Q(7) $\int_c \frac{e^z-1}{z-0}$ $|z|=1$ $a=0$ $n=1$ | $f^n(a)=\frac{\underline{|n}}{2\pi i}\int \frac{f(z)dz}{(z-a)^{n+1}}$

$f(z)=e^z-1$

$f(0)=0$ $\int_c \frac{f(z)}{z-a}dz = 2\pi i(0)=0$

(ii) ~~$f(z)=\sinh^2 z+\cos z$~~ $|z|=4$

$z-a = z-\pi i$

$\Rightarrow a=\pi i$

$f(\pi i) = \sinh^2(\pi i)+\cos(\pi i)$

$=(i\sin\pi)^2 + \cosh\pi$

$\int_c \frac{f(z)}{z-\pi i}dz = 2\pi i\cosh\pi$

(iii) $\int_c \left(\frac{3}{z+1} - \frac{4}{z-1}\right)dz$ $|z|=2$

for I_1 $f(z)=3$ $z=-1$ lies in ⊙

apply C. Integ formula

$f(-1)=3$ $\int_c \frac{3}{z+1}dz = 3(2\pi i)$

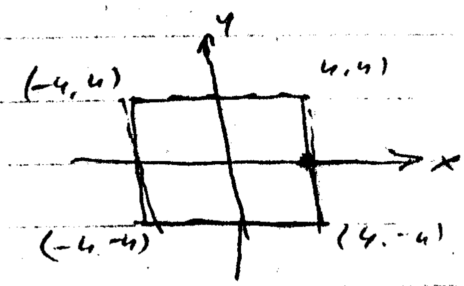
$\int_c \frac{4}{z-1}dz = 4(2\pi i)$

$I_1-I_2 = 6\pi i - 8\pi i$

$= -2\pi i$

Q (8)

(i) $\int_c \frac{e^z dz}{(z-0)^6}$



$f(z)=e^z$ $a=0,\ n=5$

$f'(z)=f''(z)=f'''(z)=f^{iv}(z)=f^{v}(z)=e^z \Rightarrow e^0=1$

$\int_c \frac{f(z)dz}{(z-a)^{n+1}} = \frac{2\pi i}{\underline{|n}} f^n(a)$ In our case $\frac{2\pi i}{\underline{|5}}(1) = \frac{\pi i}{60}$

(ii) $f(z) = e^z\cosh z$ $\left\{ \begin{array}{l} a=\pi \\ n=2 \end{array}\right.$

$f'(z) = e^z\cosh z + e^z\sinh z$

$f''(z) = 2e^z(\cosh z+\sinh z)$

$f''(\pi) = 2e^\pi(e^\pi) = 2e^{2\pi}$

Using formula $\Rightarrow$ $\frac{2\pi i}{\underline{|2}}(2e^{2\pi}) \longrightarrow 2\pi i e^{2\pi}$

9) $\int \frac{e^{az}\,dz}{(z-0)^{n+1}}$ $a=0$, $n=n$, $|z|=1$

$f(z)=e^{az}$ $f^{n}(0)=e^{0}a^{n}=a^{n}$

$f^{n}(z)=a^{n}e^{az}$

Using formula $\int_c \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{\lfloor n} f^{n}(a)$

$= \frac{2\pi i}{\lfloor n} a^{n}$

$\frac{\lfloor n}{a^{n}} = \frac{1}{2\pi i}\int_c \frac{e^{az}\,dz}{z^{n+1}}$

10 Let $g(z)$ be a complex valued fn defined and cont. on a contour C, show that $f(z)=\frac{1}{2\pi i}\int_c \frac{g(t)\,dt}{t-z}$ is analytic in every domain D containing no part of C and its derivative is given by formula $f'(z)=\frac{1}{2\pi i}\int_c \frac{g(t)\,dt}{(t-z)^2}$

Sol Suppose $f(z)=\frac{1}{2\pi i}\int_c \frac{g(t)\,dt}{t-z}$ also suppose that ~~$z_0, z \in D-C$ and~~ $g(z)$ is cont. on C we have $|g(t)|\le K$ on C for some $K>0$ & let $|z-t|<r$, the neighbourhood of t which is disjoint of C,

Suppose $z_0, z \in D-C$, we have

$f(z)-f(z_0) = \frac{1}{2\pi i}\int_c g(t)\left(\frac{1}{t-z}-\frac{1}{t-z_0}\right)dt$

$= \frac{1}{2\pi i}(z-z_0)\int_c g(t)\frac{1}{(t-z)(t-z_0)}dt$

or $\frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{2\pi i}\int_c g(t)\frac{dt}{(t-z)(t-z_0)}$ [Subtracting $\frac{1}{2\pi i}\int \frac{g(t)}{(t-z_0)^2}dt$ from both sides]

$\Rightarrow \frac{f(z)-f(z_0)}{z-z_0} - \frac{1}{2\pi i}\int_c \frac{g(t)\,dt}{(t-z_0)^2} = \frac{1}{2\pi i}\int_c \frac{g(t)}{(t-z_0)}\left(\frac{1}{t-z}-\frac{1}{t-z_0}\right)dt$

So $I = \left|\frac{f(z)-f(z_0)}{z-z_0} - \frac{1}{2\pi i}\int_c \frac{g(t)\,dt}{(t-z_0)^2}\right| \le \frac{|z-z_0|}{2\pi}\int_c \frac{|g(t)||dt|}{|(t-z_0)|^2|t-z|}$

Note that t varies over the contour C and $g(t)$ is cont. Hence $|g(t)|\le K$, ~~[illegible]~~ so $|t-z_0|>r$ for $\epsilon>0$ s.t $0<\epsilon<r/2$, $|t-z|\ge|t-z_0+z_0-z|$ $\ge |t-z_0|-|z-z_0| > r/2$

for $|z-z_0|<\epsilon<r/2$ Hence *

* $I < \frac{\epsilon}{2\pi}\int_c \frac{|\ldots|}{r\ldots}$

$<\epsilon\left(\frac{KL}{2\pi r^3}\ldots\right)$

$\because |dt|\ldots$