OPERATION RESEARCH

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Dedicated

To

My Honorable Teacher

Sir Haider Ali

&

My Parents

Lecture # 01

Operation Research:

What is Operation Research?

Operation Research (O.R) is an art of wining wars without actual fighting.

(Arthur Clark)

O.R is a scientific approach to the problems.

(H.M. Wagner)

Linear Programming:

Linear programming is a mathematical technique for determining the optimal solution and obtaining a particular objective when there are alternative uses of resources. The objective may be cost minimization or profit maximization.

The word linear means that the relationships are represented by straight line ax+by = c

The word programming that is concerned with optimal allocation of limited resources.

Linear Function:

A linear function contains terms of which is composed of only a single variable raised to the power one. Linear functions are those whose graph is a straight line. e.g. $3x+2y = 7$ (linear) , $3x^{3/2}+2y = 7$ (Non-linear)

Objective Function:

It is a linear function of decision variables. $z = x_1+x_2$ is the most typical form of objective functions are maximize f(x) or minimize f(x).

Decision Variable وہ ہوتے ہیں جو decide کرتے ہیں کہ فنکشن کہاں پر maximum ہے اور کہاں پر minimum ہے۔

Constraints:

These are the linear equation arising out of practical limitations. The mathematical forms of constraints are $f(x) \leq b$ or $f(x) \geq b$ or $f(x) = b$

Feasible Solution: A non-negative solution which satisfies all the constraints is known as feasible solution. The region comprising all feasible solutions is referred to as feasible region.

Optimal Solution:

The solution where the objective function is maximize or minimize is known as optimal solution.

General linear programming problem:

Consider the following optimize

$$z = c_1x_1 + c_2x_2 + \dots\dots + c_nx_n \quad subject\,to$$

$$a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n \quad which\ is\ (\le, \ge, =) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n \quad which\ is\ (\le, \ge, =) b_2$$

$$\begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix}$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n \quad which\ is\ (\le, \ge, =) b_m$$

$x_1, x_2, \dots., x_n \ge 0$ In summation form the above problem can be written as

$$z = \sum_{i=1}^{n} c_j x_j \quad subject\ to$$

$$\sum_{j=1}^{n} a_{ij} x_j \quad (\le, \ge, =) \quad b_i, i = 1, 2, \dots., m$$

$$and\ \ x_j \ge 0 \qquad j = 1, 2, \dots., n$$

$c_{j's}, a_{ij's}, b_{i's}$ are decision variables.

Graphical Solution of Linear Programming problem:

Question: $2x \le 4$

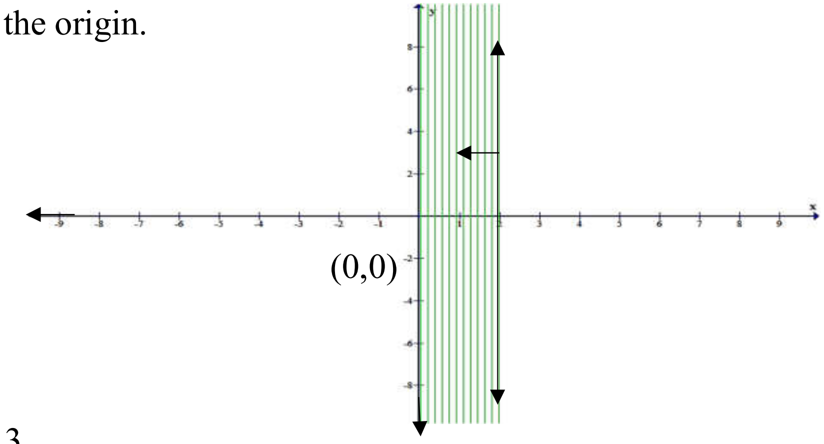
Solution: $x \le 2$

Associated equation is

$x = 2$

At (0,0) $\Rightarrow$ $0 \le 4$

If at (0,0) the equation is true then solution is toward origin & if false the solution is away from the origin.



Question: $2x - 2y \leq 3$

Solution: Associated equation is

$2x - 2y = 3$

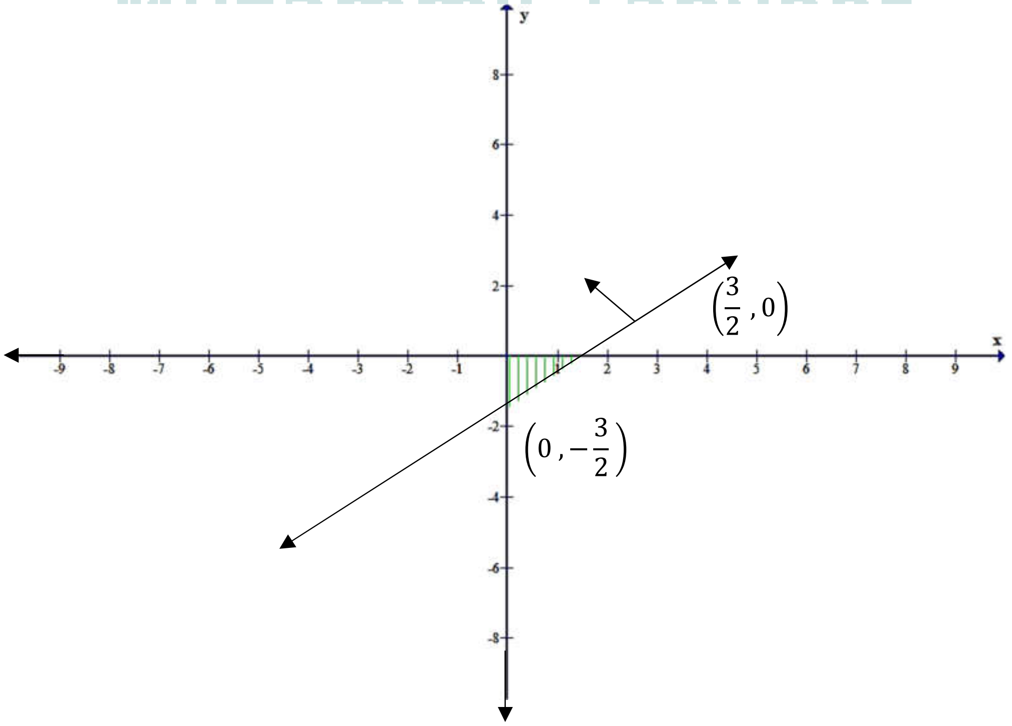
At $y = 0$ $2x = 3$ $\Rightarrow$ $x = \frac{3}{2}$ $\Rightarrow$ $\left(\frac{3}{2}, 0\right)$

At $x = 0$ $-2y = 3$ $\Rightarrow$ $y = -\frac{3}{2}$ $\Rightarrow$ $\left(0, -\frac{3}{2}\right)$

At (0,0)

$0 - 0 \leq 3$ $\Rightarrow$ $0 \leq 3$ true

Then solution is toward origin.



Corner Point:

A point of solution region where two of its boundary line are intersect is called corner point or vertex of solution region.

Example: Maximize $z = x_1 + 3x_2$ Subject to $x_1 - x_2 \leq 5$

$x_1 \geq 2$, $x_2 \leq 5$

$x_1, x_2 \geq 0$

Solution: Given that $x_1 - x_2 \leq 5$

Associated Equation is

$x_1 - x_2 = 5$

At $x_2 = 0 \Rightarrow x_1 = 5 \Rightarrow (5,0)$

At $x_1 = 0 \Rightarrow -x_1 = 5 \text{ or } x_1 = -5 \Rightarrow (0,-5)$

And $x_1 \geq 2$

Associated equation is

$x_1 = 2 \Rightarrow (2,0)$

$And\ x_2 \leq 5$

Associated equation is

$x_2 = 5 \Rightarrow (0,5)$

At (0,0) for $x_1 - x_2 \leq 5$

$0 - 0 \leq 5 \Rightarrow 0 \leq 5 \ true$

Solution toward origin

For $x_1 \geq 2$

$0 \geq 2 \ false$

Solution away from origin

For $x_2 \leq 5$

$0\leq 5$ *true* Solution toward origin

For corner point B $x_1=2\,,x_2=5\ \Rightarrow (2,5)$

For corner point A (intersection of line $x_1-x_2=5$ & $x_2=5$)

$x_1-x_2=5 \quad \Rightarrow \quad x_2=5$

$x_1-5=5 \ \Rightarrow\ x_1=10 \quad \Rightarrow (10,5)$

For maximize $z=x_1+3x_2$

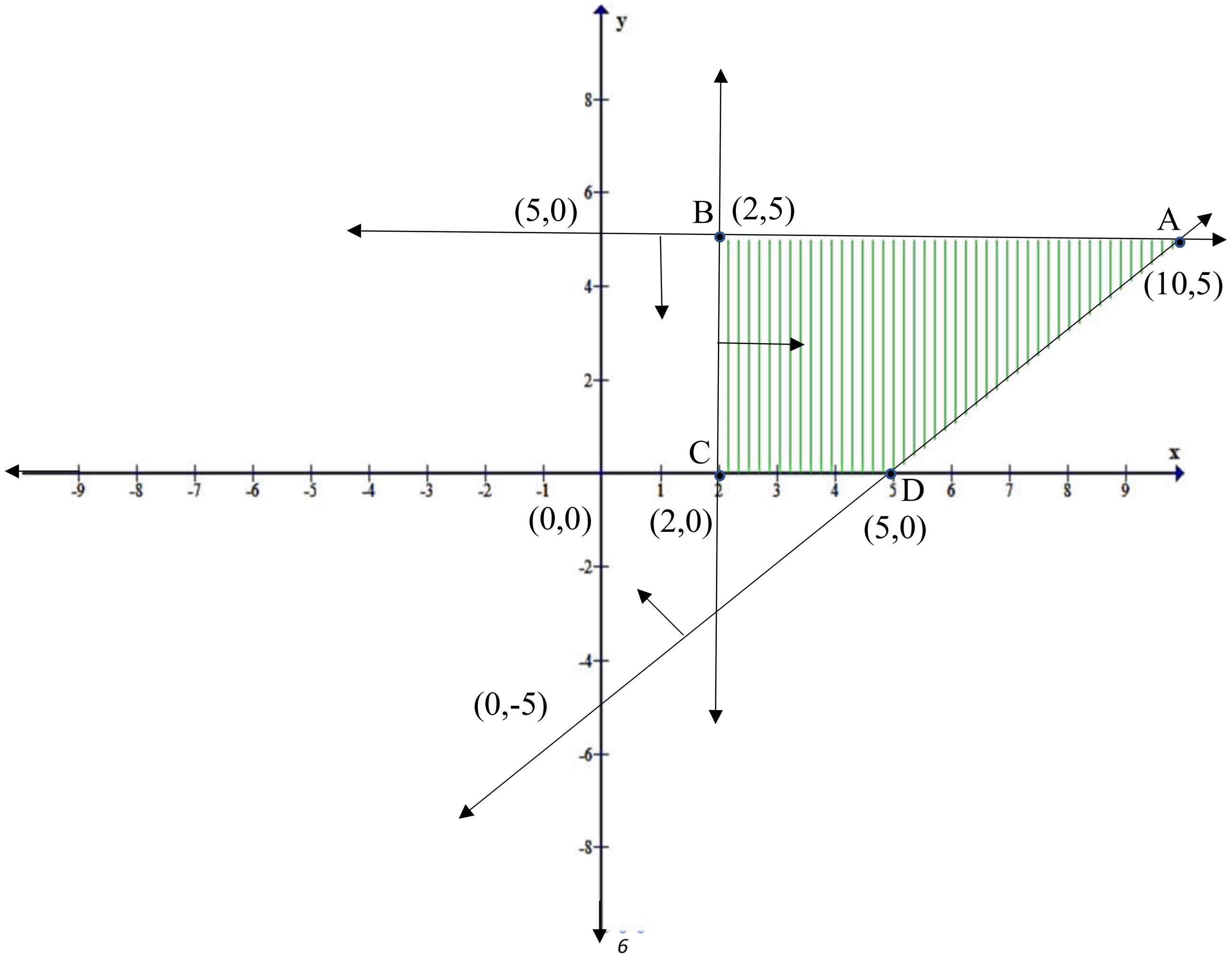
Put (2,0) $\Rightarrow$ z = 2

Put (5,0) $\Rightarrow$ z = 5

Put (2,5) $\Rightarrow$ z = 2 +15 =14

Put (10,5) $\Rightarrow$ z = 10+15 =25

So maximum at (10,5)



Example: Find the equation from graph.

Solution: We know

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$For\ (0,3)\ \&\ (4,0)$

$$y - 3 = \frac{0-3}{4-0}(x-0)$$

$$y - 3 = \frac{-3}{4}x$$

$$4(y-3) = -3x$$

$$4y - 12 = -3x$$

$$3x + 4y = 12$$

$For\ (0,-2)\ \&\ (1,0)$

$$y + 2 = \frac{0+2}{1-0}(x-0)$$

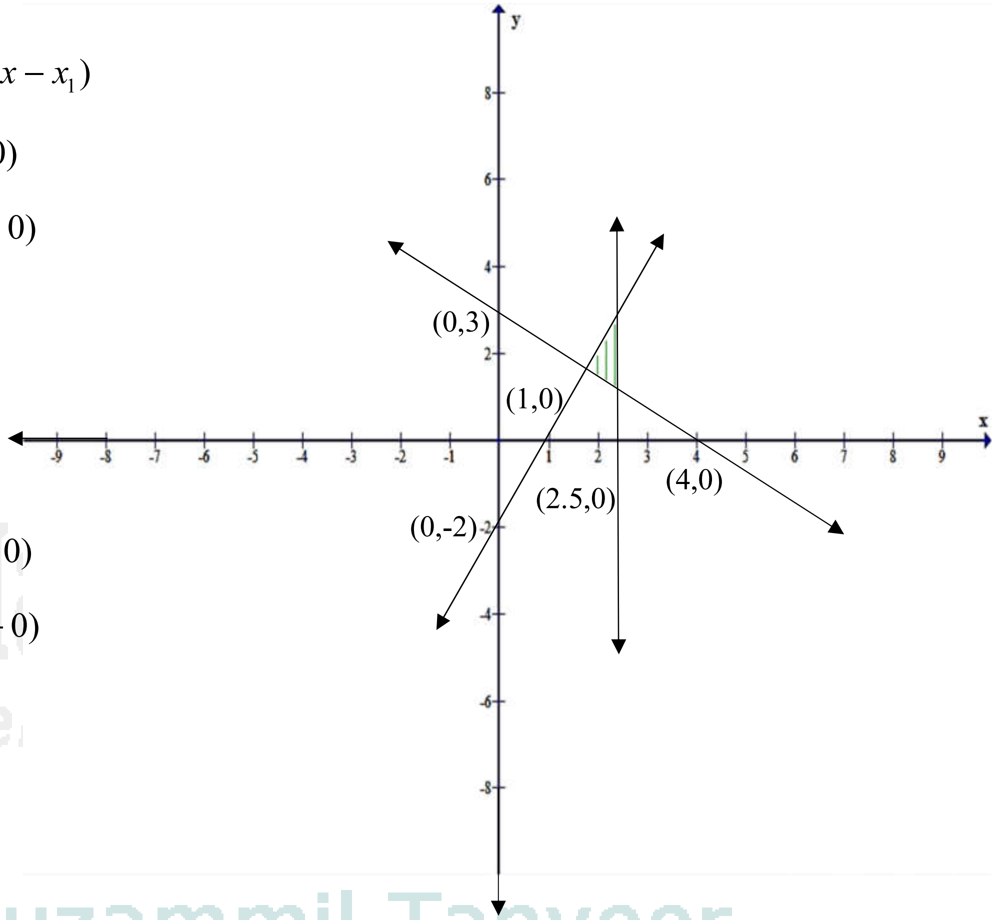
$$y + 2 = \frac{2}{1}(x)$$

$$y + 2 = 2x$$

$$2x - y = 2$$

$For\ \ (2.5,0)$

$$\Rightarrow\ x = 2.5$$



Now (0,3) & (4,0) has solution away from origin. So

$$3x+4y \geq 12$$

And (0,–2) & (1,0) has solution away from origin. So

$$2x-y \geq 2$$

Also (2.5, 0) has solution toward origin. So

$$x \leq 2.5$$

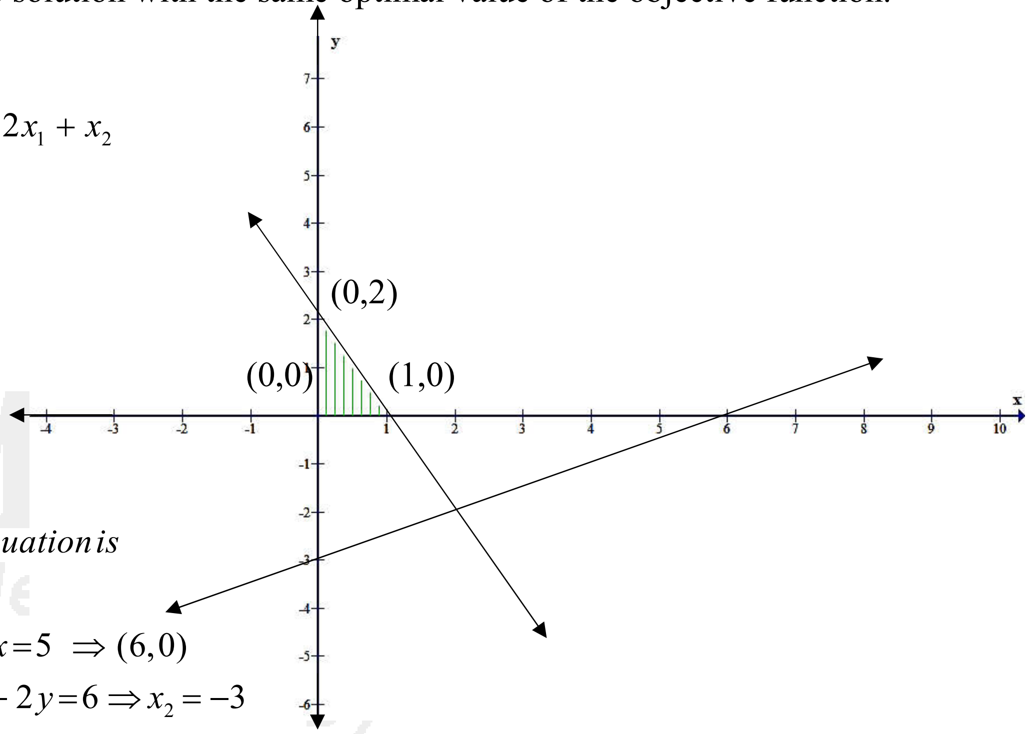
Lecture # 2

Special Cases in Graphical:

(i) Multiple Optimal Solutions:

More than one solution with the same optimal value of the objective function.

Example:



$Maximize\ \ z=2x_1+x_2$

$Subject\,to$

$x-2y\le 6$

$2x+y\le 2$

$x,y\ge 0$

$Solution:$

$Given\,that$

$x-2y\le 6$

$Associated\ equation\,is$

$x-2y=6$

$At\quad y=0\ \Rightarrow x=5\ \ \Rightarrow (6,0)$

$At\quad x=0\ \Rightarrow -2y=6 \Rightarrow x_2=-3$

$\Rightarrow (0,-3)$

$And\ \ x\ge 0$

$Associated\ equation\,is$

$2x+y=2$

$At\,y=0 \Rightarrow x=1\ \Rightarrow (1,0)$

$At\,x=0 \Rightarrow y=2 \Rightarrow (0,2)$

$And\ y\ge 0$

$z=2x_1+x_2$

$at\,(0,0)\Rightarrow z=0$

$at(1,0)\Rightarrow z=2$

$at\,(0,2)\Rightarrow z=2$

Infeasible Region:

In some case there is no feasible solution area that is there are no points which satisfy all the constraints (inequalities).

$z = 2x_1 + x_2$

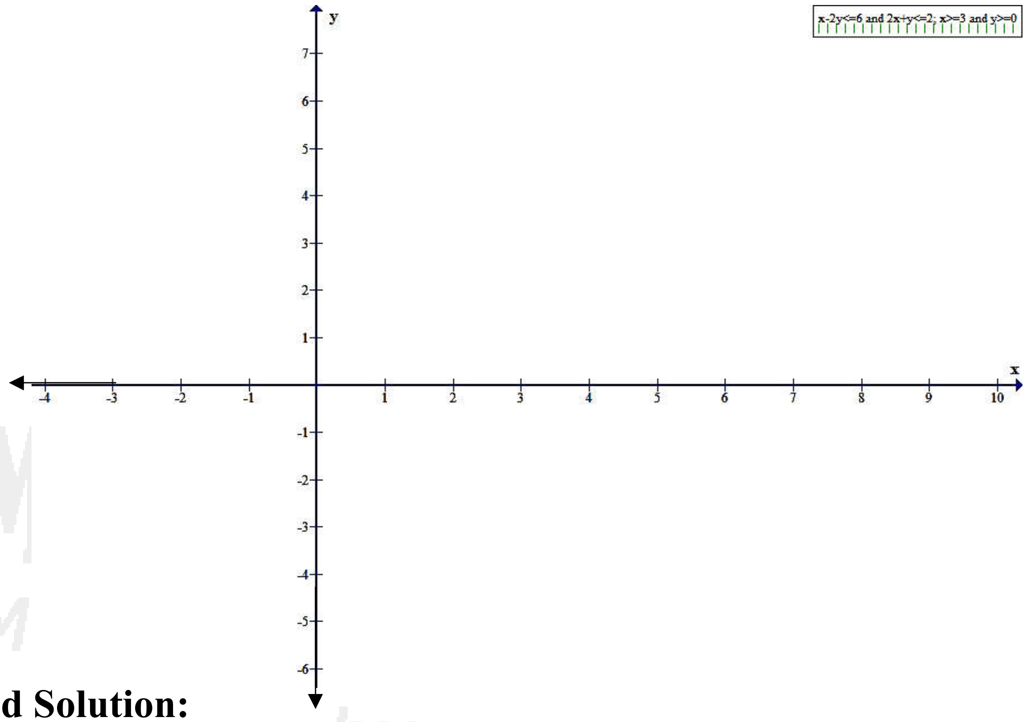
$Subject\,to$

$x - 2y \leq 6$

$2x + y \leq 2$

$x \geq 3$

$x, y \geq 0$



Unbounded Solution:

If the value of the objective function is increased indefinitely such solutions are called unbounded solution.

Example: $x - 2y \leq 6$

$2x + y \geq 2$

$x , y \geq 0$

Solution: Given that

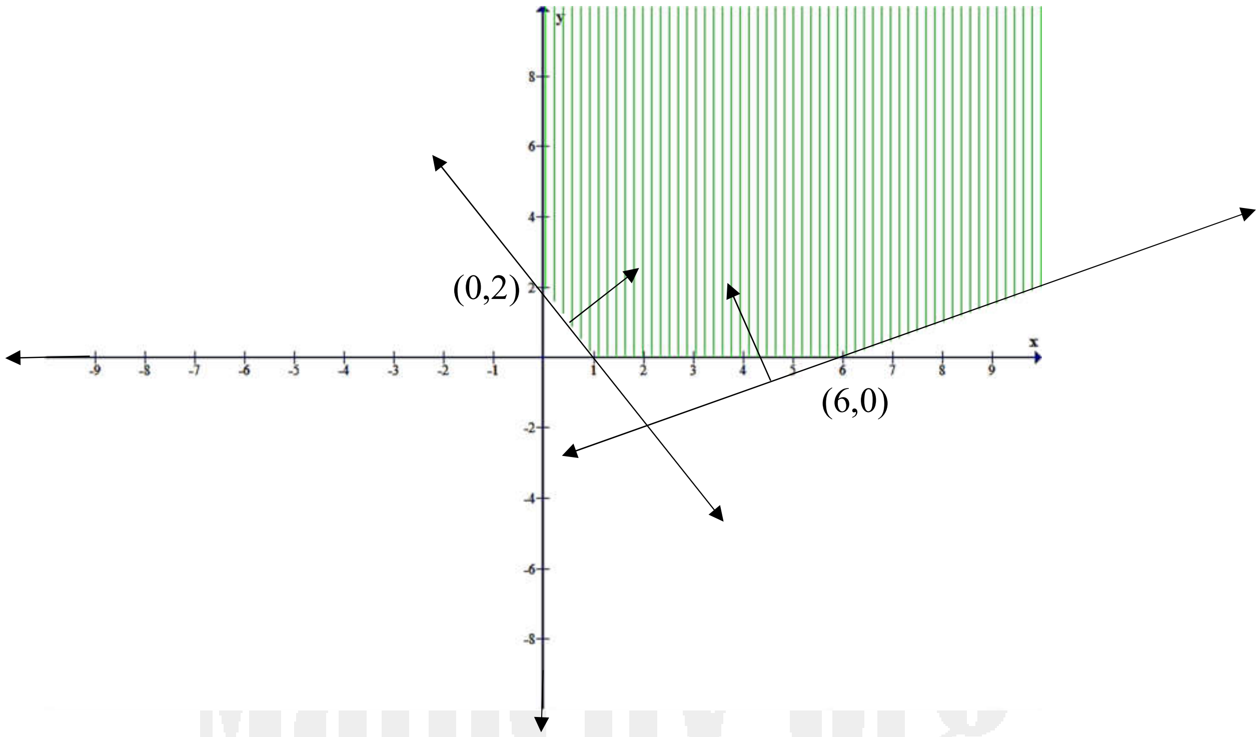
$x - 2y \leq 6$, $2x + y \geq 2$

Associated Equation is

$x - 2y = 6$, $2x + y = 2$

$x = 6$ (6,0) , $x = 1$ (1,0)

$y = -3$ (0,-3) , $y = 2$ (0,2)



Question: A person requires 10,12 & 12 units chemical A, B & C respectively for his garden. A liquid product contains 5 ,2 & 3 units of A,B & C respectively per jar.A dry product contains 3,3 and 4 units of A, B and C respectively per carton. If the liquid product sells for Rs. 3 per jar & dry product sells for Rs. 2 per carton. How many of each should be purchase to minimize the cost and meet the requirment. Only formulate the above problem.

$Minimize\, z = 3x_1 + 2x_2$

$Subject\, to$

$5x_1 + 3x_2 \geq 10 \quad .... \quad For \quad A$

$2x_1 + 3x_2 \geq 12 \quad .... \quad For \quad B$

$3x_1 + 4x_2 \geq 12 \quad .... \quad For \quad C$

$x_1, x_2 \geq 0$

Simplex Method for Solving Linear Programming Problem:

Drawback of simplex method that it solves only ≤ constraints.

Slack Variable:

$$x_1 + x_2 \leq 3$$

To change above inequality, we add some variable that variable is called slack variable.

$$x_1 + x_2 + s_1 = 3$$

It is the variable that is added to the L.H.S of a less than or equal ≤ type constraints to convert inequality into equality.

Surplus Variable:

It is a variable which is subtracted from the L.H.S of a greater than or equal to ≥ type constraints to convert inequality into equality.

e.g. $$x_1 + x_2 \geq 9$$

$$x_1 + x_2 - s_2 = 9$$

Example: Find maximum solution of the following problem by simplex method.

$$z = x_1 + x_2$$

Subject to $$2x_1 + x_2 \leq 4$$

$$x_1 + 2x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

Solution:

$$z = x_1 + x_2 + 0s_1 + 0s_2$$

Subject to $$2x_1 + x_2 + s_1 = 4$$

$$x_1 + 2x_2 + s_2 = 3$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Initial basic feasible solution:

We assume that nothing can be produced. Therefore the values of decision variable $x_1, x_2 = 0$ and also $z = 0$. So we left with unused capacities $s_1 = 4 \,\& s_2 = 3$

Variables with non-zero values are called Basic variables and with zero values are called non-basic variables.

Gauss Jordan method:

$$z - x_1 - x_2 - 0s_1 - 0s_2 = 0$$

Basic	x_1	x_2	s_1	s_2	Sol
z	-1	-1	0	0	0
s_1	2	1	1	0	4
s_2	1	2	0	1	3

Pivot element

Entering Variable:

Most -ve value in z row (for maximization)

Most +ve value in z row (for minimization)

Leaving Variable:

Minimum +ve ratio in ratio column (for both maximize and minimize)

Entering Value

Basic	x_1	x_2	s_1	s_2	Sol
z	-1	-1	0	0	0
x_1	1	1/2	1/2	0	2
s_2	1	2	0	1	3

Leaving variable

Divide by 2 to make pivot element 1

Basic	x_1	x_2	s_1	s_2	Sol
z	0	-1/2	1/2	0	2
x_1	1	1/2	1/2	0	2
s_2	0	3/2	-1/2	1	1

$z + x_1$

$s_2 - x_1$

Entering Value

Basic	x_1	x_2	s_1	s_2	Sol
z	0	-1/2	1/2	0	2
x_1	1	1/2	1/2	0	2
x_2	0	1	-1/3	2/3	2/3

Leaving variable s_2

Mult

Basic	x_1	x_2	s_1	s_2	Sol
z	0	0	1/3	1/3	7/3
x_1	1	0	2/3	-1/3	5/3
x_2	0	1	-1/3	2/3	2/3

$$z+\frac{1}{2}x_1$$

$$x_1-\frac{1}{2}x_2$$

Since all the value in z-row are non-negative. So, the solution obtained is optimal with

$$x_1=\frac{5}{3},\ x_2=\frac{2}{3}, z=\frac{7}{3}$$

$$\text{and } s_1=s_2=0$$

Lecture # 3

Special Case in Simplex method:

There are two special case in simplex method

- (i) Unbounded solution
- (ii) Multiple optimal solution

(i) Unbounded Solution:

Question: Find maximum solution of the following system of linear equation

$$z = 5x_1 + 4x_2$$

Subject to $x_1 \le 7$

$$x_1 - x_2 \le 8$$

$$x_1, x_2 \ge 0$$

Solution: $z = 5x_1 + 4x_2 + 0s_1 + 0s_2$

$$z - 5x_1 - 4x_2 - 0s_1 - 0s_2 = 0$$

$$x_1 + s_1 = 7$$

$$x_1 - x_2 + s_2 = 8$$

$$x_1, x_2, s_1, s_2 \ge 0$$

Basic	↓ x_1	x_2	s_1	s_2	Sol	R/C
z	-5	-4	0	0	0	
← s_1	1	0	1	0	7	7/1=7
s_2	1	-1	0	1	8	8/1=8

Basic	x_1	↓ x_2	s_1	s_2	Sol	R/C	
z	0	-4	5	0	35		$z+5x_1$
x_1	1	0	1	0	7	7/0 = ∞	
s_2	0	-1	-1	1	1	-1/1= -1	s_2-x_1

Since there is no minimum positive value in Ratio column. So, it is not possible to proceed further with simplex method. This is the criteria for unbounded solution in simplex method.

(ii) Multiple optimal solution:

The optimal solution may not be unique if the non-basic variable has a zero co-efficient in z row.

This implies that bringing the non-basic variable into the basic will neither increase nor decrease the value of the objective function. Thus, the problem has multiple optimal solution.

Example: Maximize $z=2x_1+3x_2$

Subject to $6x_1+9x_2\le 100$

$2x_1+x_2\le 20$

$x_1,x_2\ge 0$

Solution: $z=2x_1+3x_2+0s_1+0s_2$

$z-2x_1-3x_2-0s_1-0s_2=0$

$6x_1+9x_2+s_1=100$

$2x_1-x_2+s_2=20$

$x_1,x_2,s_1,s_2\ge 0$

Basic	x_1	x_2 ↓	s_1	s_2	Sol	R/C
z	-2	-3	0	0	0	
← s_1	6	9	1	0	100	100/9=11.11
s_2	2	1	0	1	20	20/1=20

Basic	x_1	x_2	s_1	s_2	Sol	
z	-2	-3	0	0	0	
x_2	$\frac{2}{3}$	1	$\frac{1}{9}$	0	$\frac{100}{9}$	$\frac{1}{9}s_2$
s_2	2	1	0	1	20	

Basic	x_1	x_2	s_1	s_2	Sol	
z	0	0	$\frac{1}{3}$	0	$\frac{100}{3}$	$z+3x_2$
x_2	$\frac{2}{3}$	1	$\frac{1}{9}$	0	$\frac{100}{9}$	
s_2	$\frac{4}{3}$	0	$-\frac{1}{9}$	1	$\frac{80}{9}$	s_2-x_2

$$x_1=0,\ x_2=\frac{100}{9},\ s_1=0,\ s_2=\frac{80}{9}, z=\frac{100}{3}$$

For Finding multiple solution:

Basic	$\downarrow x_1$	x_2	s_1	s_2	Sol	R/C
z	0	0	$\frac{1}{3}$	0	$\frac{100}{3}$	
x_2	$\frac{2}{3}$	1	$\frac{1}{9}$	0	$\frac{100}{9}$	$\frac{\frac{100}{9}}{\frac{2}{3}}=\frac{50}{3}$
$\leftarrow s_2$	$\frac{4}{3}$	0	$-\frac{1}{9}$	1	$\frac{80}{9}$	$\frac{\frac{80}{9}}{\frac{4}{3}}=\frac{20}{3}$

Basic	x_1	x_2	s_1	s_2	Sol
z	0	0	$\frac{1}{3}$	0	$\frac{100}{3}$
x_2	$\frac{2}{3}$	1	$\frac{1}{9}$	0	$\frac{100}{9}$
x_1	1	0	$-\frac{1}{12}$	$\frac{3}{4}$	$\frac{20}{3}$

$\therefore \frac{3}{4}s_2$

Basic	x_1	x_2	s_1	s_2	Sol
z	0	0	$\frac{1}{3}$	0	$\frac{100}{3}$
x_2	0	1	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{20}{3}$
x_1	1	0	$-\frac{1}{12}$	$\frac{3}{4}$	$\frac{20}{3}$

$x_2 - \frac{2}{3}x_1$

$$x_1=\frac{20}{3},\ x_2=\frac{20}{3},\ s_1=0,\ s_2=0, z=\frac{100}{3}$$

Lecture # 04

Degeneracy:

In some cases, there may be doubt in selecting the variable that should be introduced into the basic i.e. there is a tie between the ratio of two variables.

To resolve degeneracy, we select one of them arbitrary.

Example: Maximize $z=3x_1+9x_2$

Subject to $x_1+4x_2\le 8$, $x_1+2x_2\le 4$

$x_1,x_2\ge 0$

Solution: $z=3x_1+9x_2+0s_1+0s_2$

$z-3x_1-9x_2-0s_1-0s_2=0$

$x_1+4x_2+s_1=8$

$x_1+2x_2+s_2=4$

$x_1,x_2,s_1,s_2\ge 0$

Basic	x_1	x_2 ↓	s_1	s_2	Sol	R/C
z	-3	-9	0	0	0	
s_1	1	4	1	0	8	$\frac{8}{4}=2$
← s_2	1	2	0	1	4	$\frac{4}{2}=2$

Basic	x_1	x_2	s_1	s_2	Sol
z	$\frac{3}{2}$ $\frac{9}{2}$	0	0		18
s_1	-1	0	1	-2	0
	1				

$z+9x_2$

s_1-4x_2

$\frac{1}{2}x_2$

$$x_1=0,\ x_2=2,\ s_1=0\ ,z=18$$

Unrestricted Variable:

Sometime variables are unrestricted in sign (+,–,0). In all such cases the decision variables can be expressed as the difference between two non-negative variables.

For example, if x_1 is unrestricted in sign then we write $x_1=x_1'-x_1''$

Example: Maximize $z=2x_1+3x_2$

Subject to $-x_1+2x_2\le 4$

$x_1+x_2\le 6$

$x_1+3x_2\le 9$ where x_1,x_2 are unrestricted in sign

Solution: Here x_1 and x_2 both are unrestricted in sign so we put

$$x_1=x_1'-x_1''\ \&\ x_2=x_2'-x_2''$$

$$z=2x_1'-2x_1''+3x_2'-3x_2''+0s_1+0s_2+0s_3$$

$$z-2x_1'+2x_1''-3x_2'+3x_2''-0s_1-0s_2-0s_3=0$$

$$-x_1'-x_1''+2x_2'-2x_2''+s_1=4$$

$$x_1'-x_1''+x_2'-x_2''+s_2=6$$

$$x_1'-x_1''+3x_2'-3x_2''+s_3=9$$

$$x_1',x_2',x_1'',x_2'',s_1,s_2,s_3\ge 0$$

Basic	x_1'	x_1''	↓ x_2'	x_2''	s_1	s_2	s_3	Sol	R/C
z	-2	2	-3	3	0	0	0	0	
← s_1	-1	1	2	-2	1	0	0	4	$4/2=2$
s_2	1	-1	1	-1	0	1	0	6	$6/1=6$
s_3	1	-1	3	-3	0	0	1	9	$9/3=3$

Basic	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3	Sol	
z	-2	2	-3	3	0	0	0	0	
x_2'	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	$\frac{1}{2}$	0	0	2	$\frac{1}{2}x_2'$
s_2	1	-1	1	-1	0	1	0	6	
s_3	1	-1	3	-3	0	0	1	9	

↓

Basic	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3	Sol	
z	$-\frac{7}{2}$	$\frac{7}{2}$	0	0	$\frac{3}{2}$	0	0	6	$z+3x_2'$
x_2'	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	$\frac{1}{2}$	0	0	2	
s_2	$\frac{3}{2}$	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	1	0	4	s_2-x_2'
← s_3	$\frac{5}{2}$	$-\frac{5}{2}$	0	0	$-\frac{3}{2}$	0	1	3	s_3-x_2'

Basic	x_1'	x_1''	x_2'	x_2''	s_1	s_2	s_3	Sol	
z	$-\frac{7}{2}$	$\frac{7}{2}$	0	0	$\frac{3}{2}$	0	0	6	
x_2'	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	$\frac{1}{2}$	0	0	2	
s_2	$\frac{3}{2}$	$-\frac{3}{2}$	0	0	$-\frac{1}{2}$	1	0	4	
x_1'	1	-1	0	0	$-\frac{3}{5}$	0	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{2}{5}x_1'$

↓

Basic	$x_1^{'}$	$x_1^{''}$	$x_2^{'}$	$x_2^{''}$	s_1	s_2	s_3	Sol
z	0	0	0	0	$\frac{-3}{5}$	0	$\frac{7}{5}$	$\frac{51}{5}$
$x_2^{'}$	0	0	1	-1	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{13}{5}$
← s_2	0	0	0	0	$\frac{2}{5}$	1	$\frac{-3}{5}$	$\frac{11}{5}$
$x_1^{'}$	1	-1	0	0	$-\frac{3}{5}$	0	$\frac{2}{5}$	$\frac{6}{5}$

$z+\frac{7}{2}x_1^{'}$

$x_2^{'}+\frac{1}{2}x_1^{'}$

$s_2-\frac{3}{2}x_1^{'}$

Basic	$x_1^{'}$	$x_1^{''}$	$x_2^{'}$	$x_2^{''}$	s_1	s_2	s_3	Sol
z	0	0	0	0	$\frac{-3}{5}$	0	$\frac{7}{5}$	$\frac{51}{5}$
$x_2^{'}$	0	0	1	-1	$\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{13}{5}$
s_1	0	0	0	0	1	$\frac{5}{2}$	$\frac{-3}{2}$	$\frac{11}{2}$
$x_1^{'}$	1	-1	0	0	$-\frac{3}{5}$	0	$\frac{2}{5}$	$\frac{6}{5}$

$\frac{5}{2}s_1$

Basic	$x_1^{'}$	$x_1^{''}$	$x_2^{'}$	$x_2^{''}$	s_1	s_2	s_3	Sol
z	0	0	0	0	0	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{27}{2}$
$x_2^{'}$	0	0	1	-1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
s_1	0	0	0	0	1	$\frac{5}{2}$	$\frac{-3}{2}$	$\frac{11}{2}$
$x_1^{'}$	1	-1	0	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{9}{2}$

$z+\frac{3}{5}s_1$

$x_2^{'}-\frac{1}{5}s_1$

$x_1^{'}+\frac{3}{5}s_1$

$$x_1^{'}=\frac{9}{2},\ x_1^{''}=0\ ,x_2^{'}=\frac{3}{2},\ x_2^{''}=0$$

As $x_1 = x_1^{'} - x_1^{''}$

$$x_1=\frac{9}{2}-0$$

$$x_1=\frac{9}{2}$$

$$x_2 = x_2^{'} - x_2^{''}$$

$$x_2=\frac{3}{2}-0$$

$$x_2=\frac{3}{2}$$

And $z = \frac{27}{2}$

Lecture # 05

Two Phase method:

In two phase method the whole procedure of solving a Linear programming problem involving artificial variable is divided into two phases.

In Phase-I we form a new objective function by assigning zero to every variable into the slack and surplus variable and negative one to each of the artificial variables (for maximization case) and multiply with positive one (for minimization case). Then we try to eliminate the artificial variables from the basis. *The solution at the phase-I serves as an initial basic feasible solution for phase-II.

In phase-II original objective function is introduced and usual simplex algorithm is used to find an optimal solution.

Example: Maximize $z = 2x_1 + 3x_2 + x_3$

Subject to $x_1 + x_2 + x_3 \leq 40$

$$2x_1 + x_2 - x_3 \geq 10$$

$$-x_2 + x_3 \geq 10$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

$$x_1 + x_2 + x_3 + s_1 = 40$$

$$2x_1 + x_2 - x_3 - s_2 = 10$$

$$-x_2 + x_3 - s_3 = 10$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

$$x_1 + x_2 + x_3 + s_1 = 40$$

$$2x_1 + x_2 - x_3 - s_2 + A_1 = 10$$

$$-x_2 + x_3 - s_3 + A_2 = 10$$

$$x_1, x_2, x_3, s_1, s_2, s_3, A_1, A_2 \geq 0$$

At $x_1, x_2, x_3 = 0$, $s_1 = 40$

And $-s_2 = 10 \Rightarrow s_2 = -10$

$-s_3 = 10 \Rightarrow s_3 = -10$

s_1 is +ve and s_2, s_3 are -ve

So, we introduce Artificial variable for s_2, s_3

In phase-I we introduce a new function ‘w’

In this we write ‘Artificial variable’ to R.H.S with negative.

When A_1 or A_2 leave we will omit A_1 or A_2 Column.

Phase -I: We introduce

$$w = -A_1 - A_2$$

$$w + (A_1 + A_2) = 0 \qquad \_\_\_\_\_(1)$$

From above $A_1 = 10 - 2x_1 - x_2 + x_3 + s_2$

$$A_2 = 10 + x_2 - x_3 + s_3$$

$$A_1 + A_2 = 20 - 2x_1 + s_2 + s_3$$

Put in (1) $\quad w + 20 - 2x_1 + s_2 + s_3 = 0$

$$w - 2x_1 + s_2 + s_3 = -20$$

Basic	x_1	x_2	x_3	s_1	s_2	s_3	A_1	A_2	Sol
w	-2	0	0	0	1	1	0	0	-20
s_1	1	1	1	1	0	0	0	0	40
A_1	2	1	-1	0	-1	0	1	0	10
A_2	0	-1	1	0	0	-1	0	1	10

Basic	x_1	x_2	x_3	s_1	s_2	s_3	A_2	Sol
w	0	1	-1	0	0	1	0	-20
s_1	0	$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	0	0	35
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	5
A_2	0	-1	1	0	0	-1	1	10

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Sol
w	0	0	0	0	0	0	0
s_1	0	2	0	1	$\frac{1}{2}$	$\frac{3}{2}$	20
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	-1	1	0	0	-1	10

In this method it is obvious that when we solve phase-I then row of ‘w’ will become zero. Then we start phase-II in which we put original function of ‘z’ remaining table will not be changed.

$$z = 2x_1 + 3x_2 + x_3$$

$$z - 2x_1 - 3x_2 - x_3 = 0$$

Phase-II:

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Sol
z	-2	-3	-1	0	0	0	0
s_1	0	2	0	1	$\frac{1}{2}$	$\frac{3}{2}$	20
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	-1	1	0	0	-1	10

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Sol
z	-2	0	-1	$\frac{3}{2}$	$\frac{3}{4}$	$\frac{9}{4}$	30
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	10
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	20

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Sol
z	0	0	-1	$\frac{3}{2}$	$-\frac{1}{4}$	$\frac{5}{4}$	50
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	10
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	20

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Sol
z	0	0	0	2	0	1	70
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	10
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	0	1	1	1	-1	20

Here $x_1 = 10$, $x_2 = 10$, $x_3 = 20$ and $z = 70$ is the required solution.

Lecture # 06

Big-M Method or M-technique:

Big-M method is a modified form of two phase method. Here M is a very very large positive value.

As M-technique is modified form of “Two phase”. In Two phase method we use a new function ‘w’ in phase-I and original function ‘z’ in phase-II. But in M-technique we use new function and original function. We multiply artificial function with ‘M’ and subtract it from ‘z’.

Example: Maximize $z=2x_1+3x_2+x_3$

Subject to $x_1+x_2+x_3\leq 40$

$$2x_1+x_2-x_3\geq 10$$

$$-x_2+x_3\geq 10$$

$$x_1,x_2,x_3\geq 0$$

Solution:

$$x_1+x_2+x_3+s_1=40$$

$$2x_1+x_2-x_3-s_2+A_1=10$$

$$-x_2+x_3-s_3+A_2=10$$

$$x_1,x_2,x_3,s_1,s_2,s_3,A_1,A_2\geq 0$$

$$z=2x_1+3x_2+x_3-MA_1-MA_2$$

$$z=2x_1+3x_2+x_3-M\left(A_1+A_2\right) \qquad \_\_\_\_\_\_\_(i)$$

$$A_1=10-2x_1-x_2+x_3+s_2$$

$$A_2=10+x_2-x_3+s_3$$

$$A_1+A_2=20-2x_1+s_2+s_3 \qquad \text{put in (i)}$$

$$z=2x_1+3x_2+x_3-M\left(20-2x_1+s_2+s_3\right)$$

$$z=\left(2+2M\right)x_1+3x_2+x_3-Ms_2-Ms_3-20M$$

$$z-\left(2+2M\right)x_1-3x_2-x_3+Ms_2+Ms_3=-20M$$

Basic	↓ x_1	x_2	x_3	s_1	s_2	s_3	A_1	A_2	Sol
z	–(2+2M)	–3	–1	0	M	M	0	0	–20M
s_1	1	1	1	1	0	0	0	0	40
← A_1	2	1	–1	0	–1	0	1	0	10
A_2	0	–1	1	0	0	–1	0	1	10

Basic	x_1	x_2	↓ x_3	s_1	s_2	s_3	A_2	Sol
z	0	–2+M	–2–M	0	–1	M	0	–10M+10
s_1	0	$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{1}{2}$	0	0	35
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	5
← A_2	0	–1	1	0	0	–1	1	10

Basic	x_1	↓ x_2	x_3	s_1	s_2	s_3	Sol
z	0	–4	0	0	–1	–2	30
← s_1	0	2	0	1	$\frac{1}{2}$	$\frac{3}{2}$	20
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	–1	1	0	0	–1	10

Basic	x_1	x_2	x_3	s_1	s_2	s_3	Sol
z	0	0	0	2	0	1	70
x_2	0	1	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$	10
x_1	1	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	10
x_3	0	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$	20

$\Rightarrow \quad x_1 = 10 \ , \ x_2 = 10 \ , \ x_3 = 20$ and $z = 70$

Revised Simplex method or Dual simplex method:

Leaving variable: Most negative value in solution column

Entering variable: Minimum positive ratio in Ratio-Row

This method is reciprocal of simplex method. In this method first, we choose leaving variable the greater and negative value from solution column will leave. Then we choose entering variable by making Ratio-Row. In this we choose least positive value.

Example: Maximize $z = 80x_1 + 100x_2$

Subject to $$80x_1 + 60x_2 \geq 1500$$

$$20x_1 + 90x_2 \geq 1200$$

$$x_1, x_2 \geq 0$$

Solution: $$-80x_1 - 60x_2 \leq -1500$$

$$-20x_1 - 90x_2 \leq -1200$$

$$x_1, x_2 \geq 0$$

$$-80x_1 - 60x_2 + s_1 = -1500$$

$$-20x_1 - 90x_2 + s_2 = -1200$$

$$x_1, x_2, s_1, s_2 \geq 0 \text{ and } z - 80x_1 - 100x_2 - 0s_1 - 0s_2 = 0$$

$\because {}^{R}\!/_{R} = \frac{z}{leaving\ row}$

Basic	↓ x_1	x_2	s_1	s_2	Sol
z	-80	-100	0	0	0
← s_1	-80	-60	1	0	-1500
s_2	-20	-90	0	1	-1200
${}^{R}\!/_{R}$	${}^{-80}\!/_{80} = 1$	$\frac{-100}{-60} = {}^{5}\!/_{3}$	0	0	

Basic	x_1	↓ x_2	s_1	s_2	Sol
z	0	-40	-1	0	1500
x_1	1	$\frac{3}{4}$	$-\frac{1}{80}$	0	$\frac{75}{4}$
← s_2	0	-75	$-\frac{1}{4}$	1	-825
${}^{R}\!/_{R}$	0	$\frac{-40}{-75} = {}^{8}\!/_{15}$	$\frac{-1}{-\frac{1}{4}} = 4$	0	

Basic	x_1	x_2	s_1	s_2	Sol
z	0	0	$-\frac{13}{15}$	$\frac{-8}{15}$	1940
x_1	1	0	$-\frac{3}{200}$	$\frac{1}{100}$	$\frac{21}{2}$
x_2	0	1	$\frac{1}{300}$	$-\frac{1}{75}$	11

$$\Rightarrow \quad x_1 = \frac{21}{2}, \; x_2 = 11 \quad \text{and} \quad z = 1940$$

Note: Initial solution does not exist in Revised simplex method.

Lecture # 07

Duality in Linear Programming:

Duality is very important concept associated with linear programming. The term duality implies that every linear programming problem whether of maximize or minimize is associated with another linear programming problem based on the same data.

The original problem is this context is called primal problem whereas the other is called its dual problem.

Example: Maximize $z = x_1 + x_2 + x_3$

Subject to
$$x_1 - 2x_2 + x_3 \geq 4$$
$$2x_1 + x_2 - 3x_3 \geq 3$$
$$-x_1 + x_2 - x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

Its dual problem
$$z = 4y_1 + 3y_2 + 4y_3$$

Subject to
$$y_1 + 2y_2 - y_3 \leq 1$$
$$-2y_1 + y_2 + y_3 \leq 1$$
$$y_1 - 3y_2 - y_3 \leq 2$$
$$y_1, \mathrm{y}_2, \mathrm{y}_3 \geq 0$$

For MCQ *Dual of dual problem is called primal or oringnal problem.

Transportation problem:

The transportation problem deals with a special class of linear programming problem in which the objective is to transport a product manufactured at several plants to a number of different destination at a minimum total cost. The quantity demanded by the destination are given in the statement of the problem. The cost of shipping a unit of goods from a non-origin to a non-destination is also given. Our objective is to detemive the total minimum shipping cost.

Basic feasible solution:

A solution of M × N transportation problem is said to be basic feasible solution if the total number of allocation is equal to $m + n - 1$

Optimal Solution:

A basic feasible solution is said to be optimal when the total transportaion cost is minimum.

Methods for finding basic feasible solution:

There are three methods for finding basic feasible solution

(i) North-West corner method (NWCM) or Top left corner mehtod.
(ii) Least cost method (LCM) or Matrix minimum method.
(iii) Vogel's approximation mehtod (VAM) or Method of penalty

(i) North-West corner method:

Procedure of NWCM:

(i) First we check the problem is balance or unbalance. If supply equal demands then the problem is balance otherwise unbalance.
(ii) Select the topleft corner cell (box) of the transportaion problem and allocates as many unit as possible equal to the minimum between available supply and demand.
(iii) Adjust the supply and demand number in the respective rows and column.
(iv) If the demand for the first cell is satisfied then move horizontally to the next cell in the second column.
(v) If the supply for the first cell is satisfied then move down in the second row.
(vi) If demand is satisfied then delete (×) the column.
(vii) If supply is satisfied then delete (×) the row.
(viii) Continue the process until all supply and demand are satisfied.

Example:

Factory	Distribution				
	1	2	3	4	Supply
1	200 $\lceil 3$	50 $\lceil 1$	× $\lceil 7$	× $\lceil 9$	~~250 50~~ ×
2	× $\lceil 2$	250 $\lceil 6$	100 $\lceil 5$	× $\lceil 9$	~~350 100~~ ×
3	× $\lceil 8$	× $\lceil 3$	250 $\lceil 3$	150 $\lceil 2$	~~400 150~~ ×
Demand	~~200~~ ×	~~300 250~~ ×	~~350 250~~ ×	~~150~~ ×	1000

Total no. of allocations = m + n – 1

6 = 3+4 – 1

6 = 6

Total cost = 200 × 3 + 50 ×1 + 250×6+ 100×5+250×3+150×2 = 3700

Question:

	D_1	D_2	D_3	D_4	Supply
O_1	60 $\lceil 20$	40 $\lceil 22$	20 $\lceil 17$	× $\lceil 4$	~~120 60 20~~ ×
O_2	× $\lceil 24$	× $\lceil 37$	10 $\lceil 9$	60 $\lceil 7$	~~70 60~~ ×
O_3	× $\lceil 32$	× $\lceil 37$	× $\lceil 20$	50 $\lceil 15$	~~50~~ ×
Demand	~~60~~ ×	~~40~~×	~~30 10~~ ×	~~110 50~~×	240

Total no. of allocations = m + n – 1

6 = 3 + 4 – 1

6 = 6

Total cost = 60×20 + 40×22 + 20×17 + 10×9 +60×7+50×15 = 3680

Question:

	w_1	w_2	w_3	Supply
P_1	10 $\lceil 2$	× $\lceil 1$	× $\lceil 5$	~~10~~ ×
P_2	5 $\lceil 7$	20 $\lceil 3$	× $\lceil 7$	~~25 20~~ ×
P_3	× $\lceil 6$	2 $\lceil 5$	18 $\lceil 15$	~~20 18~~ ×
Demand	~~15 5~~ ×	~~22 2~~ ×	~~18~~ ×	55

Total no. of allocations = m + n – 1 ⇒ 5 = 3 + 3 – 1 ⇒ 5 = 5

Total cost = 10×2 + 5×7 + 20×3 + 2×5 +18×3 = 179

Question: Construct 4×5 transportation problem (balanced) and find basic feasible solution by Top-left corner method.

	w_1	w_2	w_3	w_4	w_5	Supply
P_1	$55\,\boxed{2}$	$22\,\boxed{5}$	$23\,\boxed{4}$	$\times\,\boxed{7}$	$\times\,\boxed{3}$	~~100 45 23~~ ×
P_2	$\times\,\boxed{1}$	$\times\,\boxed{3}$	$22\,\boxed{4}$	$3\,\boxed{2}$	$\times\,\boxed{6}$	~~25 3~~ ×
P_3	$\times\,\boxed{4}$	$\times\,\boxed{2}$	$\times\,\boxed{5}$	$25\,\boxed{3}$	$15\,\boxed{2}$	~~40 15~~ ×
P_4	$\times\,\boxed{3}$	$\times\,\boxed{5}$	$\times\,\boxed{4}$	$\times\,\boxed{1}$	$35\,\boxed{2}$	~~35~~ ×
Demand	~~55~~ ×	~~22~~ ×	~~45 22~~×	~~28~~ 25 ×	~~50 35~~ ×	200

Total no. of allocations = m + n – 1

8 = 4 + 5 – 1

8 = 8

Total cost = 55×2 + 22×5 + 23×4 + 22×4 +3×2 + 25×3+15×2+35×2

= 581

Lecture # 08

Special Case Degenerate / Degeneracy:

Degeneracy occurs when (Total no. of allocations ≠ m + n – 1)

Question:

	X	Y	Z	Supply
A	⌈8	⌈7	⌈3	60
B	⌈3	⌈8	⌈9	70
C	⌈11	⌈3	⌈5	80
Demand	50	80	80	210

Solution:

	X	Y	Z	Supply
A	50 ⌈8	10⌈7	× ⌈3	~~60~~ ~~10~~ ×
B	× ⌈3	70 ⌈8	× ⌈9	~~70~~ ×
C	× ⌈11	d⌈3	80 ⌈5	~~80~~ ×
Demand	~~50~~ ×	~~80~~ ~~70~~ ×	~~80~~ ×	210

Since d is very very small positive number whose contribution in the solution is negligible.

Total no. of allocations = m + n – 1

5 = 3+3 – 1

5 = 5

Total cost = 50 × 8 + 10 ×7 + 70×8+3d + 80×5 = 1430+3d

Unbalance problem:

	5	6	7	8	9	10	Supply
1	⌈9	⌈12	⌈9	⌈6	⌈9	⌈10	5
2	⌈7	⌈3	⌈7	⌈7	⌈5	⌈5	6
3	⌈6	⌈5	⌈9	⌈11	⌈3	⌈11	2
4	⌈6	⌈8	⌈11	⌈2	⌈2	⌈10	2
Demand	4	4	6	2	4	2	

- If supply less than demand then we add row
- If demand less than supply then we add column

Solution:

	5	6	7	8	9	10	Supply
1	4 ⌈9	1 ⌈12	× ⌈9	× ⌈6	× ⌈9	× ⌈10	~~5~~ ×
2	× ⌈7	3 ⌈3	3 ⌈7	× ⌈7	× ⌈5	× ⌈5	~~6~~ ~~3~~ ×
3	× ⌈6	× ⌈5	2 ⌈9	× ⌈11	× ⌈3	× ⌈11	~~2~~ ×
4	× ⌈6	× ⌈8	1 ⌈11	1 ⌈2	× ⌈2	× ⌈10	~~2~~ ~~1~~ ×
Unsatisfied Row	× ⌈0	× ⌈0	× ⌈0	1 ⌈0	4 ⌈0	2 ⌈0	~~7~~ ~~6~~ ~~2~~ ×
Demand	~~4~~ ×	~~4~~ ~~3~~ ×	~~6~~ ~~3~~ ~~1~~ ×	~~2~~ ~~1~~ ×	~~4~~ ×	~~2~~ ×	22

No. of allocations = m + n – 1

10 = 5 + 6 –1

10 = 10

Total cost = 4×9 + 1×12 + 3×3 + 3×7 + 2×9 + 1×11 + 1×2 + 1× 0+4×0+2×0

= 109

Least Cost Method:

Procedure:

(i) First of all we check the problem is balance or unbalance.

(ii) Identify the box having minimum unit transportation cost $\left(c_{ij}\right)$

(iii) If the minimum cost is not unique then choose the top left corner box for allocation.

(iv) Choose the value of the corresponding x_{ij} as much as possible subject to the supply and demand constraints.

(v) Repeat the above steps untill all restrictions are satisfied.

Example:

	5	6	7	8	9	10	Supply
1	× $\boxed{9}$	× $\boxed{12}$	5 $\boxed{9}$	× $\boxed{6}$	× $\boxed{9}$	× $\boxed{10}$	~~5~~ ×
2	× $\boxed{7}$	1 $\boxed{3}$	1 $\boxed{7}$	× $\boxed{7}$	2 $\boxed{5}$	2 $\boxed{5}$	~~6 5 3 1~~ ×
3	× $\boxed{6}$	× $\boxed{5}$	× $\boxed{9}$	× $\boxed{11}$	2 $\boxed{3}$	× $\boxed{11}$	~~2~~ ×
4	× $\boxed{6}$	× $\boxed{8}$	× $\boxed{11}$	2 $\boxed{2}$	d $\boxed{2}$	× $\boxed{10}$	~~2~~ ×
Unsatisfied Row	4 $\boxed{0}$	3 $\boxed{0}$	× $\boxed{0}$	× $\boxed{0}$	× $\boxed{0}$	× $\boxed{0}$	~~7 3~~ ×
Demand	~~4~~ ×	~~4 1~~ ×	~~6 5~~ ×	~~2~~ ×	~~4 2~~ ×	~~2~~ ×	22

Number of allocations = m + n – 1

10 = 5 + 6 – 1

Total cost = 5×9 + 1×3 + 1×7 + 2×5 + 2×5 + 2×3 + 2×2 + 2d + 4×0 + 3×0

= 85 + 2d

Question:

Find basic feasible solution of the following transportaion problem by Least cost method.

	1	2	3	4	Supply
1	× $\boxed{3}$	250 $\boxed{1}$	× $\boxed{7}$	× $\boxed{4}$	~~250~~ ×
2	200 $\boxed{2}$	× $\boxed{6}$	150 $\boxed{5}$	× $\boxed{9}$	~~350 150~~ ×
3	× $\boxed{8}$	50 $\boxed{3}$	200 $\boxed{3}$	150 $\boxed{2}$	~~400 250 200~~ ×
Demand	~~200~~ ×	~~300 50~~ ×	~~350 150~~ ×	~~150~~ ×	1000

Total no. of allocations = m + n – 1

6 = 3 + 4 – 1

6 = 6

Total cost = 250×1 + 200×2 + 150×5 + 50×3 +200×3+150×2 = 2450

Question:

Find basic feasible solution of the following transportaion problem by Least cost method.

	D_1	D_2	D_3	D_4	Supply
O_1	10 ⌈20	× ⌈22	× ⌈17	110 ⌈4	~~120~~ ~~10~~ ×
O_2	40 ⌈24	× ⌈37	30 ⌈9	× ⌈7	~~70~~ ~~40~~ ×
O_3	10 ⌈32	40 ⌈37	× ⌈20	× ⌈15	~~50~~ ~~40~~ ×
Demand	~~60~~ ~~50~~ ~~10~~ ×	~~40~~ ×	~~30~~ ×	~~110~~ ×	240

Total no. of allocations = m + n – 1

6 = 3 + 4 – 1

6 = 6

Total cost = 10×20 + 110×4 + 40×24 + 30×9 +10×32+40×37 = 7630

Question:

Find basic feasible solution of the following transportaion problem by Least cost method.

	w_1	w_2	w_3	Supply
P_1	× ⌈2	10 ⌈1	× ⌈5	~~10~~ ×
P_2	13 ⌈7	12 ⌈3	× ⌈4	~~25~~ ~~13~~ ×
P_3	2 ⌈6	× ⌈5	18 ⌈3	~~20~~ ~~2~~ ×
Demand	~~15~~ ~~13~~ ×	~~22~~ ~~12~~ ×	~~18~~ ×	55

Total no. of allocations = m + n – 1

5 = 3 + 3 – 1

5 = 5

Total cost = 10×1 + 13×7 + 12×3 + 2×6 +18×3 = 203

Question:

Find basic feasible solution of the following transportaion problem by Least cost method.

	w_1	w_2	w_3	w_4	w_5	Supply
P_1	$30\,\overline{\vert 2}$	$\times\overline{\vert 5}$	$45\,\overline{\vert 4}$	$\times\overline{\vert 7}$	$25\,\overline{\vert 3}$	~~100 70 45~~ ×
P_2	$25\,\overline{\vert 1}$	$\times\overline{\vert 3}$	$\times\overline{\vert 4}$	$\times\overline{\vert 2}$	$\times\overline{\vert 6}$	~~25~~ ×
P_3	$\times\overline{\vert 4}$	$22\,\overline{\vert 2}$	$\times\overline{\vert 5}$	$\times\overline{\vert 3}$	$18\,\overline{\vert 2}$	~~40 18~~ ×
P_4	$\times\overline{\vert 3}$	$\times\overline{\vert 5}$	$\times\overline{\vert 4}$	$28\,\overline{\vert 1}$	$7\,\overline{\vert 2}$	~~35 7~~ ×
Demand	~~55 30~~ ×	~~22~~ ×	~~45~~ ×	~~28~~ ×	~~50 32 25~~×	200

Total no. of allocations = m + n – 1

$8 = 4 + 5 - 1$

$8 = 8$

Total cost = 30×2 + 45×4 + 25×3 + 25×1 +22×2 + 18×2+28×1+7×2

= 462

Lecture # 09

Vogel's Approximation method or Method of Penalty:

The Vogel's approximation method is an iterative procedure for computing an initial basic feasible solution of a transportation problem. This method is preferred over the two methods discussed in the previous sections, because the initial basis feasible solution obtained by this method is either optimal or very close to optimal solution.

"This method is a little complex than the previously discussed methods. So, go slowly and reread the explanation at least twice".

Steps in Vogel's Approximation method (VAM):

(i) First of all, we check our problem is balance or unbalance.

(ii) Identify the boxes having minimum and next to minimum transportation cost in each row and write the difference (penalty) along the side of the table against the corresponding row.

(iii) Identify the boxes having minimum and next to minimum transportation cost in each column and write the difference (penalty) against the corresponding column.

(iv) Identify the maximum penalty. If it is along the side of the table, make maximum allocation to the box having minimum cost of transportation in that row. If it is below the table, make maximum allocation to the box having minimum cost of transportation in that column.

(v) If the penalties corresponding to two or more rows or columns are equal you are at liberty to break the tie arbitrarily.

(vi) Repeat the above steps until all restrictions are satisfied.

Question: Find basic feasible solution by Vogel's approximation method of the following transportation problem.

	4	5	6	7	Supply
1	× ⌈4	6 ⌈3	18 ⌈0	× ⌈5	~~24 6~~ ×
2	× ⌈1	13 ⌈2	× ⌈6	4 ⌈1	~~17 13~~ ×
3	15 ⌈3	× ⌈6	× ⌈2	4 ⌈3	~~19 4~~ ×
Demand	~~15~~ ×	~~19 13~~ ×	~~18~~ ×	~~8 4~~ ×	

③	1	2	2	③
1	1	1	1	2
1	③	③	-	-

2	1	2	2
2	1	-	2
-	1	-	2
-	1	-	④
-	1	-	-

Total No. of allocations = m + n – 1

6 = 3 + 4 – 1

6 = 6

Total cost = 6 × 3 + 18 × 0 +13×2 + 4×1 + 15×3 + 4×3

= 105

Question: Find basic feasible solution by North West corner method, Least corner method, Vogel's approximation method of the following transportation problem. **Solution**: (i) By NWCM

	X	Y	Z	Supply
A	50 ⌈8	10 ⌈7	× ⌈3	~~60 10~~ ×
B	× ⌈3	70 ⌈8	× ⌈9	~~70~~ ×
C	× ⌈11	d ⌈3	80 ⌈5	~~80~~ ×
Demand	~~50~~ ×	~~80 70~~ ×	~~80~~ ×	210

Since d is very very small positive number whose contribution in the solution is negligible. Total no. of allocations = m + n – 1

5 = 3+3 – 1

5 = 5

Total cost = 50 × 8 + 10 ×7 + 70×8+3d + 80×5 = 1430+3d

(ii) By L.C.M

	X	Y	Z	Supply
A	× ⌈8	× ⌈7	60 ⌈3	~~60~~ ×
B	50 ⌈3	× ⌈8	20 ⌈9	~~70~~ ~~20~~ ×
C	× ⌈11	80 ⌈3	d ⌈5	~~80~~ ×
Demand	~~50~~ ×	~~80~~ ×	~~80~~ 20 ×	210

Total no. of allocations = m + n – 1

5 = 3+3 – 1

5 = 5

Total cost = 60 × 3 + 50 ×3 + 20 × 9 + 80 × 3 + 5d = 750 + 5d

(iii) By Vogel's Approximation method:

	X	Y	Z	Supply
A	× ⌈8	× ⌈7	60 ⌈3	~~60~~ ×
B	50 ⌈3	× ⌈8	20 ⌈9	~~70~~ ~~20~~ ×
C	× ⌈11	80 ⌈3	d ⌈5	~~80~~ ×
Demand	~~50~~ ×	~~80~~ ×	~~80~~ 20 ×	210

4	④	-
⑤	1	1
2	2	2

5	4	2
-	4	2
-	⑤	4

Total No. of allocations = m + n – 1

5 = 3 + 3 – 1

5 = 5

Total cost = 60 × 3 + 50 × 3 + 20×9 + 80 × 3 + 5d

= 750 + 5d

Question: Find basic feasible solution by Vogel's approximation method of the following transportation problem.

	w_1	w_2	w_3	Supply
A	28	17	26	500
B	19	12	16	300
Demand	250	250	500	

Solution:

	w_1	w_2	w_3	Supply
A	50 28	250 17	200 26	~~500 450 250~~ ×
B	× 19	× 12	300 16	~~300~~ ×
Unrestricted row	200 0	× 0	× 0	~~200~~ ×
Demand	~~250 50~~ ×	~~250~~ ×	~~500 200~~ ×	

9	9	9	9
4	4	-	-
0	-	-	-

⑲ 12 16

9 5 ⑩

㉘ 17 26

\- 17 ㉖

Total No. of allocations = m + n – 1

5 = 3 + 3 – 1

5 = 5

Total cost = 50 × 28 + 250 × 17 + 200×26 + 300 × 16 + 200 × 0

= 15650

Lecture # 10

Optimal Solution for transportation problem:

There are two methods for finding an optimal solution. For transportation problem

(i) Stepping stone method
(ii) MODI method or Modified Distribution method or u-v method

(i) Stepping Stone method:

Question: Find initial basic feasible solution by Least Cost Method (L.C.M) and optimal solution by Stepping stone method of the following transportation problem.

Solution:

	1	2	3	4	Supply
A	× $\boxed{4}$	450 $\boxed{6}$	× $\boxed{8}$	250 $\boxed{6}$	~~700 250~~ ×
B	50 $\boxed{3}$	× $\boxed{5}$	350 $\boxed{2}$	× $\boxed{5}$	~~400 50~~ ×
C	350 $\boxed{3}$	× $\boxed{9}$	× $\boxed{6}$	250 $\boxed{5}$	~~600 250~~ ×
Demand	~~400 350~~ ×	~~450~~ ×	~~350~~ ×	~~500 250~~ ×	1700

$$\text{Total cost} = 450 \times 6 + 250 \times 6 + 50\times3 + 350 \times 2 + 350 \times 3 + 250 \times 5$$
$$= 7350$$

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
11	$4 - 6 + 5 - 3 = 0$	NIND
13	$8 - 6 + 5 - 3 + 3 - 2 = 5$	Increase
22	$5 - 6 + 6 - 5 + 3 - 3 = 0$	NIND
24	$5 - 5 + 3 - 3 = 0$	NIND
32	$9 - 6 + 6 - 5 = 4$	Increase
33	$6 - 2 + 3 - 3 = 4$	Increase

Since all the values of unoccupied cells are non-negative. So, the solution obtained is optimal.

Question: Find initial basic feasible solution by Least Cost Method (L.C.M) and optimal solution by Stepping stone method of the following transportation problem.

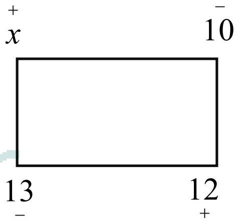
Solution:

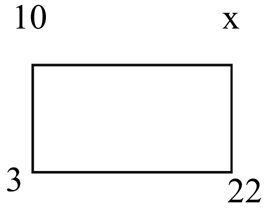
	1	2	3	Supply
A	× $\boxed{2}$	10 $\boxed{1}$	× $\boxed{5}$	10 ×
B	13 $\boxed{7}$	12 $\boxed{3}$	× $\boxed{4}$	~~13 25~~ ×
C	2 $\boxed{6}$	× $\boxed{5}$	18 $\boxed{3}$	~~20 2~~ ×
Demand	~~15 13~~ ×	~~22 12~~ ×	~~18~~ ×	

Total cost = 10 × 1 + 13 × 7 + 12×3 + 2 × 6 + 18 × 3 = 203

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
11	$2 - 1 + 3 - 7 = -3$	Decrease
13	$5 - 3 + 6 - 7 + 3 - 1 = 3$	Increase
23	$4 - 3 + 6 - 7 = 0$	NIND
32	$5 - 6 + 7 - 3 = 3$	Increase

	1	2	3	Supply
A	10 $\boxed{2}$	× $\boxed{1}$	× $\boxed{5}$	10 ×
B	3 $\boxed{7}$	22 $\boxed{3}$	× $\boxed{4}$	~~13 25~~ ×
C	2 $\boxed{6}$	× $\boxed{5}$	18 $\boxed{3}$	~~20 2~~ ×
Demand	~~15 13~~ ×	~~22 12~~ ×	~~18~~ ×	





Total cost = 10 × 2 + 3 × 7 + 22×3 + 2 × 6 + 18 × 3

= 173

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
12	$1 - 3 + 7 - 2 = 3$	Increase
13	$5 - 3 + 6 - 2 = 6$	Increase
23	$4 - 3 + 6 - 7 = 0$	NIND
32	$5 - 3 + 7 - 6 = 3$	Increase

نئے ٹیبل میں ہم جو x اس کی جگہ اور مثبت ویلیو میں چھوٹی قیمت جمع کرنا ہے۔ اور منفی ویلیو کی جگہ والی رقم سے تفریق کرنا ہے۔

Since all the values of unoccupied cells are non-negative. So, sol is optimal.

Question: Find initial basic feasible solution by Least Cost Method (L.C.M) and optimal solution by Stepping stone method of the following transportation problem.

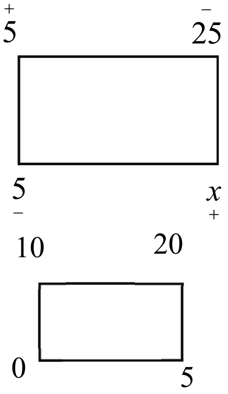
Solution:

	1	2	3	4	Supply
A	× 6	5 8	× 8	25 5	~~30 5~~ ×
B	35 5	5 11	× 9	× 7	~~40 5~~ ×
C	× 8	18 9	32 7	× 13	~~50 18~~ ×
Demand	~~35~~ ×	~~28 23 18~~ ×	~~32~~ ×	~~25~~ ×	120

Total cost = 5 × 8 + 25 × 5 + 35 × 5 + 5 × 11 + 18 × 9 + 32 × 7 = 781

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
11	6 – 8 + 11 – 5 = 4	Increase
13	8 – 7 + 9 – 8 = 2	Increase
23	9 – 7 + 9 –11 = 0	NIND
24	7 – 5 + 8 – 11 = –1	Decrease
31	8 – 9 + 11 – 5 = 5	Increase
34	13 – 5 + 8 – 9 = 7	Increase

	1	2	3	4	Supply
A	× 6	10 8	× 8	20 5	~~30 5~~ ×
B	35 5	0 11	× 9	5 7	~~40 5~~ ×
C	× 8	18 9	32 7	× 13	~~50 18~~ ×
Demand	~~35~~ ×	~~28 23 18~~ ×	~~32~~ ×	~~25~~ ×	120



Total cost = 10 × 8 + 20 × 5 + 35 × 5 + 0 × 11 + 5 × 7 + 18 × 9 + 32 × 7 = 776

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
11	6 – 5 + 7 – 5 = 3	Increase
13	8 – 7 + 9 – 8 = 2	Increase
23	9 – 7 + 9 –11 = 0	NIND
31	8 – 9 + 11 – 5 = 5	Increase
34	13 – 5 + 8 – 9 = 7	Increase

Since all the values of unoccupied cells are non-negative. So, the solution obtained is optimal.

Lecture # 11

Question: Find basic feasible solution by matrix minimum method and optimal solution by Stepping stone method of the following transportation problem.

Solution:

	1	2	3	4	Supply
A	× 3	250 1	× 7	× 4	~~250~~ ×
B	200 2	× 6	150 5	× 9	~~350 150~~ ×
C	× 8	50 3	200 3	150 2	~~400 250 200~~ ×
Demand	~~200 ×~~	~~300 50~~ ×	~~350 150~~ ×	~~150~~ ×	1000

Total cost = 250 × 1 + 200 × 2 + 150×5 + 50 × 3 + 200 × 3 + 150 × 2

= 2450

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
A1	3 – 1 + 3 – 3+5–2 = 5	Increase
A3	7 – 3 + 3 – 1 = 6	Increase
A4	4 – 2 + 3 – 1 = 4	Increase
B2	6 – 5 + 3 – 3 = 1	Increase
B4	9 – 2 + 3 – 5 = 5	Increase
C1	8– 3 + 5 – 2 = 8	Increase

Since all the values of unoccupied cells are non-negative. So, the solution obtained is optimal.

Question: Find initial basic feasible solution by Least Cost Method (L.C.M) and optimal solution by Stepping stone method of the following transportation problem.

Solution:

	1	2	3	Supply
1	× 5	× 4	100 3	100 ×
2	× 8	200 4	100 3	~~300 200~~ ×
3	300 9	d 7	× 5	~~300~~ ×
Demand	~~300~~ ×	~~200~~ ×	~~200 100~~ ×	

Total cost = 10 × 1 + 13 × 7 + 12×3 + 2 × 6 + 18 × 3 = 4100 + 7d

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
11	5 – 3 + 3 – 4 + 7–9 = –1	Decrease
12	4 – 3 + 3 – 4 = 0	NIND
21	8 – 4 + 7 – 9 = 2	Increase
33	5 – 3 + 4 – 7 = –1	Decrease

	1	2	3	Supply
1	100 $\boxed{5}$	× $\boxed{4}$	× $\boxed{3}$	100
2	× $\boxed{8}$	100 $\boxed{4}$	200 $\boxed{3}$	300
3	200 $\boxed{9}$	100+d $\boxed{7}$	× $\boxed{5}$	300
Demand	300	200	200	700

Total cost = 100 × 5 + 100 × 4 + 200×3 + 200 × 9 + (100 + d) × 7

= 4000 + 7d

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
12	4 – 7 + 9 – 5 = 1	Increase
13	3– 3 + 4 – 7+9–5 = 1	Increase
21	8 – 4 + 7 – 9 = 2	Increase
33	5 – 3 + 4 – 7 = –1	Decrease

	1	2	3	Supply
1	100 $\boxed{5}$	× $\boxed{4}$	× $\boxed{3}$	100
2	× $\boxed{8}$	200+d $\boxed{4}$	100–d $\boxed{3}$	300
3	200 $\boxed{9}$	× $\boxed{7}$	100+d $\boxed{5}$	300
Demand	300	200	200	700

Total cost = 100 × 5 + (200+d) × 4 + (100–d)×3 + 200 × 9 + (100 + d) × 5

= 3900 + 6d

Unoccupied Cell	Increase in cost of per unit reallocation	Remarks
12	4 – 4 + 3 – 5 + 9 – 5 = 2	Increase
13	3– 5 + 9 – 5 = 2	Increase
21	8 – 3 + 5 – 9 = 1	Increase
32	7 – 5 + 3 – 4 = 1	Increase

Since all the values of unoccupied cells are non-negative. So, the solution is optimal.

Modified Distribution Method or MODI Method or u-v method:

	v_j	v_1	v_2	v_3		
u_i		1	2	3	Supply	
u_1	A	× $\boxed{5}$	× $\boxed{4}$	100 $\boxed{3}$	100 ×	0
u_2	B	× $\boxed{8}$	200 $\boxed{4}$	100 $\boxed{3}$	~~300 200~~ ×	0
u_3	C	300 $\boxed{9}$	d $\boxed{7}$	× $\boxed{5}$	300 ×	3
	Demand	300 ×	200 ×	~~200 100~~ ×	700	
		6	4	3		

Total cost = 100 × 3 + 200 × 4 + 100 × 3 + 300 × 9 + 7d = 4100 + 7d

Computing u_i *and* v_j by using formula

$$u_i + v_j = c_{ij} \text{ (occupied cells)}$$

In every question we take $u_1 = 0$ and find other values with help of it.

Initially we take $u_1 = 0$

$$u_1 + v_3 = 3$$

$$\because\ u_1 = 0 \qquad 0 + v_3 = 3 \ \Rightarrow v_3 = 3$$

$$u_2 + v_3 = 3$$

$$u_2 + 3 = 3 \ \Rightarrow u_2 = 0$$

$$u_2 + v_2 = 4$$

$$0 + v_2 = 4 \ \Rightarrow v_2 = 4$$

$$u_3 + v_2 = 7$$

$$u_3 + 4 = 7 \ \Rightarrow u_3 = 3$$

$$u_3 + v_1 = 9$$

$$3 + v_1 = 9 \ \Rightarrow v_1 = 6$$

Find opportunity cost by using formula $c_{ij} - \left(u_i + v_j\right)$ (unoccupied cells)

Unoccupied Cell	$c_{ij} - (u_i + v_j)$ Opportunity cost	Remarks
11	5 – (0 + 6) = – 1	Decrease
12	4 – (0 + 4) = 0	N.I.N.D
21	8 – (0+6) = 2	Increase
33	5 – (3 + 3) = – 1	Decrease

	v_j	v_1	v_2	v_3		
u_i		1	2	3	Supply	
u_1	1	100 ⌈5	× ⌈4	× ⌈3	100	0
u_2	2	× ⌈8	100 ⌈4	200 ⌈3	300	1
u_3	3	200 ⌈9	100+d ⌈7	× ⌈5	300	4
	Demand	300	200	200	700	
		5	3	2		

Total cost = 100 × 5 + 100 × 4 + 200 × 3 + 200 × (100+d) ×7 = 4000 + 7d

Initially we take $u_1 = 0$

$$u_1 + v_1 = 5$$

$$\because\ u_1 = 0 \qquad 0 + v_1 = 5 \ \Rightarrow v_1 = 5$$

$$u_3 + v_1 = 9$$

$$u_3 + 5 = 9 \ \Rightarrow u_3 = 4$$

$$u_3 + v_2 = 7$$

$$4 + v_2 = 7 \ \Rightarrow v_2 = 3$$

$$u_2 + v_2 = 4$$

$$u_2 + 3 = 4 \ \Rightarrow u_2 = 1$$

$$u_2 + v_3 = 3$$

$$1 + v_3 = 3 \ \Rightarrow v_3 = 2$$

Unoccupied Cell	$c_{ij} - (u_i + v_j)$ Opportunity cost	Remarks
12	$4 - (0 + 3) = 1$	Increase
13	$3 - (0 + 2) = 1$	Increase
21	$8 - (1+5) = 2$	Increase
33	$5 - (4 + 2) = -1$	Decrease

	v_j	v_1	v_2	v_3		
u_i		1	2	3	Supply	
u_1	1	100 $\boxed{5}$	× $\boxed{4}$	× $\boxed{3}$	100	0
u_2	2	× $\boxed{8}$	200+d $\boxed{4}$	100–d $\boxed{3}$	300	2
u_3	3	200 $\boxed{9}$	× $\boxed{7}$	100+d $\boxed{5}$	300	4
	Demand	300	200	200	700	
		5	2	1		

Total cost = 100 × 5 + (200+d) × 4 + (100–d) × 3 + 200 × 9 +(100+d) ×5

= 3900 + 6d

Initially we take $u_1 = 0$

$$u_1 + v_1 = 5$$

$$\because\ u_1 = 0 \qquad 0 + v_1 = 5 \ \Rightarrow v_1 = 5$$

$$u_3 + v_1 = 9$$

$$u_3 + 5 = 9 \ \Rightarrow u_3 = 4$$

$$u_3 + v_3 = 5$$

$$4 + v_3 = 5 \ \Rightarrow v_3 = 1$$

$$u_2 + v_3 = 3$$

$$u_2 + 1 = 3 \ \Rightarrow u_2 = 2$$

$$u_2 + v_2 = 4$$

$$2 + v_2 = 4 \ \Rightarrow v_2 = 2$$

Unoccupied Cell	$c_{ij} - \left(u_i + v_j\right)$ Opportunity cost	Remarks
12	$4 - (0 + 2) = 2$	Increase
13	$3 - (0 + 1) = 2$	Increase
21	$8 - (2+5) = 1$	Increase
32	$7 - (4 + 2) = 1$	Increase

Since all the values of occupied cells are non-negative. Hence the solution obtained is optimal.

Question: For what value of S_4 the problem is unbalance.

$S_1 = 20$, $S_2 = 30$, $S_3 = 25$, $S_4 = ?$

$D_1 = 20$, $D_2 = 25$, $D_3 = 40$

Solution: First we check at what value of S_4 our problem is balance.

$S_4 = 10$

Unbalance for value of S_4 is

$0 \le S_4 < 10$

$S_4 > 10$

Maximization in transportation problem:

There are certain types of transportation problem where the objective is to be maximized instead of minimized. These problems can be solved by converting the maximization problem into a minimization problem.

Example: Find maximum solution of the following transportation problem.

	1	2	3	4	Supply
X	12	18	6	25	200
Y	8	7	10	18	500
Z	14	3	11	20	300
Demand	180	320	100	400	1000

Solution: Maximization transportation problem can be converted into minimization transportation problem by subtracting each transportation cost from maximum transportation cost. Here the maximum transportation cost is 25. So subtract each value from 25. The revised transportation problem is shown below.

	1	2	3	4	Supply
X	× 13	× 7	× 19	200 0	~~200~~ ×
Y	80 17	320 18	100 15	× 7	~~500 400 320~~ ×
Z	100 11	× 22	× 14	200 5	~~300 100~~ ×
Demand	~~180 80~~ ×	~~320~~ ×	~~100~~ ×	~~400 200~~ ×	1000

Total cost = 200 × 25 +80 × 8+320×7 +100 × 10 +100 × 14 + 200 × 20 = 14280

To check 14280 is maximum by u-v mehtod.

		v_1	v_2	v_3	v_4		
		1	2	3	4	Supply	
u_1	X	× 13	× 7	× 19	200 0	200	0
u_2	Y	80 17	320 18	100 15	× 7	500	11
u_3	Z	100 11	× 22	× 14	200 5	300	5
	Demand	180	320	100	400	1000	
		6	7	4	0		

Unoccupied Cell	$c_{ij} - \left(u_i + v_j\right)$ Opportunity cost	Remarks
X1	13 – (0 + 6) = 7	Increase
X2	7 – (0 + 7) = 0	N.I.N.D
X3	19 – (0+4) = 15	Increase
Y4	7 – (11 + 0) = – 4	Decrease
Z2	22–(5+7) = 10	Increase
Z3	14–(5+4) = 5	Increase

		v_1	v_2	v_3	v_4		
		1	2	3	4	Supply	
u_1	X	× 13	× 7	× 19	200 0	200	0
u_2	Y	× 17	320 18	100 15	80 7	500	7
u_3	Z	180 11	× 22	× 14	120 5	300	5
	Demand	180	320	100	400	1000	
		6	11	8	0		

Total cost = 200×25+320×7+100×10+80×18+180×14+120×20 = 14600

Unoccupied Cell	$c_{ij}-(u_i+v_j)$ Opportunity cost	Remarks
X1	13 – (0 + 6) = 7	Increase
X2	7 – (0 + 11) = –4	Decrease
X3	19 – (0+8) = 11	Increase
Y1	17 – (7 + 6) = 4	Increase
Z2	22–(11+5) = 6	Increase
Z3	14–(5+8) = 1	Increase

		v_1	v_2	v_3	v_4		
		1	2	3	4	Supply	
u_1	X	× $\boxed{13}$	200 $\boxed{7}$	× $\boxed{19}$	× $\boxed{0}$	200	0
u_2	Y	× $\boxed{17}$	120 $\boxed{18}$	100 $\boxed{15}$	280 $\boxed{7}$	500	11
u_3	Z	180 $\boxed{11}$	× $\boxed{22}$	× $\boxed{14}$	120 $\boxed{5}$	300	9
	Demand	180	320	100	400	1000	
		2	7	4	–4		

Total cost = 200×18+120×7+100×10+280×18+180×14+120×20 = 15400

Unoccupied Cell	$c_{ij}-(u_i+v_j)$ Opportunity cost	Remarks
X1	13 – (0 + 2) = 11	Increase
X3	19 – (0 + 4) = 15	Increase
X4	0 – (0 –4) = 4	Increase
Y1	17 – (11 + 2) = 4	Increase
Z2	22–(9+7) = 6	Increase
Z3	14–(9+4) = 1	Increase

Since all the unoccupied cell are non-negative. So, the solution obtianed is optimal.

Question: Find maximum solution by NWCM , also by u-v method.

Solution:

	A	B	C	D	Supply
1	⌈4	⌈6	⌈8	⌈6	700
2	⌈3	⌈5	⌈2	⌈5	400
3	⌈3	⌈9	⌈6	⌈5	600
Demand	400	450	350	500	1000

Maximum value is 9

	A	B	C	D	Supply
1	400 ⌈5	300 ⌈3	× ⌈1	× ⌈3	~~700 300~~ ×
2	× ⌈6	150 ⌈4	250 ⌈7	× ⌈4	~~400 250~~ ×
3	× ⌈6	× ⌈0	100 ⌈3	500 ⌈4	~~600 500~~ ×
Demand	~~400~~	~~450 150~~	~~350 100~~	~~500~~	1700

Total cost = 400×4+300×6+150×5+250×2+100×6+500×5 = 7750

Now by u-v method

		v_1	v_2	v_3	v_4		
		A	B	C	D	Supply	
u_1	1	400 ⌈5	300 ⌈3	× ⌈1	× ⌈3	700	0
u_2	2	× ⌈6	150 ⌈4	250 ⌈7	× ⌈4	400	1
u_3	3	× ⌈6	× ⌈0	100 ⌈3	500 ⌈4	600	−3
	Demand	400	450	350	500	1700	
		5	3	6	7		

Unoccupied Cell	$c_{ij}-(u_i+v_j)$ Opportunity cost	Remarks
C1	1 − (0 + 6) = −5	Decrease
D1	3 − (0 + 7) = −4	Decrease
A2	6 − (1+5) = 0	N.I.N.D
D2	4 − (1 + 7) = −4	Decrease
A3	6−(−3+5) = 4	Increase
B3	0−(−3+3) = 0	N.I.N.D

		v_1	v_2	v_3	v_4		
		A	B	C	D	Supply	
u_1	1	400 | 5	50 | 3	250 | 1	× | 3	700	0
u_2	2	× | 6	400 | 4	× | 7	× | 4	400	1
u_3	3	× | 6	× | 0	100 | 3	500 | 4	600	2
	Demand	400	450	350	500	1700	
		5	3	1	2		

Total cost = 400×4+50×6+250×8+400×5+100×6+500×5 = 9000

Unoccupied Cell	$c_{ij} - \left(u_i + v_j\right)$ Opportunity cost	Remarks
D1	$3 - (0 + 2) = 1$	Increase
A2	$6 - (1 + 5) = 0$	N.I.N.D
C2	$7 - (1+1) = 5$	Increase
D2	$4 - (1 + 2) = 1$	Increase
A3	$6-(2+5) = -1$	Decrease
B3	$0-(2+3) = -5$	Decrease

		v_1	v_2	v_3	v_4		
		A	B	C	D	Supply	
u_1	1	400 | 5	× | 3	300 | 1	× | 3	700	0
u_2	2	× | 6	400 | 4	× | 7	× | 4	400	6
u_3	3	× | 6	50 | 0	50 | 3	500 | 4	600	2
	Demand	400	450	350	500	1700	
		5	–2	1	2		

Total cost = 400×4+300×8+400×5+50×9+50×6+500×5 = 9250

Unoccupied Cell	$c_{ij} - \left(u_i + v_j\right)$ Opportunity cost	Remarks
B1	$3 - (0 - 2) = 5$	Increase
A2	$6 - (6 + 5) = -5$	Decrease
C2	$7 - (6+1) = 0$	N.I.N.D
D2	$4 - (6 + 2) = -4$	Decrease
A3	$6-(2+5) = -1$	Decrease
D1	$3-(0+2) = 1$	Increase

		v_1	v_2	v_3	v_4		
		A	B	C	D	Supply	
u_1	1	350 5	× 3	350 1	× 3	700	0
u_2	2	50 6	350 4	× 7	× 4	400	1
u_3	3	× 6	100 0	× 3	500 4	600	–3
	Demand	400	450	350	500	1700	
		5	3	1	7		

Total cost = 350×4+350×8+50×3+350×5+100×9+500×5 = 9500

Unoccupied Cell	$c_{ij} - \left(u_i + v_j\right)$ Opportunity cost	Remarks
B1	3 – (0+3) = 0	N.I.N.D
D1	3 – (0 + 7) = –4	Decrease
C2	7 – (1+1) = 5	Increase
D2	4 – (1 + 7) = –4	Decrease
A3	6–(–3+5) = 4	Increase
C3	3–(–3+1) = 5	Increase

		v_1	v_2	v_3	v_4		
		A	B	C	D	Supply	
u_1	1	350 5	× 3	350 1	× 3	700	0
u_2	2	50 6	× 4	× 7	350 4	400	1
u_3	3	× 6	450 0	× 3	150 4	600	1
	Demand	400	450	350	500	1700	
		5	–1	1	3		

Total cost = 350×4+350×8+50×3+350×5+450×9+150×5 = 10900

Unoccupied Cell	$c_{ij} - \left(u_i + v_j\right)$ Opportunity cost	Remarks
B1	3 – (0–1) = 4	Increase
D1	3 – (0 + 3) = 0	N.I.N.D
B2	4 – (1–1) = 4	Increase
C2	7– (1 + 1) = 5	Increase
A3	6–(1+5) = 0	N.I.N.D
C3	3–(1+1) = 1	Increase

Since all the unoccupied cell are non-negative. So, the solution obtained is optimal.

Lecture # 12

Assignment Problem:

The assignment problem is a special type of transportation problem where the objective is to minimize the cost or maximize the profit of the given problem.

Assumption in Assignment problem:

(i) Number of jobs = Number of Machine or person
(ii) Each person or machine is assigned only one job
(iii) Each person is independently capable of handling any job to be done.
(iv) Assigning criteria is clearly specified (minimizing cost or maximizing profit).

We use Hungarian method to solve assignment problem.

Question: What is feasible solution in Hungarian method?

Answer: If there are 'n' number of jobs given to 'n' different person's then it is called feasible solution in Hungarian method.

Example: Minimization

J = Jobs

P = Persons

$$\begin{array}{c|cccc|} & P_1 & P_2 & P_3 & P_4 \\ \hline J_1 & 5 & 9 & 3 & 6 \\ J_2 & 8 & 7 & 8 & 2 \\ J_3 & 6 & 10 & 12 & 7 \\ J_4 & 3 & 10 & 8 & 6 \\ \hline \end{array}$$

Procedure:

I: Identify the minimum value in each row and subtract it from every value in that row.

$$\begin{array}{|cccc|} \hline 2 & 6 & 0 & 3 \\ 6 & 5 & 6 & 0 \\ 0 & 4 & 6 & 1 \\ 0 & 7 & 5 & 3 \\ \hline \end{array}$$

II: Identify the minimum value in each column and subtract it from every value in that column.

2	2	0	3
6	1	6	0
0	0	6	1
0	3	5	3

III: If any row or any column has only one zero then make an assignment there and move to next column or row.

2	2	$\boxed{0}$	3
6	1	6	$\boxed{0}$
$\cancel{0}$	$\boxed{0}$	6	1
$\boxed{0}$	3	5	3

Where $\boxed{0}$ = job assigned

$\cancel{0}$ = job not assigned

Minimum solution = 3+2+10+3 = 18

Example: Find minimize solution of the following assignment problem by Hungarian method.

11	7	10	17	10
13	21	7	11	13
13	13	15	13	14
18	10	13	16	14
12	8	16	19	10

4	0	3	10	3
6	14	0	4	6
0	0	2	0	1
8	0	3	6	4
4	0	8	11	2

$$\begin{array}{|ccccc|}\hline 4 & 0 & 3 & 10 & 2 \\ 6 & 14 & 0 & 4 & 5 \\ 0 & 0 & 2 & 0 & 0 \\ 8 & 0 & 3 & 6 & 3 \\ 4 & 0 & 8 & 11 & 1 \\ \hline\end{array}$$

$$\begin{array}{|ccccc|}\hline 4 & \boxed{0} & 3 & 10 & 2 \\ 6 & 14 & \boxed{0} & 4 & 5 \\ \boxed{0} & \xcancel{0} & 2 & \xcancel{0} & \xcancel{0} \\ 8 & \xcancel{0} & 3 & 6 & 3 \\ 4 & \xcancel{0} & 8 & 11 & 1 \\ \hline\end{array}$$

Here we cannot proceed further because all the zeros are assigned or crossed.

Also 5 ≠ 3

So, this is not feasible. How do we get the other assignments??

We follow the following procedure.

IV: (i) Tick all the unassigned rows.

$$\begin{array}{|ccccc|c}\cline{1-5} 4 & \boxed{0} & 3 & 10 & 2 & \\ 6 & 14 & \boxed{0} & 4 & 5 & \\ \boxed{0} & \xcancel{0} & 2 & \xcancel{0} & \xcancel{0} & \\ 8 & \xcancel{0} & 3 & 6 & 3 & \checkmark \\ 4 & \xcancel{0} & 8 & 11 & 1 & \checkmark \\ \cline{1-5}\end{array}$$

(ii) If a ticked row has a zero then tick the corresponding column (If the column is not yet ticked).

$$\begin{array}{ccccc|c}
 & \checkmark & & & & \\
\hline
4 & \boxed{0} & 3 & 10 & 2 & \\
6 & 14 & \boxed{0} & 4 & 5 & \\
\boxed{0} & \xcancel{0} & 2 & \xcancel{0} & \xcancel{0} & \\
8 & \xcancel{0} & 3 & 6 & 3 & \checkmark \\
4 & \xcancel{0} & 8 & 11 & 1 & \checkmark
\end{array}$$

(iii) If a ticked column has an assignment then tick the corresponding row (If the row is not yet ticked).

$$\begin{array}{ccccc|c}
 & \checkmark & & & & \\
\hline
4 & \boxed{0} & 3 & 10 & 2 & \checkmark \\
6 & 14 & \boxed{0} & 4 & 5 & \\
\boxed{0} & \xcancel{0} & 2 & \xcancel{0} & \xcancel{0} & \\
8 & \xcancel{0} & 3 & 6 & 3 & \checkmark \\
4 & \xcancel{0} & 8 & 11 & 1 & \checkmark
\end{array}$$

(iv) Repeat steps (ii) and (iii) until no more ticking is possible.

(v) Draw lines through unticked rows and ticked column. Number of line represent the number of possible assignment.

$$\begin{array}{ccccc|c}
 & \checkmark & & & & \\
\hline
4 & \boxed{0} & 3 & 10 & 2 & \checkmark \\
6 & 14 & \boxed{0} & 4 & 5 & \\
\boxed{0} & \xcancel{0} & 2 & \xcancel{0} & \xcancel{0} & \\
8 & \xcancel{0} & 3 & 6 & 3 & \checkmark \\
4 & \xcancel{0} & 8 & 11 & 1 & \checkmark
\end{array}$$

(vi) Find out the smallest value which does not have any line passing through and called it θ. $\Rightarrow \theta = 1$

(vii) Add θ if two line is passing through.

(viii) Subtract θ if no line passing through.

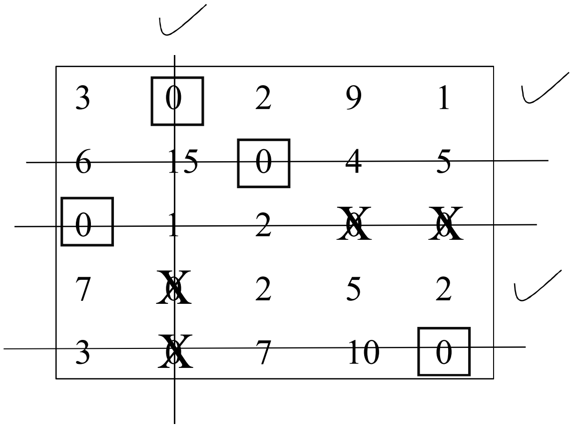
(ix) No change if the value has only one line.

$$\begin{array}{|ccccc|}\hline 3 & 0 & 2 & 9 & 1 \\ 6 & 15 & 0 & 4 & 5 \\ 0 & 1 & 2 & 0 & 0 \\ 7 & 0 & 2 & 5 & 2 \\ 3 & 0 & 7 & 10 & 0 \\ \hline \end{array}$$

(x) Repeat the above steps again.

$$\begin{array}{|ccccc|}\hline 3 & \boxed{0} & 2 & 9 & 1 \\ 6 & 15 & \boxed{0} & 4 & 5 \\ \boxed{0} & 1 & 2 & \xcancel{0} & \xcancel{0} \\ 7 & \xcancel{0} & 2 & 5 & 2 \\ 3 & \xcancel{0} & 7 & 10 & \boxed{0} \\ \hline \end{array} \qquad 5 \neq 4$$

$$\begin{array}{ccccc|c} & \checkmark & & & & \\ \hline 3 & \boxed{0} & 2 & 9 & 1 & \checkmark \\ 6 & 15 & \boxed{0} & 4 & 5 & \\ \boxed{0} & 1 & 2 & \xcancel{0} & \xcancel{0} & \\ 7 & \xcancel{0} & 2 & 5 & 2 & \checkmark \\ 3 & \xcancel{0} & 7 & 10 & \boxed{0} & \\ \hline \end{array}$$



$$\begin{array}{ccccc|c} & \checkmark & & & & \\ \hline 3 & \boxed{0} & 2 & 9 & 1 & \checkmark \\ 6 & 15 & \boxed{0} & 4 & 5 & \\ \boxed{0} & 1 & 2 & \xcancel{0} & \xcancel{0} & \\ 7 & \xcancel{0} & 2 & 5 & 2 & \checkmark \\ 3 & \xcancel{0} & 7 & 10 & \boxed{0} & \\ \hline \end{array}$$

$\theta = 1$

$$\begin{array}{ccccc} 2 & \xcancel{0} & 1 & 8 & \xcancel{0} \\ 6 & 16 & \boxed{0} & 4 & 5 \\ \boxed{0} & 2 & 2 & \xcancel{0} & \xcancel{0} \\ 6 & \boxed{0} & 1 & 4 & 1 \\ 3 & 1 & 7 & 10 & \boxed{0} \end{array}$$

$5 \neq 4$

$$\begin{array}{cccccc} & \checkmark & & & & \\ 2 & \xcancel{0} & 1 & 8 & \xcancel{0} & \checkmark \\ 6 & 16 & \boxed{0} & 4 & 5 & \\ \boxed{0} & 2 & 2 & \xcancel{0} & \xcancel{0} & \\ 6 & \boxed{0} & 1 & 4 & 1 & \checkmark \\ 3 & 1 & 7 & 10 & \boxed{0} & \checkmark \end{array}$$

$\theta = 1$

$$\begin{array}{cccccc} & \checkmark & \checkmark & & \checkmark & \\ 1 & \xcancel{0} & \xcancel{0} & 7 & \xcancel{0} & \\ 6 & 17 & \boxed{0} & 4 & 6 & \checkmark \\ \boxed{0} & 3 & 2 & \xcancel{0} & 1 & \\ 5 & \boxed{0} & \xcancel{0} & 3 & 1 & \checkmark \\ 2 & 1 & 6 & 9 & \boxed{0} & \checkmark \end{array}$$

$5 \neq 4$

$\theta = 1$

$$\begin{array}{ccccc} \boxed{0} & \xcancel{0} & \xcancel{0} & 6 & \xcancel{0} \\ 5 & 17 & \boxed{0} & 3 & 6 \\ \xcancel{0} & 4 & 3 & \boxed{0} & 2 \\ 4 & \boxed{0} & \xcancel{0} & 2 & 1 \\ 1 & 1 & 6 & 8 & \boxed{0} \end{array}$$

$5 = 5$

Jobs = Person

Minimum solution = 11+7+13+10+10 = 51

Lecture # 13

Question: Find maximum solution of the following Assignment problem.

30	37	40	28	40
40	24	27	21	36
40	32	33	30	35
25	38	40	36	36
29	62	41	34	39

Solution: Here the highest value is 62. So, we subtract each value from it.

32	25	22	34	22
22	38	35	41	26
22	30	29	32	27
37	24	22	26	26
33	0	21	28	23

10	3	0	12	0
0	16	13	19	4
0	8	7	10	5
15	2	0	4	4
33	0	21	28	23

✓					
10	3	X	8	$\boxed{0}$	
$\boxed{0}$	16	13	15	4	✓
X	8	7	6	5	✓
15	2	X	$\boxed{0}$	4	
33	$\boxed{0}$	21	24	23	

$$\begin{array}{|ccccc|}\hline 14 & 3 & \boxed{0} & 8 & \xcancel{0} \\ \xcancel{0} & 12 & 9 & 11 & \boxed{0} \\ \boxed{0} & 4 & 3 & 2 & 1 \\ 19 & 2 & \xcancel{0} & \boxed{0} & 4 \\ 37 & \boxed{0} & 21 & 24 & 23 \\ \hline\end{array}$$

Maximum solution = 40+36+40+36+62 = 214

Unbalance Problem in Assignment Problem:

Question: Minimize

$$\begin{array}{|cccc|}\hline 5 & 9 & 3 & 6 \\ 8 & 7 & 8 & 2 \\ 6 & 10 & 12 & 7 \\ \hline\end{array}$$

Solution: 3 Jobs $\neq$ 4 Person

$$\begin{array}{|cccc|}\hline 5 & 9 & 3 & 6 \\ 8 & 7 & 8 & 2 \\ 6 & 10 & 12 & 7 \\ 0 & 0 & 0 & 0 \\ \hline\end{array}$$

$$\begin{array}{|cccc|}\hline 2 & 6 & \boxed{0} & 3 \\ 6 & 5 & 6 & \boxed{0} \\ \boxed{0} & 4 & 6 & 1 \\ \xcancel{0} & \boxed{0} & \xcancel{0} & \xcancel{0} \\ \hline\end{array}$$

Minimum solution = 3+2+6+0 = 11

Also find maximum solution.

Here the maximum value is 12. So, we subtract each value from it.

7	3	9	6
4	5	4	10
6	2	0	5
12	12	12	12

4	$\boxed{0}$	6	3
$\boxed{0}$	1	$\xcancel{0}$	6
6	2	$\boxed{0}$	5
$\xcancel{0}$	$\xcancel{0}$	$\xcancel{0}$	$\boxed{0}$

Maximum solution = 9+8+12+0 =29

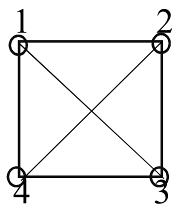
If the question arise find the difference between maximum and minimum solution of Assignment problem then we find both maximum and minimum solution and then subtract minimum solution by maximum solution.

Network Minimization:

Network minimization deals with the determination of the branches that can join all the vertex of a network, such that lenghts of the choosen branches are minimized. This minimum Network is called a minimal spanning tree.

Graph/Network:

A graph is a pair G(V,E) where V is the set of vertices/nodes and E is a set of Edges/Branches. Number of vertex in a graph is called order of the graph and number of edges in a graph is called size of graph.

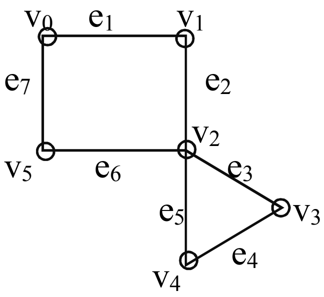


V = {1,2,3,4} E ={12,23,34,24,31,41}

Walk: A walk from u to v in a graph G is a finite alternative sequence of vertices and edges. In a walk vertices and edges can repeat.

Trail: A trail is a walk in which no edge can repeat.

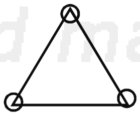
Path: A path is a walk in which no vertex and no edge can repeat.



I.	$v_0e_1v_1e_2v_2e_6v_5$	Path
II.	$v_0e_1v_1e_2v_2e_3v_3e_4v_4e_5v_2$	Trail
III.	$v_3e_4v_4e_5v_2\ e_3v_3e_4v_4$	Walk

Cylce: A closed path is called a cycle.

*smallest cycle is C_3



Connected graph:

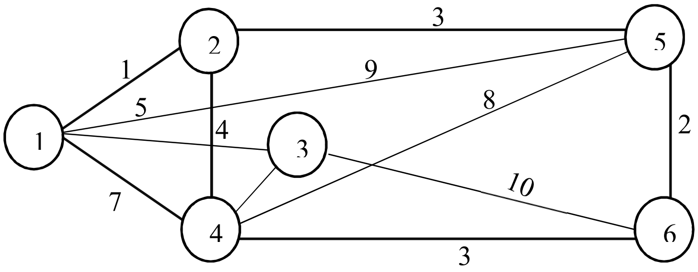
A non-emtpy graph G is connected if for any two vertices we have a path.

Tree: An acyclic (in which no cycle) connected graph is called tree.

Spanning graph: A spanning graph of a graph G that includes all the vertices of graph G.

Minimal spanning tree: A tree T is called minimal spanning tree if T is connected then T–e is disconnected for all $e \in E$.

Question: Find minimal spanning tree of the following network.



Solution: I_1 = start with node 1

$C_1 = \{1\}$, $\overline{C_1} = \{2,3,4,5,6\}$

I_2 = connect node 2 with node 1

$C_2 = \{1,2\}, \overline{C_1} = \{3,4,5,6\}$

I_3 = connect node 5 with node 2

$C_3 = \{1,2,5\}, \overline{C_3} = \{3,4,6\}$

I_4 = connect node 6 with node 5

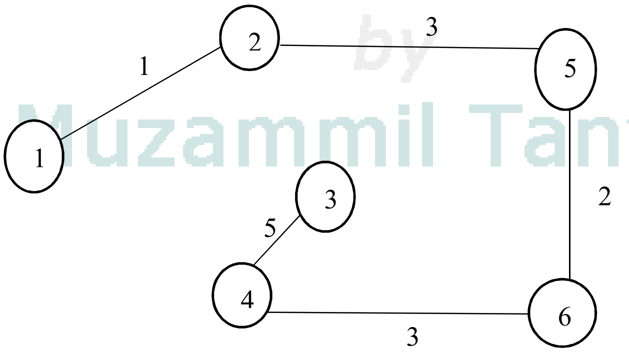
$C_4 = \{1,2,5,6\}, \overline{C_4} = \{3,4\}$

I_5 = connect node 4 with node 6

$C_5 = \{1,2,4,5,6\}, \overline{C_5} = \{3\}$

I_6 = connect node 3 with node 4

$C_6 = \{1,2,3,4,5,6\}, \overline{C_6} = \{\ \}$



Minimum distance = 1+3+2+3+5 = 14