MECHANICS

Available at: http://www.mathcity.org

mtanveer8689@gmail.com
0316-7017457

By Muzammil Tanveer

Lecture # 1

Mechanics:

Mechanics is the branch of science which studies the state of rest and motion of objects and laws governing rest, equilibrium and motion. Since material objects exist in the form of liquids gases and solids there are corresponding types of mechanics to deal with them.

(i) Kinematics
(ii) Dynamics
(iii) Statics

Kinematics:

Kinematics is the branch of mechanics which describe the motion of objects without consideration of their masses and force acting on them.

Dynamics:

Dynamics is the branch of mechanics concerned with the motion of objects under the action of force.

Statics:

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Statics is the branch of mechanics concerned with objects at rest or in equilibrium under the action of forces.

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Ch# Rigid Body Motion:

Particle:

An extremely small part of matter or an infinitesimal part of matter having negligible dimension is called a particle.

Rigid Body:

A collection of particles such that distance between any two particles remains same, irrespective to the motion of the body or force acting upon.

e.g. metallic balls, wood, stones etc.

Note: The concept of rigid body is an idealization.

Displacement:

Directed change in position of particles of a rigid body is called displacement.

Translational Motion:

The motion in which all particles of a rigid body are displaced such that the line joining initial and final position of particles are parallel to each other.

Rotational Motion:

If during the motion particles of the rigid body move in curved path or circular path or are displaced through some angle θ about (an imaginary) line called axis of rotation. Then that motion is called Rotational motion.

Chasles' Theorem:

The most general displacement of a rigid body is composed of pure translation followed by a rotation about some point (base point).

$$d\vec{r} = (\text{displacement})_{translation} + (\text{displacement})_{rotation}$$

$$d\vec{r} = \mathrm{dt}\left[\vec{V_A} + (\vec{\omega} \times \vec{r})\right]$$

$$\frac{d\vec{r}}{dt} = \vec{V_A} + (\vec{\omega} \times \vec{r})$$

where $\vec{\omega}$ is angular velocity of body and $\vec{r}$ is position vector of P with respect to A (base point).

$$\vec{V} = \vec{V_A} + (\vec{\omega} \times \vec{r}) \qquad \text{since } \frac{d\vec{r}}{dt} = \vec{V}$$

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Lecture # 2

Angular Equation motion:

We know that

$$\frac{d\omega}{dt} = \alpha$$

$$\Rightarrow \quad d\omega = \alpha dt$$

By integrating

$\Rightarrow \quad \omega = \alpha t + c \quad$ (i) ; where c is constant of integration and can be found by using initial condition $t = 0$ then $\omega = \omega_0$

$$\omega_0 = \alpha(0) + c \qquad \Rightarrow \qquad c = \omega_0$$

Put in (i) $\quad \omega = \alpha t + \omega_0 \qquad$ (ii)

called first angular equation of motion. Now from (ii)

$$\frac{d\theta}{dt} = \alpha t + \omega_0 \qquad \text{since } \omega = \frac{d\theta}{dt}$$

$$\Rightarrow d\theta = (\alpha t + \omega_0)dt$$

$$\Rightarrow \theta = \alpha\frac{t^2}{2} + \omega_0 t + c_1 \qquad \text{(iii)}$$

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When $t = 0$, $\theta = 0$ put in (iii)

$$0 = \frac{1}{2}\alpha(0)^2 + \omega_0(0) + c_1$$

$\Rightarrow c_1 = 0 \qquad$ Put in (iii)

$$\theta = \alpha\frac{t^2}{2} + \omega_0 t + 0$$

$$\theta = \alpha\frac{t^2}{2} + \omega_0 t \qquad \text{(iv)}$$

Called Second Angular equation of motion.

Now from (ii) $\alpha t = \omega - \omega_0 \qquad \Rightarrow \qquad t = \frac{\omega - \omega_0}{\alpha}$

Put in (iv) $\quad \theta = \alpha\frac{(\frac{\omega - \omega_0}{\alpha})^2}{2} + \omega_0(\frac{\omega - \omega_0}{\alpha})$

$$= (\frac{\omega - \omega_0}{\alpha})\left[\frac{1}{2}\alpha.\frac{\omega - \omega_0}{\alpha} + \omega_0\right] \qquad \Rightarrow (\frac{\omega - \omega_0}{\alpha})\left[\frac{\omega - \omega_0}{2} + \omega_0\right]$$

$$= (\frac{\omega - \omega_0}{\alpha})\left[\frac{\omega + \omega_0}{2}\right] \Rightarrow \qquad \theta = \frac{\omega^2 - {\omega_0}^2}{2\alpha} \quad \text{or } 2\alpha\theta = \omega^2 - {\omega_0}^2$$

Called the third Angular equation of motion.

Note: These equations are only useful when a rigid body is rotating with angular velocity about a point 'o'.

Question:

A rigid body is rotating about at a point say '0'. Find $\vec{V}$

(i) $\vec{\omega} = 2\hat{\imath}+3\hat{\jmath} - \hat{k}$, $\vec{r} = \hat{\imath}+\hat{\jmath} + \hat{k}$

(ii) $\vec{\omega} = \hat{\imath} - \hat{k}$, $\vec{r} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$

Solution:

(i) $\vec{\omega} = 2\hat{\imath}+3\hat{\jmath} - \hat{k}$, $\vec{r} = \hat{\imath}+\hat{\jmath} + \hat{k}$

$\vec{V} = \vec{\omega} \times \vec{r}$

$$\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{vmatrix} \Rightarrow \hat{\imath}(3+1) - \hat{\jmath}(2+1) + \hat{k}(2-3)$$

$\vec{V} = 4\hat{\imath} - 3\hat{\jmath} - \hat{k}$

(ii) $\vec{\omega} = \hat{\imath} - \hat{k}$, $\vec{r} = 2\hat{\imath} - \hat{\jmath} - 2\hat{k}$

$$\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & -1 \\ 2 & -1 & -2 \end{vmatrix} \Rightarrow \hat{\imath}(0-1) - \hat{\jmath}(-2+2) + \hat{k}(-1-0)$$

$\vec{V} = -\hat{\imath} - 0\hat{\jmath} - \hat{k}$

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Screw Motion:

The motion which consist of translation and rotation about a line along the translation is called screw motion.

In this motion linear velocity of each particle on the axis of rotation is parallel (or anti parallel) to the angular velocity.

Theorem:

Find the equation of axis of rotation in vector form in case of screw motion **Or** Show that equation of axis of rotation in case of screw motion is $\vec{r} = \vec{a} + \lambda\vec{\omega}$

where $\vec{r}$ is position vector of any point on the axis of rotation and $\vec{a}$ is any vector, λ is scalar and $\vec{\omega}$ is the angular velocity **Or** Show that instantaneous general motion is screw motion.

Proof:

Consider a rigid body in general motion. Let A be the base point of the body with linear velocity $\vec{V_A}$. Let B be any other particle then Linear velocity of B is

$\overrightarrow{V_B} = \overrightarrow{V_A} + (\vec{\omega} \times \vec{r})$ (i) ; where $\vec{r}$ is position vector of B from base point .

In general, $\vec{V}$ and $\vec{\omega}$ are not parallel (or antiparallel) but we can choose B such that Linear velocity $\vec{V}$ of B is parallel to angular velocity $\vec{\omega}$ of the rigid body.

Taking cross product of (i) with $\vec{\omega}$

$$\vec{\omega} \times \vec{V} = (\vec{\omega} \times \overrightarrow{V_A}) + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Since $\vec{\omega} \parallel \vec{V}$ so $\vec{\omega} \times \vec{V} = 0$

$$0 = (\vec{\omega} \times \overrightarrow{V_A}) + (\vec{\omega} . \vec{r})\vec{\omega} - (\vec{\omega} . \vec{\omega})\vec{r}$$

$$= (\vec{\omega} \times \overrightarrow{V_A}) + (\vec{\omega} . \vec{r})\vec{\omega} - \omega^2\vec{r}$$

Since $\vec{\omega} . \vec{\omega} = |\vec{\omega}||\vec{\omega}|cos0 = \omega^2$

$$\omega^2\vec{r} = (\vec{\omega} \times \overrightarrow{V_A}) + (\vec{\omega} . \vec{r})\vec{\omega}$$

$$\vec{r} = \left(\frac{\vec{\omega} \times \overrightarrow{V_A}}{\omega^2}\right) + \left(\frac{\vec{\omega} . \vec{r}}{\omega^2}\right)\vec{\omega}$$

$$\vec{r} = \vec{a} + \lambda\vec{\omega}$$

where $\vec{a} = \left(\frac{\vec{\omega} \times \overrightarrow{V_A}}{\omega^2}\right)$ and $\lambda = \left(\frac{\vec{\omega} . \vec{r}}{\omega^2}\right)$

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Question:

Show that $\vec{\omega} = \frac{1}{2}\,\mathrm{curl}\vec{V}$ (for rotation).

Solution:

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{V} = V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k}$$

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \Rightarrow \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{\imath} + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k} \ldots \text{(i)}$$

$$V_1\hat{\imath} + V_2\hat{\jmath} + V_3\hat{k} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\Rightarrow (\omega_2 z - \omega_3 y)\hat{\imath} + (\omega_3 x - \omega_1 z)\hat{\jmath} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$V_1 = \omega_2 z - \omega_3 y \quad , V_2 = \omega_3 x - \omega_1 z \; , \qquad V_3 = \omega_1 y - \omega_2 x$$

$$\Rightarrow \frac{\partial V_1}{\partial z} = \omega_2 \, , \frac{\partial V_1}{\partial y} = -\omega_3 \, , \frac{\partial V_2}{\partial z} = -\omega_1 \, , \frac{\partial V_2}{\partial x} = \omega_3 \, , \frac{\partial V_3}{\partial x} = -\omega_2 \, , \frac{\partial V_3}{\partial y} = \omega_1$$

Putting above values in eq. (i)

$\nabla \times \vec{V} = (\omega_1 - (-\omega_1))\hat{\imath} + (\omega_2 - (-\omega_2))\hat{\jmath} + (\omega_3 - (-\omega_3))\hat{k}$

$= (\omega_1 + \omega_1)\hat{\imath} + (\omega_2 + \omega_2)\hat{\jmath} + (\omega_3 + \omega_3)\hat{k}$

$= 2\omega_1\hat{\imath} + 2\omega_2\hat{\jmath} + 2\omega_3\hat{k}$ or $2[\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}]$

$\frac{1}{2}(\nabla \times \vec{V}) = \vec{\omega}$

Or $\vec{\omega} = \frac{1}{2}\text{curl}\vec{V}$ proved.

Question:

A rigid body is rotating about a fix point 'o'. The points A(0,-1,2) and B(2,0,0) are moving with velocities $\vec{V_A} = [7,-2,-1]$ and $\vec{V_B} = [0,6,-4]$ respectively. Find the angular velocity of the body.

Solution:

Position vector of 'A'

$\vec{r_A} = [0,-1,2]$

$\vec{V_A} = \vec{\omega} \times \vec{r_A}$

$[7,-2,-1] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -1 & 2 \end{vmatrix} = (2\omega_2 + \omega_3)\hat{\imath} - (2\omega_1 - 0)\hat{\jmath} + (-\omega_1 - 0)\hat{k}$

$[7,-2,-1] = (2\omega_2 + \omega_3)\hat{\imath} - 2\omega_1\hat{\jmath} - \omega_1\hat{k}$

$7 = 2\omega_2 + \omega_3$, $-2 = -2\omega_1 \Rightarrow \omega_1 = 1$

Position vector of 'B'

$\vec{r_A} = [2,0,0]$

$\vec{V_B} = \vec{\omega} \times \vec{r_B}$

$[0,6,-4] = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 2 & 0 & 0 \end{vmatrix} = (0 - 0)\hat{\imath} + (2\omega_3 - 0)\hat{\jmath} + (0 - 2\omega_2)\hat{k}$

$[0,6,-4] = 0\hat{\imath} + 2\omega_3\hat{\jmath} - 2\omega_2\hat{k}$

$0 = 0$, $6 = 2\omega_3 \Rightarrow \omega_3 = 3$

$-2\omega_2 = -4 \Rightarrow \omega_2 = 2$

Angular Velocity $\vec{\omega} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k} \Rightarrow \vec{\omega} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

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Lecture # 3

Question:

The instantaneous linear velocity of three particles A(a,0,0) , B(0,$\frac{a}{\sqrt{3}}$, 0) and C(0,0,2a) are $\overrightarrow{V_A}$ = (u,0,0), $\overrightarrow{V_B}(u,0,v)$ and $\overrightarrow{V_C}$(u+v,$-\sqrt{3}v$,$\frac{v}{2}$) respectively w.r.t the cartesian coordinate system. Find $|\overrightarrow{\omega}|$ of rigid body.

Solution:

Let A(a,0,0) be a base point (fix point). We know that linear velocity of any particle 'P' of a rigid body in case of general motion

$$\overrightarrow{V_P} = \overrightarrow{V_A} + (\overrightarrow{\omega} \times \overrightarrow{r_P}) \quad \because \overrightarrow{r_P} \text{ is position vector of P w.r.t A}$$

Now $\overrightarrow{r_B} = (-a,\frac{a}{\sqrt{3}},0)$ and $\overrightarrow{r_C} = (-a,0,2a)$

Then $\overrightarrow{V_B} = \overrightarrow{V_A} + (\overrightarrow{\omega} \times \overrightarrow{r_B}) \quad \because \overrightarrow{\omega} = (\omega_1, \omega_2, \omega_3)$

$\Rightarrow \quad \overrightarrow{V_B} - \overrightarrow{V_A} = (\overrightarrow{\omega} \times \overrightarrow{r_B})$

$$\Rightarrow \quad (0,0,V) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & \frac{a}{\sqrt{3}} & 0 \end{vmatrix}$$

$$= \hat{\imath}\,(0-\frac{\omega_3\, a}{\sqrt{3}}) + \hat{\jmath}(-a\omega_3 - 0) + \hat{k}(\frac{\omega_1\, a}{\sqrt{3}} + a\omega_2)$$

$$(0,0,V) = \hat{\imath}\,(-\frac{\omega_3\, a}{\sqrt{3}}) + \hat{\jmath}(-a\omega_3) + \hat{k}(\frac{\omega_1\, a}{\sqrt{3}} + a\omega_2)$$

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On comparig

$\Rightarrow \quad -\frac{\omega_3\, a}{\sqrt{3}} = 0 \;,\quad -a\omega_3 = 0 \;,\quad \frac{\omega_1\, a}{\sqrt{3}} + a\omega_2 = V$

$\Rightarrow \quad \omega_3 = 0$

Now $\overrightarrow{V_C} = \overrightarrow{V_A} + (\overrightarrow{\omega} \times \overrightarrow{r_C})$

$\overrightarrow{V_C} - \overrightarrow{V_A} = (\overrightarrow{\omega} \times \overrightarrow{r_C})$

$$\Rightarrow (V, -\sqrt{3}V, \frac{V}{2}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -a & 0 & 2a \end{vmatrix}$$

$$= \hat{\imath}\,(2a\omega_2 - 0) + \hat{\jmath}(-a\omega_3 - 2a\omega_1) + \hat{k}(0 + a\omega_2)$$

$$(V, -\sqrt{3}V, \frac{V}{2}) = \hat{\imath}\,(2a\omega_2) + \hat{\jmath}(-a\omega_3 - 2a\omega_1) + \hat{k}(a\omega_2)$$

On comparing

$2a\omega_2 = V$, $\quad -a\omega_3 - 2a\omega_1 = -\sqrt{3}V$, $\quad a\omega_2 = \frac{V}{2}$

$\Rightarrow \quad \omega_2 = \frac{V}{2a}$, $\quad -a(0) - 2a\omega_1 = -\sqrt{3}V \quad \because \ \omega_3 = 0$

$$-2a\omega_1 = -\sqrt{3}V$$

$$\omega_1 = \frac{\sqrt{3}V}{2a}$$

$\Rightarrow \ \vec{\omega} = \left(\frac{\sqrt{3}V}{2a}, \frac{V}{2a}, 0\right)$

Now $\quad |\vec{\omega}| = \sqrt{\left(\frac{\sqrt{3}V}{2a}\right)^2 + \left(\frac{V}{2a}\right)^2 + (0)^2} \ \Rightarrow \ \sqrt{\frac{3V^2}{4a^2} + \frac{V^2}{4a^2}}$

$$|\vec{\omega}| = \sqrt{\frac{3V^2+V^2}{4a^2}} = \sqrt{\frac{4V^2}{4a^2}} \quad \Rightarrow \quad |\vec{\omega}| = \frac{V}{a} \ \text{Ans}$$

Question:

The instantaneous linear velocity of three particles A(a, 2a, $-a$) , B($-a, -a, a$) and C(a,a,a) are $\vec{V_A} = \left(\frac{\sqrt{3}V}{2}, \ 0 \ , \frac{\sqrt{3}V}{2}\right)$, $\vec{V_B}\left(-\frac{V}{\sqrt{3}}, 0, -\frac{V}{\sqrt{3}}\right)$ and $\vec{V_C}\left(0, -\frac{V}{\sqrt{3}}, \frac{V}{\sqrt{3}}\right)$ respectively. Find direction of cosine of axis of rotation.

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Solution:

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Let A(a, 2a, $-a$) be a base point. We know that Linear velocity of any particle 'P' of a rigid body in general motion is

$$\vec{V_P} = \vec{V_A} + (\vec{\omega} \times \vec{r_P})$$

Now $\vec{r_B} = (-2a, -3a, 2a)$ and $\vec{r_C} = (0, -a, 2a)$

Then $\quad \vec{V_B} = \vec{V_A} + (\vec{\omega} \times \vec{r_B}) \quad \because \vec{\omega} = (\omega_1, \omega_2, \omega_3)$

$\Rightarrow \quad \vec{V_B} - \vec{V_A} = (\vec{\omega} \times \vec{r_B})$

$$\left(-\frac{5V}{2\sqrt{3}}, 0, -\frac{5V}{2\sqrt{3}}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ -2a & -3a & 2a \end{vmatrix}$$

$$= \hat{\imath}\,(2a\omega_2 + 3a\omega_3) + \hat{\jmath}(-2a\omega_3 - 2a\omega_1) + \hat{k}(-3a\omega_1 - 2a\omega_2)$$

On Comparing

$2a\omega_2 + 3a\omega_3 = -\frac{5V}{\sqrt{3}}$, $\quad -2a\omega_3 - 2a\omega_1 = 0$, $\quad -3a\omega_1 - 2a\omega_2 = -\frac{5V}{2\sqrt{3}}$

$$-\omega_3 = \omega_1$$

Now $\quad \vec{V_C} = \vec{V_A} + (\vec{\omega} \times \vec{r_C})$

$$\overrightarrow{V_C} - \overrightarrow{V_A} = (\overrightarrow{\omega} \times \overrightarrow{r_C})$$

$$\Rightarrow \left(\frac{-\sqrt{3}V}{2}, \frac{-V}{\sqrt{3}}, \frac{-V}{2\sqrt{3}}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ 0 & -a & 2a \end{vmatrix}$$

$$= \hat{\imath}\,(2a\omega_2 + a\omega_3) + \hat{\jmath}(0 - 2a\omega_1) + \hat{k}(-a\omega_1 - 0)$$

$$= \hat{\imath}\,(2a\omega_2 + a\omega_3) + \hat{\jmath}(-2a\omega_1) + \hat{k}(-a\omega_1)$$

On Comparing

$$2a\omega_2 + a\omega_3 = \frac{-\sqrt{3}V}{2}\ ,\ -2a\omega_1 = \frac{-V}{\sqrt{3}}\ ,\ -a\omega_1 = \frac{-V}{2\sqrt{3}}$$

$$\Rightarrow \quad \omega_1 = \frac{V}{2\sqrt{3}a}$$

And $\omega_3 = \frac{-V}{2\sqrt{3}a}$

$$2a\omega_2 + a\left(\frac{-V}{2\sqrt{3}a}\right) = \frac{-\sqrt{3}V}{2}$$

$$\omega_2 = \frac{-V}{2\sqrt{3}a}$$

$$\Rightarrow \overrightarrow{\omega} = \left(\frac{V}{2\sqrt{3}a}, \frac{-V}{2\sqrt{3}a}, \frac{-V}{2\sqrt{3}a}\right)$$

Now $|\overrightarrow{\omega}| = \sqrt{\left(\frac{V}{2\sqrt{3}a}\right)^2 + \left(\frac{-V}{2\sqrt{3}a}\right)^2 + \left(\frac{-V}{2\sqrt{3}a}\right)^2} \Rightarrow \sqrt{\frac{V^2}{12a^2} + \frac{V^2}{12a^2} + \frac{V^2}{12a^2}}$

$|\overrightarrow{\omega}| = \sqrt{\frac{3V^2}{12a^2}} = \sqrt{\frac{V^2}{4a^2}} \Rightarrow |\overrightarrow{\omega}| = \frac{V}{2a}$ is the magnitude of Angular velocity.

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$$\frac{\overrightarrow{\omega}}{|\overrightarrow{\omega}|} = \left(\frac{\frac{V}{2\sqrt{3}a}}{\frac{V}{2a}}, \frac{\frac{-V}{2\sqrt{3}a}}{\frac{V}{2a}}, \frac{\frac{-V}{2\sqrt{3}a}}{\frac{V}{2a}}\right) \Rightarrow \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$$

We know that direction of cosine of the axis of rotation are the component of unit vector along the angular velocity and are $\left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$.

Inertia:

Tendency of a rigid body to remain at rest or of a body in motion to stay in motion unless an external force acted upon.

Or

Measure of resistance in Linear acceleration.

Moment of Inertia (Rotational Inertia)

Measure of resistance of a rigid body in angular acceleration.

Mathematically $\quad I = mr^2$

Where r is perpendicular distance of rigid body from a line about which it is rotating and r = $\sqrt{\frac{I}{m}}$ is radius of Gyration (Latin word means rotation or rapid motion in circular path).

In case of system of particles

$$I = m_1r_1^2 + m_2r_2^2 + \ldots\ldots + m_nr_n^2$$

$$I = \sum_{i=1}^{n} m_ir_i^2$$

Radius of Gyration = $\sqrt{\frac{I}{\sum_{i=1}^{n} m_i}}$

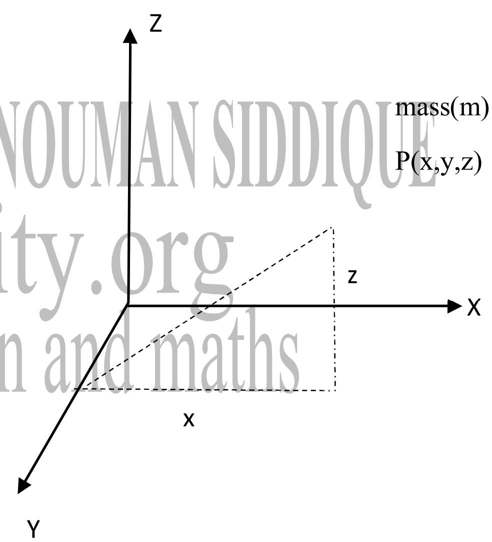
$$= \sqrt{\frac{\sum_{i=1}^{n} m_ir_i^2}{M}} \qquad \text{Where } M = \sum_{i=1}^{n} m_i$$

For Continuous or Uniform mass distribution:

$$I = \int r^2 dm$$

I_{yy} = m ($\perp$ distance from y-axis)

$\quad$ = m $(x^2 + y^2)$



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Momentum:

$\vec{P} = m\vec{V}$

Angular Momentum:

$\vec{L} = \vec{r} \times \vec{P}$

For system of particles

$\vec{L} = \sum_{i=1}^{n}(\vec{r_i} \times \vec{P_i})$

$\quad = \sum_{i=1}^{n}(\vec{r_i} \times (m_i\vec{V_i}))$

$\quad = \sum_{i=1}^{n} m_i(\vec{r_i} \times \vec{V_i}) \qquad\qquad \because \vec{V} = \vec{\omega} \times \vec{r}$

$\quad = \sum_{i=1}^{n} m_i(\vec{r_i} \times (\vec{\omega_i} \times \vec{r_i}))$

$\quad = \sum_{i=1}^{n} m_i[(\vec{r_i} . \vec{r_i})\vec{\omega} - (\vec{r_i} . \vec{\omega})\ \vec{r_i})]$

If $\vec{r_i} = (x_i, y_i, z_i)$ and $\vec{\omega} = (\omega_x, \omega_y, \omega_z)$

$\Rightarrow \quad \sum_{i=1}^{n} m_i[(\vec{r_i})^2(\omega_x, \omega_y, \omega_z) - \{(x_i, y_i, z_i).(\omega_x, \omega_y, \omega_z)\}(x_i, y_i, z_i)]$

$= \sum_{i=1}^{n} m_i[(x_i{}^2 + y_i{}^2 + z_i{}^2)(\omega_x, \omega_y, \omega_z) - (x_i\omega_x + y_i\omega_y + z_i\omega_z)(x_i, y_i, z_i)]$

$$=\sum_{i=1}^{n} m_i[(x_i^2 + y_i^2 + z_i^2)\,\omega_x, (x_i^2 + y_i^2 + z_i^2)\omega_y, (x_i^2 + y_i^2 + z_i^2)\omega_z -$$
$$(x_i^2\omega_x + x_i y_i\omega_y + x_i z_i\omega_z), (x_i\omega_x + y_i^2\omega_y + y_i z_i\omega_z), (x_i z_i\omega_x + z_i y_i\omega_y + z_i^2\omega_z)]$$

$$= \sum_{i=1}^{n} m_i\,[(x_i^2\omega_x + y_i^2\omega_x + z_i^2\omega_x) - (x_i^2\omega_x + x_i y_i\omega_y + x_i z_i\omega_z),$$
$$(x_i^2\omega_y + y_i^2\omega_y + z_i^2\omega_y) - (x_i\omega_x + y_i^2\omega_y + y_i z_i\omega_z),$$
$$(x_i^2\omega_z + y_i^2\omega_z + z_i^2\omega_z) - (x_i z_i\omega_x + z_i y_i\omega_y + z_i^2\omega_z)]$$

$$= \sum_{i=1}^{n} m_i\,[x_i^2\omega_x + y_i^2\omega_x + z_i^2\omega_x - x_i^2\omega_x - x_i y_i\omega_y - x_i z_i\omega_z\,,$$
$$x_i^2\omega_y + y_i^2\omega_y + z_i^2\omega_y - x_i\omega_x - y_i^2\omega_y - y_i z_i\omega_z\,,$$
$$x_i^2\omega_z + y_i^2\omega_z + z_i^2\omega_z - x_i z_i\omega_x - z_i y_i\omega_y - z_i^2\omega_z]$$

$$= \sum_{i=1}^{n} m_i[(y_i^2 + z_i^2)\,\omega_x - \omega_y x_i y_i - \omega_z x_i z_i\,,$$
$$(x_i^2 + z_i^2)\,\omega_x - \omega_y x_i y_i - \omega_z y_i z_i\,,$$
$$(x_i^2 + y_i^2)\,\omega_x - \omega_y x_i z_i - \omega_z y_i z_i]$$

$$= [\{\sum_{i=1}^{n} m_i\,(y_i^2 + z_i^2)\,\omega_x + (-\sum_{i=1}^{n} m_i\,x_i y_i)\omega_y + (-\sum_{i=1}^{n} m_i\,x_i z_i)\omega_z\},\,,$$
$$\{\sum_{i=1}^{n} m_i\,(x_i^2 + z_i^2)\,\omega_y + (-\sum_{i=1}^{n} m_i\,x_i y_i)\omega_x) + (-\sum_{i=1}^{n} m_i\,y_i z_i)\omega_z\}\,,$$
$$\{\sum_{i=1}^{n} m_i\,(x_i^2 + y_i^2)\,\omega_z + (-\sum_{i=1}^{n} m_i\,x_i z_i)\omega_x + (-\sum_{i=1}^{n} m_i\,y_i z_i)\omega_y\}]$$

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = [\,I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z\,, I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z, I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z]$$

$\Rightarrow\ L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

$\Rightarrow\ L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$

$\Rightarrow\ L_z = I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$

Or $\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

Diagonal element (I_{xx}, I_{yy}, I_{zz}) called Moment of Inertia about coordinate axis.

Upper and lower triangular matrix element called product of inertia about coordinate axis.

$$L = I\,\vec{\omega}$$

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Where $I_{xx} = \sum_{i=1}^{n} m_i \, ({y_i}^2 + {z_i}^2)$

$I_{yy} = \sum_{i=1}^{n} m_i \, ({x_i}^2 + {z_i}^2)$

$I_{zz} = \sum_{i=1}^{n} m_i \, ({x_i}^2 + {y_i}^2)$

$I_{xy} = (-\sum_{i=1}^{n} m_i \, x_i y_i)$

$I_{xz} = (-\sum_{i=1}^{n} m_i \, x_i z_i)$

$I_{yz} = (-\sum_{i=1}^{n} m_i \, y_i z_i)$

For continous (uniform mass distribution

$I_{xx} = \int (y^2 + z^2) dm \quad , I_{yy} = \int (x^2 + z^2) dm \, , I_{zz} = \int (x^2 + y^2) dm$

$I_{xy} = -\int xy \, dm \quad , \; I_{xz} = -\int xz \, dm \quad , I_{yz} = -\int yz \, dm$

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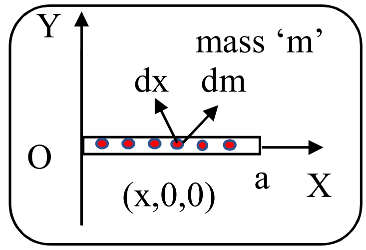
Lecture # 4

Question:

Find moment of inertia of a uniform rod about an axis passing through one of its extremities or passing through its end points and perpendicular to the length rod.

Solution:

Let the mass of the rod is 'm' and length 'a'. We take small particle of mass dm and length dx and coordinate (x,0,0).



Moment of Inertia of rod about Y-axis

$$I_{yy} = \int (x^2 + z^2)\ \text{dm}$$

$$= \int x^2\ \text{dm} \qquad \because z = 0$$

As $\rho = \dfrac{mass}{length} = \dfrac{dm}{dx} \Rightarrow dm = \rho dx$

$$I_{yy} = \int_0^a x^2 \rho\, dx = \rho \int_0^a x^2\, dx$$

$$= \rho \left| \frac{x^3}{3} \right|_0^a = \rho\left(\frac{a^3}{3} - 0\right) \qquad \because \rho = \frac{m}{a}$$

$$= \frac{m}{a}\left(\frac{a^3}{3}\right) = \frac{ma^2}{3}$$

$$I_{yy} = \frac{mass\left(length\, of\ rod\right)^2}{3}$$

If length of the rod is 2a

$$I_{yy} = \frac{mass(length\, of\ rod)^2}{3} = \frac{m(2a)^2}{3}$$

$$= \frac{4ma^2}{3}$$

$$I_{yy}^{e} = \frac{4ma^2}{3} \quad ....(A) \qquad \because I_{yy}^{e},\ e = end\ point$$

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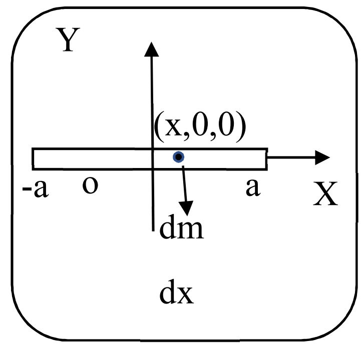
Question: Find M.I of uniform rod about an axis passing through centre of rod and $\perp$ to the length of rod.

Solution:

Let the mass of rod in 'm' and length '2a'. Take a small particle of mass 'dm' having length dx and coordiantte (x,0,0)



M.I about Y-axis

$$I_{yy} = \int (x^2 + z^2)\, dm$$

$$= \int x^2\, dm \qquad \because z = 0$$

As $\rho = \dfrac{mass}{length} = \dfrac{dm}{dx} \Rightarrow dm = \rho dx$

$$I_{yy} = \int_{-a}^{a} x^2 \rho\, dx = \rho \int_{-a}^{a} x^2\, dx$$

$$= \rho \left| \frac{x^3}{3} \right|_{-a}^{a} = \rho \left(\frac{a^3}{3} + \frac{a^3}{3} \right) \qquad \because \rho = \frac{m}{a}$$

$$= \frac{m}{2a} \left(\frac{2a^3}{3} \right) = \frac{ma^2}{3}$$

$$I_{yy}^{c} = \frac{ma^2}{3} \qquad ....(B) \qquad \because I_{yy}^{c}, c = centre$$

From A and B

$$I_{yy}^{c} = 4 I_{yy}^{e}$$

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Question:

Find M.I of rectangular lemina about an axis in the plane of lemina and passing through any of its edges.

Soluiton:

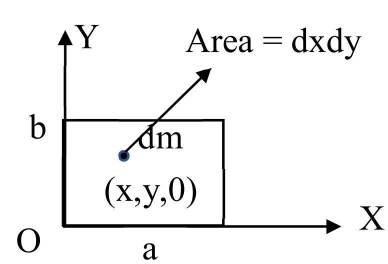
Let 'a' be the length and 'b' be the width of rectangular lemina. Take a point having mass dm and area dxdy and coordinates (x,y,0).

Moment of inertia about width

$$I_{yy} = \int (x^2 + z^2)\, dm$$

$$= \int x^2\, dm \qquad \because z = 0$$

$$= \rho \int_0^b \int_0^a x^2\, dxdy \qquad \because dm = \rho\, dxdy$$



$$=\rho\int_0^b\left|\frac{x^3}{3}\right|_0^a dy$$

$$I_{yy}=\rho\int_0^a\left(\frac{a^3}{3}-0\right)dy$$

$$=\rho\frac{a^3}{3}|y|_0^b=\rho\frac{a^3}{3}(b-0) \qquad \because \rho=\frac{m}{ab}$$

$$=\frac{m}{ab}\left(\frac{a^3b}{3}\right)=\frac{ma^2}{3}$$

$$I_{yy}^c=\frac{ma^2}{3} \quad =\frac{mass(length\,of\,\text{Re}c\tan gular\,le\min a)^2}{3}$$

Moment of inertia about length

$$I_{xx}=\int(y^2+z^2)\,dm$$

$$=\int y^2\,dm \qquad \because z=0$$

$$=\rho\int_0^b\int_0^a y^2\,dxdy \qquad \because dm=\rho\,dxdy$$

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$$=\rho\int_0^b y^2|x|_0^a dy$$

$$=\rho\int_0^b y^2(a-0)dy$$

$$=\rho\int_0^b y^2dy$$

$$=\rho a\left|\frac{y^3}{3}\right|_0^b$$

$$=\rho a\left(\frac{b^3}{3}-0\right)$$

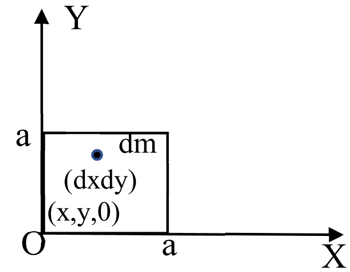
$$=\frac{m}{ab}.\frac{ab^3}{3} \qquad \because \rho=\frac{m}{ab}$$

$$I_{xx}=\frac{mb^2}{3}=\frac{mass(width\,of\,rec\tan gular\,le\min a)}{3}$$

Question: Find M.I of square lemina about an axis in the plane of lemina and passing through any of its edges.

Solution:

Let the length and the width of square lemina is 'a'. Take a small particle of mass dm and area dxdy and coordinates (x,y,0).



Moment of inertia about width

$$I_{yy}=\int (x^2+z^2)\,dm$$

$$=\int x^2\,dm \qquad \because z=0$$

$$=\rho\int_0^a\int_0^a x^2\,dxdy \qquad \because dm=\rho\,dxdy$$

$$=\rho\int_0^a\left|\frac{x^3}{3}\right|_0^a$$

$$I_{yy}=\rho\int_0^a\left(\frac{a^3}{3}-0\right)dy$$

$$=\rho\frac{a^3}{3}|y|_0^a=\rho\frac{a^3}{3}(a-0) \qquad \because \rho=\frac{m}{a.a}=\frac{m}{a^2}$$

$$=\frac{m}{a^2}\left(\frac{a^4}{3}\right)=\frac{ma^2}{3}$$

Similarly,

$$I_{xx}=\frac{ma^2}{3}$$

Question:

Find M.I of a rectagular lemina about an axis in the plane of lemina parallel to its edges and passing through its centre.

Solution:

Let the length of rectangular lemina is 'a' and width 'b' . Take a small particle of mass dm and area dx,dy and coordiante (x,y,0)

M.I about width

$$I_{yy}=\int (x^2+z^2)\,dm$$

$$=\int x^2\,dm \qquad \because z=0$$

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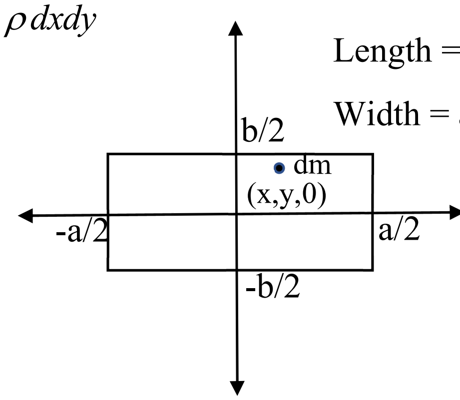
$$=\rho\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} x^2\,dxdy \qquad \because dm=\rho\,dxdy$$



Length = a

Width = ab

$$=\rho\int_{-b/2}^{b/2}\left|\frac{x^3}{3}\right|_{-a/2}^{a/2}$$

$$I_{yy}=\rho\int_{-b/2}^{b/2}\left(\frac{\left(\frac{a}{2}\right)^3}{3}-\frac{\left(\frac{-a}{2}\right)^3}{3}\right)dy$$

$$=\rho\frac{2a^3}{24}\left|y\right|_{-b/2}^{b/2}=\rho\frac{2a^3}{24}\left(\frac{b}{2}+\frac{b}{2}\right) \qquad \because \rho=\frac{m}{ab}$$

$$=\frac{m}{ab}\left(\frac{2a^3}{24}\right).b=\frac{ma^2}{12}$$

$$I_{yy}=\frac{ma^2}{12}=\frac{mass(length\,of\,rectangular\,lemina)}{12}$$

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M.I about length $I_{xx}=\int(y^2+z^2)\,dm$

$$=\int y^2\,dm \qquad \because z=0$$

$$=\rho\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} y^2\,dxdy \qquad \because dm=\rho\,dxdy$$

$$=\rho\left|x\right|_{-a/2}^{a/2}\int_{-b/2}^{b/2} y^2dy$$

$$=\rho\left(\frac{a}{2}+\frac{a}{2}\right)\left|\frac{y^3}{3}\right|_{-b/2}^{b/2}$$

$$=\rho a\left(\frac{b^3}{24}+\frac{b^3}{24}\right)$$

$$=\frac{am}{ab}\left(\frac{2b^3}{24}\right)=\frac{mb^2}{12} \qquad \because \rho=\frac{m}{ab}$$

$$I_{xx}=\frac{mb^2}{12}=\frac{mass(length\,of\,rectangular\,lemina)}{12}$$

For MCQ

$$I_{xx}^{e}=\frac{mb^2}{3} \qquad \because e=edge, end\ point$$

$$I_{xx}^{c}=\frac{mb^2}{12} \qquad \because c=centre$$

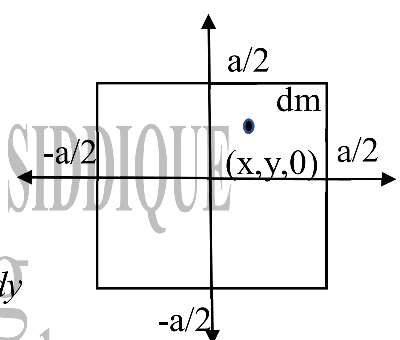
$$I_{xx}^{e}=\frac{mb^2}{3}=4\left(\frac{2b^3}{12}\right)=4I_{xx}^{c}$$

$$I_{xx}^{e}=4I_{xx}^{c}$$

Question:

Find M.I of square lemina about an axis in the plane of lemina parallel to its edges and passing through its center.

Solution: Let the length and width of square lemina is ‘a’.Take a small particle of mass dm area dxdy and coordinates (x,y,0).



M.I about width

$$I_{yy}=\int (x^2+z^2)\,dm$$

$$=\int x^2\,dm \qquad \because z=0$$

$$=\rho\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} x^2\,dxdy \qquad \because dm=\rho\,dxdy$$

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$$=\rho\int_{-a/2}^{a/2}\left|\frac{x^3}{3}\right|_{-a/2}^{a/2}$$

$$I_{yy}=\rho\int_{-a/2}^{a/2}\left(\frac{\left(\frac{a}{2}\right)^3}{3}-\frac{\left(\frac{-a}{2}\right)^3}{3}\right)dy$$

$$=\rho\frac{2a^3}{24}\left|y\right|_{-b/2}^{b/2}=\rho\frac{a^3}{12}\left(\frac{a}{2}+\frac{a}{2}\right) \qquad \because \rho=\frac{m}{a^2}$$

$$=\frac{ma^3}{a^2}\left(\frac{a}{12}\right)=\frac{ma^2}{12}$$

$$I_{yy}=\frac{ma^2}{12} \quad Similarly,\ \ I_{xx}=\frac{ma^2}{12}$$

Lecture # 5

Wallis Formula's:

(i) If 'm' is even then

$$\int_0^{\frac{\pi}{2}} cos^m x dx = \int_0^{\frac{\pi}{2}} sin^m x dx = \frac{(m-1)(m-3)\ldots\ldots\ldots 1}{m(m-2)(m-4)\ldots\ldots 2}\cdot\frac{\pi}{2}$$

(ii) If 'n' is odd then

$$\int_0^{\frac{\pi}{2}} cos^n x dx = \int_0^{\frac{\pi}{2}} sin^n x dx = \frac{(n-1)(n-3)\ldots\ldots\ldots 1}{n(n-2)(n-4)\ldots\ldots 2}$$

(iii) If both 'm' and 'n' are even then

$$\int_0^{\frac{\pi}{2}} cos^m x sin^n x dx = \frac{(m-1)(m-3)\ldots\ldots\ldots 1.(n-1)(n-3)\ldots\ldots 1}{(m+n)(m+n-2)(m+n-4)\ldots\ldots 2}\cdot\frac{\pi}{2}$$

(iv) If both 'm' is even and 'n' is odd then

$$\int_0^{\frac{\pi}{2}} cos^m x sin^n x dx = \int_0^{\frac{\pi}{2}} cos^n x sin^m x dx$$

$$= \frac{(m-1)(m-3)\ldots\ldots\ldots 1.(n-1)(n-3)\ldots\ldots 2}{(m+n)(m+n-2)(m+n-4)\ldots\ldots 2}$$

(v) If both 'm' and 'n' are odd then

$$\int_0^{\frac{\pi}{2}} cos^m x sin^n x dx = \frac{(m-1)(m-3)\ldots\ldots\ldots 2.(n-1)(n-3)\ldots\ldots 2}{(m+n)(m+n-2)(m+n-4)\ldots\ldots 2}$$

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Theorem: (Only statement used)

If f(x) is even function then

$$\int_{-\lambda}^{\lambda} f(x)dx = 2\int_0^{\lambda} f(x)dx$$

If f(x) is odd function then

$$\int_{-\lambda}^{\lambda} f(x)dx = 0$$

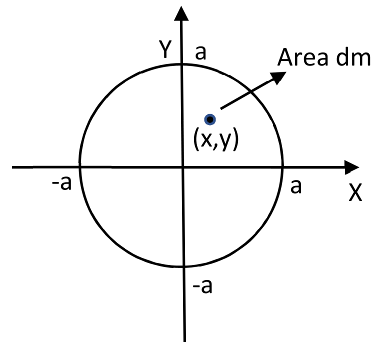
Question:

Find M.I of a circular lemina about an axis in the place of lemina and passing through its centre.

Solution:

Let 'm' be the mass of circular lemina and radius 'a'. Take a small area element having dm and coordinates (x,y).

$$I_{xx} = \int (y^2 + z^2)\, dm$$
$$= \int y^2\, dm$$



$$I_{xx} = \rho \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} y^2 dy dx$$

$\because$ dm = ρdxdy

x changes from $-a \rightarrow a$

$x^2 + y^2 = a^2$

$y^2 = a^2 - x^2$

$y = \pm\sqrt{a^2 - x^2}$

y changes from

$-\sqrt{a^2 - x^2} \rightarrow \sqrt{a^2 - x^2}$

$$= \rho \int_{-a}^{a} \left| \frac{y^3}{3} \right|_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} dx$$

$$= \frac{\rho}{3} \int_{-a}^{a} \left(\left(\sqrt{a^2 - x^2}\right)^3 - \left(-\sqrt{a^2 - x^2}\right)^3 \right) dx$$

$$= \frac{\rho}{3} \int_{-a}^{a} \left(\left(a^2 - x^2\right)^{\frac{3}{2}} + \left(a^2 - x^2\right)^{\frac{3}{2}} \right) dx$$

$$= \frac{\rho}{3} \int_{-a}^{a} \left(2\left(a^2 - x^2\right)^{\frac{3}{2}} \right) dx$$

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$$= \frac{2\rho}{3} \int_{-a}^{a} \left(\left(a^2 - x^2\right)^{\frac{3}{2}} \right) dx$$

Put x = asinθ , dx = acosθdθ

x = −a $\Rightarrow$ θ = $sin^{-1}(-1) = -\frac{\pi}{2}$

x = a $\Rightarrow$ θ = $sin^{-1}(1) = \frac{\pi}{2}$

$$I_{xx} = \frac{2\rho}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(a^2 - a^2 \sin^2\theta\right)^{\frac{3}{2}} \right) .a\cos\theta d\theta$$

$$= \frac{2\rho}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(a^2 \cos^2\theta\right)^{\frac{3}{2}} \right) .a\cos\theta d\theta$$

$$= \frac{2\rho}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a^4 \cos^4\theta d\theta$$

$$= \frac{2\rho a^4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta d\theta$$

$$= \frac{2\rho a^4}{3}.2\int_{0}^{\frac{\pi}{2}} \cos^4\theta d\theta$$

By theorem f(x) is even

$\int_{-\lambda}^{\lambda} f(x)dx = 2\int_0^{\lambda} f(x)dx$

$\because \rho = \frac{Mass}{Area} = \frac{m}{\pi a^2}$

By Wallis cosine formula

$$= \frac{4\rho a^4}{3}.\frac{3.1}{4.2}.\frac{\pi}{2}$$

$$= \frac{\rho a^4 \pi}{4}$$

$$= \frac{m}{\pi a^2}.\frac{a^4\pi}{4}$$

$$= \frac{ma^2}{4}$$

Similarly $I_{yy} = \frac{ma^2}{4}$

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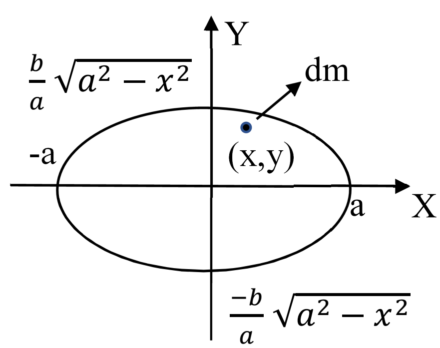
Question:

Find M.T of a elliptic lemina about an axis in the plane of lemina through its major and minor axis.

Solution:



$I_{xx} = \int (y^2 + z^2)\ \text{dm}$

$= \int y^2\ \text{dm} \qquad \because z = 0$

$$I_{xx} = \rho\int_{-a}^{a}\int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}} y^2 dydx$$

$$= \rho\int_{-a}^{a}\left|\frac{y^3}{3}\right|_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}}dx$$

x changes from $-a \to a$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y=\pm\frac{b}{a}\sqrt{a^2-x^2}$

y changes from

$-\frac{b}{a}\sqrt{a^2-x^2} \to \frac{b}{a}\sqrt{a^2-x^2}$

$$=\frac{\rho}{3}\int_{-a}^{a}\left(\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3-\left(-\frac{b}{a}\sqrt{a^2-x^2}\right)^3\right)dx$$

$$=\frac{\rho}{3}\int_{-a}^{a}\left(\frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}}+\frac{b^3}{a^3}\left(a^2-x^2\right)^{\frac{3}{2}}\right)dx$$

$$=\frac{2\rho}{3}.\frac{b^3}{a^3}\int_{-a}^{a}\left(\left(a^2-x^2\right)^{\frac{3}{2}}\right)dx$$

Put $x = a\sin\theta$, $dx = a\cos\theta d\theta$

$x=-a \quad \Rightarrow \quad \theta=\sin^{-1}(-1)=-\frac{\pi}{2}$

$x=a \quad \Rightarrow \quad \theta=\sin^{-1}(1)=\frac{\pi}{2}$

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$$I_{xx} = \frac{2\rho}{3}.\frac{b^3}{a^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\left(a^2-a^2\sin^2\theta\right)^{\frac{3}{2}}\right).a\cos\theta d\theta$$

$$=\frac{2\rho}{3}.\frac{b^3}{a^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\left(a^2\cos^2\theta\right)^{\frac{3}{2}}\right).a\cos\theta d\theta$$

$$=\frac{2\rho}{3}\frac{b^3}{a^3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a^4\cos^4\theta d\theta$$

$$=\frac{2\rho b^3 a}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta d\theta$$

$$I_{xx} = \frac{2\rho b^3 a}{3}.2\int_{0}^{\frac{\pi}{2}}\cos^4\theta d\theta$$

By theorem f(x) is even

$\int_{-\lambda}^{\lambda}f(x)dx = 2\int_{0}^{\lambda}f(x)dx$

$\because \rho=\frac{Mass}{Area}=\frac{m}{\pi ab}$

By Wallis cosine formula

$$= \frac{4\rho b^3 a}{3} \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2}$$

$$= \frac{\rho a b^3 \pi}{4}$$

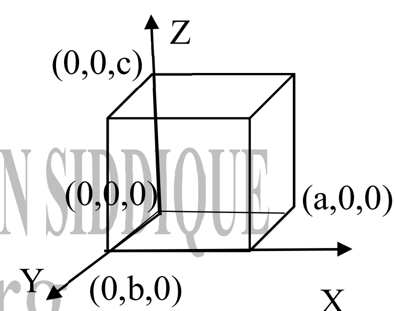
$$= \frac{m}{\pi a b} \cdot \frac{a b^3 \pi}{4}$$

$$= \frac{m b^2}{4}$$

Similarly $I_{yy} = \dfrac{m a^2}{4}$

Question:

Find M.I of rectangular parallelepiped with respect to its edges with one corner at the origin.



Solution:

$I_{xx} = \int (y^2 + z^2)\, \text{dm}$

$\rho = \dfrac{dm}{dxdydz}$

dm = ρdxdydz

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$$I_{xx} = \rho \int_0^c \int_0^b \int_0^a (y^2 + z^2)\, dxdydz$$

$$= \rho \int_0^c \int_0^b x(y^2 + z^2) \Big|_0^a dydz$$

$$= \rho \int_0^c \int_0^b (a-0)(y^2 + z^2) \Big|_0^a dydz$$

$$= \rho a \int_0^c \int_0^b (y^2 + z^2)\, dydz$$

$$= \rho a \int_0^c \left(\frac{y^3}{3} + y z^2 \right)_0^b dz$$

$$= \rho a\int_0^c\left(\left(\frac{b^3}{3}+bz^2\right)-0\right)dz$$

$$= \rho a\int_0^c(\frac{b^3}{3}+bz^2)dz$$

$$= \rho a\left|\frac{b^3}{3}z+b\frac{z^3}{3}\right|_0^c$$

$$= \rho a\left(\frac{cb^3}{3}+\frac{bc^3}{3}\right)$$

$$= \frac{m}{abc}.\frac{abc}{3}\left(b^2+c^2\right) \qquad \because \rho = \frac{Mass}{Area} = \frac{m}{abc}$$

$$= \frac{m}{3}\left(b^2+c^2\right)$$

Similarly,

$$I_{yy} = \frac{m}{3}\left(a^2+c^2\right)$$

$$I_{zz} = \frac{m}{3}\left(a^2+b^2\right)$$

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Find I_{xy}, I_{xz}, I_{yz}

$$I_{xy} = -\int xydm$$

$$= -\rho\int_0^c\int_0^b\int_0^a xy\,dxdydz$$

$$= -\rho\int_0^c\int_0^b \left.\frac{x2}{2}y\right|_0^a dydz$$

$$= -\rho\frac{a^2}{2}\int_0^c\int_0^b y\,dydz$$

$$= -\rho \frac{a^2}{2} \int_0^c \left. \frac{y^2}{2} \right|_0^b dz$$

$$= -\rho \frac{a^2}{2} \cdot \frac{b^2}{2} \int_0^c dz$$

$$= -\rho \frac{a^2}{2} \cdot \frac{b^2}{2} \left| z \right|_0^c$$

$$= \frac{m}{abc} \cdot \frac{a^2 b^2 c}{4} \qquad \because \rho = \frac{Mass}{Area} = \frac{m}{abc}$$

$$= -\frac{mab}{4}$$

Similarly,

$$I_{xz} = -\frac{mac}{4}$$

$$I_{yz} = -\frac{mbc}{4}$$

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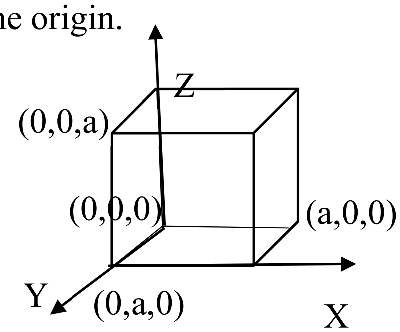
Find matrix

$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{m}{3}\left(b^2 + c^2\right) & -\frac{mab}{4} & -\frac{mac}{4} \\ -\frac{mab}{4} & \frac{m}{3}\left(a^2 + c^2\right) & -\frac{mbc}{4} \\ -\frac{mac}{4} & -\frac{mbc}{4} & \frac{m}{3}\left(a^2 + b^2\right) \end{bmatrix}$$

Question:

Find M.I of a cube w.r.t its edges with one corner at the origin.



Solution:

$I_{xx} = \int (y^2 + z^2)\, \text{dm}$

$\rho = \frac{dm}{dxdydz}$

$\text{dm} = \rho \text{dxdydz}$

$$= \rho\int_0^a\int_0^a\int_0^a\left(y^2+z^2\right)dxdydz$$

$$= \rho\int_0^a\int_0^a x(y^2+z^2)\Big|_0^a dydz$$

$$= \rho\int_0^a\int_0^a(a-0)(y^2+z^2)dydz$$

$$= a\rho\int_0^a\left|\frac{y^3}{3}+yz^2\right|_0^a dz$$

$$= a\rho\int_0^a\left(\left(\frac{a^3}{3}+az^2\right)-(0)\right)dz$$

$$= a\rho\left|\frac{a^3z}{3}+\frac{az^3}{3}\right|_0^a$$

$$= a\rho\left(\frac{a^3.a}{3}+\frac{a.a^3}{3}\right)$$

$$= a\rho\left(\frac{a^4}{3}+\frac{a^4}{3}\right)$$

$$= a\rho\left(\frac{2a^4}{3}\right)$$

$$= \frac{ma}{a^3}\left(\frac{2a^4}{3}\right)$$

$$I_{xx} = \frac{2ma^2}{3}$$

Similarly,

$$I_{yy} = \frac{2ma^2}{3}\ , I_{zz} = \frac{2ma^2}{3}$$

$$I_{xy} = -\int xy\,dm = -\rho\int_0^a\int_0^a\int_0^a dxdydz$$

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$$= -\rho\int_0^a\int_0^a \frac{x^2}{2}\left.y\right|_0^a dydz$$

$$= -\rho\frac{a^2}{2}\int_0^a\int_0^a ydydz \quad = -\rho\frac{a^2}{2}\int_0^a \left.\frac{y^2}{2}\right|_0^a dz$$

$$= -\rho\frac{a^4}{4}\int_0^a dz$$

$$= -\rho\frac{a^4}{4}\left|z\right|_0^a$$

$$= -\frac{m}{a^3}\frac{a^4}{4}(a)$$

$$I_{xy} = -\frac{ma^2}{4}$$

Similrarly,

$$I_{xz} = -\frac{ma^2}{4}, \quad I_{yz} = -\frac{ma^2}{4}$$

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Matrix form

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$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{bmatrix} = \begin{bmatrix} \frac{2ma^2}{3} & -\frac{ma^2}{4} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & \frac{2ma^2}{3} & -\frac{ma^2}{4} \\ -\frac{ma^2}{4} & -\frac{ma^2}{4} & \frac{2ma^2}{3} \end{bmatrix}$$

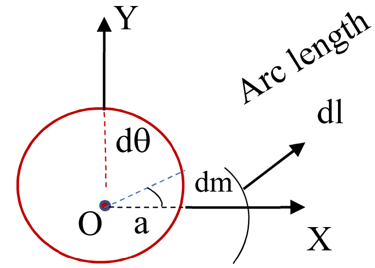
Lecture # 6

Question:

Find M.I of a circular ring about a line passing through its Centre and perpendicular to its circumference.

Solution:

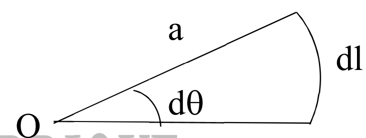
Consider the circular ring of radius 'a' in xy-plane. Consider a mass particle dm with arc length dl with central angle dθ



$$\rho = \frac{mass}{circumference\ of\ ring}$$

$$\rho = \frac{dm}{dl} = \frac{dm}{a d\theta}$$

dm = a ρdθ



M.I about z-axis = $I_{zz} = \int a^2 dm = \int_0^{2\pi} a^2 a\rho d\theta$

$= a^3\rho \int_0^{2\pi} d\theta$

$= a^3\rho \left|\theta\right|_0^{2\pi}$

$= a^3\rho(2\pi - 0)$

$= a^3\rho 2\pi$

$= \quad a^3 \frac{m}{2\pi a} . 2\pi$

Since l = rθ

$\Rightarrow$ l = aθ

$\Rightarrow$ dl = adθ

$\because \quad \rho = \frac{m}{2\pi a}$

$I_{zz} = ma^2$

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Perpendicular axis theorem:

Statement:

The M.I of Lemina (or a plane rigid body) about a normal axis (about an axis ⊥ to its plane) is equal to the sum of M.Is about two mutually ⊥ axis in the plane of lemina and passing through the intersection of lemina with the normal axis.

Proof:

Consider place Lemina in XY-plane. X-axis and Y-axis are ⊥ axis.

Z-axis is ⊥ to X-axis and Y-axis. Let P_i be the ith particle of mass m_i with

Position vector $\vec{r_i}$

M.I about z-axis

$$I_{zz}=\sum_{i=1}^{n}(x_i^2+y_i^2)dm$$

$$I_{zz}=\sum_{i=1}^{n}x_i^2dm+\sum_{i=1}^{n}y_i^2dm$$

$$I_{zz}=I_{yy}+I_{xx}$$

$$\Rightarrow\ I_{zz}=I_{xx}+I_{yy}\ \text{Proved}$$

$$Or\ \ I_{zz}=\int(x^2+y^2)\,dm=\int x^2dm+\int y^2\,dm=I_{yy}+I_{xx}$$

K.E in general motion:

Consider a rigid body in general motion with base point at the center of mass of the rigid body. Then linear velocity of the ith particle of the rigid body is

$$\vec{V_i}\ =\vec{V}+(\vec{\omega}\times\vec{r_i})\quad \text{And its K.E is } T_i=\tfrac{1}{2}\textstyle\sum_{i=1}^{n}m_i\overrightarrow{v_i^2}$$

$$=\frac{1}{2}\sum_{i=1}^{n}m_i\left[\left(\vec{V}+(\vec{\omega}\times\vec{r_i})\right).\left(\vec{V}+\left(\vec{\omega}\times\vec{r_i}\right)\right)\right]$$

$$=\frac{1}{2}\sum_{i=1}^{n}m_i\left[\left(\vec{V}^2+\vec{V}.(\vec{\omega}\times\vec{r_i})\right)+\left(\vec{V}.\left(\vec{\omega}\times\vec{r_i}\right)+\left(\vec{\omega}\times\vec{r_i}\right)^2\right)\right]$$

$$=\frac{1}{2}\sum_{i=1}^{n}m_i\left[\left(\vec{V}^2+2\vec{V}.(\vec{\omega}\times\vec{r_i})\right)+\left(\vec{\omega}\times\vec{r_i}\right)^2\right]$$

$$T\ =\frac{1}{2}\sum_{i=1}^{n}m_i\vec{V}^2+\frac{1}{2}\sum_{i=1}^{n}m_i\left(\vec{\omega}\times\vec{r_i}\right)^2+\vec{V}.\left(\vec{\omega}\times\sum_{i=1}^{n}m_i\,\vec{r_i}\right)\qquad(1)$$

If center of mass (base point) is taken at origin then $\sum_{i=1}^{n}m_i\,\vec{r_i}=0$

Then from (1)

$$T=\frac{1}{2}\sum_{i=1}^{n}m_i\vec{V}^2+\frac{1}{2}\sum_{i=1}^{n}m_i\left(\vec{\omega}\times\vec{r_i}\right)^2$$

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$$= \frac{1}{2}M\vec{V}^2+\frac{1}{2}\sum_{i=1}^{n}m_i\left(\vec{\omega}\times\vec{r_i}\right)^2 \qquad where \quad M=\sum_{i=1}^{n}m_i$$

T = $T_{translational\ K.E} + T_{rotaional\ K.E}$

$$Now \quad Let \quad \vec{\omega}\times\vec{r_i}=\vec{a}$$

$$\left(\vec{\omega}\times\vec{r_i}\right)^2=\left(\vec{\omega}\times\vec{r_i}\right).\left(\vec{\omega}\times\vec{r_i}\right)=\left(\vec{\omega}\times\vec{r_i}\right).\vec{a}$$

$$= \vec{\omega}.(\vec{r_i}\times\vec{a})$$

$$= \vec{\omega}.[\vec{r_i}\times(\vec{\omega}\times\vec{r_i})]$$

Now $\quad T_{rot} = \frac{1}{2}\sum_{i=1}^{n}m_i\left(\vec{\omega}\times\vec{r_i}\right)^2$

$$= \frac{1}{2}\sum_{i=1}^{n}m_i\vec{\omega}.[\vec{r_i}\times(\vec{\omega}\times\vec{r_i})] = \frac{1}{2}\vec{\omega}\sum_{i=1}^{n}m_i[\vec{r_i}\times(\vec{\omega}\times\vec{r_i})]$$

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$T_{rot} = \frac{1}{2}\vec{\omega}.\vec{L} \qquad where \quad \vec{L}=\sum_{i=1}^{n}m_i[\vec{r_i}\times(\vec{\omega}\times\vec{r_i})]$

Hence $T_{tran} = \frac{1}{2}M\vec{V}^2$ And $T_{rot} = \frac{1}{2}\vec{\omega}.\vec{L}$

Now $T_{rot} = \frac{1}{2}[\omega_x\ \omega_y\ \omega_z].[L_x\ L_y\ L_z] \qquad = \frac{1}{2}[\omega_xL_x+\omega_yL_y+\omega_zL_z]$

$$=\frac{1}{2}[\omega_x(\omega_xI_{xx}+\omega_yI_{xy}+\omega_zI_{xz})+\omega_y(\omega_xI_{yx}+\omega_yI_{yy}+\omega_zI_{yz})+\omega_z(\omega_xI_{zx}+\omega_yI_{zy}+\omega_zI_{zz}]$$

$$=\frac{1}{2}[(\omega_x^2)I_{xx}+\omega_x\omega_yI_{xy}+\omega_x\omega_zI_{xz}+\omega_y\omega_xI_{yx}+(\omega_y^2)I_{yy}+\omega_y\omega_zI_{yz}+\omega_z\omega_xI_{zx}+\omega_z\omega_yI_{zy}+(\omega_z^2)I_{zz}]$$

$$\because I_{xy}=I_{yx}\ ,\ I_{xz}=I_{zx}\ ,\ I_{yz}=I_{zy}$$

$$=\frac{1}{2}[(\omega_x)^2I_{xx}+(\omega_y)^2I_{yy}+(\omega_z^2)I_{zz}+2\omega_x\omega_yI_{xy}+2\omega_x\omega_zI_{xz}+2\omega_y\omega_zI_{yz}]$$

Lecture # 7

Question:

Find M.I of a rigid body about line having direction cosine λ, μ and v

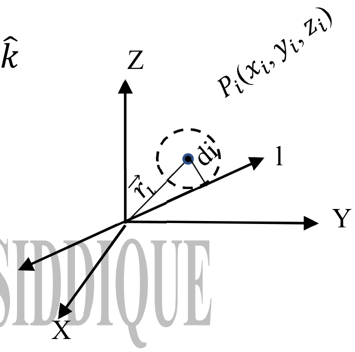
Solution:

Consider a line ‘l’ through origin having direction cosine λ, μ and v. Let $\hat{e}$ be a unit vector along line ‘l’.

Then we can write

$$\hat{e} = \lambda\hat{\imath} + \mu\hat{\jmath} + v\hat{k}$$

Let $P_i(x_i, y_i, z_i)$ be the particle of the rigid body and di be the distance of P_i From the line ‘l’ then from fig.



$$di = |\vec{r_i}|\sin\theta \quad \text{where } \vec{r_i} = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}$$

$$= |\hat{e} \times \vec{r_i}|$$

Now M.I of the rigid body about line ‘l’

$$I = \sum_{i=1}^{n} m_i\, di^2$$

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$$= \sum_{i=1}^{n} m_i \left|\hat{e} \times \vec{r_i}\right|^2 \quad ......(1)$$

Now

$$\hat{e} \times \vec{r_i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lambda & \mu & v \\ x_i & y_i & z_i \end{vmatrix}$$

$$\hat{e} \times \vec{r_i} = (\mu z_i - v y_i)\hat{i} + (v x_i - \lambda z_i)\hat{j} + (\lambda y_i - \mu x_i)\hat{k}$$

Put in (1)

$$I = \sum_{i=1}^{n} m_i\left[(\mu z_i - v y_i)^2 + (v x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2\right]$$

$$I = \sum_{i=1}^{n} m_i\left[\mu^2 z_i^2 + v^2 y_i^2 - 2\mu v z_i y_i + v^2 x_i^2 + \lambda^2 z_i^2 - 2v\lambda x_i z_i + \lambda^2 y_i^2 + \mu^2 x_i^2 - 2\lambda\mu x_i y_i\right]$$

$$I = \sum_{i=1}^{n} m_i \left[(y_i^2 + z_i^2)\lambda^2 + (x_i^2 + z_i^2)\mu^2 + (x_i^2 + y_i^2)v^2 - 2\lambda\mu x_i y_i - 2\mu v y_i z_i - 2v\lambda x_i z_i \right]$$

$$I = \lambda^2 \sum_{i=1}^{n} m_i \left(y_i^2 + z_i^2 \right) + \mu^2 \sum_{i=1}^{n} \left(x_i^2 + z_i^2 \right) + v^2 \sum_{i=1}^{n} \left(x_i^2 + y_i^2 \right) - 2\lambda\mu \left(-\sum_{i=1}^{n} m_i x_i y_i \right) + 2\mu v \left(-\sum_{i=1}^{n} m_i y_i z_i \right) + 2v\lambda \left(-\sum_{i=1}^{n} m_i x_i z_i \right)$$

$$I = \lambda^2 I_{xx} + \mu^2 I_{yy} + v^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu v I_{yz} + 2v\lambda I_{xz}$$

Which is M.I of rigid body about a line having direction cosine λ, μ and v

Momental Ellipsoid:

We know that M.I of a rigid body about a line having direction cosine λ, μ and v is given as

$$I = \lambda^2 I_{xx} + \mu^2 I_{yy} + v^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu v I_{yz} + 2v\lambda I_{xz}$$

Let $\frac{\hat{e}}{\sqrt{I}}$ be vector along 'l' and Q(x,y,z) be the point on line 'l' such that

$\overrightarrow{OQ} = \frac{\hat{e}}{\sqrt{I}}$ and $\left|\overrightarrow{OQ}\right| = \frac{1}{\sqrt{I}}$, I ≠ 0

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Also $\overrightarrow{OQ} = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}$

Now direction cosine of $\overrightarrow{OQ}$ are

$$\frac{x}{\left|\overrightarrow{OQ}\right|}, \frac{y}{\left|\overrightarrow{OQ}\right|}, \frac{z}{\left|\overrightarrow{OQ}\right|}$$

i.e. $x\sqrt{I}$, $y\sqrt{I}$, $z\sqrt{I}$

Since line 'l' and $\overrightarrow{OQ}$ have same direction cosines therefore

λ = $x\sqrt{I}$, μ = $y\sqrt{I}$ and v = $z\sqrt{I}$

Put in eq (1)

$$I = x^2 I\, I_{xx} + y^2 I\, I_{yy} + z^2 I\, I_{zz} + 2xyI\, I_{xy} + 2yzI\, I_{yz} + 2zxI\, I_{xz}$$

Divide both side by I

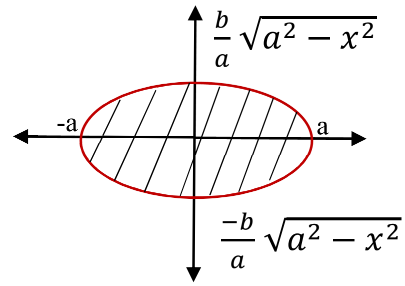
$$x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2xyI_{xy} + 2yzI_{yz} + 2zx I_{xz} = 1$$

Which represents equation of ellipsoid thus ellipsoid is called Momental ellipsoid or ellipsoid of inertia (Ass I_{xx} , I_{yy}, I_{zz} are +ve)

Question

Show that Momental ellipsoid at the center of elliptic disc is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z^2 = constant$$



Solution:

We know that

$$I_{xx} = \frac{mb^2}{4} \quad , \qquad I_{yy} = \frac{ma^2}{4}$$

$$I_{zz} = I_{xx} + I_{yy} \qquad \text{By} \perp \text{axis theorem}$$

$$= \frac{mb^2}{4}+\frac{ma^2}{4}$$

$$I_{zz} = \frac{m(a^2+b^2)}{4}$$

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$$I_{xy} = -\int xy\,dm = -\rho \int_{x=-a}^{x=a} \int_{y=-\frac{b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= \rho \int_{x=-a}^{x=a} x \left.\frac{y^2}{2}\right|_{\frac{-b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \rho \int_{x=-a}^{x=a} x\left(\frac{b^2}{a^2}\left(a^2-x^2\right)-\frac{b^2}{a^2}\left(a^2-x^2\right)\right)dx$$

$$= \rho \int_{x=-a}^{x=a} x\,(0)\,dx = 0$$

Similarly, $\qquad I_{yz} = I_{xz} = 0$

Now by Momental ellipsoid formula

$$x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2xyI_{xy} + 2yzI_{yz} + 2zx I_{xz} = 1$$

By putting the above value

$$x^2\left(\frac{mb^2}{4}\right)+y^2\left(\frac{ma^2}{4}\right)+z^2\left(\frac{m\left(a^2+b^2\right)}{4}\right)+2xy(0)+2yz(0)+2zx(0)=1$$

$$x^2\left(\frac{mb^2}{4}\right)+y^2\left(\frac{ma^2}{4}\right)+z^2\left(\frac{ma^2}{4}+\frac{mb^2}{4}\right)=1$$

Divide by $\frac{ma^2b^2}{4}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+z^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=\frac{4}{ma^2b^2}$$

$$\Rightarrow \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z^2=constant$$

Question:

Write inertia matrix if equation of Momental ellipsoid is

$$3x^2+3y^2+5z^2-xy+2yz+5xz=3$$

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Solution: Given

$$2x^2+3y^2+5z^2-xy+2yz+5xz=3$$

Divide by 3

$$\frac{2}{3}x^2+y^2+\frac{5}{3}z^2-\frac{1}{3}xy+\frac{2}{3}yz+\frac{5}{3}xz=1 \qquad \text{..................(1)}$$

As we know

$$x^2 I_{xx}+y^2 I_{yy}+z^2 I_{zz}+2xyI_{xy}+2yzI_{yz}+2zx\,I_{xz}=1 \qquad \text{..........(2)}$$

Comparing (1) and (2)

$I_{xx}=\frac{2}{3}$, $I_{yy}=1$, $I_{zz}=\frac{5}{3}$, $I_{xy}=-\frac{1}{6}$, $I_{yz}=\frac{1}{3}$, $I_{xz}=\frac{5}{6}$

Now Inertia Matrix is $\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{yz} & I_{zz} \end{bmatrix}=\begin{bmatrix} \frac{2}{3} & \frac{-1}{6} & \frac{5}{6} \\ \frac{-1}{6} & 1 & \frac{1}{3} \\ \frac{5}{6} & \frac{1}{3} & \frac{5}{3} \end{bmatrix}$

Lecture # 8

Question:

Find equation of momental ellipsoid of a uniform rectangular parallelepiped.

Solution:

As $\quad x^2 I_{xx} + y^2 I_{yy} + z^2 I_{zz} + 2xyI_{xy} + 2yzI_{yz} + 2zx I_{xz} = 1 \qquad \ldots..(1)$

We know that for a parallelopiped

$I_{xx} = \frac{m(b^2+c^2)}{3}$, $I_{yy} = \frac{m(a^2+c^2)}{3}$, $I_{zz} = \frac{m(a^2+b^2)}{3}$, $I_{xy} = -\frac{mab}{4}$, $I_{yz} = \frac{1}{3}$,
$I_{xz} = -\frac{mac}{4}$

Put these in (1)

$$\frac{m(b^2+c^2)}{3}x^2 + \frac{m(a^2+c^2)}{3}y^2 + \frac{m(a^2+b^2)}{3}z^2 + 2\left(\frac{-mab}{4}\right)xy + 2\left(\frac{-mac}{4}\right)xz + 2\left(\frac{-mbc}{4}\right)yz = 1$$

Multiplying by 6/m both side

$$2(b^2+c^2)\,x^2 + 2(a^2+c^2)\,y^2 + (a^2+b^2)\,z^2 - 3abxy - 3acxz - 3bcyz = 6/m$$

$$2(b^2+c^2)x^2 + 2(a^2+c^2)y^2 + (a^2+b^2)z^2 - 3abxy - 3acxz - 3bcyz = c$$

$\because$ c = 6/m

Which is required equation of momental ellipsoid of a uniform parallelopiped.

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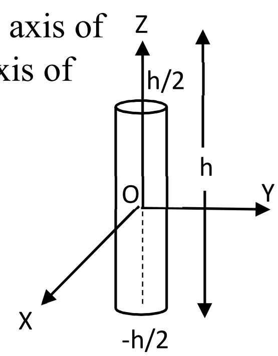
Question:

Find M.I of solid cylinder about

(i) its axis of symmetry or an axis passing through the Centre of cylinder and parallel to its length.
(ii) its central diameter or an axis passing through its Centre and perpendicular to the axis of symmetry.

Solution:

Let h be the height of cylinder 'O' be the Centre and z-axis be its axis of symmetry and x-axis & y-axis are the lines perpendicular to its axis of symmetry through its centre in cylindrical coordinate (r,θ,z)



$x = r\cos\theta$	;	$0 \le r \le a$
$y = r\sin\theta$	;	$0 \le \theta \le 2\pi$
$z = z$	;	$-h/2 \le 2 \le h/2$

$V = \pi a^2 h$

$\rho = \frac{m}{\pi a^2 h}$

M.I about the axis of symmetry

$I_{zz} = \int (x^2 + y^2)\, dm$

$I_{zz} \;=\; \rho\iiint\left(x^2+y^2\right)dV$

$$= \rho\int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a (r^2\cos^2\theta + r^2\sin^2\theta)\,r\,drd\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a r^2.r\,drd\theta dz \qquad \Rightarrow\ \rho\int_{-h/2}^{h/2}\int_0^{2\pi}\int_0^a r^3\,drd\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\int_0^{2\pi}\left.\frac{r^4}{4}\right|_0^a d\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\int_0^{2\pi}\left(\frac{a^4}{4}-0\right)d\theta dz$$

$$= \rho\left(\frac{a^4}{4}\right)\int_{-h/2}^{h/2}\int_0^{2\pi} d\theta dz$$

$$= \rho\left(\frac{a^4}{4}\right)\int_{-h/2}^{h/2}\left.\theta\right|_0^{2\pi} dz$$

$$= \rho\left(\frac{a^4}{4}\right)\int_{-h/2}^{h/2}(2\pi-0)dz$$

$$= \rho\left(\frac{a^4}{4}\right).2\pi\left|z\right|_{-h/2}^{h/2}$$

$$= \rho\left(\frac{a^4}{4}\right).2\pi\left(\frac{h}{2}+\frac{h}{2}\right)$$

$$= \frac{m}{\pi a^2 h}\left(\frac{a^4}{4}\right).2\pi h \qquad \because\ \rho = \frac{m}{\pi a^2 h}$$

$$I_{zz} = \frac{ma^2}{2}$$

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(ii). M.I about central diameter = I_{yy}

$$I_{yy} = \int (x^2 + z^2)\, dm$$

$$I_{yy} = \rho\iiint\left(x^2+z^2\right)dV = \rho\int_{-h/2}^{h/2}\int_{0}^{2\pi}\int_{0}^{a}(r^2\cos^2\theta + z^2)\,r\,drd\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\int_{0}^{2\pi}\int_{0}^{a}(r^3\cos^2\theta + rz^2)drd\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\int_{0}^{2\pi}\left(\frac{a^4}{4}\cos^2\theta + \frac{a^2}{2}z^2\right)d\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\int_{0}^{2\pi}\left(\frac{a^4}{4}\left(\frac{1+\cos 2\theta}{2}\right) + \frac{a^2}{2}z^2\right)d\theta dz$$

$$= \rho\int_{-h/2}^{h/2}\left.\left(\frac{a^4}{4}\left(\frac{\theta + \frac{\sin 2\theta}{2}}{2}\right) + \frac{a^2z^2}{2}\theta\right)\right|_0^{2\pi} d\theta dz$$

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$$= \rho\int_{-h/2}^{h/2}\left(\frac{a^4}{4}\left(\frac{2\pi + \frac{\sin 4\pi}{2}}{2}\right) + \frac{a^2z^2}{2}.2\pi - 0\right)dz$$

$$= \rho\int_{-h/2}^{h/2}\left(\frac{a^4}{4}\left(\frac{2\pi - 0}{2}\right) + a^2z^2\pi\right)dz$$

$$= \rho\int_{-h/2}^{h/2}\left(\frac{a^4\pi}{4} + a^2z^2\pi\right)dz$$

$$= \rho\frac{a^4\pi}{4}\int_{-h/2}^{h/2}dz + \rho a^2\pi\int_{-h/2}^{h/2}z^2dz$$

$$= \rho\frac{a^4\pi}{4}\left|z\right|_{-h/2}^{h/2} + \rho a^2\pi\left|\frac{z^3}{3}\right|_{-h/2}^{h/2}$$

$$= \rho\frac{a^4\pi}{4}\left(\frac{h}{2} + \frac{h}{2}\right) + \rho a^2\pi\left(\frac{h^3}{24} + \frac{h^3}{24}\right)$$

$$=\rho\frac{a^4\pi}{4}(h)+\rho a^2\pi\left(\frac{h^3}{12}\right)$$

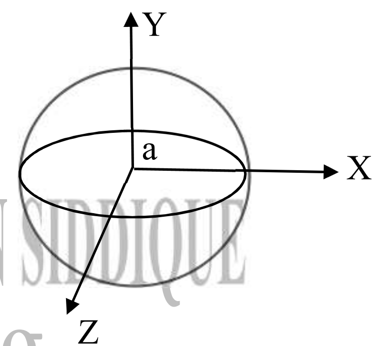
$$=\frac{m}{\pi a^2 h}.a^2\pi h\left(\frac{a^2}{4}+\frac{h^2}{12}\right) \qquad \because\ \rho=\frac{m}{\pi a^2 h}$$

$$=\left(\frac{ma^2}{4}+\frac{mh^2}{12}\right)$$

$$I_{yy}=\frac{ma^2}{4}+\frac{mh^2}{12}$$

Question:

Find M.I of a sphere about its diameter or about axis passing through the centre of sphere.



Solution:

$$I_{zz}=\int(x^2+y^2)\,\mathrm{dm}$$

$$I_{zz}\ =\ \rho\iiint\left(x^2+y^2\right)dV\ \ldots(1)$$

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x = rsinθcosϕ

y = r sinθsinϕ

z = rcosθ

Put in Eq (1)

$$I_{zz}=\rho\int_0^{2\pi}\int_0^{\pi}\int_0^{a}\left(r^2\sin^2\theta\cos^2\phi+r^2\sin^2\theta\sin^2\phi\right)r^2\sin\theta drd\theta d\phi$$

$$I_{zz}=\rho\int_0^{2\pi}\int_0^{\pi}\int_0^{a}r^2\sin^2\theta\left(\cos^2\phi+\sin^2\phi\right)r^2\sin\theta drd\theta d\phi$$

$$I_{zz}=\rho\int_0^{2\pi}\int_0^{\pi}\int_0^{a}r^4\sin^3\theta drd\theta d\phi$$

$$I_{zz}=\rho\int_0^{2\pi}\int_0^{\pi}\left.\frac{r^5}{5}\right|_0^a\sin^3\theta d\theta d\phi$$

$$\because\ dV=h_1h_2h_3\,du_1du_2du_3 \quad \because\ h_1=1,\ h_2=r,\ h_3=r\sin\theta$$

$$dV=(1)(r)(r\sin\theta)drd\theta d\phi$$

$$dV=r^2\sin\theta drd\theta d\phi$$

$$=\rho\int_0^{2\pi}\int_0^{\pi}\left(\frac{a5}{5}-0\right)\sin^3\theta d\theta d\phi$$

$$=\rho\frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta d\theta d\phi$$

$$=\rho\frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi}\sin\theta\sin^2\theta d\theta d\phi$$

$$=\rho\frac{a^5}{5}\int_0^{2\pi}\int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta d\phi$$

Put $\cos\theta = u$

$-\sin\theta\, d\theta = du$

Where $\theta = 0 \Rightarrow u = 1$

$\theta = \pi \Rightarrow u = -1$

$$=\rho\frac{a^5}{5}\int_0^{2\pi}\int_1^{-1}(1-u^2)(-du)d\phi \quad\Rightarrow\quad =\rho\frac{a^5}{5}\int_0^{2\pi}\int_{-1}^{1}(1-u^2)du d\phi$$

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$$=\rho\frac{a^5}{5}\int_0^{2\pi}\left(u-\frac{u^3}{3}\right|_{-1}^{1} d\phi$$

$$=\rho\frac{a^5}{5}\int_0^{2\pi}\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]d\phi \quad\Rightarrow\quad =\rho\frac{a^5}{5}\left(\frac{4}{3}\right)\int_0^{2\pi}d\phi$$

$$=\frac{m}{\frac{4}{3}\pi a^3}\frac{a^5}{5}\left(\frac{4}{3}\right)\left(\phi\right|_0^{2\pi} \quad\Rightarrow\quad =\frac{m}{\pi a^3}\frac{a^5}{5}(2\pi-0)$$

$$I_{zz}=\frac{2ma^2}{5}$$

Lecture # 9

Theorem:

Show that in matrix notation

$$\left[\overrightarrow{\dot{L}}\right]=[I]\left[\overrightarrow{\dot{\omega}}\right]+\left[\vec{\omega}\times\vec{L}\right] \text{ where } '\bullet' \text{ mean derivative w.r.t } 't'$$

Proof:

We know that for a rotating rigid body

$$\vec{L} = \sum_{i=1}^{n}\left(\vec{r_i} \times \vec{P_i}\right)$$

$$= \sum_{i=1}^{n}\left(\vec{r_i} \times (m_i\vec{V_i})\right)$$

$$= \sum_{i=1}^{n} m_i\left(\vec{r_i} \times \vec{V_i}\right) \quad \ldots\ldots\ldots(1) \quad \because \vec{V} = \vec{\omega} \times \vec{r}$$

$$= \sum_{i=1}^{n} m_i\left(\vec{r_i} \times (\vec{\omega_i} \times \vec{r_i})\right) \quad \ldots\ldots\ldots(2)$$

In components form

$$\Rightarrow L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$\Rightarrow L_y = I_{xy}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$\Rightarrow L_z = I_{xz}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$$

Or $\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

$$\Rightarrow \quad [L] = [I]\,[\omega]$$

Now from (1) Diff. w.r.t 't'

$$\overrightarrow{\dot{L}} = \sum_{i=1}^{n} m_i[\overrightarrow{\dot{r_i}}\times\vec{V_i} + \vec{r_i}\times\overrightarrow{\dot{V_i}}]$$

$$= \sum_{i=1}^{n} m_i[\vec{V_i}\times\vec{V_i} + \vec{r_i}\times\left(\vec{\omega_i}\times\vec{r_i}\right)^{\bullet}] \quad \because \overrightarrow{\dot{r_i}} = \vec{V_i} \text{ and } \vec{V_i} = \vec{\omega}\times\vec{r_i}$$

$$= \sum_{i=1}^{n} m_i[\vec{r_i}\times\left(\overrightarrow{\dot{\omega}}\times\vec{r_i} + \vec{\omega}\times\overrightarrow{\dot{r_i}}\right)] \quad \because \overrightarrow{\dot{r_i}} = \vec{V_i}$$

$$= \sum_{i=1}^{n} m_i[\vec{r_i}\times\left(\overrightarrow{\dot{\omega}}\times\vec{r_i}\right)] + \sum_{i=1}^{n} m_i[\vec{r_i}\times\left(\vec{\omega}\times\vec{r_i}\right)]$$

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$$= \sum_{i=1}^{n} m_i [\, \vec{r}_i \times \left(\overrightarrow{\dot{\omega}} \times \vec{r}_i \right)] + \sum_{i=1}^{n} m_i [\, \vec{r}_i \times \left(\vec{\omega} \times (\vec{\omega} \times \vec{v}_i) \right)] \qquad ...(A)$$

From (2) and (3) we note that $= \sum_{i=1}^{n} m_i(\vec{r_i} \times (\vec{\omega} \times \vec{v_i}))$ can be written in matrix notation as [I] [ω] .In the same way we can write $\sum_{i=1}^{n} m_i [\, \vec{r}_i \times \left(\overrightarrow{\dot{\omega}} \times \vec{r}_i \right)]$ as

[I] $[\overrightarrow{\dot{\omega}}]$.

Now $\vec{r}_i \times \left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_i) \right) = \vec{r}_i \times \left[\left((\vec{\omega} . \vec{r}_i)\vec{\omega} - (\vec{\omega} \times \vec{\omega})\vec{r}_i \right) \right]$

$$= \vec{r}_i \times \left[\left((\vec{\omega} . \vec{r}_i)\vec{\omega} - (\vec{\omega})^2 \vec{r}_i \right) \right]$$

$$= \vec{r}_i \times \left[\left((\vec{\omega} . \vec{r}_i)\vec{\omega} - (\vec{\omega})^2 \vec{r}_i \right) \right]$$

$$= (\vec{\omega} . \vec{r}_i)(\vec{r}_i \times \vec{\omega}) - (\vec{\omega})^2 (\vec{r}_i \times \vec{r}_i)$$

$$= -(\vec{\omega} . \vec{r}_i)(\vec{\omega} \times \vec{v}_i) \qquad ...(4) \qquad \because \vec{r}_i \times \vec{r}_i = 0 \text{ and } \because \vec{r}_i \times \vec{\omega} = -(\vec{\omega} \times \vec{v}_i)$$

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Similarly (or replace $\vec{r_i} \times \vec{\omega}$)

$$\vec{\omega} \times \left(\vec{r}_i \times (\vec{r}_i \times \vec{\omega}) \right) = -(\vec{\omega} . \vec{r}_i)\,(\vec{r}_i \times \vec{\omega})$$

$$\vec{\omega} \times \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \right) = -(\vec{\omega} . \vec{r}_i)\,(\vec{\omega} \times \vec{r}_i) \qquad ..(5) \qquad \because \vec{r}_i \times \omega = -\left(\vec{\omega} \times \vec{r}_i \right)$$

Comparing (4) and (5)

$$\vec{r}_i \times \left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_i) \right) = \vec{\omega} \times \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \right)$$

Now Apply $\sum_{i=1}^{n} m_i$ both side

$$\sum_{i=1}^{n} m_i (\vec{r}_i \times \left(\vec{\omega} \times (\vec{\omega} \times \vec{r}_i) \right)) = \sum_{i=1}^{n} m_i (\vec{\omega} \times \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \right))$$

$$= \vec{\omega} \times (\sum_{i=1}^{n} m_i \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) \right))$$

$$= \vec{\omega} \times \vec{L} \qquad \because \vec{L} = \sum_{i=1}^{n} m_i (\vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i \right)$$

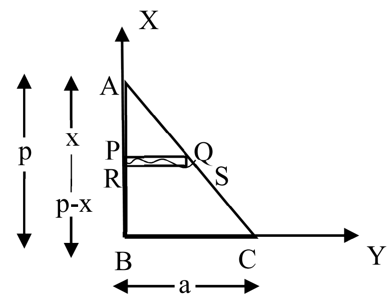
So $\sum_{i=1}^{n} m_i(\vec{r}_i \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}_i)))$ can be written in matrix form as $[\vec{\omega} \times \vec{L}]$

Hence from (A)

$\dot{\vec{L}} = [I][\dot{\vec{\omega}}] + [\vec{\omega} \times \vec{L}]$ proved

Question:

Find M.I of a tringular lemina about one of its edges.



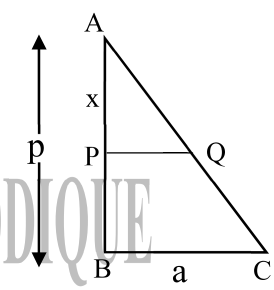
Solution:

M.I about y-axis

$I_{yy} = \int mass\ (\perp distance)^2$

$= \int (p-x)^2 \text{dm}$

$= \rho \int (p-x)^2 \text{dA}$ …..(1) $\because\ dm = \rho dA$



PQRS become a rectangle

Area = length × width

dA = $|PQ|$dx

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Since the Δ APQ and Δ ABC are similar so

$\frac{|\overrightarrow{PQ}|}{|\overrightarrow{BC}|} = \frac{|\overrightarrow{AP}|}{|\overrightarrow{AB}|}$

$\frac{|\overrightarrow{PQ}|}{a} = \frac{x}{p} \Rightarrow |\overrightarrow{PQ}| = \frac{ax}{p}$

dA $= |\overrightarrow{PQ}|$ dx $= \frac{ax}{p}$ dx

Area $= \frac{1}{2}$ap

$\rho = \frac{m}{A} = \frac{m}{\frac{1}{2}ap} = \frac{2m}{ap}$

Put in (1) $I_{yy} = \rho \int_{x=0}^{x=p} (p-x)^2 \frac{ax}{p} dx$

$$= \frac{\rho a}{p} \int_{x=0}^{x=p} x(p^2 + x^2 - 2px)dx \Rightarrow \frac{\rho a}{p} \int_{x=0}^{x=p} (xp^2 + x^3 - 2px^2)dx$$

$$\frac{\rho a}{p} \left| xp^2 + x^3 - 2px^2 \right|_0^p \Rightarrow \frac{\rho a}{p}\left(\frac{p^2}{2}.p^2 + \frac{p^4}{4} - \frac{2p^4}{3}\right) - (0)$$

$$\frac{\rho a}{p}\left(\frac{p^2}{2}.p^2 + \frac{p^4}{4} - \frac{2p^4}{3}\right) = \frac{\rho a}{p}\left(\frac{6p^4 + 3p^4 - 8p^4}{12}\right) \Rightarrow I_{yy} = \frac{mp^2}{6}$$

Lecture # 10

Equimomental System:

Two systems are said to be Equimomental if they have same M.I about any line space.

Principle axis:

Three mutually perpendicular axis are called Principle axis if product of inertia realtive to these axis if product of inertia relative to these axis is zero and corresponding M.I's are called Principle moment of inertias i.e. A set of three mutually $\perp$ axis relative to which Inertia matrix is diagonal.

Theorem:

Two systems are said to be equimomental iff

(i) They have same masses
(ii) They have same centroid (center of mass of an object with uniform density)
(iii) They have same Principle axis and same Principle M.I at cenroid

Proof:

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Suppose that condition (i) , (ii) and (iii) holds. Let S_1 and S_2 be two systems each having same masses 'm' same centroid 'C' and same Principal axis at centroid.

Consider a line 'l' having direction cosines λ , μ and v passing throug centroid C.

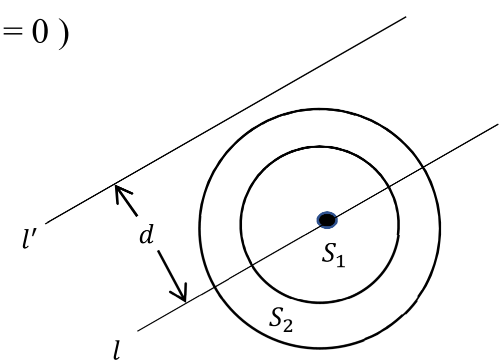
The M.I of system S_1 about line 'l' is

$$I_l^{(1)} = \lambda^2 I_{xx} + \mu^2 I_{yy} + v^2 I_{zz} + 2\lambda\mu I_{xy} + 2\mu v I_{yz} + 2\lambda v I_{xz} \qquad .....(1)$$

Since given axis are Principle axis $I_{xy} = I_{xz} = I_{yz} = 0$)

Therefore from (1) we have

$$I_l^{(1)} = \lambda^2 I_{xx} + \mu^2 I_{yy} + v^2 I_{zz} \qquad ....(2)$$



Similarly M.I of system S_2 about line 'l' is

$$I_l^{(2)} = \lambda^2 I_{xx} + \mu^2 I_{yy} + v^2 I_{zz} \qquad ....(3)$$

From (2) and (3)

$$I_l^{(1)} = I_l^{(2)} = I$$

Let l' be the line parallel to ‘l’ at a distance ‘d’. then M.I of S_1 about line ‘l'’ is

$$I_{l'}^{(1)} = I + md^2 \quad ..(4) \quad (By \ \ Parallel \ \ Axis\, theorem)$$

Similary M.I of S_2 about line ‘l'’ is

$$I_{l'}^{(2)} = I + md^2 \quad ..(5) \quad (By \ \ Parallel \ \ Axis\, theorem)$$

From (4) and (5)

Hence both system S_1 and S_2 are equimomental

Conversely:

(i) Let S_1 and S_2 are equimomental

Consider a line ‘l_1’ passing through centroids G_1 and G_2 of system S_1 and S_2 respectively. Since system S_1 and S_2 are equimomental therefore

$$I_l^{(1)} = I_l^{(2)} = I$$

Let ‘l_2’ be the line parallel to ‘l_1’ at a distance ‘d_1’. Then M.I of S_1 about line ‘l_2’ is

$$I_{l_2}^{(1)} = I + m_1 d_1^{\,2} \quad ..(6) \quad (By \ \ Parallel \ \ Axis\, theorem)$$

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Similarly M.I of S_2 about line ‘l_2’ is

$$I_{l_2}^{(2)} = I + m_2 d_1^{\,2} \quad ..(7) \quad (By \ \ Parallel \ \ Axis\, theorem)$$

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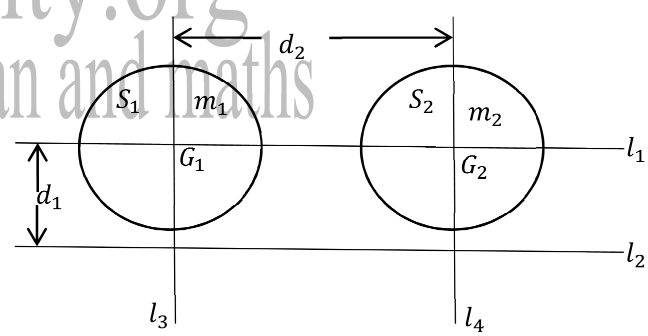
As S_1 and S_2 are equimomental so



$$I_{l_2}^{(1)} = I_{l_2}^{(2)}$$

$$\Rightarrow \quad I + m_1 d_1^{\,2} = I + m_2 d_1^{\,2}$$

$$\Rightarrow \quad m_1 = m_2$$

They have same masses.

(ii) Now consider a line ‘l_3’ passing through centroid G_1 and $\perp$ to ‘l_1’. Also consider a line ‘l_4’ passing through G_2 and $\perp$ to ‘l_1’. This means ‘l_3’ and ‘l_4’ are parllel

Now M.I of S_1 about line ‘l_4’ is

$$I_{l_4}^{(1)} = I_{l_3}^{(1)} + md_2^{\,2} \quad (By \ \ Parallel \ \ Axis\, theorem)$$

Similarly M.I of S_2 about line ‘l_3’ is

$$I_{l_3}^{(2)} = I_{l_4}^{(2)} - md_2^{\,2} \quad (By \ \ Parallel \ \ Axis\, theorem)$$

As S_1 and S_2 are equimomental. So

$$I_{l_4}^{(1)} = I_{l_4}^{(2)}$$

$$\Rightarrow \; I_{l_3}^{(1)} + md_2^{\,2} = I_{l_4}^{(2)} - md_2^{\,2}$$

$$\Rightarrow \; 2md_2^{\,2} = 0$$

$$\Rightarrow \; d_2 = 0$$

$\Rightarrow$ l_3 and l_4 coincides. $G_1 = G_2$

$\Rightarrow$ same centroid

(iii) Now the two systems have same centroid and same masses if we take any line through the common centroid say G

i.e. G = $G_1 = G_2$

The two systems have the same M.I about that line. It means that we have same Principle axis and same M.I at point G.

Hence all the conditions are satisfied which complete the proof.

Question:

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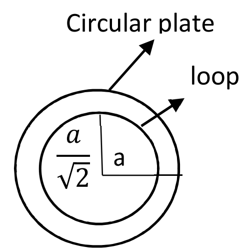
Show that a circular plate of mass 'm' and radius 'a' is equimomental with a loop of same mass 'm' and radius $\frac{a}{\sqrt{2}}$

Solution: We know M.I of circular plate about an axis passing through its centroid and $\perp$ to its plane $= \frac{ma^2}{2}$



M.I of loop about same axis $= m\left(\frac{a}{\sqrt{2}}\right)^2 = \frac{ma^2}{2}$

Hence circular plate and loop are equimomental.

Lecture # 11

Theorem (Existence of Principle axis theorem):

Statement:

For a rigid body ∃ a set of three mutually perpendicular axis relative to which product of inertia are zero and angular velocity $\vec{\omega}$ and a linear momentum $\vec{L}$ are in same direction.

Proof:

Consider an axis through a point '0' such that $\vec{\omega}$ and $\vec{L}$ are parallel to it. Then we can write

$$\vec{L} = \lambda\vec{\omega} \qquad \text{; where } \lambda \text{ is contant of proportionality}$$

$$\left.\begin{aligned} L_1 &= \lambda\omega_1 \\ L_2 &= \lambda\omega_2 \\ L_3 &= \lambda\omega_3 \end{aligned}\right\} \qquad \ldots\ldots\ldots(1)$$

Next we know that (In general)

$$\left.\begin{aligned} L_1 &= I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 \\ L_2 &= I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 \\ L_3 &= I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 \end{aligned}\right\} \ldots.(2)$$

From (1) and (2) we can write

$$\left.\begin{aligned} (I_{11}-\lambda)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 &= 0 \\ I_{21}\omega_1 + (I_{22}-\lambda)\omega_2 + I_{23}\omega_3 &= 0 \\ I_{31}\omega_1 + I_{32}\omega_2 + (I_{33}-\lambda)\omega_3 &= 0 \end{aligned}\right\} \ldots.(3)$$

This is homogenous system of equation which unknown $\omega_1, \omega_2, \omega_3$. It has non-trivial soltution when determinant of the matrix made by the coefficient in set of equation (3) is given

i.e. $\left|I_{ij}\right| = 0$

$$\begin{vmatrix} I_{11}-\lambda & I_{12} & I_{13} \\ I_{21} & I_{22}-\lambda & I_{23} \\ I_{31} & I_{32} & I_{33}-\lambda \end{vmatrix} = 0$$

Which is cubic equation in λ which has three relation $\lambda = I_1, I_2, I_3$. These relation are actually eigen value of the inertia matrix and these values are principal moment of inertia of the rigid body. The corresponding eigen vectors give the direction of principal axis.

Working rule of finding Principle M.I and Principle Axis:

(i) Find inertia matrix at given point (say origin)

(ii) If inertia matrix is diagonal then corresponding axis are Principle. If inertia matrix is not diagonal the eigen values matrix is not diagonal the eigenvalues of this matrix give values of Principle moment of Inertia.

(iii) Eigenvectors corresponding to each eigenvalue gives the direction of Principle axis.

Question:

Consider a system of particle having masses m,2m,3m,4m located at points (a,a,a) . (a,-a,-a),(-a,a,-a),(-a,-a,a) respectively. Find Principle moment of inertia of the system about origin.

Solution:

$$\begin{aligned}
I_{xx} &= \sum_{i=1}^{n} m_i(y_i^2 + z_i^2) \\
&= m(a^2 + a^2) + 2m(a^2 + a^2) + 3m(a^2 + a^2) + 4m\,(a^2 + a^2) \\
&= 2ma^2 + 4ma^2 + 6ma^2 + 8ma^2 \\
&= 20ma^2
\end{aligned}$$

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$$\begin{aligned}
I_{yy} &= \sum_{i=1}^{n} m_i(x_i^2 + z_i^2) \\
&= m(a^2 + a^2) + 2m(a^2 + a^2) + 3m(a^2 + a^2) + 4m\,(a^2 + a^2) \\
&= 2ma^2 + 4ma^2 + 6ma^2 + 8ma^2 \\
&= 20ma^2
\end{aligned}$$

$$\begin{aligned}
I_{zz} &= \sum_{i=1}^{n} m_i(x_i^2 + y_i^2) \\
&= m(a^2 + a^2) + 2m(a^2 + a^2) + 3m(a^2 + a^2) + 4m\,(a^2 + a^2) \\
&= 2ma^2 + 4ma^2 + 6ma^2 + 8ma^2 \\
&= 20ma^2
\end{aligned}$$

$$I_{xy} = -\sum_{i=1}^{n} m_i \; x_i y_i$$

$$= -[m(a^2) + 2m(-a^2) + 3m(-a^2) + 4m(a^2)]$$

$$= -[ma^2 - 2ma^2 - 3ma^2 + 4ma^2]$$

$$I_{xy} = 0 = I_{yx}$$

$$I_{yz} = -\sum_{i=1}^{n} m_i \; y_i z_i$$

$$= -[m(a^2) + 2m(a^2) + 3m(-a^2) + 4m(-a^2)]$$

$$= -[ma^2 + 2ma^2 - 3ma^2 - 4ma^2]$$

$$I_{yz} = -[-4ma^2] = 4ma^2 = I_{zy}$$

$$I_{xz} = -\sum_{i=1}^{n} m_i \; x_i z_i$$

$$= -[m(a^2) + 2m(-a^2) + 3m(a^2) + 4m(-a^2)]$$

$$= -[ma^2 - 2ma^2 + 3ma^2 - 4ma^2]$$

$$I_{xz} = -[-2ma^2] = 2ma^2 = I_{zx}$$

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$$\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 20ma^2 & 0 & 2ma^2 \\ 0 & 20ma^2 & 4ma^2 \\ 2ma^2 & 4ma^2 & 20ma^2 \end{bmatrix}$$

For Eigenvalue

$$\begin{vmatrix} I_{xx} - \lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - \lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 20ma^2 - \lambda & 0 & 2ma^2 \\ 0 & 20ma^2 - \lambda & 4ma^2 \\ 2ma^2 & 4ma^2 & 20ma^2 - \lambda \end{vmatrix} = 0$$

By expanding we get

$$\left(20ma^2 - \lambda\right)\left(384m^2a^4 + \lambda^2 - 40ma^2\lambda - 4m^2a^4\right) = 0$$

$$\left(20ma^2-\lambda\right)=0 \ \ and \ \left(380m^2a^4+\lambda^2-40ma^2\lambda\right)=0$$

$$\lambda=20ma^2 \quad , \quad \lambda^2-40ma^2\lambda+380m^2a^4=0$$

$$\lambda=\frac{-(-40ma^2)\pm\sqrt{1600m^2a^4-1520m^2a^4}}{2(1)}$$

$$\lambda=\frac{40ma^2\pm\sqrt{80m^2a^4}}{2(1)}$$

$$\lambda=\frac{40ma^2\pm4\sqrt{5}ma^2}{2}$$

$$\lambda=20ma^2\pm2\sqrt{5}ma^2$$

$$\lambda=20ma^2+2\sqrt{5}ma^2 \quad , \quad \lambda=20ma^2-2\sqrt{5}ma^2$$

$$\lambda=2(10+\sqrt{5})ma^2 \quad , \quad \lambda=2(10-\sqrt{5})ma^2$$

$$\lambda=20ma^2 \quad , \quad \lambda=20ma^2\pm2\sqrt{5}ma^2$$

are the Principle M.I about origin.

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Lecture # 12

Spherical Top:

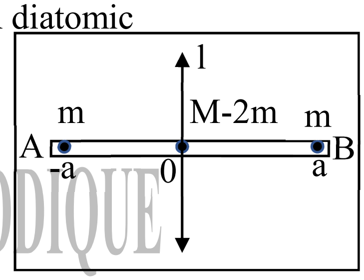
A rigid body is said to be spherical top if its all Principle moment of inertias are equal.

i.e. $I_{11} = I_{22} = I_{33}$. A sphere is an example of spherical top with axis passing through the centre of sphere.

Symmetrical Top: A rigid body is said to be symmetircal top if $I_{11} = I_{22} \neq I_{33}$. A cylinder is an example of symmetrical top (taking third axis along the axis of cylinder).

Asymmetrical top: A rigid body is called Asymmetrical top if $I_{11} \neq I_{22} \neq I_{33}$. A rigid body in general is example of Asymmetircal top.

Rotor: A rigid body is called rotor if its two Principle M.I's are equal and third M.I is zero i.e. $I_{11} = I_{22}$ and $I_{33} = 0$ or $I_{11} = 0$ and $I_{22} = I_{33}$. A diatomic molecule is an example of rotor.



Question: Find equimomental system of particle for a rod of mass M.

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Solution: Consider a uniform rod of length 2a. If '0' be the centre of mass. Let the masses m, M-2m , m are located at point A,O,B respectively.

The system of particle will be equimomental with the rod. If its M.I about any line is equal to M.I of the rod about the same line.

Let 'l' be the line through '0' perpendicular to this rod.

Then M.I of rod about y-axis (axis passing through cenroid of rod) = I_1

$$I_1 = \frac{1}{3}(\text{Mass})\ (half\ of\ length)^2 = \frac{1}{3}\,\text{M}a^2$$

M.I of system of particle about y-axis

$$I_2 = \text{m}(-a)^2 + (\text{M-2m})\ (0)^2 + \text{m}(a)^2 = 2\text{m}a^2$$

If both system are equimomental then $I_1 = I_2 \Rightarrow \frac{1}{3}\,\text{M}a^2 = 2\text{m}a^2 \Rightarrow \text{m} = \frac{M}{6}$

Hence if we take two particles each having mass $\frac{M}{6}$ at end points of rod and a particle mass $\frac{2}{3}$M at the centre of rod then this system of three particles is equimomental with the given rod of mass M

Lecture # 13

Euler's Dynamical equations:

Consider a rigid body is rotating in OXYZ (Cartesian) coordiante system such that OX, OY & OZ are Principle axes. Then we know that

$$\vec{L} = I\,\vec{\omega} \qquad \text{Or in matrix form}$$

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$\because$ OX, OY and OZ are Principle axes. So $I_{xy} = I_{yz} = I_{xz} = 0$

$$\Rightarrow \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} = \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} I_{11}\omega_1 \\ I_{22}\omega_2 \\ I_{33}\omega_3 \end{bmatrix}$$

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In vector form angular momentum is given below

$$\Rightarrow \vec{L} = I_{11}\omega_1\hat{i} + I_{22}\omega_2\hat{i} + I_{33}\omega_3\hat{k}$$

We know that rate of change of vector function in fix and rotating coordiante system is related as follow

$$\left(\frac{dV}{dt}\right)_f = \left(\frac{d\vec{V}}{dt}\right)_r + \left(\vec{\omega} \times \vec{V}\right)$$

Putting $\vec{V} = \vec{L}$ Where V is any vector function

$$\left(\frac{d\vec{L}}{dt}\right)_f = \left(\frac{d\vec{L}}{dt}\right)_r + \left(\vec{\omega} \times \vec{L}\right)$$

$$\left(\frac{d\vec{L}}{dt}\right)_f = \left(I_{11}\dot{\omega}_1\hat{i} + I_{22}\dot{\omega}_2\hat{i} + I_{33}\dot{\omega}_3\hat{k}\right) + \left(\vec{\omega} \times \vec{L}\right)$$

Here $\left(\frac{d\vec{L}}{dt}\right)_f = \tau$ *is external torque acting on the system*

$$\because \quad \frac{d\vec{L}}{dt} = \frac{d\left(\vec{r}\times\vec{P}\right)}{dt} = \vec{V}\times\vec{P} + \vec{r}\frac{d\vec{P}}{dt}$$

$$= \quad \vec{V}\times\vec{P} + \vec{r}\times\vec{F} \quad = \quad \vec{V}\times m\vec{V} + \vec{r}\times\vec{F}$$

$$= 0 + \quad \vec{r}\times\vec{F} = \tau$$

So $\vec{\tau} = \left(I_{11}\dot{\omega}_1\hat{i} + I_{22}\dot{\omega}_2\hat{i} + I_{33}\dot{\omega}_3\hat{k}\right) + \left(\vec{\omega}\times\vec{L}\right)$ ….(A)

$$\vec{\omega}\times\vec{L} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ I_{11}\omega_1 & I_{22}\omega_2 & I_{33}\omega_3 \end{vmatrix}$$

$$\vec{\omega}\times\vec{L} = \left(I_{33}\omega_2\omega_3 - I_{22}\omega_2\omega_3\right)\hat{i} - \left(I_{33}\omega_1\omega_3 - I_{11}\omega_1\omega_3\right)\hat{j} + \left(I_{22}\omega_1\omega_2 - I_{11}\omega_1\omega_2\right)\hat{k}$$

$\vec{\omega}\times\vec{L} = -\left(I_{22} - I_{33}\right)\omega_2\omega_3\hat{i} - \left(I_{33} - I_{11}\right)\omega_1\omega_3\hat{j} - \left(I_{11} - I_{22}\right)\omega_1\omega_2\hat{k}$ Put in (A)

$$\vec{\tau} = \left(I_{11}\dot{\omega}_1\hat{i} + I_{22}\dot{\omega}_2\hat{i} + I_{33}\dot{\omega}_3\hat{k}\right) - \left(I_{22} - I_{33}\right)\omega_2\omega_3\hat{i} - \left(I_{33} - I_{11}\right)\omega_1\omega_3\hat{j} - \left(I_{11} - I_{22}\right)\omega_1\omega_2\hat{k}$$

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$$\tau_1 = I_{11}\dot{\omega}_1 - (I_{22} - I_{33})\omega_2\omega_3$$

$$\tau_2 = I_{22}\dot{\omega}_2 - (I_{33} - I_{11})\omega_1\omega_3$$

$\tau_3 = I_{33}\dot{\omega}_3 - (I_{11} - I_{33})\omega_1\omega_2$ Are Eulers Dynamic equations

Results: If there is no external torque then

$$I_{11}\dot{\omega}_1 - (I_{22} - I_{33})\omega_2\omega_3 = 0 \qquad \text{(i)}$$

$$I_{22}\dot{\omega}_2 - (I_{33} - I_{11})\omega_1\omega_3 = 0 \qquad \text{(ii)}$$

$$I_{33}\dot{\omega}_3 - (I_{11} - I_{33})\omega_1\omega_2 = 0 \qquad \text{(iii)}$$

Multiplying (i) by ω_1 , (ii) by ω_2 and (iii) by ω_3 and adding

$$I_{11}\omega_1\dot{\omega}_1 - \left(I_{22} - I_{33}\right)\omega_1\omega_2\omega_3 + I_{22}\omega_2\dot{\omega}_2 - \left(I_{33} - I_{11}\right)\omega_1\omega_2\omega_3 + I_{33}\omega_3\dot{\omega}_3 - \left(I_{11} - I_{22}\right)\omega_1\omega_2\omega_3 = 0$$

$$I_{11}\omega_1\omega_1^{\bullet} - I_{22}\omega_1\omega_2\omega_3 + I_{33}\omega_1\omega_2\omega_3 + I_{22}\omega_2\omega_2^{\bullet} - I_{33}\omega_1\omega_2\omega_3 + I_{11}\omega_1\omega_2\omega_3 + I_{33}\omega_3\omega_3^{\bullet} - I_{11}\omega_1\omega_2\omega_3 + I_{22}\omega_1\omega_2\omega_3 = 0$$

$$I_{11}\omega_1\omega_1^{\bullet} + I_{22}\omega_2\omega_2^{\bullet} + I_{33}\omega_3\omega_3^{\bullet} = 0$$

$$\Rightarrow \frac{1}{2}\left(I_{11}.2\omega_1\omega_1^{\bullet} + I_{22}.2\omega_2\omega_2^{\bullet} + I_{33}.2\omega_3\omega_3^{\bullet}\right) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2\right) = 0 \Rightarrow \left(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2\right) = \text{constant}$$

Now we know that Rotational K.E is $\mathrm{T} = \frac{1}{2}\vec{\omega}\cdot\vec{L} \quad = \frac{1}{2}\left(\omega_1 L_1 + \omega_2 L_2 + \omega_3 L_3\right)$

$$= \frac{1}{2}\left(\omega_1\left(I_{11}\omega_1\right) + \omega_2\left(I_{22}\omega_2\right) + \omega_3\left(I_{33}\omega_3\right)\right)$$

$$= \frac{1}{2}\left(I_{11}\omega_1^2 + I_{22}\omega_2^2 + I_{33}\omega_3^2\right) = \text{constant}$$

In the absence of external torque the rotational K.E of system is constant.

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Now multiplying (i) by $I_{11}\omega_1$, (ii) by $I_{22}\omega_2$ and (iii) by $I_{33}\omega_3$ and adding

$$I_{11}^2\omega_1\omega_1^{\bullet} - I_{11}\left(I_{22} - I_{33}\right)\omega_1\omega_2\omega_3 + I_{22}^2\omega_2\omega_2^{\bullet} - I_{22}\left(I_{33} - I_{11}\right)\omega_1\omega_2\omega_3 + I_{33}^2\omega_3\omega_3^{\bullet} - I_{33}\left(I_{11} - I_{22}\right)\omega_1\omega_2\omega_3 = 0$$

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$$I_{11}^2\omega_1\omega_1^{\bullet} - I_{11}I_{22}\omega_1\omega_2\omega_3 + I_{11}I_{33}\omega_1\omega_2\omega_3 + I_{22}^2\omega_2\omega_2^{\bullet} - I_{22}I_{33}\omega_1\omega_2\omega_3 + I_{11}I_{22}\omega_1\omega_2\omega_3 + I_{33}^2\omega_3\omega_3^{\bullet} - I_{11}I_{33}\omega_1\omega_2\omega_3 + I_{22}I_{33}\omega_1\omega_2\omega_3 = 0$$

$$I_{11}^2\omega_1\omega_1^{\bullet} + I_{22}^2\omega_2\omega_2^{\bullet} + I_{33}^2\omega_3\omega_3^{\bullet} = 0$$

$$\Rightarrow \frac{1}{2}\left(I_{11}^2.2\omega_1\omega_1^{\bullet} + I_{22}^2.2\omega_2\omega_2^{\bullet} + I_{33}^2.2\omega_3\omega_3^{\bullet}\right) = 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\omega_3^2\right) = 0$$

$$\left(I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\omega_3^2\right) = \text{constant}$$

Now we know $\quad \vec{L} \; = I_{11}\omega_1\hat{i} + I_{22}\omega_2\hat{i} + I_{33}\omega_3\hat{k}$

$$\left|\vec{L}\right| = \sqrt{I_{11}^2\omega_1^2 + I_{22}^2\omega_2^2 + I_{33}^2\omega_3^2} = \text{constant}$$

In the absence of external torque magnitude of angular momentum is constant.

Lecture # 14

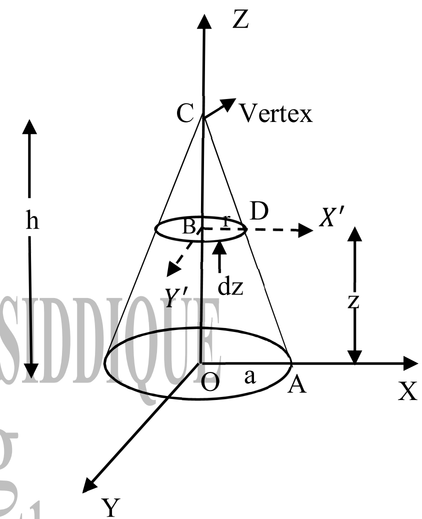
Question:

Find M.I of a solid right circular cone.

(i) About its axis of symmetry
(ii) About an axis passing through its base and also through its center OR About the diameter of its base.

Solution: (i)

Let M be the mass ‘a’ the radius and ‘h’ be the height of right circular cone. We choose the z-axis along the axis of symmetry and consider a typical disc of radius r and width dz at a distance z from the base.



Since ΔOAC and ΔBCD are similar triangles.

So $\frac{OA}{BD} = \frac{OC}{BC} \Rightarrow \frac{a}{r} = \frac{h}{h-z}$

r $= \frac{a(h-z)}{h}$

Now mass of disk = dm = ρdv = $\rho\pi r^2$dz

Also $\rho = \frac{Mass\ of\ cone}{Volume\ of\ cone} = \frac{3M}{\pi a^2 h}$

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We know that

M.I of disk about z-axis (as shown in fig)

$$= \frac{1}{2} r^2 \text{dm}$$

Now M.I of cone about z-axis (as shown in fig)

$$= \int \frac{1}{2} r^2 dm = \frac{1}{2}\int_0^h r^2 \rho\pi r^2 dz$$

$$= \frac{1}{2}\rho\pi\int_0^h r^4 dz$$

$$= \frac{1}{2}\rho\pi\int_0^h \frac{a^4(h-z)^4}{h^4} dz = \frac{1}{2}\frac{\rho\pi a^4}{h^4}\int_0^h (-(z-h))^4 dz$$

$$= \frac{1}{2}\frac{\rho\pi a^4}{h^4}\frac{(z-h)^5}{5}\Bigg|_0^h$$

$$= \frac{1}{2}\frac{3M}{\pi a^2 h}\cdot\frac{\pi a^4}{h^4}\left[\frac{(h-h)^5}{5} - \frac{(0-h)^5}{5}\right]$$

$$= \frac{3Ma^2}{2h^5}\left[-\frac{(-h)^5}{5}\right] = \frac{3Ma^2}{2h^5}\left[\frac{(h)^5}{5}\right]$$

$$= \frac{3Ma^2}{10}$$

(ii). Note that for disk

$$I_{X'} = I_{Y'} \qquad ....(1)$$

Now by ⊥ axis theorem

$$I_{X'} + I_{Y'} = I_{Z'} \qquad \because \; I_{X'} = I_{Y'} \;\; \& \;\; I_{Z'} = I_Z$$

$$\Rightarrow \qquad 2I_{X'} = I_Z$$

$$\Rightarrow \qquad I_{X'} = \frac{1}{2}I_Z = \frac{1}{2}\left(\frac{1}{2}r^2 dm\right)$$

$$\Rightarrow \qquad I_{X'} = \left(\frac{1}{4}r^2 dm\right)$$

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Now by // axis theorem

M.I of disk about x-axis (as shown in fig)

$$= I_{X'} + z^2 dm$$

$$= \frac{1}{4}r^2 dm + z^2 dm$$

$$= \frac{1}{4}r^2(\rho\pi r^2 dz) + z^2(\rho\pi r^2 dz) \qquad \because \; dm = \rho\pi r^2 dz$$

$$= \frac{1}{4}\rho\pi r^4 dz + z^2\rho\pi r^2 dz$$

$$= \frac{1}{4}\rho\pi\left(\frac{a^4(h-z)^4}{h^4}\right)dz + z^2\rho\pi\left(\frac{a^2(h-z)^2}{h^2}\right)dz$$

$$= \frac{\rho\pi a^4}{4h^4}(h-z)^4 dz + \frac{\rho\pi a^2}{4h^2}.z^2(h^2+z^2-2hz)dz$$

$$= \frac{\rho\pi a^4}{4h^4}(-(z-h))^4 dz + \frac{\rho\pi a^2}{4h^2}(h^2z^2+z^4-2hz^3)dz$$

Now M.I of cone about x-axis (as shown in fig)

$$= \frac{1}{2}\int_0^h \left[\frac{\rho\pi a^4}{4h^4}(z-h)^4 dz + \frac{\rho\pi a^2}{h^2}(h^2z^2+z^4-2hz^3)dz\right]$$

$$= \frac{\rho\pi a^4}{4h^4}\int_0^h (z-h)^4 dz + \frac{\rho\pi a^2}{h^2}\int_0^h (h^2z^2+z^4-2hz^3)dz$$

$$= \frac{\rho\pi a^4}{4h^4}\frac{(z-h)^5}{5}\Bigg|_0^h + \frac{\rho\pi a^2}{h^2}\left[h^2\frac{z^3}{3}+\frac{z^5}{5}-2h\frac{z^4}{4}\right]_0^h$$

$$= \frac{\rho\pi a^4}{4h^4}\left[0-\left(\frac{-h^5}{5}\right)\right] + \frac{\rho\pi a^2}{h^2}\left[h^2\frac{h^3}{3}+\frac{h^5}{5}-2h\frac{h^4}{4}\right]$$

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$$= \frac{\pi a^4}{4h^4}\frac{3M}{\pi a^2 h}\left[0-\left(\frac{-h^5}{5}\right)\right] + \frac{3M}{\pi a^2 h}\frac{\pi a^2}{h^2}\left[\frac{h^5}{3}+\frac{h^5}{5}-\frac{h^5}{2}\right]$$

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$$= \frac{\pi a^4}{4h^4}\frac{3M}{\pi a^2 h}\left[\left(\frac{h^5}{5}\right)\right] + \frac{3M}{\pi a^2 h}\frac{\pi a^2}{h^2}\left[\frac{h^5}{3}+\frac{h^5}{5}-\frac{h^5}{2}\right]$$

$$= \frac{3Ma^2}{20} + \frac{3M}{h^3}\left[\frac{20h^5+12h^5-30h^5}{60}\right]$$

$$= \frac{3Ma^2}{20} + \frac{3M}{h^3}\left[\frac{2h^5}{60}\right]$$

$$= \frac{3Ma^2}{20} + \frac{3M}{h^3}\left(2h^2\right)$$

$$= \frac{M}{20}\left[3a^2+2h^2\right]$$