"Functional Analysis" 1

CHAPTER No:1 {Normed Linear spaces}

Def: (1.1): Norm: A norm on a linear space X is a real valued function $\|\cdot\|$ (ie $\|\cdot\|: X \to \mathbb{R}$) whose value at x, denoted by $\|x\|$, have the following properties.

ⓐ $\|x_1+x_2\| \leq \|x_1\| + \|x_2\|$; $\forall\ x_1, x_2 \in X$.

ⓑ $\|\alpha x\| = |\alpha|\,\|x\|$; for any scaler α and $x \in X$.

ⓒ $\|x\| \geq 0$; $\forall\ x \in X$.

ⓓ $\|x\| = 0$ iff $x = 0$; $\forall\ x \in X$.

The pair $(X, \|\cdot\|)$ is called a normed linear space or normad vector space.

Remark (1.2): If x is a vector, its length is $\|x\|$, the length $\|x_1 - x_2\|$ of the vector difference $x_1 - x_2$ is the distance b/w the end points of the vectors x_1 and x_2.

Examples (1.3):

① the real linear space $\mathbb{R}$ is a normed linear space with norm $\|\cdot\|: \mathbb{R} \to \mathbb{R}$ defined by $\|x\| = |x|$; $\forall\ x \in \mathbb{R}$.

Pf: ⓐ For any $x_1, x_2 \in \mathbb{R}$, we have

$$\|x_1 + x_2\| = |x_1 + x_2| \quad \text{(by def:)}$$
$$\leq |x_1| + |x_2|$$
$$= \|x_1\| + \|x_2\|$$

ie $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$ $\forall\ x_1, x_2 \in \mathbb{R}$.

(b) For any scaler α and $x \in \mathbb{R}$, we have

$$\|\alpha x\| = |\alpha x| = |\alpha||x| = |\alpha|\,\|x\|$$

(c) For any $x \in \mathbb{R}$, we have:

$$\|x\| = |x| \geq 0 \quad \Rightarrow \quad \|x\| \geq 0.$$

(d) For any $x \in \mathbb{R}$, we have:

$$\|x\| = |x|$$

Thus $\|x\| = 0 \Leftrightarrow |x| = 0 \Leftrightarrow x = 0.$

ie $\|x\| = 0 \Leftrightarrow x = 0.$

(2) The complex linear space $\mathbb{C}$ is a normed linear space with the norm defined by:

$$\|z\| = |z| \;;\; \forall\, z \in \mathbb{C}.$$

Pf: (a) For any $z_1, z_2 \in \mathbb{C}$, we have:

$$\begin{aligned}\|z_1 + z_2\| &= |z_1 + z_2| \quad \text{(by definition)}\\ &\leq |z_1| + |z_2| \quad \text{(property of complex nos:)}\\ &= \|z_1\| + \|z_2\| \quad \text{(by def:)}\end{aligned}$$

ie $\|z_1 + z_2\| \leq \|z_1\| + \|z_2\| \;;\; \forall\, z_1, z_2 \in \mathbb{C}.$

(b) For any scaler α and $z \in \mathbb{C}$, we have:

$$\begin{aligned}\|\alpha z\| &= |\alpha z| \quad \text{(by def:)}\\ &= |\alpha||z| \quad \text{(property of complex nos:)}\\ &= |\alpha|\,\|z\| \quad \text{(by def:)}\end{aligned}$$

(c) For any $z \in \mathbb{C}$, we have:

$$\|z\| = |z| = 0 \text{ iff } z = 0.$$

(d) For any $z \in \mathbb{C}$, we have:

$$\|z\| = |z| \geq 0. \quad \text{(property of complex nos:)}$$

ie $\|z\| \geq 0 \;;\forall\, z \in \mathbb{C}.$

Hence the complex linear space $\mathbb{C}$ is a normed linear space with the norm defined above.

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(3) The spaces $\mathbb{R}^n$ (n-dimensional Euclidean space) and $\mathbb{C}^n$ (n-dimensional unitary space) of all n-tuples of real and complex numbers are normal linear spaces with the norms defined by:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \; ; \quad 1 \le p \le \infty$$

$$\text{i.e. } \|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p} \longrightarrow (1)$$

where $x_i = (x_1, x_2, \ldots, x_n)$

OR $\|x\| = \max\{|x_i| \; ; \; i = 1, 2, \ldots, n\}$.

$$= \max\{|x_1|, |x_2|, \ldots, |x_n|\} \longrightarrow (2)$$

where $x_i = (x_1, x_2, \ldots, x_n)$, $1 \le i \le n$

Note: we can define more than one norm on a linear space.

Next, we introduce some special normed linear spaces.

(4) $l^p(x)$, when $\mathbb{C}^n$ or $\mathbb{R}^n$ is considered as normed linear spaces with the norm (1) of Example (3), we denote the space by $l^p(x)$.

Notice that we shall use $l^p(x)$ for both the

moreover, the question of whether the space under discussion is real or complex will either be clear from the context or we shall make a specific statement if necessary.

(5) $l^p = l_p$ = The space of all sequences $x = \{x_n\}$ with $\sum_{i=1}^{\infty} |x_i|^p < \infty$, $p \ge 1$; then this space l^p is a n.l.s with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \; ; \; \forall x \in l^p$$

(6) $l^\infty = l_\infty$ is the space of all bounded sequences $x = \{x_i\}$, Then l^∞ is a n.l.s with the norm:

$$\|x\|_\infty = \sup |x_i| \quad ; \quad 1 \le i \le \infty$$
$$= \sup \{ |x_1|, |x_2|, \ldots \}.$$

(7) $C[a,b]$ is the space of all continuous real valued functions defined on $[a,b]$ ie $f : [a,b] \longrightarrow \mathbb{R}$, which is continuous.

Then $C[a,b]$ is a n.l.s with norms:

(i) $\|f\| = \sup |f(x)| \; ; \; \forall \, f \in C[a,b], \, x \in [a,b]$.

(ii) $\|f\| = \int_a^b |f(x)| \, dx \; ; \; \forall \, f \in C[a,b]$.

(8) C = This is the space of all convergent sequences in l^∞.

C_0 = This is also the space of all sequences in l^∞ converging to zero.

then C and C_0 are normed linear spaces with norm as in l^∞.

Note that $C_0 \subset C \subset l^\infty$.

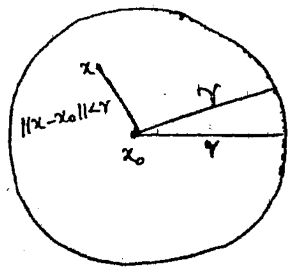
<u>Definition (1.4)</u>:- Let X be a normed linear space

(a) An open sphere (or open ball) with centre x_0 and radius $r > 0$ is the set:

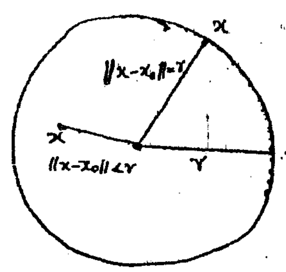
$$B(x_0; r) = \{ x \in X : \|x - x_0\| < r \}$$

A closed sphere (or ball) with centre x_0 and radius $r > 0$ is the set:

$$\bar{B}(x_0; r) = \{ x \in X : \|x - x_0\| \le r \}$$



$B(x_0; r)$

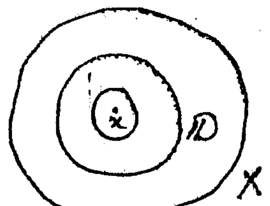


$\bar{B}(x_0; r)$

By the surface (or boundary) of this ball, we mean the set:

$$S(x_0; r) = \{x \in X : \|x - x_0\| = r\}$$

(b) A set D in X is said to be open if for every $x \in D$, there exists a ball with centre x ~~and~~ which is contained in D.



(c) A set D in X is said to be closed if for any sequence $\{x_n\}$ in D with $x_n \to x$ implies that $x \in D$.

(d) A set D is said to be bounded in X if there exists a constant M such that $\|x\| \leq M$; $\forall\, x \in D$.

(e) A set D is said to be compact if whenever $\{x_n\}$ is in D, there exists a cgt subsequence of $\{x_n\}$ whose limit is in D.

(f) A sequence $\{x_n\}$ is called bounded, if there exists a real constant $K > 0$ such that $\|x_n\| \leq K \;\; \forall\, n$.

Proposition (1.5):

ⓐ Every norm linear space X is a metric space w.r.t the metric $d(x,y) = \|x-y\|$; $\forall\, x,y \in X$.

ⓑ $\big|\|x\| - \|y\|\big| \leq \|x-y\|$; $\forall\; x,y \in X$.

Proof: ⓐ let X be a norm linear space. Define a mapping $d: X \times X \to \mathbb{R}$ by:

$$d(x,y) = \|x-y\| \; ; \; \forall\, x,y \in X.$$

we show that d is a metric on X.

Since (i) $d(x,y) = \|x-y\| \geq 0$ (by def:)

ie $d(x,y) \geq 0$.

(II) $d(x,y) = \|x-y\| = 0$ iff $x-y=0$ (by def:)

iff $x=y$

ie $d(x,y) = 0$ iff $x=y$.

(iii) $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

ie $d(x,y) = d(y,x)$.

(iv) $d(x,z) = \|x-z\| = \|x-y+y-z\| \leq \|x-y\| + \|y-z\|$
$= d(x,y) + d(y,z)$

so $d(x,z) \leq d(x,y) + d(y,z)$.

Hence d is a metric on norm linear space X, known as metric induced by a norm and hence X with d is a metric space.

ⓑ $\big|\|x\| - \|y\|\big| \leq \|x-y\|$; $\forall\, x,y \in X$.

Pf: we can write : $x = x-y+y$

$\Rightarrow \|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|. \longrightarrow (1)$

similarly we can write : $y = y-x+x$.

$\Rightarrow \|y\| = \|y-x+x\| \leq \|y-x\|+\|x\|$

$\Rightarrow -\|y-x\| \leq \|x\|-\|y\|$

$\Rightarrow -\|x-y\| \leq \|x\|-\|y\| \longrightarrow ②$

Combining ① and ②, we have:

$$-\|x-y\| \leq \|x\|-\|y\| \leq \|x-y\|.$$

$$\Rightarrow \quad \big|\|x\|-\|y\|\big| \leq \|x-y\|.$$

Definition (1.6):

let X be a normed linear space and let $\{x_n\}$ be a sequence in X. Then

ⓐ we say that the sequence $\{x_n\}$ of elements of X converges to the limit $x \in X$ if for every $\varepsilon > 0$, there exists a +ve integer N such that

$\|x_n - x\| < \varepsilon$ for $n \geq N$.

In other words, we say that $\{x_n\}$ is convergent to the limit $x \in X$ iff $\lim\limits_{n\to\infty} \|x_n - x\| = 0$.

Symbolically we write $x_n \to x$ as $n \to \infty$.

ⓑ we say that the sequence $\{x_n\}$ in X is a Cauchy sequence if for every $\varepsilon > 0$, there exists a +ve integer N such that:

$\|x_m - x_n\| < \varepsilon$ for $m, n \geq N$.

In other words, $\{x_n\}$ is a Cauchy sequence iff $\lim\limits_{\substack{m\to\infty \\ n\to\infty}} \|x_m - x_n\| = 0$.

Exercise (1.7): let X be a norm linear space.

① If The limit of a sequence $\{x_n\}$ in X exists Then it is unique.

② Every Convergent sequence in X is a Cauchy sequence, but The Converse is not true, in general.

③ A Cauchy sequence is Convergent iff it has a Convergent subsequence.

④ Every Cauchy sequence in X is bounded.

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Proposition (1.8): let X be a norm linear space

ⓐ Norm is a Continuous function

ie if $x_n \to x$ Then $\|x_n\| \to \|x\|$

OR If $\{x_n\}$ is a Convergent sequence in X, Then $\|x_n\|$ is a Convergent sequence in $\mathbb{R}$.

ⓑ Addition and scaler multiplication are jointly Continuous in X ie If $x_n \to x$ and $y_n \to y$ Then $x_n + y_n \to x + y$.

and if $x_n \to x$ and $\alpha_n \to \alpha$, Then $\alpha_n x_n \to \alpha x$.

ⓒ If $x_n \to x$ and $y_n \to y$, Then $ax_n + by_n \to ax + by$ where a and b are Constants.

Proof: ⓐ Since $x_n \to x$. So by definition:

$$\lim_{n \to \infty} \|x_n - x\| = 0 \quad \text{①}$$

Now $\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|$ [using (1.5) (b)]

so $\lim_{n \to \infty} \big|\|x_n\| - \|x\|\big| \leq \lim_{n \to \infty} \|x_n - x\| = 0$ (by ①)

Thus $\|x_n\| \to \|x\|$ ie norm is a Continuous function.

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(b) Since $x_n \to x$ and $y_n \to y$. So by definition

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\| = 0$$

Now
$$\|(x_n + y_n) - (x+y)\| = \|x_n - x + y_n - y\|$$
$$\leq \|x_n - x\| + \|y_n - y\|.$$

So
$$\lim_{n\to\infty} \|(x_n+y_n) - (x+y)\| \leq \lim_{n\to\infty} \|x_n - x\| + \lim_{n\to\infty} \|y_n - y\|$$
$$= 0 + 0 \quad \text{(from above)}.$$
$$= 0$$

Hence $x_n + y_n \longrightarrow x+y$

Next we show that $\alpha_n x_n \longrightarrow \alpha x$.

Since $\alpha_n \to \alpha$, so by definition; we have

$$\lim_{n\to\infty} |\alpha_n - \alpha| = 0$$

Now
$$\|\alpha_n x_n - \alpha x\| = \|\alpha_n x_n - \alpha_n x + \alpha_n x - \alpha x\|$$
$$\leq \|\alpha_n x_n - \alpha_n x\| + \|\alpha_n x - \alpha x\|.$$
$$= \|\alpha_n (x_n - x)\| + \|x(\alpha_n - \alpha)\|$$
$$= |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\,\|x\|.$$

So
$$\lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| \leq \lim_{n\to\infty} |\alpha_n|\|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha|\,\|x\|.$$
$$= |\alpha_n| \lim_{n\to\infty}\|x_n - x\| + \lim_{n\to\infty} |\alpha_n - \alpha|\,\|x\|.$$
$$= 0 + 0 \quad \text{(from above)}$$
$$= 0$$

Hence $\alpha_n x_n \longrightarrow \alpha x$

ie scalar multiplication and addition are jointly continuous.

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(c) Since $x_n \to x$ and $y_n \to y$, so we have:

$$\lim_{n\to\infty} \|x_n - x\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\| = 0.$$

Now $\|(ax_n + by_n) - (ax+by)\| = \|ax_n - ax + by_n - by\|$
$\leq \|ax_n - ax\| + \|by_n - by\|$

Now

$$\lim_{n\to\infty} \|(ax_n + by_n) - (ax+by)\| \leq \lim_{n\to\infty}\left(\|ax_n - ax\| + \|by_n - by\|\right)$$

$$= \lim_{n\to\infty} \|a(x_n - x)\| + \lim_{n\to\infty} \|b(y_n - y)\|$$

$$= |a| \lim_{n\to\infty} \|x_n - x\| + |b| \lim_{n\to\infty} \|y_n - y\|$$

$$= 0 + 0 \text{ (from above)}$$
$$= 0$$

Thus $ax_n + by_n \longrightarrow ax + by$

which completes the proof.

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Bounded Linear operators:

Before defining a bounded linear operator, we recall ~~some~~ definitions and results from "Algebra."

Definition: Let X and Y be linear spaces with the same scaler field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$).
Let A be a function with $D(A)$ in X and range $R(A)$ in Y [ie $A: D(A) \subset X \longrightarrow R(A) \subset Y$]

Then A is called a linear operator if $D(A)$ is a subspace of X and if:

$$\left.\begin{array}{ll} \text{(a)} & A(x_1 + x_2) = Ax_1 + Ax_2 \,;\, \forall\, x_1, x_2 \in D(A) \\ \text{(b)} & A(\alpha x) = \alpha A(x) \,;\, \forall\, \alpha \in \mathbb{K} \text{ and } x \in D(A) \end{array}\right\} \to (1)$$

Clearly condition (1) is equivalent to:

$$A(\alpha x_1 + \beta x_2) = \alpha A x_1 + \beta A x_2 \,;\, \forall\, \alpha, \beta \in \mathbb{K} \text{ and } x_1, x_2 \in D(A).$$

If $D(A) = X$, we often say that A is a linear operator on (or from) X to Y.

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Remark: (1) It follows immediately by induction from (a) and (b) of above definition that

$$A(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) = \alpha_1 A x_1 + \alpha_2 A x_2 + \cdots + \alpha_n A x_n.$$

(2) If $\alpha = 0$ in above definition, then we have $A(0) = 0$.

(3) An important subset of the domain of A is the null space of A denoted by $\mathcal{N}(A)$ and is defined by:

$$\mathcal{N}(A) = \{x \in D(A) : Ax = 0\}.$$

It is readialy verified that $\mathcal{N}(A)$ is a subspace.
[illegible] $x_1, x_2 \in$ [illegible], then $Ax_1 = 0$, $Ax_2 = 0$. Let α, β [illegible]
[illegible] $\Rightarrow \alpha x_1 + \beta x_2 \in \mathcal{N}(A)$

Examples:-

(1) The identity operator $I : X \longrightarrow X$ defined by $I(x) = x \;;\; \forall x \in X$ is clearly a linear operator from X into itself.

(2) Zero operator $T : X \longrightarrow Y$ defined by: $T(x) = 0 ; \forall x \in X$ is clearly linear operator.
Note that a zero operator is also called Null operator OR Trivial operator.

(3) Consider the linear space P of all polynomials $p(x)$ with real coefficients, defined on $[0,1]$. Then the mapping D defined by:
$D(p) = \dfrac{dp}{dx}$, is a linear operator from P into itself.

(4) The mapping T defined by: $T(f) = \int_0^1 f(x)\,dx$ is clearly seen to be a linear operator of $C[0,1]$, the space of continuous real functions defined on the closed unit interval $[0,1]$ into the real linear space of all real nos: ie $T : C[0,1] \longrightarrow \mathbb{R}$.

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Definition: (a) A mapping $T: D(T) \subset X \longrightarrow Y$ is said to be injective or one-to-one if different points in the domain has different images.

ie if for any $x_1, x_2 \in D(T)$, we have:

$x_1 \neq x_2 \Rightarrow Tx_1 \neq Tx_2$

or equivalently $Tx_1 = Tx_2 \Rightarrow x_1 = x_2$.

(b) A mapping $T: D(T) \subset X \longrightarrow Y$ is said to be surjective or onto if $R(T) = Y$ ie if every element of Y is the image of at least one element in X.

(c) If T is both injective and surjective, Then it is called bijective.

Notations: If a linear operator A has an inverse, Then it is denoted by A^{-1}. The statement "A^{-1} exists" is the same as "A has an inverse".

It is known that A^{-1} exists iff A is one-to-one

ie $Ax_1 = Ax_2 \Longrightarrow x_1 = x_2$.

Thm (A): let A be a linear operator, Then A^{-1} exists iff $Ax = 0 \Rightarrow x = 0$.

when A^{-1} exists, Then A^{-1} is a linear operator.

Thm (B): If A is a linear operator from a linear space X into a linear space Y. Then A^{-1} exists iff A is one-to-one and onto.

Theorem: Let A be a linear operator, then A^{-1} exists iff $Ax=0 \Rightarrow x=0$. When A^{-1} exists, it is also a linear operator.

Proof: Before proving the above result, we remember the following fact:

"The inverse of an operator A exists iff A is one-to-one i.e if $Ax_1 = Ax_2 \Rightarrow x_1 = x_2$; $\forall x_1, x_2 \in D(A)$."

Now we prove the required result.

First let us suppose that A^{-1} exists. Suppose x is an arbitrary vector in $D(A)$ such that $Ax=0$.

But as A is a linear operator, so that $A(0)=0$ i.e. $Ax = A(0)$. But A^{-1} exists, so A is one-to-one therefore $Ax = A(0) \Rightarrow x=0$.

Conversely, let us suppose that $Ax=0 \Rightarrow x=0$. We are to prove that A^{-1} exists and for this we will show that A is one-to-one.

For this let $Ax_1 = Ax_2$, where $x_1, x_2 \in D(A)$

$\Rightarrow Ax_1 - Ax_2 = 0 \Rightarrow A(x_1 - x_2) = 0$ (as A is linear)

$\Rightarrow x_1 - x_2 = 0$ [by supposition]

$\Rightarrow x_1 = x_2$

which shows that A is one-to-one.

Consequently A^{-1} exists. Hence proved.

Finally we show that when A^{-1} exists, then it is also a linear operator.

Now let $x_1, x_2 \in D(A)$, then we can find y_1, y_2 in $R(A)$ such that $Ax_1 = y_1$ and $Ax_2 = y_2$

Since A^{-1} exists, so that $x_1 = A^{-1}y_1$ and $x_2 = A^{-1}y_2$.

Now $y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2)$ [$\because A$ is Linear]

Since A^{-1} exists, so $A^{-1}(y_1 + y_2) = x_1 + x_2 = A^{-1}(y_1) + A^{-1}(y_2)$

Again let $\alpha \in \mathbb{K}$ and consider αy_1.

Now $\alpha y_1 = \alpha(Ax_1) = A(\alpha x_1)$ [$\because A$ is linear]

$\Rightarrow A^{-1}(\alpha y_1) = \alpha x_1 = \alpha A^{-1}(y_1)$ [$\because A^{-1}$ exists]

$\Rightarrow A^{-1}(\alpha y_1) = \alpha A^{-1}(y_1)$

Hence A^{-1} is also a linear operator.

Theorem: A^{-1} exists iff $N(A)=\{0\}$, when A is linear operator.

Proof: First we recall that A^{-1} exists iff A is one-one. Now suppose that A^{-1} exists, we prove that $N(A)=\{0\}$. For this let $x\in N(A)$, so by def:, $Ax=0$. But as A is a linear operator, so $A(0)=0$ therefore $Ax=A(0)$. Since A^{-1} exists, so A is one-one. Hence $x=0$. Therefore $N(A)=\{0\}$.

Conversely suppose that $N(A)=\{0\}$ and we show that A^{-1} exists and to show that A^{-1} exists, we show that A is one-to-one. For this let $Ax_1 = Ax_2$, where $x_1, x_2 \in D(A)$

$\Rightarrow Ax_1 - Ax_2 = 0$

$\Rightarrow A(x_1 - x_2) = 0$ [$\because A$ is linear]

$\Rightarrow x_1 - x_2 \in N(A) = \{0\}$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Thus A is one-to-one, Consequently A^{-1} exists. This completes the required proof.

Definition (1.9): Let X and Y be two normed linear spaces over a field $\mathbb{K}$ and $T: X \longrightarrow Y$ be a linear operator, Then

ⓐ we say that T is continuous at $x_0 \in X$ if for every $\epsilon > 0$, There exists $\delta > 0$ such that
$\|Tx - Tx_0\| <$ whenever $\|x - x_0\| < \delta$.

ⓑ we say that T is continuous on X if it is continuous for every point of X.

OR T is continuous on X iff for any sequence $\{x_n\}$ in X with $x_n \longrightarrow x$ implies $Tx_n \longrightarrow Tx$.

ⓒ T is continuous at the origin iff $x_n \longrightarrow 0$ implies $Tx_n \longrightarrow 0$.

ⓓ we say that T is uniformly continuous on X if for ~~every~~ any $x_1, x_2 \in X$ and $\epsilon > 0$, There exists $\delta > 0$ such that:
$\|Tx_1 - Tx_2\| < \epsilon$ whenever $\|x_1 - x_2\| < \delta$.

Proposition (1.10): ⓐ A uniformly continuous function is continuous.

ⓑ A continuous function on a compact space is uniformly continuous.

Definition (1.11): An operator $T: X \longrightarrow Y$ is said to be bounded if there exists a constant $M > 0$: such that $\|Tx\| \leq M\|x\|$; $\forall x \in X$.

Theorem (1.12) Let $T: X \to Y$ be a linear operator from a n.l. space X into a n.l.s Y; Then

(a) If T is continuous at some point $x_0 \in X$, then T is uniformly continuous.

(b) T is (uniformly) continuous iff T is bounded.

Proof: (a) Let T be continuous at some point $x_0 \in X$, then by definition, for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$\|Tx - Tx_0\| < \varepsilon$ whenever $\|x - x_0\| < \delta$ ⟶ (1)

Let y_1, y_2 be any two points in X.

Let $w = y_1 - y_2 + x_0$, Then $w \in X$ because X is a linear space (closed under addition).

Suppose $\|w - x_0\| < \delta$, Then by (1), we have:

$\|Tw - Tx_0\| < \varepsilon$

i.e. $\|y_1 - y_2 + x_0 - x_0\| < \delta$ implies $\|T(y_1 - y_2 + x_0) - Tx_0\| < \varepsilon$

i.e. $\|y_1 - y_2\| < \delta$ implies $\|Ty_1 - Ty_2 + Tx_0 - Tx_0\| < \varepsilon$. ($\because T$ is linear operator)

i.e. $\|Ty_1 - Ty_2\| < \varepsilon$ whenever $\|y_1 - y_2\| < \delta$.

Thus T is uniformly continuous on X.

Note: The converse of this result is also true, because by proposition (1.10 (a)), we have: "Every unifi. continuous function is continuous".

(b) Suppose that T is bounded. So by definition there exists a constant $M > 0$ such that

$\|Tx\| \leq M\|x\|$, $\forall x \in X$.

Now consider any point $x_0 \in X$. let $\epsilon > 0$ be given. then for every $x \in X$ such that $\|x - x_0\| < \delta$ where $\delta = \epsilon/M$, we have:

$\|Tx - Tx_0\| = \|T(x - x_0)\|$ ($\because T$ is linear)

$\leq M\|x - x_0\|$ ($\because T$ is bounded)

$\leq M.\delta$

$= M.\epsilon/M$

$= \epsilon$

ie $\|Tx - Tx_0\| < \epsilon$ whenever $\|x - x_0\| < \delta$

Since x_0 was an arbitrary point of X, This result shows that T is continuous on X.

$\Rightarrow$ T must be continuous at some point of X.

Therefor by part ⓐ, it is uniformly continuous.

Conversely, if T is continuous at origin, then there exits $\delta > 0$ such that:

$\|Tu\| \leq 1$ if $\|u\| \leq \delta$ ($\because T0 = 0$)

Given any $x \in X$, we may write:

$x = cu$, where $\|u\| = \delta$ and $c = \delta^{-1}\|x\| > 0$ ie c is Constt.

$\because X$ is L.S

then $Tx = T(cu) \Rightarrow \|Tx\| = \|T(cu)\| = c\|Tu\|$

$\leq c$ ($\because \|Tu\| \leq 1$)

$= \delta^{-1}\|x\|$

If we put $m = \delta^{-1}$, Then we have,

$\|Tx\| \leq m\|x\|$ $\forall x \in X$, which shows that T is bounded.

Proof (b): suppose that T is continuous on X, then the statement " T is continuous at some point of X " is obviously true.

Conversely, suppose that T is continuous at some point $x_0 \in X$, then by definition for every $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$\|Tx - Tx_0\| < \varepsilon \quad \text{whenever} \quad \|x - x_0\| < \delta, \ \forall x \in X. \longrightarrow (i)$$

we show that T is continuous on X.

For this let y be any arbitrary point of X, then we can write: $x - y = (x - y + x_0) - x_0$

Clearly $x - y + x_0 \in X$ ($\because X$ is a linear space)

Now

Since the condition (i) is true $\forall x \in X$ and since $x - y + x_0 \in X$; so by (i), we can write:

$$\|T(x - y + x_0) - Tx_0\| < \varepsilon \quad W \quad \|(x - y + x_0) - x_0\| < \delta$$

$$\Rightarrow \|Tx - Ty + Tx_0 - Tx_0\| < \varepsilon \quad W \quad \|x - y + x_0 - x_0\| < \delta$$

$$\Rightarrow \|Tx - Ty\| < \varepsilon \quad W \quad \|x - y\| < \delta$$

$\Rightarrow$ T is continous at y. But y was an arbitrary point of X, so T is continuous on every point of X, consequently T is continuous on X.

Pf: (c):- suppose that T is bounded, then by definition there exists a +ve constant M such that:

$$\|Tx\| \leq M\|x\| \ ; \ \forall x \in X \longrightarrow (*)$$

we show that T is continuous on X.

For this let $\{x_n\}$ be a sequence in X such that $x_n \to x$. In order to show that T is continuous on X, we show that $Tx_n \to Tx$.

Now since $\{x_n\}$ is a sequence in X, so by condition (*), we have:

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Theorem (1.13): let X and Y be norm linear spaces and $T: X \to Y$ be a linear operator, then:

ⓐ T is continuous on X iff it is uniformly contin on X.

ⓑ T is contin: on X iff it is continuous at some point of X.

ⓒ T is continuous on X iff it is bounded.

Proof: ⓐ suppose that T is continuous on X, then it is continuous at every point of X.
let $x_0 \in X$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|T(x) - T(x_0)\| < \epsilon$ whenever $\|x - x_0\| < \delta$; (i)
we shall show that T is uniformly continuous on X.
For this let x_1, x_2 be any two points of X and let $w = x_1 - x_2 + x_0$, then $w \in X$ ($\because X$ is a linear space)
so replacing x by w in (i), we get:

$$\|T(w) - T(x_0)\| < \epsilon \text{ whenever } \|w - x_0\| < \delta$$

i.e $\|T(x_1 - x_2 + x_0) - T(x_0)\| < \epsilon$ W $\|x_1 - x_2 + x_0 - x_0\| < \delta$

i.e $\|Tx_1 - Tx_2 + Tx_0 - Tx_0\| < \epsilon$ W $\|x_1 - x_2\| < \delta$

i.e $\|Tx_1 - Tx_2\| < \epsilon$ W $\|x_1 - x_2\| < \delta$

which shows that T is uniformly continuous on X.

Conversely, suppose that T is uniformly continuous on X. then by definition, for every $\epsilon > 0$, there exists $\delta > 0$ such that:

$$\|Tx_1 - Tx_2\| < \epsilon \text{ W } \|x_1 - x_2\| < \delta \;;\; \forall x_1, x_2 \in X$$

$\Rightarrow T$ is continuous at $x_2 \in X$. But $x_2 \in X$ is an arbitrary point of X, so T is continuous on X. this completes the proof.

$$\|T(x_n - x)\| \leq M\|x_n - x\|$$

$$\Rightarrow \|Tx_n - Tx\| \leq M\|x_n - x\| \quad (\because T \text{ is linear})$$

$$\Rightarrow \lim_{n\to\infty} \|Tx_n - Tx\| \leq \lim_{n\to\infty} M\|x_n - x\|$$

$$= M \lim_{n\to\infty} \|x_n - x\|$$

$$= 0 \quad (\because x_n \to x)$$

$$\Rightarrow \lim_{n\to\infty} \|Tx_n - Tx\| = 0 \quad (\because \text{norm is always greater or equal to zero})$$

$$\Rightarrow Tx_n \longrightarrow Tx.$$

Hence T is continuous on X.

Conversely, suppose that T is continuous on X, we shall show that T is bounded. On contrary let us suppose that T is unbounded, then we can find a sequence $\{x_n\}$ in X such that:

$$\|Tx_n\| > n\|x_n\| \quad \forall n. \Rightarrow \frac{\|Tx_n\|}{n\|x_n\|} > 1 \quad \forall n.$$

Let us choose $y_n = \dfrac{x_n}{n\|x_n\|}$, then $y_n \in X$ as X is a linear space.

$$\Rightarrow T(y_n) = T\left(\frac{x_n}{n\|x_n\|}\right)$$

$$\Rightarrow \|Ty_n\| = \left\|T\left(\frac{x_n}{n\|x_n\|}\right)\right\| = \frac{\|Tx_n\|}{n\|x_n\|} > 1 \quad (\text{by above})$$

i.e. $\|Ty_n\| > 1$.

Since $y_n = \dfrac{x_n}{n\|x_n\|} \Rightarrow \|y_n\| = \left\|\dfrac{x_n}{n\|x_n\|}\right\| = \dfrac{\|x_n\|}{n\|x_n\|} = \dfrac{1}{n}$

$$\Rightarrow \|y_n\| = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$\Rightarrow y_n \to 0 \text{ as } n \to \infty$$

$$\Rightarrow Ty_n \to T(0) = 0 \quad (\because T \text{ is continuous on } X)$$

$$\Rightarrow Ty_n \to 0 \Rightarrow \|Ty_n\| \to 0$$

$$\Rightarrow \|Tx_n\| \to 0 \quad \left(\because \|Ty_n\| = \frac{\|Tx_n\|}{n\|x_n\|}\right)$$

So $\|Ty_n\| = \dfrac{\|Tx_n\|}{n\|x_n\|} = 0 < 1$, which is a contradiction to the fact that $\|Ty_n\| > 1$, so our supposition was wrong and hence T is bounded. #

Definition (1.14):

let X and Y be two normed linear spaces and let $T: X \longrightarrow Y$ be a bounded (Continuous) linear operator, then the norm of T is defined as:

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

The norm of T is also defined by the following formulae.

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| \quad \text{and} \quad \|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}.$$

Theorem (1.15): let $T: X \longrightarrow Y$ be a Continuous (bounded) linear operator from a n.l. space X into a n.l. space Y, Then

ⓐ $\|T\| < \infty$ ⓑ $\|Tx\| \leq \|T\|\|x\|$; $\forall x \in X$.

Proof: ⓐ Since T is a bounded linear operator, so by definition, There exists a constant say $M > 0$ such That $\|Tx\| \leq M\|x\|$; $\forall x \in X$

Then $\sup_{\|x\|=1} \|Tx\| \leq M \sup_{\|x\|=1} \|x\|$

$$\Rightarrow \sup_{\|x\|=1} \|Tx\| \leq M \cdot 1 \Rightarrow \sup_{\|x\|=1} \|Tx\| \leq M.$$

$$\Rightarrow \|T\| \leq M < \infty \quad (\text{by def. of } \|T\|)$$

$$\Rightarrow \|T\| < \infty.$$

ⓑ If $x = 0$, Then the inequality is obvious.

If $x \neq 0$, Then put $y = \frac{x}{\|x\|}$ so That $\|y\| = 1$

Thus $Ty = T\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|} \cdot Tx$. ($\because T$ is linear $\to T(\alpha x) = \alpha Tx$)

$$\Rightarrow \|Ty\| = \frac{\|Tx\|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \|T\| \quad \to (1).$$

$\|y\| = \left\|\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|} \cdot \|x\| = 1$

Also $\|Ty\| = \frac{\|Tx\|}{\|x\|}$ gives

$$\|Tx\| = \|Ty\|\,\|x\| \leq \|T\|\,\|x\| \quad (\text{by } ①)$$

$$\Rightarrow \|Tx\| \leq \|T\|\,\|x\| \;;\; \forall\, x \in X. \quad \#\text{proved}$$

Proposition: Let T be a bounded linear operator. Then $x_n \to x \Rightarrow Tx_n \to Tx$.

Proof: Since $x_n \to x$. So that $\|x_n - x\| \to 0$ as $n \to \infty$
or $\lim_{n\to\infty} \|x_n - x\| = 0$.

Now $\|Tx_n - Tx\| = \|T(x_n - x)\|$ $\quad$ ($\because T$ is linear)

$\leq \|T\|\,\|x_n - x\|$ $\quad$ ($\because T$ is bounded)

$\longrightarrow 0$ as $n \to \infty$

ie $\lim_{n\to\infty} \|Tx_n - Tx\| = 0$

Hence $Tx_n \to Tx$ as $n \to \infty$.

Theorem (1.16): Suppose $T: X \to Y$ be a linear operator where X and Y are n.l. spaces. Then T^{-1} exists and is continuous on its domain of definition iff there exists a constant $m > 0$ such that:

$$m\|x\| \leq \|Tx\| \;;\; \forall\, x \in X.$$

Proof:- Suppose there exists a constant $m > 0$ such that $m\|x\| \leq \|Tx\| \;;\; \forall\, x \in X \longrightarrow ①$

In order to prove that T^{-1} exists, it is enough to show that $Tx = 0 \Rightarrow x = 0$. (Thm A).

Suppose that $Tx = 0$, then ① becomes:

$$m\|x\| \leq \|0\| = 0 \Rightarrow m\,\|x\| = 0 \Rightarrow \|x\| = 0$$
$$\Rightarrow x = 0.$$

i.e $Tx=0$ implies $x=0$

Thus T^{-1} exists.

Now To prove The Continuity of T^{-1}, we define $Tx=y$, where $x\in X$ and $y\in Y$.

Since T^{-1} exists, so $T^{-1}y=x$.

Hence from ①, we have:

$m\|T^{-1}y\|\le\|y\| \Rightarrow \|T^{-1}y\|\le\frac{1}{m}\|y\|$

for all y in the range of T, which is the domain of T^{-1}.

~~So by Thm (1.12)~~

so that T^{-1} is bounded and by Thm (1.12) T^{-1} is Continuous.

Conversely, if T^{-1} exists and is Continuous, Then by Thm (1.12), T^{-1} is bounded and so we have:

$\|T^{-1}y\|\le\frac{1}{m}\|y\|$; $\forall$ y in The range of T.

i.e $m\|T^{-1}y\|\le\|y\|$

But $Tx=y$ or $T^{-1}y=x$. so that

$m\|x\|\le\|Tx\|$; $\forall x\in X$.

which Completes The required Proof.

Definition (1.17):

① let X and Y be two normed linear spaces. A mapping T on X into Y is called an Isomorphism if the following conditions are satisfied.

ⓐ $T(x+y) = Tx + Ty$ ⓑ $T(\alpha x) = \alpha (Tx)$ $\forall x, y \in X$ and any scalar α.

ⓒ T is one-to-one.

ie if T is linear and one-to-one.

The spaces X and Y are said to be Isomorphic if There exists an Isomorphism of X onto Y (ie $T(X) = Y$).

② An Isometric Isomorphism between two normed linear spaces X and Y is an Isomorphism $T: X \longrightarrow Y$ such That

$\|Tx\| = \|x\|$; $\forall x \in X$.

The spaces X and Y are said to be Isometrically Isomorphic or Congruent if there exists an Isometric Isomorphism of X onto Y.

③ A topological Isomorphism between two normed linear spaces X and Y is an Isomorphism $T: X \longrightarrow Y$ such That T and T^{-1} are continuous in Their respective domain.

The spaces X and Y are said to be topologically Isomorphic if there is a homeomorphism T of X onto Y That is also linear operator. For This reason X and Y may be called linearly homeomorphic.

Remark (1.18):

① In order to show that X and Y are congruent, it is necessary and sufficient that there exists a linear operator T with domain X and range Y such that $\|Tx\| = \|x\|$; $\forall x \in X$.

② Two norm linear spaces may be Isomorphic but not necessarly congruent. (Find example).

③ Topological Isomorphism is an equivalence relation ie it is reflexive, symmetric and Transitive.

Theorem (1.19): If X and Y are norm linear spaces they are topologically Isomorphic iff there exists a linear operator T with domain X and range Y and +ve constants m, M such that:

$$m\|x\| \le \|Tx\| \le M\|x\| \;;\; \forall x \in X \longrightarrow ①$$

Proof: suppose that there exists a linear operator T with domain X and range Y and +ve constants m, M such that ① is satisfied.

we may write ① into two inequalities ie

$$m\|x\| \le \|Tx\| \;;\; \forall x \in X \hookrightarrow ②$$

and $\|Tx\| \le M\|x\|$; $\forall x \in X$. $\hookrightarrow ③$

Now by Thm (1.16) "T^{-1} exists and is continuous iff $m\|x\| \le \|Tx\|$; $\forall x \in X$ ie ② is satisfied".

Also by Thm (1.12), "T is continuous iff $\|Tx\| \le M\|x\|$ $\forall x \in X$ ie ③ is satisfied".

Hence combining the two results, we get:

T^{-1} exists and both T, T^{-1} are continuous if There exists Constants $m>0$, $M>0$ such that

$$m\|x\| \leq \|Tx\| \leq M\|x\| \; ; \forall x \in X.$$

which implies that X and Y are topologically Isomorphic iff There exists a linear operator T with domain X and range Y and positive Constants m & M such that:

$$m\|x\| \leq \|Tx\| \leq M\|x\| \; ; \forall x \in X$$

which Completes the proof of the Theorem.

T^{-1} exists ... shown ... bijective ... contin: T, T^{-1} ... shown ... homeo: T ...

Definition (1.20):

Let X be a linear space (or vector space). A norm $\|\cdot\|_1$ on X is said to be equivalent to a norm $\|\cdot\|_2$ on X iff there exists constants m, M both positive such that:

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \; ; \forall x \in X.$$

Theorem (1.21): Let X be a linear space and suppose two norms $\|x\|_1$ and $\|x\|_2$ are defined on X. These norms define the same topology on X iff there exists +ve Constants m, M such that

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_1 \; ; \forall x \in X \quad (\text{ie They are equiv;})$$

Proof: Let X_1, X_2 be the normed linear spaces that becomes with the norms $\|x\|_1$ and $\|x\|_2$ respectively. ie $X_1 = (X, \|x\|_1)$, $X_2 = (X, \|x\|_2)$.

let us define $Tx = x$ and consider T as an operator with domain X_1 and range X_2 (ie $T: X_1 \to X_2$ is linear with domain $D(T) = X_1$ & range $R(T) = X_2$).

Suppose that there exists +ve constants m, M such that $m\|x\|_1 \le \|x\|_2 \le M\|x\|_1$; $\forall x \in X$.

Since $Tx = x$, so that

$$m\|x\|_1 \le \|Tx\|_2 \le M\|x\|_1 \;;\; \forall x \in X.$$

Hence by Thm (1.19):

$m\|x\|_1 \le \|Tx\|_2 \le M\|x\|_1$; $\forall x \in X$ iff

X_1 and X_2 are topologically Isomorphic iff

T^{-1} exists and both T and T^{-1} are Continuous iff

the open sets in X_1 are the same as the open sets in X_2 (by def: of Continuity of $\underbrace{X_1 \;\&\; X_2}$ $T \& T^{-1}$).

Thus proving that the two norms define the same topology on X; since elements (open sets) of both the topologies are same.

which completes the required proof.

Finite dimensional norm linear spaces:

Theorem (1.22): Any two norm linear spaces of same finite dimension with the same scaler field are topologically Isomorphic.

Proof: let X_1, X_2 be two norm linear spaces of the same finite dimension with the same scaler field. we need to show that X_1 is topologically Isomorphic to X_2.
the case when $n=0$ is trivial. so we may assume that $n \geq 1$. It will suffice to prove that "if X is an n-dimensional n.l-space, it is topologically Isomorphic to $l'(n)$.
In order to prove that $l'(n)$ and X are topologically Isomorphic, we need to show that there exists a linear operator T with domain $l'(n)$ and range X and +ve constants m, M such that:

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\| \;; \forall \eta \in l'(n)$$

(see Thm 1-20).

let $\{x_1, x_2, x_3, ---, x_n\}$ be a basis for X.
Define an operator $T: l'(n) \longrightarrow X$ by:

$$T(\eta) = \eta_1 x_1 + \eta_2 x_2 + \cdots + \eta_n x_n = \sum_{j=1}^{n} \eta_j x_j \quad (*)$$

for all $\eta = (\eta_1, \eta_2, ---, \eta_n) \in l'(n)$

then T is linear. we show that for all $\eta = (\eta_1, \eta_2, ---, \eta_n) \in l'(n)$, there exists $m>0$, $M>0$ such that

$$m\|\eta\| \leq \|T\eta\| \leq M\|\eta\|$$

that is

$$\|T\eta\| \le M\|\eta\| \longrightarrow ①$$

$$m\|\eta\| \le \|T\eta\| \longrightarrow ②$$

If $\eta = 0$, then ① and ② are obviously true.

If $\eta \neq 0$, then by ⊛,

$$\|T\eta\| = \left\|\sum_{j=1}^{n} \eta_j x_j\right\|$$

$$\le \sum_{j=1}^{n} \|\eta_j x_j\|$$

$$= \sum_{j=1}^{n} |\eta_j|\,\|x_j\|$$

Let us take $M = \max\{\|x_1\|, \|x_2\|, \cdots, \|x_n\|\}$

then $\|T\eta\| \le M(|\eta_1| + |\eta_2| + \cdots + |\eta_n|)$

$$= M\|\eta\| \quad [\because \eta = (\eta_1, \eta_2, \cdots, \eta_n) \in l(n)]$$

which implies that ① is true for $\eta \neq 0$.

From ②, note that:

$$m\|\eta\| \le \|T\eta\| \text{ iff } m \le \frac{\|T\eta\|}{\|\eta\|}$$

$$\text{iff } m \le \frac{\|T(\eta_1, \eta_2, \cdots, \eta_n)\|}{\|\eta\|}$$

$$\text{iff } m \le \|T(\beta)\|$$

where $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$, where

$$\beta_i = \frac{\eta_i}{\|\eta\|}, \quad \|\eta\| = |\eta_1| + |\eta_2| + \cdots + |\eta_n|$$

Then $\|\beta\| = 1$, because $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$

$$\Rightarrow \|\beta\| = \sum_{j=1}^{n} |\beta_j| = |\beta_1| + |\beta_2| + \cdots + |\beta_n|$$

$$= \frac{|\eta_1|}{\|\eta\|} + \frac{|\eta_2|}{\|\eta\|} + \cdots + \frac{|\eta_n|}{\|\eta\|}$$

$$= \frac{|\eta_1| + |\eta_2| + \cdots + |\eta_n|}{\|\eta\|}$$

$$= \frac{\|\eta\|}{\|\eta\|} = 1.$$

In order to prove (2), it is enough to show that there exists a constant $m > 0$ such that $m \leq \|T\beta\|$ for all $\beta \in l'(n)$ with $\|\beta\| = 1$.

We define a mapping $f : l'(n) \longrightarrow R$ by

$$f(\eta) = \|T\eta\| \text{ for all } \eta \in l'(n) \quad (**)$$

Then f is continuous function, because for any $\gamma \in l'(n)$, we have:

$$|f(\eta) - f(\gamma)| = \big|\, \|T\eta\| - \|T\gamma\| \,\big|$$
$$\leq \|T\eta - T\gamma\| \quad (\text{by Prop: (1.5)})$$
$$= \|T(\eta - \gamma)\| \quad (\because T \text{ is linear})$$
$$\leq c\,\|\eta - \gamma\| \quad , \text{where } c > 0$$

i.e. $|f(\eta) - f(\gamma)| \leq c\,\|\eta - \gamma\|$, $c > 0$.

putting $\delta = \varepsilon/c$, we have:

$$\|\eta - \gamma\| < \delta \Rightarrow |f(\eta) - f(\gamma)| < \varepsilon$$

Thus f is continuous at $\gamma \in l'(n)$.

But γ is choosen arbitrary in $l'(n)$. Hence f is continuous on $l'(n)$.

Now we know (from Analysis) that "the surface of the unit sphere in $l'(n)$ is compact", that is

$$K = \{\eta \in l'(n) : \|\eta\| = 1\} \text{ is Compact in } l'(n).$$

Hence the restriction of f to K namely g ie $f|_K = g$ is also Continuous, because f is Continuous.
Also we know that "A real valued function on a compact set attains its maximum and minimum".
Thus g attains its minimum, that is there exists $\eta \in K$ such that $g(\eta) \le g(\beta)$; $\forall\ \beta \in K$, which yields $\|T\eta\| \le \|T\beta\|$ (by (**)).

$$\Rightarrow \quad 0 \le m \le \|T\beta\| \text{, where } \|T\eta\| = m$$

If $m = 0$, then $\|T\eta\| = 0$ iff $T\eta = 0$ iff $\sum_{j=1}^{n} \eta_j x_j = 0$

where $\eta = (\eta_1, \eta_2, ---, \eta_n)$.

But each η_i cannot be zero, because

$\|\eta\| = 1$ (by def: of K).

So $\{x_1, x_2, \cdots, x_n\}$ is linearly dependent, which is a contradiction to the fact that $\{x_1, x_2, ---, x_n\}$ is linearly independent. Hence our supposition was wrong and so there exists a constant $m > 0$ such that $m \le \|T\beta\|$; $\forall\ \beta \in l'(n)$ with $\|\beta\| = 1$.
This implies that there exists a constant $m > 0$ such that $m\|\eta\| \le \|T\eta\|$, which is the inequality (II).
So we have proved that there exists a linear operator T with domain $l'(n)$ and range n-dimensional norm linear

space X and constants $m>0$, $M>0$ such that :

$$m\|\eta\| \le \|T\eta\| \le M\|\eta\| \quad ; \forall \eta \in l'(n).$$

Hence by Theorem (1.19), $l'(n)$ and X are topologically Isomorphic and Consequently X_1 and X_2 are topologically Isomorphic.

This Completes the required proof of the theorem.

Remark: Let X and Y are topologically Isomorphic norm linear spaces and if one of them is complete (as a metric space), then other is also complete.

Theorem (1.23): A finite dimensional norm linear space is complete.

Proof: By above remark, If X and Y are two topologically Isomorphic norm linear spaces, and if one of them is complete, then does the other.

Note that the space $l'(n)$ is topologically Isomorphic to the space $l'(1)$ (ie the real or complex field) which is complete. Thus the finite dimensional norm linear space $l'(n)$ is complete.

More generally, if X is any finite dimensional norm linear space, then we know that every finite dimensional norm linear space X is topologically Isomorphic to $l'(n)$ and hence X is complete. (by above Remark)

Theorem (1.24): If X is a norm linear space, Then every finite dimensional subspace of X is necessarily closed.

Proof: let X be a norm linear space and M be a finite dimensional subspace of X, Then by above result, it is Complete.
Then by a result stating that " Every Complete subspace of a metric space is closed", we have That M is closed.

Definition (1.25):
A metric space X is said to be Compact (or sequentially compact) if every sequence in X has a convergent subsequence.
A subset M of X is said to be Compact if every sequence in M has a convergent subsequence whose limit is an element of M.

Theorem (1.26) Continuous mapping Theorem
let X and Y be metric spaces and $T: X \to Y$ be Continuous mapping, then the image of a Complete subset M of X under T is Compact.

Theorem (1.27): If X is a finite dimensional normed linear space then each closed and bounded set in X is compact.

Proof: let X be a finite dimensional norm linear space and M be a closed and bounded set in X. we show that M is compact in X.

we know that "Any two norm linear spaces of the same finite dimencion with the same scaler field are topologically Isomorphic", so there exists a topological Isomorphism $T: X \xrightarrow{\text{onto}} l^1(n)$.

Since $M \subset X$, then $T(M) = K$, closed and bounded in $l^1(n)$. [$\because T$ is a homeomorphism], and so K is compact. [using Hiene-Borel theorem], because in space $l^1(n)$, we have from analysis that "each closed and bounded set in $l^1(n)$ is always compact".
Since T^{-1} exits (ie $T^{-1}: l^1(n) \to X$) and is continuous so using the fact that "Continuous image of compact set is compact", we can say that $T^{-1}(K) = M$ is compact in X. which completes the proof.

Theorem (1-28): If X is finite dimensional n.l. space, then each compact subset M of X is closed and bounded.

Proof: we know that "each compact set in a metric space is closed and bounded". Hence if M is a compact set in X, it must be closed and bounded. ($\because$ every n.l. space is a metric space)

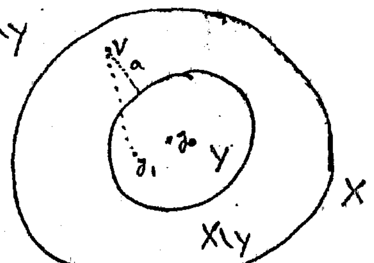
Remark: Combining Thm (1.27) and Thm (1.28), we have the following theorem.

Theorem (1.29): If X is a finite dimensional norm linear space, then each subset M of X is compact iff it is closed and bounded.

(1.30) Lemma (F. Riesz's Lemma):

Let Y be a subspace of a norm linear space X (of any dimension) such that Y is closed and a proper subset of X, then for every real number θ in the interval $(0,1)$, there exists a vector $x \in X$ such that $\|x\| = 1$ and $\|x-y\| \geq \theta$ $\forall\, y \in Y$.

Proof: We consider any vector $v \in X \setminus Y$ and denote its distance from Y by a, that is



$$a = \inf_{y \in Y} \|v-y\|$$

Clearly $a > 0$ [because norm is always non-negative but $v \in X \setminus Y$]

Since Y is closed. We know $\theta \in (0,1)$. By definition of an infimum, there is a $y_0 \in Y$ such that

$$a \leq \|v-y_0\| \leq \frac{a}{\theta} \quad \left(\text{since } \theta \in (0,1), \text{ so } a < \frac{a}{\theta}\right)$$

Let $x = c(v-y_0)$, where $c = \dfrac{1}{\|v-y_0\|}$

Then $\|x\| = \|c(v-y_0)\| = c\|v-y_0\| = \dfrac{1}{\|v-y_0\|} \cdot \|v-y_0\|$

i.e. $\|x\| = 1$.

and we remain to show that $\|x-y\| \geq \theta \ \forall y \in Y$.

Now $\|x-y\| = \|c(v-y_0)-y\| = c\,\|(v-y_0) - c^{-1}y\|$

$$= c\,\|v-(y_0+c^{-1}y)\|$$

$$= c\,\|v-y_1\| \ , \text{ where } y_1 = y_0 + c^{-1}y .$$

The form of y_1 shows that $y_1 \in Y$ ($\because Y$ is a subspace)

Hence $\|v-y_1\| \geq a$ $\left(\because a = \inf\limits_{y\in Y} \|v-y\| \text{, by graph}\right)$.

Now
$$\begin{aligned}
\|x-y\| &= c\,\|v-y_1\| \\
&\geq c \cdot a \\
&= \frac{1}{\|v-y_0\|}\cdot a \qquad \left(\because c = \frac{1}{\|v-y_0\|}\right) \\
&\geq \frac{a}{a/\theta} \qquad \left(\because a \leq \|v-y_0\| \leq \frac{a}{\theta}\right) \\
&= \theta
\end{aligned}$$

So that $\|x-y\| \geq \theta$, where $\theta \in (0,1)$.

Since $y \in Y$ was choosen arbitrary; therefore $\|x-y\| \geq \theta$; $\forall y \in Y$. This completes the proof.

Theorem (1.31) (Converse of 1.27)

Let X be a norm linear space and suppose that the surface of the unit sphere $S = \{x \in X : \|x\| = 1\}$ in X is compact, then X is finite dimensional.

Proof: Let X be a norm linear space. We need to show that X is finite dimensional.

We assume that $\dim X = \infty$, But S is compact in X, and we show that this leads to a contradiction.

we choose any $x_1 \in S$. Define $X_1 = \langle x_1 \rangle$
ie x_1 generates a one dimensional space X_1 of X.
Then X_1 is closed (by Thm 1.12.4) and is a proper subspace of X, because $\dim X = \infty$.

Hence by Riez's Lemma, There is $x_2 \in S$ such that $\|x_2 - x_1\| \geq \theta = \frac{1}{2}$

Define $X_2 = \langle x_1, x_2 \rangle$, a two dimensional space generated by x_1, x_2 in S. So X_2 is a proper closed subspace of X. Again by Riez's Lemma, There is an $x_3 \in S$ such that for all $x \in X$, we have:

$$\|x_3 - x\| \geq \tfrac{1}{2}.$$

In particular, if $x = x_1$, then $\|x_3 - x_1\| \geq \frac{1}{2}$.

and if $x = x_2$, then $\|x_3 - x_2\| \geq \frac{1}{2}$.

Proceeding by induction, we obtain an infinite sequence $\{x_n\}$ in S such that

$$\|x_m - x_n\| \geq \tfrac{1}{2} \quad (m \neq n).$$

obviously $\{x_n\}$ Cannot have a Convergent Subsequence be cause $\{x_n\}$ itself ~~Cannot~~ is not a Convergent Sequence. This facts Contradicts the Compactness of S ($\because S$ is Compact iff every sequence in S converges to a point in S)

Hence our supposition that $\dim X = \infty$ was false, and so $\dim X < \infty$.

— * — * — * — * — * — * — * —

Theorem (1.32): On a finite dimensional norm linear space, any two norms are equivalent.

Proof: Before proving this result, we state the following lemma.

"If $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set in n.l. space, then there exists a constant $c>0$ such that for each scalers $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq c \sum_{i=1}^{n} |\alpha_i| \text{"}.$$

Now we prove the required result.

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for X.

If $x \in X$, so it can be uniquely expressed as

$$x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n = \sum_{i=1}^{n} \xi_i x_i \qquad \text{①}$$

where ξ_i are scalers.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms defined on X.

By above lemma, there exists a constant $c>0$ such that:

$$\|x\|_1 = \left\| \sum_{i=1}^{n} \xi_i x_i \right\|_1 \geq c \sum_{i=1}^{n} |\xi_i| \qquad \text{②}$$

Since $\|\cdot\|_2$ is a norm on X, so

$$\begin{aligned}
\|x\|_2 &= \left\| \sum_{i=1}^{n} \xi_i x_i \right\|_2 \\
&\leq \sum_{i=1}^{n} \|\xi_i x_i\|_2 \qquad (\because \|x_1+x_2\| \leq \|x_1\| + \|x_2\|) \\
&= \sum_{i=1}^{n} |\xi_i| \, \|x_i\|_2 \\
&\leq K \sum_{i=1}^{n} |\xi_i|
\end{aligned}$$

[illegible] expand [illegible]

where $K = \max \|x_i\|_2$, $1 \le i \le n$.

$c > 0$
$K = \max \|x_i\|_2$
so $K > 0$
$\therefore c/K > 0$.

so that $\frac{c}{K} \|x\|_2 \le \frac{c}{K} \cdot K \sum_{i=1}^{n} |\xi_i|$

$$\Rightarrow m \|x\|_2 \le c \sum_{i=1}^{n} |\xi_i| \quad , \text{ where } m = c/K.$$

$\therefore m > 0$

$$\Rightarrow m \|x\|_2 \le \|x\|_1 \longrightarrow (3)$$

If we interchange the role of $\|\cdot\|_1$ and $\|\cdot\|_2$, we get:

$\|x\|_1 \le M \|x\|_2$, where $M > 0$. $\longrightarrow (4)$

Combining (3) and (4), we get:

$m\|x\|_2 \le \|x\|_1 \le M \|x\|_2$ where $M > 0$, $m > 0$.

Hence $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Proposition (1.33): If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, p and q are Holder Conjugate (or simply Conjugate) of each other, then for $a \ge 0$, $b \ge 0$, we have the following inequality $a^{1/p} \cdot b^{1/q} \le \frac{a}{p} + \frac{b}{q}$.

Proof: If $a = 0$ or $b = 0$, proposition is clearly satisfied.

we assume the case when both $a > 0$, $b > 0$.

Now if $k \in (0,1)$, define $f(t)$ for $t \ge 1$ by:

$$f(t) = k(t-1) - t^k + 1. \longrightarrow (1)$$

Note that $f(1) = 0$ and $f(t) \ge 0$ for all other values of t.

we have $0 \le f(t) = k(t-1) - t^k + 1$

$$\Rightarrow t^k \le kt + (1-k) \longrightarrow (2)$$

If $a \geq b$, then put $t = \frac{a}{b}$ and $k = \frac{1}{p}$.

so that ② becomes:

$$\left(\frac{a}{b}\right)^{1/p} \leq \frac{1}{p}\left(\frac{a}{b}\right) + \left(1 - \frac{1}{p}\right)$$

$$\Rightarrow a^{1/p} \cdot b^{-1/p} \leq \frac{1}{p} \cdot \frac{a}{b} + \frac{1}{q} \quad \left(\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{q} = 1 - \frac{1}{p}\right)$$

$$\Rightarrow a^{1/p} \cdot b^{1-1/p} \leq \frac{a}{p} + \frac{b}{q} \quad (\text{mult: by } b)$$

$$\Rightarrow a^{1/p} \cdot b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

Now if $a < b$, then put $t = b/a$, $k = 1/q$

so that ② becomes:

$$\left(\frac{b}{a}\right)^{1/q} \leq \frac{1}{q} \cdot \frac{b}{a} + (1 - 1/q)$$

$$\Rightarrow a^{-1/q} \cdot b^{1/q} \leq \frac{1}{q} \cdot \frac{b}{a} + \frac{1}{p} \left(\because \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \frac{1}{p} = 1 - 1/q\right)$$

$$\Rightarrow a^{1-\frac{1}{q}} \cdot b^{1/q} \leq \frac{b}{q} + \frac{a}{p} \quad (\text{mult: by } a)$$

$$\Rightarrow a^{1/p} \cdot b^{1/q} \leq \frac{a}{p} + \frac{b}{q}.$$

which completes the proof.

(1.34) Hölder Inequality:

If $1 < p < \infty$ and $x = (x_1, x_2, \dots, x_n)$,

$y = (y_1, y_2, \dots, y_n)$; Then

$$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \, \|y\|_q \quad \text{, where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proof: we know that for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \text{ and } \|y\|_q = \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}$$

If $x = 0$ or $y = 0$, Then The inequality is obvious. we assume that x, y are both non-zero, Then assume that:

$$a_i = \left(\frac{|x_i|}{\|x\|_p} \right)^p, \quad b_i = \left(\frac{|y_i|}{\|y\|_q} \right)^q \longrightarrow (1)$$

Then by proposition (1.33), we have:

$$a_i^{1/p} \cdot b_i^{1/q} \le \frac{a_i}{p} + \frac{b_i}{q} \longrightarrow (2)$$

Therefore
$$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} = \frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_q}$$
$$= a_i^{1/p} \cdot b_i^{1/q} \quad [\text{by } (1)]$$
$$\le \frac{a_i}{p} + \frac{b_i}{q} \quad [\text{by } (2)].$$
$$= \frac{\left(\frac{|x_i|}{\|x\|_p} \right)^p}{p} + \frac{\left(\frac{|y_i|}{\|y\|_q} \right)^q}{q} \quad [\text{by } (1)]$$

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ie $$\frac{|x_i y_i|}{\|x\|_p \|y\|_q} \le \frac{\left(\frac{|x_i|}{\|x\|_p}\right)^p}{p} + \frac{\left(\frac{|y_i|}{\|y\|_q}\right)^q}{q}.$$

Taking finite summation of both sides,

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \le \sum_{i=1}^{n} \frac{\left(\frac{|x_i|}{\|x\|_p}\right)^p}{p} + \sum_{i=1}^{n} \frac{\left(\frac{|y_i|}{\|y\|_q}\right)^q}{q}$$

$$= \sum_{i=1}^{n} \frac{\frac{|x_i|^p}{\|x\|_p^p}}{p} + \sum_{i=1}^{n} \frac{\frac{|y_i|^q}{\|y\|_q^q}}{q}. \quad \text{③}$$

Thus by above recall and ③, we have:

$$\sum_{i=1}^{n} \frac{|x_i y_i|}{\|x\|_p \|y\|_q} \le \frac{\frac{\|x\|_p^p}{\|x\|_p^p}}{p} + \frac{\frac{\|y\|_q^q}{\|y\|_q^q}}{q}$$

$$= \frac{1}{p} + \frac{1}{q}.$$

$$= 1 \quad \text{(Given)}$$

$\Rightarrow \sum_{i=1}^{n} |x_i y_i| \le \|x\|_p \|y\|_q$ for $x = (x_1, x_2, \ldots, x_n)$
$y = (y_1, y_2, \ldots, y_n)$

which completes the proof.

Remark: when $p = q = 2$; we have:

$\sum_{i=1}^{n} |x_i y_i| \le \|x\|_2 \|y\|_2$, which is called the Cauchy's inequality.

(1.35) Minkowski's Inequality:

If $1 \le p < \infty$ and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, then $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

Proof: For $p = 1$, the inequality is simply the triangle inequality. So we assume that $1 < p < \infty$.

We have $x + y = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

thus $\|x+y\|_p = \left(\sum_{i=1}^{n} |x_i + y_i|^p \right)^{1/p}$

$$\text{So } \|x+y\|_p^p = \sum_{i=1}^{n} |x_i + y_i|^p$$

$$= \sum_{i=1}^{n} |x_i+y_i| \, |x_i+y_i|^{p-1}$$

$$\le \sum_{i=1}^{n} (|x_i| + |y_i|) \, |x_i+y_i|^{p-1}$$

$$= \sum_{i=1}^{n} |x_i| \, |x_i+y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i+y_i|^{p-1}$$

$$\le \|x\|_p \, \|x+y\|_p^{p-1} + \|y\|_p \, \|x+y\|_p^{p-1}$$

(By applying Holder inequality)

$$= (\|x\|_p + \|y\|_p) \, \|x+y\|_p^{p-1}.$$

i.e. $\|x+y\|_p^p \le (\|x\|_p + \|y\|_p) \, \|x+y\|_p^{p-1}$.

$\Rightarrow \|x+y\|_p^{p-(p-1)} \le \|x\|_p + \|y\|_p$ (Dividing by $\|x+y\|_p^{p-1}$)

$\Rightarrow \|x+y\|_p \le \|x\|_p + \|y\|_p$

which is the desired Minkowski's inequality.

Remark: For $p = 2$, we have from above:

$$\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$$

which is the famous Schwarz's inequality.

The end chapter # 1

Written by: TAHIR HUSSAIN JAFFERY

Peshawar University.

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