Mittag-Lefflers' Expansion Theorem

Let $f(z)$ be a meromorphic function whose only singularities in the finite part of the plane are simple poles at $a_1, a_2, a_3, \cdots, a_n, \cdots,$ which can be arranged as $0 < |a_1| < |a_2| < \cdots < |a_n| < \cdots$ with residues $b_1, b_2, b_3, \cdots, b_n, \cdots,$ respectively. Suppose also that $f(z)$ is bounded i.e; $|f(z)| < M$.

Consider a sequence of closed contours $C_1, C_2, C_3, \cdots, C_n, \cdots,$ in the form of concentric circles of radii $R_1, R_2, R_3, \cdots, R_n, \cdots,$ respectively, having centers at origin, such that, (i) C_n encloses $a_1, a_2, a_3, \cdots, a_n$ and no other poles and (ii) $\lim_{n\to\infty} R_n = \infty$. Then for all values of z, except poles

$$f(z) = f(0) + \sum_{m=1}^{\infty} b_m\left(\frac{1}{z-a_m} + \frac{1}{a_m}\right).$$

Or more generally,

$$f(z) = f(0) + \sum_{m=-\infty}^{\infty} b_m\left(\frac{1}{z-a_m} + \frac{1}{a_m}\right).$$

Proof: Consider the integral $$I = \frac{1}{2\pi i}\int_{C_n} \frac{f(t)\mathrm{d}t}{t(t-z)},$$

where z is a point with in C_n other than poles of $f(z)$. The poles of integrand are at

(i) $t = a_m,\ m = 1,2,3,\cdots,n$ (simple poles)

(ii) $t = 0$ (simple pole)

(iii) $t = z$ (simple poles)

The residues at poles are

$$R_m = R_m(f, a_m) = \lim_{t\to a_m} \frac{(t-a_m)f(t)}{t(t-z)}, \qquad m = 1,2,3,\cdots,n$$

$$= \frac{b_m}{a_m(a_m - z)} \quad \because b_m \text{ is residue of } f(z) \text{ at } z = a_m \Rightarrow \lim_{t\to a_m}(t-a_m)f(t) = b_m$$

$$R(f,0) = \lim_{t\to 0}\frac{f(t)}{(t-z)} = \frac{f(0)}{-z}$$

$$R(f,z) = \lim_{t\to z}\frac{f(t)}{t} = \frac{f(z)}{z}$$

By Cauchy residues theorem

$$I = \frac{1}{2\pi i}\left[2\pi i\left(\sum_{m=1}^{n} R_m + R(f,0) + R(f,z)\right)\right]$$

$$I=\sum_{m=1}^{n}\frac{b_m}{a_m(a_m-z)}-\frac{f(0)}{z}+\frac{f(z)}{z} \quad \text{.......... (1)}$$

Now consider:-

$$I=\frac{1}{2\pi i}\int_{C_n}\frac{f(t)dt}{t(t-z)}$$

$$|I|=\left|\frac{1}{2\pi i}\int_{C_n}\frac{f(t)\mathrm{d}t}{t(t-z)}\right|=\frac{1}{2\pi|i|}\left|\int_{C_n}\frac{f(t)\mathrm{d}t}{t(t-z)}\right|\le\frac{1}{2\pi}\int_{C_n}\left|\frac{f(t)\mathrm{d}t}{t(t-z)}\right|=\frac{1}{2\pi}\int_{C_n}\frac{|f(t)|\,|\mathrm{d}t|}{|t|\,|t-z|} \quad \text{.......... (2)}$$

But $\big||t|-|z|\big|\le|t-z|\Rightarrow\frac{1}{|t-z|}\le\frac{1}{\big||t|-|z|\big|}$ .... (3)

Using (3), (2) becomes

$$|I|\le\frac{M}{2\pi\big(R_n\,|R_n-|z||\big)}\int_{C_n}|dt|, \qquad \because |f(t)|\le M \text{ and for } C_n \text{ (circle of radius } R_n \text{ with centre at origin)}, \ |t|=R_n$$

$$=\frac{M}{2\pi\big(R_n\,|R_n-|z||\big)}(2\pi R_n)=\frac{M}{|R_n-|z||}\ \to 0 \ \text{ as } R_n\to\infty \text{ (or equivalently as } n\to\infty) \qquad \because \int_{C_n}|dt|=2\pi R_n$$

$\Rightarrow$ $I\to 0$ as $R_n\to\infty$ (or equivalently as $n\to\infty$)

So, (1) takes the form, as long as $n\to\infty$

$$0=\sum_{m=1}^{\infty}\frac{b_m}{a_m(a_m-z)}-\frac{f(0)}{z}+\frac{f(z)}{z}$$

$$\frac{f(z)}{z}=\frac{f(0)}{z}-\sum_{m=1}^{\infty}\frac{b_m}{a_m(a_m-z)}$$

$$f(z)=f(0)+\sum_{m=1}^{\infty}\frac{b_m z}{a_m(z-a_m)}$$

$$f(z)=f(0)+\sum_{m=1}^{\infty}b_m\left[\frac{1}{a_m}+\frac{1}{z-a_n}\right] \qquad \because \frac{z}{a_m(z-a_m)}=\frac{1}{a_m}+\frac{1}{z-a_m}$$

Or more generally, $f(z)=f(0)+\sum_{m=-\infty}^{\infty}b_m\left(\frac{1}{z-a_m}+\frac{1}{a_m}\right)$ **Hence the theorem.**

Question 1: **Prove that** $\boldsymbol{\pi \tan(\pi z) = -\sum_{n=0}^{\infty}\left[\frac{1}{z-n-1/2}+\frac{1}{z+n+1/2}\right]}$ **and deduce that**

$$\boldsymbol{\frac{\pi^2}{\cos^2(\pi z)} = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n-1/2)^2}}$$

Solution: Let, $f(z) = \tan(\pi z) \quad \Rightarrow \quad \lim_{z\to 0} f(z) = \lim_{z\to 0}\tan(\pi z) = 0$

Since, $\lim_{z\to 0} f(z) = 0$(= defined), therefore, there is removable singularity at $z = 0$ (and hence no pole at $z = 0$).

Poles of $\tan(\pi z)$ are determined by $\cos(\pi z) = 0 \quad \Rightarrow \quad \pi z = \left(n+\frac{1}{2}\right)\pi \quad \Rightarrow \quad z = n+\frac{1}{2},\ n = 0, \pm 1, \pm 2, \pm 3, \cdots$

$$R(f, n+1/2) = \lim_{z\to n+\frac{1}{2}}\frac{\big(z-(n+1/2)\big)\sin(\pi z)}{\cos(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to n+\frac{1}{2}}\frac{\big(z-(n+1/2)\big)\pi\cos(\pi z)+\sin(\pi z)}{-\pi\sin(\pi z)} = \frac{0+(-1)^n}{-\pi(-1)^n} = -\frac{1}{\pi}$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n\left[\frac{1}{z-a_n}+\frac{1}{a_n}\right]$

$$\tan(\pi z) = 0 + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{\pi}\right)\left[\frac{1}{z-(n+1/2)}+\frac{1}{n+1/2}\right] \qquad \longrightarrow (1)$$

$$\pi\tan(\pi z) = -\left(\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)}+\frac{1}{n+1/2}\right]+\sum_{n=-1}^{-\infty}\left[\frac{1}{z-(n+1/2)}-\frac{1}{n+1/2}\right]\right)$$

Replace n by $-n$ in second summation, we get

$$\pi\tan(\pi z) = -\left(\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)}+\frac{1}{n+1/2}\right]+\sum_{n=1}^{\infty}\left[\frac{1}{z-(-n+1/2)}-\frac{1}{(-n+1/2)}\right]\right)$$

Now, replace n by $n+1$ in second summation, to get

$$\pi\tan(\pi z) = -\left(\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)}+\frac{1}{n+1/2}\right]+\sum_{n=0}^{\infty}\left[\frac{1}{z-(-n-1/2)}-\frac{1}{(-n-1/2)}\right]\right)$$

$$= -\sum_{n=0}^{\infty}\left[\frac{1}{z-n-1/2}+\frac{1}{n+1/2}+\frac{1}{z+n+1/2}-\frac{1}{n+1/2}\right]$$

$$= -\sum_{n=0}^{\infty}\left[\frac{1}{z-n-1/2}+\frac{1}{z+n+1/2}\right]$$

Hence proved.

Differentiate (1) with respect to z, we get

$$\pi\sec^2(\pi z) = \frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{(z-n-1/2)^2} \quad \Rightarrow \quad \frac{\pi^2}{\cos^2(\pi z)} = \sum_{n=-\infty}^{\infty}\frac{1}{(z-n-1/2)^2}$$

Hence proved.

Question 2: **Prove that** $\dfrac{\pi}{\sin(\pi z)} = \dfrac{1}{z} + \sum_{n=1}^{\infty} \dfrac{(-1)^n 2z}{z^2 - n^2}$

Solution: Let, $f(z) = \dfrac{\pi}{\sin(\pi z)} - \dfrac{1}{z} = \dfrac{z\pi - \sin(\pi z)}{z \sin(\pi z)}$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z\pi - \sin(\pi z)}{z\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$
$$= \lim_{z\to 0} \frac{\pi - \pi\cos(\pi z)}{\pi z\cos(\pi z) + \sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$
$$= \lim_{z\to 0} \frac{\pi^2 \sin(\pi z)}{-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) + \pi\cos(\pi z)} = 0$$

Since, $\lim_{z\to 0} f(z) = 0$(= defined), therefore, there is removable singularity at $z = 0$ (and hence no pole at $z = 0$).

The poles of $f(z)$ are determined by $\sin(\pi z) = 0, \quad z \neq 0 \quad \Rightarrow \quad \pi z = n\pi \quad \Rightarrow \quad z = n, \quad n = \pm1, \pm2, \pm3, \cdots$

$$R(f,n) = \lim_{z\to n} \frac{(z-n)(\pi z - \sin(\pi z))}{z\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$
$$= \lim_{z\to n} \frac{(z-n)(\pi - \pi\cos(\pi z)) + (\pi z - \sin(\pi z))}{\pi z\cos(\pi z) + \sin(\pi z)}$$
$$= \frac{n\pi}{n\pi(-1)^n} = (-1)^n \qquad \because \frac{1}{(-1)^n} = (-1)^n, \ \forall\, n \in \mathbb{Z}$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n \left[\dfrac{1}{z - a_n} + \dfrac{1}{a_n}\right]$

$$\frac{\pi}{\sin(\pi z)} - \frac{1}{z} = 0 + \sum_{n=-\infty}^{\infty}{}' (-1)^n \left[\frac{1}{z-n} + \frac{1}{n}\right]$$

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=-1}^{-\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{n}\right]$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=1}^{\infty} (-1)^{-n} \left[\frac{1}{z+n} - \frac{1}{n}\right] \qquad \left(\begin{array}{c}\text{by replacing } n \\ \text{by } -n \text{ in second} \\ \text{summation}\end{array}\right)$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n}\right] \qquad \because (-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n, \ \forall\, n \in \mathbb{Z}$$
$$= \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{z-n} + \frac{1}{z+n}\right] = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(-1)^n 2z}{z^2 - n^2}$$

Hence proved.

Question 3: **Prove that** $\tan z = \sum_{n=0}^{\infty} \dfrac{2z}{(n+1/2)^2\pi^2 - z^2}$

Solution: Let, $f(z) = \tan z \quad \Rightarrow \quad f(0) = 0$

Poles of $\tan z$ are determined by $\cos z = 0 \quad \Rightarrow \quad z = \left(n + \dfrac{1}{2}\right)\pi, \quad n = 0, \pm1, \pm2, \pm3, \cdots$

$$R(f,(n+1/2)\pi) = \lim_{z\to(n+1/2)\pi} \frac{(z-(n+1/2)\pi)\sin z}{\cos z} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to(n+1/2)\pi} \frac{(z-(n+1/2)\pi)\cos z + \sin z}{-\sin z} = \frac{0+(-1)^n}{-(-1)^n} = -1$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n\left[\frac{1}{z-a_n} + \frac{1}{a_n}\right]$

$$\tan z = 0 + \sum_{n=-\infty}^{\infty}(-1)\left[\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi}\right]$$

$$\tan z = -\left(\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi}\right] + \sum_{n=-1}^{-\infty}\left[\frac{1}{z-(n+1/2)\pi} - \frac{1}{(n+1/2)\pi}\right]\right)$$

Replace n by $-n$ in second summation, we get

$$\tan z = -\left(\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{z-(-n+1/2)\pi} - \frac{1}{(-n+1/2)\pi}\right]\right)$$

Now, replace n by $n+1$ in second summation, to get

$$\tan z = -\left(\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi}\right] + \sum_{n=0}^{\infty}\left[\frac{1}{z-(-n-1/2)\pi} - \frac{1}{(-n-1/2)\pi}\right]\right)$$

$$= -\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)\pi} + \frac{1}{(n+1/2)\pi} + \frac{1}{z+(n+1/2)\pi} - \frac{1}{(n+1/2)\pi}\right]$$

$$= -\sum_{n=0}^{\infty}\left[\frac{1}{z-(n+1/2)\pi} + \frac{1}{z+(n+1/2)\pi}\right] = -\sum_{n=0}^{\infty}\frac{1}{z^2-[(n+1/2)\pi]^2}$$

$$= \sum_{n=0}^{\infty}\frac{2z}{(n+1/2)^2\pi^2 - z^2}$$

Hence proved.

Question 4: **Prove that** $\frac{1}{e^z-1} = -\frac{1}{2} + \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-4n^2\pi^2}$

Solution: Let, $f(z) = \frac{1}{e^z-1} - \frac{1}{z} = \frac{z-e^z+1}{z(e^z-1)}$

$$\lim_{z\to0} f(z) = \lim_{z\to0}\frac{z-e^z+1}{z(e^z-1)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to0}\frac{1-e^z}{ze^z+e^z-1} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to0}\frac{-e^z}{ze^z+e^z+e^z} = -\frac{1}{2}$$

Since, $\lim_{z\to0} f(z) = \frac{-1}{2}$ (= defined), therefore, there is removable singularity at $z = 0$ (and hence no pole at $z = 0$).

The poles of $f(z)$ are determined by $e^z = 1 = e^{2n\pi i}$, $z \neq 0 \Rightarrow z = 2n\pi i$, $n = \pm1, \pm2, \pm3, \cdots$

$$R(f,2n\pi i) = \lim_{z\to 2n\pi i}\frac{(z-2n\pi i)(z-e^z+1)}{z(e^z-1)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to 2n\pi i}\frac{(z-2n\pi i)(1-e^z)+\ (z-e^z+1)}{ze^z+e^z-1} = \frac{2n\pi i}{2n\pi i} = 1$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n\left[\frac{1}{z-a_n}+\frac{1}{a_n}\right]$

$$\frac{1}{e^z-1} - \frac{1}{z} = -\frac{1}{2} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-2n\pi i}+\frac{1}{2n\pi i}\right]$$

$$\Rightarrow \quad \frac{1}{e^z-1} = -\frac{1}{2}+\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{1}{z-2n\pi i}+\frac{1}{2n\pi i}\right]+\sum_{n=-1}^{-\infty}\left[\frac{1}{z-2n\pi i}+\frac{1}{2n\pi i}\right]$$

$$= -\frac{1}{2}+\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{1}{z-2n\pi i}+\frac{1}{2n\pi i}\right]+\sum_{n=1}^{\infty}\left[\frac{1}{z+2n\pi i}-\frac{1}{2n\pi i}\right] \qquad \left(\begin{array}{c}\text{by replacing } n\\ \text{by } -n \text{ in second}\\ \text{summation}\end{array}\right)$$

$$= -\frac{1}{2}+\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{1}{z-2n\pi i}+\frac{1}{2n\pi i}+\frac{1}{z+2n\pi i}-\frac{1}{2n\pi i}\right] = -\frac{1}{2}+\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{1}{z-2n\pi i}+\frac{1}{z+2n\pi i}\right]$$

$$= -\frac{1}{2}+\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2+4n^2\pi^2}$$

Hence proved.

Question 5: **Prove that** $\dfrac{\sin(\alpha z)}{\sin(\pi z)} = \dfrac{2}{\pi}+\sum_{n=1}^{\infty}(-1)^n\dfrac{n\sin(\alpha n)}{z^2-n^2}$

Solution: This is exactly the solved example 6 on page 293 of the book "Fundamentals of Complex Analysis" by "Dr. Iqbal". You may see solution there.

Question 6: Construct a function $f(z)$ which is holomorphic except at the poles $z=\pm 1,\pm 2,\pm 3,\cdots$ and is such that $f(z)-\cot(\pi z)$ tends to zero at each of these poles.

Solution: Consider the function , $f(z) = z\cot(\pi z) = \dfrac{z\cos(\pi z)}{\sin(\pi z)}$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{z\cos(\pi z)}{\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to 0}\frac{-\pi z\sin(\pi z)+\cos(\pi z)}{\pi\cos(\pi z)} = \frac{1}{\pi}$$

Since, $\lim_{z\to 0} f(z) = \frac{1}{\pi}$ (= defined), therefore, there is removable singularity at $z=0$ (and hence no pole at $z=0$).

The poles of $f(z)$ are determined by $\sin(\pi z) = 0,\ \ z\neq 0 \ \Rightarrow\ \pi z = n\pi \ \Rightarrow\ z = n,\ \ n = \pm 1,\pm 2,\pm 3,\cdots$

$$R(f,n) = \lim_{z\to n}\frac{(z-n)z\cos(\pi z)}{\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to n}\frac{(z-n)(-\pi z\sin(\pi z)+\cos(\pi z))+z\cos(\pi z)}{\pi\cos(\pi z)} = \frac{n(-1)^n}{\pi(-1)^n} = \frac{n}{\pi}$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n \left[\frac{1}{z - a_n} + \frac{1}{a_n}\right]$

$$f(z) = z\cot(\pi z) = \frac{1}{\pi} + \sum_{n=-\infty}^{\infty}{}' \frac{n}{\pi}\left[\frac{1}{z-n} + \frac{1}{n}\right] = \frac{1}{\pi} + \frac{1}{\pi}\sum_{n=-\infty}^{\infty}{}' n\left[\frac{1}{z-n} + \frac{1}{n}\right]$$ **is the required function.**

Question 7: **Prove that** $\boldsymbol{\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2}}$ **and deduce the following results:**

$$(i)\ \sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \quad \text{and} \quad (ii)\ \sqrt{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdots$$

Solution: Let , $f(z) = \pi\cot(\pi z) - \frac{1}{z} = \frac{\pi\cos(\pi z)}{\sin(\pi z)} - \frac{1}{z} = \frac{\pi z\cos(\pi z) - \sin(\pi z)}{z\sin(\pi z)}$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\pi z\cos(\pi z) - \sin(\pi z)}{z\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{z\to 0}\frac{-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) - \pi\cos(\pi z)}{\pi z\cos(\pi z) + \sin(\pi z)} = \lim_{z\to 0}\frac{-\pi^2 z\sin(\pi z)}{\pi z\cos(\pi z) + \sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{z\to 0}\frac{-\pi^2[\pi z\cos(\pi z) + \sin(\pi z)]}{-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) + \pi\cos(\pi z)} = 0$$

Since, $\lim_{z\to 0} f(z) = 0 (= \text{defined})$, therefore, there is removable singularity at $z = 0$ (and hence no pole at $z = 0$).

The poles of $f(z)$ are determined by $\sin(\pi z) = 0,\quad z \neq 0 \quad\Rightarrow\quad \pi z = n\pi \quad\Rightarrow\quad z = n,\quad n = \pm 1, \pm 2, \pm 3, \cdots$

$$R(f, n) = \lim_{z\to n}\frac{(z-n)[\pi z\cos(\pi z) - \sin(\pi z)]}{z\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{z\to n}\frac{(z-n)[-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) - \pi\cos(\pi z)] + (\pi z\cos(\pi z) - \sin(\pi z))}{\pi z\cos(\pi z) + \sin(\pi z)}$$

$$= \frac{n\pi(-1)^n}{n\pi(-1)^n} = 1$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n \left[\frac{1}{z - a_n} + \frac{1}{a_n}\right]$

$$\pi\cot(\pi z) - \frac{1}{z} = 0 + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-n} + \frac{1}{n}\right]$$

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=-1}^{-\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n}\right] \qquad \left(\begin{array}{c}\text{by replacing } n \\ \text{by } -n \text{ in second} \\ \text{summation}\end{array}\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n}\right] \qquad \because (-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n,\ \forall\ n \in \mathbb{Z}$$

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\left[\frac{1}{z-n}+\frac{1}{z+n}\right]=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$ **Hence proved**.

From (1), we have $$\frac{\pi\cos(\pi z)}{\sin(\pi z)}=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

Integrating above equation (on both sides), with respect to z, between the limits 0 to z, we get

$$\Big[\log(\sin(\pi z))\Big]_{z=0}^{z}=\Big[\log z\Big]_{z=0}^{z}+\sum_{n=1}^{\infty}\Big[\log(z^2-n^2)\Big]_{z=0}^{z}$$

$$\Rightarrow\ \log(\sin(\pi z))-\lim_{z\to 0}\log(\sin(\pi z))=\log z-\lim_{z\to 0}\log z+\sum_{n=1}^{\infty}\left[\log(z^2-n^2)-\log(-n^2)\right]$$

$$\Rightarrow\ \log\left(\frac{\sin(\pi z)}{z}\right)-\lim_{z\to 0}\log\left(\frac{\sin(\pi z)}{z}\right)=\sum_{n=1}^{\infty}\log\left(\frac{z^2-n^2}{-n^2}\right)$$

$$\Rightarrow\ \log\left(\frac{\sin(\pi z)}{z}\right)-\log\left(\lim_{z\to 0}\frac{\pi\sin(\pi z)}{\pi z}\right)=\sum_{n=1}^{\infty}\log\left(\frac{z^2-n^2}{-n^2}\right)$$

$$\Rightarrow\ \log\left(\frac{\sin(\pi z)}{z}\right)-\log\pi=\sum_{n=1}^{\infty}\log\left(1-\frac{z^2}{n^2}\right)$$

$$\Rightarrow\ \log\left(\frac{\sin(\pi z)}{\pi z}\right)=\log\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

$$\Rightarrow\ \frac{\sin(\pi z)}{\pi z}=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)\qquad ----------------------\to\ (1)$$

$$\Rightarrow\ \sin(\pi z)=\pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$ **Hence proved**.

Put $z=1/2$ in (1), we get,

$$\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}}=\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n)^2}\right)$$

$$\Rightarrow\frac{2}{\pi}=\left(1-\frac{1}{(2)^2}\right)\left(1-\frac{1}{(4)^2}\right)\left(1-\frac{1}{(6)^2}\right)\left(1-\frac{1}{(8)^2}\right)\left(1-\frac{1}{(10)^2}\right)\left(1-\frac{1}{(12)^2}\right)\left(1-\frac{1}{(14)^2}\right)\cdots\quad ------\to\ (2)$$

Put $z=1/4$ in (1), we get,

$$\frac{\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}}=\prod_{n=1}^{\infty}\left(1-\frac{1}{(4n)^2}\right)$$

$$\Rightarrow \frac{4}{\sqrt{2}\,\pi}=\left(1-\frac{1}{(4)^2}\right)\left(1-\frac{1}{(8)^2}\right)\left(1-\frac{1}{(12)^2}\right)\left(1-\frac{1}{(16)^2}\right)\left(1-\frac{1}{(20)^2}\right)\left(1-\frac{1}{(24)^2}\right)\cdots \longrightarrow \quad (3)$$

Dividing (2) by (3), we get

$$\left(\frac{2}{\pi}\right)\left(\frac{\sqrt{2}\,\pi}{4}\right)=\left(1-\frac{1}{(2)^2}\right)\left(1-\frac{1}{(6)^2}\right)\left(1-\frac{1}{(10)^2}\right)\left(1-\frac{1}{(14)^2}\right)\cdots$$

$$\frac{\sqrt{2}}{2}=\left(\frac{(2)^2-1}{(2)^2}\right)\left(\frac{(6)^2-1}{(6)^2}\right)\left(\frac{(10)^2-1}{(10)^2}\right)\left(\frac{(14)^2-1}{(14)^2}\right)\cdots$$

$$\frac{1}{\sqrt{2}}=\left[\frac{(2-1)(2+1)}{(2)(2)}\right]\left[\frac{(6-1)(6+1)}{(6)(6)}\right]\left[\frac{(10-1)(10+1)}{(10)(10)}\right]\left[\frac{(14-1)(14+1)}{(14)(14)}\right]\cdots$$

Taking reciprocal on both sides of above expression, we get

$$\sqrt{2}=\left[\frac{(2)(2)}{(2-1)(2+1)}\right]\left[\frac{(6)(6)}{(6-1)(6+1)}\right]\left[\frac{(10)(10)}{(10-1)(10+1)}\right]\left[\frac{((14)(14)}{(14-1)(14+1)}\right]\cdots$$

$$\sqrt{2}=\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{10}{9}\cdot\frac{10}{11}\cdot\frac{14}{13}\cdot\frac{14}{15}\cdots$$

Hence proved.

Question 8: Deduce from the result of question 6 that $\boldsymbol{\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}}$

Solution: In question 6, we have proved that: $\pi\cot(\pi z)=\frac{1}{\pi}+\frac{1}{\pi}{\sum_{n=-\infty}^{\infty}}' n\left[\frac{1}{z-n}+\frac{1}{n}\right]$

$$\Rightarrow z\cot(\pi z)=\frac{1}{\pi}+\frac{1}{\pi}\left(\sum_{n=1}^{\infty}n\left[\frac{1}{z-n}+\frac{1}{n}\right]+\sum_{n=-1}^{-\infty}n\left[\frac{1}{z-n}+\frac{1}{n}\right]\right)$$

$$=\frac{1}{\pi}+\frac{1}{\pi}\left(\sum_{n=1}^{\infty}n\left[\frac{1}{z-n}+\frac{1}{n}\right]+\sum_{n=1}^{\infty}(-n)\left[\frac{1}{z+n}-\frac{1}{n}\right]\right) \qquad \left(\begin{array}{c}\text{by replacing } n\\ \text{by } -n \text{ in second}\\ \text{summation}\end{array}\right)$$

$$=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}n\left[\frac{1}{z-n}+\frac{1}{n}-\frac{1}{z+n}+\frac{1}{n}\right]$$

$$=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}n\left[\frac{n(z+n)+(z+n)(z-n)-n(z-n)+(z+n)(z-n)}{n(z+n)(z-n)}\right]$$

$$=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{nz+n^2+z^2-n^2-nz+n^2+z^2-n^2}{z^2-n^2}=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2}$$

$$\Rightarrow \pi z\cot(\pi z)=1+\sum_{n=1}^{\infty}\frac{2z^2}{z^2-n^2} \quad\Rightarrow\quad \pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

Hence proved.

Question 9: Expand $\boldsymbol{\cot(\pi z)}$ by Mittag Leffler's theorem and deduce the following results:

$$(i)\ \ \frac{\pi^2}{\sin^2(\pi z)}=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2} \quad \text{and} \quad (ii)\ \ \frac{\pi^2}{\cos^2(\pi z)}=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n-1/2)^2}.$$

Solution: Let, $f(z) = \pi\cot(\pi z) - \frac{1}{z} = \frac{\pi\cos(\pi z)}{\sin(\pi z)} - \frac{1}{z} = \frac{\pi z\cos(\pi z) - \sin(\pi z)}{z\sin(\pi z)}$

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{\pi z\cos(\pi z) - \sin(\pi z)}{z\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to 0}\frac{-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) - \pi\cos(\pi z)}{\pi z\cos(\pi z) + \sin(\pi z)} = \lim_{z\to 0}\frac{-\pi^2 z\sin(\pi z)}{\pi z\cos(\pi z) + \sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to 0}\frac{-\pi^2[\pi z\cos(\pi z) + \sin(\pi z)]}{-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) + \pi\cos(\pi z)} = 0$$

Since, $\lim_{z\to 0} f(z) = 0$(= defined), therefore, there is removable singularity at $z = 0$ (and hence no pole at $z = 0$).

The poles of $f(z)$ are determined by $\sin(\pi z) = 0, \quad z \neq 0 \quad \Rightarrow \quad \pi z = n\pi \quad \Rightarrow \quad z = n, \quad n = \pm 1, \pm 2, \pm 3, \cdots$

$$R(f, n) = \lim_{z\to n}\frac{(z-n)[\pi z\cos(\pi z) - \sin(\pi z)]}{z\sin(\pi z)} \qquad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{z\to n}\frac{(z-n)[-\pi^2 z\sin(\pi z) + \pi\cos(\pi z) - \pi\cos(\pi z)] + (\pi z\cos(\pi z) - \sin(\pi z))}{\pi z\cos(\pi z) + \sin(\pi z)}$$

$$= \frac{n\pi(-1)^n}{n\pi(-1)^n} = 1$$

Mittag Lefflers' Formula: $f(z) = f(0) + \sum_{n=-\infty}^{\infty} b_n\left[\frac{1}{z - a_n} + \frac{1}{a_n}\right]$

$$\pi\cot(\pi z) - \frac{1}{z} = 0 + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-n} + \frac{1}{n}\right] \quad \text{-----------------}\rightarrow \quad (1)$$

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=-1}^{-\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right]$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n}\right] + \sum_{n=1}^{\infty}\left[\frac{1}{z+n} - \frac{1}{n}\right] \qquad \left(\begin{array}{c}\text{by replacing } n \\ \text{by } -n \text{ in second} \\ \text{summation}\end{array}\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n}\right] \qquad \because (-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n, \ \forall\ n \in \mathbb{Z}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty}\left[\frac{1}{z-n} + \frac{1}{z+n}\right] = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2}$$

$\Rightarrow \cot(\pi z) = \frac{1}{\pi z} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{2z}{z^2 - n^2}$ **is the required expansion of** $\boldsymbol{\cot(\pi z)}$

From (1), we have: $\qquad \pi\cot(\pi z) = \frac{1}{z} + \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{z-n} + \frac{1}{n}\right]$

Differentiating above equation (on both sides) with respect to "z", we get

$$-\pi^2 \operatorname{cosec}^2(\pi z) = -\frac{1}{z^2} + \sum_{n=-\infty}^{\infty}{}' \left[-\frac{1}{(z-n)^2}\right]$$

$$\Rightarrow \ -\frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \left[-\frac{1}{(z-n)^2}\right],$$ where the term $-\frac{1}{z^2}$ corresponds to $n = 0$ in the summation sign.

$$\Rightarrow \ \frac{\pi^2}{\sin^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$ **Hence proved**. $\longrightarrow$ (2)

Replace z by $z - 1/2$ in (2), we get

$$\frac{\pi^2}{\sin^2(\pi z - \pi/2)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-1/2-n)^2} \Rightarrow \frac{\pi^2}{\cos^2(\pi z)} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n-1/2)^2}$$ **Hence Proved**.