**Form**: $\int_0^\infty x^{\alpha-1} f(x)dx$

**Working Rule:**

**Case I:** For the poles of $f(z)$, which do not lie on positive part of the real axis.

**Step I:** Replace $x$ by $z$ and $x^{\alpha-1}$ by $\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}$ and consider $\varphi(z) = \mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]} f(z)$

**Step II:** Calculate all poles of $f(z)$ and compute residues of $\varphi(z)$ at all these poles. (Note that, each pole of $f(z)$ is also a pole of $\varphi(z)$, but the converse is not true, as $\varphi(z)$ may have singularities other than the poles of $f(z)$.)

**StepIII**: $\int_0^\infty x^{\alpha-1} f(x)dx = \frac{-\pi}{\sin(\pi\alpha)} \mathrm{e}^{-\alpha\pi i} \sum_i R_i$ ,

where $\sum_i R_i =$ sum of residues of all the poles which do not lie on positive part of real axis.

**Case II:** For the poles of $f(z)$, which do lie on positive part of the real axis.

**Step I:** Replace $x$by $z$ and $x^{\alpha-1}$ by $\mathrm{e}^{(\alpha-1)\log(z)}$ and consider $\varphi(z) = \mathrm{e}^{[(\alpha-1)\log(z)]} f(z)$

**Step II:** Calculate all poles of $f(z)$ and compute residues of $\varphi(z)$ at all these poles. (Note that, each pole of $f(z)$ is also a pole of $\varphi(z)$, but the converse is not true, as $\varphi(z)$ may have singularities other than the poles of $f(z)$.)

**StepIII**: $\int_0^\infty x^{\alpha-1} f(x)dx = -\pi\cot(\pi\alpha) \sum_i R_i'$ ,

where $\sum_i R_i' =$ sum of residues of all the poles which do lie on positive part of real axis.

**Question 1**: **Prove that** $\int_0^\infty \frac{x^{\alpha-1}\mathrm{d}x}{x+\mathrm{e}^{i\beta}} = \pi\frac{\mathrm{e}^{(\alpha-1)\beta i}}{\sin(\alpha\pi)}$, $\quad 0 < \alpha < 1, \quad -\pi < \beta < \pi$

**Solution:** Pole of $f(z) = \frac{1}{z+\mathrm{e}^{i\beta}}$ is $z = -\mathrm{e}^{i\beta}$ (having order one), which do not lie on positive part of the real axis due to the fact that $-\pi < \beta < \pi$

Let $I = \int_0^\infty \frac{x^{\alpha-1}\mathrm{d}x}{x+\mathrm{e}^{i\beta}} = \int_C \frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}\mathrm{d}z}{z+\mathrm{e}^{i\beta}}$ and $\varphi(z) = \frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}}{z+\mathrm{e}^{i\beta}}$

$$R(\varphi, -\mathrm{e}^{i\beta}) = \frac{\mathrm{e}^{[(\alpha-1)(\log(-z)+i\pi)]}}{z+\mathrm{e}^{i\beta}}(z+\mathrm{e}^{i\beta}) = \mathrm{e}^{[(\alpha-1)(\log \mathrm{e}^{i\beta}+i\pi)]} = \mathrm{e}^{[(\alpha-1)(i\beta+i\pi)]} = \mathrm{e}^{i(\alpha-1)\beta}.\mathrm{e}^{i(\alpha-1)\pi}$$
$$= \mathrm{e}^{i(\alpha-1)\beta}\mathrm{e}^{i\alpha\pi}\mathrm{e}^{-i\pi} = -\mathrm{e}^{i(\alpha-1)\beta}\mathrm{e}^{i\pi\alpha} \qquad \because \mathrm{e}^{-i\pi} = -1$$

$$I = \frac{-\pi}{\sin(\pi\alpha)}\mathrm{e}^{i\alpha\pi}\sum_i R_i = \frac{-\pi}{\sin(\pi\alpha)}\mathrm{e}^{-i\alpha\pi}[-\mathrm{e}^{i(\alpha-1)\beta}\mathrm{e}^{i\pi\alpha}] = \pi\frac{\mathrm{e}^{(\alpha-1)\beta i}}{\sin(\alpha\pi)}$$ **Hence proved**.

**Question 2**: **Prove that** $\int_0^\infty \frac{x^\alpha\, dx}{(x+a)(x+b)} = \frac{\pi}{\sin(\alpha\pi)}\left[\frac{a^\alpha - b^\alpha}{a-b}\right], \quad -1 < \alpha < 1, \quad a > 0.$

**Solution:** Poles of $f(z) = \frac{1}{(z+a)(z+b)}$ are $z = -a, -b$ (both of them have order one), which do not lie on positive part of the real axis due to the fact that $a > 0,\ b > 0$

Let, $I = \int_0^\infty \frac{x^\alpha\, dx}{(x+a)(x+b)} = \int_C \frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{(z+a)(z+b)} dz$ and $\varphi(z) = \frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{(z+a)(z+b)}$

The poles of $\varphi(z)$ $\big($or equivalently that of $f(z)\big)$ are $z = -a$ and $z = -b$ and both of them have order 1.

$$R_1(\varphi, -a) = \lim_{z\to -a} \frac{e^{[(\alpha)(\log(-z)+i\pi)]}(z+a)}{(z+a)(z+b)} = \frac{e^{\alpha(\log(a)+i\pi)}}{b-a} = \frac{e^{\alpha \log a}\, e^{i\pi\alpha}}{b-a} = \frac{e^{\log a^\alpha} e^{i\pi\alpha}}{-(a-b)} = \frac{-a^\alpha e^{i\pi\alpha}}{a-b}$$

$$R_2(\varphi, -b) = \lim_{z\to -b} \frac{e^{[(\alpha)(\log(-z)+i\pi)]}(z+b)}{(z+a)(z+b)} = \frac{e^{[(\alpha)(\log(b)+i\pi)]}}{a-b} = \frac{b^\alpha e^{i\pi\alpha}}{a-b}$$

$$I = \frac{-\pi}{\sin((a+1)\pi)} e^{-(\alpha+1)\pi i} \sum_{i=1}^{2} R_i$$

$\because \sin\big((\alpha+1)\pi\big) = \sin(\alpha\pi + \pi) = -\sin(\alpha\pi)$ and $e^{-(\alpha+1)\pi i} = e^{-\alpha\pi i} e^{-\pi i} = e^{-\alpha\pi i}(-1) = -e^{-\alpha\pi i}$

$$\therefore I = \frac{-\pi}{\sin(\alpha\pi)} e^{-\alpha\pi i}\left[\frac{b^\alpha e^{i\pi\alpha}}{a-b} - \frac{a^\alpha e^{i\pi\alpha}}{a-b}\right] = \frac{-\pi}{\sin(\alpha\pi)} e^{-\alpha\pi i}\left[\frac{e^{\alpha\pi i}(b^\alpha - a^\alpha)}{a-b}\right]$$

$$= \frac{\pi}{\sin(\alpha\pi)}\left[\frac{(a^\alpha - b^\alpha)}{a-b}\right]$$ **Hence proved**.

**Question 3**: **Prove that** $\int_0^\infty \frac{x^\alpha}{x^4+1} dx = \frac{\pi}{4\sin\left(\frac{\pi(\alpha+1)}{4}\right)}$, **where** $-1 < \alpha < 3$

**Solution:** Poles of $f(z) = \frac{1}{z^4+1}$ are $z_k = e^{(\pi i + 2k\pi i)/4}$, $k = 0,1,2,3$ (all of them have order one), which do not lie on positive part of the real axis.

Let, $I = \int_0^\infty \frac{x^\alpha}{x^4+1} dx = \int_C \frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{z^4+1} dz$ and $\varphi(z) = \frac{e^{[(\alpha)(\log(-z)+i\pi)]}}{z^4+1}$

Let $\beta$ be a root of $z^4 = -1$, then

$$R(\varphi, \beta) = \lim_{z\to\beta} \frac{(z-\beta)e^{[(\alpha)(\log(-z)+i\pi)]}}{z^4+1}, \qquad \left(\frac{0}{0}\right) \text{form}$$

Apply L, Hospital's rule

$$\boldsymbol{R(\varphi,\beta) = \lim_{z\to\beta} \frac{e^{[(\alpha)(\log(-z)+i\pi)]} + (z-\beta)\frac{d}{dz}\left(e^{[(\alpha)(\log(-z)+i\pi)]}\right)}{4z^3} = \frac{e^{[(\alpha)(\log(-\beta)+i\pi)]}}{4\beta^3} = \frac{\beta e^{[(\alpha)(\log(-\beta)+i\pi)]}}{-4} \quad \because \beta^4 = -1}$$

Sum of residues at all poles is

$$\sum_{\beta} \boldsymbol{R_\beta} = \frac{\mathbf{-1}}{\mathbf{4}}\left[\mathbf{e}^{\pi i/4}\mathbf{e}^{\alpha[\log(-\mathbf{e}^{\pi i/4})+\pi i]} + \mathbf{e}^{3\pi i/4}\mathbf{e}^{\alpha[\log(-\mathbf{e}^{3\pi i/4})+\pi i]} + \mathbf{e}^{5\pi i/4}\mathbf{e}^{\alpha[\log(-\mathbf{e}^{5\pi i/4})+\pi i]} + \mathbf{e}^{7\pi i/4}\mathbf{e}^{\alpha[\log(-\mathbf{e}^{7\pi i/4})+\pi i]}\right]$$

Now, polar forms of $-\mathrm{e}^{\frac{\pi i}{4}}, -\mathrm{e}^{\frac{3\pi i}{4}}, -\mathrm{e}^{\frac{5\pi i}{4}}$ and $-\mathrm{e}^{\frac{7\pi i}{4}}$, respectively, are given by

$$-\mathrm{e}^{\frac{\pi i}{4}} = \frac{-1}{\sqrt{2}}(1+i) = \mathrm{e}^{\frac{-3\pi i}{4}}, \qquad -\mathrm{e}^{\frac{3\pi i}{4}} = \frac{-1}{\sqrt{2}}(-1+i) = \mathrm{e}^{\frac{-\pi i}{4}},$$

$$-\mathrm{e}^{\frac{5\pi i}{4}} = \frac{-1}{\sqrt{2}}(-1-i) = \mathrm{e}^{\frac{\pi i}{4}}, \qquad -\mathrm{e}^{\frac{7\pi i}{4}} = \frac{-1}{\sqrt{2}}(1-i) = \mathrm{e}^{\frac{3\pi i}{4}}$$

$$\Rightarrow \sum_{\beta} \boldsymbol{R_\beta} = \frac{-1}{4}\left[\mathbf{e}^{\pi i/4}\mathbf{e}^{\alpha[\log(\mathrm{e}^{-3\pi i/4})+\pi i]} + \mathbf{e}^{3\pi i/4}\mathbf{e}^{\alpha[\log(\mathrm{e}^{-\pi i/4})+\pi i]} + \mathbf{e}^{5\pi i/4}\mathbf{e}^{\alpha[\log(\mathrm{e}^{\pi i/4})+\pi i]} + \mathbf{e}^{7\pi i/4}\mathbf{e}^{\alpha[\log(\mathrm{e}^{3\pi i/4})+\pi i]}\right]$$

$$= \frac{-1}{4}\left[\mathrm{e}^{\frac{\pi i}{4}}\mathrm{e}^{\alpha\left(\frac{-3\pi i}{4}+\pi i\right)} + \mathrm{e}^{\frac{3\pi i}{4}}\mathrm{e}^{\alpha\left(\frac{-\pi i}{4}+\pi i\right)} + \mathrm{e}^{\frac{5\pi i}{4}}\mathrm{e}^{\alpha\left(\frac{\pi i}{4}+\pi i\right)} + \mathrm{e}^{\frac{7\pi i}{4}}\mathrm{e}^{\alpha\left(\frac{3\pi i}{4}+\pi i\right)}\right]$$

$$= \frac{-1}{4}\left[\frac{1}{\sqrt{2}}(1+i)\mathrm{e}^{\alpha\left(\frac{\pi i}{4}\right)} + \frac{1}{\sqrt{2}}(i-1)\mathrm{e}^{\alpha\left(\frac{3\pi i}{4}\right)} + \frac{1}{\sqrt{2}}(-1-i)\mathrm{e}^{\alpha\left(\frac{5\pi i}{4}\right)} + \frac{1}{\sqrt{2}}(1-i)\mathrm{e}^{\alpha\left(\frac{7\pi i}{4}\right)}\right]$$

$$= \frac{-1}{4\sqrt{2}}\left[(1+i)\mathrm{e}^{\frac{\alpha\pi i}{4}}\left(1-\mathrm{e}^{\alpha\pi i}\right) + (-1+i)\mathrm{e}^{\frac{3\alpha\pi i}{4}}\left(1-\mathrm{e}^{\alpha\pi i}\right)\right]$$

$$= \frac{-1}{4\sqrt{2}}\left(1-\mathrm{e}^{\alpha\pi i}\right)\left((1+i)\mathrm{e}^{\frac{\alpha\pi i}{4}} + (-1+i)\mathrm{e}^{\frac{3\alpha\pi i}{4}}\right)$$

$$= \frac{-1}{4\sqrt{2}}\,\mathrm{e}^{\frac{\alpha\pi i}{2}}\left(\mathrm{e}^{-\frac{\alpha\pi i}{2}} - \mathrm{e}^{\frac{\alpha\pi i}{2}}\right)\left[\left(\mathrm{e}^{\frac{\alpha\pi i}{4}} - \mathrm{e}^{\frac{3\alpha\pi i}{4}}\right) + i\left(\mathrm{e}^{\frac{\alpha\pi i}{4}} + \mathrm{e}^{\frac{3\alpha\pi i}{4}}\right)\right]$$

$$= \frac{-1}{4\sqrt{2}}\mathrm{e}^{\frac{\alpha\pi i}{2}}\left(-2i\sin\left(\frac{\alpha\pi}{2}\right)\right)\mathrm{e}^{\frac{\alpha\pi i}{4}}\mathrm{e}^{\frac{\alpha\pi i}{4}}\left[\left(\mathrm{e}^{-\frac{\alpha\pi i}{4}} - \mathrm{e}^{\frac{\alpha\pi i}{4}}\right) + i\left(\mathrm{e}^{-\frac{\alpha\pi i}{4}} + \mathrm{e}^{\frac{\alpha\pi i}{4}}\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\left(i\sin\left(\frac{\alpha\pi}{2}\right)\right)\mathrm{e}^{\alpha\pi i}\left[2i\sin\left(\frac{\alpha\pi}{4}\right) - 2i\cos\left(\frac{\alpha\pi}{4}\right)\right]$$

$$= i^2\,\mathrm{e}^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\left[\frac{1}{\sqrt{2}}\cos\left(\frac{\alpha\pi}{4}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{\alpha\pi}{4}\right)\right]$$

$$= -\,\mathrm{e}^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\left[\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\alpha\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\alpha\pi}{4}\right)\right] = -\,\mathrm{e}^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)$$

$$I = \frac{-\pi}{\sin\big((\alpha+1)\pi\big)}e^{-(\alpha+1)\pi i}\sum_{\beta} R(f,\beta)$$

$$\because \sin\big((\alpha+1)\pi\big) = \sin(\alpha\pi+\pi) = -\sin(\alpha\pi) \quad \text{and} \quad \mathrm{e}^{-(\alpha+1)\pi i} = \mathrm{e}^{-\alpha\pi i}\mathrm{e}^{-\pi i} = \mathrm{e}^{-\alpha\pi i}(-1) = -\mathrm{e}^{-\alpha\pi i}$$

$$I = \frac{-\pi\left(-\mathrm{e}^{-\alpha\pi i}\right)}{-\sin(\alpha\pi)}\left[-\mathrm{e}^{\alpha\pi i}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)\right] = \frac{\pi}{\sin(\alpha\pi)}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)$$

$$= \frac{\pi}{2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)}\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{2\cos\left(\frac{\alpha\pi}{2}\right)}\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)$$

$$I = \frac{\pi}{2\sin\left(\frac{\alpha\pi}{2}+\frac{\pi}{2}\right)}\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right) \qquad \because\ \sin\left(\frac{\alpha\pi}{2}+\frac{\pi}{2}\right) = \cos\left(\frac{\alpha\pi}{2}\right)$$

$$= \frac{\pi}{4\sin\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right)}\cos\left(\frac{\alpha\pi}{4}+\frac{\pi}{4}\right) = \frac{\pi}{4\sin\left(\frac{\pi(\alpha+1)}{4}\right)}$$ **Hence proved**.

**Question 4: Prove that** $\displaystyle\int_0^\infty \frac{x^b\,\mathrm{d}x}{1+x^2} = \frac{\pi}{2}\sec\left(\frac{b\pi}{2}\right), \quad -1 < b < 1$

**Solution:** Poles of $f(z) = \frac{1}{1+z^2}$ are $z = \pm i$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $$I = \int_0^\infty \frac{x^b\,\mathrm{d}x}{1+x^2} = \int_C \frac{\mathrm{e}^{b[\log(-z)+\pi i]}\mathrm{d}z}{1+z^2} \quad \text{and} \quad \varphi(z) = \frac{\mathrm{e}^{b[\log(-z)+\pi i]}}{1+z^2}$$

$$R_1(f,i) = \lim_{z\to i}\left[\frac{(z-i)\mathrm{e}^{b[\log(-z)+\pi i]}}{(z-i)(z+i)}\right] = \lim_{z\to i}\left[\frac{\mathrm{e}^{b[\log(-z)+\pi i]}}{(z+i)}\right] = \frac{\mathrm{e}^{b[\log(-i)+\pi i]}}{2i} = \frac{\mathrm{e}^{b\left[\log\left(\mathrm{e}^{\frac{-\pi i}{2}}\right)+\pi i\right]}}{2i} = \frac{\mathrm{e}^{b\left[\frac{-\pi i}{2}+\pi i\right]}}{2i} = \frac{\mathrm{e}^{\frac{b\pi i}{2}}}{2i}$$

$$R_1(f,-i) = \lim_{z\to -i}\left[\frac{(z+i)\mathrm{e}^{b[\log(-z)+\pi i]}}{(z-i)(z+i)}\right] = \lim_{z\to -i}\left[\frac{\mathrm{e}^{b[\log(-z)+\pi i]}}{(z-i)}\right] = \frac{\mathrm{e}^{b[\log(i)+\pi i]}}{2i} = \frac{\mathrm{e}^{b\left[\log\left(e^{\frac{\pi i}{2}}\right)+\pi i\right]}}{2i} = \frac{\mathrm{e}^{b\left[\frac{\pi i}{2}+\pi i\right]}}{2i} = \frac{\mathrm{e}^{\frac{3b\pi i}{2}}}{2i}$$

$$\sum_{i=1}^{2} R_i = \left[\frac{\mathrm{e}^{\frac{b\pi i}{2}}}{2i} - \frac{\mathrm{e}^{\frac{3b\pi i}{2}}}{2i}\right] = \frac{1}{i}\mathrm{e}^{b\pi i}\left[\frac{\mathrm{e}^{\frac{-b\pi i}{2}} - \mathrm{e}^{\frac{b\pi i}{2}}}{2}\right] = \frac{1}{i}\mathrm{e}^{b\pi i}\left(-i\sin\left(\frac{b\pi}{2}\right)\right) = -\mathrm{e}^{b\pi i}\sin\left(\frac{b\pi}{2}\right)$$

$$I = \frac{-\pi}{\sin\big((b+1)\pi\big)}\mathrm{e}^{-(b+1)\pi i}\sum_{i=1}^{2} R_i$$

$$\because \sin\big((b+1)\pi\big) = \sin(b\pi+\pi) = -\sin(b\pi) \quad \text{and} \quad \mathrm{e}^{-(b+1)\pi i} = \mathrm{e}^{-b\pi i}\mathrm{e}^{-\pi i} = \mathrm{e}^{-b\pi i}(-1) = -\mathrm{e}^{-\alpha\pi i}$$

$$\Rightarrow I = \frac{-\pi}{-\sin(b\pi)}\left(-\mathrm{e}^{-b\pi i}\right)\left[-\mathrm{e}^{b\pi i}\sin\left(\frac{b\pi}{2}\right)\right] = \frac{\pi}{\sin(b\pi)}\sin\left(\frac{b\pi}{2}\right) = \frac{\pi}{2\sin\left(\frac{b\pi}{2}\right)\cos\left(\frac{b\pi}{2}\right)}\sin\left(\frac{b\pi}{2}\right) = \frac{\pi}{2\cos\left(\frac{b\pi}{2}\right)}$$

$$= \frac{\pi}{2}\sec\left(\frac{b\pi}{2}\right)$$ **Hence proved**.

**Question 5**: **prove that** $\displaystyle\int_0^\infty \frac{\sqrt{x}}{x^2+x+1}\mathrm{d}x = \frac{\pi}{\sqrt{3}}$

**Solution:** Poles of $f(z) = \frac{1}{z^2+z+1}$ are $z = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm\sqrt{3}\,i}{2}$ (both of them have order one), which do not lie on positive part of the real axis.

Let $$I = \int_0^\infty \frac{\sqrt{x}}{x^2+x+1}\mathrm{d}x = \int_C \frac{\mathrm{e}^{\frac{1}{2}[\log(-z)+\pi i]}\,\mathrm{d}z}{z^2+z+1} \quad \text{and} \quad \varphi(z) = \frac{\mathrm{e}^{\frac{1}{2}[\log(-z)+\pi i]}}{z^2+z+1}$$

$$R_1\left(\varphi,\frac{-1+\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1+\sqrt{3}i}{2}}\frac{e^{\frac{1}{2}[\log(-z)+\pi i]}\left(z+\frac{1-\sqrt{3}i}{2}\right)}{\left(z+\frac{1-\sqrt{3}i}{2}\right)\left(z+\frac{1+\sqrt{3}i}{2}\right)}=\frac{e^{\frac{1}{2}\left[\log\left(\frac{1-\sqrt{3}i}{2}\right)+\pi i\right]}}{2\left(\frac{\sqrt{3}i}{2}\right)}=\frac{e^{\frac{1}{2}\left[\log\left(e^{\frac{-\pi i}{3}}\right)+\pi i\right]}}{\sqrt{3}i}$$

$$=\frac{e^{\frac{1}{2}\left[\frac{-\pi i}{3}+\pi i\right]}}{\sqrt{3}i}=\frac{e^{\frac{1}{2}\left[\frac{2\pi i}{3}\right]}}{\sqrt{3}i}=\frac{e^{\frac{\pi i}{3}}}{\sqrt{3}i}\qquad \because \frac{1-\sqrt{3}i}{2}=e^{\frac{-\pi i}{3}}\text{ (in polar form)}$$

$$R_2\left(\varphi,\frac{-1-\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1-\sqrt{3}i}{2}}\frac{e^{\frac{1}{2}[\log(-z)+\pi i]}\left(z+\frac{1+\sqrt{3}i}{2}\right)}{\left(z+\frac{1-\sqrt{3}i}{2}\right)\left(z+\frac{1+\sqrt{3}i}{2}\right)}=\frac{e^{\frac{1}{2}\left[\log\left(\frac{1+\sqrt{3}i}{2}\right)+\pi i\right]}}{-2\left(\frac{\sqrt{3}i}{2}\right)}=\frac{e^{\frac{1}{2}\left[\log\left(e^{\frac{\pi i}{3}}\right)+\pi i\right]}}{-\sqrt{3}i}=\frac{e^{\frac{1}{2}\left[\frac{\pi i}{3}+\pi i\right]}}{-\sqrt{3}i}$$

$$=\frac{e^{\frac{1}{2}\left[\frac{4\pi i}{3}\right]}}{-\sqrt{3}i}=\frac{e^{\frac{2\pi i}{3}}}{-\sqrt{3}i}\qquad \because \frac{1+\sqrt{3}i}{2}=e^{\frac{\pi i}{3}}\text{ (in polar form)}$$

$$I=\frac{-\pi}{\sin\left(\frac{3\pi}{2}\right)}e^{-\frac{3\pi i}{2}}\sum_{i=1}^{2}R_i=\frac{-\pi}{-1}e^{-\frac{3\pi i}{2}}\left[\frac{e^{\frac{\pi i}{3}}}{\sqrt{3}i}-\frac{e^{\frac{2\pi i}{3}}}{\sqrt{3}i}\right]$$

$$=\pi\left[\cos\left(-\frac{3\pi}{2}\right)+i\sin\left(-\frac{3\pi}{2}\right)\right]\left[\frac{\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)-\cos\left(\frac{2\pi}{3}\right)-i\sin\left(\frac{2\pi}{3}\right)}{\sqrt{3}i}\right]$$

$$=\pi\frac{i}{\sqrt{3}i}\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}+\frac{1}{2}-i\frac{\sqrt{3}}{2}\right]=\frac{\pi}{\sqrt{3}}$$

**Hence proved.**

**Question 6**: **Prove that** $\int_0^\infty \frac{x^{\alpha-1}}{x^2+x+1}dx=\frac{2\pi}{3}\frac{\cos\left[\frac{2\alpha\pi+\pi}{6}\right]}{\sin(\pi\alpha)}$, **where** $0<\alpha<2$

**Solution:** Poles of $f(z)=\frac{1}{z^2+z+1}$ are $z=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}\,i}{2}$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $I=\int_0^\infty \frac{x^{\alpha-1}}{x^2+x+1}dx=\int_C \frac{e^{[(\alpha-1)(\log(-z)+i\pi])]}}{z^2+z+1}dz$ and $\varphi(z)=\frac{e^{[(\alpha-1)[\log(-z)+i\pi]]}}{z^2+z+1}$

$$R_1\left(\varphi,\frac{-1+\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1+\sqrt{3}i}{2}}\frac{\left(e^{(\alpha-1)(\log(-z)+i\pi)}\right)\left(z+\frac{1-\sqrt{3}i}{2}\right)}{\left(z+\frac{1-\sqrt{3}i}{2}\right)\left(z-\left(\frac{-1-\sqrt{3}i}{2}\right)\right)}=\frac{e^{(\alpha-1)\left[\log\left(\frac{1-\sqrt{3}i}{2}\right)+i\pi\right]}}{\frac{-1+\sqrt{3}i}{2}-\left(\frac{-1-\sqrt{3}i}{2}\right)}$$

$$R_1\left(\varphi,\frac{-1+\sqrt{3}i}{2}\right)=\frac{e^{(\alpha-1)\left[\log\left(e^{\frac{-\pi i}{3}}\right)+i\pi\right]}}{\sqrt{3}i}=\frac{e^{(\alpha-1)\left[\frac{-\pi i}{3}+\pi i\right]}}{\sqrt{3}i}=\frac{e^{\frac{2\pi(\alpha-1)i}{3}}}{\sqrt{3}i}\qquad \because \frac{1-\sqrt{3}i}{2}=e^{\frac{-\pi i}{3}}\text{ (in polar form)}$$

$$R_2\left(\varphi,\frac{-1-\sqrt{3}i}{2}\right)=\lim_{z\to\frac{-1-\sqrt{3}i}{2}}\frac{e^{(\alpha-1)[\log(-z)+i\pi]}\left(z+\frac{1+\sqrt{3}i}{2}\right)}{\left(z+\frac{1+\sqrt{3}i}{2}\right)\left(z-\left(\frac{-1+\sqrt{3}i}{2}\right)\right)}=\frac{e^{(\alpha-1)\left[\log\left(\frac{1+\sqrt{3}i}{2}\right)+i\pi\right]}}{\left(\frac{-1-\sqrt{3}i}{2}-\left(\frac{-1+\sqrt{3}i}{2}\right)\right)}$$

$$=\frac{e^{(\alpha-1)\left[\log\left(e^{\frac{\pi i}{3}}\right)+i\pi\right]}}{-\sqrt{3}i}=\frac{e^{(\alpha-1)\left[\frac{\pi i}{3}+i\pi\right]}}{-\sqrt{3}i}=\frac{e^{\frac{4\pi(\alpha-1)i}{3}}}{-\sqrt{3}i}\qquad \because \frac{1+\sqrt{3}i}{2}=e^{\frac{\pi i}{3}}\text{ (in polar form)}$$

$$I=\frac{-\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}\cdot\sum_{i=1}^{2}R_i=\frac{-\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}\left[\frac{e^{\frac{2\pi(\alpha-1)i}{3}}}{\sqrt{3}i}-\frac{e^{\frac{4\pi(\alpha-1)i}{3}}}{\sqrt{3}i}\right]$$

$$=\frac{-\pi}{\sqrt{3}i\sin(\alpha\pi)}e^{-\alpha\pi i}e^{\frac{2\pi(\alpha-1)i}{3}}\left[1-e^{\frac{2\pi(\alpha-1)i}{3}}\right]$$

$$=\frac{-\pi}{\sqrt{3}i\sin(\alpha\pi)}\;e^{-\alpha\pi i+\frac{2\pi(\alpha-1)i}{3}}\;e^{\frac{2\pi(\alpha-1)i}{6}}\left[e^{-\frac{\pi(\alpha-1)i}{3}}-e^{\frac{2\pi(\alpha-1)i}{3}}\right]$$

$$=\frac{-\pi}{\sqrt{3}i\sin(\alpha\pi)}e^{\frac{-\alpha\pi i-2\pi i}{3}}e^{\frac{\pi(\alpha-1)i}{3}}\left[-2i\sin\left(\frac{\pi(\alpha-1)}{3}\right)\right]$$

$$=\frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}e^{\frac{-\alpha\pi i-2\pi i}{3}+\frac{\pi(\alpha-1)i}{3}}\sin\left(\frac{\pi(\alpha-1)}{3}\right)=\frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}e^{-\pi i}\sin\left(\frac{\pi(\alpha-1)}{3}\right)$$

$$=\frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}\cos\left(\frac{\pi(\alpha-1)}{3}+\frac{\pi}{2}\right)\qquad \because \sin\theta=\cos\left(\theta+\frac{\pi}{2}\right)$$

$$=\frac{2\pi}{\sqrt{3}\sin(\alpha\pi)}\cos\left(\frac{2\pi\alpha-2\pi+3\pi}{6}\right)=\frac{2\pi}{3}\frac{\cos\left[\frac{2\alpha\pi+\pi}{6}\right]}{\sin(\pi\alpha)}$$

**Hence proved**

**Question 7**: **Prove that** $\displaystyle\int_0^\infty \frac{x^\alpha dx}{(1+x^2)^2}=\frac{\pi(1-\alpha)}{4\cos\left(\frac{\pi\alpha}{2}\right)},\qquad -1<\alpha<3$

**Solution:** Poles of $f(z)=\frac{1}{(1+z^2)^2}$ are $z=\pm i$ (both of them have order two), which do not lie on positive part of the real axis.

Let, $\displaystyle I=\int_0^\infty\frac{x^\alpha dx}{(1+x^2)^2}=\int_C\frac{e^{\alpha[\log(-z)+\pi i]}dz}{(1+z^2)^2}$ and $\displaystyle\varphi(z)=\frac{e^{\alpha[\log(-z)+\pi i]}}{(1+z^2)^2}$

$$R_1(\varphi,i)=\lim_{z\to i}\frac{d}{dz}\left[\frac{(z-i)^2e^{\alpha[\log(-z)+\pi i]}}{(z-i)^2(z+i)^2}\right]=\lim_{z\to i}\frac{d}{dz}\left[\frac{e^{\alpha[\log(-z)+\pi i]}}{(z+i)^2}\right]$$

$$R_1(\varphi,i)=\lim_{z\to i}\frac{(z+i)^2e^{\alpha[\log(-z)+\pi i]}\frac{\alpha}{z}-e^{\alpha[\log(-z)+\pi i]}\cdot 2(z+i)}{(z+i)^4}=\frac{(2i)^2e^{\alpha[\log(-i)+\pi i]}\frac{\alpha}{i}-e^{\alpha[\log(-i)+\pi i]}\cdot 2(2i)}{(2i)^4}$$

$$=\frac{4\alpha ie^{\alpha\left[\log\left(e^{\frac{-\pi i}{2}}\right)+\pi i\right]}-4ie^{\alpha\left[\log\left(e^{\frac{-\pi i}{2}}\right)+\pi i\right]}}{16}=\frac{4\alpha ie^{\alpha\left[\frac{-\pi i}{2}+\pi i\right]}-4ie^{\alpha\left[\frac{-\pi i}{2}+\pi i\right]}}{16}=\frac{4\alpha ie^{\frac{\alpha\pi i}{2}}-4ie^{\frac{\alpha\pi i}{2}}}{16}$$

$$R_1(\varphi, i) = \frac{4ie^{\frac{\alpha\pi i}{2}}(\alpha-1)}{16} = \frac{ie^{\frac{\alpha\pi i}{2}}(\alpha-1)}{4}$$

$$R_2(\varphi, -i) = \lim_{z\to -i}\frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{(z+i)^2 \mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z-i)^2(z+i)^2}\right] = \lim_{z\to -i}\frac{\mathrm{d}}{\mathrm{dz}}\left[\frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z-i)^2}\right]$$

$$= \lim_{z\to -i}\left[\frac{(z-i)^2\mathrm{e}^{\alpha[\log(-z)+\pi i]}\frac{\alpha}{z} - \mathrm{e}^{\alpha[\log(-z)+\pi i]}2(z-i)}{(z-i)^4}\right]$$

$$= \frac{(-2i)^2\mathrm{e}^{\alpha[\log(i)+\pi i]}\frac{\alpha}{-i} - \mathrm{e}^{\alpha[\log(i)+\pi i]}2(-2i)}{(-2i)^4} = \frac{-4\alpha i\mathrm{e}^{\alpha\left[\log\left(\mathrm{e}^{\frac{\pi i}{2}}\right)+\pi i\right]} + 4i\mathrm{e}^{\alpha\left[\log\left(\mathrm{e}^{\frac{\pi i}{2}}\right)+\pi i\right]}}{16}$$

$$= \frac{-4\alpha i\mathrm{e}^{\alpha\left[\frac{\pi i}{2}+\pi i\right]} + 4i\mathrm{e}^{\alpha\left[\frac{\pi i}{2}+\pi i\right]}}{16} = \frac{-4\alpha i\mathrm{e}^{\frac{3\alpha\pi i}{2}} + 4i\mathrm{e}^{\frac{3\alpha\pi i}{2}}}{16} = \frac{-4i\mathrm{e}^{\frac{\alpha\pi i}{2}}(\alpha-1)}{16} = \frac{-i\mathrm{e}^{\frac{3\alpha\pi i}{2}}(\alpha-1)}{4}$$

$$I = \frac{-\pi}{\sin(\alpha+1)\pi}\mathrm{e}^{-(\alpha+1)\pi i}\cdot\sum_{i=1}^{2}R_i$$

$\because \sin\big((\alpha+1)\pi\big) = \sin(\alpha\pi+\pi) = -\sin(\alpha\pi)$ and $\mathrm{e}^{-(\alpha+1)\pi i} = \mathrm{e}^{-\alpha\pi i}\mathrm{e}^{-\pi i} = \mathrm{e}^{-\alpha\pi i}(-1) = -\mathrm{e}^{-\alpha\pi i}$

$$\Rightarrow I = \frac{-\pi}{-\sin(\alpha\pi)}\left(-\mathrm{e}^{-\alpha\pi i}\right)\left[\frac{ie^{\frac{\alpha\pi i}{2}}(\alpha-1)}{4} - \frac{ie^{\frac{3\alpha\pi i}{2}}(\alpha-1)}{4}\right] = \frac{-\pi}{\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}\mathrm{e}^{\alpha\pi i}i(\alpha-1)\left[\frac{\mathrm{e}^{\frac{-\alpha\pi i}{2}} - \mathrm{e}^{\frac{\alpha\pi i}{2}}}{4}\right]$$

$$= \frac{-\pi}{\sin(\alpha\pi)}i(\alpha-1)\left[\frac{-2i\sin\left(\frac{\alpha\pi}{2}\right)}{4}\right] = \frac{-\pi(\alpha-1)}{\sin(\alpha\pi)}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{2} = \frac{\pi(1-\alpha)}{2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)}\frac{\sin\left(\frac{\alpha\pi}{2}\right)}{2}$$

$$= \frac{\pi(1-\alpha)}{4\cos\left(\frac{\alpha\pi}{2}\right)}$$ **Hence proved.**

**Question 8**: **Question 1 of book is repeated here** (**see the book**). **No need to repeat same question**.

**Questio 9**: **Prove that** $\displaystyle\int_0^\infty \frac{x^\alpha \mathrm{d}x}{1+2x\cos\theta+x^2} = \frac{\pi}{\sin(\alpha\pi)}\frac{\sin(\alpha\theta)}{\sin\theta}$ $\quad 0<\alpha<1, -\pi<\theta<\pi$

**Solution:** Poles of $f(z) = \frac{1}{1+2z\cos\theta+z^2}$ are $z = \frac{-2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2} = -\cos\theta \pm i\sin\theta = -\mathrm{e}^{-i\theta}, -\mathrm{e}^{i\theta}$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $\displaystyle I = \int_0^\infty \frac{x^\alpha \mathrm{d}x}{1+2x\,\mathrm{cso}\theta+x^2} = \int_C \frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}\mathrm{d}z}{1+2z\cos\theta+z^2}$ and $\displaystyle\varphi(z) = \frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{1+2z\cos\theta+z^2}$

$$R_1\left(f, -\mathrm{e}^{-i\theta}\right) = \lim_{z\to-\mathrm{e}^{-i\theta}}\frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z+\mathrm{e}^{i\theta})(z+\mathrm{e}^{-i\theta})} = \lim_{z\to-\mathrm{e}^{-i\theta}}\frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{z+\mathrm{e}^{i\theta}} = \frac{\mathrm{e}^{\alpha[\log(\mathrm{e}^{-i\theta})+\pi i]}}{-\mathrm{e}^{-i\theta}+\mathrm{e}^{i\theta}} = \frac{\mathrm{e}^{\alpha[-i\theta+\pi i]}}{2i\sin\theta} = \frac{\mathrm{e}^{\alpha i[\pi-\theta]}}{2i\sin\theta}$$

$$R_2\left(f, -\mathrm{e}^{i\theta}\right) = \lim_{z\to-\mathrm{e}^{i\theta}}\frac{\left(z+\mathrm{e}^{i\theta}\right)\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{(z+\mathrm{e}^{i\theta})(z+\mathrm{e}^{-i\theta})} = \lim_{z\to-\mathrm{e}^{i\theta}}\frac{\mathrm{e}^{\alpha[\log(-z)+\pi i]}}{z+\mathrm{e}^{-i\theta}} = \frac{\mathrm{e}^{\alpha[\log(\mathrm{e}^{i\theta})+\pi i]}}{-\mathrm{e}^{i\theta}+\mathrm{e}^{-i\theta}} = \frac{\mathrm{e}^{\alpha[i\theta+\pi i]}}{-2i\sin\theta} = \frac{-\mathrm{e}^{\alpha i[\pi+\theta]}}{2i\sin\theta}$$

$$I=\frac{-\pi}{\sin(\alpha+1)\pi}\mathrm{e}^{-(\alpha+1)\pi i}\sum_{i=1}^{2}R_i$$

$\because \sin\big((\alpha+1)\pi\big)=\sin(\alpha\pi+\pi)=-\sin(\alpha\pi)$ and $\mathrm{e}^{-(\alpha+1)\pi i}=\mathrm{e}^{-\alpha\pi i}\mathrm{e}^{-\pi i}=\mathrm{e}^{-\alpha\pi i}(-1)=-\mathrm{e}^{-\alpha\pi i}$

$$\Rightarrow I=\frac{-\pi}{-\sin\alpha\pi}\left(-\mathrm{e}^{-\alpha\pi i}\right)\left[\frac{\mathrm{e}^{\alpha i[\pi-\theta]}}{2i\sin\theta}-\frac{\mathrm{e}^{\alpha i[\pi+\theta]}}{2i\sin\theta}\right]=\frac{-\pi}{-2i\sin(\alpha\pi)\ \sin\theta}\left(-\mathrm{e}^{-\alpha\pi i}\right)\mathrm{e}^{\alpha\pi i}\left[\mathrm{e}^{-\alpha i\theta}-\mathrm{e}^{\alpha i\theta}\right]$$

$$=\frac{-\pi}{2i\sin(\alpha\pi)\ \sin\theta}[-2\,i\sin(\alpha\theta)]=\frac{\pi}{\sin(\alpha\pi)}\frac{\sin(\alpha\theta)}{\sin\theta}$$ **Hence proved.**

**Question 10**: **Prove that** $\boldsymbol{\int_0^\infty \frac{x^\alpha}{x^4+1}dx=\frac{\pi}{4}\mathrm{coses}\left(\frac{\pi(\alpha+1)}{4}\right)}$, **where** $\boldsymbol{-1<\alpha<3}$

**Solution:** From result of question 3, the result of question 10 is obvious.

**Question 11**: **prove that** $\boldsymbol{\int_0^\infty \frac{x^{1/3}\mathrm{d}x}{(x+a)(x+b)}=\frac{2\pi}{\sqrt{3}}\left[\frac{a^{1/3}-b^{1/3}}{a-b}\right]}$, $\boldsymbol{a>0}$ , $\boldsymbol{b>0}$, $\boldsymbol{a\neq b}$

**Solution:** Poles of $f(z)=\frac{1}{(z+a)(z+b)}$ are $z=-a,-b$ (both of them have order one), which do not lie on positive part of the real axis due to the fact that $a>0$, $b>0$.

Let, $I=\int_0^\infty \frac{x^{1/3}\mathrm{d}x}{(x+a)(x+b)}=\int_C \frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}}{(z+a)(z+b)}\mathrm{d}z$ and $\varphi(z)=\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}}{(z+a)(z+b)}$

$$R_1(\varphi,-a)=\lim_{z\to-a}\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}(z+a)}{(z+a)(z+b)}=\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(a)+i\pi)\right]}}{-a+b}=\frac{\mathrm{e}^{\frac{1}{3}(\log(a)+i\pi)}}{b-a}=\frac{\mathrm{e}^{\frac{1}{3}\log a}\mathrm{e}^{\frac{i\pi}{3}}}{b-a}=\frac{-a^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}$$

$$R_2(\varphi,-b)=\lim_{z\to-b}\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(-z)+i\pi)\right]}(z+b)}{(z+a)(z+b)}=\frac{\mathrm{e}^{\left[\left(\frac{1}{3}\right)(\log(b)+i\pi)\right]}}{a-b}=\frac{b^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}$$

$$I=\frac{-\pi}{\sin\left(\left(\frac{1}{3}+1\right)\pi\right)}\mathrm{e}^{-(\frac{1}{3}+1)\pi i}\sum_{i=1}^{2}R_i=\frac{-\pi}{\sin\left(\frac{1}{3}\pi+\pi\right)}\mathrm{e}^{-\frac{\pi i}{3}}\mathrm{e}^{-\pi i}\left[\frac{b^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}-\frac{a^{\frac{1}{3}}\mathrm{e}^{\frac{i\pi}{3}}}{a-b}\right]$$

$$=\frac{-\pi}{\sin\left(\frac{\pi}{3}+\pi\right)}\mathrm{e}^{-\frac{\pi i}{3}}\mathrm{e}^{-\pi i}\left[\frac{\mathrm{e}^{\frac{i\pi}{3}}\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)}{a-b}\right]=\frac{-\pi}{-\sin\left(\frac{\pi}{3}\right)}\mathrm{e}^{-\frac{\pi i}{3}}(-1)\left[\frac{\mathrm{e}^{\frac{i\pi}{3}}\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)}{a-b}\right]$$

$$=\frac{-\pi}{\frac{\sqrt{3}}{2}}\mathrm{e}^{-\frac{\pi i}{3}}\left[\frac{\mathrm{e}^{\frac{i\pi}{3}}\left(b^{\frac{1}{3}}-a^{\frac{1}{3}}\right)}{a-b}\right]=\frac{2\pi}{\sqrt{3}}\left[\frac{a^{1/3}-b^{1/3}}{a-b}\right]$$ **Hence proved.**

**Question 12**: **Prove that** $\boldsymbol{\int_0^\infty \frac{x^\alpha\mathrm{d}x}{(x^2-b^2)(x^2-c^2)}=\frac{\pi}{2}\left[\frac{b^{\alpha-1}-c^{\alpha-1}}{b^2-c^2}\right][\mathrm{cosec}\,(\pi\alpha)-\cot(\pi\alpha)]}$, $\boldsymbol{a>0}$, $\boldsymbol{b>0}$

**Note:** This question is not in the book

**Solution:** Poles of $f(z)=\frac{1}{(z^2-b^2)(z^2-c^2)}$ are $z=\pm b,\pm c$ (all of them have order one). $z=b$ and $z=c$ do lie on positive part of the real axis, while $z=-b$ and $z=-c$ do not lie on the positive part of the real axis due to the fact that $a>0,\ b>0$.

For the poles $z=b,c$, we let, $I=\int_0^\infty \frac{x^\alpha dx}{(x^2-b^2)(x^2-c^2)}=\int_C \frac{e^{\alpha\log z}}{(z^2-b^2)(z^2-c^2)}dz,\ \varphi_1(z)=\frac{e^{\alpha\log z}}{(z^2-b^2)(z^2-c^2)}$

For the poles $z=-b,-c$, we let, $I=\int_0^\infty \frac{x^\alpha dx}{(x^2-b^2)(x^2-c^2)}=\int_C \frac{e^{\alpha[\log(-z)+\pi i]}}{(z^2-b^2)(z^2-c^2)}dz,\ \varphi_2(z)=\frac{e^{\alpha[\log(-z)+\pi i]}}{(z^2-b^2)(z^2-c^2)}$

$$R_1(\varphi_1,b)=\lim_{z\to b}\frac{(z-b)e^{\alpha\log z}}{(z-b)(z+b)(z^2-c^2)}=\lim_{z\to b}\frac{e^{\alpha\log z}}{(z+b)(z^2-c^2)}=\frac{e^{\alpha\log b}}{(2b)(b^2-c^2)}=\frac{e^{\log b^\alpha}}{(2b)(b^2-c^2)}$$
$$=\frac{b^\alpha}{2b(b^2-c^2)}=\frac{b^{\alpha-1}}{2(b^2-c^2)}$$

$$R_2(\varphi_1,c)=\lim_{z\to c}\frac{(z-c)e^{\alpha\log z}}{(z-c)(z+c)(z^2-b^2)}=\lim_{z\to c}\frac{e^{\alpha\log z}}{(z+c)(z^2-b^2)}=\frac{e^{\alpha\log c}}{(2c)(c^2-b^2)}=\frac{-e^{\log c^\alpha}}{(2b)(b^2-c^2)}$$
$$=\frac{-c^\alpha}{2c(b^2-c^2)}=\frac{-c^{\alpha-1}}{2(b^2-c^2)}$$

$$R_3(\varphi_2,-b)=\lim_{z\to -b}\frac{(z+b)e^{\alpha[\log(-z)+\pi i]}}{(z-b)(z+b)(z^2-c^2)}=\lim_{z\to -b}\frac{e^{\alpha[\log(-z)+\pi i]}}{(z-b)(z^2-c^2)}=\frac{e^{\alpha[\log b+\pi i]}}{(-2b)(b^2-c^2)}=\frac{e^{\alpha\log b}e^{\alpha\pi i}}{(-2b)(b^2-c^2)}$$
$$=\frac{e^{\log b^\alpha}e^{\alpha\pi i}}{(-2b)(b^2-c^2)}=\frac{-b^\alpha e^{\alpha\pi i}}{2b(b^2-c^2)}=\frac{-b^{\alpha-1}e^{\alpha\pi i}}{2(b^2-c^2)}$$

$$R_4(\varphi_2,-c)=\lim_{z\to -c}\frac{(z+c)e^{\alpha[\log(-z)+\pi i]}}{(z-c)(z+c)(z^2-b^2)}=\lim_{z\to -c}\frac{e^{\alpha[\log(-z)+\pi i]}}{(z-c)(z^2-b^2)}=\frac{e^{\alpha[\log c+\pi i]}}{(-2c)(c^2-b^2)}=\frac{e^{\alpha\log c}e^{\alpha\pi i}}{(2c)(b^2-c^2)}$$
$$=\frac{e^{\log c^\alpha}e^{\alpha\pi i}}{(2c)(b^2-c^2)}=\frac{c^\alpha e^{\alpha\pi i}}{2c(b^2-c^2)}=\frac{c^{\alpha-1}e^{\alpha\pi i}}{2(b^2-c^2)}$$

$$I=-\pi\cot\big((\alpha+1)\pi\big)[R_1+R_2]-\frac{\pi}{\sin\big((\alpha+1)\pi\big)}e^{-(\alpha+1)\pi i}[R_3+R_4]$$

$$\because \cot\big((\alpha+1)\pi\big)=\cot(\alpha\pi+\pi)=\cot(\alpha\pi),\quad e^{-(\alpha+1)\pi i}=e^{-\alpha\pi i}e^{-\pi i}=-e^{-\alpha\pi i},\quad \sin\big((\alpha+1)\pi\big)=-\sin(\alpha\pi)$$

$$\int_0^\infty \frac{x^\alpha dx}{(x^2-b^2)(x^2-c^2)}=-\pi\cot(\alpha\pi)\left[\frac{b^{\alpha-1}-c^{\alpha-1}}{2(b^2-c^2)}\right]-\frac{\pi}{\sin(\alpha\pi)}e^{-\alpha\pi i}e^{\alpha\pi i}\left[\frac{c^{\alpha-1}-b^{\alpha-1}}{2(b^2-c^2)}\right]$$
$$=\frac{\pi}{2}\left[\frac{b^{\alpha-1}-c^{\alpha-1}}{b^2-c^2}\right][\text{cosec}\,(\pi\alpha)-\cot(\alpha\pi)]$$ **Hence proved**.

**Question 13**: **Proved that** $\int_0^\infty \frac{x^{\alpha-1}dx}{x^2+b^2}=\frac{\pi}{2}\frac{b^{\alpha-2}}{\sin\left(\frac{\alpha\pi}{2}\right)},\quad b>0$

**Note:** This question is not in the book

**Solution:** Poles of $f(z)=\frac{1}{z^2+b^2}$ are $z=\pm bi$ (both of them have order one), which do not lie on positive part of the real axis.

Let, $$I=\int_0^\infty \frac{x^{\alpha-1}\mathrm{d}x}{x^2+b^2}=\int_C \frac{\mathrm{e}^{(\alpha-1)[\log(-z)+\pi i]}\mathrm{d}z}{z^2+b^2} \quad \text{and} \quad \varphi(z)=\frac{\mathrm{e}^{(\alpha-1)[\log(-z)+\pi i]}}{z^2+b^2}$$

$$R_i(f,b)=\lim_{z\to bi}\frac{(z-bi)\mathrm{e}^{(\alpha-1)[\log(-z)+\pi i]}}{(z-bi)(z+bi)}=\lim_{z\to bi}\frac{\mathrm{e}^{(\alpha-1)[\log(-z)+\pi i]}}{z+bi}=\frac{\mathrm{e}^{(\alpha-1)[\log(-bi)+\pi i]}}{2bi}=\frac{\mathrm{e}^{(\alpha-1)\left[\log \mathrm{b}-\frac{\pi i}{2}+\pi i\right]}}{2bi}$$
$$=\frac{\mathrm{e}^{(\alpha-1)\left[\log \mathrm{b}+\frac{\pi i}{2}\right]}}{2bi}=\frac{\mathrm{e}^{(\alpha-1)\log \mathrm{b}}\mathrm{e}^{(\alpha-1)\frac{\pi i}{2}}}{2bi}=\frac{\mathrm{e}^{\log b^{(\alpha-1)}}\mathrm{e}^{(\alpha-1)\frac{\pi i}{2}}}{2bi}=\frac{b^{(\alpha-1)}\mathrm{e}^{(\alpha-1)\frac{\pi i}{2}}}{2bi}$$

$$\because\ \mathrm{e}^{[\log(-bi)]}=\log b+\log(-i)=\log b+\log\left(\mathrm{e}^{\frac{-\pi i}{2}}\right)=\log b+\frac{-\pi i}{2}$$

$$R_2(f,-bi)=\lim_{z\to -bi}\frac{(z+bi)\mathrm{e}^{(\alpha-1)[\log(-z)+\pi i]}}{(z-bi)(z+bi)}=\lim_{z\to -bi}\frac{\mathrm{e}^{(\alpha-1)[\log(-z)+\pi i]}}{z-bi}=\frac{\mathrm{e}^{(\alpha-1)[\log(bi)+\pi i]}}{-2bi}=\frac{\mathrm{e}^{(\alpha-1)\left[\log \mathrm{b}+\frac{\pi i}{2}+\pi i\right]}}{-2bi}$$
$$=\frac{\mathrm{e}^{(\alpha-1)\left[\log \mathrm{b}+\frac{3\pi i}{2}\right]}}{-2bi}=\frac{\mathrm{e}^{(\alpha-1)\log \mathrm{b}}\mathrm{e}^{(\alpha-1)\frac{3\pi i}{2}}}{-2bi}=\frac{\mathrm{e}^{\log b^{(\alpha-1)}}\mathrm{e}^{(\alpha-1)\frac{3\pi i}{2}}}{-2bi}=\frac{b^{(\alpha-1)}\mathrm{e}^{(\alpha-1)\frac{3\pi i}{2}}}{-2bi}$$

$$\because\ \mathrm{e}^{[\log(bi)]}=\log b+\log(i)=\log b+\log\left(\mathrm{e}^{\frac{\pi i}{2}}\right)=\log b+\frac{\pi i}{2}$$

$$I=\frac{-\pi}{\sin\alpha\pi}\mathrm{e}^{-\alpha\pi i}\sum_i R_i=\frac{-\pi}{\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}\left[\frac{b^{(\alpha-1)}\mathrm{e}^{(\alpha-1)\frac{\pi i}{2}}}{2bi}-\frac{b^{(\alpha-1)}\mathrm{e}^{(\alpha-1)\frac{3\pi i}{2}}}{2bi}\right]$$
$$=\frac{-\pi}{2i\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}b^{(\alpha-2)}\left[\mathrm{e}^{\frac{\alpha\pi i}{2}}\mathrm{e}^{\frac{-\pi i}{2}}-\mathrm{e}^{\frac{3\alpha\pi i}{2}}\mathrm{e}^{\frac{-3\pi i}{2}}\right]=\frac{-\pi}{2i\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}b^{(\alpha-2)}\left[-i\mathrm{e}^{\frac{\alpha\pi i}{2}}-i\mathrm{e}^{\frac{3\alpha\pi i}{2}}\right]$$
$$=\frac{-\pi}{2i\sin(\alpha\pi)}\mathrm{e}^{-\alpha\pi i}\mathrm{e}^{\alpha\pi i}b^{(\alpha-2)}(-i)\left[\mathrm{e}^{\frac{-\alpha\pi i}{2}}+\mathrm{e}^{\frac{\alpha\pi i}{2}}\right]=\frac{\pi}{2\sin(\alpha\pi)}b^{(\alpha-2)}\left[2\cos\left(\frac{\alpha\pi}{2}\right)\right]$$
$$=\frac{\pi}{\sin(\alpha\pi)}b^{(\alpha-2)}\cos\left(\frac{\alpha\pi}{2}\right)=\frac{\pi}{2\sin\left(\frac{\alpha\pi}{2}\right)\cos\left(\frac{\alpha\pi}{2}\right)}b^{(\alpha-2)}\cos\left(\frac{\alpha\pi}{2}\right)$$
$$=\frac{\pi}{2}\frac{b^{\alpha-2}}{\sin\left(\frac{\alpha\pi}{2}\right)}$$

**Hence proved**.