Advanced Analysis: Handwritten Notes

provided by

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Equivelent Set:-

Two Sets A and B are equivelent if there exist a bijective mapping between them and it is denoted by

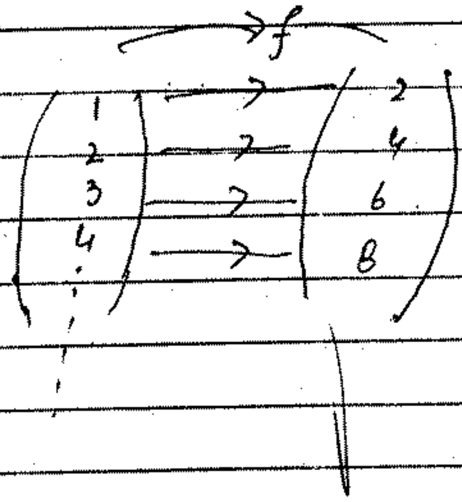
$$A \sim B.$$

In such type of sets only number of elements are same i.e they have equal number of elements.

Example

$$P \sim E$$

$$f(n) = 2n$$



$f(1) = 2(1) = 2.$
$f(2) = 2(2) = 4.$
$f(3) = 2(3) = 6.$
$f(4) = 2(4) = 8$
$\vdots \qquad \vdots$

Hence relation is 1-1 and onto Hence the mapping is bijective. Hence

$$P \sim E.$$

By the similar we can show that set of Natural number is equivelant to the set of odd number by defining the mapping i.e

$$f : N \longrightarrow O \text{ defined by}$$

$$f(n) = 2n+1$$

which is one-one & onto. Hence f is bijective Then

$$N \sim O.$$

By the Similar we define a mapping b/w Set of natural number and set of integer Such that

$$f : N \longrightarrow Z.$$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

which is bijective and.

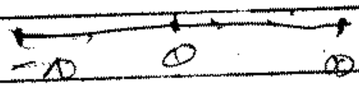
$$N \sim Z$$

//

Q. Show That unit interval $[0,1]$ is equivalent to $[a,b]$

Proof :- First we defined a mapping b/w $[0,1]$ and $[a,b]$ i.e

$$f : [0,1] \longrightarrow [a,b]$$



$$f(x) = a + (b-a)x$$

$$f(0) = a$$

$$f(1) = b$$

So unit close interval is equivelent to every closed interval of real line.

NOTE : $I \sim [a,b]$

$A = \{1,2,3\}$
$A \times A = \{($ ${}^{L}_{1}R_2$ $(A \sim A)$
$I(a) = x$

Theorem

A equivalent relation $(\sim)$ is an equivalence relation.

Proof:

Reflexive:

For any non-empty set A $\exists$ a mapping I, such that

$I : A \to A$ defined by

$I(a) = a$

which is bijective. Hence

$A \sim A$.

Symmetric Property:-

Let A and B be to non-empty set and $A \sim B$ Then we have to show that

$A \sim B$.

$f : A \sim B$ which is bijective.

$\Rightarrow f^{-1} : B \to A$ is also bijective.

NOTE :- inverse of bijective function is also bijective.

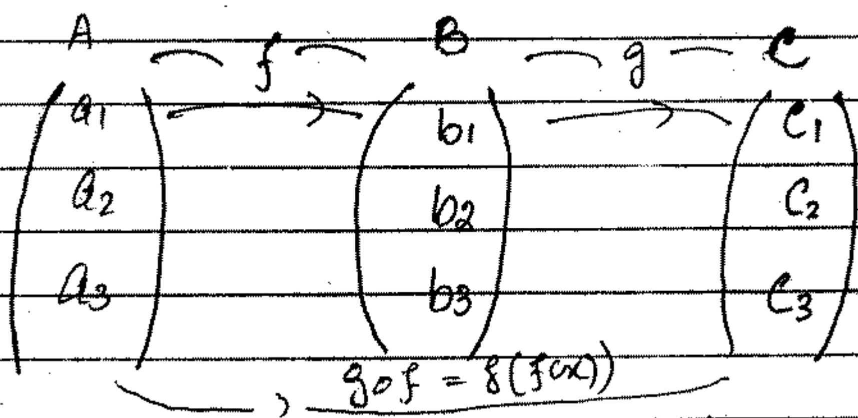
Hence $B \sim A$.

Transitive Property:-

For any three non-empty set A, B, C if $A \sim C$ and $A \sim B$ and $B \sim C$ be two bijective mapping f and g such that

$f : A \to B$ bijective

$g : B \to C$ bijective.



$g \circ f = g(f(x))$

Composition function of f and g.

$g(f(a_1)) = g(b_1) = c_1$

$g \circ f : A \longrightarrow C$ is bijective.

Hence $A \sim C$ and Equivelante Relation.

Hence Equivelent Relation is equivelence Relation.

Finite SET:-

A Set is Said to be finit if it consist of finite number of element or if a set is equivelent to set of fixed natural number.

$N = \{1, 2, 3, ---, n\}$

$A = \{2, 4, 6, --- n\}$

$N \sim A$.

Hence A is finite.

Infinite Set:-
A set is said to be infinite if it is equivalent to its proper subset
i.e $P \sim O$
& $P \sim Z$
$I \sim [a, b]$
Then
O, P, I are infinite set.

Denomerable Set:-
A set A is said to be denomerable if it is equivalent to the set of natural number. i.e
$O \sim N$
$E \sim N$
$Z \sim N$.
Then
O, E, Z are denomerable.
Denomerable set is infinite.

$A = \{1, 2, 3, 4\}$ $f : A \to N$

(6)

Countable set : A set which is finite or equivelent to set of natural number is called countable set;

NOTE:-
Every Denomerable set is Countable but every countable set is not denomerable.//

Theorem A Countable union of finite sets is Countable.

Proof : let $\mathcal{A} = \{S_i : i \in P\}$ be Countable Collection of finite set and let

$C = \bigcup S_i$ if C is empty The C is Countable. Suppose $C \neq \phi$. Define $A_1 = S_1$ $A_2 = S_2 \setminus S_1 = A_3 = S_3 \setminus S_2$ $A_4 = S_4 \setminus S_3 \cdots$. Then the set A_i are finite and pairwise disjoint say

$A_1 = \{a_{11}, a_{12}, a_{13}, \cdots a_{1n}\}$
$A_2 = \{a_{21}, a_{22}, \cdots, a_{2n}\}$
$A_3 = \{a_{31}, a_{32}, \cdots a_{3n}\}$

Then union $B = \bigcup_i A_i$ can be written as sequence as follows

$B = \{a_{11}, a_{12}, a_{13}, \cdots a_{1n}, a_{21}, \cdots, a_{2n}\}$
That first we write down the elements of A_1, then the elements of A_2, and so on.
$f : B \to P$ as follow $f(a_{ij}) = n_1 + n_2 + \cdots$

Example (7)

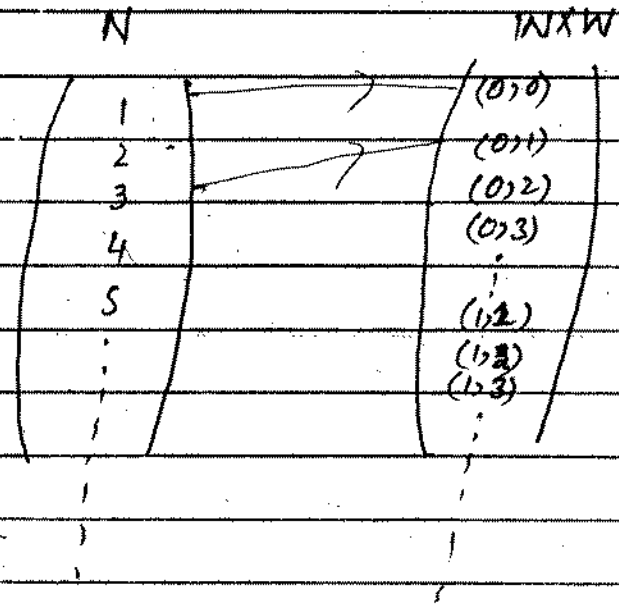
Show That $W \times W$ is denomerable

Sol:-

we define a function

$f: N \longrightarrow W \times W$

$f(a) = (r, s)$ where $a = 2^r(2s+1)$



$\Rightarrow f$ is bijective

$N \sim W \times W$ and
$W \times W$ is denomerable.

Theorem:- Every Subset of countable set is countable.

Proof:- let A be any Countable set. Then There are two possibilities

if

i) A is finite.

ii) A is denomerable mean which equivelent ~~infinite~~ to the set of natural number and infinite.

Now if A is finite and B is any Subset of A. Then B belong the Subset of finit set is finite. Now if A is denomerable. if B is a Subset of A and let.

$$A = \{a_1, a_2, a_3, \dots, a_n, \dots\}$$

$$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \dots, a_{n_r}\}$$

If indexing set $\{n_1, n_2, n_3, \dots n_r, \dots\}$ is finite Then B is also finite $\Rightarrow$ B is Countable.

if the indexing set $\{n_1, n_2, n_3, \dots n_r, \dots\}$ is infinite Then B is infinite. Then B being ~~[illegible]~~ infinite sequence of distinct term is denomerable.

$\Rightarrow$ B is countable.

Theorem:- *

Every infinite set contain a denomerable Subset.

Proof

let A be an infinite Set. Also let $\mathcal{A}$ represent the collection of Subsets of A.

we define a mapping $f : \mathcal{A} \to A$ Such that

$$f(A) = a_1 \text{ where } a_1 \in A.$$

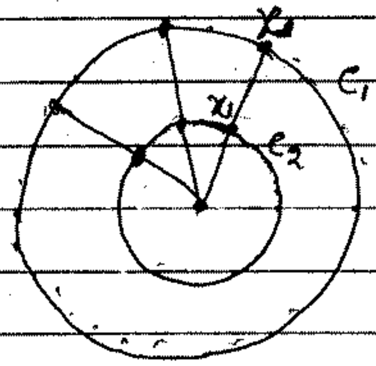
$$f(A - \{a_1\}) = a_2 \text{ where } a_2 \in A - \{a_1\}$$

$$f(A - \{a_1, a_2\}) = a_3 \text{ where } a_3 \in A \setminus \{a_1, a_2\}$$

$$f(A - \{a_1, a_2, a_3\}) = a_4 \quad \text{where}$$
$$a_4 \in A \sim \{a_1, a_2, \dots, a_3\}$$

Since $\{a_1, a_2, a_3, \dots\}$ is a subset of A.
Hence B being the infinite sequence of distinct element is denomerable.

Geometrical interperation of Equivelent Set:-



Consider two circles C_1 and C_2 and defined a mapping f from

$f: C_1 \longrightarrow C_2$ s.t

$f(x) = x_1$ (Point of intersection of the line joing x and the Centre of circle C_1.

Obviously 'f' is bijective and hence

$C_1 \sim C_2$.

Theorem:- Set of real number R is infinite

Proof To show that R is infinite we show a proper subset of R is equivelent to R.

let

$A = \,]-\pi/2, \pi/2[\, \subset R$

we define a mapping
$f : A \longrightarrow R$ s.t.

$f(x) = \tan x$.

let $x_1, x_2 \in A$.
and $f(x_1) = f(x_2)$
$\tan x_1 = \tan x_2$

$\Longrightarrow x_1 = x_2$.

f is one-one.
For each $\tan x \in R$ $\exists$ an element $x \in A$ such that
$f(x) = \tan x$.
$\Rightarrow$ f is onto.
$\Longrightarrow$ f is bijective
$\Rightarrow A \sim R$
$\Longrightarrow$ R is infinite.

Show N is equivalent to $N \times N$.

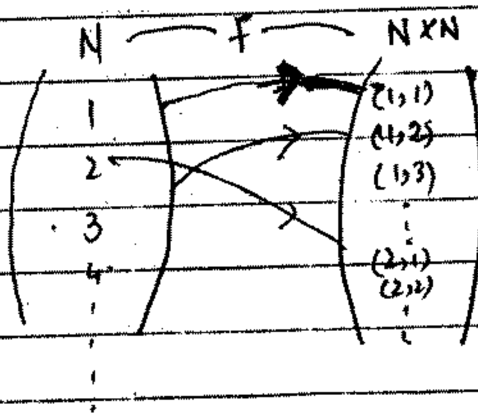
Proof

To show that $N \sim N \times N$
we define a mapping f i.e

$f : N \longrightarrow N \times N$ defined by

$f(a) = (r, s)$
where

$a = 2^{s-1}(2s - 1)$.



f is bijective Hence
$N \sim N \times N$.

Theorem:- Every subset of denomerable is either denomerable or finite.

Proof let A be any denomerable set. Then $N \sim A$.
let $A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$
let B be any subset of A if

$B = \{\}$ Then B is finite
let B be any non-empty sub set of A
i.e
$$B = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots, a_{n_r}, \ldots\}$$
If the indexing set i.e $\{n_1, n_2, n_3, \ldots, n_r, \ldots\}$ is finit. Then B is finite.
If the indexing set i.e $\{n_1, n_2, n_3, \ldots, n_r, \ldots\}$ is infinite. Then B is infinite and clearly B is the sequence of distinct term. So
$$B \sim N$$
$\Rightarrow$ B is denomerable.

Theorem:- Union of family of pairwise disjoine denomerable set is denomerable.

Proof:- let $\{A_1, A_2, A_3, \ldots\}$ be the family of pairwise disjoint denomerable sets

let $A_1 = \{a_1, a_2, a_3, \ldots\}$
$A_2 = \{b_1, b_2, b_3, \ldots\}$
$A_3 = \{c_1, c_2, c_3, \ldots\}$
$\vdots \quad \vdots \quad \vdots$

we have to show that $\cup A_i$ is denomerable where

$$\cup A_i = \{a_1, a_2, a_3, ---, b_1, b_2, b_3, ---, c_1, c_2, c_3, ---\ ----\}$$

As all A_i's are Pairwise disjoint so

$$A_i \cap A_j = \phi. \text{ for some } i,j \in I \text{ \& } i \neq j$$

$\Rightarrow$ Union of A_i's is infinite sequence of ~~disjoint~~ distinct term so

$\Rightarrow$ $\cup A_i$'s is denomerable.

Duality :-

let E be an equation of set algebra Then E^* of E is the equation obtain by replacing $\cup, \cap, \mho$ and ϕ in E by $\cap, \cup, \phi, \mho$ respectively.

for e.g

$$(\mho \cap A) \cup (B \cap A) = A \text{ ------ } (1)$$

By replacing $\mho, \cap, \cup$ by $\phi, \cup, \cap$ Then

$$(\phi \cup A) \cap (B \cup A) = A \text{ ------ } (2)$$

$\Rightarrow$ equation (2) is dual of (1).

Choice function:-

Let $\{A_i\}$ where $i \in I$ be family of non-empty subsets of a set A. Then a function f

$$f : \{A_i\}_{i \in I} \longrightarrow A$$

is called a choice function if for every $i \in I$

$$f(A_i) \in A.$$

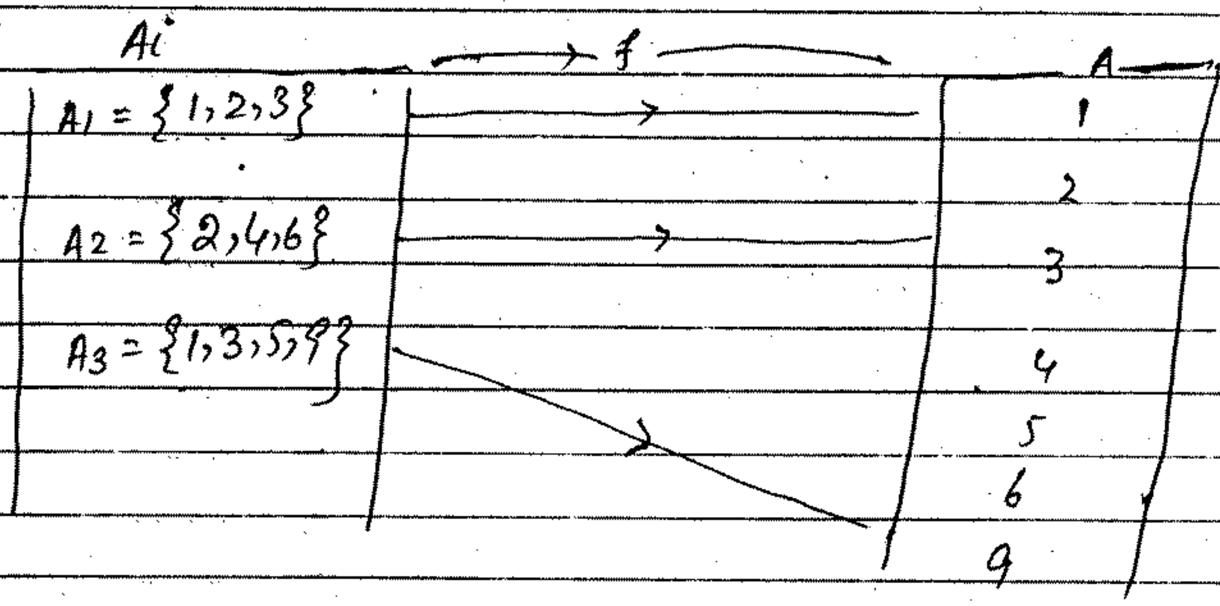
i.e image of each set is an element of the set. A.

Let $A = \{1, 2, 3, 4, 5, 6, 9\}$

$A_1 = \{1, 2, 3\}$

$A_2 = \{2, 4, 6\}$

$A_3 = \{1, 3, 5, 9\}$



Then f is choice function.

Characterstic Function:- let U be a universel set and $A \subset U$ Then charatelstics function

$$\chi_A : U \longrightarrow \{0,1\}$$

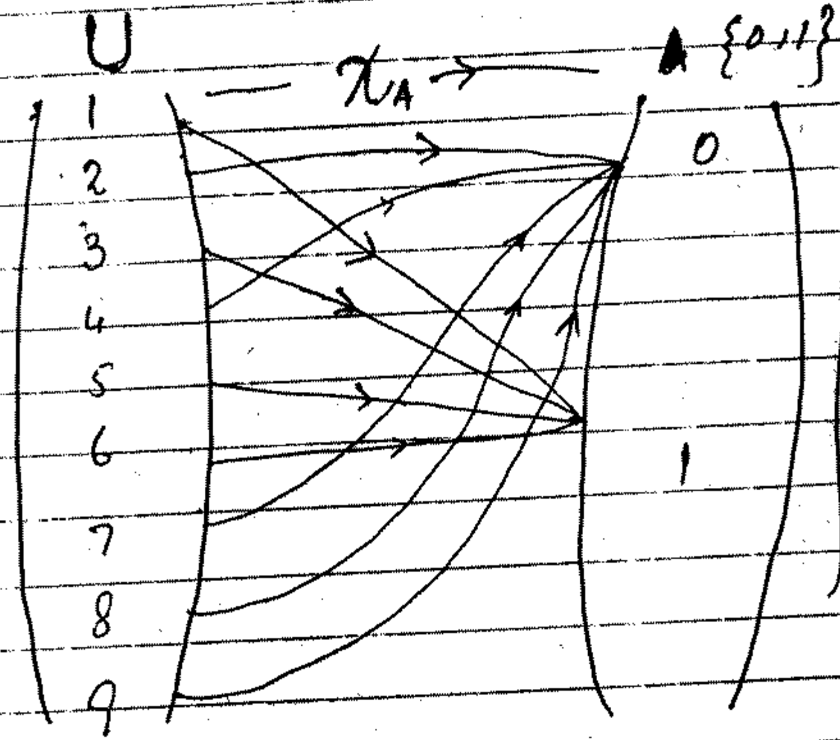
$$\chi_A(x) = \begin{cases} 0 & \text{if } x \notin A. \\ 1 & \text{if } x \in A. \end{cases}$$

e.g.

$U = \{1,2,3,4,5,6,7,8,9\}$

$A = \{1,3,5,6\}$

$$\chi_A : U \longrightarrow \{0,1\}$$



NOTE:-

1) $\chi_{(A\cup B)} = \chi_A + \chi_B$.

$\chi_{A\cap B} = \chi_A \cdot \chi_B$.

represent a charatestic function

Theorem:

Let X be non-empty set and $C(X)$ denote all characterestic functions Then $C(X)$ is equivalent to 2^X i.e

$$C(X) \sim 2^X$$

Proof Let $A \subseteq X$ and

$f: 2^X \to C(X)$ defined by

$$f(A) = \chi_A.$$

To show that f is one-one. Let $A, B \in 2^X$ and suppose that

$$f(A) = f(B)$$

$$\chi_A = \chi_B.$$

$$\Rightarrow A = B.$$

$\Rightarrow f$ is 1-1.

To show that f is onto

For each $\chi_A \in C(X)$ $\exists$ an element $A \in 2^X$ such that

$$f(A) = \chi_A$$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective.

Hence

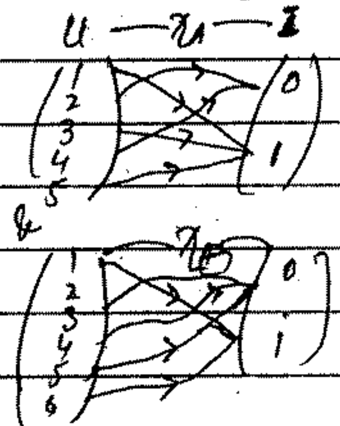
$$2^X \sim C(X)$$

for e.g.

$U = \{1,2,3,4,5\}$

$A = \{1,3,5\}$

$B = \{1,5,6\}$



If $A \neq B$ then $\chi_A \neq \chi_B$.

x x x

Imp

Theorem: Set of rational number Q is denomerable.

Proof: As the set of rational number Q can be written as

$$Q = \overset{+}{Q} \cup \overset{-}{Q} \cup \{0\}$$

we defined a function

$$f : \overset{+}{Q} \longrightarrow N \times N \quad \text{Such that}$$

$$f\left(\frac{p}{q}\right) = (p, q)$$

For $\frac{p}{q}, \frac{p_1}{q_1} \in Q^{+}$.

If $f\left(\frac{p}{q}\right) = f\left(\frac{p_1}{q_1}\right)$

$$\Rightarrow (p, q) = (p_1, q_1)$$

$$\Rightarrow p = p_1 \ \& \ q = q_1$$

$$\Rightarrow \frac{p}{q} = \frac{p_1}{q_1}$$

Hence f is 1-1.

For every $(p, q) \in N \times N$ There exist pere image $f(p/q)$ Such that

$$f(p/q) = (p, q).$$

Hence f is onto.

Hence f is bijective.

So

$$Q^+ \sim N \times N.$$

also

$$N \times N \sim N$$

Therefore by transitive property

$$Q^+ \sim N.$$

Hence Q^+ denomerable.

Again we define a function

$$g : Q^+ \longrightarrow \bar{Q}$$

$$g\left(\frac{p}{q}\right) = -\frac{p}{q}$$

obviously g is bijective. Hence

$$Q^+ \sim \bar{Q} \Rightarrow \bar{Q} \sim N$$

Hence

$\bar{Q}$ is denomerable.

$\bar{Q} \cup \{0\}$ is also denomerable being the infinit sequence of distinct term.

Also we know that union of two denomerable set is denomerable so

$$Q^+ \cup \bar{Q} \cup \{0\} = Q \text{ is denomerable.}$$

Imp

Theorem: If A and B are denomerable Then $A \times B$ (Their Cartesion Product) is denomerable.

Proof: Since A and B are denomerable sets. So both can be express a infinite sequence of distinct term.

let $A = \{a_1, a_2, a_3, \text{------}\}$

$B = \{b_1, b_2, b_3, \text{------}\}$

Then

$$A \times B = \left\{ \begin{array}{l} (a_1, b_1), (a_1, b_2), (a_1, b_3) \cdots\cdots, (a_n, b_1) \text{--} \\ (a_2, b_1), (a_2, b_2), (a_2, b_3) \text{------} \\ (a_3, b_1), (a_3, b_2), (a_3, b_3) \text{------} \\ \text{------------} \\ \text{------------} \\ \text{------------} \end{array} \right\}$$

we define a function 'f'

$f : N \times N \longrightarrow A \times B$. defind as.

$$f(i, j) = (a_i, b_j)$$

let $(i_1, j_1), \& (i_2, j_2) \in N \times N$ also if

$$f(i_1, j_1) = f(i_2, j_2).$$

$$\Rightarrow (a_{i_1}, a_{j_1}) = (a_{i_2}, a_{j_2})$$

$$\Rightarrow a_{i_1} = a_{i_2}, \; \& \; a_{j_1} = a_{j_2}$$

$$\Rightarrow (i_1, j_1) = (i_2, j_2)$$

Hence f is 1-1.

Also for each $(a_i, b_j) \in A \times B$ There exist $(i, j) \in N \times N$ such that

$$f(i,j) = (a_i, b_j)$$

$\Rightarrow f$ is onto. Hence f is bijective and

$$A \times B \sim N \times N.$$

and

$$N \times N \sim N$$

Then by transitive property.

$$A \times B \sim N.$$

$\Rightarrow A \times B$ is denomerable.

Preposition :-

Show That the set of point with rational coordinates in the Plane is denomerable.

Proof :-

let $x \in Q$ Then $x = \frac{p}{q}$ where $p, q \in Z$ & $q \neq 0$

Then p and q are also rationals.

$$A = \{(p, q),\ p, q \in Q\}$$

$$= Q \times Q.$$

we defines a mapping

$f: Q \longrightarrow Q \times Q$ defined as.

$$f\left(\frac{p}{q}\right) = (p, q)$$

Obviously f is bijective. Hence

$Q \times Q \sim Q \Rightarrow Q \times Q$ is deomerable. i.e the set of points with rational coordinate in the plane.

---

(20)

imp

Theorem:- The unit interval $[0,1]$ is non-denumerable.

Proof:- Suppose that the unit interval $[0,1]$ is denumerable. Then it can be written infinit sequence of distinct terms.

$$[0,1] = \{x_1, x_2, x_3, x_4, x_5, ----\}$$

where

$x_1 = 0 \cdot a_{11} a_{12} a_{13} a_{14} a_{15} ---$

$x_2 = 0 \cdot a_{21} a_{22} a_{23} a_{24} a_{25} ----$

⋮

$x_n = 0 \cdot a_{n1} a_{n2} a_{n3} \cdot a_{n4} \cdot a_{n5} ----$

⋮

Now consider $y \in [0,1]$

$y = 0 \cdot b_1 b_2 b_3 b_4 b_5 ----$

where $b_1 \neq a_{11}$ and $b_1 \neq 0$

$b_2 \neq a_{22}$ and $b_2 \neq 0$

$b_3 \neq a_{33}$ and $b_3 \neq 0$

⋮

$b_n \neq a_{nn}$ and $b_n \neq 0$.

$\Rightarrow y \neq x_1, \ y \neq x_2, \ y \neq x_3, ---, y \neq x_n, ----$

$a_{ij} \in \{0, 1, 2, 3, --- 9\}$

$b_j \in \{1, 2, 3, ----, 9\}$

$\Rightarrow y \notin [0,1]$ which is contradiction. (Impossible). Hence our supposition is wrong. Hence unit interval is non-denumerable.

—— x ——

Cardinal Number:

Let A be any set equivalent to B i.e $A \sim B$.

$\sim$ is an equivalence relation. If C is collection of sets in which equivalent relation is defined Then the equivalence class generated by any set $A \in C$ is called cardinal number of A and is denoted by cardinal of A i.e $\#A$. It represent size of the set.

OR

Let A be any set and "α" denote the collection of sets which are equivalent to A. Then 'α' is called cardinal number of A.

e.g:- Cardinality of the following sets ϕ, $\{1\}$, $\{1,2\}$, $\{1,2\}$, $\{a,b,c\}$ is 0, 1, 2, 3.

As cardinality of every finite set is the number of element of that set so these cardinal numbers are finite cardinal numbers.

Generalization of real numbers.

Remark:-

The set A ~~set~~ which is equivalent to $[0,1]$ has cardinal $\mathbf{C}$ and is said to have power of power of continuum.

Cardinality Theorem :- If A and B are finit set Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof As A and B finit sets. let

$A = \{a_1, a_2, a_3, \ldots, a_m, c_1, c_2, c_3, \ldots c_r\}$

$B = \{b_1, b_2, b_3, \ldots, b_n, c_1, c_2, c_3, \ldots c_r\}$

$A \cap B = \{c_1, c_2, c_3, \ldots, c_r\}$

$n(A) = m + r, \quad n(B) = n + r.$

$n(A \cap B) = r.$

$$\begin{aligned} n(A \cup B) &= m + n + r. \\ &= m + n + r + r - r \\ &= (m + r) + (n + r) - r \quad (\text{By adding sub } 'r') \end{aligned}$$

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Case : If A & B are disjoint

ie $A \cap B = \phi$

$\therefore n(A \cap B) = 0.$

Therefore

$$\begin{aligned} n(A \cup B) &= n(A) + n(B) - n(A \cap B) \\ &= n(A) + n(B) - 0 \end{aligned}$$

$$n(A \cup B) = n(A) + n(B)$$

Case If $A \subset B$.
Then $A \cap B = A$

$$n(A \cap B) = n(A)$$

As
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= \cancel{n(A)} + n(B) - \cancel{n(A)}$$

$$n(A \cup B) = n(B)$$

If $B \subset A$ Then
$$n(A \cup B) = n(A).$$

— x — x — x — x

Arithematic of Cardinals:-

let A and B be two disjoint sets and
$$\# A = \alpha \; ; \; \# B = \beta .$$
Then
$$\#(A \cup B) = \# A + \# B.$$
$$= \alpha + \beta$$

$$\#(A \cup B) = \alpha + \beta.$$

Similarly

$$\#(A \times B) = \# A \cdot \# B$$
$$\#(A \times B) = \alpha \cdot \beta$$

— x — x — x —

* Transfinit Cardinal Numbers:- Cardinal numbers of infinite sets are called transfinite cardinal number

Cardinality of N is $\aleph_0$ (alph-naught)

That is

$$\text{Card } N \text{ or } \# N = \aleph_0$$

$$\text{Card } (\mathbb{Z}) \text{ or } \# \mathbb{Z} = \aleph_0 \text{ as } \mathbb{Z} \sim N.$$

$$\text{Card } (\mathbb{Q}) = \aleph_0 \text{ as } \mathbb{Q} \sim N.$$

$$\text{Card } [0,1] = C$$

$$\text{Card } \mathbb{R} = C \quad \because \mathbb{R} \sim [0,1]$$

Further more cardinality of any interval on real line C. bcs every interval on Real line is equivalent to unit interval.

Example:- If n is a finite number Then

$$n + \aleph_0 = \aleph_0$$

Sol:- Let $A = \{1, 2, 3, \text{---}, n\}$

$$B = \{n+1, n+2, n+3, \text{---}\}$$

$$A \cup B = \{1, 2, 3, \ldots\} = N.$$

$$Card(A \cup B) = Card(N)$$

$$card(A) + card(B) = \aleph_0$$

$$n + \aleph_0 = \aleph_0$$

NOTE:- If we add finite number into transfinite Then again it is transfinite.

Remarks

It may be noted That addition and multiplication of finite cardinal number crosspond to ordinary addition and multiplication of natural number.

But in the case infinite cardinal number the addition and multiplication does not crosspond to ordinary addition and multiplication of natural number.

Theorem:- Imp Show that ordinary addition & multiplication does not hold infinite cardinal number.

Also Cansalation Law do not hold in infinite cardinal number.

Proof:- Let

$$A = \{1, 3, 5, 7, \ldots\}$$
$$B = \{2, 4, 6, \ldots\}$$

$$\# A = \aleph_0$$
$$\& \ \# B = \aleph_0$$

Also $\# N = \aleph_0$

$$\#(A \cup B) = \#A + \#B$$
$$\# N = \aleph_0 + \aleph_0$$

$$\aleph_0 = \aleph_0 + \aleph_0$$

which is a contradiction to ordinary addition.

$$\#(A \times B) = \#A \cdot \#B$$

$$\#(N \times N) = \aleph_0 \cdot \aleph_0 \quad \text{Since } A \sim N \text{ \& } B \sim N$$
$$\# N = \aleph_0 \cdot \aleph_0$$
$$\aleph_0 = \aleph_0 \cdot \aleph_0$$

which is a contradiction to ordinary multiplication.

It can be observed in the above example that $\aleph_0$ is neither zero (0) nor 1. (ONE).

$$\aleph_0 + 1 = \aleph_0$$

$$\aleph_0 + 1 = \aleph_0 + \aleph_0 \quad \because \aleph_0 = \aleph_0 + \aleph_0$$

$$\not\Rightarrow \aleph_0 = 1$$

$\Rightarrow$ Cancellation laws does not hold (under addition). in infinite cardinal number.

$$\aleph_0 \cdot 2 = \aleph_0$$
$$\aleph_0 \cdot 2 = \aleph_0 \cdot \aleph_0 \quad \because \aleph_0 = \aleph_0 \cdot \aleph_0$$

$\Rightarrow$ No = 1.

so Cancelation laws under multiplication do not hold in infinite cardinal numbers.

—x— —x— —x— —x—

Theorem: For any cardinals α, β, r we have

i) $\alpha+(\beta+r) = (\alpha+\beta)+r$

ii) $\alpha(\beta r) = (\alpha\beta)r$

iii) $\alpha+\beta = \beta+\alpha.$

iv) $\alpha\beta = \beta r$ ✓

v) $\alpha(\beta+r) = \alpha\beta + \alpha r.$

Two show that $\#A = \#B$ then show that $A \sim B$ i.e. $f: A \to B$ is bijective mapping.

Proof: let A, B, C be three disjoint set such that.

$$\#A = \alpha, \quad \#B = \beta, \quad \#C = r.$$

$$\#(A \cup (B \cup C)) = \alpha + (\beta + r).$$

$$\#((A \cup B) \cup C) = (\alpha+\beta)+r.$$

L.H.S $\quad \alpha + (\beta + r)$

$= \# A \cup (B \cup C)$

$= \#((A \cup B) \cup C)$

$= (\alpha+\beta) + r \quad = R.H.S.$

Hence

$$\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma.$$

iv) we have to show that

$$\#(A\times B) = \#(B\times A).$$

we define a function

$$f: A\times B \longrightarrow B\times A.$$

s.t.

$$f(a,b) = (b,a).$$

let (a_1,b_1) and $(a_2,b_2) \in A\times B$

and let

$$f(a_1,b_1) = f(a_2,b_2).$$

$$(b_1,a_1) = (b_2,a_2).$$

$$\Rightarrow b_1=b_2 \ \& \ a_1=a_2.$$

Hence

$$(a_1,b_1) = (a_2,b_2).$$

so function is 1-1.

for every (b,a) there exist an element $(a,b) \in A\times B$ such that

$$f(a,b) = (b,a)$$

Since fn is onto

Hence f is bijective $\Rightarrow$

$$A\times B \cong B\times A$$

Hence

$$\#(A\times B) = \#(B\times A)$$

iii) $\alpha+\beta = \beta+\alpha$.

Let A, B be disjoint sets. Such that

$$\# A = \alpha, \quad \# B = \beta.$$
$$\#(A \cup B) = \#A + \#B = \alpha+\beta.$$
$$\#(B \cup A) = \beta+\alpha.$$

L.H.S

$$\alpha+\beta = \#A + \#B$$
$$= \cancel{\#B + \#A} \quad \#(A \cup B)$$
$$= \#(B \cup A)$$
$$= \#B + \#A.$$
$$= \beta+\alpha$$

Hence

$$\alpha+\beta = \beta+\alpha.$$

ii) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

we have to prove that

$$\# A \times (B \times C) = \#(A \times B) \times C.$$

we define a mapping

$f: A \times (B \times C) \longrightarrow (A \times B) \times C$. defeind as

$$f(a, (b, c)) = ((a, b), c)$$

we are to show that f is bijective.

for this first we show that f is 1-1.

let $(a_1, (b_1, c_1))$ and $(a_2, (b_2, c_2)) \in A \times (B \times C)$

Also let $f(a_1, (b_1, c_1)) = f(a_2, (b_2, c_2))$

$$((a_1, b_1), c_1) = ((a_2, b_1), c_2)$$

$(a_1, b_1) = (a_2, b_2)$ & $c_1 = c_2 \Rightarrow a_1 = a_2, b_1 = b_2, c_1 = c_2$.

$$\Rightarrow (a_1, (b_1, c_1)) = (a_2, (b_2, c_2))$$

For every $((a,b),c)$ There exist an element $(a,(b,c)) \in A\times(B\times C)$ such that

$$f(a,(b,c)) = ((a,b),c).$$

Hence f is bijective.

Then

$$\# A\times(B\times C) = \# (A\times B)\times C.$$

Hence $\alpha(\beta r) = (\alpha\beta)r.$

(V) . $\alpha(\beta+r) = \alpha\beta + \alpha r$

Proof :-

$$\# A\times(B\cup C) = \alpha(\beta+r)$$
$$\# A\times B = \alpha\beta.$$
$$\# A\times C = \alpha r.$$

$$\alpha(\beta+r) = \#[A\times(B\cup C)]$$
$$= \#[A\times B \cup A\times C]$$
$$= \# A\times B + \# A\times C.$$
$$= \alpha\beta + \alpha r$$

Prove that $C + C = C$?

As we have

$$\# [0, 1/2] = C$$

$$\# (1/2, 1] = C$$

$$[0, 1/2] \cup (1/2, 1] = [0, 1]$$

$$\# ([0, 1/2] \cup (1/2, 1]) = \# [0, 1]$$

$$\# [0, 1/2] + \# (1/2, 1] = \# [0, 1]$$

$$C + C = C \quad \text{Hence proved.}$$

Definition: If a set A is equivalent to a subset of B then A is said to proceeds B or simply A proceeds B and write as $A \preccurlyeq B$.

If A is equivalent to proper subset of B. Then we say A strictly proceed B i.e $A \prec B$. and if A is equivalent to exactly B. Then we say $A \sim B$.

Ordering of Cardinal Numbers:

Let A & B two sets then cardinal of A. i.e $\text{Card } A \leq \text{Card } B$ if A is equivelent to a subset of B.

OR

If There exist 1-1 ftn $f: A \to B$ Then $\text{Cr } A \leq \text{Card } B$.

Theorem: Show That $\aleph_0 < C$

Define a function $f: N \longrightarrow [0,1]$

$$f(n) = \frac{1}{n}.$$

For 1-1 let $n_1, n_2 \in N$

$$f(n_1) = f(n_2).$$

$$\frac{1}{n_1} = \frac{1}{n_2}.$$

$\Longrightarrow n_1 = n_2$

$\Longrightarrow$ f is 1-1.

& It is not onto bcs $0 \in [0,1]$ and There is no such $n \in N$ such that $f(n) = 0$. Hence N strictly proceeds closed unit interval $[0,1]$ i.e

$$N \prec [0,1]$$

$$\text{Card } N < \text{Card } [0,1]$$

$$\aleph_0 < C.$$

Imp.

Cantor's Theorem:-

For any set A, $A \prec 2^A$ and hence for cardinal number $\alpha = \# A$, we have

$$\alpha < 2^{\alpha}$$

Proof:- Define a function

$$f : A \longrightarrow 2^A \text{ defined as}$$

$$f(a) = \{a\} \quad \forall\ a \in A.$$

Since every singleton set $\{a\}$ for all $a \in A$ is a subset of A. Therefore it can be observed A is equivalent a collection of all singleton in 2^A (Power set) which form a subset of 2^A. Therefore

$$A \prec 2^A. \quad \text{—— (1)}$$

we will have to show that

$$A \not\sim 2^A.$$

On contrary suppose there exist a function

$$g : A \longrightarrow 2^A \text{ which is bijective}$$

we say $a \in A$. is a bad element if $a \notin g(a)$ i.e 'a' does not belonges to its image.

Let B be the set of all bad elements. Then $B = \{a \mid a \in A \wedge a \notin g(a)\}$

Since 'g' is onto and $B \in 2^A$ then there exist an element say $b \in A$ such that

$$g(b) = B.$$

Case I If b. is bad element

$$b \notin g(b) = B$$

$$\Rightarrow b \notin B$$

which is controduction as B is the set of all bad elements

Case-II

If b is not bad element-

$$\Rightarrow b \in g(b) = B.$$

$\Rightarrow b \in B$ which is again a controduction as b is not a bad element.

This conclude There exist no onto function. That is our assumption is wrong. Hence

$$A \nsim 2^A$$

∴ From (1)

$$A \precsim 2^A$$

$$\text{Card } A < \text{Car } 2^A$$

$$\alpha < 2^\alpha$$

Hence proved

— x — x — x —

*

Theorem: Prove That $2^{\aleph_0} = C$

Proof:

Since $Q \sim N$

$\text{Card } Q = \aleph_0$

and

$\text{Card } 2^{Q} = 2^{\aleph_0}$

we define $f: R \longrightarrow 2^{Q}$ defined as such that

For $a, b \in R$ $f(a) = \{x : x \in Q \ \& \ x < a\}$

& If $a \neq b$

$\Rightarrow a < b$ (say).

Then There exist $r \in Q$ such that

$a < r < b$.

$\Rightarrow r < b$

$\Rightarrow r \in f(b)$. also $r > a$

$\Rightarrow r \notin f(a)$.

$\Rightarrow f(a) \neq f(b)$.

$\Rightarrow f$ is 1-1.

Therefore

$R \precsim 2^{Q}$

$\Rightarrow \text{Card } R \leq \text{Card } 2^{Q}$. $P(X) = 2^{X}$

$C \leq 2^{\aleph_0}$ —— (1) $P(N) = 2^{N}$

$C(N) =$

Now Suppose that N is a set of natural number and $C(N)$ be the set of all Characteristic functions of Subsets of N. we also know that $C(N) \sim 2^{N}$

$$C(N) \sim 2^N$$

i.e

$$\text{Card } C(N) = \text{Card } 2^N$$

$$= 2^{\aleph_0}$$

we define

$$F : C(N) \longrightarrow [0,1]$$

$$F(f) = 0.f(1)f(2)f(3)\ldots\ldots$$

If $f \neq g$ for $f, g \in C(N)$ then clearly

$$F(f) \neq F(g)$$

Hence

F is 1-1.

Therefore

$$C(N) \preceq [0,1]$$

$$\text{Card } C(N) \leq \text{Card } [0,1] \quad \text{—(2)}$$

$$2^{\aleph_0} \leq c$$

From (1) & (2)

$$c = 2^{\aleph_0} \text{ //}$$

EXPONENT OF CARDINAL NUMBER:-

Let A and B be two set. Then B^A represents the set of all functions from A to B. we defined

$$\# B^A = \beta^\alpha .$$

e.g:-

$$\text{let} \quad A = \{a, b, c\}$$

$$B = \{0, 1\}$$

$$\Rightarrow \quad \# A = 3, \quad \# B = 2$$

$$\alpha = 3, \quad \beta = 2 .$$

Now we check All functions from A to B.

$$f_1 = \{(a,0), (b,0), (c,0)\}$$

$$f_2 = \{(a,1), (b,1), (c,1)\}$$

$$f_3 = \{(a,0), (b,1), (c,1)\}$$

$$f_4 = \{(a,1), (b,0), (c,1)\}$$

$$f_5 = \{(a,1), (b,1), (c,0)\}$$

$$f_6 = \{(a,1), (b,0), (c,0)\}$$

$$f_7 = \{(a,0), (b,1), (c,0)\}$$

$$f_8 = \{(a,0), (b,0), (c,1)\}$$

$$B^A = \{f_1, f_2, f_3, \ldots, f_8\}$$

$\# = 8$ — (1), also $\beta^\alpha = 2^3 = 8$. — (2)

from (1) & (2)

Prepositions:

Let A, B and C are sets having Cardinal number

$$\# A = \alpha, \quad \# B = \beta$$
$$\text{and} \quad \# C = \gamma$$

Then show that:

i) $(\alpha)^\beta \cdot (\alpha)^\gamma = (\alpha)^{\beta+\gamma}$.

ii) $(\alpha\beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$

iii) $((\alpha^\beta))^\gamma = \alpha^{\beta\gamma}$

Proof: As α^β, α^γ & $(\alpha)^{\beta+\gamma}$ are the cardinalities of A^B, A^C and $A^{B\cup C}$ respectively so $\alpha^\beta \cdot \alpha^\gamma$ be the cardinality of $A^B \times A^C$

Now we define a function

$$F: A^{B\cup C} \longrightarrow A^B \times A^C \quad \text{defined as}$$

$$F(f) = (f|_B, f|_C)$$

Hence obviously "F" is 1-1 & onto so F is bijective. So

$$A^{B\cup C} \simeq A^B \times A^C$$
$$\Rightarrow \# A^{B\cup C} = \# (A^B \times A^C)$$

$$\Rightarrow \quad (\alpha)^{\beta+r} = \alpha^{\beta} \cdot \alpha^{r}$$

--- x --- x --- x ---

ii) $(\alpha\beta)^r = \alpha^r \cdot \beta^r$

Proof $\quad \# A = \alpha, \quad \# B = \beta, \quad \# C = r$

$$\# (A \times B)^C = (\alpha\beta)^r, \quad \# A^C = \alpha^r$$

and

$$\# B^C = \beta^r.$$

$(A\times B)^C$ be the set of all functions from $C \longrightarrow A\times B$

i.e

$$f : C \longrightarrow A \times B.$$

Now we define

$$g : C \longrightarrow A$$
$$\Rightarrow g \in A^C$$

and $\quad h : C \longrightarrow B$

$$\Rightarrow h \in B^C.$$

Now we defined

$$F : (A\times B)^C \longrightarrow A^C \times B^C$$

such that

$$F(f) = (g, h)$$

For 1-1

Let $g_1, g_2 \in A^C, \; h_1, h_2 \in B^C$

and $f_1, f_2 \in (A\times B)^C$

if

$$F(f_1) = F(f_2)$$
$$\Rightarrow (g_1, h_1) = (g_2, h_2).$$

$\Rightarrow \quad g_1 = g_2$ and $h_1 = h_2$.

$\Rightarrow \quad f_1 = f_2$.

$\Rightarrow$ F is one-one.

For each $(g, h) \in A^C \times B^C$ $\exists$ a function $f \in (A \times B)^C$ such that

$$F(f) = (g, h)$$

$\Rightarrow$ F is onto.

$\Rightarrow$ F is bijective

Hence

$$(A \times B)^C \sim A^C \times B^C.$$

$$\# (A \times B)^C = \# (A^C \times B^C)$$

$(\alpha\beta)^\gamma = \alpha^\gamma \cdot \beta^\gamma$ which is required result.

iii) $$\left((\alpha^\beta)\right)^\gamma = \alpha^{\beta\gamma}$$

$$\# (A^B)^C = \left((\alpha^\beta)\right)^\gamma, \quad \# (A)^{B \times C}.$$

$(A^B)^C$ is the set of ftns such that

$$f : C \longrightarrow A^B.$$

$$\Rightarrow f \in (A^B)^C.$$

and

$(A)^{B \times C}$ is the set of functions such that

$$h : B \times C \longrightarrow A.$$

$$\Rightarrow h \in (A)^{B \times C}$$

Now we defined a function

$$F : (A^B)^C \longrightarrow (A)^{B\times C}$$

Such that

$$F(f) = h$$

let

$$f_1, f_2 \in (A^B)^C \quad \text{and} \quad h_1, h_2 \in (A)^{B\times C}.$$

Then

if

$$F(f_1) = F(f_2)$$

$$h_1 = h_2.$$

$$\Rightarrow f_1 = f_2$$

$$\Rightarrow \quad F \text{ is one-one.}$$

Also for each $h \in (A)^{B\times C}$ There exist a function $f \in (A^B)^C$ such that

$$F(f) = h.$$

Hence F is onto.

so F is bijective. Then $(A^B)^C \sim (A)^{B\times C}$

$$\Rightarrow \quad \#(A^B)^C = \#(A)^{B\times C}.$$

$$\Rightarrow \left((\alpha^\beta)\right)^r = (\alpha)^{\beta r}.$$

x x x

Preposition.

Let A and B be two set Such that $\# A = \alpha$, $\# B = \beta$

if $\alpha \leq \beta$ Then show That

i) $\alpha^r \leq \beta^r$

ii) $\gamma^\alpha \leq \gamma^\beta$.

proof

ii) TO Show That $\alpha^r \leq \beta^r$.

$$\alpha^r = \# A^C, \quad \beta^r = \# B^C$$

where

A^C is the set of all functions Such that $f: C \longrightarrow A$.

$$\Rightarrow f \in A^C$$

Since

$$\alpha \leq \beta$$

$$\Rightarrow A \subseteq B$$

let

Cancelled

Preposition:- let A and B be two sets such that $\# A = \alpha$, $\# B = \beta$.
if $\alpha \leq \beta$. Then show that

i) $r^{\alpha} \leq r^{\beta}$ where $r = \# C$

ii) $\alpha^{r} \leq \beta^{r}$

Proof

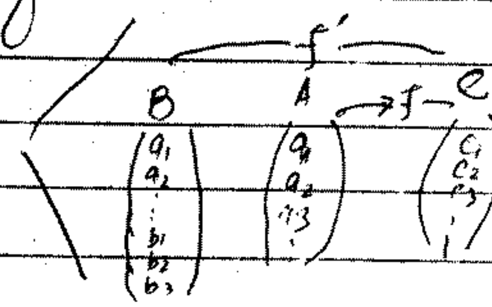
i) To show that $r^{\alpha} \leq r^{\beta}$

since $r^{\alpha} = \# C^{A}$, $r^{\beta} = \# C^{B}$.

where C^{A} is the set of all functions such that

$f : A \longrightarrow C$

$\Rightarrow f \in C^{A}$.



Since $\alpha \leq \beta$

$\Rightarrow A \subseteq B$.

let

$f' : B \rightarrow C \Rightarrow f' \in C^{B}$.

if $f \neq g \Rightarrow f' \neq g'$ $\because A \subset B$

So we defined

$F : C^{A} \longrightarrow C^{B}$ as

$F(f) = f'$ which is one-one.

$C^{A} \underline{\propto} C^{B}$

$\# C^{A} \leq \# C^{B}$

$r^{\alpha} \leq r^{\beta}$.

ii) $\alpha^r \le \beta^r$

Proof $\alpha^r = \# A^C$, $\beta^r = \# B^C$

where A^C is the set of all function such that

$$f : C \longrightarrow A.$$

$$\Longrightarrow f \in A^C$$

Since

$$\alpha \le \beta.$$

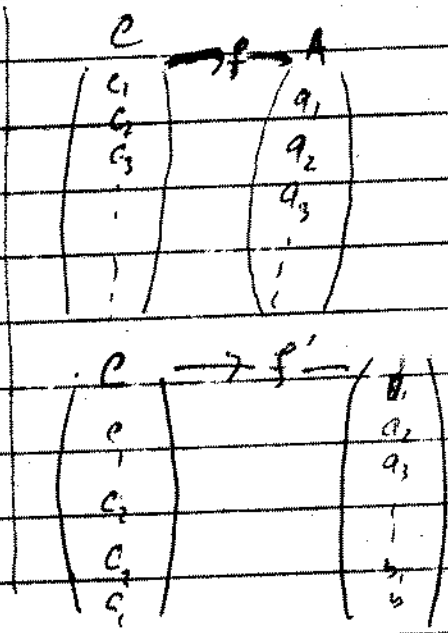
Therefore

$$A \subseteq B.$$

let

$$f' : C \longrightarrow B.$$

$$\Longrightarrow f' \in B^C$$



if $f \neq g \Longrightarrow f' \neq g'$

So we defined

$$F : A^C \longrightarrow B^C \text{ as.}$$

$$F(f) = f' \text{ which is one-one}$$

$$A^C \precsim B^C$$

$$\# A^C \le \# B^C.$$

$$\alpha^r \le \beta^r$$

— x — x — x — x —

** Imp This Theorem is appliable only for infinit set

Theorem:- Schroeder Bernstein Theorem:-

If $X \supset Y \supset X_1$ and $X \sim X_1$ Then $X \sim Y$.

Proof

Since $X \sim X_1$, Then $\exists$ a function "f" such that

$$f: X \longrightarrow X_1 \text{ is bijective.}$$

Since Y is contained in X. Then restriction of "f" from Y to X_1 is one-one (not onto). There exist a subset Y_1 of X_1. (i.e $X_1 \supset Y_1$) and $f: Y \longrightarrow Y_1$ is bijective.

$$\Rightarrow Y \sim Y_1$$

By Similar reasoning we will get a $X_2 \subset Y_1$ Then by given hypothesis $X_1 \supset Y_1 \supset X_2$ such that:

$X_1 \sim X_2$ we have.

$$X \supset Y \supset X_1 \supset Y_1 \supset X_2 \supset Y_2 \cdots$$

proceeding in the same way we will get sequence of equivalent sets

$$X \sim X_1 \sim X_2 \sim X_3 \sim X_4 \sim \cdots$$

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$$Y \sim Y_1 \sim Y_2 \sim Y_3 \sim \cdots$$

Now as

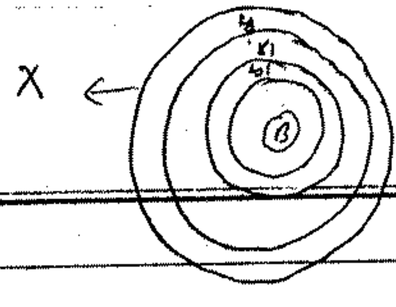
$$X \sim X_1 \text{ and } Y \sim Y_1$$

$\Rightarrow X - Y \sim X_1 - Y_1$. By property of Cardinality

Similarly

$$X_1 - Y_1 \sim X_2 - Y_2 \text{ , } X_2 - Y_2 \sim X_3 - Y_3$$

and so on.



Now we can write

$$X = (X - Y) \cup (Y - X_1) \cup (X_1 - Y_1) \cup \cdots \cup B$$

where

$$B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap \cdots$$

$$Y = (Y - X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cup \cdots \cup B$$

we know that

$$X_n - Y_n \sim X_{n+1} - Y_{n+1}$$

Then there exist a function g such

$$g : X_n - Y_n \longrightarrow X_{n+1} - Y_{n+1} \text{ is}$$

bijective.

we defined a function 'h'

Such that

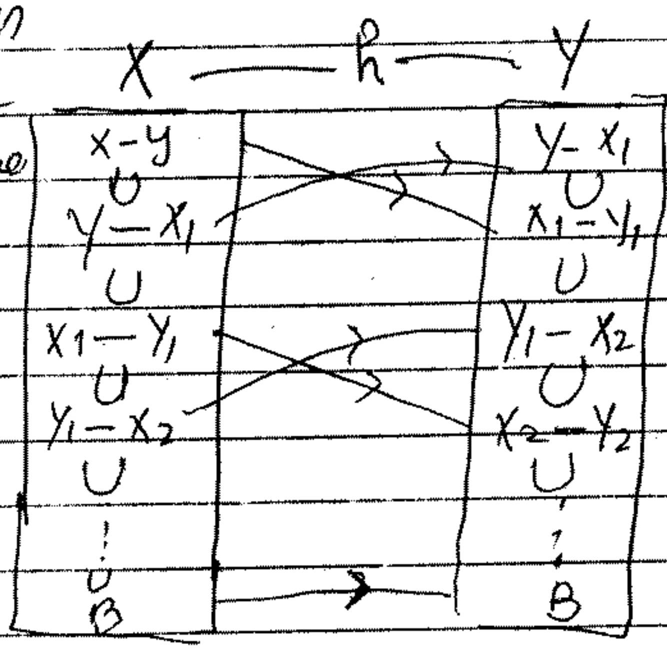
$$h : X \longrightarrow Y$$

$$h(x) = \begin{cases} x & \text{if } x \in Y_i - X_{i+1} \text{ or } B \\ g(x) & \text{if } x \in X_i - Y_i \text{ and } \\ & x \in X - Y. \end{cases}$$

Which shows 'h' is both one-one and onto hence bijective

Hence

$$X \sim Y.$$



—— X —— X —— X ——

Ordered Set:-

Any set in which order is defined is called ordered set.

* Partially ordered Set:-

If A is a non-empty set and R be the relation defined in A. Such that

(1) R is reflexive. i.e $\forall a \in A$ si aRa.

(2) R is antisymmetric. i.e

if aRb and bRa

Then $a = b$

3) R is transitive i.e

if aRb and bRc Then

$aRc \quad \forall a, b, c \in A$.

Set 'A' together with partial ordered relation is called partially ordered set. Further more if relation 'R' in 'A' defines a partial order in A. Then order pair $(a, b) \in R$ i.e aRb. It is denoted by $a \preceq b$ (a proceed b or b dominate a). we also write $a \prec b$ if $a \neq b$ which reades as a strictly proceed b or b strictly dominate b. Some time the term Poset (partially order set) is used for (partially order set).

Ex:

Consider the set P (set of natural number) of +ve intergers we say $a|b$ (a divides b) if There exist an integer $c \in P$ such that $ac = b$. for ex. $2|4$, $3|12$

This relation of divisibility is partially ordering of P.

Ex. Consider the Z set of integer defined $a R b$ if there is +ve integer r such that $b = a^r$

Reflexive:

Since for all $a \in Z$. we can write

$a = a^1 \Rightarrow aRa$ i.e. where $r = 1 \in Z^+$

R is reflexive.

Antisymmetric :-

If aRb and bRa

$\Rightarrow b = a^r$ and $a = b^s$

where $r, s \in Z^+$

$$a = b^s$$
$$= (a^r)^s$$
$$= a^{rs}$$

There arises three cases

(i) if $rs = 1$ then $r = 1$ & $s = 1$

$\Rightarrow a = b$

ii) $a = 1$ then $b = 1$

$\Rightarrow a = b$.

iii) If $a = -1$ Then $b = -1$ where $s \in O$.
i.e s is odd.

In all case R is anti symmetric.

<u>Transitive</u> :-
If aRb and bRc. For $a, b, c \in \mathbb{Z}$.
Then $\exists\ r, s \in \mathbb{Z}^+$ s.t

$$b = a^r \text{ and } c = b^s.$$

$$\Rightarrow c = (a^r)^s$$
$$= a^{rs}$$
$$c = a^q \text{ where } rs = q \in \mathbb{Z}$$

$\Rightarrow aRc$ Hence R is Transitive

Hence R is partial ordering relation and set of integer is partially order set under the relation R.

Q. ✓ (imp)
Let $\mathcal{A}$ be collection of ~~Subset~~ sets Then the relation of set inclusion is a partial order relation.

Sol:- <u>Reflexive</u>:
Since every set is subset of their own. Therefore For any set $A \in \mathcal{A}$

$$A \subseteq A$$
$$\Rightarrow ARA.$$
$$\Rightarrow R \text{ is reflexive.}$$

ii) Anti Symmetry:

Let $A, B \in \mathcal{A}$ and

$A R B$ and $B R A$.

$\Rightarrow A \subseteq B$ and $B \subseteq A$.

$\Rightarrow A = B$.

Hence R is antisymmetry.

iii) Transitive property:

Let $A, B, C \in \mathcal{A}$ and

$A R B$ and $B R C$.

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C$.

$\Rightarrow A R C$.

$\Rightarrow R$ Transitive.

$\Rightarrow R$ is partial ordering relation & $\mathcal{A}$ is partially ordered set.

* — * — * — * — *

Examples: Let $A = \{a, b, c, d, e\}$

Then the relation defined by

$x \preceq y$ if $x = y$ or if one can go from x to y in the diagram moving in identical direction.

$$\begin{tikzcd}
b & a \arrow[l] \\
d \arrow[u] \arrow[r] & c \arrow[u] \\
 & e \arrow[u]
\end{tikzcd}$$

Sol:- The relation defined on A is a

$$R = \{(a,a), (b,b), (c,c), (d,d), (e,e), (e,c), (c,a), (e,a), (b,a), (d,b), (b,c)\}$$

Reflexive

Since $\forall\ x \in A$

$x = x$.

$\Rightarrow x \preceq x$.

$\Rightarrow x R x$

Antisymmetric :-

Now for $x, y \in A$.

and $x R y$ & $y R x$. i.e

$x \preceq y$ & $y \preceq x$.

$\Rightarrow x = y$.

$\Rightarrow R$ is antisymmetric

Transitive :-

For $x, y, z \in A$.

If $x R y$ & $y R z$ Then we are to show that $x R z$. i.e

$x \preceq y$ and $y \preceq z$.

$\Rightarrow x \preceq z$

$\Rightarrow x R z$.

$\Rightarrow (x, z) \in A$.

$\Rightarrow R$ is transitive.

Hence the relation R is partial ordering relation. so A is partial order set.

Ex:-

let $A = \{2, 3, 4, 5, 6\}$

and R be defined by x/y.

Then R is partial order.

Sol:-

R is define by x/y.

so

$R = \{(2,2), (3,3), (4,4), (5,5), (6,6),$
$(2,4), (2,6), (3,6)\}$

1) Reflexive:-

since

$\forall \; x \in A$

$x/x \Rightarrow x \preceq x$.

$\Rightarrow x R x$.

$\Rightarrow R$ is reflexive

2) Antisymmetry:-

for $x, y \in A$.

If $x \preceq y$ and $y \preceq x$

$\Rightarrow x = y$.

3) Transitive:

For $x, y, z \in A$

if xRy and yRz then

we are two show that xRz.

since

xRy and yRz.

$\Rightarrow x \preceq y$ and $y \preceq z$.

$\Rightarrow x/y$ and y/z.

$\Rightarrow x/z \Rightarrow x \preceq z$

$\Rightarrow xRz \Rightarrow R$ is Transitive

Hence R is partial ordering relation and A is partial ordered set.

— $\ast$ — $\ast$ — $\ast$ — $\ast$ —

~~Qtn~~ let A be any sub set of R. Then the relation R in A is defined by $x \leq y$ is a partial order in A. It is called natural order or usual order relation.

Sol:- let A be any non-empty sub set of R.

Reflexive

Since $\forall\ x \in A$ such that

$x \leq x$

$\Rightarrow x \alpha x$

$\Rightarrow x R x$.

$\Rightarrow R$ is reflexive.

Antisymmetry:-

let $x, y \in A$.

If $x \leq y$ and $y \leq x$

$\Rightarrow x \alpha y$ and $y \alpha x$

$\Rightarrow x = y$

$\Rightarrow R$ is Anti Symmetric

Transitive:

let $x, y, z \in A$

if $x \leq y$ and $y \leq z$

$\Rightarrow x \leq z \Rightarrow x \alpha z$

$$x \preceq z$$
$$x \leq z$$
$$xRz$$

Hence

R is Transitive.

Hence $A \subset R$ is partially ordered set under natural order relation.

$\dot{x}$ ——— $\dot{x}$ ———

NOTE:- Similarly set of natural number can also be showed that it is partially orderd set.

$$N = \{1, 2, 3, 4, \text{---}\}$$

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow \cdots$$

<u>Remark:</u> let A be a partially order set and R be a partial order relation. If $(a,b) \notin R$ we say 'a' and 'b' are not comparable. i.e $a \not\preceq b$ and $b \not\preceq a$ also $a \neq b$.

Inverse Order:-

let R be a partial order defined in a set A then R^{-1} in A is also partial order

$$R^{-1} = \{(y,x) \mid (x,y) \in R\}$$

Ex:-

let $A = \{a,b,c,d,e\}$

& $R = \{(a,a),(b,b),(c,c),(d,d),(e,e)$
$(e,a),(d,b),(b,a),(d,a),(d,c)\}$

is a partial order then show that R^{-1} is also a partial order.

Sol:-

Now $R^{-1} = \{(a,a),(b,b),(c,c),(d,d),(e,e)$
$(a,e),(b,d),(a,b),(a,d),(c,d)\}$

i) <u>Reflexive</u> Since $\forall\ x \in A$

$\cancel{x \preceq x} \Rightarrow (x,x) \in R^{-1}$

$\Rightarrow x \preceq x$
$= xRx$

Anti Symmetric :-

Since, for $x, y \in A$

if $(x,y), (y,x) \in R^{-1}$

$\Rightarrow x = y$

iii) Transitive property :-

For any $x, y, z \in A$

If $(x,y), (y,z) \in R^{-1}$ Then

$(x,z) \in R^{-1}$

$\Rightarrow x \propto z$

$\Rightarrow x R z$

Hence R^{-1} is partial order.

x — x — x — x

Totally Order Set :-

Let R be a partial order relation defined in A. Then R is said to be totaly order if for every pair $x, y \in A$, $x \propto y$ or $y \propto x$ or $x = y$

i.e

If every element of set A is comparable with other element of the set for example the natural order is total order.

Ex:- i) set of natural number under natural order relation is totally ordered.

ii) set of Real number under relation natural order relation.

Totaly orderd set is partially orderd But converse may not True.

for e.g.

$A = \{1, 2, 3, 4, 5, 6\}$

is P.O under the relation " | " (Divisibility) but not totally order be $2 \nmid 3$. i.e 2 & 3 are not comparable under that relation.

** Every open and closed interval is totally order under the natural order relation.

First Element:-

let "A" be an order set Then $a \in A$ is called First Element of A. if $a \preceq x \; \forall \; x \in A$

i.e every element of A dominate "a".

For e.g. $\{1, 2, 3, 4\}$ Then 1^{st} element under ~~set~~ natural order relation is 1.

Last Element

let "A" be an order set, Then "b" $\in A$ is called last element of A. If

$x \preceq b \; \forall \; x \in A$.

i.e every element of A proceeds "b" or "b" dominate evevery element of A.

Remarks:- Every totally orderd set need not be a first element or last or both

for e.g:- $\{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \}$

$\{1, 2, 3, \ldots \}$, $\{\frac{1}{n}\}$.

ii) If a set A has first and last element Then 'A' needs not be totally orderd.
for e.g:-
$$\{2, 4, 6, 8, 24\}$$
under the divisebelity '|'

First ~~elt~~ element = 2
Last element = 24.
$4 \nmid 6$ or $6 \nmid 4$.

Hence A is not totally ordered.
for e.g.
$$\{-2, 2, 4\}$$ has
1st & last element but not partially ordered.

Assignment:
Form a set have 1st & last element which partially orderd.

$A = \{3, 6, 9, 36\}$ under relation Divisibility
→ which is partially order

First element = 3.

Last element = 36. ✓

x — x

Theorem:-

In an ordered set first and last element are unique.

Proof:- For First element:

Let A be an ordered set and $a, b \in A$ are first elements of A.

Now if 'a' is 1st element Then

$a \preceq x \quad \forall x \in A$.

In particular $b \in A$.

$a \preceq b$ —— (1)

if b is the 1st element Then

$b \preceq x \quad \forall x \in A$.

In particular $a \in A$.

$\therefore b \preceq a$ —— (2)

From (1) & (2).

$a = b$.

LAST element:-

Let A be an ordered set and $a, b \in A$ are last element of A.

if 'a' is last element of A - Then.

$x \preceq a \quad \forall x \in A$.

In particular $b \in A$.

$\therefore b \preceq a$ —— (1)

Now if b is the last element of A - Then

$x \preceq b \quad \forall x \in A$.

In particular $a \in A$.

$\therefore a \preceq b$ —— (2)

From (1) & (2)

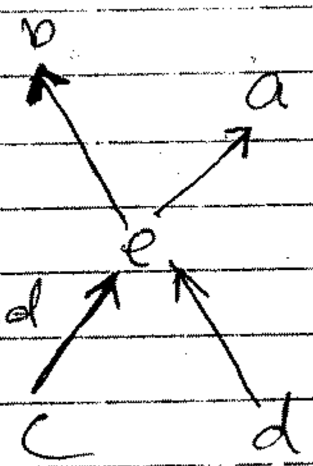
$a = b$.

Hence The 1st and last element of an ordered set are unique.

Maximal element:-
Let A be an ordered set. Then $a \in A$ is called a maximal element of A. if $a \preceq x$
$\cancel{x \preceq a} \Rightarrow a = x$ i.e
if no element of A strictly dominate 'a'.

Manimal element:-
An element $b \in A$ is called a minal element of A. if
$x \preceq b \Rightarrow x = b$ i.e
if no element of A strictly proceeds to b. Then b is called minimal element.

if
$W = \{a, b, c, d, e\}$ be ordered by following diagramme.



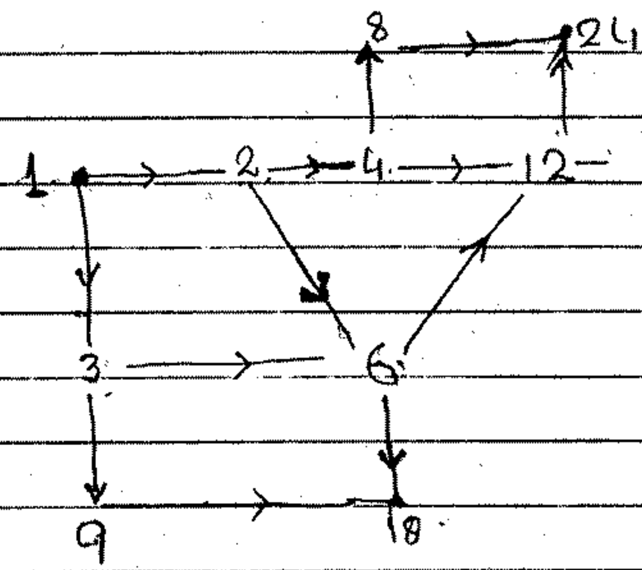
Maximal element $= a, b$
minimal element $= c, d$
1st element = no

Last element = no.

$24 \preceq x \Rightarrow x = 24$ $\qquad$ $x \preceq 1 \Rightarrow x = 1$

let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be order by the relation $x | y$.

NOTE: Finit POSET has more then maximal and minimal elemement



Maximal elemement : 18, 24.
Minional element :- 1
1^{st} element :- 1
Last element :- no,

(c) The linearly ordered set

$\{x, y, z, u, v\}$

$x \longrightarrow y \longrightarrow z \longrightarrow u \longrightarrow v$. (Chaine)

1st element $= x$
Last element $= v$
Max Element $= v$ ✓
min - Element $= x$.

Imp (a) let S be finite Linearly ordered set i.e

$$S = \{a_1, a_2, ---, a_m\}$$

1st element = minimal element

1st element $=$ mini $\{a_1, a_2, a_3, ---, a_m\}$

Last element = maximal element

$$= \max\{a_1, a_2, a_3, ---, a_m\}$$

Since 'S' is totally or linearly order. ∴ in totaly order set The elements becomes a chaine.

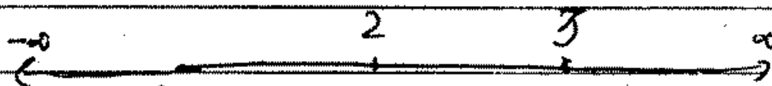
* LOWER BOUND:-

If S is an ordered set and 'A' is the subset of 'S' Then an element $a \in S$ is said to lower bound of A if

$$a \leq x \quad \forall \; x \in A.$$

e.g.

$$A = [2, 3]$$
$$S = R$$



Set of lower bound of A is

$$= (-\infty, 2]$$

* UPPER BOUND:-

If S is an Ordered set and 'A' is the sub set of 'S' Then an element $b \in S$ is said to be upper bound of A if

$$x \leq b \quad \forall \; x \in A.$$

e.g

$$A = [2, 3]$$
$$S = R$$

Then

Set of ~~....~~ upper bound of A

$$= [3, \infty).$$

NOTE :- Lower bound & upper bound of closed & open interval are same.

Infimum:-

let S be an order set and B be its sub set and element $a \in S$ is called infimum of B. if 'a' is 'maximum' or last element of the set of lower bounds of B.

if $B = (7, 11)$

$S = \mathbb{R}$

Set of lower bounds of $B = (-\infty, 7]$

$\Longrightarrow$ Infimum of $B = 7$.

or $\inf B = 7$.

Suprimum:-

let S be an ordered set and B be its sub set and element $b \in S$ is called suprimum of B. if b is the 'minimum' or 1st element of the set of upper bounds of B.

if $S = \mathbb{R}$

$B = (7, 11)$

Set of upper bounds of $B = [11, \infty)$

$\sup B = 11$

Remarks:~ i) Every bounded Set needs not contain Suprimum & infimum for e.g. if $S=\mathbb{R}$ & $A=(0,1)$ i.e 0 is infimum & 1 is suprimum of A respectively but not contain in A.

ii) Every upper bound is not the Suprimum & Every lower bound is not the infimum of A.

iii) If 'a' is the 1st element of an ordered set Then 'a' is only ~~Then a is only~~ the minimal element. i.e let $S=\{1,2,3,\ldots,50\}$ Then $1\to 2\to 3\to 4\to \cdots\to 50$. Then 1 is 1st element & also the minimal element.

iv) In case of partially ordered set the maximal and minimal elements may not unique but In case of totally ordered set maximal and minimal elements are unique & They are exactly equal to the last and 1st element.

Theorem:~

In a totally ordered set maximal and minimal elements are unique.

Proof:~ For minimal elements. let A be totally ordered set & $a,b\in A$ are minimal element. Since A is totally ordered set & $a,b\in A$. Therefore

$$a\preceq b \text{ or } b\preceq a$$

i) $a\preceq b$

$\Rightarrow a=b$ as b is minimal element of A.

if $b\preceq a$

$b=a \Rightarrow$ as a is the minimal element of A.

For maximal elements.

let A be totally orderd set & $a, b \in A$. are maximal elements of A. Since 'A' is totally ordered Then.

$a \preceq b$ or $b \preceq a$.

if

$a \preceq b$

$\Rightarrow a = b$ as a is the maximal element of A.

if

$b \preceq a$

$b = a$ as b is the maximal element of A.

Similar Sets

Two ordered sets 'A' & 'B' are similar if There exist a mapping

$f : A \longrightarrow B$ such that

i) f is bijective

ii) f preserve The order.

It may be noted that the function defined above is called similarities mapping.

Remarks

If two set are similar Then They are equivelent but converse may not true.

Ex:- Let $N = \{1, 2, 3, ----\}$ &

$E = \{2, 4, 6, ----\}$

Then we define a mapping

$f(n) = 2n$ where $n \in N$.

Since f is bijective and also for any to elements $x, y \in N$ s.t $x \leq y$

$\Rightarrow f(x) \leq f(y)$.

$\Rightarrow$ f preserve the order. So N and E are similar set and f is similarities mapping.

Imp Ex. Let 'N' be the set of natural number and 'Z' be the set of integers. Then $f: N \longrightarrow Z$ defined by.

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

i) Since f is bijective.

ii) if $4, 5 \in N$ such that $4 \leq 5$

Then $f(4) = 2, \quad f(5) = -2$.

So $\quad 2 \nleq -2$

i.e

$f(4) \nleq f(5)$

Hence f not preserve the order. So 'N' & 'Z' are not similar.

$A = \{a_1, a_2, \ldots, a_n\}$ P.O.Set

Theorem: Finit partially ordered set has atleast one maximal and one minimal element.

Proof: Let A be a finit partially ordered set and let $a_1 \in A$ is not a maximal element Then $\exists\ a_2 \in A$ Such that

$$a_1 \preceq a_2$$

Similarly if a_2 is not maximal element Then $\exists\ a_3 \in A$ such that

$$a_2 \preceq a_3$$

Continuing the same process after finit steps we obtained $a_n \in A$ s.t

$$a_n \preceq x \Rightarrow a_n = x.$$

i.e a_n is the maximal element.

For minimal elemen

Let B be a finit partially ordered set and $b_1 \in B$ is not a minimal element Then $\exists$ an element $b_2 \in B$ such that

$$b_2 \preceq b_1.$$

Similarly if b_2 is not the minimal element of B Then There exist $b_3 \in B$ such that

$$b_3 \preceq b_2.$$

Continuing in the same process after finit steps we obtained $b_n \in B$ s.t

$$x \preceq b_n$$

$$\Rightarrow x = b_n$$

i.e b_n is the minimal element.

//

Theorem:
let $f: A \to B$ be Similarity mapping from an orderd set A to an orderd set B. Then $a \in A$ is maximal (minimal, 1st, last) element of A iff $f(a)$ is the maximal (minimal, 1st, last) element of B.

Proof:- a) for Maximal
Since f is similarity mapping & $a \in A$ is maximal element of A.
$\Rightarrow a \preceq x \Rightarrow a = x$
i.e $a \not\prec x \quad \forall\, x \in A$.
$\Rightarrow f(a) \not\prec f(x)$ $\because f$ is similarity mapping.
$f(a) = f(x)$
$\Rightarrow f(a)$ is maximal element of B.

Conversely
let $f(a)$ is maximal element of B.
i.e
$f(a) \preceq f(x) \Rightarrow f(a) = f(x)$.
$\Rightarrow f(a) \not\prec f(x)$.
Since f is similarity mapping. Therefore
$a \not\prec x \quad \forall\, x \in A$.
$\Rightarrow a$ is maximal element of A.

---

b) For 1st Element
let f be a similarity mapping & $a \in A$ is 1st element of A.
$\Rightarrow a \preceq x \;\forall\, x \in A$.
$\Rightarrow f(a) \preceq f(x) \quad \forall\, x \in A$. ~~where~~
Since f is similarity mapping & $f(x) \in B$
$\Rightarrow f(a) \preceq f(x)$
$\Rightarrow f(a)$ is 1st element of B.

Conversely suppose that $f(a)$ is 1st element of B. Then

$$f(a) \preceq f(x) \quad \forall \; f(x) \in B.$$

Since f is similarity mapping

Therefore

$$a \preceq x \quad \forall \; x \in A.$$

$\Rightarrow$ a is the 1st element of A.

Hence proved.

(2)

For LAST ELEMENT:-

Let f be similarity mapping & $b \in A$ be the last element of A. Then

$$x \preceq b \quad \forall \; x \in A.$$

Since f is similarity mapping

Therefore

$$f(x) \preceq f(b) \text{ Then}$$

$f(b)$ is the last element of B

Conversely suppose that $f(b) \in B$ is the last element of Then

$$f(x) \preceq f(b) \quad \forall \; f(x) \in B.$$

Since f is similarity mapping

Therefore

$$f(x) \preceq f(b)$$

$$\Rightarrow x \preceq b \quad \forall \; x \in A.$$

$\Rightarrow$ b is the last element of A.

Ex:- Imp.

Given example of an orderd set (A, R) which is similar to an orderd set (A, R^{-1}) i.e (A with inverse order).

Sol:- Consider Q the of rational numbers with natural order i.e. $(Q, \leq)$

Then if

$$f : (Q, \leq) \longrightarrow (Q, \geq)$$

Such that

$$f(x) = -x.$$

$\Rightarrow$ f is bijective

ii) let $x, y \in (Q, \leq)$ and

$$x \leq y \quad — (1)$$

Then $f(x) = -x, \; f(y) = -y.$

i.e $(1) \Rightarrow -x \geq -y.$

$$\Rightarrow f(x) \geq f(y).$$

i.e $f(x) \preceq f(y) \Rightarrow f$ preserve the ~~mapping~~ order.

Hence f is similarity mapping.

Hence $(Q, \leq)$ and $(Q, \geq)$ are similar.

Ex // Note,

Set of natural number i.e $(N, \leq)$ and $(N, \geq)$ are not similar.

i) Negative numbers are not present
ii) 1st element exist in $(N, \leq)$ but There is not ~~1st~~ 1st element in $(N, \geq)$ so There exist no similarity mapping.

Theorem:- The relation $(\simeq)$ is an equivalence relation in any collection of sets.

Proof:-

Reflexive:-

Since $I: A \longrightarrow A$ is a mapping defined by $I(a) = a$. which is bijective and preserve the order
$\Rightarrow$ I is similarity mapping & $A \simeq A$.

Symmetric Property:-

Suppose $f: A \longrightarrow B$ be similarity mapping.

$\Rightarrow f^{-1}: B \longrightarrow A$ is bijective mapping as inverse of bijective mapping is bijective. Now we have to prove f^{-1} preserve the order i.e if $a_1, a_2 \in A$

$a_1 \preceq a_2 \iff f^{-1}(a_1) \preceq f^{-1}(a_2)$

$\Rightarrow f(a_1) \preceq f(a_2)$ $\because$ f is similarity mapping.

$f(a_1) = b_1$, $f(a_2) = b_2$

$b_1 \preceq b_2$

Now f^{-1} is a mapping from B to A.

if

$f^{-1}(b_1) \preceq f^{-1}(b_2)$

$\Rightarrow f^{-1}(f(a_1)) \preceq f^{-1}(f(a_2))$

$\Rightarrow a_1 \preceq a_2$. i.e

f^{-1} preserve the order.

So $B \simeq A$.

Hence '$\simeq$' is Symmetric.

<u>Transitive:</u>

Suppose $f: A \rightarrow B$ and $g: B \rightarrow C$ are similarity mapping. Then

$g \circ f: A \longrightarrow C$ is bijective.

Now $a_1, a_2 \in A$ and $a_1 \preceq a_2$

Then $f(a_1) \preceq f(a_2)$. $\because$ f is similarity mapping.

$\Rightarrow g(f(a_1)) \preceq g(f(a_2))$ $\because$ g is similarity mapping.

$\Rightarrow g \circ f(a_1) \preceq g \circ f(a_2)$.

$\Rightarrow$ $g \circ f$ preserve the order

So $g \circ f$ is similarity mapping & $A \simeq C$.

Hence similar relation "$\simeq$" is an equivalence relation.

Order Type: (Ordinal Number).

Let A be a totally ordered set and ξ the family of those sets which are similar to 'A'. Then 'ξ' is called the ordered type of A. If A and B be two set (Totally ordered set) s.t $A \simeq B$, we say order Type of A is equal to order Type of B.
For e.g.

$$\text{Order type of } N = \omega$$
$$\text{order Type of } Z = \pi$$
$$\text{order Type of } Q = \eta$$

Well ordered Set:-

An ordered set A is said to be well ordered if every subset of A has first element.
e.g.

Set of Natural = N

Set of Even number = E.

Set of odd number = O

Counter example

:- Set of integers = Z.

as $\{-1, -2, -3, \ldots\}$ has no first element

Every open & close interval on real line are not well order.

Theorem:
A well ordered set is Totally ordered.

Proof:
let 'A' be well ordered set. also let $a, b \in A$. Then

$\{a, b\} \subseteq A$.

$\{a, b\}$ be must have 1st element as A is well ordered. i.e a & b are comparable.
Since a & b are arbitrary elements of A. Hence A is totally ordered set. (All the elements of A are comparable).

Theorem
A set which has the 1st element and the last element must be Totally ordered but Converse does not hold.
Sol: The statment is false we show it by counterexample
let $A = \{2, 4, 6, 8, 24\}$
$\Rightarrow$ 2 is the 1st and 24 is the last element of A. ~~but~~ under the relation divisibility But
$4/6 \Rightarrow 4 \nmid 6$ Hence A is not tolly totally ordered.
set of Naturall which totally But It has no last element.

- If A is well order then $\cdots C A_n C \cdots C A_3 C A_2 C A_1 C A$

$\Downarrow$.

Theorem:

Every Sub set of a well order set is well ordered.

Proof:

let A be a well ordered set and B be a Subset of A. To prove that B is well ordered consider $B_1 \subset B$.

$\Rightarrow B_1 \subseteq A \quad \because B \subset A$.

As A is well ordered so B_1 has 1st element. Therefore B is well order also as B is arbitrary Sub set of A. so every Subset of A is well ordered.

Theorem:

If A is well-ordered and B is Similar to A Then B is also well ordered.

let $f: A \longrightarrow B$ be a Similarity mapping since $A \sim B$. and $B_1 \subseteq B$. Since 'f' is onto such that

$$f(A_1) = B_1$$

Now 'A' is well ordered and $A_1 \subset A$. Therefore A_1 has 1st element say 'a_0' Then $f(a_0) \in B_1$ is 1st element of B_1. Since B_1 is arbitrary Therefore B is well ordered.

Theorem:- All finite totally ordered set with same number of elements are well order and similar to each other.

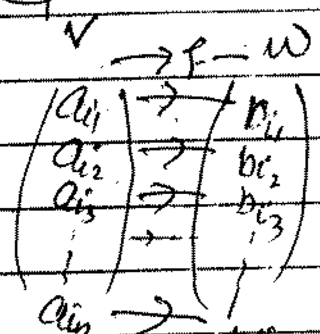
Proof:- Let $V = \{a_1, a_2, a_3, \ldots, a_n\}$ be a totally ordered set now we arrange as element according to their order Then

$$V = \{a_{i_1}, a_{i_2}, a_{i_3}, \ldots, a_{i_n}\}$$

as V is finit totally order Therefore If we take any subset of V It contain 1st element and hence V is well order.

Now if $W = \{b_{i_1}, b_{i_2}, \ldots, b_{i_n}\}$
Then

$$f : V \longrightarrow W \text{ defined}$$



$$f(a_{i_n}) = b_{i_n}$$

obviously f is bijective as well as preserve the order $\because$ if

$$a_{i_1} \preceq a_{i_2}$$

$$\Rightarrow f(a_{i_1}) \preceq f(a_{i_2})$$

$$\Rightarrow b_{i_1} \preceq b_{i_2}$$

$\Rightarrow f$ is similarity mapping. Hence All the finit totally ordered set with same number of element are similar to each other.

Principal of Mathematical induction.

let S be sub set of the set N of natural number with following two properties

i) $1 \in S$

ii) If $n \in S$ Then $n+1 \in S$

Then $S = N$ i.e S is actually the set of ~~##~~ countable number.

** INITIAL SEGMENT :-

let 'A' be a well ordered set the initial segment $S(a)$ of an element $a \in A$ consist of all elements in A which strictly proceed a. In other ward $S(a) = \{x : x \in A \wedge x \prec a\}$

e.g:- $N \longrightarrow$ well ordered set

$9 \in N$

$S(9) = \{1, 2, 3, \ldots, 8\}$

Theorem:-

~~Let~~ If S be subset of well ordered set A with the following properties

i) $a_0 \in S$ where a_0 is the 1st element

ii) $S(a) \subseteq S \Rightarrow a \in S \quad \forall a \in A$

Then $S = A$.

S to T A

$N = \{1, 2, 3, ---\}$
$1 \in S$
$S(2) = \{1\} \subseteq S \Rightarrow 2 \in S$
$S(3) = \{1, 2\} \subseteq S \Rightarrow 3 \in S$

Proof :- On contrary suppose that $S \neq A$. i.e $A \setminus S = T$ is non-empty. Since A is well ordered so T has 1st element say (t_0). each element $x \in S(t_0)$, strictly proceed t_0. and Therefore can't belonges to T, as 't_0' is first element of T. so $x \in S$ Hence $S(t_0) \subseteq S \Rightarrow t_0 \in S$ by (ii) hypothesis. which is controduction to the fact that $t_0 \in T$. Hence our assumption is wrong i.e $S \neq A$ is not true In otherward $\boxed{S = A}$

Imp. **Theorem:** If $S(A)$ denote the collection of initial segments of elements in well ordered set A and let $S(A)$ be ordered by set inclusion Then A is similar to $S(A)$. i.e

$$A \simeq S(A).$$

Proof :- Let $f : A \longrightarrow S(A)$ defined by

$$f(a) = S(a) \quad \forall \quad a \in A.$$

By definition 'f' is onto.

Suppose $x \neq y$, $x, y \in A$.

$\Rightarrow$ one of them we say

$$x \prec y$$

$$\Rightarrow x \in S(y).$$

$$\text{but } x \notin S(x)$$

$$\Rightarrow S(x) \neq S(y)$$

$$\Rightarrow f \text{ is one-one.}$$

Now we show that 'f' preserve the order i.e if $x \prec y \Leftrightarrow S(x) \subseteq S(y)$

Now of $x \prec y$ also suppose

$a \in S(x)$ i.e segment of x

$$\Rightarrow a \prec x \prec y.$$

$$\Rightarrow a \prec y$$

$$\Rightarrow a \in S(y)$$

$$\Rightarrow S(x) \subset S(y).$$

Now Suppose That

$$x \not\prec y$$

Then

$$y \prec x \Rightarrow y \in S(x)$$

$\Rightarrow y \notin S(y)$

But by definition of initial segment

Hence

$$S(x) \not\subset S(y).$$

In other words $x \prec y \Leftrightarrow S(x) \subset S(y)$.

$\Rightarrow f$ is similarity mapping

Hence

$$A \simeq S(A).$$

Theorem: let A be well ordered set and B be subset of A. Also $f: A \to B$ is similarity mapping of A onto B. Then for every $a \in A$ $\quad a \preceq f(a)$.

Proof: let

$$D = \{x \mid x \in A \wedge f(x) \prec x\}$$

There are two cases to be discussed

Case-I

If $D = \varphi$ Then the result is trivial.

Case II. If $D \neq \varphi$. Then since A is well order so D has 1st element say (d_0).

Now $d_0 \in D$

$$\Rightarrow f(d_0) \prec d_0 \qquad \because f(d_0), d_0 \in A \text{ \& } f \text{ is similarity mapping}$$

$$\Rightarrow f[f(d_0)] \prec f(d_0).$$

$$\Rightarrow f(d_0) \in D \qquad \because D = \{x \mid x \in A \text{ and } f(x) \prec x\}$$

Now as $d_0 \in D$ and $f(d_0) \in D$

also $f(d_0) \prec d_0$

which is contradiction as 'd_0' is the 1st element of D. Hence our assumption is wrong and $D = \varphi$ and result is true i.e

$$a \preceq f(a).$$

x — x — x — x — x —

(A) similar (B) similar (A) then $g \circ f : A \to A$ (similar)

*

Theorem let A and B be similar well-ordered sets Then There exist only one similarity mapping of A into B.

Proof let

$f : A \to B$ and $g : A \to B$ be two similarity mapping that is $f \neq g$.

Then $\exists$ an element $x \in A$ such that

$$f(x) \neq g(x)$$

where $f(x), g(x) \in B$.

$\Rightarrow$ Either $f(x) \prec g(x)$ or $g(x) \prec f(x)$

Say $f(x) \prec g(x)$

$\because$ $g : A \to B$ is similarity mapping

Then

$g^{-1} : B \to A$ is also similarity mapping.

furthermore

$g^{-1} \circ f : A \to A$ is also similarity mapping as composition mapping of two similarity mappings is also similarity mapping but

$$f(x) \prec g(x). \quad \text{and } f(x), g(x) \in B.$$

$$\Rightarrow g^{-1}(f(x)) \prec g^{-1}(g(x))$$

$$\Rightarrow g^{-1}(f(x)) \prec g^{-1}g(x) = x$$

$$\Rightarrow g^{-1}(f(x)) \prec x.$$

$$\Rightarrow g^{-1} \circ f(x) \prec x$$

which is contradiction to the Theorem "If A is well order set & $B \subset A$ & $f : A \to B$ is similarity mapping Then $\forall\, a \in A$ $a \prec f(a)$"

Hence our assumption that $f \neq g$ is wrong. Hence similarity mapping is unique. //

NOTE :- IDENTITY Relation is P.O.R $\Leftrightarrow$ E.R.

Theorem:- A well order set can not be similar to any initial segment.

Proof:- Let A be well ordered set and $S(a)$ be one of its initial segment.

Suppose $f: A \longrightarrow S(a)$ be similarity mapping. Then

$$f(a) \in S(a)$$

$\Rightarrow f(a) \prec a$ By Def of initial segment.

Which is contradiction to the theorem "If A is well ordered set and $B \subset A$ and $f: A \longrightarrow B$ be a similarity mapping then $\forall\, a \in A$ $a \prec f(a)$".

Hence there is no similarity mapping b/w a well order set and its initial segment.

Imp Theorem:-

Two different initial segments of well order set cannot be similar.

Proof:- Let $S(a)$ and $S(b)$ be different initial segments i.e $a \neq b$

Then either $a \prec b$ or $b \prec a$

Suppose

$a \prec b \Rightarrow a \in S(b)$

Let $x \in S(a)$

$\Rightarrow x \prec a$

$\Rightarrow x \prec a \prec b$

$\Rightarrow x \prec b$

$\Rightarrow x \in S(b)$.

$\Rightarrow S(a) \subseteq S(b)$.

$N = \{1, 2, 3, \dots\}$

$S(9) = \{1, 2, 3, \dots 8\}$

$S(10) = \{1, 2, 3, \dots 9\}$

Since $\# S(9) \neq \# S(10)$

Then these can never be bijection

$\Rightarrow S(a) \subseteq S(b)$

Hence $S(b)$ is well ordered being a Subset of well ordered set and $S(a)$ of its initial segment. Therefore can't be similar to its initial segment. As by Theorem " A well ordered set Can't be similar to its initial segment."

Theorem: let A and B be two well ordered set and let an initial segment $S(a)$ of A is similar to an initial segment $S(b)$ of B. Then $S(a)$ is similar to a unique initial segment $S(b)$ of B.

Proof: let $a \in A$, and $b, b' \in B$ also Suppose that

$$S(a) \simeq S(b) \quad \text{Then } S(b) \simeq S(a) \quad -(1)$$

and

$$S(a) \simeq S(b'). \quad -(2)$$

Then from (1) & (2) by Transitive property

$$S(b) \simeq S(b').$$

Since $b, b' \in B$. Then $S(b) \simeq S(b')$ is controduction bcs by the Theorem

or any Two differential segment of a well orderd set is not similar.

Hence $S(a)$ is similar to a unique initial segment $S(b)$ of well order set B.

Imp.

Theorem: let A be well order set and S be subset of A with the following properties

If $a \prec b$, $b \in S$ Then $a \in S$

Then

$S = A$ or S is an initial segment of a.

Proof: Suppose $S \neq A$. Then $A \setminus S = T \neq \varphi$

Also 'T' has first element being the subset of well order set A. say $a_0 \in T$ is first element of T. where $a_0 \notin S$. we show that

$N = \{1, 2, 3, 4, ---\}$
$1 \prec 2$, $2 \in S$
$\Rightarrow 1 \in S$
$S = \{1, 2\}$

$$S = S(a_0)$$

Suppose $x \in S(a_0)$

$\Rightarrow x \prec a_0$

$\Rightarrow x \notin A / S = T$. $\because a_0$ is 1st element of T.

$\Rightarrow x \in S$

$\Rightarrow S(a_0) \subseteq S$ — (1)

Now Suppose $y \notin S(a_0)$
Then
$$a_0 \propto y$$
also Suppose $y \in S$ and $a_0 \preceq y$
$$\Rightarrow a_0 \in S \quad \text{(By given property)}$$
which is controduction as a_0 is 1st element of T. Hence our assumption that $y \in S$ is wrong. we have ~~$a_0 \in S(a_0)$~~

$$y \notin S(a_0)$$
$$\Rightarrow y \notin S$$
$$\Rightarrow S \subseteq S(a_0) \quad (2)$$

From (1) & (2)

$$S = S(a_0).$$

Now we are to show that

$$A = S$$

Suppose that $A \neq S$ i.e $A \setminus S = T$
Since A is well order Therefore $T \subset A$ has 1st element a_0 (say). each element

$$x \in S(a_0)$$
$$\Rightarrow x \propto a_0$$
$$\Rightarrow x \notin T \quad \because a_0 \text{ is 1st element}$$
$$\Rightarrow x \in S$$
$$\Rightarrow S(a_0) \subseteq S.$$
$$\Rightarrow a_0 \in S$$

which is controduction

NOTE $f: A \to B$ where A, B are finite if f is 1-1 Then it is bijective.

to the fact that $a_0 \in T$ is the 1st element of T. Hence our assumption is wrong.

So $A = S$

Theorem

Let A and B be two well ordered set such that an initial segment $S(a)$ of A is similar to initial segment $S(b)$ of B. Then each initial segment of $S(a)$ is similar to an initial segment of $S(b)$. i.e

$$a' \angle a \Rightarrow S(a') \simeq S(b') \text{ where } b' \propto b.$$

where 'b'' is image of a'.

Proof:

Let $f: S(a) \longrightarrow S(b)$ be similarity mapping where $S(a)$ & $S(b)$ are initial segment of A and B are respectively.

$$\text{Let } f(a') = b'$$

Note that f' restricted to $S(a')$ is 1-1 and preserve the order.

Hence

$$S(a') \simeq f(S(a')) \text{ i.e}$$

$$S(a') \simeq f(S(a')) \text{ —— } (1)$$

Furthermore since f' is similarity mapping and if $a^* \angle a'$

$$\Rightarrow f(a^*) \propto f(a')$$

$$\Rightarrow f(a^*) \prec f(a') = b'$$

i.e

$$f(a^*) \prec b'$$

$$a^* \in S(a) \iff f(a^*) \in f(S(a'))$$
$$= S(b')$$

$$\Rightarrow f(S(a')) = S(b') \quad \text{--- (2)}$$

From (1) & (2)

$$S(a') \simeq S(b').$$

* Theorem: Let A & B be well ordered sets. Let S and T are the subsets of A and B respectively such that

$$S = \{x : x \in A \quad S(x) \simeq S(y) \,;\, y \in B\}$$

&

$$T = \{y : y \in B \quad S(y) \simeq S(x) \,;\, x \in A\}$$

Then

$$S \simeq T.$$

Proof: Let $x \in S$. Then by the Theorem "Let A, B be well ordered and initial segment of a i.e $S(a)$ is similar to initial segment of b i.e $S(b)$. Then $S(a)$ is similar to unique $S(b)$. i.e

$$S(a) \simeq S(b)\text{''}.$$

Then $S(x)$ is similar to a unique $S(y)$ of the set B.

Thus to each $x \in S$ There Correspond to unique $y \in$ "T" such that

$$S(x) \simeq S(y)$$. and vice versa

Hence

$f : S \longrightarrow T$ defined by

$$f(x) = y \text{ if } S(x) \simeq S(y).$$

$\Rightarrow$ $f(x)$ is one-one & onto Hence bijective.

Now let $x, x' \in S$. such that

$x' \prec x$ Then we are to show that

$f(x') \prec f(x)$

$\Rightarrow y' \prec y$

let $g : S(x) \longrightarrow S(y)$ be similarity mapping of $S(x)$ into $S(f(x)) = S(y)$.

By Theorem.

"let A & B be well ordered set $S(a)$ and $S(b)$ be initial segment of a and b respectively such that $f : S(a) \longrightarrow S(b)$ is similarity mapping on $S(b)$ where $a' \prec a \Rightarrow f(a') \prec f(a)$ and $b' \prec b$.

Then $g\Big|_{S(x')}$ is similarity mapping of $S(x')$ onto $S(g(x'))$. of B. but There exist only one similarity mapping of $S(x')$ into B. Then Consequently

$$g(x') = f(x') = y'$$

$$\text{Since } g(x') \in S(y) \Longrightarrow y' \in S(y)$$

$$\Longrightarrow y' \prec y$$

Hence f is similarity mapping. Then $|S \simeq T|$

Immediate Successor & Immediate Predecessor:-

If $a, b \in A$ where A is order set Then b is said to immediate successor if $a \preceq b$ such There exist no element $c \in A$ such that $a \preceq c \preceq b$. Then it is denoted by

$$a << b$$

where 'a' is immediate predessore.

Theorem:- Every element in a well order set has a unique immediate successor except the last element.

Proof let $a \in A$ and $M(a)$ be the set of all those element of 'A' which strictly succeeds 'a'. If 'a' is not the last element Then $M(a) \neq \phi$

Since set 'A' is well order Then $M(a)$ has 1st element say 'b'. we claim that b is an immediate successor of 'a'. otherwise there is an element $c \in A$ such that

$$a \prec c \prec b \text{ Then}$$

$$c \in M(a) \quad \because a \prec c.$$

and This Controducts the fact that 'b' is the 1st element of $M(a)$. Hence 'b' is immediat successor.

we claim that 'b' is the unique immediate successor of 'a' otherwise there is another immediate successor of 'a' say $d \in M(a)$.

and since 'b' is the 1st element of $M(a)$
so $a \alpha b \alpha d$.
This contraducts that assumption that 'd' is the immediate successor of $M(a)$. Thus 1st element 'b' is the unique immediate successor of 'a'.

Book.
Theorem: Every element in well ordered set has a unique immediat predessor excep the ~~last~~ 1st element.

Proof: let $a \in A$ and $S(a)$ be the set of all those element of 'A' which strictly preceed 'a'. If 'a' is not the 1st element Then $S(a) \neq \phi$.
~~1st~~ Since set A well ordered Then $S(a)$ has ~~last~~ element say 'b'. we claim that b is an immidiate ~~predecessor~~ successor of a. otherwise There exist an element $c \in A$ such that

$$b \alpha c \alpha a.$$
$$\Rightarrow c \in S(a).$$

and This controducts the fact that 'b' is last element of $S(a)$ bss $b \alpha c$. Hence 'b' is immediate predessor.
we claime that 'b' is the unique predessor of 'a' otherwise there is another immediat predessore of 'a' say $d \in S(a)$ and since 'b' is the ~~first~~ last element of $S(a)$
so ~~$b \alpha d \alpha$~~ d $b \alpha d \alpha a$
This contraducts that assumption that 'd' is the the immediate predessor of

S(a). Thus last element b is the unique immediate predessor of a in A.

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* **Shorter and Longer Sets:** Suppose 'A' & 'B' are well ordered set and suppose A is similar to an initial segment of B. Then A is said to be shorter than 'B' and B is said to be longer than 'A'.

i.e

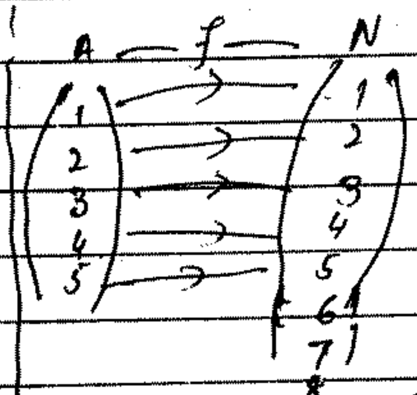
$$A \simeq S(b) \subset B$$

Then

A is shorter then B

~~and~~ or

B is longer then A.



$A \simeq S(6)$, where $f(x) = x$.

Then

A is shorter &

N is longer.

Theorem:- let $\mathcal{A}$ be a collection of pairwise non-Similar well ordered sets, Then There exist a set A_0 in $\mathcal{A}$ Such that A_0 is shorter than every other set in $\mathcal{A}$.

Proof:- let B be the set in $\mathcal{A}$. defined

$$\beta = \{ x : x \in \mathcal{A},\ x \text{ is shorter Then } B \}$$

$\mathcal{A} = \{A_1, A_2, A_3, A_4, \ldots\}$

$\beta = \{x : x \in \mathcal{A} : x \subseteq S(b) \text{ for some } b \in B\}$

$\beta' = \{S(b) : \forall\, b \in B\}$

If $\beta = \varphi$ Then 'B' Satisfies the requirements of Theorem i.e B is shorter.

But if $\beta \neq \varphi$. If we show that β has Shortest set A_0 Then considering the way β, we defined A_0 will also be the shortest set $\mathcal{A}$.

Since every set $A \in \beta$ is shorter Then B So each set A is similar to a unique ~~one~~ intial segment $S(b)$ of B. i.e $A \simeq S(b)$.

let β' be the collection of all those initial segments of B, each of which is Similar to a set of β. Then β' contains an intial segment $S(a_0)$ which shorter than every other initial segment of B. Since B is well order. Consequently the set $A_0 \in \beta$ which is Similar to $S(a_0)$ is shorter Then to any other set of β.

Therefore A_0 is the Shortest set in $\mathcal{A}$. and A_0 satisfying the requiredment of the Theorem.

//

Ordinal Numbers:-

consider a collection L of well order set. each set A in L is assigned a symbol in such a way that any two well-ordered sets A and B are assigned the symbol iff A and B are similar. This symbole is called the ordinal number of A & B, and we will write

$$\lambda = \mathrm{ord}(A) = \mathrm{ord}(B) \text{ iff } A \simeq B$$

are similar

Transfinite Ordinal numbers:-

The ordinal numbers of each of the well order set $\varphi, \{1\}, \{1,2\}, \{1,2,3\}, \ldots$ is denoted by $0, 1, 2, 3, \ldots$ respectively and are called finite ordinal numbers all other ordinal numbers are called Transfinite Ordinal numbers. The ordinal number of set of natural number is denoted by $\omega = \text{ord}(N)$.
'ω' is also called 1st limit ordinal.

—x— x— x— x—

* Inequality And Ordinal Numbers:

An equality relation is defined by the ordinal numbers as follows

let 'λ' and 'μ' be ordinal numbers and let A and B be well-ordered sets such that

$$\lambda = \text{ord}(A) \ \& \ \mu = \text{ord}(B)$$

Then

$\lambda < \mu$ if A is shorter then B, or A is similar to an initial segment of B.

* Ordinal addition:-

An operation of addition is defined for ordinal number as follow let 'λ' and 'μ' be ordinal numbers and let 'A' & 'B' disjoint sets such that

$$\lambda = \text{ord}(A) \text{ and } \mu = \text{ord}(B)$$

Then

$$\lambda + \mu = \text{ord}[A ; B]$$

Ordinal Multiplication:-

An operation on multiplication is defined for ordinal numbers as follow.

let 'λ' and 'μ' be ordinal numbers and let A and B well order sets such that

$$\lambda = ord(A) \text{ and } \mu = ord(B).$$

Then

$$\lambda\mu = ord(A \times B).$$

Ex let $A = \{a, b\}$ and $B = P = N$ be well order sets then

$$ord(A) = 2, \quad ord(P) = \omega \quad \text{Then}$$

$$A \times B = \{a, b\} \times P.$$

$$A \times B = \{(a,1), (a,2), (a,3), \ldots, (b,1), (b,2), (b,3), \ldots\}$$

Then $ord(A \times B) = ord\{(a,1), (a,2), \ldots; (b,1), (b,2), \ldots\}$

Now as

$$\{(a,1), (a,2), (a,3), \ldots\} \simeq N$$

$$\& \{(b,1), (b,2), (b,3), \ldots\} \simeq N$$

$$ord(A \times B) = \omega * \omega$$

$$= \omega 2.$$

Structure of ordinal Number:-

we now write down many of the ordinal number according to their numbers (structures). First we come to finite ordinal which are $0, 1, 2, 3, \ldots$.

& Then comes to 1st limit ordinal ω and its successors are $\omega, \omega+1, \omega+2, \omega+3, \ldots$

Hence next comes to second limit ordinal number $\omega 2$. and its successor are $\omega 2, \omega 2+1, \omega 2+2, \omega 2+3, \ldots$. The next limit ordinal number is $\omega 3$. we proceed as follows $\omega 3, \omega 3+1, \omega 3+2, \ldots$

$\omega 4, \omega 4+1, \omega 4+2, \ldots$

— — — —

— — — — —

— — — —

$$\omega \cdot \omega = (\omega)^2 = (\omega)^2, (\omega)^2+1, \omega^2+2, \ldots$$

$$(\omega)^3 \ldots (\omega)^4 \ldots (\omega)^\omega \ldots$$

$$\ldots (\omega^\omega)^1 \ldots (\omega^\omega)^2 \ldots (\omega^\omega)^\omega$$

$$, \ldots \left(\left(\omega^\omega\right)^\omega\right)^\omega, \ldots$$

After all these ordinal we have the ordinal ε_0 we can continoue

$\varepsilon_0, \varepsilon_0+1, \varepsilon_0+2, \varepsilon_0+3, \ldots$

we note each of the ordinal numbes we have enumirated is still the ordinal number of countable set. bse union of countable set is countable.

//

Theorem:

Let $\alpha = ord(A)$ let $\mu < \alpha$. Then There is a unique initial segment $S(a)$ of A. Such that

$$\mu = ord(S(a)).$$

Proof: Let $\mu = ord(B)$
and since $\alpha = ord(A)$
and $\mu < \alpha$ Then B is shorter Then A. i.e B is Similar to an initial Segment $S(a)$ of A. Therefore $\mu = ord(S(a))$.
Since two different initials segments of a well order set cannot be Similar. Therefore $S(a)$ is the only initial segment whose ordinal number is μ. i.e

$$\mu = ord(S(a)).$$

* — * — * — *

* Imp

Theorem:

Let $S(\alpha)$ be the set of ordinals less than the ordinal α Then

$$\alpha = ord(S(\alpha))$$

$S(9) = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$ord\ S(9) = 9$
$S(A) = \{S(a) : a \in A\}$

Proof:
Let $\alpha = ord(A)$
and let $S(A)$ denote the collection of all initial segments of A. ordered by set inclusion.
Since $A \simeq S(A)$

So $\operatorname{ord} S(A) = \lambda$.

Next we show that

$$S(\lambda) \simeq S(A).$$

let

$$\mu \in S(\lambda).$$

Then

$$\mu \prec \lambda$$

Then there is unique initial segment $S(a)$ of A such that

$$\mu = \operatorname{ord}(S(a)) \quad \text{by the previous theorem}$$

Now we defined a function

$$f : S(\lambda) \longrightarrow S(A).$$

By

$$f(\mu) = S(a) \text{ iff } \mu = \operatorname{ord}(S(a)).$$

So f is clearly 1-1. furthere we show that f is onto.

Suppose $S(b) \in S(A)$. Then $S(b)$ is shorter than A. & therefore ordinal number of $S(b) = \nu < \operatorname{ord}(A) = \lambda$

$\nu < \lambda$.

$\Rightarrow \nu \in S(\lambda)$. $\because$ $S(\lambda)$ is the set of ordinal number which are less then λ.

Such that

$$f(\nu) = S(b) \quad \text{where } \nu = \operatorname{ord}(S(b))$$

$\Rightarrow f$ is onto.

Hence f is bijective.

Now we will proove that f preserves order. For that suppose $\mu, \nu \in S(\lambda)$ such that $\mu \prec \nu$

where $\mu = ord(S(a))$

$\nu = ord(S(b))$

i.e $f(\mu) = S(a)$

and

$f(\nu) = S(b)$.

Since $\mu \prec \nu$ Therefore

$S(a)$ is an initial segment of $S(b)$. Hence $S(a)$ is proper subset of $S(b)$. In otherwards under the ordering of $S(A)$,

$S(a) \prec S(b)$. $\because S(a) \subset S(b)$

$\Rightarrow f(\mu) \prec f(\nu)$. $\Rightarrow f$ Preserves the order

$\Rightarrow f$ is Similarity mapping

i.e

$S(\alpha) \simeq S(A)$

$\Rightarrow ord(S(\alpha)) = ord(S(A))$

$= \alpha$

Hence

$\alpha = ord(S(\alpha))$

//

* ORDINARY DIFFERENTIAL EQUATION:- OF HIGHER ORDER HYPER GEOMETRIC D.En

In mathematics the hypergeometric D.E is a 2nd order linear ordinary D.E whose solutions are given by the hypergeometric series.

* Hypergeometric Series:-

A hypergeometric series is a power series in which the ratios of successive coefficient is a rational function the series of convergent will define a hypergeometric function.

Hypergeometric function generalize many special functions including the Bessel function, The incomplet GAMMA Function, The ERROR Function.

Symbolically hypergeometric function can be written as

$$F(a, b, c; z) = F\left[\begin{matrix} a, b \\ c, z \end{matrix}\right]$$

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} z^n$$

where $(a)_n$ read as a factorial function of order n or generlize factorial function or rising function.

and

$$(a)_n = a(a+1)(a+2)(a+3) + \cdots + (a+n-1).$$

Now we take $a = c$ and $b = 1$ we get

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(c)_n (1)_n}{(c)_n \, n!} z^n$$

$$F(a,b,c;z) = \sum_{n=1}^{\infty} \frac{z^n}{n!}$$

* GAMMA FUNCTION:- The GAMMA Function represented by the capitalized Greek letter "Γ" which is an extension of factorial function to the real and complex numbers z with positive (+ve) real part.

NOTATION $\Gamma(z)$ = Gamma z. It is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, dt.$$

Properties:-

$\Gamma(1) = 1$

using

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \, dt.$$

Put $z = 1$

$$\Gamma(1) = \int_0^{\infty} e^{-t} t^{1-1} \, dt.$$

$$= \int_0^{\infty} e^{-t} \, dt.$$

$$\Gamma(1) = \lim_{h\to\infty}\int_0^h e^{-t}\,dt.$$

$$= \lim_{h\to\infty}\left|-e^{-t}\right|_0^h$$

$$= \lim_{h\to\infty}\left|-\frac{1}{e^h}+\frac{1}{e^0}\right|$$

$$= \lim_{h\to\infty}\left|-\frac{1}{e^\infty}+1\right|$$

$$= \cancel{\lim_{h\to\infty}}\,|-0+1|$$

$$= \cancel{\lim_{h\to\infty}}\,(1)$$

Hence $\Gamma(1) = 1$ Ans.

---

ii)

$$\Gamma(z+1) = z\,\Gamma(z).$$

$$\Gamma(z+1) = \int_0^\infty e^{-t}.\,t^{z+1-1}\,dt.$$

$$= \int_0^\infty e^{-t}.\,t^z\,dt.$$

$$= \lim_{h\to\infty}\int_0^\infty e^{-t}t^z\,dt$$

$$= \lim_{h\to\infty}\left[t^z\frac{e^{-t}}{-1}\Big|_0^h - \int_0^h \frac{e^{-t}}{-1}z.\,t^{z-1}\,dt\right]$$

$$\cancel{= \lim_{h\to\infty}\Big[-}$$

$$= 0 + \lim_{h\to\infty} z \int_0^h e^{-t} t^{z-1}\, dt.$$

$$= z \lim_{h\to\infty} \int_0^h e^{-t} t^{z-1}\, dt.$$

$$= z \int_0^\infty e^{-t} t^{z-1}\, dt.$$

Hence

$$\Gamma(z+1) = z\,\Gamma z$$

prove $\Gamma(z+1) = z!$

Since

$$\Gamma(z+1) = z \cdot \Gamma(z)$$

$$= z \cdot (z-1)\Gamma(z-1).$$

$$= z \cdot (z-1)(z-2)\Gamma(z-2)$$

$$= z \cdot (z-1)(z-2)(z-3)\Gamma(z-3)$$

$$\vdots$$

$$z \cdot (z-1)(z-2)(z-3) \cdots \cdot 1\Gamma.$$

$$= z \cdot (z-1)(z-2)(z-3) \cdots 1.$$

$$\Gamma(z+1) = z!$$

Show that $0! = 1$.

we know that

$$n! = n(n-1)!$$

$$\Rightarrow (n-1)! = \frac{n}{n!}$$

Put $n = 1$

$$0! = \frac{1}{1!} = \frac{1}{1} = 1$$

$$\boxed{0! = 1}$$

//

Bemta function:-

Notation: $B(x,y)$
It is defined as.

$$B(x,y) = \int_0^1 (t)^{x-1} (1-t)^{y-1} dt \text{ where } x,y \in Z.$$

* Relation b/w Gamma & Beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(y)\Gamma(x)}{\Gamma(y+x)}.$$

✓ $B(x,y) = B(y,x)$
i.e Beta function is Symmetric.

Some useful Result:

i) Binomial Theorem:-

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{n!}$$

ii) $\Gamma(\alpha+n) = (\alpha+n-1)\Gamma(\alpha+n-1)$

$$= (\alpha+n-1)(\alpha+n-2)\Gamma(\alpha+n-2).$$

$$= (\alpha+n-1)(\alpha+n-2) \cdots \alpha\Gamma(\alpha).$$

$$\Gamma(\alpha+n) = (\alpha)_n \Gamma(\alpha). \implies \boxed{(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}}$$

Question If $|z|<1$ & $Re(\gamma) > Re(\beta) > 0$

Then $F(\alpha, \beta, \gamma; z) = \dfrac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \displaystyle\int_0^1 (t)^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt$

Sol:- Since

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n \, n!} z^n \quad \text{--- (1)}$$

take

$$\frac{(\beta)_n}{(\gamma)_n} = \frac{\Gamma(\beta+n)}{\Gamma(\beta)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma+n)} \quad \because (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\gamma+n)} \quad \begin{array}{|l} x = \beta+n \\ y = \gamma-\beta \\ x+y = \gamma+n \end{array}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot B(\beta+n, \gamma-\beta) \quad \text{By defn of Beta function}$$

$$\frac{(\beta)_n}{(\gamma)_n} = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \cdot \int_0^1 (t)^{\beta+n-1} (1-t)^{\gamma-\beta-1}$$

using the above value in eq (1)

$$F(\alpha, \beta, \gamma ; z) = \sum_{n=0}^{\infty} \frac{\Gamma\gamma}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 (t)^{\beta+n-1} (1-t)^{\gamma-\beta-1} dt \; \frac{(\alpha)_n z^n}{n!},$$

$$= \frac{\Gamma\gamma}{\Gamma\beta\,\Gamma(\gamma-\beta)} \int_0^1 \sum_{n=0}^{\infty} (t)^{\beta-1} \cdot t^n (1-t)^{\gamma-\beta-1} dt \, \frac{(\alpha)_n}{n!} z^n.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} (tz)^n dt.$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \times \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt.$$

$$\because \quad (1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n.$$

Hence.

$$F(\alpha, \beta, \gamma ; z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1} (1-t)^{\gamma-\beta-1} (1-tz)^{-\alpha} dt. \quad //$$

= x = x = x = x = x =

Q11 $Re(c-a-b) > 0$ & $Re(c) > Re(b) > 0$

Then

$$F(a,b,c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\cdot\Gamma(c-b)}.$$

Solution Since

$$F(a,b,c;1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} (1)^n.$$

$$F(a,b,c;1) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} \qquad \text{——— (1)}$$

take

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}.$$

$$= \frac{\Gamma(c)\,\Gamma(b+n)\,\Gamma(c-b)}{\Gamma(b)\,\Gamma(c-b)\,\Gamma(c+n)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} B(b+n, c-b) \qquad \because B(x,y) = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$$

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \cdot \int_0^1 (t)^{b+n-1} \cdot (1-t)^{c-b-1}\, dt.$$

using in eqn (1)

$$F(a,b,c;1) = \sum_{n=0}^{\infty} \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 (t)^{b+n-1} (1-t)^{c-b-1} \cdot \frac{(a)_n}{n!}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 \sum_{n=0}^{\infty} (t)^{b-1} \cdot t^n (1-t)^{c-b-1} \cdot \frac{(a)_n}{n!}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(a)_n t^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-a}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-a-b-1}\, dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \cdot B\left(b,\ c-a-b\right)$$

$$= \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \cdot \frac{\Gamma(b) \cdot \Gamma(c-a-b)}{\Gamma(\cancel{b}+c-a-\cancel{b})}$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)}\,\Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)} \cdot \Gamma(c-b-a)}{\Gamma(c-a)}$$

$$= \frac{\Gamma(c)\,\Gamma(c-b-a)}{\Gamma(c-b) \cdot \Gamma(c-a)}$$ is the required result.

(110)

Ass

Q. $F(-n, 1-b-n, a; 1) = \dfrac{(a+b-1)_{2n}}{(a)_n (a+b-1)_n}$.

Sol:

Since
$F(a,b,c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$,

$\therefore$

$$F(-n, 1-b-n, a; 1) = \frac{\Gamma(a)\Gamma(a-(-n)-(1-b-n))}{\Gamma(a-(-n))\Gamma(a-(1-b-n))}$$

$$= \frac{\Gamma(a)\Gamma(a+n-1+b+n)}{\Gamma(a+n)\Gamma(a-1+b+n)}.$$

$$= \frac{\Gamma(a)\Gamma(a+b+2n-1)}{\Gamma(a+n)\Gamma(a+b-1+n)}$$

$$= \frac{\Gamma(a)\Gamma(a+b-1+2n)}{\Gamma(a+n)\Gamma(a+b-1+n)}$$

$$= \frac{1}{(a)_n} \cdot \frac{\Gamma(a+b-1+2n)}{\Gamma(a+b-1+n)}$$

$\because (a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$.

$$= \frac{1}{(a)_n} \cdot \frac{\Gamma(a+b-1+2n)}{\Gamma(a+b-1)} \cdot \frac{\Gamma(a+b-1)}{\Gamma(a+b-1+n)}.$$

$$F(-n, 1-b-n, a; 1) = \frac{1}{(a)_n} \frac{(a+b-1)_{2n}}{(a+b-1)_n} = \frac{(a+b-1)_{2n}}{(a)_n (a+b-1)_n}$$

Ans //

Q11 $F(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n}$.

Sol:-

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n (b)_n (1)^n}{(c)_n n!}$$

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n (b)_n}{(c)_n n!} \quad \text{——} \quad (1)$$

take

$$\frac{(b)_n}{(c)_n} = \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)}$$

$$= \frac{\Gamma(c) \cdot \Gamma(b+n) \cdot \Gamma(c-b)}{\Gamma(b) \cdot \Gamma(c-b) \cdot \Gamma(c+n)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} B(b+n, c-b) \quad \because B(x,y) = \frac{\Gamma x \cdot \Gamma y}{\Gamma(x+y)}$$

$$= \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c-b)} \int_0^1 t^{(b+n-1)} \cdot (1-t)^{(c-b-1)} dt$$

$$= \frac{\Gamma(c)}{\Gamma(b) \cdot \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \cdot t^n dt$$

using in (1)

$$F(-n, b, c; 1) = \sum_{n=0}^{\infty} \frac{(-n)_n}{n!} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} t^n dt$$

$$= \frac{\Gamma(c)}{\Gamma(b)\cdot\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} \sum_{n=0}^{\infty} \frac{(-n)_n t^n}{n!} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-t)^{-(-n)} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b+n-1} dt.$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot B(b, c-b+n) \quad \text{By Def of Beta ftn}$$

$$= \frac{\Gamma(c)}{\cancel{\Gamma(b)}\cdot\Gamma(c-b)} \cdot \frac{\cancel{\Gamma(b)}\cdot\Gamma(c-b+n)}{\Gamma(\cancel{b}+c-\cancel{b}+n)} \quad \because B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= \frac{\Gamma(c)}{\Gamma(c-b)} \cdot \frac{\Gamma(c-b+n)}{\Gamma(c+n)}.$$

$$= \frac{(c-b)_n}{(c)_n}$$

Hence

$$F(-n, b, c; 1) = \frac{(c-b)_n}{(c)_n} \;//.$$

$$(a+1-c)_n = (a+1-c)(a+1-c+1)(a+1-c+2) + \cdots + (a+1-c+(n-1))$$
$$= (a+1-c)(a+2-c)(a+3-c)\cdots(a+n-c). \quad (113)$$

Imp Q.

Let $F(-n, a+n, c; 1) = \dfrac{(-1)^n (a+1-c)_n}{(c)_n}$.

Sol.

As

$$F(a, b, c; 1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}.$$

Then

$$F(-n, a+n, c; 1) = \frac{\Gamma(c)\,\Gamma(c\cancel{+n}-a\cancel{-n})}{\Gamma(c+n)\cdot\Gamma(c-a-n)}.$$

$$= \frac{\Gamma(c)\cdot\Gamma(c-a)}{\Gamma(c+n)\cdot\Gamma(c-a-n)}.$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)}\cdot\frac{(c-a-1)\Gamma(c-a-1)}{\Gamma(c-a-n)}.$$

$$= \frac{\Gamma(c)}{\Gamma(c+n)}\cdot\frac{(c-a-1)(c-a-2)\cdots(c-a-n)\cancel{\Gamma(c-a-n)}}{\cancel{\Gamma(c-a-n)}}$$

$$= \frac{1}{(c)_n}\cdot(-1)^n (a+1-c)(a+2-c)\cdots(a+n-c)$$

$$= \frac{(-1)^n}{(c)_n}(a+1-c)_n = \frac{(-1)^n (a+1-c)_n}{(c)_n}$$

(114) $(a)_1 = a$, $(a)_2 = a(a+1)$.

*

Q#

$$\frac{d}{dx} F(a, b, c; x) = \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Sol:

We know that,

$$F(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n \, n!} x^n$$

$$= 1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)\,3!}x^3 + \cdots$$

Differentiating both sides

$$\frac{d}{dx} F(a,b,c;x) = \frac{ab}{c} + \frac{\cancel{2}ab(a+1)(b+1)}{c(c+1)\cancel{2!}}x + \frac{3ab(a+1)(a+2)(b+1)(b+2)}{c(c+1)(c+2)\,3!}x^2 + \cdots$$

$$= \frac{ab}{c}\left[1 + \frac{(a+1)(b+1)}{(c+1)}x + \frac{(a+1)(a+2)(b+1)(b+2)}{(c+1)(c+2)2!}x^2 + \cdots\right]$$

$$= \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a+1)_n (b+1)_n}{(c+1)_n \, n!} x^n$$

$$= \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Hence

$$\frac{d}{dx} F(a,b,c;x) = \frac{ab}{c} F(a+1, b+1, c+1; x)$$

Q11 Show That

$$\frac{d^n}{dx^n}\left[x^{a+n-1}F(a,b,c;x)\right] = (a)_n x^{a-1} F(a+n,b,c;x)$$

Let

$$g(x) = x^{a+n-1} F(a,b,c;x)$$

$$= x^{a+n-1}\left[\sum_{n=0}^{n} \frac{(a)_n (b)_n}{(c)_n\, n!} x^n\right]$$

$$= x^{a+n-1}\left[1 + \frac{ab}{c}x + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!}x^3 + \cdots\right]$$

$$g(x) = x^{a+n-1} + \frac{ab}{c}x^{a+n} + \frac{a(a+1)b(b+1)}{c(c+1)2!}x^{a+n+1} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!}x^{a+n+2} + \cdots$$

$$\frac{d}{dx}g(x) = (a+n-1)x^{a+n-2} + \frac{ab}{c}(a+n)x^{a+n-1} + \frac{a(a+1)b(b+1)(a+n+1)}{c(c+1)2!}x^{a+n}$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)(a+n+2)}{c(c+1)(c+2)3!}x^{a+n+1} + \cdots$$

$$\frac{d^2}{dx^2}\left[g(x)\right] = \Big[(a+n-1)(a+n-2)x^{a+n-3} + \frac{ab}{c}(a+n)(a+n-1)x^{a+n-2}$$

$$+ \frac{a(a+1)b(b+1)}{c(c+1)2!}(a+n+1)(a+n)x^{a+n-1}$$

$$+ \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}(a+n+2)(a+n+1)x^{a+n} + \cdots\Big]$$

Generally:

$$\frac{d^n}{dx^n}\left[g(x)\right] = (a+n-1)(a+n-2)(a+n-3)\cdots(a+n-(n-1))\,a\,x^{a-1}$$

$$+ \frac{ab}{c}(a+n)(a+n-1)(a+n-2)\cdots(a+1)\,x^{a}$$

$$+ \frac{a(a+1)b(b+1)}{c(c+1)2!}(a+n+1)(a+n)(a+n-1)(a+n-2)\cdots(a+2)\,x^{a+1} + \cdots\Big]$$

$$\frac{d^n}{dx}\left[g(x)\right] = a(a+1)(a+2)\cdots(a+(n-1))\,x^{a-1}$$

$$+ a(a+1)(a+2)\cdots(a+n-1)(a+n)\frac{b}{c}x^{a}$$

$$+ a(a+1)(a+2)(a+3)\cdots(a+n-1)(a+n)(a+n+1)\frac{b(b+1)}{c(c+1)2!}x^{a+1} + \cdots$$

$$\frac{d^n[g(x)]}{dx^n} = (a)_n x^{a-1} + (a)_n \frac{(a+n)}{c} x^{a} + (a)_n \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)\,2!} x^{a+1} \cdots$$

$$= (a)_n x^{a-1}\left[1 + \frac{(a+n)}{c} b x^{1} + \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)\,2!} x^{2} + \cdots\right]$$

$$= (a)_n x^{a-1}\left[\sum_{n=0}^{\infty} \frac{(a+n)_n\,(b)_n}{(c)_n\, n!} x^{n}\right]$$

$$= (a)_n x^{a-1} F(a, b, c; x).$$

Hence

$$\frac{d^n}{dx^n}\left[x^{a+n-1} F(a,b,c;x)\right] = (a)_n x^{a-1} F(a+n, b, c; x)$$

(118) Results Prove That

i) $B(x+1, y) = \frac{x}{x+y} B(x, y)$.

ii) $B(x, y+1) = \frac{y}{x+y} B(x, y)$.

Proof:

$$B(x+1, y) = \frac{\Gamma(x+1)\Gamma y}{\Gamma(x+y+1)}.$$

$$= \frac{x\Gamma(x)\Gamma(y)}{(x+y)\Gamma(x+y)} \qquad \because \Gamma(z+1) = z\Gamma(z)$$

$$= \frac{x}{x+y} \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$$B(x+1, y) = \frac{x}{x+y} B(x, y)$$

ii) $B(x, y+1) = \frac{y}{x+y} B(x, y)$.

L.H.S

$$B(x, y+1) = \frac{\Gamma(x) \cdot \Gamma(y+1)}{\Gamma(x+y+1)}.$$

$$= \frac{\Gamma(x) \cdot y\,\Gamma(y)}{(x+y)\Gamma(x+y)}$$

$$B(x, y+1) = \frac{y}{(x+y)} \cdot \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

$$B(x, y+1) = \frac{y}{x+y} B(x, y).$$

— x — x — x — x — x —

Prove That.

Imp Q.

$$\Gamma(z) = 2 \int_0^\infty e^{-t^2} \cdot t^{2z-1}\, dt.$$

Proof

$$\text{Since } \Gamma(z) = \int_0^\infty e^{-t} \cdot t^{z-1}\, dt \quad \text{—— (1)}$$

Put $t = u^2 \implies dt = 2u\,du$.

when $t \to 0$ Then $u \to 0$

when $t \to \infty$ Then $u \to \infty$

(1) $\implies$

$$\Gamma(z) = \int_0^\infty e^{-u^2} (u^2)^{z-1} (2u\,du).$$

$$= 2 \int_0^\infty e^{-u^2} u^{2z-2} \cdot u\,du.$$

$$= 2 \int_0^\infty e^{-u^2} u^{2z-1}\, du.$$

$$= 2 \int_0^\infty e^{-t^2} (t)^{2z-1}\, dt \quad (\because u \text{ is dummy variable})$$

— x — x — x —

Theorem :- $\int_0^{\pi/2} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, d\theta$

$$= \frac{\Gamma(x)\,\Gamma(y)}{2\,\Gamma(x+y)}.$$

Proof :- Consider

$$I = \iint_R e^{-t^2} \cdot e^{-u^2} \cdot t^{2x-1} \, u^{2y-1} \, dt\,du$$

where 'R' is the region of the tu-plane consisting of first quardrant only.

R: $0 \le t < \infty$, $0 \le u < \infty$

$$I = \int_0^\infty \int_0^\infty e^{-t^2} \cdot e^{-u^2} \cdot t^{2x-1} \, u^{2y-1} \, dt\,du.$$

$$I = \int_0^\infty \left(\int_0^\infty e^{-t^2} \cdot t^{2x-1} \, dt \right) e^{-u^2} u^{2y-1} \, du.$$

$$I = \int_0^\infty \frac{1}{2}\,\Gamma(x)\, e^{-u^2} u^{2y-1} \, du \quad \because \; \Gamma(x) = 2\int_0^\infty e^{-t^2} \cdot t^{2x-1} \, dt$$

$$= \frac{1}{2}\,\Gamma(x) \cdot \frac{1}{2}\,\Gamma(y) \quad \text{——— (1)}$$

Now we solve I in polar coordinates

put

$t = r\cos\theta, \quad u = r\sin\theta.$

$dA = r\,dr\,d\theta.$

where

$R: 0 \le r < \infty$ and $0 \le \theta < \pi/2.$

$$I = \int_0^\infty \int_0^{\pi/2} e^{-r^2\cos^2\theta} \cdot e^{-r^2\sin^2\theta} \cdot (r\cos\theta)^{2x-1} (r\sin\theta)^{2y-1} r\,dr\,d\theta$$

$$I = \int_0^\infty \int_0^{\pi/2} e^{-r^2(\cos^2\theta + \sin^2\theta)} \cdot r^{2(x+y)-1} \cos^{2x-1}\theta \sin^{2y-1}\theta\, dr\,d\theta$$

$$I = \int_0^\infty \int_0^{\pi/2} e^{-r^2} r^{2(x+y)-1} \cos^{2x-1}\theta \sin^{2y-1}\theta \, dr\,d\theta$$

$$I = \int_0^{\pi/2} \frac{1}{2}\Gamma(x+y) \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta. \quad \because 2\int_0^\infty r^{2(x+y)-1} e^{-r^2} dr = \Gamma(x+y).$$

$$I = \frac{1}{2}\Gamma(x+y) \int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta \quad \text{--- (2)}$$

Comparing eqn (1) & (2) we get

$$\frac{1}{2}\Gamma(x+y)\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta = \frac{1}{2}\cdot\frac{1}{2}\Gamma(x)\Gamma(y)$$

$$\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta\, d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}.$$

//

Theorem.

Imp Q. $\Gamma(1/2) = \sqrt{\pi}$

Proof: Since $\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta$

$$= \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$$

put $x = y = 1/2$.

Then $\int_0^{\pi/2} \cos^0\theta \cdot \sin^0\theta \, d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

$\Rightarrow \int_0^{\pi/2} (1)(1) \, d\theta = \frac{\Gamma(1/2)\Gamma(1/2)}{2\Gamma(1)}$ $\because x^0 = 1$

$\Rightarrow \int_0^{\pi/2} d\theta = \frac{(\Gamma(1/2))^2}{2}$

$\Rightarrow \theta \Big|_0^{\pi/2} = \frac{(\Gamma(1/2))^2}{2}$

$\Rightarrow \frac{\pi}{\cancel{2}} = \frac{(\Gamma(1/2))^2}{\cancel{2}}$

$\pi = (\Gamma(1/2))^2$

$\Rightarrow \sqrt{\pi} = \Gamma(1/2)$

merchant

$\Rightarrow \Gamma(1/2) = \sqrt{(\pi)}$. //

T. from stat notes

Assignment

Prove That

$$\Gamma(x)\cdot\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Proof: Since we know that

$$\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta = \frac{\Gamma(x)\,\Gamma(y)}{2\Gamma(x+y)}$$

put $y = 1-x$.

$$\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{2(1-x)-1}\theta \, d\theta = \frac{\Gamma(x)\cdot\Gamma(1-x)}{2\Gamma(x+(1-x))}.$$

$$\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{1-2x}\theta \, d\theta = \frac{\Gamma(x)\cdot\Gamma(1-x)}{2\Gamma(1)}$$

$$\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{1-2x}\theta \, d\theta = \frac{\Gamma(x)\cdot\Gamma(1-x)}{2}.$$

$$\Rightarrow 2\int_0^{\pi/2} \cos^{2x-1}\theta \cdot \sin^{1-2x}\theta \, d\theta = \Gamma(x)\cdot\Gamma(1-x)$$

Prove That $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$.

Proof:

$$\Gamma(x) = 2\int_0^\infty e^{-t^2} . t^{2x-1} dt.$$

Put $x = 1/2$

$$\Gamma(1/2) = 2\int_0^\infty e^{-t^2} . t^{2(1/2)-1} dt.$$

$$\Gamma(1/2) = 2\int_0^\infty e^{-t^2} dt.$$

$$\sqrt{\pi} = 2\int_0^\infty e^{-t^2} dt.$$

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-t^2} dt$$

Hence $\int_0^\infty e^{-t^2} dt = \sqrt{\pi}/2$.

Prove That.

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}.$$

Proof: Since

$$B(x,y) = \int_0^1 t^{x-1} . (1-t)^{y-1} dt.$$

Put $t = \cos^2\theta \Rightarrow dt = -2\cos\theta \sin\theta d\theta$

when $t \to 0$ Then $\theta \to \pi/2$.

when $t \to 1$ Then $\theta \to 0$

(126)

$$B(x,y) = \int_{\pi/2}^{0} (\cos^2\theta)^{x-1} (1-\cos^2 x)^{y-1} (-2\cos\theta\sin\theta)\,d\theta.$$

$$= 2\int_0^{\pi/2} \cos^{2x-2}\theta \cdot \sin^{2y-2}\theta \cdot \cos\theta\sin\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta.$$

$$= \cancel{2}\cdot\frac{\Gamma(x)\cdot\Gamma(y)}{\cancel{2}\,\Gamma(x+y)}$$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

— x — x — x — x —

imp 4. Express $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta$ in term of Gamma ftn or Beta functi and simplify of possible.

Proof:- $$\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta.$$

(127)

$$= \int_0^{\pi/2} \cos^{-1/2}\theta \sin^{1/2}\theta \, d\theta \quad \text{--- (1)}$$

Since $\displaystyle\int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta = \frac{\Gamma(x)\Gamma(y)}{2\Gamma(x+y)}$ --- (2)

put $2x-1 = -1/2 \Rightarrow 2x = 1/2 \Rightarrow x = 1/4$

and $2y-1 = 1/2 \Rightarrow 2y = 3/2 \Rightarrow y = 3/4$

putting value of x & y in eqn (2)

$$\int_0^{\pi/2} \cos^{-1/2}\theta \sin^{1/2}\theta \, d\theta = \frac{\Gamma(1/4)\cdot\Gamma(3/4)}{2\Gamma(1/4+3/4)}$$

$$= \frac{\Gamma(1/4)\cdot\Gamma(1-1/4)}{2\Gamma(1)}$$

$$= \frac{\Gamma(1/4)\cdot\Gamma(1-1/4)}{2}$$

$$= \frac{\frac{\pi}{\sin\pi(1/4)}}{2} \quad \because \Gamma x\cdot\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

$$= \frac{\pi}{2\sin\pi/4}$$

Hence $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$ // $\displaystyle= \frac{\pi}{2\cdot\frac{1}{\sqrt{2}}} = \frac{\pi}{\sqrt{2}}$

Bessal Function:-

Notation: $J_n(x)$

It is defined as

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n}\, m!\, \Gamma(m+n+1)}$$

is called Bessal function of index 'n' order 'n' and we obtain the Bessal function by solving Bessal differential equation of order 'n',

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

where 'n' is non-negative integer.

* Show That

$$J_{-n}(x) = (-1)^n J_n(x).$$

Proof:- we know

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^n (x)^{2m+n}}{2^{2m+n}\, m!\, \Gamma(m+n+1)}$$

Put $n = -n$.

$$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-n}}{2^{2m-n}\, m!\, \Gamma(m-n+1)}$$

$\Gamma(m-n+1)$ diverges if

$m - n + 1 \leq 0$

$\Rightarrow m \leq n-1$

$$\boxed{\begin{aligned}\Gamma(z+1) &= (z)\Gamma(z) \\ \Gamma(z+2) &= (z+1)z\Gamma z.\end{aligned}}$$

(129)

for This value

$$J_{-n}(x) = 0 \qquad \text{so } m \geq n.$$

$$\Rightarrow J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m (x)^{2m-n}}{2^{2m-n}(m!)\Gamma(m-n+1)}$$

let

$$k = m-n \Rightarrow m = k+n.$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(x)^{2(k+n)-n}}{2^{2(k+n)-n}\cdot(k+n)!\,\Gamma(k+n-n+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+n}(x)^{2k+2n-n}}{2^{2k+2n-n}\cdot(k+n)!\,\Gamma(k+1)}$$

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(-1)^n\cdot(x)^{2k+n}}{2^{2k+n}(k+n)(k+(n-1))\cdots(k+1)k!\,\Gamma(k+1)}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k(x)^{2k+n}}{2^{2k+n}k!\,\Gamma(k+n+1)}$$

$$\boxed{J_{-n}(x) = (-1)^n J_n(x).}$$ required Result.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\sqrt{2} = (2-1)/2-1)$
$= (2-1)(2-)$

(130)

Show that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \sin x.$$

Sol: As

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+n}}{2^{2k+n} . k!\, \Gamma(k+n+1)}$$

Put $n = 1/2$.

$$J_{1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1/2}}{2^{2k+1/2} .\, k!\, \Gamma(k+1/2+1)}$$

$$J_{1/2} = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+1/2}}{2^{2k+1/2} .\, k!\, (\Gamma(k+3/2)}.$$

$$= \frac{x^{1/2}}{2^{1/2}\, \Gamma(3/2)} + \frac{(-1)^1 (x)^{5/2}}{2^{5/2} 1!\, \Gamma(5/2)} + \frac{(-1)^2 (x)^{9/2}}{2^{9/2} . 2!\, \Gamma(9/2)} + \cdots$$

$$J_{1/2}(x) = \frac{\left(\frac{x}{2}\right)^{1/2}}{\Gamma(3/2)} - \frac{(x/2)^{5/2}}{\Gamma(5/2)} + \frac{(x/2)^{9/2}}{2!\, \Gamma(9/2)}$$

$$- \frac{(x/2)^{11/2}}{3!\, \Gamma(9/2)} + \cdots$$

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2} \left[\frac{1}{\frac{1}{2}\Gamma(1/2)} - \frac{(x/2)^2}{\frac{3}{2} \cdot \frac{1}{2}\Gamma(1/2)} + \frac{(x/2)^4}{2! \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(1/2)} - [illegible] \cdots \right]$$

$$= \frac{\left(\frac{x}{2}\right)^{1/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\left[1 - \frac{x^2}{2^2 \cdot 3/2} + \frac{x^4}{2^4 \cdot 2! \cdot 5/2 \cdot 3/2} - \frac{x^6}{2^6 \cdot 3! \cdot \frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}} + \cdots\right]$$

$$= \frac{x^{1/2}}{\frac{1}{2}\cdot 2^{1/2}\sqrt{\pi}}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right]$$

$$= \frac{2^{1/2}\cdot x^{1/2}}{x\sqrt{\pi}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \sqrt{\frac{2}{x\pi}}\,\sin x.$$

Hence

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

(132)

Show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\, \cos x.$$

Sol:- As by the definition Bessal function

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (x)^{2m+n}}{2^{2m+n} \cdot m!\,\Gamma(m+n+1)}$$

put

$n = -1/2$.

$$J_{(-1/2)}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-1/2}}{2^{2m-1/2} \cdot m!\,\Gamma(m-1/2+1)}.$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m-1/2}}{2^{2m-1/2}\, m!\,\Gamma(m+1/2)}.$$

$$= \frac{x^{-1/2}}{2^{-1/2} \cdot \Gamma(1/2)} + \frac{(-1)^1 (x)^{3/2}}{2^{3/2} (1!)\,\Gamma(3/2)} + \frac{(-1)^2 (x)^{7/2}}{2^{7/2} (2!)\,\Gamma(5/2)}$$

$$+ \frac{(-1)^3 \cdot (x)^{11/2}}{2^{11/2} \cdot (3!)\,\Gamma(7/2)} + \cdots$$

$$= \frac{x^{-1/2}}{2^{-1/2} \cdot 1}\left[\frac{1}{\Gamma(1/2)} - \frac{x^2}{2^2 \cdot \frac{1}{2}\Gamma(1/2)} + \frac{x^4}{2^4 \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \Gamma(1/2)} - \cdots\right]$$

$$= \frac{2^{1/2}}{x^{1/2}} \cdot \frac{1}{\Gamma(1/2)}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right]$$

$$= \sqrt{\frac{2}{x}} \cdot \frac{1}{\sqrt{\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cos x .$$

Hence $J_{(-1/2)}(x) = \sqrt{\frac{2}{\pi x}} \cos x .$

— x — x — x — x — x —

V-Imp

* Differential Recurrence Relation of Bessal function:-

As $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m! \Gamma(m+n+1)}$ — ①

Multiplying (x^n) on both sides

$$x^n J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+n}}{2^{2m+n} m! (\Gamma(m+n+1))}$$

$$x^n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x)^{2m+2n}}{2^{2m+n} \cdot m! \cdot \Gamma(m+n+1)}$$

On Differentiating

$$x^n J_n'(x) + n x^{n-1} J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot 2(\cancel{m}+n)(x)^{2m+2n-1}}{2^{2m+n} \cdot m! \, (\cancel{m+n}) \cdot \Gamma(m+n)}$$

Short $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$ / $\frac{d}{dx} x^5 J_5(x) = x^5 J_4(x)$

$= x^n J_{n-1}(x)$

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$$x^n J_n'(x) + n x^{n-1} J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+2n-1}}{2^{2m+n-1} \cdot m! \, \overline{|(m+n)}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n-1} \cdot (x)^n}{2^{2m+n-1} \cdot m! \, \overline{|(m+n)}}.$$

$$= x^n \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n-1}}{2^{2m+n-1} \cdot m! \, \overline{|(m+n-1+1)}}$$

$$= x^n J_{n-1}(x)$$

$$x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x).$$

$$J_n'(x) = J_{n-1}(x) - n x^{-1} J_n(x) \text{ --- } (2)$$

Now Multiplying x^{-n} on both sides of (1)

$$x^{-n} J_n(x) = x^{-n} \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m+n}}{2^{2m+n} \cdot m! \, \overline{|(m+n+1)}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \cdot x^{2m}}{2^{2m+n} \cdot m! \, \overline{|(m+n+1)}}.$$

on Differentiating

$$\therefore\; x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = \sum_{m=1}^{\infty} \frac{(-1)^m \cdot 2m \cdot x^{2m-1}}{2^{2m+n} \cdot m! \cdot \Gamma(m+n+1)}$$

put

$$m-1 = k \implies m = k+1$$

$$x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot 2(k+1) \cdot x^{2k+1}}{2^{2k+n+2} \cdot (k+1)!\, \Gamma(k+n+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot (-1)^1 \cdot \cancel{(k+1)}\, x^{2k+1}}{2^{2k+n+1} \cdot \cancel{(k+1)}\, k!\, \Gamma(k+n+2)}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+n+1} \cdot x^{-n}}{2^{2k+n+1} \cdot k!\, \Gamma(k+n+1+1)}$$

$$= -x^{-n} \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+n+1}}{2^{2k+n+1} \cdot k!\, \Gamma(k+\underline{n+1}+1)} \quad \Big\}\ \text{Here } n = n+1$$

$$\therefore\; x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x).$$

i) $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$ when multiplying by x^n & differentiate.

ii) $xJ_n'(x) = nJ_n(x) - xJ_n(x)$ when muliply by x^{-n} & diff

iii) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$.

iv) $J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

$$\Rightarrow +J_n'(x) - nx^{-1}J_n(x) = -J_{n+1}(x)$$

$$J_n'(x) = nx^{-1}J_n(x) - J_{n+1}(x) \quad \text{③}$$

adding ② & ③ we have.

$$\boxed{2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)}$$

This relation is called differential recurence relation.

now subtracting ③ from ②

$$0 = J_{n-1}(x) - 2nx^{-1}J_n(x) + J_{n+1}(x)$$

$$2nx^{-1}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

$$\boxed{J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]}$$

This relation is called pure recurence or non-differential relation.

V.Imp Spherical Bessal Function:

General Representation of Spherical Bessal function is

$$J_{n+1/2}(z) = A(z)\sin z + B(z)\cos z$$

where $A(z)$ and $B(z)$ are polynomial in z.

* Properties of Bessal function:-

1) $$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{\sin z - z\cos z}{z}\right]$$

Sol:- we know

$$J_n(z) = \frac{z}{2n}\left[J_{n-1}(z) + J_{n+1}(z)\right]$$

put $n = 1/2$.

$$J_{1/2}(z) = \frac{z}{2(1/2)}\left[J_{\frac{1}{2}-1}(z) + J_{\frac{1}{2}+1}(z)\right]$$

$$J_{1/2}(z) = z\left[J_{-1/2}(z) + J_{3/2}(z)\right]$$

$$= z\,J_{-1/2}(z) + z\,J_{3/2}(z)$$

$$z\,J_{3/2}(z) = J_{1/2}(z) - z\,J_{-1/2}(z).$$

$$J_{3/2}(z) = z^{-1}\left[J_{1/2}(z) - z\,J_{-1/2}(z)\right]$$

$$J_{3/2}(z) = z^{-1}\left[J_{1/2}(z) - z\,J_{-1/2}(z)\right]$$

$$= z^{-1}\left[\sqrt{\frac{2}{\pi z}}\sin z - z\sqrt{\frac{2}{\pi z}}\cos z\right]$$

$$J_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{\sin z - z\cos z}{z}\right]$$

Assingment

$J_{-3/2}$, $J_{5/2}$, $J_{-5/2}$

(Q). Show that $\int x^n J_{n-1}(x)\,dx$

Sol. $\int x^n J_{n-1}(x)\,dx = x^n J_n(x) + C$

$$= \int \frac{d}{dx}\left(x^n J_n(x)\right) dx$$

$$= x^n J_n(x) + C$$

$$\frac{d}{dx}\left(x^{-n} J_n(x)\right)$$

b) $\int x^{-n} J_{n+1}(x)\, dx.$

$$= -\int \frac{d}{dx}\left(x^{-n} J_n(x)\right\} dx.$$

$$= -x^{-n} J_n + C.$$

e.g. $\int x^{-3} J_4(x)\, dx.$

$$= -x^{-3} J_3(x) + C.$$

---

* Find $J_{-3/2}(x)$.

we know that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

put $n = -1/2$.

$$J_{-1/2}(x) = \frac{x}{2(-1/2)}\left[J_{-1/2-1}(x) + J_{-1/2+1}(x)\right]$$

?

$$J_{-1/2}(x) = -x\left[J_{-3/2}(x) + J_{1/2}(x)\right]$$

$$J_{-1/2}(x) = -x J_{-3/2}(x) - x J_{1/2}(x).$$

$$x J_{-3/2}(x) = J_{-1/2}(x) - x J_{1/2}(x).$$

$$J_{-3/2}(x) = \frac{1}{x}\left[J_{-1/2}(x) - x J_{1/2}(x)\right]$$

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$$J_{-3/2}(x) = \frac{1}{x}\left[J_{-\frac{1}{2}}(x) - x\, J_{1/2}(x)\right]$$

$$J_{-3/2}(x) = \frac{1}{x}\left[\sqrt{\frac{2}{\pi x}}\cos x - x\left[\sqrt{2/\pi x}\left(\frac{\sin x - x\cos x}{x}\right)\right]\right]$$

$$= \frac{1}{x}\sqrt{\frac{2}{\pi x}}\cdot\cos x - \sqrt{\frac{2}{\pi x}}\sin x + x\cos x\cdot\sqrt{2/\pi x}$$

$$= \left(\frac{1}{x} + x\right)\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x.$$

$$J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left(x\cos x + \frac{1}{x}\cos x - \sin x\right)$$

Find $J_{5/2}(x)$. ?

we know That

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

Put $n = 3/2$.

$$J_{3/2}(x) = \frac{x}{\cancel{2}(3/\cancel{2})}\left[J_{1/2}(x) + J_{5/2}(x)\right]$$

$$J_{3/2}(x) = \frac{x}{3}J_{1/2}(x) + \frac{x}{3}J_{5/2}(x).$$

$$J_{5/2}(x) = \frac{3}{x}\left[J_{3/2}(x) - \frac{x}{3}J_{(1/2)}(x)\right]$$

$$= \frac{3}{x}J_{3/2}(x) - J_{1/2}(x).$$

$$J_{5/2}(x) = \frac{3}{x}\cdot\left[\sqrt{2/\pi x}\left(\frac{\sin x - x\cos x}{x}\right)\right] - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3(\sin x - x\cos x)}{x^2} - \sin x\right]$$

$$J_{5/2}(x) = \sqrt{2/\pi x}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right]$$

---

* Find $J_{-5/2}(x)$. ?

we know that

$$J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$$

put $n = -3/2$.

$$J_{-3/2}(x) = \frac{x}{2(-3/2)}\left[J_{-5/2}(x) + J_{-1/2}(x)\right]$$

$$J_{-3/2}(x) = -\frac{x}{3}\left[J_{-5/2}(x) + J_{-1/2}(x)\right]$$

$$J_{-5/2}(x) = \frac{3}{x}\left[-\frac{x}{3}J_{-1/2} - J_{-3/2}(x)\right]$$

$$J_{-5/2}(x) = \left[-J_{-1/2} - \frac{3}{x}J_{-3/2}(x)\right]$$

$$= -\sqrt{\frac{2}{\pi x}}\cos x - \frac{3}{x}\left[\sqrt{\frac{2}{\pi x}}\left[x\cos x + \frac{1}{x}\cos x - \sin x\right]\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[-\cos x - \frac{3}{x}\left(x\cos x + \frac{1}{x}\cos x - \sin x\right)\right]$$

$$= \sqrt{\frac{2}{\pi x}}\left[\cancel{\cos x} - 4\cos x - \frac{3}{x^2}\cos x - 3\frac{\sin x}{x}\right] \text{Ans.}$$

* * i) $\int z^n J_{n-1}(z)\,dz = z^n J_n(z)$ — If power is greater than order of bessel function
ii) $\int z^{-n} J_{n+1}(z)\,dz = -z^{-n} J_n(z)$

(142)

Q11 Evaluate $\int x^3 J_3(x)\,dx$.

$$= \int x^5 \cdot x^{-2} J_3(x)\,dx$$

$$= x^5 \cdot \int x^{-2} J_3(x)\,dx - \int (5x^4)(-x^{-2} J_2(x))\,dx$$

$$= x^5 \cdot (-x^{-2} J_2(x)) + 5\int x^2 J_2(x)\,dx$$

$$= -x^3 J_2(x) + 5\int x^{-1} \cdot x^3 J_2(x)\,dx$$

$$= -x^3 J_2(x) + 5\left[x^3 \int x^{-1} J_2(x)\,dx - \int 3x^2 \cdot (-x^{-1} J_1(x))\,dx\right]$$

$$= -x^3 J_2(x) + 5\left[-x^3 \cdot x^{-1} J_1(x) + 3\int x^2 x^{-1} J_1(x)\,dx\right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15\int x J_1(x)\,dx$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15\left[\int x^1 \cdot x^{-0} J_1(x)\,dx\right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15\left[x \int x^{-0} J_1(x) - \int -x^{-0} J_0(x)\right]$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) + 15x(-x^0 J_0(x)) + 15\int x^0 J_0(x)$$

$$= -x^3 J_2(x) - 5x^2 J_1(x) - 15x J_0(x) + 15\int J_0(x)\,dx$$

Q11. a) $\int x^4 J_1(x)\,dx.$

b) $\int x^4 J_4(x)\,dx.$

c) $\int x^3 J_2(x)\,dx.$

i) $\frac{d}{dx} x^n J_n(x) = x^n J_{n-1}(x).$

ii) $\int x^n J_{n-1}(x) = x^n J_n(x)$

iii) $\int x^{-n} J_{n+1}\,dx = x^{-n} J_n(x).$

Sol:

a) $\int x^4 J_1(x)\,dx$

$$= \int x^2 x^2 J_1(x)\,dx$$

$$= x^2 \int x^2 J_1(x)\,dx - \int 2x \cdot x^2 J_2(x)\,dx$$

$$= x^2 \cdot x^2 J_2(x) - 2\int x^3 J_2(x)\,dx.$$

$$x^4 J_2(x) - 2x^3 J_3(x)$$

$$\cancel{= x^4 J_2(x) - 2\left[\int x \cdot x^2 J_2(x)\,dx\right]}$$

$$\cancel{= x^4 J_2(x) - 2\left[x \cdot x^2 J_2(x) - \int x^2 J_2(x)\,dx\right]}$$

$$\cancel{= x^4 J_2(x) - 2\left[x^3 J_2(x) - \int x^3 x^{-1} J_2(x)\,dx\right]}$$

$$\cancel{= x^4 J_2(x) - 2(x^3 J_2(x)) + 2\left[x^3(-x^{-1} J_1(x)) + \int 3x^2 x^{-1} J_1(x)\,dx\right]}$$

$$\cancel{= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6\int x J_1(x)\,dx.}$$

$$\cancel{= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6\left[\int x \cdot x^0 J_1(x)\,dx\right]}$$

$$\cancel{= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) + 6\left[x(-x^0 J_0(x)) + \int x^0 J_0(x)\,dx\right]}$$

$$= x^4 J_2(x) - 2x^3 J_2(x) - 2x^2 J_1(x) - 6 J_0(x) - 6\int J_0(x)\,dx$$

Ans //

(144) Solve

b) $\int x^4 J_4(x)\,dx.$

$= \int x^7 \cdot x^{-3} J_4(x)\,dx.$

$= x^7(-x^{-3}J_3(x)) + 7\int x^6 \cdot x^{-3} J_3(x)\,dx.$

$= -x^4 J_3(x) + 7\int x^3 J_3(x)\,dx.$

$= -x^4 J_3(x) + 7\int x^5 x^{-2} J_3(x)\,dx.$

$= -x^4 J_3(x) + 7\left[x^5(-x^{-2}J_2(x)) + 5\int x^4 \cdot x^{-2} J_2(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) + 35\int x^2 J_2(x)\,dx$

$= -x^4 J_3(x) - 7x^3 J_2(x) + 35\left[\int x^3 x^{-1} J_2(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) + 35\left[-x^3 \cdot x^{-1} J_1(x) + 3\int x^2 x^{-1} J_1(x)\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105\int x J_1(x)\,dx$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105\left[\int x \cdot x^{0} J_1(x)\,dx\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) + 105\left[-x \cdot x^{0} J_0(x) + \int J_0(x)\right]$

$= -x^4 J_3(x) - 7x^3 J_2(x) - 35x^2 J_1(x) - 105 x J_0(x) + \int J_0(x)\,dx$

//

i) $\int x^n J_{n-1}(x)\,dx = x^n J_n(x)$ (145)

ii) $\int x^{-n} J_{n+1}\,dx = -x^{-n} J_n$

c) $\int$ nve

$$\int x^3 J_2(x)\,dx$$

$$= x^3 J_3(x).$$

a) $$\int x^4 J_1(x)\,dx.$$

$$= \int x^2 \cdot x^2 J_1(x)\,dx.$$

$$= x^2 \int x^2 J_1(x) - \int 2x \cdot \int x^2 J_1(x)\,dx.$$

$$= x^2 \int x^2 J_1(x) - 2\int x \cdot x^2 J_2(x)\,dx$$

$$= x^4 J_2(x) - 2\int x^3 J_2(x)$$

$$= x^4 J_2(x) - 2x^3 J_3(x)$$

$$\Gamma(3) = 2\,\Gamma(2) = 2\cdot 1\cdot \Gamma(1) = 2\cdot 1\cdot$$

imp

Q11 Show that

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

Sol:

we know

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\cdot x^{2k+n}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}$$

Put $n = 1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k\cdot x^{2k+1}}{2^{2k+1}\cdot k!\cdot \Gamma(k+2)}$$

$$= \frac{x}{2\cdot 0!\,\Gamma(2)} + \frac{(-1)^1 x^3}{2^3\cdot 1!\,\Gamma(3)} + \frac{(-1)^2 x^5}{2^5\cdot 2!\,\Gamma(4)}$$

$$+ \frac{(-1)^3 x^7}{2^7\cdot 3!\cdot \Gamma(5)} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^3\cdot 2} + \frac{x^5}{2^6\cdot 3\cdot 2} - \frac{x^7}{2^7\, 3!\cdot 4\cdot 3\cdot 2\cdot 1} + \cdots$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2\cdot 4} + \frac{x^5}{2^2\cdot 4^2\cdot 6} - \frac{x^7}{2^2\cdot 4^2\cdot 6^2\cdot 8} + \cdots$$

which is required result.

$F_n \longrightarrow$ q.n.r

(148)

★ Hypergeometric function:-

we define the General hypergeometric function with 'm' upper perameter and 'n' Lower parameter written in

$F_n (a_1, a_2, a_3, ---, a_m, b_1, b_2, b_3, --, b_n; z)$

by

$$mF_n (a_1, a_2, a_3, --, a_m; b_1, b_2, b_3, --, b_n; z)$$

$$mF_n = \sum_{r=0}^{\infty} \frac{(a_1)_r (a_2)_r (a_3)_r --- (a_m)_r}{(b_1)_r (b_2)_r (b_3)_r --- (b_n)_r} \frac{z^n}{n!}$$

The notation $mF_n \left[\begin{matrix} a_1, a_2, a_3, ---, a_m; x \\ b_1, b_2, b_3, ---, b_n; \end{matrix} \right]$

is also often used.

Now we discussed Two cases

If

$$m = n = 1$$

Then The general hypergeometric ftn will be called the "confluent" hypergeometric function. Thes Confluent Hyper Geometric function.

${}_1F_1 (a; b; x)$ is often denoted by

$M(a; b; x)$.

If $m = 2, n = 1$ Then ftn is hypergeometric ${}_2F_1(a, b; c; z)$

Some special cases of ${}_mF_n$.

i) when There are no upper and low upper parameters.

$${}_0F_0(; ; \pm x) = e^{\pm x}$$

As

$${}_0F_0(; ; x) = \sum_{r=0}^{\infty} \frac{(x)^r}{r!}$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \text{---}$$

$$= e^{x} \quad \text{--- } (1)$$

Also

$${}_0F_0(; ; -x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!}$$

$$= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!}$$

$$= e^{-x} \quad \text{--- } (2)$$

Combining (1) & (2)

$${}_0F_0(; ; \pm x) = e^{\pm x}$$

$(1/2)_2 = 1/2\,(1/2+1)$ $\quad (a)_n = a(a+1)(a+2)\cdots(a+(n-1))$

$= 3/6$

$= 3/2\,(3/2+1)$ $\quad (1/2)_1 = 1/2+0 = 1/2$

(150.)

2) $\quad {}_1F_0(a;\ ;x) = (1-x)^{-a}$

Proof

$${}_1F_0(a;\ ;x) = \sum_{r=0}^{\infty} \frac{(a)_r}{r!}(x)^r$$

$$= (1-x)^{-a}$$

3) $\quad x \cdot {}_2F_1(1/2; 1/2; 3/2; x)$

$$= x + \frac{x^2}{2\cdot 3} + \frac{3x^3}{8\cdot 5}$$

4) $\quad x \cdot {}_2F_1(1,1;2;-x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$

$= \log(1+x)$.

(3) L.H.S.

Sol:

$$x \cdot {}_2F_1(1/2; 1/2; 3/2; x).$$

$$= x \cdot \left[\sum_{r=0}^{\infty} \frac{(1/2)_r (1/2)_r}{(3/2)_r} \frac{x^r}{r!}\right]$$

$$= x\left[\frac{(1/2)_0 (1/2)_0}{(3/2)_0\, 0!} x^0 + \frac{(1/2)_1 (1/2)_1}{(3/2)_1\, 1!} x + \frac{(1/2)_2 (1/2)_2}{(3/2)_2\, 2!} x^2 + \cdots\right]$$

$$\cancel{= x\left[1 + \frac{1}{6}x + \frac{3/4 \cdot 3/4}{9/10 \cdot 2} x^2\right.}$$

$$= x\left[1 + \frac{1}{6}x + \frac{3}{3\cdot 5}x^2 + \cdots\right]$$

$$= x + \frac{x^2}{2\cdot 3} + \frac{3}{3\cdot 5}x^3 + \cdots$$

$(1)_2 = (1+0)(1+1) = 2$
$(2)_2 = (2)(2+1) = 6$

(151)

Q11 $x\,{}_2F_1(1,1;2;-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

L.H.S

$x\,{}_2F_1(1,1;2;-x)$

$$= x\sum_{r=0}^{\infty} \frac{(1)_r (1)_r}{(2)_r\, r!} (-x)^r$$

$$= x\left[1 + \frac{(1)_1(1)_1}{(2)_1\, 1!}(-x)^1 + \frac{(1)_2(1)_2}{(2)_2\, 2!}(-x)^2 + \cdots\right]$$

$$= x\left[1 + \frac{1}{2}(-x)^1 + \frac{\cancel{2}\cdot\cancel{2}}{\cancel{6}\cdot 2!}(-x)^2 + \cdots\right]$$

$$= x\left[1 - \frac{1}{2}x + \frac{\cancel{4}\,1}{\cancel{12}\,3}x^2 + \cdots\right]$$

$$= x\left[1 - \frac{1}{2!}x + \frac{x^2}{3} - \cdots\right]$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$= \log(1+x) \quad \text{Ans}$$

$(1/2)_2 = \frac{1}{2}(1/2+1) = \frac{1}{2}\cdot\frac{3}{2} = 3/4$

$(3/2)_2 = 3/2\,(3/2+1) = 3/2 \cdot 5/2 = 15/4$

(152).

(5) $x\,{}_2F_1(1/2, 1; 3/2; x^2)$.

$$= x \sum_{r=0}^{\infty} \frac{(1/2)_r (1)_r}{(3/2)_r\, r!} (x^2)^r.$$

$$= x\left[1 + \frac{(1/2)_1 (1)_1}{(3/2)_1\, 1!}(x^2)^1 + \frac{(1/2)_2 (1)_2}{(3/2)_2\, 2!}(x^2)^2 + \cdots\right]$$

$(1+0)(1+1)$

$$= x\left[1 + \frac{1/2}{3/2} x^2 + \frac{(3/4)(2)}{15/4 \cdot 2!} x^4 + \cdots\right]$$

$$= x\left(1 + \frac{1}{3}x^2 + \frac{1}{5} x^4 + \cdots\right)$$

$$= x + \frac{1}{3}x^3 + \frac{1}{5} x^5 + \cdots$$

5) $x\,{}_2F_1(1/2, 1; 3/2; x^2)$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$= \tanh^{-1} x.$$

6) $x\,{}_2F_1(1/2, 1; 3/2; -x^2) = \tan^{-1} x.$

7) $x\,{}_2F_1(1/2, 1/2; 3/2; x^2) = \sin^{-1} x$

8) $x\,{}_0F_1(\;; 3/2; +x^2/4) = \sin x.$

9) $x\,{}_0F_1(\;; 3/2; -x^2/4) = \cos x.$

Objective Problem [redacted]

$$J_0'(x) = -J_1(x).$$

Sol: we know that

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x).$$

Put $n = 0$

$$J_0'(x) = -J_1(x) + \frac{0}{x} J_n(x)$$

$$\boxed{J_0'(x) = -J_1(x)}$$

Show that

$$\frac{d}{dx}\left(x J_1(x)\right) = x J_0(x).$$

Sol: we know that

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n}\, k!\, \Gamma(k+n+1)}.$$

Put $n=1$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+1}}{2^{2k+1}\cdot k!\,\Gamma(k+2)}.$$

Multiplying by x.

$$x J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+2}}{2^{2k+1}\cdot k!\,\Gamma(k+2)}.$$

Differentiating on both sides

$$\frac{d}{dx} x J_1(x) = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k+2}}{2^{2k+1}\cdot k!\,\Gamma(k+2)}.$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2(k+1)\, x^{2k+1}}{2^{2k+1}\cdot k!\,\Gamma(k+2)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k\, 2(k+1)\, x^{2k+1}}{2^{k+1}\cdot k!\,\cancel{(k+1)}\,\Gamma(k+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{2^{2n} \cdot n! \, \Gamma(n+1)}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{2^{2n} \cdot n! \, \Gamma(n+1)}$$

$$= x \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+0}}{2^{2n+0} \, n! \, \Gamma(n+1+0)}$$

$$= x \, J_0(x)$$

Hence

$$\frac{d}{dx} x J_1(x) = x J_0(x)$$

Q: Find $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Sol: we know that

$$2n \, J_n(x) = x\left(J_{n-1}(x) + J_{n+1}(x)\right) \quad \text{(A)}$$

put $n = 2$

$$2(2) J_2(x) = x\left(J_1(x) + J_3(x)\right)$$

$$x J_3(x) = 4 J_2(x) - x J_1(x) \quad \text{(1)}$$

Again putting $n=1$ in (A)

$$2(1)\,J_1(x) = x\left(J_0(x) + J_2(x)\right)$$

$$x J_2(x) = 2J_1(x) - x J_0(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - \frac{x}{x} J_0(x)$$

$$J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$$

using this value in equation (1)

$$x J_3(x) = 4\left(\frac{2}{x} J_1(x) - J_0(x)\right) - x J_1(x)$$

$$J_3(x) = \frac{4}{x}\left(\frac{2}{x} J_1(x) - J_0(x)\right) - J_1(x).$$

Q11 Show that

$$J_n''(x) = \frac{1}{2}\left[J_{n-1}(x) - 2J_n(x) + J_{n+2}(x)\right]$$

we know that

$$J_n'(x) = n x^{-1} J_n(x) - J_{n+1}(x)$$

Differentiating we have.

Q11

Show $J_n''(x) = \frac{1}{2}\left[J_{n-1}(x) - 2J_n(x) + J_{n+2}(x)\right]$

As we know that

$$2J_n'(x) = J_{n-1}(x) + J_{n+1}(x).$$

Differentiating we have

$$2J_n''(x) = J_{n-1}'(x) + J_{n+1}'(x).$$

$$2J_n''(x) = \frac{1}{2}\left[J_{n-2}(x) + J_n(x)\right] + \frac{1}{2}\left[J_n(x) - J_{n+2}\right]$$

$$= \frac{1}{2}\left[J_{n-2} + 2J_n(x) - J_{n+2}\right]$$

$$J_n''(x) = \frac{1}{4}\left[J_{n-2} + 2J_n(x) - J_{n+2}\right] \longrightarrow (A)$$

$$y_1(x) = c_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{3!}x^2 - \frac{(n-2)(n-4)(n+1)(n+3)(n+5)}{4!}x^4 + \cdots\right]$$

$$y_2(x) = c_1\left(x - n - \right.$$

* Legender Polynomial:

The equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)y = 0$$

or

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$ is called

The Legender equation.

By the method of series Solution (Forbenion solution).

we get two independ solution of above equation of the form

$$y_1(x) = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 + \frac{(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + \cdots\right]$$

&

$$y_2(x) = C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots\right]$$

If n is even Then $y_1(x)$ (1st eqn) get terminate and we get $y(n)$ as a polynomial of degree n. while 2nd series is an infinit series.

Now If n is odd Then 2nd eqn get terminate and we get $y(n)$ a polynomial of degree n and 1st series an infinite series. These polynomial are known as Legender polynomial.

The cofficents of such a legender polynomial are found from the recurrence relation

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)}C_j$$

Initial Condition for Legender polynomial

$$P_n(1) = 1. \quad \forall \; n \in N.$$

If $n = 2$

$$P_2(x) = C_0\left[1 - \frac{2(2+1)}{2!}x^2\right]$$

$$= C_0\left[1 - \frac{2(3)}{2}x^2\right]$$

$$P_2(x) = C_0\left[1 - 3x^2\right]$$

As $P_2(1) = 1$ Therefore

$$1 = C_0\left[1 - 3(1)^2\right]$$

$$1 = C_0(1-3)$$

$$1 = -2C_0 \implies C_0 = -1/2.$$

So required Legender polynomial is

$$P_2(x) = -\frac{1}{2}\left[1 - 3x^2\right]$$

$$P_2(x) = -\frac{1}{2}\left[1 - 3x^2\right]$$

---x---x---

Ans. for $n = 3$

$$P_3(x) = C_1\left[x - \frac{(3-1)}{3!}(3+2)x^3\right]$$

$$P_3(x) = C_1\left[x - \frac{2(5)}{3!}x^3\right]$$

$$P_3(x) = C_1\left[x - \frac{5}{3}x^3\right]$$

$$P_3(1) = C_1\left[1 - \frac{5}{3}\right]$$

as

$$P_3(1) = 1$$

Therefore

$$1 = C_1(-2/3) \Rightarrow C_1 = -3/2$$

Hence

$$P_3(x) = -\frac{3}{2}\left[x - \frac{5}{3}x^3\right]$$

Find the legender polynomial for $n = 4$

$$P_4(x) = C_0\left[1 - \frac{4(4+1)}{2!}x^2 + \frac{4(4-2)(4+1)(4+3)}{4!}x^4\right]$$

$$P_4(x) = C_0\left[1 - 10x^2 + \frac{35}{3}x^4\right]$$

$$P_4(1) = C_0\left[1 - 10 + \frac{35}{3}\right]$$

$$P_4(1) = C_0\left[8/3\right]$$

$$1 = C_0(8/3) \Rightarrow C_0 = 3/8$$

$$P_4(x) = 3/8\left[1 - 10x^2 + \frac{35}{3}x^4\right]$$

(162)

For $P_4(x)$

As
$$P_4(x) = C_0 + C_2 x^2 + C_4 x^4.$$
As we know
That
$$C_n = \frac{(2n)!}{2^n (n!)^2}$$
put $n = 4$

$$C_4 = \frac{8!}{2^4 (4!)^2}$$

$$C_4 = \frac{\cancel{8} \cdot 7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot 16 \cdot 4 \cdot \cancel{3} \cdot 2 \cdot 1 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$C_4 = \frac{35}{8}$$

Now

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j$$

put $j = 2$

$$C_4 = \frac{(2-4)(2+4+1)}{(2+1)(2+2)} C_2$$

$$\frac{35}{8} = \frac{(-\cancel{2})(7)}{(3)(\cancel{4})} C_2$$

$$\frac{35}{8} = -\frac{7}{6} C_2 \implies C_2 = -\frac{\cancel{6}}{\cancel{7}} \times \frac{\cancel{35}}{\cancel{8}}$$

$$= -\frac{15}{4}$$

Put $J = 0$

$$C_2 = \frac{(0-4)(0+4+1)}{(0+1)(0+2)} C_0$$

$$-\frac{15}{4} = -\frac{\overset{10}{\cancel{20}}}{\cancel{2}} C_0$$

$$C_0 = -\frac{\overset{3}{\cancel{15}}}{4} \times -\frac{1}{\underset{2}{\cancel{10}}} = +3/8$$

$$C_0 = 3/8$$

So

$$P_4(x) = \frac{3}{8} - \frac{15}{4}x^2 + \frac{35}{8}x^4$$

---x--- ---x--- ---x--- ---x---x

$P_5(x)$

As

$$P_5(x) = \cancel{C_1 x} \quad C_1 x + C_3 x^3 + C_5 x^5$$

Now

$$C_5 = \frac{(10!)}{2^5 \cdot (5!)^2}$$

$$C_5 = \frac{\cancel{10}\cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{32} \cdot \cancel{5} \cdot 4 \cdot \cancel{3} \cdot 2 \cdot 1 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}$$

$$C_5 = \frac{63}{8}$$

Now we know that

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j$$

for $j = 3$,

$$C_5 = \frac{(3-5)(3+5+1)}{(3+1)(3+2)} C_3 .$$

$$C_5 = \frac{(-2)(9)}{(4)(5)} C_3$$

$$\frac{63}{8} = \frac{(-2)(9)}{(4)(5)} C_3$$

$$C_3 = -\frac{\cancel{63}^{7}}{\cancel{8}_{2}} \frac{(\cancel{4}^{1})(5)}{(2)(\cancel{9})}$$

$$C_3 = -\frac{35}{4} .$$

for $j = 1$

$$C_3 = \frac{(1-5)(1+5+1)}{(1+1)(1+2)} C_1$$

$$-\frac{35}{4} = \frac{(-\cancel{4}^{2})(7)}{(\cancel{2})(3)} C_1$$

$$C_1 = +\frac{\cancel{35}^{5}}{4} \frac{(+3)(\cancel{7})}{(2)(\cancel{7})} = \frac{15}{8}$$

So

$$P_5(x) = \frac{15}{8}x - \frac{35}{4}x^3 + \frac{63}{8}x^5$$

.x. .x. .x.

$P_8(x)$

So The legendre eqn is

$$P_8(x) = C_0 + C_2 x^2 + C_4 x^4 + C_6 x^6 + C_8 x^8$$

$$C_8 = \frac{(16!)}{2^8 \cdot (8!)(8!)}$$

$$C_8 = \frac{\cancel{16}\cdot\cancel{15}\cdot\cancel{14}\cdot 13\cdot\cancel{12}\cdot 11\cdot\cancel{10}\cdot 9\cdot \cancel{8!}}{8\;\cancel{256}\cdot\cancel{8}\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot 4\cdot\cancel{3}\cdot\cancel{2}\cdot 1\cdot\cancel{8}\cdot\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}$$

$$C_8 = \frac{715}{32}.$$

now we know That

$$C_{j+2} = \frac{(j-n)(j+n+1)}{(j+1)(j+2)} C_j$$

Put $j = 6$

$$C_8 = \frac{(-2)(15)}{(7)(8)} C_6$$

$$C_6 = -\frac{\cancel{(715)}^{143}}{\cancel{(32)}_{4}} \cdot \frac{(7)\cancel{(8)}^{1}}{(2)\cancel{(18)}_{3}} = -\frac{143 \times 7}{4 \times 2 \times 3}.$$

$$C_6 = \cancel{\frac{2275}{24}}$$

$$C_6 = -\frac{1001}{24}.$$

Put $r = 4$.

$$C_6 = \frac{(4-8)(4+8+1)}{(4+1)(4+2)} C_4.$$

$$-\frac{1001}{24} = \frac{(-4)(13)}{(5)(6)} C_4$$

$$C_4 = \frac{1001}{\cancel{24}_{4}} \cdot \frac{(5)\cancel{(6)}^{1}}{(4)(13)}$$

$$C_4 = \frac{5005}{208}$$

Put $j = 2$

$$C_4 = \frac{(2-8)(2+8+1)}{(2+1)(2+2)} C_2$$

$$\frac{5005}{208} = \frac{(-6)(11)}{(3)(4)} C_2$$

$$C_2 = -\frac{5005}{\cancel{208}_{104}} \frac{(\cancel{3}^{1})(\cancel{4}^{\cancel{2}\,1})}{(\cancel{6}_{\cancel{2}})(11)}$$

$$C_2 = -\frac{5005}{1144}$$

Now put $j = 0$

$$C_2 = \frac{(-8)(9)}{(1)(2)} C_0$$

$$C_0 = -\frac{5005}{1144} \frac{(-1)(\cancel{2})}{(\cancel{8}_{4})(9)}$$

$$C_0 = +\frac{5005}{41184}$$

Now

$$P_8(x) = \frac{5005}{41184} - \frac{5005}{1144} x^2 + \frac{5005}{208} x^4 - \frac{1001}{24} x^6 + \frac{715}{32} x^8 \;//$$

(168) Generating function :-

Suppose we have a function

$$F(x,t) = \sum_{m=0}^{\infty} f_m(x)\, t^m$$

we called $F(x,t)$ a generating function for a set of functions $\{f_m(x)\}$ provided the above series converges. For e.g.

Consider

$$\frac{1}{1-xt} = (1-xt)^{-1}$$

$$= 1 + xt + x^2t^2 + x^3t^3 + \cdots$$

$$= \sum_{m=0}^{\infty} x^m t^m$$

$\Rightarrow$

$F(x,t)$ generate a function a set of function $\{x^m\}$

x ——— x ——— x

Parel.

Q. Show the function is generating fn. (10)

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} \text{ gives } \sum_{n=0}^{\infty} P_n(x) t^n.$$

i.e. This function generates Legendre polynomial $P_n(x)$.

Soln:

$$\frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-1/2}$$

$$= (1-(2xt-t^2))^{-1/2}$$

$$= (1-d)^{-1/2} \text{ where } d = 2xt-t^2.$$

$$= 1 + (-1/2)(-d) + \frac{(-1/2)(-1/2-1)(-d)^2}{2!} + \cdots$$

$$= 1 + d/2 + \frac{3/4}{2!} d^2 + \cdots$$

$$= 1 + d/2 + \frac{(3/4) d^2}{2} + \cdots$$

$$= 1 + \frac{(2xt-t^2)}{2} + \frac{3}{8}(2xt-t^2)^2 + \cdots$$

$$= 1 + xt - \frac{1}{2}t^2 + \frac{3}{8}(4x^2t^2 + t^4 - 4xt^3) + \cdots$$

$$= x^0t^0 + xt + \left(-\frac{1}{2} + \frac{3}{2}x^2\right)t^2 + \cdots$$

$$= P_0(x)t^0 + P_1(x)t + \frac{1}{2}(3x^2-1)t^2 + \cdots$$

($P_1(x) = x$) , ($P_n(1) = 1$ ...)

$$= P_0(x)t^0 + P_1(x)t + P_2(x)t^2 + \cdots$$

$$F(x,t) = \sum_{n=0}^{\infty} P_n(x)t^n.$$

Hence given function is generating function.

* Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n.$$

This formula is used to find the Legendre polynomial of different degree.

* Show that $P_n(x)$ is $\begin{cases} \text{even if } n \text{ is even} \\ \text{odd if } n \text{ is odd} \end{cases}$

Proof: If n is even.

Then

$$P_n(x) = C_0 + C_2x^2 + C_4x^4 + \cdots + C_nx^n$$

Put $x = -x$

$$P_n(-x) = C_0 + C_2(-x)^2 + C_4(-x)^4 + \cdots + C_n(-x)^n$$

$$= C_0 + C_2x^2 + C_4x^4 + \cdots + C_nx^n$$

$$P_n(-x) = P_n(x)$$

$\Rightarrow$ $P_n(x)$ is even.

If n is odd Then

$$P_n(x) = C_1 + C_3x^3 + C_5x^5 + \cdots + C_nx^n$$

Put $x = -x$.

$$P_n(-x) = C_1 + C_3(-x)^3 + C_5(-x)^5 + \cdots + C_n(-x)^n$$

$$= C_1 - C_3x^3 - C_5x^5 + \cdots - C_n(x)^n$$

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(-x) = -P_n(x) \quad \because (-1)^n = -1$$

as n is odd

Q. Show that

$$F(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Then $P_n(x)$ is even if n is even and $P_n(x)$ is odd if n is odd.

Now as

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Change 'x' by $-x$, gives

$$\frac{1}{\sqrt{1+2xt+t^2}} = \sum_{n=0}^{\infty} P_n(-x)t^n.$$

Now change 't' by '$-t$'

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(-x)(-t)^n$$

$$\sum_{n=0}^{\infty} P_n(x)t^n = \sum_{n=0}^{\infty} P_n(-x)(-1)^n t^n.$$

$$\boxed{\begin{aligned} &P_0(x)t^0 + P_1(x)t^1 + P_2(x)t^2 + \cdots \\ &= P_0(-x)(-1)^0 t^0 + P_1(-x)(-1)^1 t^1 \\ &\quad + P_2(-x)(-1)^2 t^2 + \cdots \end{aligned}}$$

On comparing the coefficients of "t^n" we have

$$P_n(x) = P_n(-x)(-1)^n$$

Now if n is even

Then $P_n(x) = P_n(-x)$. $\because (-1)^n = 1$ as n is even.

$\Rightarrow P_n(x)$ is even.

Now if n is odd

$$P_n(x) = P_n(-x)(-1)^n$$

$\Rightarrow P_n(x) = -P_n(-x)$ $\because (-1)^n = -1$ as n is odd.

$\Rightarrow P_n(x)$ is odd.

$\dot{x}$ — $\dot{x}$ — $\dot{x}$

Q: Show that

$$(2n+1)P_n(x) = \underline{P'_{n+1}(x)} - \underline{P'_{n-1}(x)}$$

Sol: Since $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

Put $n = n+1$

$$P_{n+1} = \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}$$

$$P'_{n+1} = \frac{1}{2^{n+1}(n+1)!}\frac{d^{n+2}}{dx^{n+2}}(x^2-1)^{n+1}$$

$$= \frac{1}{2^{n+1}\cdot(n+1)!}\frac{d^n}{dx^n}\left[\frac{d^2}{dx^2}(x^2-1)^{n+1}\right]$$

$$= \frac{1}{2^{n+1}\cdot(n+1)!}\cdot\frac{d^n}{dx^n}\left[\frac{d}{dx}(n+1)(x^2-1)^n 2x\right]$$

$$= \frac{2(n+1)}{2^{n+1}\cdot(n+1)!}\cdot\frac{d^n}{dx^n}\left[n(x^2-1)^{n-1}(x)(2x) + (x^2-1)^n(1)\right]$$

$$= \frac{2(n+1)}{2^{n+1}\cdot(n+1)!}\cdot\frac{d^n}{dx^n}\left[2x^2 n(x^2-1)^{n-1} + (x^2-1)^n\right]$$

$$= \frac{2}{2^{n+1}\cdot n!}\frac{d^n}{dx^n}\left[2x^2 n(x^2-1)^{n-1} + (x^2-1)^n\right]$$

$$= \frac{1}{2^n . n!}\frac{d^n}{dx^n}\left[2x^2 n(x^2-1)^{n-1}\right] + \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P'_{n+1}(x) = \frac{2n}{2^n n!}\frac{d^n}{dx^n}\left(x^2(x^2-1)^{n-1}\right] + P_n(x) \quad (1)$$

Now By Rodring formula

$$P_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^{n-1}$$

By taking derivative

$$P'_{n-1}(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n} (x^2-1)^{n-1} \quad (2')$$

Now subtracting (2) from (1) (2') we get.

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{2n}{2^n . n!} \frac{d^n}{dx^n}\left[x^2(x^2-1)^{n-1}\right] + P_n(x) - \frac{1}{2^n(n-1)!} \frac{d^n}{dx^n}(x^2-1)^{n-1}$$

$$= \frac{2n}{2^n n!} \frac{d^n}{dx^n}\left[x^2(x^2-1)^{n-1}\right] - \frac{1}{2^n(n-1)!} \frac{d^n}{dx^n}(x^2-1)^{n-1} + P_n(x)$$

$$= \frac{1}{2^{n-1}(n-1)!} \frac{d^n}{dx^n}\left[\frac{2n\, x^2(x^2-1)^{n-1}}{2n} - (x^2-1)^{n-1}\right] + P_n(x)$$

$$= \frac{1}{2^{n-1} (n-1)!} \frac{d^n}{dx^n}\left[(x^2-1)^{n-1} . (x^2-1)\right] + P_n(x)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = \frac{1}{2^{n-1}\cdot(n-1)!}\frac{d^n}{dx^n}(x^2-1)^n + P_n(x)$$

$$= \frac{2n}{2^n(n!)}\frac{d^n}{dx^n}(x^2-1)^n + P_n(x)$$

$$= 2n\,P_n(x) + P_n(x)$$

$$= (2n+1)P_n(x) \checkmark$$

i.e

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

Q// $(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$

for $n = 1, 2, 3)$

Sol: we know that

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$$

Differentiating w.r.t to 't'

$$(-1/2)(1-2xt+t^2)^{-3/2}(-2x+2t) = \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$\frac{(x-t)}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$\frac{x-t}{(1-2xt+t^2)^{1/2}(1-2xt+t^2)} = \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$(x-t) \cdot \frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n-1}$$

$$(x-t) \sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2) \sum_{n=0}^{\infty} P_n(x) n t^{n-1}$$

$$x \sum_{n=0}^{\infty} P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} P_n(x) \cdot n \cdot t^{n-1} - 2x \sum_{n=0}^{\infty} P_n(x) \cdot n t^n + \sum_{n=0}^{\infty} P_n(x) \cdot n t^{n+1}$$

P.T.O

$$\Rightarrow x\sum_{n=0}^{\infty} P_n(x)\, t^n + 2nx \sum_{n=0}^{\infty} P_n(x) t^n$$

$$= \sum_{n=0}^{\infty} P_n(x)\cdot n\cdot t^{n-1} + \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n+1} + \sum_{n=0}^{\infty} P_n(x) t^{n+1}$$

$$\Rightarrow \cancel{(1+2n)\, x \sum_{n=0}^{\infty} P_n(x) t^n}$$

$$\Rightarrow \sum_{n=0}^{\infty} (1+2n)\, x\, P_n(x)\, t^n$$

$$= \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n-1} + \sum_{n=0}^{\infty} P_n(x)\cdot n t^{n+1} + \sum_{n=0}^{\infty} P_n(x)\, t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+1)\, x\, P_n(x)\, t^n = \sum_{n=0}^{\infty} P_n(x)\, n t^{n-1} + \sum_{n=0}^{\infty} (n+1) P_n(x)\, t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty} (2n+1)\, x\, P_n(x)\, t^n = \sum_{n=1}^{\infty} P_n(x)\, n\, t^{n-1} + \sum_{n=0}^{\infty} (n+1) P_n(x)\, t^{n+1}$$

$$\Rightarrow \sum_{n=0}^{\infty}(2n+1)x P_n(x) t^n$$

$$= \sum_{n=0}^{\infty} P_{n+1}(x)(n+1)t^n \qquad \text{By taking } n=n+1 \ \& \ n=n-1$$

$$+ \sum_{n=1}^{\infty} n P_{n-1}(x) t^n$$

Now on Comparing Coefficients of t^n for $n=1$ and onward we have

$$\Rightarrow (2n+1)x P_n(x)$$

$$= P_{n+1}(x)(n+1) + n P_{n-1}(x)$$

Hence

$$(2n+1)x P_n(x) = P_{n+1}(x)(n+1) + n P_{n-1}(x).$$

(180)

Theorem:

* The hypergeometric function is solution of D.E

$$z(1-z)\frac{d^2y}{dz^2}+\{\gamma-(\alpha+\beta+1)z\}\frac{dy}{dz}-\alpha\beta y=0 \quad (1)$$

which is called Hypergeometric differential equation α, β, γ are constants.

Proof:

we take a trial solution

$$y=\sum_{k=0}^{\infty} C_k z^{k+\gamma} \quad (2)$$

$$\Rightarrow \frac{dy}{dz}=\sum_{k=0}^{\infty} C_k(k+\gamma)z^{k+\gamma-1}$$

$$\Rightarrow \frac{d^2y}{dz^2}=\sum_{k=0}^{\infty} C_k(k+\gamma)(k+\gamma-1)z^{k+\gamma-2}$$

using in eqn (1)

$$z(1-z)\left(\sum_{k=0}^{\infty} C_k(k+\gamma)(k+\gamma-1)z^{k+\gamma-2}\right)$$

$$+\{\gamma-(\alpha+\beta+1)z\}\sum_{k=0}^{\infty} C_k(k+\gamma)z^{k+\gamma-1}$$

$$-\alpha\beta\sum_{k=0}^{\infty} C_k z^{k+\gamma}=0$$

$\rightarrow$

$$\Rightarrow \sum_{n=0}^{\infty} C_n(n+r)(n+r-1) z^{n+r-1}$$

$$- \sum_{n=0}^{\infty} C_n(n+r)(n+r-1) z^{n+r}$$

$$+ \gamma \sum_{n=0}^{\infty} C_n (n+r) z^{n+r-1}$$

$$- (\alpha+\beta+1) \sum_{n=0}^{\infty} C_n (n+r) z^{n+r-1}$$

$$- \alpha\beta \sum_{n=0}^{\infty} C_n z^{n+r} = 0$$

On Comparing Least power of 'z' i.e z^{r-1} by putting $n=0$. we get

$$C_0 r(r-1) + \gamma C_0 (r) = 0$$

$$C_0 r(r-1+\gamma) = 0$$

$$\Rightarrow C_0 \neq 0 \quad \text{so} \quad r(r-1+\gamma) = 0$$

$$\Rightarrow r = 0 \quad \text{or} \quad r = 1-\gamma$$

Now Comparing Coefficient of z^{n+r-1}.

$$C_n(n+r)(n+r-1) - C_{n-1}(n+r-1)(n+r-2)$$
$$+\gamma C_n(n+r) - (\alpha+\beta+1) C_{n-1}(n+r-1)$$
$$-\alpha\beta\, C_{n-1} = 0 \quad (3)$$

Case-I

when $\gamma = 0$

so

$$C_n\left[n(n-1) + \gamma n\right] = C_{n-1}\left[(n-1)(n-2) + (\alpha+\beta+1)(n-1) + \alpha\beta\right]$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n-2 + \alpha+\beta+1) + \alpha\beta\right]}{n(n-1) + \gamma n.}$$

$$C_n = \frac{C_{n-1}\left[(n+\alpha-1+\beta)^{(n-1)} + \alpha\beta\right]}{n\left[n-1+\gamma\right]}$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n+\alpha-1) + \beta(n-1) + \alpha\beta\right]}{n\left[n-1+\gamma\right]}$$

$$C_n = \frac{C_{n-1}\left[(n-1)(n+\alpha-1) + \beta(n+\alpha-1)\right]}{n(n-1+\gamma)}$$

$$C_n = C_{n-1}\left[\frac{(n+\alpha-1)(n+\beta-1)}{n(n+\gamma-1)}\right]$$

NOTE 1st the Geometric series is always 1. ∴ $C_0 = 1$

for $k = k-1$

$$C_{k-1} = C_{k-2}\left[\frac{(k-2+\alpha)(k-2+\beta)}{(k-1)(k-2+\gamma)}\right]$$

$$\vdots$$

$$C_2 = C_1\left[\frac{(1+\alpha)(1+\beta)}{(2)(1+\gamma)}\right]$$

and

$$C_1 = C_0\left[\frac{\alpha\cdot\beta}{1\cdot\gamma}\right]$$

$$C_1 = C_0\left(\frac{\alpha\beta}{\gamma}\right)$$

$$C_1 = \frac{\alpha\beta}{\gamma}$$

$$C_2 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{2\gamma(\gamma+1)}$$

$$C_3 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{3!\,\gamma(\gamma+1)(\gamma+3)}.$$

~~then~~ Choose $C_0 = 1$

using back subsituation we get

$$C_k = \frac{(k-1+\alpha)(k-2+\alpha)\cdots(k-(k-1)+\alpha)\alpha\,.\,(k-1+\beta)(k-2+\beta)\cdots(k-(k-1)+\beta)\beta}{k\cdot(k-1)(k-2)\cdots 3\cdot 2\cdot 1\cdot(k-1+\gamma)(k-2+\gamma)\cdots(k-(k-1)+\gamma)\gamma}$$

$$C_k = \frac{(\alpha)_k(\beta)_k}{k!\,(\gamma)_k}.$$ ✓

using in solution we have

$$y = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{k!\,(\gamma)_k} z^k \quad ; \gamma = 0$$

<u>Case II</u>

when $\gamma = 1 - \Gamma$

using in equation ③ we get

$$C_k\left[(k+1-\Gamma)(k-\Gamma) + \Gamma(k+1-\Gamma)\right]$$

$$= C_{k-1}\left[(k-\Gamma)(k-1-\Gamma) + (\alpha+\beta+1)(k-\Gamma) + \alpha\beta\right] = 0$$

$$C_k = \frac{C_{k-1}\left[(k-\Gamma)(k-1-\Gamma) + (\alpha+1)(k-\Gamma) + \beta(k-\Gamma) + \alpha\beta\right]}{(k+1-\Gamma)(k-\Gamma) + \Gamma(k+1-\Gamma)}$$

$$C_k = \frac{C_{k-1}\left[(k-\Gamma)\left[k-\cancel{1}-\Gamma+k-\Gamma+\alpha+\cancel{1}\right] + \beta(k-\Gamma+\alpha)\right]}{(k+1-\Gamma)(k-\cancel{\Gamma}+\cancel{\Gamma})}$$

$$C_k = C_{k-1}\left[\frac{(k-r+\alpha)(k-r+\beta)}{(k+1-r)(k)}\right]$$

put

$k = k-1$

$$C_{k-1} = C_{k-2}\left[\frac{(k-1-r+\alpha)(k-1-r+\beta)}{(k-1+1-r)(k-1)}\right]$$

$$\vdots$$

$$C_2 = C_1\left[\frac{(2-r+\alpha)(2-r+\beta)}{2(2-r+1)}\right]$$

$$C_1 = C_0\left[\frac{(1-r+\alpha)(1-r+\beta)}{1\cdot(1-r+1)=(2-r)}\right]$$

Now choose $C_0 = 1$

$$C_k = \frac{(k-r+\alpha)(k-1-r+\alpha)(k-2-r+\alpha)\cdots(1-r+\alpha)\,(k-r+\beta)(k-1-r+\beta)(k-2-r+\beta)\cdots(1-r+\beta)}{k(k-1)(k-2)\cdots 3\cdot2\cdot1\cdot(k+1-r)(k-r)(k-1-r)\cdots(3-r)(2-r)}$$

$$C_k = \frac{(1-r+\alpha)_k\cdot(1-r+\beta)_k}{k!\;(2-r)_k}$$

using in solution

$$y = \sum_{k=0}^{\infty}\frac{(1-r+\alpha)_k\,(1-r+\beta)_k}{(2-r)_k}\,z^{k+1-r} \quad \because \gamma = 1-r$$

*

Bessel Functions:-

Bessel's Differential Equation.

The differential equation

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \ , \ n \geq 0$$

The solution of Bessel's D.E is called Bessel function.

* Find the solution of Bessell's D.E

$$x^2 y'' + x y' + (x^2 - n^2) y = 0 \ , \ n \geq 0 \quad \text{(1)}$$

Proof:- Let $y = \sum_{k=0}^{\infty} C_k x^{k+r}$ be a series solution of bessell's D.E, we may write

$$y = \sum_{k=-\infty}^{\infty} C_k x^{k+r} \quad \text{where } C_k = 0 \text{ for } k<0$$

$$\Rightarrow \quad y' = \sum_{k=0}^{\infty} C_k (k+r) x^{k+r-1}$$

$$y'' = \sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r-2}$$

using in eqn (1) we get

$$x^2 \sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r-2}$$
$$+ x \sum_{k=0}^{\infty} C_k (k+r) x^{k+r-1}$$
$$+ (x^2 - n^2) \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

$$x^2 \sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r-2}$$
$$+ x \sum_{k=0}^{\infty} C_k (k+r) x^{k+r-1}$$
$$+ x^2 \sum_{k=0}^{\infty} C_k x^{k+r} - n^2 \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

$$\sum_{k=0}^{\infty} C_k (k+r)(k+r-1) x^{k+r} + \sum_{k=0}^{\infty} C_k (k+r) x^{k+r}$$
$k<0$ $$+ \sum_{k=0}^{\infty} C_k x^{k+r+2} - n^2 \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

Comparing coefficent of x^{k+r} we get

$$C_k (k+r)(k+r-1) + C_k (k+r)$$
$$+ C_{k-2} - n^2 C_k = 0$$

$$C_k (k+r)(k+r \cancel{-1} \cancel{+1}) + C_{k-2} - n^2 C_k = 0$$

$$C_k (k+r)(k+r) + C_{k-2} - n^2 C_k = 0$$

$$\Rightarrow \quad C_k\left[(k+r)^2-n^2\right] = -C_{k-2} \quad (2)$$

This is known as recurrence relation or identical relation//

Put $k=0$ in eqn (2)

$$C_0\left[(0+r)^2-n^2\right] = -C_{-2}$$

$$C_0\left[r^2-n^2\right] = 0 \quad \because C_{-2}=0 \ \forall k<0$$

Either $C_0=0$ or $r^2-n^2=0$

$k=2$ in eqn (2)

$$C_2\left[(2+r)^2-n^2\right] = -C_0$$

$k=4$

$$C_4\left[(4+r)^2-n^2\right] = -C_2$$

So all C_k for even value of k depends upon C_0. To get a non-trivial solution $C_0 \neq 0$

So $r^2-n^2=0$

$$r^2=n^2$$

$$r=\pm n$$

from eq (2) we get

$$C_k = \frac{(-1)\, C_{k-2}}{(k+r)^2 - n^2} \quad (3)$$

$$C_k = \frac{(-1)\, C_{k-2}}{(k+r+n)(k+r-n)}$$

$$C_k = \frac{(-1)\, C_{k-2}}{(k+r+n)(k+r-n)}$$

Now for $k = 1$

$$C_1 = \frac{(-1)\, C_{-1}}{(1+r+n)(1+r-n)}$$

$C_1 = 0 \quad \because C_{-1} = 0$

$$C_3 = \frac{(-1)\, C_1}{(3+r+n)(3+r-n)}$$

$C_3 = 0 \quad \because C_1 = 0$

so

$C_1 = C_3 = C_5 = C_7 = \cdots = 0$

* Case I

when $r = n$ put in eqn (3)

$$C_k = \frac{(-1)\, C_{k-2}}{k(2n+k)}$$

(190)

$$C_{k-2} = \frac{(-1)\, C_{k-4}}{(k-2)(2n+k-2)}$$

$$C_4 = \frac{(-1)\, C_2}{(4)(2n+4)}$$

$$C_2 = \frac{(-1)\, C_0}{(2)(2n+2)}$$

let

$$C_0 = \frac{1}{2^n \Gamma(n+1)}$$

$$C_2 = \frac{(-1)}{(2)(2n+2)} \cdot \frac{1}{2^n \Gamma(n+1)}$$

$$= \frac{(-1)}{2^{n+1} \cdot 2!\, (n+1)\Gamma(n+1)} = \frac{(-1)}{2^{n+2} \cdot 1!\, \Gamma(n+2)}$$

$$C_2 = \frac{(-1)}{2^{n+1} \cdot 2!\, \Gamma(n+2)} \qquad \because \Gamma(z+1) = z \cdot \Gamma(z).$$

Now

$$C_4 = \frac{(-1)\, C_2}{(4)(2n+4)}$$

$$C_4 = \frac{(-1)^2}{4(2n+4)\cdot 2^{n+1}\cdot 2!\,\Gamma(n+2)}$$

$$C_4 = \frac{(-1)^2}{2^2(2n+2^2)\cdot 2^{n+1}\cdot 2!\,\Gamma(n+2)}$$

$$C_4 = \frac{(-1)^2}{2^{n+4}(n+2)\cdot 2!\,\Gamma(n+2)}$$

$$C_4 = \frac{(-1)^2}{2^{n+4}(2!)\,\Gamma(n+2+1)}$$

$$\vdots$$

$$C_{k=2m} = \frac{(-1)^{2m/2=m}}{2^{n+2m}(m!)\cdot\Gamma(n+m+1)}$$

using these values of C_k and r in the solution

$$y = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n}\cdot m!\,\Gamma(m+n+1)} = J_n(x)$$

Case-II

when $\gamma = -n$ put in eqn (3)

$$C_k = \frac{(-1)\, C_{k-2}}{(k-n)^2 - n^2}$$

$$C_k = \frac{(-1)\, C_{k-2}}{k^2 \cancel{+n^2} - 2kn \cancel{+n^2}} = \frac{(-1)\, C_{k-2}}{k^2 - 2kn}$$

$$C_k = \frac{(-1)\, C_{k-2}}{k(k-2n)}$$

$$C_{k-2} = \frac{(-1)\, C_{k-4}}{(k-2)(k-2-2n)}$$

$$C_4 = \frac{(-1)\, C_2}{(4)(4-2n)}$$

$$C_2 = \frac{(-1)\, C_0}{\cancel{(2)}\,(2-2n)}$$

Let $C_0 = \dfrac{1}{2^{-n}\,\Gamma(-n+1)}$ $\quad\because n = -n.$

So

$$C_2 = \frac{(-1)}{2^2(1-n)} \cdot \frac{1}{2^{-n}\,\Gamma(-n+1)}.$$

$$C_2 = \frac{(-1)}{2^{2-n}(1-n)\,\Gamma(-n+1)}$$

$$C_2 = \frac{(-1)}{2^{2-n}\cdot\Gamma(-n)} \qquad \because (1-n)\,\Gamma(-n+1) = \Gamma(-n).$$

$$C_4 = \frac{(-1)^2\; C_2}{4(4-4n)}.$$

$$C_4 = \frac{(-1)^2}{4(4-2n)\cdot 2^{2-n}\,\Gamma(-n+0)}.$$

$$C_4 = \frac{(-1)^2}{2^3(2-n)\cdot 2^{2-n}\cdot\Gamma(-n+0)}$$

$$C_4 = \frac{(-1)^2}{2^{4-n}\cdot 2!\;(2-n)\,\Gamma(-n+0)}$$

$$C_4 = \frac{(-1)^2}{2^{4-n}\cdot 2!\;(-n+2)\,\Gamma(-n+0)}.$$

* Find the solution of the Legender equation

$$(1-x^2)\,y'' - 2xy' + n(n+1)y = 0 \quad (1)$$

Proof: We assume a solution of eqn (1) as

$$y = \sum_{k=0}^{\infty} C_k x^{k+\beta} \quad (2) \quad \text{and } C_k = 0 \text{ for } k < 0$$

$\Rightarrow$

$$y' = \sum_{k=0}^{\infty} C_k (k+\beta)\, x^{k+\beta-1}$$

$$y'' = \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta-2}.$$

using values in equation (1) we get

$$(1-x^2)\sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta-2}$$
$$-2x\sum_{k=0}^{\infty} C_k (k+\beta)\, x^{k+\beta-1}$$
$$+ n(n+1)\sum_{k=0}^{\infty} C_k x^{k+\beta}\Big) = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta-2} - \sum_{k=0}^{\infty} C_k (k+\beta)(k+\beta-1)\, x^{k+\beta}$$
$$- \sum 2C_k (k+\beta)\, x^{k+\beta} + n(n+1)\sum C_k x^{k+\beta} = 0$$

Comparing coefficient of $x^{k+\beta}$, we get

$$C_{k+2}(k+\beta+2)(k+1+\beta) - C_k(k+\beta)(k+\beta-1)$$
$$-2C_k(k+\beta) + n(n+1)C_k = 0$$

$$C_{k+2}(k+\beta+2)(k+\beta+1) + C_k\left[n(n+1)-(k+\beta)(k+\beta-1) - 2(k+\beta)\right]=0$$

$$C_{k+2}(k+\beta+2)(k+\beta+1) + C_k\left[n(n+1)-(k+\beta)[k+\beta-1+2]\right]=0$$

$$C_{k+2}(k+\beta+2)(k+\beta+1) + C_k\left[n(n+1)-(k+\beta)(k+\beta+1)\right]=0 \quad (3)$$

This is known as identical eqn. or recurrence equation.

Put $k=-2$.

$$C_0\beta(\beta-1) + C_{-2}\left[n(n+1)-(-2+\beta)(-2+\beta+1)\right]=0$$

$$C_0\beta(\beta-1)=0 \quad \because C_{-2}=0$$

Assum $C_0 \neq 0$

Then either $\beta=0$ or $\beta-1=0 \Rightarrow \beta=1$

Case-I put $\beta=0$ in eqn (3) we get

$$C_{k+2}(k+2)(k+1) + C_k\left(n(n+1)-k(k+1)\right]=0$$

$$C_{k+2} = \frac{k(k+1)-n(n+1)}{(k+1)(k+2)}C_k$$

Put $K=0$

$$C_2 = \frac{-n(n+1)}{1\cdot 2} C_0$$

$$C_2 = \frac{-n(n+1)}{2!} C_0$$

Put $K=1$

$$C_3 = \frac{1(2)-n(n+1)}{2\cdot 3} \cdot C_1$$

$$C_3 = \frac{2-n^2-n}{2\cdot 3} C_1$$

$$C_3 = \frac{2-2n+n-n^2}{2\cdot 3} C_1$$

$$C_3 = \frac{2(1-n)+n(1-n)}{2\cdot 3} C_1$$

$$C_3 = \frac{(2+n)(1-n)}{3!} C_1$$

Put $K=2$.

$$C_4 = \frac{2\cdot 3-n(n+1)}{3\cdot 4} C_2$$

$$= \frac{6-n^2-n}{3\cdot 4} C_2$$

$$= \frac{-n^2-n+6}{3\cdot 4} C_2$$

$$= -\frac{(n^2+n-6)}{3\cdot 4} C_2.$$

$$C_4 = -\frac{(3+n)(n-2)}{4\cdot 3} C_2.$$

$$C_4 = -\frac{(3+n)(n-2)}{4\cdot 3} \cdot \left(\frac{-n(n+1)}{2!} C_0\right)$$

$$C_4 = \frac{n(n-2)(n+1)(n+3)}{24 = 4!}$$

Put $k = 3$

$$C_5 = \frac{3\cdot 4 - n(n+1)}{4\cdot 5} C_3 = \frac{12 - n^2 - n}{4\cdot 5} C_3$$

$$C_5 = \frac{-n^2 - n + 12}{4\cdot 5} C_3 = -\frac{(n^2+n-12)}{4\cdot 5} C_3$$

$$C_5 = -\frac{(4+n)(n-3)}{5\cdot 4} C_3.$$

$$C_5 = -\frac{(n+4)(n-3)}{5\cdot 4} \cdot -\frac{(n+2)(n-1)}{3!} C_1$$

$$C_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} C_1,$$

using all values in equation (2) we get

$$y = C_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 \cdots\right]$$

$$+ C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 + \cdots\right]$$

which is the required solution of (1)

This is a solution with two arbitrary constant so not consider $\beta = 1$ //

CH # 01 Lebesgue Measure

Indroducti-

Interval: An interval is the set of real numbers e.g $I = [a,b] = \{x : x \in \mathbb{R} \wedge a \le x \le b\}$
The length of an interval is equal to the difference of the end points of the interval e.g $I = [a,b]$

$$l(I) = b - a.$$

The intervals $[a,b]$ and (a,b) have same length.

* **SET FUNCTION:**

A function f is said to be a set function if its domain is a collection of sets for e.g if M is a collection of sets then the function

$f : M \to \{-1, 1\}$ defined as $f(A) = \begin{cases} -1 & \text{if } \operatorname{ord} A = \infty \\ 1 & \text{if } \operatorname{ord} A < \infty \end{cases}$

* **Non-negative Function:**

A set function is said to be non-negative function if its range is non-negative extended real number.
e.g if we define a function

$$f : M \longrightarrow [0, \infty] \text{ as}$$

$$f(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$$

Additivity of a Set Function:

A set function

$f : M \longrightarrow [0, \infty]$ is said to be additive if for $A, B \in M$

$$f(A \cup B) = f(A) + f(B).$$

* Finitely Additive Set function:

The set function

$f : M \longrightarrow [0, \infty]$ is said to be finitely additive set function if

$$f\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} f(A_i).$$

* Countable Additive Set function:

If 'M' is a Collection of sets and $f : M \longrightarrow [0, \infty]$ is said to countable additive or sigma additive if for $A_1, A_2, A_3, \ldots, A_n, \ldots \in M$

NOTE Every additive ftn is sub additive but converse not true.

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also $A_1 \cup A_2 \cup A_3 \cdots A_n \cdots \in M$

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} f(A_i)$$

* Sub additivity of set function.

A set function

$$f : M \to [0, \infty]$$

is said to be sub additive set function if $A, B \in M$ also $A \cup B \in M$ such that

$$f(A \cup B) \leq f(A) + f(B).$$

* Finitly sub additive of set function:

A set function

$$f : M \to [0, \infty]$$

is said to be Finitly sub additive set function if $f\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} f(A_i)$

* Countable sub additive of set function:

A set function

$$f : M \to [0, \infty]$$

is said to be Countably sub additive set function if for $A_1, A_2, A_3, \ldots, A_n, \ldots \in M$ also $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n \cup \cdots \in M$

$$f\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} f(A_i)$$

-x-

* **Pre- Measure:-** A set function $f: M \to [0, \infty]$ is said to be pre-measure if $f(\varphi) = 0$

e.g.

Let M be the collection of all open interval and $f: M \to [0, \infty]$ defined as

$$f(I) = l(I).$$

let $]a, a[\in M$

clearly $]a, a[= \varphi$

$$\Rightarrow f(\varphi) = f(]a, a[)$$
$$= l(]a, a[)$$
$$= 0$$

Hence f is pre measure.

Outer Measure:

A set function

$f: 2^X \longrightarrow [0, \infty]$ is said to be outer measure if

i) $f(\phi) = 0$

ii) If $A_1, A_2, \ldots, A_n, \ldots \in 2^X$

& $A \subseteq \bigcup A_i$ or $\mu^*(E) \leq \mu^*(E_2)$ if $E_1 \subseteq E_2$ (monotonicity)

$$f(A) \leq \sum_{i}^{\infty} f(A_i)$$

* **Monoton set function:** A set function $f: M \longrightarrow [0, \infty]$ is said to be monoton if for

$$A \subseteq B$$
$$f(A) \leq f(B).$$

Remark:

i) Every outer measure is pre-measure but converse may not true.

ii) An outer measure is a monoton sub additivly.

— x — x — x — x —

Ex: A set function

$f : 2^X \longrightarrow [0, \infty]$ defined as

$f(A) = \text{Ord}(A)$.

Sol:- Now first if

i) $A = \phi$.

$\text{ord}(A) = \text{ord}(\phi)$

$= 0$

i.e

$f(A) = 0$ for $A = \phi$.

ii) Let $A_1, A_2, A_3, \ldots, A_i, \ldots \in 2^X$.

&

$A \subseteq \bigcup A_i$

Then clearly

$\text{ord}(A) \leq \text{ord}(\bigcup A_i)$

$\Rightarrow f(A) \leq f(\bigcup A_i) = \sum f(A_i)$

$\Rightarrow f(A) \leq \sum f(A_i)$ $\because$ Sub additive property

which shows that 'f'

is an outer measure.

Ex. A set function

$$f : 2^X \longrightarrow [0, \infty] \text{ defined as}$$

$$f(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \end{cases}$$

Sol:

Now first if $A = \phi$ Then by definition of this function

$$f(A) = f(\phi) = 0$$

ii) Now let $A_1, A_2, A_3, \ldots, A_i, \ldots \in 2^X$ and

$$A \subseteq \bigcup A_i$$

Now if $A = \phi$ Then $f(A) = 0$
also if A is improperly contained in $\bigcup A_i$ Then

$$f(\bigcup A_i) = 0 \quad \text{--- (1)}$$

But if A is properly contained in $\bigcup A_i$
i.e There must exist at least one non-empty set A_i such that

$$f(A) \leq \sum f(A_i) \quad \text{--- (2)}$$

Combining (1) & (2) we get

$$f(A) \leq \sum f(A_i).$$

Now if $A \neq \phi$ Then $f(A) = 1$
and $A \subseteq \bigcup A_i$

Then there must exist at least one non-empty set A_i such that

$$f(A) \le \sum f(A_i)$$

Hence f is outer measure.

OR

$$f(A) = \begin{cases} 0 & \text{if } y \notin A = \phi \\ 1 & \text{if } y \in A \neq \phi. \end{cases}$$

Assignment

If $\mu : 2^X \longrightarrow [0, \infty]$ defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable.} \end{cases}$$

Sol:- i) if $A = \phi \Rightarrow A$ is finite and countable

$$\mu(A) = \mu(\phi) = 0$$

$$\mu(A) = 0$$

ii) Now let $A_1, A_2, A_3, \dots, A_i, \dots$ and

$$A \subseteq \cup A_i$$

if A is countable and each A_i is countable so $\cup A_i$ is also countable then

$$f(A) = 0 \quad \text{—— (1)}$$

and
if A is ~~un~~countable

Then ~~at~~ at least one non-empty set A_i is non-countable

Then
$$f(A) < \Sigma f(A_i) \quad -(2)$$
from
(1) & (2) we get.

$$f(A) \leq \Sigma f(A_i)$$

Now if A is uncountable Then There is at least one non-empty set A_i which is uncountable
Hence

$$f(A) \leq \Sigma f(A_i)$$

$\Rightarrow$ "f" is outermeasure.

— ?. — ?. — ?. — ?. — ?. —

(207) ~~Imp in paper~~

* Sum of Two Outer Measure

Let f and g be two outer measures from $2^x \xrightarrow{P(x)} [0,\infty]$. Then the function $(f+g) : 2^x \longrightarrow [0,\infty]$ define as

$$(f+g)A = f(A) + g(A)$$

Sol:- Now if $A = \varphi$.

Then

$$(f+g)\varphi = f(\varphi) + g(\varphi)$$

$$= 0 + 0 \quad \because f, g \text{ are outer measures}$$

$$(f+g)\varphi = 0$$

(ii) Let $A_1, A_2, A_3, \ldots, A_i \ldots \in 2^x$

and $A \subseteq \bigcup A_i$

Then

$$(f+g)A = f(A) + g(A)$$

$$\leq \sum f(A_i) + \sum g(A_i) \quad \text{since } f \text{ and } g \text{ are outer measure}$$

$$= \sum (f(A_i) + g(A_i))$$

$$= \sum (f+g)A_i$$

$$\Longrightarrow (f+g)A \leq \sum (f+g)A_i$$

$\Longrightarrow$ "$f+g$" is outer measure.

* **Scalar Multiplication of an outer measure ; is outer measure**

Let 'f' be an outer measure defined as

$f : 2^{X} \longrightarrow [0, \infty]$ & 'c' be a non-negative real number

Define $cf : 2^{X} \longrightarrow [0, \infty]$

as $Cf(A) = cf(A)$

To check 'cf' is outer measure

if $A = \phi$.

$Cf(A) = Cf(\phi) = C(0) = 0$ ∵ f is outer measure

$\Rightarrow C f(\phi) = 0$

$\Rightarrow$ Cf is pre-measure

(ii) Let $A_1, A_2, A_3, \ldots, A_i, \ldots \in 2^{X}$.

such that

$A \subseteq \cup A_i$

$$Cf(A) = c f(A)$$
$$\leq c \sum f(A_i)$$

$$= \sum cf(A_i)$$

$$\Rightarrow Cf(A) \leq \sum Cf(A_i)$$

Hence $cf : 2^{X} \longrightarrow [0, \infty]$ is outer measure.

✓ *

Difference of Two outer measure needs not be an outer measure.

Sol:

Let $f : 2^X \longrightarrow [0, \infty]$ and

$g : 2^X \longrightarrow [0, \infty]$ are two outer measure defined as

$$f(A) = \begin{cases} 0 & \text{if } A = \varphi \\ 1 & \text{if } A \neq \varphi. \end{cases}$$

and

$$g(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is uncountable} \end{cases}$$

Then $g - f : 2^X \longrightarrow [0, \infty]$ defined by

$$(g - f)(A) = g(A) - f(A)$$

Now if $A = \phi$.
Then

$(g-f)\phi = g(\phi) - f(\phi)$

$= 0 - 0$

$= 0$ $\because$ f & g are outer measure

Now

If A is Countable but $A \neq \phi$.

$(g-f)A = g(A) - f(A)$

$= 0 - 1$

$(g-f)A = -1$ i.e $(g-f)A$ not belonges $[0,\infty]$

Hence $(g-f)$ is not an outer measure.

Remark: Difference of two outer measur is pre measure but may or may not outer measure.

* **Lebegue Outer measure:.**
of a set of ~~Real number~~.

Let A be set of real numbers
and let $\{I_n\}$ be a countable collection
of open intervals covering A
i.e

$$A \subseteq \bigcup I_n$$

and Let

$\sum l(I_n)$ be the sum of
lengths of all these intervals.
we define lebegue outermeasure
of A denoted by

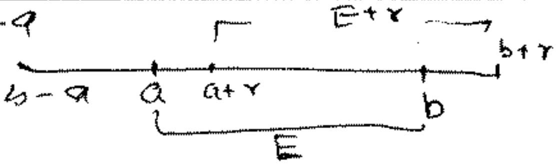
$m^*(A)$ by infimum of all
numbers $\sum l(I_n)$
Specifically

$$m^*(A) = \inf\left\{ \sum l(I_n) \,/\, A \subseteq \bigcup I_n \right\}$$

—— x —— x —— x —— x ——

$E = (a, b) \quad = b - a$

$E + r = (a + r, b + r) = b - a$



length of interval not change.

* Translation of a Set:

Let E be subset of $\mathbb{R}$ and $r \in \mathbb{R}$ Then

$$E + r = \{x + r : x \in E\}$$ is called translation of the set E.

* ~~Proof~~ Lebesgue Outer measure is translationally invariant i.e

$$E \subset \mathbb{R}, \ r \in \mathbb{R} \quad \text{Then}$$

$$m^*(E + r) = m^*(E).$$

Proof: Let $\{I_n\}$ be any countable cover of E i.e $E \subseteq \bigcup I_n$ Then

$\{I_n + r\}$ will be countable open cover of $E + r$. Now $I = (a, b) \in I_n$ Then $I + r = (a + r, b + r) \in I_n + r$

$$\begin{aligned} l(I + r) &= l(a + r, b + r) \\ &= b + r - (a + r) \\ &= b + \cancel{r} - a - \cancel{r} \\ &= b - a \end{aligned}$$

$$l(I + r) = l(I)$$

$$\Rightarrow \sum l(I_n + r) = \sum l(I_n)$$

$$\Rightarrow m^*(E + r) \le m^*(E) \quad \text{——} (1)$$

P.T.O

$$(E+r) - r = E$$

i.e translation of $E+r$ by the real number $(-r)$ is E so by the same argument

$$m^*(E) \leq m^*(E+r) \quad \text{--- (2)}$$

From (1) & (2)

$$m^*(E+r) = m^*(E)$$

$\Rightarrow$ outer measure is translationally invarient.

Theorem:- (Countable Subadditivity)

Let $\{I_n\}$ be countable collection of real numbers Then

$$m^*(\bigcup A_n) \leq \sum m^*(A_n)$$

Proof:- Case I.
If one of the set $\{A_n\}$ has outer measure '∞'. Then

$$m^*(\bigcup A_n) = \sum m^* A_n$$

Case II
if all A_n have finite outer measure Then There is a Countable collection $\{I_{n,i}\}_i$ of open intervals Such that for any $\varepsilon > 0$

$\varepsilon \sum \frac{1}{2^n} \longrightarrow 1$ G.series cg. to 1.
$\Rightarrow = \varepsilon(1) = \varepsilon$

$$\sum_i l(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n} \quad (1)$$

Then $\{I_{n,i}\}_{n,i}$ is Countable collection of open intervals as each $\{I_{n,i}\}_i$ is Countable

&

$$\bigcup A_n \subseteq \bigcup \{I_{n,i}\}_{n,i}$$

$$\Rightarrow m^* \bigcup A_n \leq \sum_{n,i} l(I_{n,i})$$

$$\Rightarrow m^* \bigcup A_n \leq \sum_n \sum_i l(I_{n,i})$$

$$\leq \sum_n \left(m^*(A_n) + \frac{\varepsilon}{2^n} \right)$$

$\because$ eq (1)

$$\Rightarrow m^* \bigcup A_n \leq \sum_n m^*(A_n) + \sum_n \frac{\varepsilon}{2^n}$$

$$\Rightarrow m^* \bigcup A_n \leq \sum_n m^*(A_n) + \varepsilon$$

Since ε is an arbitrary

$$\Rightarrow m^* \bigcup A_n \leq \sum_n m^* A_n$$

as $\varepsilon \to 0$

— x — x — x —

Problem: If $m^*(A) = 0$ Then
$$m^*(A \cup B) = m^*(B).$$

Sol:-

Since ~~$A \subseteq A \cup B$~~ & $B \subset A \cup B$

so

$$\Rightarrow m^*(B) \le m^*(A \cup B) \quad \text{①}$$

Now By Theorem

$$m^*(A \cup B) \le m^*(A) + m^*(B)$$

$$\Rightarrow m^*(A \cup B) \le m^*(B) \quad \text{②} \qquad \because m^*(A) = 0$$

From ① & ② we get

$$m^*(A \cup B) = m^*(B)$$

Monotone property:- If A & B be any two sets of real numbers s.t $A, B \in 2^{\mathbb{R}}$ & if $A \subseteq B$ Then.

$$m^*(A) \le m^*(B).$$

Proof let $\{I_n\}$ be countable collection of open intervals which covers B.

i.e

$$B \subseteq \bigcup I_n$$

$$m^*(B) = \inf\left\{ \sum l(I_n) \mid B \subseteq I_n \right\}$$

$$\Rightarrow m^*(B) = \inf\{\sum l(I_n) \mid B \subseteq \cup I_n\} \geq \inf\{\sum l(I_n) \mid A \subseteq \cup I_n\}$$

$$\Rightarrow m^*(B) \geq m^*(A)$$

$$\Rightarrow m^*(A) \leq m^*(B).$$

V-Imp.

Theorem Outer measure of an interval is its length.

Proof:- we begin with case in which we have a finit closed interval say $I = [a, b]$, for any $\varepsilon > 0$ the closed interval $[a,b]$ is contained in

$(a-\varepsilon, b+\varepsilon)$ i.e $[a,b] \subseteq (a-\varepsilon, a+\varepsilon)$

Then

$$m^*[a,b] \leq l(a-\varepsilon, b+\varepsilon) = b+\varepsilon-(a-\varepsilon)$$
$$= b-a+2\varepsilon$$

$$\Rightarrow m^*[a,b] \leq b-a+2\varepsilon$$

since ε is arbitrary

as $\varepsilon \longrightarrow 0$

$$m^*[a,b] \leq b-a \quad \text{---} \; (1)$$

so we only have to prove

$$b-a \leq m^*[a,b]$$

or

$$m^*[a,b] \geq b-a.$$

but it is equivalent to show that

$$\sum l(I_n) \geq b-a \quad \text{---} \; (2)$$

$\{I_n\}$

For any countable cover $\{I_n\}_n$
By Hein-Borel Theorem

$\{I_n\}$ is reducible to finite sub cover $\{I_n\}$ for so it is sufficient to prove (2) for a finite subcover.

Now $a \in \bigcup_1^N I_n$. Then $\exists$ an open interval $(a_1, b_1) \in \{I_n\}_1^N$ which contains a. This is $a_1 < a < b_1$

$b_1 \leq b$ as $b_1 \notin (a_1, b_1)$

Then there is an open interval in the collection $\{I_n\}_1^N$ $(a_2, b_2) \in \{I_n\}_1^N$ s.t.

$b_1 \in (a_2, b_2)$ i.e. $\Rightarrow a_2 < b_1 < b_2$

Proceeding in the same way, we reach on an open interval (a_k, b_k) s.t. (Processing in the same way we obtain a sequence $(a_1, b_1), (a_2, b_2)$ —)

$a_k < b < b_k$ i.e. $b \in (a_k, b_k)$.

It is because $\{I_n\}$ finite sub cover.

Also in our process we have each a_i, since $\{I_n\}_1^N$ is a finite collection from the collection s.t. $a_i < b_{i-1} < b_i$; so our process must stop

$a_i < b_{i-1} < b_i$

as $\bigcup_1^N I_n \supseteq \bigcup_{i=1}^K (a_i, b_i)$

$$\Longrightarrow \sum_{n=1}^{m} l(I_n) \geq \sum_{i=1}^{K} l(b_i - a_i)$$

since $\bigcup_{i=1}^K (a_i, b_i) \subset \bigcup_{n=1}^m I_n$, $m \geq K$

$$= (b_K - a_K) + (b_{K-1} - a_{K-1}) + \cdots + (b_2 - a_2) + (b_1 - a_1)$$

$$= b_K - (a_K - b_{K-1}) - (a_{K-1} - b_{K-2}) - \cdots - (a_2 - b_1) - a_1 \geq b_K - a_1$$

$\geq b_K - a_1$

and $b_k - a_i > b-a$ as $b_k > b$ and $a_i < a$ so $b_k - a_i > b-a$

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hence

$$\Rightarrow \sum_{n=1}^{m} l(I_n) \geq b-a \quad \text{--- (3)}$$

From (1), (2) & (3)

$$m^*[a,b] \geq b-a \quad \text{--- (4)}$$

From (1) & (4) we have

$$m^*[a,b] = b-a.$$

Case II

Now let 'I' be any arbitrary finite interval. Then for $\varepsilon > 0$ there is a closed interval 'J' s.t

$$J \subseteq I$$

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and $l(J) \geq l(I) - \varepsilon$

Then

$$l(I) - \varepsilon \leq l(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = l(\bar{I}) = l(I)$$

$$\Rightarrow l(I) - \varepsilon \leq m^*(I) \leq l(I).$$

As 'ε' is an arbitrary and approaching to zero Therefore so

$$l(I) \leq m^*(I) \leq l(I)$$
$$\Rightarrow m^*(I) = l(I)$$

Case III

Let 'I' be an infinite interval Then given any positive real number 'Δ' we have a closed interval

(2.19) $\boxed{m^*(A) = \inf\left\{\sum l(I_n) : A \subseteq I_n\right\}}$

$J \subseteq I$

s.t

$$l(J) = \Delta$$

$$\Rightarrow m^*(I) \geq m^*(J) = l(J) = \Delta$$

$$\Rightarrow m^*(I) \geq \Delta$$

$$\Rightarrow m^*(I) = \infty \quad \because \Delta \text{ is any positive real numbers.}$$

$$\Rightarrow m^*(I) = l(I)$$

i.e $l(I) = \infty$

---

* Lebesgue Measure of set of Real Numbers.

Let E be a set of real numbers we say E is Lebesgue measurable if for any set A of real number we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Since we always have

$$m^*(A) = m^*(A \cap R)$$
$$= m^*(A \cap (E \cup \tilde{E}))$$
$$= m^*[(A \cap E) \cup (A \cap \tilde{E})]$$
$$\leq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

$\because$ By countable sub additivity

So we only have to prove

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$.

Lemma: If E_1 and E_2 are measurable. Then $E_1 \cup E_2$ is also measurable.

Proof: Let A be any set of real numbers. Then we have to show that

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim (E_1 \cup E_2))$$

Now

$$m^*(A \cap (E_1 \cup E_2)) = m^*((A \cap E_1) \cup (A \cap E_2))$$

$$\leq m^*(A \cap E_1) + m^*(A \cap E_2)$$

$$= m^*(A \cap E_1) + m^*(A \cap (E_2 \cap \tilde{E}_1))$$

Adding $m^*(A \cap \sim (E_1 \cup E_2))$ on both sides of inequality we have.

$$m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim (E_1 \cup E_2))$$
$$\leq m^*(A \cap E_1) + m^*(A \cap (E_2 \cap \tilde{E}_1)) + m^*(A \cap (\tilde{E}_1 \cap \tilde{E}_2))$$

$$= m^*(A \cap E_1) + m^*((A \cap \tilde{E}_1) \cap E_2) + m^*((A \cap \tilde{E}_1) \cap \tilde{E}_2)$$

$$\leq m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1)$$

$\because E_2$ is measurable.

$$\Rightarrow m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim (E_1 \cup E_2))$$
$$\leq m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1)$$

$$\leq m^*(A) \quad \because E_1 \text{ is measurable}$$

$$\Rightarrow m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim (E_1 \cup E_2)) \quad \text{--- (1)}$$

NOTE If E_1 is measurable then $\tilde{E_1}$ is measurable.

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Generaly:

$$m^*(A) \le m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim(E_1 \cup E_2)) \quad (2)$$

From (1) & (2) we have

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \sim(E_1 \cup E_2))$$

$\Rightarrow$ $E_1 \cup E_2$ is measurable.

Theorem let E_1 & E_2 are two measurable sets then $E_1 \cap E_2$ is also measurable.

proof: let X be any set of real numbers. we have to prove that

$$m^*(X) \ge m^*(X \cap (E_1 \cap E_2)) + m^*(X \cap \sim(E_1 \cap E_2))$$

By Demorgan's Law.

$$X \cap \sim(E_1 \cap E_2) = X \cap (\tilde{E_1} \cup \tilde{E_2})$$

$$= (X \cap \tilde{E_1}) \cup (X \cap \tilde{E_2})$$

$$\Rightarrow m^*(X \cap \sim(E_1 \cap E_2)) = m^*((X \cap \tilde{E_1}) \cup (X \cap \tilde{E_2}))$$

$$\le m^*(X \cap \tilde{E_1}) + m^*(X \cap \tilde{E_2})$$

adding $m^*(X \cap (E_1 \cap E_2))$ on both sides of inequality we get

$$m^*(X \cap (E_1 \cap E_2)) + m^*(X \cap \sim(E_1 \cap E_2))$$
$$\leq m^*(X \cap \tilde{E}_1) + m^*(X \cap \tilde{E}_2) + m^*(X \cap (E_1 \cap E_2))$$
$$\leq m^*((X \cap E_1) \cap E_2) + m^*(X \cap \tilde{E}_1) + m^*(X \cap (E_1 \cap \tilde{E}_2))$$
$$= m^*((X \cap E_1) \cap E_2) + m^*(X \cap \tilde{E}_1) + m^*((X \cap E_1) \cap \tilde{E}_2)$$
$$\leq m^*(X \cap E_1) + m^*(X \cap \tilde{E}_1) \quad \because E_2 \text{ is measurable}$$
$$\leq m^*(X) \quad \because E_1 \text{ is measurable}$$

$$m^*(X \cap (E_1 \cap E_2)) + m^*(X \cap \sim(E_1 \cap E_2)) \leq m^*(X) \quad (1)$$

Generally.
$$m^*(X \cap (E_1 \cap E_2)) + m^*(X \cap \sim(E_1 \cap E_2)) \geq m^*(X) \quad (2)$$

From (1) & (2) we get

$$m^*(X) = m^*(X \cap (E_1 \cap E_2)) + m^*(X \cap (E_1 \cap E_2)^c)$$

$\Rightarrow$ $E_1 \cap E_2$ is measurable

Q# If A and B are measurable Then $A-B$ is measurable.

Proof ~~For any set~~ Given that A & B are measurable. Also

$A-B = A \cap B^c$

Since B is measurable. So B^c is also measurable as Complement of measurable set is also measurable.

Also we know that intersection of two measurable set is also measurable. so $A \cap B^c$ is measurable

$\Rightarrow$ $A-B$ is measurable.

* Boolean Algebra or Simply algebra of sets:-

A Collection 'α' of Subsets of Real numbers is called Algebra of sets if

i) $A \in \alpha \Rightarrow \tilde{A} \in \alpha$.

ii) $A, B \in \alpha \Rightarrow A \cup B \in \alpha$.

* Sigma Algebra:- (σ Algebra):

An algebra of sets is called σ-algebra if every union of Countable collection of members of 'α' belongs to 'α'.

Result: The collection of measurable sets is an algebra of sets:

Proof:- Let 'α' be the collection of measurable sets. Then for any $A \in \alpha$.

$\Rightarrow$ $\tilde{A}$ is also measurable as complement of measurable set is also measurable.

$\Rightarrow$ $\tilde{A} \in \alpha$.

Secondly let $A, B \in \alpha$.

i.e A & B are measurable

$\Rightarrow$ $A \cup B$ is measurable.

$\Rightarrow$ $A \cup B \in \alpha$.

Hence the collection of measurable sets is an algebra of sets. //

Imp **Lemma:** Let A be any set of real numbers and $E_1, E_2, E_3, \ldots, E_n$ be the collection of disjoint measurable sets. Then

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof: we prove the lemma by induction on 'i'

for $i=1$

$$m^*(A \cap E_1) = \sum_{i=1}^{1} m^*(A \cap E_1)$$

so it is true.

Suppose the result is true for $i = n-1$ i.e

$$m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

Now

E_i's are disjoint, $\left(A \cap \bigcup_{i=1}^{n} E_i\right)$ is any subset of real numbers and 'E_n' is measurable

Then

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = m^*\left(A \cap \bigcup_{i=1}^{n} E_i \cap E_n\right) + m^*\left(A \cap \bigcup_{i=1}^{n} E_i \cap \tilde{E}_n\right).$$

$$= m^*(A \cap E_n) + m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right)$$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i) \quad \text{By Hypothesis}$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i)$$

i.e The result is true for $i = n$.

Hence

$$m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Now for $i = 1, 2, 3, \ldots$,

$$\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \supset \left(A \cap \bigcup_{i=1}^{n} E_i\right)$$

$$\Rightarrow m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) \leq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right).$$

$$\Rightarrow \sum_{i=1}^{n} m^*(A \cap E_i) \leq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right)$$

Since R.H.S of inequality is independent of n. Therefore

$$\sum_{i=1}^{\infty} m^*(A \cap E_i) \leq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \quad (1)$$

also by Countable sub additivity

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(A \cap E_i) \quad (2)$$

From (1) & (2) we have

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Hence The proof.

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V. Imp.

Theorem:-

The Collection M of measurable sets is σ - Algebra.

Proof

i) As if $A \in M$ Then $\tilde{A} \in M$ Since Complement of measurable set is again measurable.

ii) If $A, B \in M$ Then $A \cup B \in M$. As union of measurable set is again measurable.

iii) we have to prove union of Countable collection of measurable sets is also measurable.

Let A be any set of real number and E can be expressed as

$$E = \bigcup_{i=1}^{\infty} E_i,$$

where E_i's are pairwise disjoint measurable set.

Our Requirement is to show

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E})$$

Let

$$F_n = \bigcup_{i=1}^{n} E_i$$

Then F_n is measurable ∵ union of finite measurable set measurable.

Now

$$F_n \subset E$$

$\sim$ - Complement

$\Rightarrow \tilde{F_n} \supseteq \tilde{E}$

Since F_n is measurable.

Therefore

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap \tilde{F_n})$$

$$m^*(A) \geq m^*(A \cap F_n) + m^*(A \cap \tilde{E}).$$

$\because \tilde{F_n} \supseteq \tilde{E}$

$$\Rightarrow m^*(A) \geq m^*\left(A \cap \bigcup_{i=1}^{n} E_i\right) + m^*(A \cap \tilde{E})$$

$$m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E}).$$

Since L.H.S is independent of 'n'

$$\Rightarrow m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$\Rightarrow m^*(A) \geq m^*\left(A \cap \bigcup_{i=1}^{\infty} E_i\right) + m^*(A \cap \tilde{E})$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap \tilde{E}) \quad (1)$$

Generally

$$= m^*(A) \leq m^*(A \cap E) + m^*(A \cap \tilde{E}) \quad (2)$$

From (1) & (2) we get

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \tilde{E}).$$

$\Rightarrow$ E is measureable.

Hence from (i), (ii) and (iii) The Collection M of Measurable set σ algebra

— x —

Borel Subsets of Real line:-

let β be the intersection of all the δ-Algebra of Subsets $\mathbb{R}$ containing Every open subset of $\mathbb{R}$. The members of β are called Borel subsets of $\mathbb{R}$.

or

The members of the Smallest δ-Algebra of Subsets of $\mathbb{R}$ are called Borel Subsets of $\mathbb{R}$.

v. imp **Lemma** The open interval (a, ∞) is measurable

Proof:- let A be any set of real number Then $A_1 = A \cap (a, \infty)$, $A_2 = A \cap (-\infty, a]$ So we must have to prove

$$m^*(A) \geq m^*(A_1) + m^*(A_2).$$

Now if $m^*(A) = \infty$ Then There is nothing to prove.

But if $m^*(A) < \infty$

Then for $\varepsilon > 0$ $\exists$ countable Collection $\{I_n\}$ of open interval such that

$$\Sigma l(I_n) \leq m^*(A) + \varepsilon \longrightarrow (1)$$

let $I_n' = I_n \cap (a, \infty)$
&
$I_n'' = I_n \cap (-\infty, a]$
Then
I_n' and I_n'' are intervals and

$$I_n = I_n' \cup I_n''$$

$$\Rightarrow \sum l(I_n) = \sum l(I_n') + \sum l(I_n'')$$

Now
$$A_1 \subset \cup I_n' \quad \& \quad A_2 \subset \cup I_n''$$

$$\Rightarrow m^*(A_1) \leq \sum l(I_n') = m^*(I_n') \quad \text{--- (2)}$$
&
$$m^*(A_2) \leq \sum l(I_n'') = m^*(I_n'') \quad \text{--- (3)}$$

From (1)

$$m^*(A) + \varepsilon \geq \sum l(I_n)$$
$$= \sum l(I_n') + \sum l(I_n'')$$

$$m^*(A) \geq \sum l(I_n') + \sum l(I_n'') \quad ; \varepsilon \to 0$$
$$\geq m^*(A_1) + m^*(A_2)$$

$$\Rightarrow m^*(A) \geq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) \quad (4)$$

Generally
$$m^*(A) \leq m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a]) \quad (5)$$
From (4) & (5) we get
$$m^*(A) = m^*(A \cap (a, \infty)) + m^*(A \cap (-\infty, a])$$
Hence the open interval (a, ∞) is measurable.

Imp

* Every Borel set is measurable.

Proof:-

By the definition of Borel set it is sufficient to prove that every open set is measurable and since every open set is union of countable collection of disjoint open interval so it is sufficient to prove that every open interval is (a,b) is measurable

we have (a,∞) is measurable.

$\Rightarrow \sim(a,\infty)$ is also measurable.

as $\sim(a,\infty) = (-\infty\ a]$ is measurable for every $a \in R$

Now

$$(-\infty, b) = U(-\infty, b-\tfrac{1}{n}] \text{ where } n \to \infty$$

$\Rightarrow (-\infty, b)$ is measurable since countable union of ~~countable~~ measurable set is measurable

Now

$$(a,b) = (a,\infty) \cap (-\infty, b).$$

$\Rightarrow$ (a,b) is measurable being the intersection of two measurable set is again measurable. Hence our result ~~ie~~ is proved

ie Every Borel set is measurable.

Every countable set of Borel set is of measure zero.

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* Lebesgue Measure of Measurable set:-

If E is measurable then its Lebesgue measure is denoted by $m(E)$ will be taken as its outer measure.

* Let $\langle E_i \rangle$ be sequence of measurable sets then $m \cup E_i \leq \sum m E_i$ if E_i's are disjoint then.
$m \cup E_i = \sum m E_i$

Proof:- we have

$$m^* \cup E_i \leq \sum m^* E_i$$

Since E_i's are measurable therefore

$$m^* \cup E_i = m \cup E_i$$

$$\Rightarrow m \cup E_i \leq \sum m E_i \quad \text{——— (1)}$$

Now if $\langle E_i \rangle$ is finite sequence of mutually disjoint measureable then we have

$$m^* \left(A \cap \bigcup_{i=1}^{n} E_i \right) \leq \sum_{i=1}^{n} m^* (A \cap E_i)$$

Now

$$A = R$$

$$m^* \left(R \cap \bigcup_{i=1}^{n} E_i \right) \leq \sum_{i=1}^{n} m^* (R \cap E_i)$$

$$m^* \left(\bigcup_{i=1}^{n} E_i \right) \leq \sum_{i=1}^{n} m^* (E_i)$$

As E_i's are measurable and disjoint

$$\Rightarrow m \bigcup_{i=1}^{n} E_i = \sum_{i=1}^{n} m E_i$$

Now $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$

$$\Rightarrow \bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{\infty} E_i$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) \leq m \bigcup_{i=1}^{\infty} E_i$$

$\because (a,b) \subset [a,b]$

$m^*(a,b) = b-a$

$m^*[a,b] = b-a$

$$\Rightarrow \sum_{i=1}^{n} m E_i \leq m \bigcup_{i=1}^{\infty} E_i$$

$\because m \bigcup_{i=1}^{n} E_i = \sum_{i=1}^{n} m E_i$

Since R.H.S an independent of 'n'

$$\Rightarrow \sum_{i=1}^{\infty} m E_i \leq m \bigcup_{i=1}^{\infty} E_i \quad \text{--- (2)}$$

also By Countable Subadditivity

$$m \bigcup_{i=1}^{\infty} E_i \leq \sum_{i=1}^{\infty} m E_i \quad \text{--- (3)}$$

From (2) & (3) we get

$$m \bigcup_{i=1}^{\infty} E_i = \sum_{i=1}^{\infty} m E_i \quad //$$

* Proposition:-

Let $\langle E_n \rangle$ be a decreasing sequence of measurable sets i.e

$E_1 \supset E_2 \supset E_3 \supset E_4 \supset \cdots$

and let mE_1 is finite Then

$m \bigcap_{i=1}^{\infty} E_i = \lim_{n \to \infty} mE_n$

Proof:-

Let $E = \bigcap_{i=1}^{\infty} E_i$ & let $F_i = E_i - E_{i+1}$

Then

$F_1 = E_1 - E_2 \,,\, F_2 = E_2 - E_3, \cdots$

i.e

F_i's are mutually disjoint sets.

$m \bigcup F_i = \sum m F_i$

$m \bigcup F_i = \sum m (E_i - E_{i+1}) \longrightarrow (1)$

~~$m(E_i - E) = \sum m($~~

$\bigcup F_i = (E_1 - E_2) \cup (E_2 - E_3) \cup (E_3 - E_4) \cup \cdots$

$= E_1 - E$ (where $E = \bigcap E_i$)

$m(\bigcup F_i) = m(E_1 - E) \longrightarrow (2)$

From (1) & (2) we have

$m(E_1 - E) = \sum m(E_i - E_{i+1})$

(235) $E_1 = E \cup (E_1 - E)$.

$$mE_1 - mE = \sum m(E_i - E_{i+1})$$

$$= m(E_1 - E_2) + m(E_2 - E_3) + \cdots$$

$$\Rightarrow \cancel{mE_1} - mE = \cancel{mE_1} - \cancel{mE_2} + \cancel{mE_2} - mE_3 + mE_3 - mE_4 + \cdots$$

$$\cancel{-} mE = \cancel{-} \lim_{n \to \infty} mE_n$$

$$mE = \lim_{n \to \infty} mE_n.$$

$$\boxed{m \bigcap_{i=1}^{\infty} E_i = \lim_{n \to \infty} mE_n}$$

Ass:

Show that Condition $m(E_1) < \infty$ is necessary in the above proposition.

$\overset{E_3}{(}\overset{E_2}{(}\overset{E_1}{(}\quad)))$

Preposition:

Let $\langle E_n \rangle$ be an infinite increasing sequence of measusable set i.e

$$E_1 \subset E_2 \subset E_3 \subset \cdots$$

Then

$$m \bigcup E_i = \lim_{n\to\infty} mE_n.$$

Proof:

If $mE_n = \infty$ for some n

Then

$$m \bigcup_{i=1}^{\infty} E_i = \infty$$

&

$$\lim_{n\to\infty} mE_n = \infty.$$

$$\Rightarrow m \bigcup_{i=1}^{\infty} E_i = \lim_{n\to\infty} mE_n.$$

Consider the case

$(E_n) < \infty$ for all $n \in N$.

$$= E_1 \cup (E_2 \cap \tilde{E}_1) \cup (E_3 \cap \tilde{E}_2)$$

m the R.H.S are mutually

nce

$$= mE_1 + m(E_2 - E_1) + m(E_3 - E_2) + \cdots \quad (1)$$

$$E_{i+1} = E_i \cup (E_{i+1} \cap \tilde{E}_i)$$

$$m(E_{i+1}) = mE_i + m(E_{i+1} \cap \tilde{E}_i)$$

$$= mE_i + m(E_{i+1} - E_i)$$

$$\Rightarrow m(E_{i+1} - E_i) = mE_{i+1} - mE_i$$

(1) $\Rightarrow$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \cancel{mE_1} + \cancel{mE_2} - \cancel{mE_1} + \cancel{mE_3} - \cancel{mE_2} + \cancel{mE_4} - \cancel{mE_3} + \cdots$$

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$$

Imp **Problem:-**

If E is measurable then ~~Each~~ each translation of E is also measurable.

Solution:- Let E be any subset of real numbers. Then for $y \in \mathbb{R}$, we have to prove

$$m^*(A) = m^*(A \cap E+y) + m^*(A \cap \sim E+y) \quad (1)$$

we have from set Theory

$$(A-y \cap E)+y = A \cap E+y$$

&

$$(A-y \cap \tilde{E})+y = A \cap \sim E+y$$

$f : [0, \infty] \longrightarrow [0, \infty]$

Now E is mble and $A-y$ is the subset of real numbers.

Then

$$m^*(A-y) = m^*(A-y \cap E) + m^*(A-y \cap \sim E) \quad (2)$$

Since outermeasure is translationary invarient

$$m^*[(A-y) \cap E] + y]$$

$$= m^*(A-y \cap E)$$

$$m^*[(A-y \cap \tilde{E}) + y] = m^*(A-y \cap \tilde{E})$$

$\because$ invarient property.

also

$$m^*(A-y) = m^*(A).$$

eq (2) $\Rightarrow$

$$m^*(A-y) = m^*[(A-y \cap E) + y] + m^*[(A-y \cap \tilde{E}) + y]$$

$$m^*(A) = m^*(A \cap E+y) + m^*(A \cap \sim E+y)$$

$\Rightarrow$ $E+y$ is mble. each

Hence each translation of E is measurable

* Measurable Function:-

let 'f' be an extended real valued function whose domain is measurable then the following statements are equivelent.

i) The set $\{x : f(x) > \alpha\}$ is mble

ii) The set $\{x : f(x) \geq \alpha\}$ is mble

iii) The set $\{x : f(x) < \alpha\}$ is mble

iv) The set $\{x : f(x) \leq \alpha\}$ is mble. where 'α' is any real number.

Preposition

The above four statment are equivelent:

Proof:- (i) $\Leftrightarrow$ (iv)

From (iv) $\{x : f(x) \leq \alpha\}$

$$= D - \{x : f(x) > \alpha\}$$ where D is the mble domain of f.

The R.H.S is measurable being the difference of two measurable set. There L.H.S is also measurable. i.e

$\{x : f(x) \leq \alpha\}$ is mble.

NOW

$$\{x : f(x) > \alpha\} = D - \{x : f(x) \leq \alpha\}$$

as R.H.S is mble being difference of mble set is mble. so L.H.S is also mble.

$\bigcup_{i=n}^{\infty} I_n$ is mble $\Rightarrow$ $(\bigcup_{i=1}^{n} I_n)^c$ is mble.

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$\alpha-1 \quad \alpha-\frac{1}{n^2} \quad \alpha^{-1} \quad \alpha$

$x : f(x) > \alpha$

Now

(i) $\Longleftrightarrow$ (ii)

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : \underset{(i)}{f(x) > \alpha - \tfrac{1}{n}}\}$$

R.H.S is mble being countable intersection of mble set. Therefore L.H.S is also mble.

Now

$$\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : \underset{(ii)}{f(x) \geq \alpha + \tfrac{1}{n}}\}$$

R.H.S is mble being countable union of mble set. Therefore L.H.S is mble.

(ii) $\Longleftrightarrow$ (iii)

Consider $\{x : f(x) < \alpha\}$

$$= D - \{x : f(x) \geq \alpha\}$$

Since D and $\{x : f(x) \geq \alpha\}$ is measurable Therefore R.H.S is measurable

$$\Rightarrow \{x : f(x) < \alpha\} \text{ is mble.}$$

Now $\{x : f(x) \geq \alpha\} = D - \{x : f(x) < \alpha\}$

..S is mble being the difference

$\alpha-1/3 \quad \alpha-1/2 \quad \alpha-1 \quad \alpha$

of two mable set.
Hence L.H.S is measurable.

(iii) $\Longleftrightarrow$ (iv)

$$\{x : f(x) \leq \alpha\} = \bigcup_{n=1}^{\infty} \{x : f(x) < \alpha - 1/n\}$$

R.H.S mable being the countable union of measurable set is measurable
Hence R.H.S is mable.

Now
$$\{x : f(x) < \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) \leq \alpha + 1/n\}$$

R.H.S is measurable being the countable union of measurable set is measurable.

1 $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv)

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Proposition :-

The four statements are equivalent to

$$\{x : f(x) = \alpha\} \quad \text{——— (v)}$$

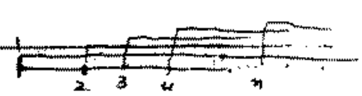
Proof :- (iii) & (iv) $\Longrightarrow$ (v)

Case - I.

$$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}$$

which exist if "α" is finite

Case - II



when $\alpha = \infty$ then

$$\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \geq n\}$$

The R.H.S is measurable being a countable intersection of measurable sets ~~R.H.S~~ so L.H.S is measurable.

when

$\alpha = -\infty$

$$\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) \leq -n\}$$

R.H.S is measurable being a countable intersection of measurable set is measurable so L.H.S is measurable.

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NOTE: f is said to be mble function if the following four statements are equivalent:

i) $\{x : f(x) > \alpha\}$ is mble

ii) $\{x : f(x) \geq \alpha\}$ is mble

iii) $\{x : f(x) < \alpha\}$ is measurable

iv) $\{x : f(x) \leq \alpha\}$ is measurable.

$\Updownarrow$ $\{x : f(x) = \alpha\}$

$c + f(x) = c + f(x)$ $c+f$, cf

Proposition

Let c be any constant and 'f' be a measurable function. Then '$c+f$' and 'cf' are also measurable.

Proof:-

Let 'α' be any real number, then

$$\{x : c + f(x) > \alpha\} = \{x : f(x) > \alpha - c\}$$

R.H.S is measurable since 'f' is measurable

$\therefore$ L.H.S is measurable. i.e '$c+f$' is measurable.

Now $$\{x : cf(x) > \alpha\} = \{x : f(x) > \frac{\alpha}{c}\} \text{ if } c > 0$$

$$= \{x : f(x) < \frac{\alpha}{c}\} \text{ if } c < 0$$

In both cases

R.H.S is 'f' measurable. so L.H.S measurable

i.e

'cf' is measurable.

$cf(x) >$

$f(x) > \frac{\alpha}{c}$

$(c+f)(x)$

$= c + f(x) > \alpha$

$f(x) > \alpha - c$

$cf(x)$

$= cf(x)$

— x — x — x —

(247)

$\alpha - 7 \quad \alpha$

$f(\xi^2)$

Proposition: Let f and g are measurable functions Then $f+g$, $f-g$ and fg is measurable.

$(f+g)x = f(x) + g(x)$

Proof: Let α be any real number and $f(x) + g(x) < \alpha$.

$$\Rightarrow f(x) < \alpha - g(x) \text{ . (two real numbers)}$$

$\Rightarrow \exists$ a rational number r such that

$$f(x) < r < \alpha - g(x).$$

$$\{ x : f(x) + g(x) < \alpha \}$$

$$= \bigcup_{r \in Q} \{ x : f(x) < r \} \cap \{ x : g(x) < \alpha - r \}$$

Since f and g measurable functions and Q is countable. Therefore R.H.S being countable union of measurable is measurable i.e R.H.S is measurable

$\Rightarrow$ L.H.S is measurable

$\Rightarrow$ $f+g$ is measurable

Now as g is measurable.
$\Rightarrow$ cg is also measurable.
In particular if $c = -1$

$\Rightarrow -g$ is measurable

$\Rightarrow f + (-g) = f - g$ is also measurable.

Now for any real number 'α'
Case-I if $\alpha > 0$

$$\{x : f^2(x) < \alpha\} = \cancel{\{x : f(x) < \sqrt{\alpha}\} \cup \{x : f(x) > -\sqrt{\alpha}\}}$$

$$= \{x : f(x) < \sqrt{\alpha}\} \cup \{x : f(x) > -\sqrt{\alpha}\}$$

if $\alpha \leq 0$

as R.H.S is measurable being the union of measurable set. So L.H.S is measurable.

Now

$\alpha < 0$

$$\{x : f^2(x) > \alpha\} = \text{Domain of } D$$

where D is the domain of which is measurable so $\{x : f^2(x) > \alpha\}$ if $\alpha < 0$ is measurable.

Hence f^2 is measurable for all 'α'.

$$\text{Now } fg = \frac{1}{2}\left[(f+g)^2 - (f+g)^2\right]$$

Since f & g are measure therefore f^2 and g^2 and $f+g$ are also

$f \vee g\ (x) = \max(f(x)$ or $g(x))$

measurable. also cf is measurable. Therefore R.H.S is measurable. Hence L.H.S is measurable.

So $f \cdot g$ is measurable.

$\cdot x \cdot$ ——— $\cdot x \cdot$ ——— $\cdot x \cdot$ ——— $\cdot x \cdot$

<u>Proposition</u>: let f & g be measurable function Then

(i) $f \underset{(or)}{\vee} g$ ii) $f \underset{(and)}{\wedge} g$

let α be any real number

& $f \vee g = \max\{f(x), g(x)\}$

$\Rightarrow \{x : (f \vee g)(x) > \alpha\}$

$= \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$

Since R.H.S is measurable being union of two measurable set. So L.H.S is measurable Therefore

$\{x : (f \vee g)(x) > \alpha\}$ is measurable

ii) Let α be any real number

& $f \wedge g = \underline{\min}\{f(x), g(x)\}$

$\{x : (f \wedge g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cap \{x : g(x) > \alpha\}$

R.H.S measurable. being intersection

mp

Almost Everywhere:

A property is said to be "Almost everywhere" if the set of points where does not hold is a set of measure zero

Imp

Preposition

If f is measurable function and $f = g$ (almost Everywhere) Then g is mable.

Proof: let D be the domain of both function f and g and let E be the set where $f \neq g$ Then by hypothesis

$$m^* E = 0.$$

let α be any real number Then the set

$$\{x \in D \mid g(x) > \alpha\} = \{x \in D \mid f(x) > \alpha\} \cup \{x \in E \mid g(x) > \alpha\}$$

Now the 1st set on R.H.S is mable by hypothesis i.e f is mable and the 2nd set being the sub set of E of measure zero is also mable. Hence R.H.S is mble being the union of two mble sets therefore L.H.S is also mable.

(251) $\{x : f(x) < \alpha$ $\quad$ $f : [0,\infty] \to (0,\infty]$ $\quad$ $\{x : f(x) > a\}$ $\quad$ $\{0,1\}$ $\quad$ $x > f^{-1}(a)$

* The characteristic function χ_A of A is measurable iff A is measurable.

Proof: Suppose that χ_A is mble. Then the the set

$$\{x \mid \chi_A(x) > 0\} = A$$ By definition of characteristic fun.

Hence A is mble.

Conversely suppose that A is mble.

$$\{x \mid \chi_A > 0\} = \begin{cases} A & \text{if } 0 \le \alpha < 1 \\ R & \alpha < 0 \\ \varphi & \alpha \ge 1 \end{cases}$$

Now A is mble by assumption R and φ are mble Hence χ_A is mble.

— x — — x — — x —

* Let f be mble function. Then for any interval I, $f^{-1}(I)$ is mble.

Proof:- Let $I = (a, b) = (a, \infty) \cap (-\infty, b)$ $\quad$ $f : [0,\infty] \to [0$ $\quad$ $\{x \mid f(x) < a\}$ $\quad$ $\{f^{-1}(x) > 0\}$

$$f^{-1}(I) = f^{-1}[(a,\infty) \cap (-\infty, b)]$$

$$= f^{-1}(a,\infty) \cap f^{-1}(-\infty, b)$$

$$= \{x \mid f(x) > a\} \cap \{x \mid f(x) < b\}.$$

Both the sets on R.H.S is mble Hence

$f^{-1}[I]$ is mble

— x — x —

$\pm f(x) > \alpha$,
$f(x) > \alpha$, $f(x) < -\alpha$ (252)

A rable *. $|f|$ is mble if f is mble.

'e

Proof: Let α be any real number. Then $\alpha = 0$

If

$$\{x \mid |f| > \alpha\} = \{x : f(x) > \alpha\} \cup \{x \mid f(x) < -\alpha\}$$

R.H.S mble being the union of two mble set. Hence $|f|$ is mble.

If $\alpha < 0$.

$$\{x : |f(x)| > \alpha\} = D$$

R.H.S mble Hence $|f|$ is mble.

If $\alpha > 0$

$$\{x : |f(x)| > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : f(x) < -\alpha\}$$

Again $|f|$ is mble

---

*. $\frac{1}{f}$ is mble if f is mble:

$$\{x : \frac{1}{f(x)} > \alpha\} = D - \{x : f(x) \geq \alpha'\} \quad \alpha' = \frac{1}{\alpha}$$

$$= \{x : f(x) < \alpha'\}$$

Since R.H.S mble Therefore R.H.S is mble. That

$\frac{1}{f}$ is mble.

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Problem: Let D and E be mble sets and "f" be real value function with domain $(D \cup E)$. Then "f" is mble iff its restriction to D and E are mble.

Proof Let α be any real value and f be mble function.

$$\{x \mid f|_D(x) > \alpha\} = \{x \in D \cup E \mid f(x) > \alpha\} \cap D$$

Since 1st set on R.H.S is mble as f is mble and D is measurble by hypothesis. Hence restriction of $f|_D$ is measurable.
Similarly $f|_E$ is mble

Conversely let $f|_D$ and $f|_E$ are mble. Then for any real number α.

$$\{x : f(x) > \alpha \mid x \in D \cup E\}$$

$$= \{x : f|_D(x) > \alpha\} \cup \{x : f|_E(x) > \alpha\}$$

Since R.H.S measureable. Therefore L.H.S is measurable. Hence

'f' is measurable of Domain

$D \cup E$.

Problem let D is a mble set and f be a function with Domain D Then f is mble iff the function g defined by

$$f(x) = g(x) \text{ if } x \in D \text{ and}$$

$$g(x) = 0 \text{ if } x \notin D \text{ is measurable.}$$

Proof: let α be any real number and f is measurable. we will have to prove that 'g' is measurable. let E be the Domain of g: Then E is mble
& $\alpha > 0$

$$\{x : x \in E \mid g(x) > \alpha\} = \{x \in D \mid f(x) > \alpha\}$$

Since the R.H.S is mble Therefore L.H.S is mble

now if $\alpha < 0$

$$\{x : x \in E \mid g(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup (E - D).$$

Again R.H.S mble Therefore L.H.S mble.
Hence g is mble.

Conversely let g is measureable Then the set

$$\{x \in D \mid f(x) > \alpha\} = \{x \in E \mid g(x) > \alpha\} \cap D.$$

R.H.S measureable. Therefore 'f' is mble.

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*) If $f: E \longrightarrow R$ is continous and E is mble Then 'f' is also mble.

Proof. let α be any Real number Then we have to show that

$$B = \{x \in E \; f(x) > \alpha\}. \text{ is mble}$$

let

x be an arbitrary point of B

Then $f(x) > \alpha$ and f is continous at x. So There is open interval U_x contains x such that.

$$f(y) > \alpha \;\; \forall \; y \in U_x.$$

Then

$$E \cap U_x \subset B$$

let

$$U = \bigcup_{x \in B} (U_x)$$

$$E \cap U = E \cap \bigcup_{x \in B} (U_x).$$

$$= \bigcup_{x \in B} (E \cap U_x).$$

Since $E \cap U_x \subseteq B$ for every $x \in B$.

$$\Rightarrow E \cap U = \bigcup_{x \in B} (E \cap U_x) \subset B \text{ — } ①$$

But $B \subseteq E \cap U$ — ②

From ① & ② we have.

$$B = E \cap U$$

U is open being union of open interval and hence measurable. Since E is also mble Therefore $E \cap U$ is mble Hence

$B = \{x : f(x) > \alpha\}$ is mble.

NOTE: If f and g are boral mble ftn Then gof is also boral mble.

Boral Measurable:~

// A function is called Boral measurable if for every real number 'α' The set $\{x \mid f(x) > \alpha\}$ is boral set.

Theorem If f is mble and B a Boral set Then $f^{-1}(B)$ is mble. (set).

Proof:- By definition of boral set we prove $f^{-1}(B)$ is mble for an open set B.

Now every open set is a union of Countable collection of open interval Then

$$B = \cup I_i$$

$$f^{-1}(B) = f^{-1}(\cup I_i) = \cup f^{-1}(I_i).$$

$$= \cup f^{-1}(a_i, b_i)$$

$$= \cup (f^{-1}(-\infty, b) \cap (a_i, \infty))$$

$$f^{-1}(B) = \{f^{-1}(-\infty, b) \cap f^{-1}(a_i, \infty)\}$$
$$= \{x : f(x) < b_i\} \cap \{x : f(x) > a_i\}$$

Since R.H.S mble being the ~~...~~ intersection of two mble set. So R.H.S i.e

$f^{-1}(B)$ is mble.

* If f is Borel mble and B is boral set Then $f^{-1}(B)$ is also Borel set.

Proof :: By definition of Borel set it is sufficient to prove that $f^{-1}(B)$ is a Borel set for any open set

∵ An open set the countable union of disjoint open inteval we have

$$B = \cup I_i$$
$$f^{-1}(B) = f^{-1} \cup I_i$$
$$= \cup f^{-1}(I_i)$$

Since set of all Borel sets is a sigma algebra

so

$$I = (a, b) = (a, \infty) \cap (-\infty, b)$$
$$f^{-1}(I) = f^{-1}(a, \infty) \cap f^{-1}(-\infty, b)$$

$$= \{x \mid f(x) > a\} \cap \{x \mid f(x) < b\} \quad \text{---} (1)$$

Now

$$\{x \mid f(x) < b\} = \{x \mid f(x) \geq b\}^c$$

$$= \left[\bigcap_{n=1}^{\infty} \{x \mid f(x) > b - 1/n\}\right]^c$$

Since intersection of Borel set is boral set so by definition of Borel set 2nd set on R.H.S of (1) is borel set being the complement of countable intersection of borel sets. Hence $f^{-1}(I)$ is boral set.

x —— x —— x.

* Let f be bounded function and E be measureable set of finite measure then simple function φ and ψ

$$\text{Inf} \int \psi \, dx = \sup \int \varphi \, dx$$

iff f is measurable.

Proof Let f is bounded by M and suppose that f is measurable then the sets

$$E_k = \left\{x : \frac{kM}{n} \geq f(x) \geq \frac{(k-1)M}{n}\right\}, \; -n \leq k \leq n$$

are measurable disjoint and have union E.

Thus $\sum_{k=1}^{n} mE_k = mE$

The simple function defined by

$$\psi_n(x) = \frac{M}{n}\sum_{K=-n}^{n} K\,\chi_{E_K}(x).$$

and

$$\varphi_n(x) = \frac{M}{n}\sum_{K=1}^{n} (K-1)\,\chi_{E_K}(x).$$

Satisfy

$$\varphi_n(x) \le f(x) \le \psi_n(x)$$

Thus

$$\inf\int_E \psi_n(x) \le \int_E \psi(x)\,dx = \frac{M}{n}\sum_{K=-n}^{K} K\, m E_K.$$

and

$$\sup\int_E \varphi(x)\,dx \ge \int_E \varphi_n(x)\,dx = \frac{M}{n}\sum_{n=-n}^{n}(K-1)\, m E_K$$

$$0 \le \inf\int_E \psi_n(x)\,dx - \sup\int_E \varphi(x)\,dx \le \frac{M}{n}\sum_{K=-n}^{n} m E_K = \frac{M}{n} m A$$

Since n is arbitrary we have

$$\inf\int_E \psi(x)\,dx - \sup\int_E \varphi(x)\,dx = 0$$

$$\inf\int_E \psi(x)\,dx = \sup\int_E \varphi(x)\,dx.$$

and the condition is sufficient

Problem: Let D and E be mble sets and f be real value function and ... $(D \cup E)$ then f is mble

Conversely suppose that

$$\inf \int \psi(x)\,dx = \sup \int \varphi(x)\,dx.$$

Then given n, there are simple function $\varphi_n(x)$ and $\psi_n(x)$

$$\varphi_n(x) \le f(x) \le \psi_n(x).$$

and

$$\int \psi_n(x)\,dx - \int \varphi_n(x)\,dx < \frac{1}{n}$$

Then the function.

$\psi^* = \inf \psi_n$ and $\varphi^* = \sup \varphi_n$ are mble by hypothesis and.

$$\varphi^*(x) \le f(x) \le \psi^*(x)$$

Now the set

$$\Delta = \{x ;\ \varphi^*(x) < \psi^*(x)\}$$

is the union of the sets

$$\Delta v = \{x :\ \varphi^*(x) < \psi^*(x) - 1/v\}$$

But each of the set contained in the set

$$Av = \{x ;\ \varphi_n(x) < \psi_n(x) - 1/v\}$$ and this latter set has measure less then v/n Since n is arbitrary $m\,\Delta v = 0$ and so $m\,\Delta v = 0$ Thus $\varphi^* = \psi^*$ excepet on a set of measure zero. Thus f is measurable.