Exercise 2.2 (Solutions)

Mathematics (Science Group): 10th

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$Q.1$ Find the cube root of $-1, 8, -27, 64.$

(i) Cube roots of -1

Solution: Let $x = (-1)^{\frac{1}{3}}$

$x^3 = -1$

$x^3 + 1 = 0$

$x^3 + (1)^3 = 0$

$\because (a^3 + b^3) = (a+b)(a^2 - ab + b^2)$

$(x+1)(x^2 - (x)(1) + 1^2) = 0$

$(x+1)(x^2 - x + 1) = 0$

$x + 1 = 0 \qquad (x^2 - x + 1) = 0$

$x = -1 \qquad x^2 - x + 1 = 0$

Then we solve $x^2 - x + 1 = 0$ by formula

$ax^2 + bx + c = 0$

$a = 1,\ b = -1,\ c = 1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2 \times 1}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

Cube roots of -1

$1, \frac{1-\sqrt{-3}}{2} \qquad 1, \frac{1+\sqrt{-3}}{2}$

$1, -\left(\frac{-1+\sqrt{-3}}{2}\right) \qquad -1\left(\frac{-1-\sqrt{-3}}{2}\right)$

$1, -\omega \qquad 1, -\omega^2$

$x = -1\omega \qquad x = -1(\omega)^2$

$x = -\omega \qquad x = -\omega^2$

Cube roots of -1 are $-1, -\omega, -\omega^2$

(ii) Cube roots of 8

Solution: Let $x = (8)^{\frac{1}{3}}$

$x^3 = 8$

$x^3 - 8 = 0$

$x^3 - 2^3 = 0$

$\because (a^3 - b^3) = (a-b)(a^2 + ab + b^2)$

$(x-2)(x^2 + (x)(2) + 2^2) = 0$

$(x-2)(x^2 + 2x + 2^2) = 0$

$(x-2)(x^2 + 2x + 4) = 0$

$x - 2 = 0 \qquad x^2 + 2x + 4 = 0$

$x = 2 \qquad x^2 + 2x + 4 = 0$

Then we solve $x^2 + 2x + 4 = 0$ by formula

$a = 1,\ b = 2,\ c = 4$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$x=\frac{-2\pm\sqrt{-12}}{2}$$

$$x=\frac{-2\pm\sqrt{4\times(-3)}}{2}$$

$$x=\frac{-2\pm\sqrt{4}\sqrt{-3}}{2}$$

$$x=\frac{-2\pm2\sqrt{-3}}{2}$$

$$x=\frac{2(-1\pm\sqrt{-3})}{2}$$

$$x=2\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$

$$x=2\left(\frac{-1+\sqrt{-3}}{2}\right)\qquad x=2\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$x=2\omega\qquad x=2\omega^2$$

Cube roots of 8 *are* $2, 2\omega, 2\omega^2$

(iii) *Cube roots of* -27

Solution: *Let* $x=(-27)^{\frac{1}{3}}$

$$x^3=-27$$

$$x^3+27=0$$

$$x^3+3^3=0$$

$$\because (a^3+b^3)=(a+b)(a^2-ab+b^2)$$

$$(x+3)(x^2-(x)(3)+3^2)$$

$$(x+3)(x^2-3x+9)=0$$

$$x+3=0\qquad x^2-3x+9=0$$

$$x=-3\qquad x^2-3x+9=0$$

Then we solve $x^2-3x+9=0$ *by formula*

$$a=1,\ b=-3,\ c=9$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}$$

$$x=\frac{3\pm\sqrt{9-36}}{2}$$

$$x=\frac{3\pm\sqrt{-27}}{2}$$

$$x=\frac{3\pm\sqrt{9\times(-3)}}{2}$$

$$x=\frac{3\pm\sqrt{9}\sqrt{-3}}{2}$$

$$x=\frac{3\pm3\sqrt{-3}}{2}$$

$$x=\frac{-3(-1\pm\sqrt{-3})}{2}$$

$$x=-3\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$

$$x=-3\left(\frac{-1+\sqrt{-3}}{2}\right)\qquad x=-3\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$x=-3\omega\qquad x=-3\omega^2$$

Cube roots of -27 *are* $-3, -3\omega, -3\omega^2$

(ii) *Cube roots of* 64

Solution: *Let* $x=(64)^{\frac{1}{3}}$

$$x^3=64$$

$$x^3-64=0$$

$$x^3-4^3=0$$

$$\because (a^3-b^3)=(a-b)(a^2+ab+b^2)$$

$$(x-4)(x^2+(x)(4)+4^2)=0$$

$$(x-4)(x^2+4x+16)=0$$

$$x-4=0\qquad x^2+4x+16=0$$

$$x=4\qquad x^2+4x+16=0$$

Then we solve $x^2+4x+16=0$ *by formula*

$a=1,\ b=4,\ c=16$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-4\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$

$$x=\frac{-4\pm\sqrt{16-64}}{2}$$

$$x=\frac{-4\pm\sqrt{-48}}{2}$$

$$x=\frac{-4\pm\sqrt{16\times(-3)}}{2}$$

$$x=\frac{-4\pm4\sqrt{-3}}{2}$$

$$x=\frac{4(-1\pm\sqrt{-3})}{2}$$

$$x=4\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$

$$x=4\left(\frac{-1+\sqrt{-3}}{2}\right)\qquad x=4\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$x=4\omega\qquad x=4\omega^2$$

Cube roots of 64 *are* $4, 4\omega, 4\omega^2$

Q.2 Evaluate

$(i)\left(1-\omega-\omega^2\right)^7$

Solution : $\left(1-\omega-\omega^2\right)^7$

$$=\left[1-\left(\omega-\omega^2\right)\right]^7$$

$$=\left[1-(-1)\right]^7$$

$$=(1+1)^7$$

$$=2^7=128$$

$(ii)\left(1-3\omega-3\omega^2\right)^5$

Solution : $\left(1-3\omega-3\omega^2\right)^5$

$$=\left[1-3\left(\omega+\omega^2\right)\right]^5$$

$$=\left[1-3(-1)\right]^5$$

$$=(1+3)^5$$

$$=4^5=1025$$

$(iii)\left(9+4\omega+4\omega^2\right)^3$

Solution : $\left(9+4\omega+4\omega^2\right)^3$

$$=\left[9+4\left(\omega+\omega^2\right)\right]^3$$

$$=\left[9+4(-1)\right]^3\quad\left(\because\ \omega+\omega^2=-1\right)$$

$$=(9-4)^3$$

$$=5^3=125$$

$(iv)\left(2+2\omega+2\omega^2\right)\left(3-3\omega+3\omega^2\right)$

Solution : $\left(2+2\omega+2\omega^2\right)\left(3-3\omega+3\omega^2\right)$

$$=\left(2(1+\omega)-2\omega^2\right)\left(3+3\omega^2-3\omega\right)$$

$$=\left[2(1+\omega)-2\omega^2\right]\left[3\left(1+\omega^2\right)-3\omega\right]$$

$$\left\{\because\ 1+\omega+\omega^2=0\right\}$$

$$\left\{1+\omega=-\omega^2\quad 1+\omega^2=-\omega\right\}$$

$$=\left[2(-\omega)^2-2\omega^2\right]\left[3(-\omega)-3\omega\right]$$

$$=\left(-2\omega^2-2\omega^2\right)\left(-3\omega-3\omega\right)$$

$$=\left(-4\omega^2\right)\left(-6\omega\right)$$

$$=24\omega^3=24(1)=24$$

$(v)\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$

Solution : $\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$

As $\dfrac{-1+\sqrt{-3}}{2}=\omega\qquad\dfrac{-1-\sqrt{-3}}{2}=\omega^2$

$$-1+\sqrt{-3}=2\omega\qquad -1-\sqrt{-3}=2\omega^2$$

Then it becomes

$$=(2\omega)^6+(2\omega^2)^6$$
$$=2^6\omega^6+2^6\omega^{12}$$
$$=2^6\left[(\omega^3)^2+(\omega^3)^4\right]$$
$$=2^6\left[(1)^2+(1)^4\right]$$
$$=64(1+1)$$
$$=64(2)=128$$

$(vi)\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

$Solution:\left(\dfrac{-1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^9$

$As\ \dfrac{-1+\sqrt{-3}}{2}=\omega\quad \dfrac{-1-\sqrt{-3}}{2}=\omega^2$

Then it becomes

$$=(\omega)^9+(\omega^2)^9$$
$$=\omega^9+\omega^{18}$$
$$=(\omega^3)^3+(\omega^3)^6$$
$$=(1)^3+(1)^6$$
$$=1+1=2$$

$(vii)\ \omega^{37}+\omega^{38}-5$

$Solution:\omega^{37}+\omega^{38}-5$

$$=\omega^{36}\omega+\omega^{36}\omega^2-5$$
$$=(\omega^3)^{12}\omega+(\omega^3)^{12}\omega^2-5$$
$$=(1)^{12}\omega+(1)^{12}\omega^2-5$$
$$=1\omega+1\omega^2-5$$
$$=(\omega+\omega^2)-5$$
$$=(-1)-5$$
$$=-1-5=-6$$

$(viii)\ \omega^{-13}+\omega^{-17}$

$Solution:\omega^{-13}+\omega^{-17}$

$$=\frac{1}{\omega^{13}}+\frac{1}{\omega^{17}}$$
$$=\frac{1}{\omega^{12}\omega}+\frac{1}{\omega^{15}\omega^2}$$
$$=\frac{1}{(\omega^3)^4\omega}+\frac{1}{(\omega^3)^5\omega^2}$$
$$=\frac{1}{(1)^4\omega}+\frac{1}{(1)^5\omega^2}$$
$$=\frac{1}{\omega}+\frac{1}{\omega^2}$$
$$=\frac{\omega^2+\omega}{(\omega)(\omega^2)}=\frac{-1}{\omega^3}$$
$$=-\frac{1}{1}=-1$$

Q.3 Prove that

$$x^3+y^3=(x+y)(x+\omega y)(x+\omega^2 y)$$

$Solution: Let,$

$$R.H.S=(x+y)(x+\omega y)(x+\omega^2 y)$$
$$=(x+y)(x^2+\omega^2xy+\omega xy+\omega^3y^2)$$
$$=(x+y)\left[x^2+(\omega^2+\omega)xy+\omega^3y^2\right]$$
$$\because 1+\omega+\omega^2=0,\ \omega+\omega^2=-1,\ \omega^3=1$$
$$=(x+y)\left[x^2+(-1)xy+1y^2\right]$$
$$=(x+y)(x^2-xy+y^2)$$

$As\ (a^3+b^3)=(a+b)(a^2-ab+b^2)\ so,$

$$=x^3+y^3=L.H.S$$

Q.4 Prove that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z)$$

$Solution: Let\ R.H.S$

$$=(x+y+z)(x+\omega y+\omega^2z)(x+\omega^2y+\omega z)$$
$$=(x+y+z)(x^2+\omega^2xy+\omega xz+\omega yx+\omega^3y^2+\omega^2yz+\omega^2xz+\omega^4zy+\omega^3z^2)$$

$$=(x+y+z)\left[\left(x^2+\omega^3y^2+\omega^3z^2+\left(\omega^2+\omega\right)xy+\left(\omega^2+\omega^4\right)yz+\left(\omega+\omega^2\right)zx\right)\right]$$
$$=(x+y+z)\left[x^2+1y^2+1z^2+(-1)xy+\left(\omega^2+\omega^3\omega\right)yz+(-1)zx\right]$$
$$=(x+y+z)\left[x^2+1y^2+1z^2+(-1)xy+\left(\omega^2+1\omega\right)yz+(-1)zx\right]$$
$$=(x+y+z)\left(x^2+y^2+z^2-xy-yz-zx\right)$$
$$=x^3+y^3+z^3-3xyz=L.H.S$$

Using formula:

$$=(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)=a^3+b^3+c^3-3abc$$

Question No 5

$$=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\text{..........}2n \textit{ factors}=1$$

Solution : *L H S*

$$=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\text{..........}2n \textit{ factors}$$
$$=(1+\omega)(1+\omega^2)(1+\omega\omega^3)(1+\omega^6\omega^2)\text{..........}2n \textit{ factors}$$
$$=(1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\text{..........}2n \textit{ factors} \qquad \therefore \omega^3=1$$
$$=\left[(1+\omega)(1+\omega^2)\right]\left[(1+\omega)(1+\omega^2)\right]\text{.........}n \textit{ factors}$$
$$=\left[(1+\omega)(1+\omega^2)\right]^n$$
$$=\left[(-\omega^2)(-\omega)\right]^n$$
$$=\left[\omega^3\right]^n$$
$$=[1]^n$$
$$=1$$
$$L.H.S=R.H.S$$

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