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CH: 5

Exercise # 5.1

Q: Sum the following series upto n terms

Q:1 $1^2+3^2+5^2+7^2+\cdots$

Sol: 1st find the general term (T_k) for the series

As $1+3+5+7+\cdots$ is Arithmetic progression

Then $T_k = a+(k-1)d$ $\quad (a=1 \ \& \ d=2)$

$\Rightarrow T_k = 1+(k-1)2$

$\Rightarrow T_k = 1+2k-2$

$\Rightarrow T_k = 2k-1$

But the given series is $1^2+3^2+5^2+7^2+\cdots$

Then $T_k = (2k-1)^2 \Rightarrow T_k = 4k^2-4k+1$

For sum upto n terms

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(4k^2-4k+1)$$

$$= 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + \sum_{k=1}^{n}1$$

$$= 4\frac{n(n+1)(2n+1)}{6} - 4\frac{n(n+1)}{2} + n$$

$$= n\left\{\frac{2(n+1)(2n+1)}{3} - 2(n+1)+1\right\}$$

$$= n\left\{\frac{2(n+1)(2n+1)-6(n+1)+3}{3}\right\}$$

$$= n\left\{\frac{2(2n^2+3n+1)-6n-6+3}{3}\right\}$$

$$= n\left\{\frac{4n^2+6n+2-6n-6+3}{3}\right\}$$

$$= \frac{n}{3}\left\{4n^2-1\right\} \quad \text{Ans}$$

Q:2 $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\cdots$

Sol: clearly the general term (T_k) will be

$1^2+2^2+3^2+\cdots+k^2$

i.e $T_k = 1^2+2^2+3^2+\cdots+k^2$

$$\Rightarrow T_k = \frac{k(k+1)(2k+1)}{6}$$

$$\Rightarrow T_k = \frac{k(2k^2+3k+1)}{6}$$

$$\Rightarrow T_k = \frac{2k^3+3k^2+k}{6}$$

For sum upto n terms

$$\Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{6}\sum_{k=1}^{n}(2k^3+3k^2+k)$$

$$\Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{6}\left\{2\sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k\right\}$$

$$\Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{6}\left\{2\left(\frac{n(n+1)}{2}\right)^2 + 3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$\Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{6}\left\{\frac{2n^2(n+1)^2}{4} + \frac{n(2n^2+n+2n+1)}{2} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{6}\left\{\frac{n^2(n^2+2n+1)}{2} + \frac{n(2n^2+3n+1)}{2} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{n}{6}\left\{\frac{n(n^2+2n+1)}{2} + \frac{2n^2+3n+1}{2} + \frac{n+1}{2}\right\}$$

$$= \frac{n}{6}\left\{\frac{(n^3+2n^2+n)+(2n^2+3n+1)+(n+1)}{2}\right\}$$

$$= \frac{n}{12}\left\{n^3+4n^2+5n+2\right\} \quad \text{Ans}$$

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Q:3 $2^2+4^2+6^2+\cdots$

Sol As $2+4+6+\cdots$ is Arithmetic progression whose general term (T_k) is

$T_k = a+(k-1)d$ where $a=2$ and $d=2$

$\Rightarrow T_k = 2+(k-1)2$

$\Rightarrow T_k = 2+2k-2$

$\Rightarrow T_k = 2k$

But the given series is $2^2+4^2+6^2+\cdots$

Then $T_k = (2k)^2$

For sum upto n terms, we have

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k)^2$$

$$= \sum_{k=1}^{n} 4k^2$$

$$= 4\sum_{k=1}^{n} k^2$$

$$= 4\,\frac{n(n+1)(2n+1)}{6}$$

$$= \frac{2n(n+1)(2n+1)}{3} \text{ Ans}$$

Q:4 $1^3+3^3+5^3+\cdots$

Sol $T_k = \{a+(k-1)d\}^3$

$= \{1+(k-1)2\}^3$

$= \{1+2k-2\}^3$

$= (2k-1)^3$

$= 8k^3-1-12k^2+6k$

$\Rightarrow T_k = 8k^3-12k^2+6k-1$

For sum upto n terms

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(8k^3-12k^2+6k-1)$$

$$\Rightarrow \sum_{k=1}^{n} T_k = 8\sum_{k=1}^{n} k^3 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1$$

$$= 8\left(\frac{n(n+1)}{2}\right)^2 - 12\,\frac{n(n+1)(2n+1)}{6} + 6\,\frac{n(n+1)}{2} - n$$

$$= \frac{8n^2(n+1)^2}{4} - \frac{12n(n+1)(2n+1)}{6} + 3n(n+1) - n$$

$$= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n$$

$$= 2n^2(n^2+2n+1) - 2n(2n^2+3n+1) + 3n(n+1) - n$$

$$= 2n^4+4n^3+2n^2-4n^3-6n^2-2n+3n^2+3n-n$$

$$= 2n^4-n^2$$

$$= n^2(2n^2-1) \text{ Ans}$$

Q:5 $1^3+5^3+9^3+\cdots$

$1+5+9+\cdots$ is A.P. $a=1$ $d=4$

Sol T_k for $1+5+9+\cdots$ is

$T_k = a+(k-1)d$

$\Rightarrow T_k = 1+(k-1)4$

$\Rightarrow T_k = 1+4k-4$

$\Rightarrow T_k = 4k-3$

But the given series is $1^3+5^3+9^3+\cdots$

Then $T_k = (4k-3)^3$

$\Rightarrow T_k = 64k^3-27-144k^2+108k$

Now for sum upto n terms

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(64k^3-144k^2+108k-27)$$

$$= 64\sum_{k=1}^{n} k^3 - 144\sum_{k=1}^{n} k^2 + 108\sum_{k=1}^{n} k - 27\sum_{k=1}^{n} 1$$

$$= 64\,\frac{n^2(n+1)^2}{4} - 144\,\frac{n(n+1)(2n+1)}{6} + 108\,\frac{n(n+1)}{2} - 27n$$

$$= 16n^2(n+1)^2 - 24n(n+1)(2n+1) + 54n(n+1) - 27n$$

151

$$\Rightarrow \sum_{k=1}^{n} T_k = n\left\{16n(n+1)^2 - 24(n+1)(2n+1) + 54(n+1) - 27\right\}$$
$$= n\left\{16n(n^2+2n+1) - 24(2n^2+3n+1) + 54n + 54 - 27\right\}$$
$$= n\left\{16n^3 + 32n^2 + 16n - 48n^2 - 72n - 24 + 54n + 27\right\}$$
$$= n\left\{16n^3 - 16n^2 - 2n + 3\right\} \quad \text{Ans}$$

Q:6 $1.2 + 2.3 + 3.4 + \cdots + 99.100$

Sol clearly the general term is

$$T_k = k(k+1)$$
$$\Rightarrow T_k = k^2 + k$$

For sum upto n terms (later we will put $n = 99$)

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+k)$$
$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

take $\frac{n(n+1)}{2}$ as common

$$\sum_{k=1}^{n} T_k = \frac{n(n+1)}{2}\left\{\frac{2n+1}{3} + 1\right\}$$
$$= \frac{n(n+1)}{2}\left\{\frac{2n+1+3}{3}\right\}$$
$$= \frac{n(n+1)}{2}\left\{\frac{2n+4}{3}\right\}$$
$$= \frac{n(n+1)}{2} \; 2\left(\frac{n+2}{3}\right)$$
$$\sum_{k=1}^{n} T_k = \frac{n(n+1)(n+2)}{3}$$

Now put $n = 99$

$$\Rightarrow \sum_{k=1}^{99} T_k = \frac{99(99+1)(99+2)}{3}$$
$$= \frac{99(100)(101)}{3} = 33(100)(101)$$
$$= 333300 \quad \text{Ans}$$

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Q:7 $1^2 + 3^2 + 5^2 + \cdots + 99^2$

Sol As $1 + 3 + 5 + \cdots$ is A.P

$$T_k = a + (k-1)d$$
$$\Rightarrow T_k = 1 + (k-1)2$$
$$\Rightarrow T_k = 1 + 2k - 2$$
$$\Rightarrow T_k = 2k - 1$$

But the given series is $1^2 + 3^2 + 5^2 + \cdots$

Then $T_k = (2k-1)^2$

$$\Rightarrow T_k = 4k^2 - 4k + 1$$

For sum upto n terms (later we will put $n = 99$)

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(4k^2 - 4k + 1)$$
$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$\Rightarrow \sum_{k=1}^{n} T_k = 4\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n(n+1)}{2} + n$$
$$= \frac{2}{3}n(2n^2+3n+1) - 2n(n+1) + n$$
$$= n\left\{\frac{2}{3}(2n^2+3n+1) - 2(n+1) + 1\right\}$$
$$= n\left\{\frac{2(2n^2+3n+1) - 6(n+1) + 3}{3}\right\}$$
$$= \frac{n}{3}\left\{4n^2 + 6n + 2 - 6n - 6 + 3\right\}$$
$$\sum_{k=1}^{n} T_k = \frac{n}{3}\left\{4n^2 - 1\right\}$$

Now put $n = 99$

$$\Rightarrow \sum_{k=1}^{99} T_k = \frac{99}{3}\left\{4(99)^2 - 1\right\}$$
$$= 33\left\{4(99)^2 - 1\right\} \quad \text{Ans}$$

Q.8 Find the sum of

$2+(2+5)+(2+5+8)+\cdots$ to n terms

Sol The general term (say T_k) is $T_k = 2+5+8+\cdots$

$T_k = \frac{k}{2}\{2a+(k-1)d\}$

$T_k = \frac{k}{2}\{2(2)+(k-1)3\}$

$T_k = \frac{k}{2}\{4+3k-3\}$

$\Rightarrow T_k = \frac{k}{2}(3k+1)$

$\Rightarrow T_k = \frac{k}{2}(3k+1)$

For sum upto n terms

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{k}{2}(3k+1)$$

$$= \frac{1}{2}\sum_{k=1}^{n}(3k^2+k)$$

$$= \frac{1}{2}\left\{3\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k\right\}$$

$$= \frac{1}{2}\left\{3\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{2}\left\{\frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{2}\frac{n(n+1)}{2}\{(2n+1)+1\}$$

$$= \frac{n(n+1)}{4}(2n+2)$$

$$= \frac{n(n+1)}{4}2(n+1)$$

$$= \frac{n(n+1)^2}{2} \quad \text{Ans}$$

Q.9 Sum $2+5+10+17+\cdots + n$ terms

Sol The given series can be written as

$$= (1^2+1)+(2^2+1)+(3^2+1)+(4^2+1)+\cdots+(k^2+1)$$

i.e $T_k = k^2+1$

For sum upto n terms

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^2+1)$$

$$= \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}1$$

$$= \frac{n(n+1)(2n+1)}{6} + n$$

$$= n\left\{\frac{(n+1)(2n+1)}{6}+1\right\}$$

$$= n\left\{\frac{2n^2+n+2n+1+6}{6}\right\}$$

$$= \frac{n}{6}\{2n^2+n+2n+7\}$$

$$= \frac{n}{6}\{2n^2+3n+7\} \quad \text{Ans}$$

Q.10 $1.3.5 + 2.4.6 + 3.5.7 + \cdots + $ upto n. terms

Sol First consider

$1+2+3+\cdots$ & $3+4+5+\cdots$ & $5+6+7+\cdots$

All are Arithmetic series

$a=1 \quad d=1$	$a=3 \quad d=1$	$a=5 \quad d=1$
$T_k = a+(k-1)d$	$T_k = a+(k-1)d$	$T_k = a+(k-1)d$
$= 1+(k-1)1$	$= 3+(k-1)1$	$= 5+(k-1)1$
$= 1+k-1$	$= 3+k-1$	$= 5+k-1$
$= k$	$= k+2$	$= k+4$

So the given series is

$$1.3.5 + 2.4.6 + 3.5.7 + \cdots + k(k+2)(k+4)$$

i.e $T_k = k(k+2)(k+4)$

$\Rightarrow T_k = k(k^2+4k+2k+8)$

$\Rightarrow T_k = k(k^2+6k+8)$

$\Rightarrow T_k = k^3+6k^2+8k$

For sum upto n terms

$$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(k^3+6k^2+8k)$$

$$= \sum_{k=1}^{n} k^3 + 6\sum_{k=1}^{n} k^2 + 8\sum_{k=1}^{n} k$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 6\,\frac{n(n+1)(2n+1)}{6} + 8\,\frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{4} + 6\,\frac{n(n+1)(2n+1)}{6} + 4n(n+1)$$

Take $n(n+1)$ as common

$$= n(n+1)\left\{\frac{n(n+1)}{4} + 2n+1 + 4\right\}$$

$$= n(n+1)\left\{\frac{n(n+1)+4(2n+1)+16}{4}\right\}$$

$$= n(n+1)\left\{\frac{n^2+n+8n+4+16}{4}\right\}$$

$$= \frac{n(n+1)}{4}\{n^2+9n+20\}$$ Ans

OR

$$= \frac{n(n+1)}{4}\{n^2+5n+4n+20\}$$

$$= \frac{n(n+1)}{4}\{n(n+5)+4(n+5)\}$$

$$= \frac{n(n+1)}{4}\{(n+5)(n+4)\}$$

$$= \frac{n(n+1)(n+4)(n+5)}{4}$$ Ans

Q:11 $1.3.5 + 3.5.7 + 5.7.9 + \dots\dots\dots$ n terms

Sol 1st considre the series of the first number of each term

i.e $1 + 3 + 5 + \dots$ which is Arithmetic $a=1$ & $d=2$

$T_k = a+(k-1)d$

$= 1+(k-1)2$

$= 1+2k-2$

$= 2k-1$

Now considre the series of the 2nd numbers

$3+5+7+\dots$

$T_k = a+(k-1)d$

$= 3+(k-1)2$

$= 3+2k-2$

$= 2k+1$

Similarly $5+7+9+\dots$

$T_k = a+(k-1)d$

$= 5+(k-1)2$

$= 5+2k-2$

$= 2k+3$

Then the general term of the series $1.3.5+3.5.7+5.7.9+\dots$ is

$T_k = (2k-1)(2k+1)(2k+3)$

$\Rightarrow T_k = \{(2k)^2-1^2\}(2k+3)$

$\Rightarrow T_k = (4k^2-1)(2k+3)$

$\Rightarrow T_k = 8k^3+12k^2-2k-3$

For sum upto n terms

$$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(8k^3+12k^2-2k-3)$$

$$= 8\sum_{k=1}^{n} k^3 + 12\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k - 3\sum_{k=1}^{n} 1$$

$$= 8\,\frac{n^2(n+1)^2}{4} + 12\,\frac{n(n+1)(2n+1)}{6} - 2\,\frac{n(n+1)}{2} - 3n$$

$$= 2n^2(n+1)^2 + 2n(n+1)(2n+1) - n(n+1) - 3n$$

$$= 2n^2(n^2+1+2n) + 2n(2n^2+3n+1) - n^2 - n - 3n$$

$$= 4n^4+2n^2+4n^3+4n^3+6n^2+2n - n^2-n-3n$$

$$= 4n^4+8n^3+7n^2-2n$$ Ans

158

Q12 Find the sum of 2n terms of the series whose nth term is

$T_n = 4n^2+5n+1$

Sol Let T_k be the general term

$T_k = 4k^2+5k+1$

For sum upto 2n terms

$\Rightarrow \sum_{k=1}^{2n} T_k = \sum_{k=1}^{2n}(4k^2+5k+1)$

$= 4\sum_{k=1}^{2n}k^2 + 5\sum_{k=1}^{2n}k + \sum_{k=1}^{2n}1$

$= 4\,\frac{2n(2n+1)(2(2n)+1)}{6} + 5\,\frac{2n(2n+1)}{2} + 2n$

$= \frac{8n(2n+1)(4n+1)}{6} + 5n(2n+1) + 2n$

$= \frac{8n(8n^2+2n+4n+1)+30n(2n+1)+12n}{6}$

take 2n as common

$= 2n\,\frac{\{4(8n^2+2n+4n+1)+15(2n+1)+6\}}{6}$

$= \frac{2n}{6}\{4(8n^2+6n+1)+15(2n+1)+6\}$

$= \frac{n}{3}\{32n^2+24n+4+30n+15+6\}$

$= \frac{n}{3}\{32n^2+54n+25\}$ Ans

Q13 Find the sum of n terms of the series whose nth term is

(i) $n^2(2n+3)$

Sol Given $T_n = n^2(2n+3)$

Let T_k be the general term

$T_k = k^2(2k+3)$

$\Rightarrow T_k = 2k^3+3k^2$

For sum upto n terms

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(2k^3+3k^2) = 2\sum_{k=1}^{n}k^3 + 3\sum_{k=1}^{n}k^2$

$= 2\,\frac{n^2(n+1)^2}{4} + 3\,\frac{n(n+1)(2n+1)}{6}$

$\Rightarrow \sum_{k=1}^{n} T_k = \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{2}$

$= \frac{n(n+1)}{2}\{n(n+1)+2n+1\}$

$= \frac{n(n+1)}{2}\{n^2+3n+1\}$

(ii) $3(4^n+2n^2)-4n^3$

Sol $T_n = 3(4^n+2n^2)-4n^3$

Let T_k be the general term

$\Rightarrow T_k = 3(4^k+2k^2)-4k^3$

$\Rightarrow T_k = 3\cdot 4^k+6\cdot k^2-4k^3$

For sum upto n terms

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n}(3\cdot 4^k+6\cdot k^2-4k^3)$

$= 3\sum_{k=1}^{n}4^k + 6\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k^3$

$= 3\{4^1+4^2+4^3+\dots+4^n\} + 6\,\frac{n(n+1)(2n+1)}{6} - 4\,\frac{n^2(n+1)^2}{4}$

$= 3\left\{a\left(\frac{r^n-1}{r-1}\right)\right\} + n(n+1)(2n+1) - n^2(n+1)^2$

$= 3\left\{4\left(\frac{4^n-1}{4-1}\right)\right\} + n(n+1)(2n+1) - n^2(n+1)^2$

$= 3\left\{\frac{4(4^n-1)}{3}\right\} + n(n+1)(2n+1) - n^2(n+1)^2$

$= 4(4^n-1) + \underbrace{n(n+1)}_{\text{common}}\{(2n+1)-n(n+1)\}$

$= 4\cdot 4^n-4 + n(n+1)\{2n+1-n^2-n\}$

$= 4^{n+1}-4 + n(n+1)(n+1-n^2)$

$= 4^{n+1}-4 + n(n+1)(-1)(-n-1+n^2)$

$= 4^{n+1}-4 - n(n+1)(n^2-n-1)$ A

Exercise # 5.2

Sum to n terms the following series

Q.1 $1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots\cdots$ (It is Arithmetical geometric series)

Sol Let S_n be the sum

$S_n = 1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots\cdots + n.2^n \longrightarrow (1)$

Multiply by 2, we get (Because the ratio of the G.P is 2)

$2S_n = 2(1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots\cdots + n.2^n)$

$\Rightarrow 2S_n = 1.2^2 + 2.2^3 + 3.2^4 + \cdots\cdots + n.2^{n+1} \longrightarrow (2)$

Eqn (2) − Eqn (1)

$$\begin{aligned}\Rightarrow\ S_n &= 1.2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots\cdots + n.2^n \\ \underline{-2S_n} &= \underline{\ \ \ \ \ \ \ -1.2^2 \pm 2.2^3 \pm 3.2^4 + \cdots\cdots \pm (n-1)2^n \pm n.2^{n+1}}\end{aligned}$$

$\Rightarrow S_n - 2S_n = 1.2 + 1.2^2 + 1.2^3 + \cdots\cdots + 1.2^n - n2^{n+1}$

$\Rightarrow -S_n = (2 + 2^2 + 2^3 + \cdots\cdots + 2^n) - n.2^{n+1}$

$\Rightarrow -S_n = a\left(\dfrac{1-r^n}{1-r}\right) - n\,2^{n+1}$

$\Rightarrow -S_n = 2\left(\dfrac{1-2^n}{1-2}\right) - n\,2^{n+1}$

$\Rightarrow -S_n = \dfrac{2(1-2^n)}{-1} - n\,2^{n+1}$

$\Rightarrow -S_n = -2(1-2^n) - n.2^{n+1}$

Xing by −1, we get

$\Rightarrow S_n = 2(1-2^n) + n.2^{n+1}$

$\Rightarrow S_n = 2 - 2.2^n + n.2^{n+1}$

$S_n = 2 - 2^{n+1} + n.2^{n+1}$

$\Rightarrow S_n = 2 + 2^{n+1}(-1+n)$

$\Rightarrow \boxed{S_n = 2 + 2^{n+1}(n-1)}$ Ans

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Q.2 $1 + 2.2 + 3.2^2 + 4.2^3 + \cdots\cdots$

Sol It is Arithmetical geometric series

Let S_n be the sum

$\Rightarrow S_n = 1 + 2.2 + 3.2^2 + 4.2^3 + \cdots\cdots + n.2^{n-1} \longrightarrow (1)$

Multiply b.s by 2 ($\because$ the ratio of the G.P is 2)

$\Rightarrow 2S_n = 2 + 2.2^2 + 3.2^3 + 4.2^4 + \cdots\cdots (n-1)2^{n-1} + n.2^n \longrightarrow (2)$

Eqn (1) − Eqn (2)

$$\begin{aligned} S_n &= 1 + 2.2 + 3.2^2 + 4.2^3 + \cdots\cdots + n.2^{n-1} \\ \underline{-2S_n} &= \underline{\ \ \ \ -2 \pm 2.2^2 \pm 3.2^3 \pm \cdots \pm (n-1)2^{n-1} \mp n.2^n} \end{aligned}$$

$S_n - 2S_n = 1 + 1.2 + 1.2^2 + 1.2^3 + \cdots\cdots + 1.2^{n-1} - n.2^n$

$\Rightarrow -S_n = (1 + 2 + 2^2 + 2^3 + \cdots\cdots 2^{n-1}) - n\,2^n$

It is G. series

$\Rightarrow -S_n = \dfrac{a(1-r^n)}{1-r} - n.2^n$

$\Rightarrow -S_n = \dfrac{1(1-2^n)}{1-2} - n.2^n$

$\Rightarrow -S_n = \dfrac{(1-2^n)}{-1} - n.2^n$

Multiply by −1, we get

$\Rightarrow S_n = (1-2^n) + n.2^n$

$\Rightarrow S_n = 1 + n.2^n - 2^n$

$\Rightarrow \boxed{S_n = 1 + (n-1)2^n}$ Ans

Q.3 $1 + 4x + 7x^2 + 10x^3 + \cdots\cdots$

Sol It is Arithmetical geometrical series.

Let S_n be the sum

$S_n = 1 + 4x + 7x^2 + 10x^3 + \cdots (3n-2)x^{n-1} \longrightarrow (1)$

Multiply by x, we get

$xS_n = x + 4x^2 + 7x^3 + \cdots\cdots (3n-5)x^{n-1}\ \ (3n-2)x^n \longrightarrow (2)$

Eqn (1) − Eqn (2), we get

$\Rightarrow S_n = 1 + 4x + 7x^2 + 10x^3 + \cdots + (3n-2)x^{n-1}$

$-xS_n = -x + 4x^2 + 7x^3 + \cdots (3n-5)x^{n-1} + (3n-2)x^n$

$S_n - xS_n = 1 + 3x + 3x^2 + 3x^3 + \cdots + 3x^{n-1} - (3n-2)x^n$

$\Rightarrow S_n - xS_n = 1 + (3x + 3x^2 + 3x^3 + \cdots + 3x^{n-1}) - (3n-2)x^n$

$\Rightarrow S_n(1-x) = 1 + a\left(\frac{1-r^{n-1}}{1-r}\right) - (3n-2)x^n$

$\Rightarrow S_n(1-x) = 1 + 3x\left(\frac{1-x^{n-1}}{1-x}\right) - (3n-2)x^n$

Divide by $1-x$

$\Rightarrow S_n = \frac{1}{1-x} + \frac{3x(1-x^{n-1})}{(1-x)^2} - \frac{(3n-2)x^n}{1-x}$ Ans

Q:4 $1 + 2x + 3x^2 + 4x^3 + \cdots$

Sol Let $S_n = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} \longrightarrow ①$

Multiply by x ($\because$ ratio of the geometric part is x)

$\Rightarrow xS_n = x + 2x^2 + 3x^3 + 4x^4 + \cdots (n-1)x^{n-1} + nx^n \longrightarrow ②$

Eqn ① − Eqn ②

$S_n = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1}$

$-xS_n = -x - 2x^2 - 3x^3 - \cdots - (n-1)x^{n-1} - nx^n$

$S_n - xS_n = (1 + x + x^2 + x^3 + \cdots x^{n-1}) - nx^n$

$S_n(1-x) = a\left(\frac{1-r^n}{1-r}\right) - nx^n$

$\Rightarrow S_n(1-x) = \frac{1(1-x^n)}{1-x} - nx^n$

Divide by $(1-x)$

$\Rightarrow S_n = \frac{1-x^n}{(1-x)^2} - \frac{nx^n}{(1-x)}$ Ans

Q:5 $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots$

Sol Let S_n be the sum

$S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots + \frac{(3n-2)}{5^{n-1}} \longrightarrow ①$

Multiply both sides by $\frac{1}{5}$, we get

$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n} \longrightarrow ②$

Eqn ① − Eqn ②

$S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots + \frac{(3n-2)}{5^{n-1}}$

$-\frac{1}{5}S_n = -\frac{1}{5} - \frac{4}{5^2} - \frac{7}{5^3} - \cdots - \frac{(3n-5)}{5^{n-1}} - \frac{(3n-2)}{5^n}$

$\Rightarrow S_n - \frac{1}{5}S_n = 1 + \left(\frac{4}{5} - \frac{1}{5}\right) + \left(\frac{7}{5^2} - \frac{4}{5^2}\right) + \left(\frac{10}{5^3} - \frac{7}{5^3}\right) + \cdots \left(\frac{3n-2}{5^{n-1}} - \frac{3n-5}{5^{n-1}}\right) - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \cdots + \frac{3}{5^{n-1}} - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + \left(\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \cdots + \frac{3}{5^{n-1}}\right) - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + a\left(\frac{1-r^n}{1-r}\right) - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + \frac{\frac{3}{5}\left(1 - \left(\frac{1}{5}\right)^{n-1}\right)}{1 - \frac{1}{5}} - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + \frac{\frac{3}{5}\left(1 - \frac{1}{5^{n-1}}\right)}{4/5} - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + \frac{3}{4}\left(1 - \frac{1}{5^{n-1}}\right) - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = 1 + \frac{3}{4} - \frac{3}{4}\cdot\frac{1}{5^{n-1}} - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{3}{4}\cdot\frac{1}{5^n \cdot 5^{-1}} - \left(\frac{3n-2}{5^n}\right)$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left\{\frac{3}{4\cdot 5^{-1}} - (3n-2)\right\}$

60

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left\{\frac{3.5}{4}\right.$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left(\frac{15}{4} - (3n-2)\right)$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left(\frac{15}{4} - 3n+2\right)$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left(\frac{15}{4}+2 - 3n\right)$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left(\frac{15+8}{4} - 3n\right)$

$\Rightarrow \frac{4}{5}S_n = \frac{7}{4} - \frac{1}{5^n}\left(\frac{23}{4} - 3n\right)$

$\Rightarrow S_n = \frac{5}{4}\left\{\frac{7}{4} - \frac{1}{5^n}\left(\frac{23}{4} - 3n\right)\right\}$

$\Rightarrow S_n = \frac{5}{4}\left\{\frac{7}{4} - \frac{1}{5^n}\left(\frac{23-12n}{4}\right)\right\}$

$\Rightarrow S_n = \frac{5}{4}\cdot\frac{7}{4} - \frac{5}{4}\cdot\frac{1}{5^n}\left(\frac{23-12n}{4}\right)$

$\Rightarrow S_n = \frac{35}{16} - \frac{5}{16}\left(\frac{23-12n}{5^n}\right)$ Ans

Note
$1+7+13+\cdots$
$a=1, d=6$

Q:6 $1 - 7x + 13x^2 - 19x^3 + \cdots\cdots$

Sol Since it is Arithmetical geometric series

Let $S_n = 1 - 7x + 13x^2 - 19x^3 + \cdots\cdots + (-x)^{n-1}(6n-5)$ ①

The ratio of the geometrical series is $-x$

so multiply by $-x$, we get

$-xS_n = -x + 7x^2 - 13x^3 + 19x^4 + \cdots (-x)^{n-1}(6n-11) + (-x)^n(6n-5)$ ②

Eqn ① - Eqn ②

$$\begin{aligned} S_n &= 1 - 7x + 13x^2 - 19x^3 + \cdots\cdots + (-x)^{n-1}(6n-5) \\ -xS_n &= -x + 7x^2 - 13x^2 + \cdots\cdots + (-x)^{n-1}(6n-11) + (-x)^n(6n-5) \end{aligned}$$

$\Rightarrow S_n + xS_n = 1 - 6x + 6x^2 - 6x^3 + \cdots\cdots + 6(-x)^{n-1} - (-x)^n(6n-5)$

$\Rightarrow S_n(1+x) = 1 + \{6(-x) + 6(-x)^2 + 6(-x)^3 + \cdots\cdots + (-x)^{n-1}\} - (6n-5)(-x)^n$

$\Rightarrow S_n(1+x) = 1 + \frac{a(1-r^n)}{1-r} - (6n-5)(-x)^n$

$= 1 + \frac{-6x(1-(-x)^{n-1})}{1-(-x)} - (6n-5)(-x)^n$

CH-5
P-5

$\Rightarrow S_n(1+x) = 1 - \frac{6x(1-(-x)^{n-1})}{1+x} - (6n-5)(-x)^n$

$\div$ by $(1+x)$

$\Rightarrow S_n = \frac{1}{1+x} - \frac{6x(1-(-x)^{n-1})}{(1+x)^2} - \frac{(6n-5)(-x)^n}{1+x}$ Ans

Q:7 $1^2 + 3^2x + 5^2x^2 + 7^3x^3 + \cdots\cdots$ $x < 1$

Sol Let S be the sum of the infinite series

$\Rightarrow S = 1^2 + 3^2x + 5^2x^2 + 7^3x^3 + \cdots\cdots$

$S = 1 + 9x + 25x^2 + 49x^3 + \cdots\cdots$ ① multiply by x

$\Rightarrow xS = x + 9x^2 + 25x^3 + 49x^4 + \cdots\cdots$ ②

Eqn ① - Eqn ②

$$\begin{aligned} S &= 1 + 9x + 25x^2 + 49x^3 + \cdots\cdots \\ xS &= \quad x + 9x^2 + 25x^3 + \cdots\cdots \end{aligned}$$

$S - xS = 1 + 8x + 16x^2 + 24x^3 + \cdots\cdots$

$\Rightarrow S(1-x) = 1 + 8x + 16x^2 + 24x^3 + \cdots\cdots$ ③

Again multiply by x

$\Rightarrow Sx(1-x) = x + 8x^2 + 16x^3 + 24x^4 + \cdots\cdots$ ④

Eqn ④ - Eqn ③

$$\begin{aligned} S(1-x) &= 1 + 8x + 16x^2 + 24x^3 + \cdots\cdots \\ Sx(1-x) &= \quad x + 8x^2 + 16x^3 + \cdots\cdots \end{aligned}$$

$S(1-x) - Sx(1-x) = 1 + 7x + 8x^2 + 8x^3 + 8x^4 + \cdots\cdots$

$S(1-x)\{1-x\} = 1 + 7x + 8x^2 + 8x^3 + 8x^4 + \cdots\cdots$

$\Rightarrow S(1-x)^2 = 1 + 7x + 8x^2(1 + x + x^2 + \cdots\cdots)$

Apply $S_\infty = \frac{a}{1-r}$ formula

$\Rightarrow S(1-x)^2 = 1 + 7x + 8x^2\left(\frac{1}{1-x}\right)$

$\div$ by $(1-x)^2$

$\Rightarrow S = \frac{1+7x}{(1-x)^2} + \frac{8x^2}{(1-x)^3}$ Ans

Q:8 $1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$

Sol Let S be the sum of infinite geometric series

$\Rightarrow S = 1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$ ①

multiply both sides by $\frac{1}{3}$, we get

$\Rightarrow \frac{1}{3}S = \frac{1}{3}+\frac{4}{3^2}+\frac{9}{3^3}+\frac{16}{3^4}+\cdots$ ②

Eqn ① − Eqn ②

$$S = 1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\frac{25}{3^4}+\cdots$$
$$\underline{-\frac{1}{3}S = \quad -\frac{1}{3}-\frac{4}{3^2}-\frac{9}{3^3}-\frac{16}{3^4}+\cdots}$$

$$S-\frac{1}{3}S = 1+\left(\frac{4}{3}-\frac{1}{3}\right)+\left(\frac{9}{3^2}-\frac{4}{3^2}\right)+\left(\frac{16}{3^3}-\frac{9}{3^3}\right)+\left(\frac{25}{3^4}-\frac{16}{3^4}\right)+\cdots$$

$\Rightarrow \frac{3S-S}{3} = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\cdots$

$\Rightarrow \frac{2}{3}S = 1+\frac{3}{3}+\frac{5}{3^2}+\frac{7}{3^3}+\cdots$ ③

Again multiply by $1/3$, we get

$\Rightarrow \frac{1}{3}\left(\frac{2}{3}S\right) = \frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\frac{7}{3^4}+\cdots$

$\Rightarrow \frac{2}{9}S = \frac{1}{3}+\frac{3}{3^2}+\frac{5}{3^3}+\frac{7}{3^4}+\cdots$ ④

Eqn ③ − Eqn ④

$\Rightarrow \frac{2}{3}S - \frac{2}{9}S = 1+\left(\frac{3}{3}-\frac{1}{3}\right)+\left(\frac{5}{3^2}-\frac{3}{3^2}\right)+\left(\frac{7}{3^3}-\frac{5}{3^3}\right)+\cdots$

$\Rightarrow \frac{6S-2S}{9} = 1+\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots$

$\Rightarrow \frac{4S}{9} = 1+\left(\frac{2}{3}+\frac{2}{3^2}+\frac{2}{3^3}+\cdots\right)$ Apply $S_\infty = \frac{a}{1-r}$

$\Rightarrow \frac{4S}{9} = 1+\frac{2}{3}\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)$

$\frac{4S}{9} = 1+\frac{2}{3}\left(\frac{a}{1-r}\right)$

$\frac{4S}{9} = 1+\frac{2}{3}\left(\frac{1}{1-\frac{1}{3}}\right)$

162

$\Rightarrow \frac{4S}{9} = 1+\frac{2}{3}\left(\frac{1}{2/3}\right)$

$\Rightarrow \frac{4S}{9} = 1+1$

$\Rightarrow \frac{4S}{9} = 2$

$\Rightarrow S = \frac{18}{4}$

$\Rightarrow S = 9/2$

Hence $\boxed{1+\frac{4}{3}+\frac{9}{3^2}+\frac{16}{3^3}+\cdots = 9/2}$ Ans

Q:9 Find the nth term of the Arithmetico-geometric sequence

$5, \frac{7}{3}, 1, \frac{11}{27}, \ldots$

Sol The above sequence can be written as

$5, \frac{7}{3}, \frac{9}{3^2}, \frac{11}{3^3}, \ldots$

The A.P is $5, 7, 9, 11, \ldots$ & G.P is $1, \frac{1}{3}, \frac{1}{3^2}, \ldots$

$a = 5,\ d = 2$

$a_n = a+(n-1)d$

$= 5+(n-1)2$

$= 2n+3$

$a = 1,\ r = 1/3$

$a_n = ar^{n-1}$

$\Rightarrow a_n = 1\left(\frac{1}{3}\right)^{n-1}$

$\Rightarrow a_n = \left(\frac{1}{3}\right)^{n-1}$

Now the nth term of Arithmetico-geometric sequence is

$a_n = (a_n \text{ of A.P})(a_n \text{ of G.P})$

$a_n = (2n+3)\left(\frac{1}{3}\right)^{n-1}$ Ans

Q:10 Find the sum of the following Arithmetico-geometric series

$$5+\frac{7}{3}+1+\frac{11}{27}+\cdots$$

Sol The given series can be written as

$$\frac{5}{1}+\frac{7}{3}+\frac{9}{3^2}+\frac{11}{3^3}+\cdots$$

The Arithmetic series is $5+7+9+\cdots$ & Geometric series is $1+\frac{1}{3}+\frac{1}{3^2}+\cdots$

$\Rightarrow a=5$, $d=2$; $a=1$, $r=1/3$

As we know that the sum of infinite Arithmetical geometrical series is $S_\infty=\frac{a}{1-r}+\frac{dr}{(1-r)^2}$

$$\Rightarrow S_\infty=\frac{5}{1-\frac{1}{3}}+\frac{2\left(\frac{1}{3}\right)}{\left(1-\frac{1}{3}\right)^2}$$

$$\Rightarrow S_\infty=\frac{5}{2/3}+\frac{2/3}{(2/3)^2}$$

$$\Rightarrow S_\infty=\frac{15}{2}+\frac{1}{(2/3)}$$

$$\Rightarrow S_\infty=\frac{15}{2}+\frac{3}{2}$$

$$\Rightarrow S_\infty=\frac{18}{2}$$

$$\Rightarrow \boxed{S_\infty=9} \text{ Ans}$$

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Exercise # 5.3

Q: Find the sum of the following series

Q:1 $1.3.5+2.4.6+3.5.7+\cdots$ to n terms

Sol The general term of the series is $U_n=n(n+2)(n+4)$

By formula

$$\sum U_n=\frac{(n\text{th term})\times\text{next factor}}{(\text{number of factors in each term}+1)\times\text{c.d}}+C$$

$$\Rightarrow \sum U_n=\frac{n(n+2)(n+4)\times(n+6)}{(3+1)\,2}+C$$

$$\Rightarrow \sum U_n=\frac{n(n+2)(n+4)(n+6)}{8}+C$$

or

$$S_n=\frac{n(n+2)(n+4)(n+6)}{8}+C \longrightarrow (A)$$

where C is independent of n and can be calculated by taking any value of n

Let $n=1$

$$\Rightarrow S_1=\frac{1(1+2)(1+4)(1+6)}{8}+C$$

$$\Rightarrow 1.3.5=\frac{1(3)(5)(7)}{8}+C$$

$$\Rightarrow 15=\frac{15\times7}{8}+C \Rightarrow 1=\frac{7}{8}+C$$

$$\Rightarrow 1-\frac{7}{8}=C$$

$$\Rightarrow \frac{1}{8}=C$$

P.T.V in eqn (A), we get

$$S_n=\frac{n(n+2)(n+4)(n+6)}{8}+\frac{1}{8}$$

Q:2 $\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\cdots\cdots+$ infinity

Sol Here $a_n=\frac{1}{(3n-2)(3n+1)}$

By formula

$$S_n = c - \frac{\text{nth term with a factor from beginning neglected}}{(\text{number of factors in each term} - 1)\times \text{c.d}}$$

$$S_n = c - \frac{\frac{1}{(3n+1)}}{(2-1)3}$$

$$\Rightarrow S_n = c - \frac{1}{3(3n+1)} \longrightarrow (A)$$

To find c put $n=1$

$$\Rightarrow S_1 = c - \frac{1}{3\{3(1)+1\}}$$

$$\Rightarrow \frac{1}{1.4} = c - \frac{1}{12} \Rightarrow \frac{1}{4}+\frac{1}{12}=c$$

$$\Rightarrow \frac{3+1}{12}=c \Rightarrow \boxed{\frac{1}{3}=c}$$

Eqn (A) $\Rightarrow S_n = \frac{1}{3} - \frac{1}{3(3n+1)}$

If $n\longrightarrow\infty$

$$S_\infty = \frac{1}{3} - \frac{1}{3\{3(\infty)+1\}}$$

$$S_\infty = \frac{1}{3} - \frac{1}{\infty} \Rightarrow S_\infty = \frac{1}{3} - 0 \Rightarrow \boxed{S_\infty = \frac{1}{3}} \text{ Ans}$$

Q:3 $\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\cdots\cdots\cdots$ infinity

Sol Here $a_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$

By formula

$$S_n = c - \frac{(\text{nth term with a factor from beginning neglected})}{(\text{number of factors in each term} - 1)\times \text{c.d}}$$

$$\Rightarrow S_n = c - \frac{\frac{1}{(2n+1)(2n+3)}}{(3-1)\times 2}$$

$$\Rightarrow S_n = c - \frac{1}{4(2n+1)(2n+3)} \longrightarrow (A)$$

to find c, put $n=1$

$$\Rightarrow S_1 = c - \frac{1}{4\{2(1)+1\}\{2(1)+3\}}$$

$$\Rightarrow \frac{1}{1.3.5} = c - \frac{1}{4(3)(5)}$$

$$\Rightarrow \frac{1}{15} = c - \frac{1}{60} \Rightarrow c = \frac{1}{15}+\frac{1}{60} \Rightarrow c = \frac{4+1}{60} \Rightarrow c = 5/60$$

$$\Rightarrow \boxed{c = 1/12}$$

Eqn (A) $\Rightarrow S_n = \frac{1}{12} - \frac{1}{4\{(2n+1)(2n+3)\}}$

If $n\longrightarrow\infty$

$$\Rightarrow S_\infty = \frac{1}{12} - \frac{1}{4\{2(\infty)+1\}\{2(\infty)+3\}}$$

$$\Rightarrow S_\infty = \frac{1}{12} - \frac{1}{\infty} \Rightarrow S_\infty = \frac{1}{12} - 0 \Rightarrow \boxed{S_\infty = \frac{1}{12}} \text{ Ans}$$

Q:4 $1.4.7 + 4.7.10 + 7.10.13 + \cdots\cdots$ to n terms.

Sol $a_n = (3n-2)(3n+1)(3n+4)$

By formula

$$S_n = \frac{a_n \times \text{next factor}}{(\text{number of factors in each term}+1)\times \text{c.d}} + C$$

$$\Rightarrow S_n = \frac{(3n-2)(3n+1)(3n+4)(3n+7)}{(3+1)\,3} + C$$

$$\Rightarrow S_n = \frac{(3n-2)(3n+1)(3n+4)(3n+7)}{12} + C \longrightarrow (A)$$

To find c put $n=1$

$$\Rightarrow S_1 = \frac{\{3(1)-2\}\{3(1)+1\}\{3(1)+4\}\{3(1)+7\}}{12} + c$$

$$\Rightarrow 1.4.7 = \frac{1\cdot 4\cdot 7\cdot 10}{12} + C$$

164

$$\Rightarrow 28 = \frac{280}{12} + c$$

$$\Rightarrow c = 28 - \frac{280}{12}$$

$$\Rightarrow c = \frac{336-280}{12} \Rightarrow c = \frac{56}{12} \Rightarrow c = \frac{14}{3}$$

Eqn (A) $\Rightarrow$ $S_n = \dfrac{(3n-2)(3n+1)(3n+4)(3n+7)}{12} + \dfrac{14}{3}$ Ans

Q:5 $1.5.9 + 2.6.10 + 3.7.11 + \cdots\cdots$ to n terms

Sol For this series $a_n = n(n+4)(n+8)$

As $S_n = \dfrac{a_n \times \text{next term}}{(\#\text{ of factors in each term} + 1) \times c.d} + c$

$$\Rightarrow S_n = \frac{n(n+4)(n+8) \times (n+12)}{(3+1)\times 4} + c$$

$$\Rightarrow S_n = \frac{n(n+4)(n+8)(n+12)}{16} + c \longrightarrow (A)$$

Let $n=1$

$$\Rightarrow S_1 = \frac{1(1+4)(1+8)(1+12)}{16} + c$$

$$\Rightarrow 1.5.9 = \frac{1(5)(9)(13)}{16} + c$$

$$\Rightarrow 45 = \frac{585}{16} + c \Rightarrow c = 45 - \frac{585}{16}$$

$$\Rightarrow c = \frac{720-585}{16} \Rightarrow \boxed{c = \frac{135}{16}}$$

Eqn (A) $\Rightarrow$ $S_n = \dfrac{n(n+4)(n+8)(n+12)}{16} + \dfrac{135}{16}$ Ans

Q:6 $\dfrac{4}{1.2.3} + \dfrac{5}{2.3.4} + \dfrac{6}{3.4.5} + \cdots\cdots + n$ terms

Sol $a_n = \dfrac{n+3}{n(n+1)(n+2)}$

$$\Rightarrow a_n = \frac{n}{n(n+1)(n+2)} + \frac{3}{n(n+1)(n+2)}$$

$$\Rightarrow a_n = \frac{1}{(n+1)(n+2)} + \frac{3}{n(n+1)(n+2)}$$

As $S_n = C - \dfrac{n\text{th term with 1st factor neglected}}{(\text{number of factors in each term} - 1) \times c.d}$

$$\Rightarrow S_n = c - \left\{ \frac{\frac{1}{n+2}}{(2-1)1} + \frac{\frac{3}{(n+1)(n+2)}}{(3-1)1} \right\}$$

$$\Rightarrow S_n = c - \left\{ \frac{1}{n+2} + \frac{3}{2(n+1)(n+2)} \right\}$$

$$\Rightarrow S_n = c - \left\{ \frac{2(n+1) + 3}{2(n+1)(n+2)} \right\}$$

$$\Rightarrow S_n = c - \left\{ \frac{2n+5}{2(n+1)(n+2)} \right\} \longrightarrow (A)$$

To find c put $n=1$

$$\Rightarrow S_1 = c - \left\{ \frac{2(1)+5}{2(1+1)(1+2)} \right\}$$

$$\Rightarrow \frac{4}{1.2.3} = c - \left\{ \frac{7}{12} \right\} \Rightarrow \frac{4}{6} = c - \frac{7}{12} \Rightarrow c = \frac{4}{6} + \frac{7}{12} = \frac{8+7}{12}$$

$$\Rightarrow c = \frac{15}{12} \Rightarrow \boxed{c = 5/4}$$

Eqn (A) $\Rightarrow$

$$S_n = \frac{5}{4} - \left\{ \frac{2n+5}{2(n+1)(n+2)} \right\} \text{ Ans}$$

Q.7 $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\frac{7}{4.5.6}+\cdots\cdots$ infinity

Sol: $a_n = \frac{2n-1}{n(n+1)(n+2)}$

$\Rightarrow a_n = \frac{2n}{n(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}$

$\Rightarrow a_n = \frac{2}{(n+1)(n+2)} - \frac{1}{n(n+1)(n+2)}$

Now $S_n = c - \left\{ \frac{\frac{2}{n+2}}{(2-1)1} - \frac{\frac{1}{(n+1)(n+2)}}{(3-1)1} \right\}$

$\Rightarrow S_n = c - \left\{ \frac{2}{n+2} - \frac{1}{2(n+1)(n+2)} \right\} \longrightarrow (A)$

Put $n=1$

$\Rightarrow S_1 = c - \left\{ \frac{2}{1+2} - \frac{1}{2(1+1)(1+2)} \right\}$

$\Rightarrow \frac{1}{1.2.3} = c - \left\{ \frac{2}{3} - \frac{1}{12} \right\}$

$\Rightarrow \frac{1}{6} = c - \frac{2}{3} + \frac{1}{12} \Rightarrow \frac{1}{6} = c + \frac{-8+1}{12}$

$\Rightarrow \frac{1}{6} = c - \frac{7}{12} \Rightarrow c = \frac{1}{6} + \frac{7}{12}$

$c = \frac{2+7}{12} = 9/12 = 3/4$

$\Rightarrow \boxed{c = 3/4}$

Eq (A) $\Rightarrow S_n = \frac{3}{4} - \left\{ \frac{2}{n+2} - \frac{1}{2(n+1)(n+2)} \right\}$

If $n \to \infty$

$S_\infty = \frac{3}{4} - \left\{ \frac{2}{\infty} - \frac{1}{\infty} \right\} \Rightarrow S_\infty = \frac{3}{4} - \{0-0\}$

$\Rightarrow \boxed{S_\infty = 3/4}$ Ans

Q.8 Find the sum of n terms whose general term is $(n^2+5n+4)(n^2+5n+8)$.

Sol

$a_n = (n^2+5n+4)(n^2+5n+8)$

$\Rightarrow a_n = n^4+5n^3+8n^2+5n^3+25n^2+40n+4n^2+20n+32$

$\Rightarrow a_n = n^4+10n^3+37n^2+60n+32$

For sum upto n term

$\sum_{n=1}^{n} a_n = \sum (n^4+10n^3+37n^2+60n+32)$

$\Rightarrow S_n = \sum_{n=1}^{n} n^4 + 10\sum_{n=1}^{n} n^3 + 37\sum_{n=1}^{n} n^2 + 60\sum_{n=1}^{n} n + 32\sum_{n=1}^{n} 1$

$\Rightarrow S_n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + 10\frac{n^2(n+1)^2}{4} + 37\frac{n(n+1)(2n+1)}{6} + 60\frac{n(n+1)}{2} + 32n$

take n as common

$\Rightarrow S_n = n\left\{ \frac{(n+1)(2n+1)(3n^2+3n-1)}{30} + \frac{5n(n+1)^2}{2} + \frac{37(n+1)(2n+1)}{6} + 30(n+1) + 32 \right\}$

$\Rightarrow S_n = n\left\{ \frac{(n+1)(2n+1)(3n^2+3n-1) + 75n(n+1)^2 + 185(n+1)(2n+1) + 900(n+1) + 960}{30} \right\}$

$\Rightarrow S_n = \frac{n\{(n+1)(2n+1)(3n^2+3n-1) + 75n(n+1)^2 + 185(n+1)(2n+1) + 900(n+1)\}}{30} + \frac{960n}{30}$

$\Rightarrow S_n = \frac{n(n+1)}{30}\left\{ \frac{(2n+1)(3n^2+3n-1) + 75n(n+1) + 185(2n+1) + 900}{1} \right\} + 32n$

$\Rightarrow S_n = \frac{n(n+1)}{30}\{6n^3+6n^2-2n+3n^2+3n-1+75n^2+75n+370n+185+900\} + 32n$

$\Rightarrow S_n = \frac{n(n+1)}{30}\{6n^3+84n^2+376n+1084\} + 32n$

$\Rightarrow S_n = \frac{n(n+1)}{30}\, 2\{3n^3+42n^2+188n+542\} + 32n$

$\Rightarrow S_n = \frac{n(n+1)}{15}\{3n^3+42n^2+188n+542\} + 32n$ Ans

Q.9 Find the sum of n terms of a series whose general term is $n^2(n^2+1)$.

Sol

$a_n = n^2(n^2+1)$

$\Rightarrow a_n = n^4 + n^2$

Apply Σ sign to b.s, we get

$\Rightarrow \sum a_n = \sum (n^4 + n^2)$

$\Rightarrow S_n = \sum n^4 + \sum n^2$

$\Rightarrow S_n = \dfrac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + \dfrac{n(n+1)(2n+1)}{6}$

take $\dfrac{n(n+1)(2n+1)}{6}$ as common

$\Rightarrow S_n = \dfrac{n(n+1)(2n+1)}{6}\left\{\dfrac{3n^2+3n-1}{5} - 1\right\}$

$\Rightarrow S_n = \dfrac{n(n+1)(2n+1)}{6}\left\{\dfrac{3n^2+3n-1-5}{5}\right\}$

$= \dfrac{n(n+1)(2n+1)}{6}\left\{\dfrac{3n^2+3n-6}{5}\right\}$

$= \dfrac{n(n+1)(2n+1)}{6}\, 3\left\{\dfrac{n^2+n-2}{5}\right\}$

$= \dfrac{n(n+1)(2n+1)(n^2+n-2)}{10}$

$= \dfrac{n(n+1)(2n+1)(n^2+2n-n-2)}{10}$

$= \dfrac{1}{10} n(n+1)(2n+1)\{n(n+2) - 1(n+2)\}$

$= \dfrac{1}{10} n(n+1)(2n+1)(n+2)(n-1)$

$\Rightarrow S_n = \dfrac{1}{10} n(n-1)(n+1)(n+2)(2n+1)$ Ans

Q.10 Sum upto infinity the series, whose general term is $\dfrac{1}{n(n+1)(n+3)}$

Sol

$a_n = \dfrac{1}{n(n+1)(n+3)}$

Note: Difference between the factors is not same.

$\times$ and $\div$ by $n+2$

$\Rightarrow a_n = \dfrac{n+2}{n(n+1)(n+2)(n+3)}$

$\Rightarrow a_n = \dfrac{n}{n(n+1)(n+2)(n+3)} + \dfrac{2}{n(n+1)(n+2)(n+3)}$

$a_n = \dfrac{1}{(n+1)(n+2)(n+3)} + \dfrac{2}{n(n+1)(n+2)(n+3)}$

Now $S_n = c - \left\{\dfrac{a_n \text{ with 1st. factor neglected}}{(\text{No. of factors in each term} - 1)\text{ c.d}}\right\}$

$\Rightarrow S_n = c - \left\{\dfrac{\frac{1}{(n+2)(n+3)}}{(3-1)\,1} + \dfrac{\frac{2}{(n+1)(n+2)(n+3)}}{(4-1)\,1}\right\}$

$\Rightarrow S_n = c - \left\{\dfrac{1}{2(n+2)(n+3)} + \dfrac{2}{3(n+1)(n+2)(n+3)}\right\} \longrightarrow (A)$

to find c put $n=1$

$\Rightarrow S_1 = c - \left\{\dfrac{1}{2(3)(4)} + \dfrac{2}{3(2)(3)(4)}\right\}$

$\Rightarrow \dfrac{1}{1(2)(4)} = c - \left[\dfrac{1}{24} + \dfrac{1}{36}\right] \Rightarrow \dfrac{1}{8} = c - \dfrac{5}{72} \Rightarrow c = \dfrac{1}{8} + \dfrac{5}{72}$

$\Rightarrow \boxed{c = \dfrac{7}{36}}$

Eqn (A) $\Rightarrow S_n = \dfrac{7}{36} - \left\{\dfrac{1}{2(n+2)(n+3)} + \dfrac{2}{3(n+1)(n+2)(n+3)}\right\}$

If $n \to \infty$

$S_\infty = \dfrac{7}{36} - \left[\dfrac{1}{\infty} + \dfrac{2}{\infty}\right]$

$\Rightarrow S_\infty = \dfrac{7}{36} - [0+0] \Rightarrow \boxed{S_\infty = 7/36}$ Ans

168

Exercise # 5.4

Find the sum of the following

Q.1 $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots\cdots$ to n terms

Sol. The general term is

$$a_n = \frac{1}{(2n-1)(2n+1)}$$

By partial fraction

$$\frac{1}{(2n-1)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n+1} \longrightarrow (i)$$

multiply by $(2n-1)(2n+1)$, we get

$$\Rightarrow 1 = A(2n+1) + B(2n-1) \longrightarrow (ii)$$

put $n = -1/2$, we get

$$1 = A\{2(-\tfrac{1}{2})+1\} + B\{2(-\tfrac{1}{2})-1\}$$

$$1 = A(0) + B(-2) \Rightarrow 1 = -2B \Rightarrow \boxed{B = -1/2}$$

Now put $n = 1/2$, we get

$$\Rightarrow 1 = A\{2(\tfrac{1}{2})+1\} + B\{2(\tfrac{1}{2})-1\}$$

$$\Rightarrow 1 = A(2) + B(0) \Rightarrow 1 = 2A \Rightarrow \boxed{A = 1/2}$$

Put the values in eqn (i), we get

$$\Rightarrow \frac{1}{(2n-1)(2n+1)} = \frac{1/2}{2n-1} + \frac{-1/2}{2n+1}$$

$$\Rightarrow \frac{1}{(2n-1)(2n+1)} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}$$

For sum upto n terms take Σ on b.s

$$\Rightarrow \sum \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\sum\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= \frac{1}{2}\left\{\left(1-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \cdots\cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right\}$$

$$= \frac{1}{2}\left\{1 - \frac{1}{2n+1}\right\}$$

$$= \frac{1}{2}\left\{\frac{2n+1-1}{2n+1}\right\} = \frac{1}{2}\left\{\frac{2n}{2n+1}\right\} = \frac{n}{2n+1} \text{ Ans}$$

Q.2 $\frac{1}{1.3.5} + \frac{1}{3.5.7} + \frac{1}{5.7.9} + \cdots\cdots$ to infinity

Sol General term for this series is

$$a_n = \frac{1}{(2n-1)(2n+1)(2n+3)}$$

By partial fraction

$$\frac{1}{(2n-1)(2n+1)(2n+3)} = \frac{A}{2n-1} + \frac{B}{2n+1} + \frac{C}{2n+3} \longrightarrow (i)$$

X'ing by $(2n-1)(2n+1)(2n+3)$, we get

$$\Rightarrow 1 = A(2n+1)(2n+3) + B(2n-1)(2n+3) + C(2n-1)(2n+1) \longrightarrow (ii)$$

Put $n = -1/2$, we get

$$\Rightarrow 1 = A\{2(-\tfrac{1}{2})+1\}\{2(-\tfrac{1}{2})+3\} + B\{2(-\tfrac{1}{2})-1\}\{2(-\tfrac{1}{2})+3\} + C\{2(-\tfrac{1}{2})-1\}\{2(-\tfrac{1}{2})+1\}$$

$$\Rightarrow 1 = A(0)(2) + B(-2)(2) + C(-2)(0)$$

$$\Rightarrow 1 = -4B \Rightarrow \boxed{-\tfrac{1}{4} = B}$$

Now put $n = 1/2$ in eqn (ii), we get

$$\Rightarrow 1 = A\{2(\tfrac{1}{2})+1\}\{2(\tfrac{1}{2})+3\} + 0 + 0$$

$$\Rightarrow 1 = A(2)(4) \Rightarrow 1 = 8A \Rightarrow \boxed{\tfrac{1}{8} = A}$$

Finally put $n = -3/2$ in eqn (ii), we get

$$\Rightarrow 1 = A\{2(-\tfrac{3}{2})+1\}\{2(-\tfrac{3}{2})+3\} + B\{2(-\tfrac{3}{2})-1\}\{2(-\tfrac{3}{2})+3\} + C\{2(-\tfrac{3}{2})-1\}\{2(-\tfrac{3}{2})+1\}$$

$$\Rightarrow 1 = A(-2)(0) + B\{-4\}\{0\} + C\{-4\}\{-2\}$$

$$\Rightarrow 1 = 0 + 0 + 8C \Rightarrow \boxed{\tfrac{1}{8} = C}$$

Put the values in eqn (i), we get

$$\frac{1}{(2n-1)(2n+1)(2n+3)} = \frac{1/8}{2n-1} + \frac{-1/4}{2n+1} + \frac{1/8}{2n+3}$$

So
$$a_n = \frac{1}{8(2n-1)} - \frac{1}{4(2n+1)} + \frac{1}{8(2n+3)}$$

Take Σ on both sides we get

$$\Sigma a_n = \sum\left(\frac{1}{8(2n-1)} - \frac{1}{4(2n+1)} + \frac{1}{8(2n+3)}\right)$$

Put $n = 1, 2, 3, \ldots$

$$\Rightarrow S_n = \left\{\left(\frac{1}{8} - \frac{1}{12} + \frac{1}{40}\right) + \left(\frac{1}{24} - \frac{1}{20} + \frac{1}{56}\right) + \left(\frac{1}{40} - \frac{1}{28} + \frac{1}{72}\right) + \cdots \cdots \left(\frac{1}{\infty} - \frac{1}{\infty} + \frac{1}{\infty}\right)\right\} \text{ Ans}$$

Q:3 $\frac{1}{1\cdot4} + \frac{1}{4\cdot7} + \frac{1}{7\cdot10} + \cdots\cdots$ infinity

Sol The general term is

$$a_n = \frac{1}{(3n-2)(3n+1)}$$

By partial fraction

$$\frac{1}{(3n-2)(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1} \longrightarrow (i)$$

multiply by $(3n-2)(3n+1)$, we get

$$\Rightarrow 1 = A(3n+1) + B(3n-2) \longrightarrow (ii)$$

put $n = -1/3$ in eqn (ii)

$$\Rightarrow 1 = A\{3(-\tfrac{1}{3})+1\} + B\{3(-\tfrac{1}{3})-2\}$$

$$\Rightarrow 1 = A(0) + B(-3) \Rightarrow 1 = -3B \Rightarrow \boxed{B = -1/3}$$

Now put $n = \frac{2}{3}$ in eqn (ii), we get

$$\Rightarrow 1 = A\{3(\tfrac{2}{3})+1\} + B\{3(\tfrac{2}{3})-2\}$$

$$\Rightarrow 1 = A(3) \Rightarrow \boxed{A = 1/3}$$

eqn (i) $\Rightarrow$

$$\frac{1}{(3n-2)(3n+1)} = \frac{1/3}{3n-2} + \frac{-1/3}{3n+1}$$

So

$$a_n = \frac{1}{3}\left\{\frac{1}{3n-2} - \frac{1}{3n+1}\right\}$$

Apply Σ sign to b.s, we get

$$\Sigma a_n = \frac{1}{3}\Sigma\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right) \qquad \text{put } n = 1,2,3,\cdots$$

$$\Rightarrow S_\infty = \frac{1}{3}\left\{\left(\frac{1}{1}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \left(\frac{1}{7}-\frac{1}{10}\right) + \cdots\right\}$$

$$\Rightarrow S_\infty = \frac{1}{3}(1)$$

$$\Rightarrow S_\infty = \frac{1}{3} \quad \text{Ans}$$

Engr. Majid Amin

Q:4 $\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \cdots\cdots$ to infinity

Sol The nth term (general term) can be written as

$$a_n = \frac{1}{n(n+1)(n+2)}$$

By partial Fraction

$$\frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2} \longrightarrow (i)$$

Multiply by $n(n+1)(n+2)$

$$\Rightarrow 1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1) \longrightarrow (ii)$$

Put $n = 0$

$$\Rightarrow 1 = A(0+1)(0+2) + 0 + 0$$

$$\Rightarrow 1 = 2A \Rightarrow \boxed{A = 1/2}$$

Put $n = -1$ in eqn (ii)

$$\Rightarrow 1 = 0 + B(-1)(-1+2) + 0$$

$$\Rightarrow 1 = -B \Rightarrow \boxed{B = -1}$$

Now put $n = -2$ in eqn (ii), we get

$$\Rightarrow 1 = 0 + 0 + C(-2)(-2+1)$$

$$\Rightarrow 1 = 2C \Rightarrow \boxed{C = 1/2}$$

Eqn (i) $\Rightarrow$

$$\frac{1}{n(n+1)(n+2)} = \frac{1/2}{n} + \frac{-1}{n+1} + \frac{1/2}{n+2}$$

$$\Rightarrow \frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

Apply Σ sign, we get

$$\Rightarrow \sum \frac{1}{n(n+1)(n+2)} = \sum\left\{\frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}\right\}$$

put $n = 1, 2, 3, \cdots$

$$\Rightarrow \Sigma a_n = \left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{3}+\frac{1}{8}\right) + \left(\frac{1}{6}-\frac{1}{4}+\frac{1}{10}\right) + \left(\frac{1}{8}-\frac{1}{5}+\frac{1}{12}\right) + \cdots \frac{1}{\infty}$$

$$S_\infty = \frac{1}{4} + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right) - \frac{1}{3} + \left(\frac{1}{8}+\frac{1}{8}-\frac{1}{4}\right) + \cdots$$

$$= \frac{1}{4} + \left(\frac{2}{6}-\frac{1}{3}\right) + \left(\frac{2}{8}-\frac{1}{4}\right) + \cdots$$

$$= \frac{1}{4} + \left(\frac{1}{3}-\frac{1}{3}\right) + \left(\frac{1}{4}-\frac{1}{4}\right) + \cdots$$

$$\Rightarrow S_\infty = 1/4 \quad \text{Ans}$$

Q:5 Find the sum of series $\sum_{k=1}^{n} \frac{1}{9k^2+3k-2}$

Sol $T_k = \frac{1}{9k^2+3k-2}$

$T_k = \frac{1}{9k^2+6k-3k-2}$

$= \frac{1}{3k(3k+2)-1(3k+2)}$

$= \frac{1}{(3k+2)(3k-1)}$

By partial Fraction

$\frac{1}{(3k+2)(3k-2)} = \frac{A}{3k+2} + \frac{B}{3k-1} \longrightarrow (i)$

multiply by $(3k+2)(3k-2)$

$\Rightarrow 1 = A(3k-1) + B(3k+2) \longrightarrow (ii)$

put $k = 1/3$

$\Rightarrow 1 = A\{3(\frac{1}{3})-1\} + B\{3(\frac{1}{3})+2\}$

$1 = A(0) + B(3) \Rightarrow 1 = 3B \Rightarrow \boxed{B = 1/3}$

Now put $k = -2/3$ in eqn (ii), we get

$\Rightarrow 1 = A\{3(-\frac{2}{3})-1\} + B\{3(-\frac{2}{3})+2\}$

$1 = A(-3) + B(0) \Rightarrow 1 = -3A \Rightarrow \boxed{A = -1/3}$

eqn (i) $\Rightarrow \frac{1}{(3k+2)(3k-2)} = \frac{-1/3}{3k+2} + \frac{1/3}{3k-1}$

$\Rightarrow T_k = -\frac{1}{3(3k+2)} + \frac{1}{3(3k-1)}$

$\Rightarrow T_k = \frac{1}{3}\left\{\frac{1}{3k-1} - \frac{1}{3k+2}\right\}$

Apply $\sum_{k=1}^{n}$ sign, we get

$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{3}\left(\frac{1}{3k-1} - \frac{1}{3k+2}\right)$

$= \frac{1}{3}\left\{\left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{8}-\frac{1}{11}\right) + \cdots \left(\frac{1}{3n-1}\right) - \frac{1}{3n+2}\right\}$

$= \frac{1}{3}\left\{\frac{1}{2} - \frac{1}{3n+2}\right\}$

$\Rightarrow \sum T_k = \frac{1}{3}\left\{\frac{3n+2-2}{2(3n+2)}\right\}$

$= \frac{1}{3}\left\{\frac{3n}{2(3n+2)}\right\}$

$= \frac{n}{2(3n+2)}$ Ans

Q:6 Find the sum of the series $\sum_{k=2}^{n} \frac{1}{k^2-k}$

Sol Here $T_k = \frac{1}{k^2-k}$

$\Rightarrow T_k = \frac{1}{k(k-1)}$

By partial Fraction

$\frac{1}{k(k-1)} = \frac{A}{k} + \frac{B}{k-1} \longrightarrow (i)$

Multiply by $k(k-1)$

$\Rightarrow 1 = A(k-1) + Bk \longrightarrow (ii)$

Put $k=0$ in eqn (ii)

$\Rightarrow 1 = A(0-1) + B(0) \Rightarrow 1 = -A \Rightarrow \boxed{A=-1}$

Put $k=1$ in eqn (ii)

$\Rightarrow 1 = A(1-1) + B(1) \Rightarrow 1 = 0 + B \Rightarrow \boxed{B=1}$

eqn (i) $\Rightarrow \frac{1}{k(k-1)} = \frac{-1}{k} + \frac{1}{k-1} \Rightarrow T_k = \frac{-1}{k} + \frac{1}{k-1}$

Apply $\sum_{k=2}^{\infty} T_k = \sum_{k=2}^{\infty}\left(-\frac{1}{k} + \frac{1}{k-1}\right)$

put $k = 2, 3, 4, \cdots \infty$

$= \left(-\cancel{\frac{1}{2}} + \frac{1}{1}\right) + \left(-\frac{1}{3} + \cancel{\frac{1}{2}}\right) + \cdots + \left(-\frac{1}{n} + \frac{1}{n-1}\right)$

$= 1 - \frac{1}{n}$

$\sum_{k=2}^{\infty} T_k = \frac{n-1}{n}$ Ans

Q:7 Find the sum of the series

$$\sum_{k=1}^{n} \frac{1}{k^2+7k+12}$$

Sol Here $T_k = \frac{1}{k^2+7k+12}$

$\Rightarrow T_k = \frac{1}{k^2+3k+4k+12}$

$\Rightarrow T_k = \frac{1}{k(k+3)+4(k+3)}$

$\Rightarrow T_k = \frac{1}{(k+3)(k+4)}$

By partial fraction

$\Rightarrow \frac{1}{(k+3)(k+4)} = \frac{A}{k+3} + \frac{B}{k+4} \longrightarrow (i)$

Multiply by $(k+3)(k+4)$

$\Rightarrow 1 = A(k+4) + B(k+3) \longrightarrow (ii)$

Put $k=-4$

$\Rightarrow 1 = A(-4+4) + B(-4+3) \Rightarrow 1 = B(-1) \Rightarrow \boxed{B=-1}$

Now put $k=-3$ in eqn (ii), we get.

$\Rightarrow 1 = A(-3+4) + B(-3+3)$

$\Rightarrow 1 = A(1) + 0 \Rightarrow \boxed{A=1}$

Eqn (i) $\Rightarrow \frac{1}{(k+3)(k+4)} = \frac{1}{k+3} + \frac{-1}{k+4}$

$\Rightarrow T_k = \frac{1}{k+3} - \frac{1}{k+4}$

Apply $\sum_{k=1}^{n}$ sign

$\Rightarrow \sum_{k=1}^{n} T_k = \sum_{k=1}^{n}\left(\frac{1}{k+3} - \frac{1}{k+4}\right)$

$= \left(\frac{1}{4} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{6}} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n+3} - \frac{1}{n+4}\right)$

$\Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{4} - \frac{1}{n+4}$

$= \frac{1(n+4)-4}{4(n+4)}$

$= \frac{n+4-4}{4(n+4)}$

$= \frac{n}{4(n+4)}$

Q:8 Find the sum of n terms of the series

$$\frac{1}{(x+1)(3x+1)} + \frac{1}{(3x+1)(5x+1)} + \frac{1}{(5x+1)(7x+1)} + \cdots$$

Sol The general term will be

$T_k = \frac{1}{[(2k-1)x+1][(2k+1)x+1]}$

By partial fraction

$\frac{1}{[(2k-1)x+1][(2k+1)x+1]} = \frac{A}{[(2k-1)x+1]} + \frac{B}{[(2k+1)x+1]} \longrightarrow (i)$

Multiply by $[(2k-1)x+1][(2k+1)x+1]$

$\Rightarrow 1 = A[(2k+1)x+1] + B[(2k-1)x+1] \longrightarrow (ii)$

put $x = -\frac{1}{2k-1}$, we get

$\Rightarrow 1 = A\left[(2k+1)\left(\frac{-1}{2k-1}\right)+1\right] + B\left[(2k-1)\left(\frac{-1}{2k-1}\right)+1\right]$

$\Rightarrow 1 = A\left[-\left(\frac{2k+1}{2k-1}\right)+1\right] + 0$

$\Rightarrow 1 = A\left[\frac{-(2k+1)+(2k-1)}{2k-1}\right]$

$\Rightarrow 1 = A\left[\frac{-2k-1+2k-1}{2k-1}\right] \Rightarrow 1 = A\left(\frac{-2}{2k-1}\right)$

$\Rightarrow 1 = A\left(\frac{-2}{-1(-2k+1)}\right) \Rightarrow 1 = A\left(\frac{2}{1-2k}\right)$

$$\Rightarrow A = \frac{1-2k}{2}$$

Now put $x = -\frac{1}{2k+1}$ in eqn (ii), we get

$$\Rightarrow 1 = A\left[(2k+1)\left(\frac{-1}{2k+1}\right)+1\right] + B\left[(2k-1)\left(\frac{-1}{2k+1}\right)+1\right]$$

$$\Rightarrow 1 = A\left[0\right] + B\left[-\frac{2k-1}{2k+1}+1\right]$$

$$\Rightarrow 1 = B\left(\frac{-(2k-1)+2k+1}{2k+1}\right)$$

$$\Rightarrow 1 = B\left(\frac{-2k+1+2k+1}{2k+1}\right) \Rightarrow 1 = B\left(\frac{2}{2k+1}\right) \Rightarrow B = \frac{2k+1}{2}$$

Put the values of A and B in eqn (i)

$$\frac{1}{[(2k-1)x+1][(2k+1)x+1]} = \frac{\frac{1-2k}{2}}{[(2k-1)x+1]} + \frac{\frac{1+2k}{2}}{(2k+1)x+1}$$

$$\Rightarrow T_k = \frac{1}{2}\left[\frac{1-2k}{(2k-1)x+1} + \frac{1+2k}{(2k+1)x+1}\right]$$

Apply $\sum_{k=1}^{n}$

$$\Rightarrow \sum_{k=1}^{n} T_k = \frac{1}{2}\sum_{k=1}^{n}\left[\frac{1-2k}{(2k-1)x+1} + \frac{1+2k}{(2k+1)x+1}\right]$$

put $k = 1, 2, 3, \ldots$

$$\sum_{k=1}^{n} T_k = \frac{1}{2}\left\{\left(\frac{-1}{x+1}+\frac{3}{3x+1}\right) + \left(-\frac{3}{3x+1}+\frac{5}{5x+1}\right) + \cdots + \left(\frac{1-2n}{(2n-1)x+1} + \frac{2n+1}{(2n+1)x+1}\right)\right\}$$

$$= \frac{1}{2}\left\{-\frac{1}{x+1} + \frac{2n+1}{(2n+1)x+1}\right\}$$

$$\Rightarrow S_n = \frac{1}{2}\left\{\frac{-\{(2n+1)x+1\}+(2n+1)(x+1)}{(x+1)\{(2n+1)x+1\}}\right\}$$

$$\Rightarrow S_n = \frac{1}{2}\left\{\frac{-(2n+1)x-1+(2n+1)x+(2n+1)}{(x+1)\{(2n+1)x+1\}}\right\}$$

$$\Rightarrow S_n = \frac{1}{2}\left\{\frac{-1+(2n+1)}{(x+1)\{(2n+1)x+1\}}\right\}$$

$$S_n = \frac{1}{2}\left\{\frac{2n}{(x+1)\{(2n+1)x+1\}}\right\}$$

$$\Rightarrow S_n = \frac{n}{(x+1)\{(2n+1)x+1\}} \quad \text{Ans}$$

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